

# Estimates for the maximal singular integral in terms of the singular integral: the case of even kernels

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## Abstract

Let  $T$  be a smooth homogeneous Calderón-Zygmund singular integral operator in  $\mathbb{R}^n$ . In this paper we study the problem of controlling the maximal singular integral  $T^*f$  by the singular integral  $Tf$ . The most basic form of control one may consider is the estimate of the  $L^2(\mathbb{R}^n)$  norm of  $T^*f$  by a constant times the  $L^2(\mathbb{R}^n)$  norm of  $Tf$ . We show that if  $T$  is an even higher order Riesz transform, then one has the stronger pointwise inequality  $T^*f(x) \leq CM(Tf)(x)$ , where  $C$  is a constant and  $M$  is the Hardy-Littlewood maximal operator. We prove that the  $L^2$  estimate of  $T^*$  by  $T$  is equivalent, for even smooth homogeneous Calderón-Zygmund operators, to the pointwise inequality between  $T^*$  and  $M(T)$ . Our main result characterizes the  $L^2$  and pointwise inequalities in terms of an algebraic condition expressed in terms of the kernel  $\frac{\Omega(x)}{|x|^n}$  of  $T$ , where  $\Omega$  is an even homogeneous function of degree 0, of class  $C^\infty(S^{n-1})$  and with zero integral on the unit sphere  $S^{n-1}$ . Let  $\Omega = \sum P_j$  be the expansion of  $\Omega$  in spherical harmonics  $P_j$  of degree  $j$ . Let  $A$  stand for the algebra generated by the identity and the smooth homogeneous Calderón-Zygmund operators. Then our characterizing condition states that  $T$  is of the form  $R \circ U$ , where  $U$  is an invertible operator in  $A$  and  $R$  is a higher order Riesz transform associated with a homogeneous harmonic polynomial  $P$  which divides each  $P_j$  in the ring of polynomials in  $n$  variables with real coefficients.

## 1. Introduction

Let  $T$  be a smooth homogeneous Calderón-Zygmund singular integral operator on  $\mathbb{R}^n$  with kernel

$$(1) \quad K(x) = \frac{\Omega(x)}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where  $\Omega$  is a (real valued) homogeneous function of degree 0 whose restriction to the unit sphere  $S^{n-1}$  is of class  $C^\infty(S^{n-1})$  and satisfies the cancellation property

$$\int_{|x|=1} \Omega(x) d\sigma(x) = 0,$$

with  $\sigma$  being the normalized surface measure on  $S^{n-1}$ . Recall that  $Tf$  is the principal value convolution operator

$$(2) \quad Tf(x) = P.V. \int f(x-y)K(y)dy \equiv \lim_{\epsilon \rightarrow 0} T^\epsilon f(x),$$

where  $T^\epsilon$  is the truncation at level  $\epsilon$  defined by

$$T^\epsilon f(x) = \int_{|y-x|>\epsilon} f(x-y)K(y)dy.$$

As we know, the limit in (2) exists for almost all  $x$  for  $f$  in  $L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ .

Let  $T^*$  be the maximal singular integral

$$T^*f(x) = \sup_{\epsilon > 0} |T^\epsilon f(x)|, \quad x \in \mathbb{R}^n.$$

In this paper we consider the problem of controlling  $T^*f$  by  $Tf$ . The most basic form of control one may think of is the  $L^2$  estimate

$$(3) \quad \|T^*f\|_2 \leq C\|Tf\|_2, \quad f \in L^2(\mathbb{R}^n).$$

Another way of saying that  $T^*f$  is dominated by  $Tf$ , apparently much stronger, is provided by the pointwise inequality

$$(4) \quad T^*f(x) \leq C M(Tf)(x), \quad x \in \mathbb{R}^n,$$

where  $M$  denotes the Hardy-Littlewood maximal operator. Notice that (4) may be viewed as an improved version of classical Cotlar's inequality

$$T^*f(x) \leq C(M(Tf)(x) + Mf(x)), \quad x \in \mathbb{R}^n,$$

because the term involving  $Mf$  is missing in the right-hand side of (4).

We prove that if  $T$  is an even higher order Riesz transform, then (4) holds. Recall that  $T$  is a higher order Riesz transform if its kernel is given by a function  $\Omega$  of the form

$$\Omega(x) = \frac{P(x)}{|x|^d}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

with  $P$  a homogeneous harmonic polynomial of degree  $d \geq 1$ . If  $P(x) = x_j$ , then one obtains the  $j$ -th Riesz transform  $R_j$ . If the homogeneous polynomial  $P$  is not required to be harmonic, but has still zero integral on the unit sphere, then we call  $T$  a polynomial operator.

Thus, if  $T = R$  is an even higher order Riesz transform, one has the weak  $L^1$  type inequality

$$(5) \quad \|R^*f\|_{1,\infty} \leq C\|Rf\|_1,$$

which combined with the classical weak  $L^1$  type estimate

$$\|R^*f\|_{1,\infty} \leq C\|f\|_1,$$

yields the sharp inequality

$$\|R^*f\|_{1,\infty} \leq C \min\{\|f\|_1, \|Rf\|_1\}.$$

In [MV06] one proved (4) for the Beurling transform in the plane and also that (5) fails for the Riesz transforms  $R_j$ . Therefore the assumption that the operator is even is crucial.

The question of estimating  $T^*f$  by  $Tf$  was first raised in [MV06]. The problem originated in an attempt to gain a better understanding of how one can obtain almost everywhere existence of principal values of truncated singular integrals from  $L^2$  boundedness, for underlying measures more general than the Lebesgue measure in  $\mathbb{R}^n$ . This in turn is motivated by a problem of David and Semmes [DS91], which consists in deriving uniform rectifiability of a  $d$ -dimensional Ahlfors regular subset of  $\mathbb{R}^n$  from the  $L^2$  boundedness of the Riesz kernel of homogeneity  $-d$  with respect to  $d$ -dimensional Hausdorff measure on the set. For more details on that see the last section of [MV09].

Our main result states that for even operators, inequalities (3) and (4) are equivalent to an algebraic condition involving the expansion of  $\Omega$  in spherical harmonics. This condition may be very easily checked in practice and so, in particular, we can produce extremely simple examples of even polynomial operators for which (3) and (4) fail. For these operators no control of  $T^*f$  by  $Tf$  seems to be known. To state our main result we need to introduce a piece of notation.

Recall that  $\Omega$  has an expansion in spherical harmonics; that is,

$$(6) \quad \Omega(x) = \sum_{j=1}^{\infty} P_j(x), \quad x \in S^{n-1},$$

where  $P_j$  is a homogeneous harmonic polynomial of degree  $j$ . If  $\Omega$  is even, then only the  $P_j$  of even degree  $j$  may be nonzero.

An important role in this paper will be played by the algebra  $A$  consisting of the bounded operators on  $L^2(\mathbb{R}^n)$  of the form

$$\lambda I + S,$$

where  $\lambda$  is a real number and  $S$  a smooth homogeneous Calderón-Zygmund operator.

Our main result reads as follows.

**THEOREM.** *Let  $T$  be an even smooth homogeneous Calderón-Zygmund operator with kernel (1) and assume that  $\Omega$  has the expansion (6). Then the following are equivalent.*

- (i)  $T^*f(x) \leq C M(Tf)(x), \quad x \in \mathbb{R}^n.$
- (ii)  $\|T^*f\|_2 \leq C \|Tf\|_2, \quad f \in L^2(\mathbb{R}^n).$

- (iii) *The operator  $T$  can be factorized as  $T = R \circ U$ , where  $U$  is an invertible operator in the algebra  $A$  and  $R$  is a higher order Riesz transform associated with a harmonic homogeneous polynomial  $P$  which divides each  $P_j$  in the ring of polynomials in  $n$  variables with real coefficients.*

Two remarks are in order.

*Remark 1.* Observe that condition (iii) is algebraic in nature. This is one of the reasons that makes the proof difficult. Condition (iii) can be reformulated in a more concrete fashion as follows. Assume that the expansion of  $\Omega$  in spherical harmonics is

$$\Omega(x) = \sum_{j=j_0}^{\infty} P_{2j}(x), \quad P_{2j_0} \neq 0.$$

Then (iii) is equivalent to the following condition.

- (iv) *For each  $j$  there exists a homogeneous polynomial  $Q_{2j-2j_0}$  of degree  $2j-2j_0$  such that  $P_{2j} = P_{2j_0} Q_{2j-2j_0}$  and  $\sum_{j=j_0}^{\infty} \gamma_{2j} Q_{2j-2j_0}(\xi) \neq 0$ ,  $\xi \in S^{n-1}$ .*

Here, for a positive integer  $j$ , we have set

$$(7) \quad \gamma_j = i^{-j} \pi^{\frac{n}{2}} \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{n+j}{2})}.$$

The quantities  $\gamma_j$  appear in the computation of the Fourier multiplier of the higher order Riesz transform  $R$  with kernel given by a homogeneous harmonic polynomial  $P$  of degree  $j$ . One has (see [Ste70, p. 73])

$$\widehat{Rf}(\xi) = \gamma_j \frac{P(\xi)}{|\xi|^j} \hat{f}(\xi), \quad f \in L^2(\mathbb{R}^n).$$

As we will show later, the series  $\sum_{j=j_0}^{\infty} \gamma_{2j} Q_{2j-2j_0}(x)$  is convergent in  $C^\infty(S^{n-1})$ . If  $U$  is defined as the operator in the algebra  $A$  whose Fourier multiplier is  $\gamma_{2j_0}^{-1}$  times the sum of the preceding series, then  $T = R \circ U$  for the higher order Riesz transform  $R$  given by the polynomial  $P_{2j_0}$ . This shows that (iii) follows from (iv).

To show that (iii) implies (iv) we prove that  $P = \lambda P_{2j_0}$  for some real number  $\lambda \neq 0$ . Since  $P$  divides  $P_{2j_0}$  by assumption, we only need to show that the degree  $d$  of  $P$  must be  $2j_0$ . Now, let  $\mu(\xi)$  denote the Fourier multiplier of  $U$ , so that  $\mu$  is a smooth function with no zeros on the sphere. The Fourier multiplier of  $T$  is

$$\sum_{j=j_0}^{\infty} \gamma_{2j} P_{2j}(\xi) = \gamma_d P(\xi) \mu(\xi), \quad \xi \in S^{n-1}.$$

If  $d$  is less than  $2j_0$ , then  $P$  is orthogonal to all  $P_{2j}$  [Ste70, p. 69], and so

$$\int_{|\xi|=1} P(\xi)^2 \mu(\xi) d\xi = 0,$$

which yields  $P(\xi) = 0$ ,  $|\xi| = 1$ , a contradiction.

*Remark 2.* Condition (iii) is rather easy to check in practice. For instance, take  $n = 2$  and consider the polynomial of fourth degree

$$P(x, y) = xy + x^4 + y^4 - 6x^2y^2.$$

The polynomial operator associated with  $P$  does not satisfy (i) or (ii), because the definition of  $P$  above is also the spherical harmonics expansion of  $P$  and  $xy$  clearly does not divide  $x^4 + y^4 - 6x^2y^2$ . Section 7 contains other examples of polynomial operators that do not satisfy (i) or (ii).

On the other hand, the polynomial operator associated with

$$P(x, y) = xy + x^3y - y^3x$$

does satisfy (i) and (ii), but this is not the case for the operator determined by

$$P(x, y) = xy + 2(x^3y - y^3x),$$

although  $xy$  obviously divides  $x^3y - y^3x$ . See Section 8 for the details.

Thus the condition on  $\Omega$  so that  $T$  satisfies (i) or (ii) is rather subtle.

Having clarified the statement of the theorem and some of its implications, we now say a few words on the proofs and the organization of the paper.

We devote Sections 2, 3 and 4 to the proof of “(iii) implies (i)”, which we call the sufficient condition. In Section 2 we prove that the even higher order Riesz transforms satisfy (i). Section 3 is devoted to the proof of the sufficient condition for polynomial operators. The argument is an extension of that used in the previous section. The drawback is that we lose control on the dependence of the constants on the degree of the polynomial. The main difficulty we have to overcome in Section 4 to complete the proof of the sufficient condition in the general case, is to find a second approach to the polynomial case which gives some estimates with constants independent of the degree of the polynomial. This allows us to use a compactness argument to finish the proof. It is an intriguing fact that the approach in Section 3 cannot be dispensed with, because it provides certain properties which are vital for the final argument and do not follow otherwise.

In Sections 5 and 6 we prove the necessary condition; that is, “(ii) implies (iii)”. In Section 5 we deal with the polynomial case. Analysing the inequality (ii) via Plancherel at the frequency side, we obtain various inclusion relations among zero sets of certain polynomials. This requires a considerable combinatorial effort for reasons that will become clear later on. In Maple, we found a

formidable ally in formulating the right identities which were needed, which we proved rigorously afterwards. In a second step we solve the division problem which leads us to (iii) by a recurrent argument with some algebraic geometry ingredients, the Hilbert's Nullstellensatz in particular. The question of independence on the degree of the polynomial appears again, this time related to the coefficients of certain expansions. We deal with this problem in Section 6. Section 7 is devoted to the proof of the intricate combinatorial lemmas used in the previous sections. In Section 8 we discuss some examples and we ask a couple of questions that we have not been able to answer.

Our methods are a combination of classical Fourier analysis techniques and Calderón-Zygmund theory with potential theoretic ideas coming from our previous work [MO90], [MPV05], [MV88], [MNOV96], [Ver87] and [Ver01].

As was discovered in [MV06], there is remarkable difference between the odd and even cases for the problem we consider. To keep this article a reasonable length, we decided to only deal here with the even case, which is more difficult, because one needs special  $L^\infty$  estimates for singular integrals, which hold only in the even case. The results for the odd case will be published elsewhere [MOPV10].

The  $L^\infty$  estimates mentioned above are not obvious even for the simplest even homogeneous Calderón-Zygmund operator, the Beurling transform, which plays an important role in planar quasiconformal mapping theory. An application of our estimates to planar quasiconformal mappings is given in [MOV09].

## 2. Even higher order Riesz transforms

In this section we prove that if  $T$  is an even higher order Riesz transform, then

$$(8) \quad T^* f(x) \leq C M(Tf)(x), \quad x \in \mathbb{R}^n.$$

Let  $B$  be the open ball of center 0 and radius 1,  $\partial B$  its boundary and  $\overline{B}$  its closure. In proving (8) we will encounter the following situation. We are given a function  $\varphi$  defined by different formulae in  $B$  and  $\mathbb{R}^n \setminus \overline{B}$ , which is differentiable up to order  $N$  on  $B \cup (\mathbb{R}^n \setminus \overline{B})$  and whose derivatives up to order  $N - 1$  extend continuously up to  $\partial B$ . The question is to compare the distributional derivatives of order  $N$  with the expressions one gets on  $B$  and  $\mathbb{R}^n \setminus \overline{B}$  by taking ordinary derivatives. The next simple lemma is a sample of what we need.

**LEMMA 1.** *Let  $\varphi$  be a continuously differentiable function on  $B \cup (\mathbb{R}^n \setminus \overline{B})$  which extends continuously to  $\partial B$ . Then we have the identity*

$$\partial_j \varphi = \partial_j \varphi(x) \chi_B(x) + \partial_j \varphi(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x),$$

where the left-hand side is the  $j$ -th distributional derivative of  $\varphi$ .

*Proof.* Let  $\psi$  be a test function. Then

$$\langle \partial_j \varphi, \psi \rangle = - \int \varphi \partial_j \psi = - \int_B \varphi \partial_j \psi - \int_{\mathbb{R}^n \setminus \overline{B}} \varphi \partial_j \psi.$$

Now apply Green-Stokes' theorem to the domains  $B$  and  $\mathbb{R}^n \setminus \overline{B}$  to move the derivatives from  $\psi$  to  $\varphi$ . The boundary terms cancel precisely because of the continuity of  $\varphi$  on  $\partial B$ , and we get

$$\langle \partial_j \varphi, \psi \rangle = \int (\chi_B \partial_j \varphi + \chi_{\mathbb{R}^n \setminus \overline{B}} \partial_j \varphi) \psi \, dx. \quad \square$$

We need an analog of the previous statement for second order derivatives and radial functions, which is the case we take up in the next corollary.

**COROLLARY 2.** *Assume that  $\varphi$  is a radial function of the form*

$$\varphi(x) = \varphi_1(|x|)\chi_B(x) + \varphi_2(|x|)\chi_{\mathbb{R}^n \setminus \overline{B}}(x),$$

where  $\varphi_1$  is continuously differentiable on  $[0, 1)$  and  $\varphi_2$  on  $(1, \infty)$ . Let  $L$  be a second order differential operator with constant coefficients. Then the distribution  $L\varphi$  satisfies

$$L\varphi = L\varphi(x)\chi_B(x) + L\varphi(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x),$$

provided  $\varphi_1, \varphi_1', \varphi_2$  and  $\varphi_2'$  extend continuously to the point 1 and the two conditions

$$\varphi_1(1) = \varphi_2(1), \quad \varphi_1'(1) = \varphi_2'(1)$$

are satisfied.

*Proof.* The proof reduces to applying Lemma 1 twice. Before the second application one should remark that the hypothesis  $\varphi_1'(1) = \varphi_2'(1)$  gives the continuity of all first order partial derivatives of  $\varphi$ . □

We proceed now to describe in detail the main argument for the proof of (8). By translating and dilating, one reduces the proof of (8) to

$$(9) \quad |T^1 f(0)| \leq C M(Tf)(0),$$

where

$$T^1 f(0) = \int_{|y|>1} f(y)K(y) \, dy$$

is the truncated integral at level 1. Recall that the kernel of our singular integral is

$$K(x) = \frac{\Omega(x)}{|x|^n} = \frac{P(x)}{|x|^{n+d}},$$

where  $P$  is an even homogeneous harmonic polynomial of degree  $d \geq 2$ . The idea is to obtain an identity of the form

$$(10) \quad K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T(b)(x)$$

for some measurable bounded function  $b$  supported on  $B$ . Once (10) is at our disposition we get, for  $f$  in some  $L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ ,

$$\begin{aligned} T^1 f(0) &= \int \chi_{\mathbb{R}^n \setminus \overline{B}}(y) K(y) f(y) dy \\ &= \int T(b)(y) f(y) dy \\ &= \int_B b(y) T f(y) dy, \end{aligned}$$

and so (9) follows with  $C = V_n \|b\|_\infty$ ,  $V_n$  being the volume of the unit ball of  $\mathbb{R}^n$ .

Let us turn our attention to the proof of (10). Set  $d = 2N$  and let  $E$  be the standard fundamental solution of the  $N$ -th power  $\Delta^N$  of the Laplacean. Consider the function

$$(11) \quad \varphi(x) = E(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x) + (A_0 + A_1 |x|^2 + \dots + A_{d-1} |x|^{2d-2}) \chi_B(x),$$

where the constants  $A_0, A_1, \dots, A_{d-1}$  are chosen as follows. Since  $\varphi(x)$  is radial, the same is true of  $\Delta^j \varphi$  for each positive integer  $j$ . Thus, in order to apply the preceding corollary  $N$  times, one needs  $2N = d$  conditions, which (uniquely) determine  $A_0, A_1, \dots, A_{d-1}$ . Therefore, for some constants  $\alpha_1, \alpha_2, \dots, \alpha_{N-1}$ ,

$$(12) \quad \Delta^N \varphi = (\alpha_0 + \alpha_1 |x|^2 + \dots + \alpha_{N-1} |x|^{2(N-1)}) \chi_B(x) = b(x),$$

where the last identity is a definition of  $b$ . Since

$$\varphi = E \star \Delta^N \varphi,$$

taking derivatives of both sides we obtain

$$(13) \quad P(\partial) \varphi = P(\partial) E \star \Delta^N \varphi.$$

To compute  $P(\partial)E$  we take the Fourier transform

$$\widehat{P(\partial)E}(\xi) = P(i\xi) \hat{E}(\xi) = \frac{P(\xi)}{|\xi|^d}.$$

On the other hand, as is well known ([Ste70, p. 73],

$$P.V. \frac{P(x)}{|x|^{n+d}} (\xi) = \gamma_d \frac{P(\xi)}{|\xi|^d}.$$

See (7) for the precise value of  $\gamma_d$ , which is not important now. We conclude that, for some constant  $c_d$  depending on  $d$ ,

$$P(\partial)E = c_d P.V. \frac{P(x)}{|x|^{n+d}}.$$

Thus

$$P(\partial)\varphi = c_d P.V. \frac{P(x)}{|x|^{n+d}} \star \Delta^N \varphi = c_d T(b).$$



The only thing left is the computation of  $P(\partial) \varphi$ . We have, by Corollary 2,  $P(\partial) \varphi = c_d K(x) \chi_{\mathbb{R}^n \setminus \bar{B}} + P(\partial)(A_0 + A_1 |x|^2 + \dots + A_{d-1} |x|^{2d-2})(x) \chi_B(x)$ , and so, to complete the proof of (10), we only have to show that

$$(14) \quad P(\partial)(|x|^{2j}) = 0, \quad 1 \leq j \leq d - 1.$$

Notice that the degree of  $P$  may be much smaller than the degree of  $|x|^{2j}$ , and so the previous identity is not obvious. Taking the Fourier transform we obtain

$$\widehat{P(\partial)(|x|^{2j})} = c_j P(\xi) \Delta^j \delta,$$

where  $\delta$  is the Dirac delta at the origin and  $c_j$  is a constant depending on  $j$ . Let  $\psi$  be a test function. Then, since  $P$  is harmonic,

$$\begin{aligned} \langle P(\xi) \Delta^j \delta, \psi \rangle &= \langle \Delta^j \delta, P(\xi) \psi(\xi) \rangle \\ &= \langle \Delta^{j-1} \delta, 2 \nabla P(\xi) \cdot \nabla \psi(\xi) + P(\xi) \Delta \psi(\xi) \rangle. \end{aligned}$$

Iterating the previous computation, we obtain that

$$\langle P(\xi) \Delta^j \delta, \psi \rangle = \langle \delta, D(\xi) \rangle = D(0),$$

where  $D$  is a linear combination of products of the form  $\partial^\alpha \psi(\xi) \partial^\beta P(\xi)$ , with multi-indices  $\beta$  of length  $|\beta| \leq j \leq d - 1$ . Therefore  $\partial^\beta P(\xi)$  is a homogeneous polynomial of degree at least  $d - j \geq 1$ , and so  $\partial^\beta P(0) = 0$ . This yields  $D(0) = 0$  and completes the proof of (14) and, thus, of (10).

In fact, (14) follows immediately from an identity of Lyons and Zumbun [LZ94] which will be discussed in the next section. However, we prefer to present here the above independent natural argument for the reader's convenience.

### 3. Proof of the sufficient condition: the polynomial case

In this section we assume that  $T$  is an even polynomial operator. This amounts to saying that for some even integer  $2N$ ,  $N \geq 1$ , the function  $|x|^{2N} \Omega(x)$  is a homogeneous polynomial of degree  $2N$ . Such a polynomial may be written as [Ste70, p. 69]

$$|x|^{2N} \Omega(x) = P_2(x) |x|^{2N-2} + \dots + P_{2j}(x) |x|^{2N-2j} + \dots + P_{2N}(x),$$

where  $P_{2j}$  is a homogeneous harmonic polynomial of degree  $2j$ ,  $1 \leq j \leq N$ . In other words, the expansion of  $\Omega(x)$  in spherical harmonics is

$$\Omega(x) = P_2(x) + P_4(x) + \dots + P_{2N}(x), \quad |x| = 1.$$

As in the previous section, we want to obtain an expression for the kernel  $K(x)$  off the unit ball  $B$ . For this we need the differential operator  $Q(\partial)$  defined by the polynomial

$$Q(x) = \gamma_2 P_2(x) |x|^{2N-2} + \dots + \gamma_{2j} P_{2j}(x) |x|^{2N-2j} + \dots + \gamma_{2N} P_{2N}(x).$$

If  $E$  is the standard fundamental solution of  $\Delta^N$ , then

$$Q(\partial)E = P.V. K(x),$$

which may be easily verified by taking the Fourier transform of both sides.

Now take the function  $\varphi$  of the previous section. We have  $\varphi = E \star \Delta^N \varphi$  and thus

$$Q(\partial)\varphi = Q(\partial)E \star \Delta^N \varphi = P.V. K(x) \star b = T(b),$$

where  $b$  is defined as  $\Delta^N \varphi$ . On the other hand, by Corollary 2,

(15)

$$Q(\partial)\varphi = K(x)\chi_{\mathbb{R}^n \setminus \bar{B}} + Q(\partial)(A_0 + A_1|x|^2 + \dots + A_{2N-1}|x|^{4N-2})(x)\chi_B(x).$$

Contrary to what happened in the previous section, the term

$$S(x) := -Q(\partial)(A_0 + A_1|x|^2 + \dots + A_{2N-1}|x|^{4N-2})(x)$$

does not necessarily vanish, the reason being that now  $Q$  does not need to be harmonic.

Our goal is to find a function  $\beta \in L^\infty(\mathbb{R}^n)$ , satisfying the decay estimate

$$(16) \quad |\beta(x)| \leq \frac{C}{|x|^{n+1}}, \quad |x| \geq 2$$

and

$$(17) \quad S(x)\chi_B(x) = T(\beta)(x).$$

Once this is achieved, the proof of (i) is just a variation of the argument presented in Section 2, which we now explain. By (15), the definition of  $S(x)$  and (17), we get

$$(18) \quad K(x)\chi_{\mathbb{R}^n \setminus \bar{B}}(x) = T(b)(x) + T(\beta)(x).$$

Set  $\gamma = b + \beta$ . We show (9) by arguing as follows. For  $f$  in any  $L^p(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , we have

$$\begin{aligned} T^1 f(0) &= \int \chi_{\mathbb{R}^n \setminus \bar{B}}(y) K(y) f(y) dy \\ &= \int T(\gamma)(y) f(y) dy \\ &= \int \gamma(y) T f(y) dy \\ &= \int_{2B} \gamma(y) T f(y) dy + \int_{\mathbb{R}^n \setminus 2B} \gamma(y) T f(y) dy. \end{aligned}$$

Thus, by the decay inequality (16) with  $\beta$  replaced by  $\gamma$ ,

$$\begin{aligned} |T^1 f(0)| &\leq C \left( \|\gamma\|_\infty \frac{1}{|2B|} \int_{2B} |T f(y)| dy + \int_{\mathbb{R}^n \setminus 2B} \frac{|T f(y)|}{|y|^{n+1}} dy \right) \\ &\leq C M(T f)(0). \end{aligned}$$

To construct  $\beta$  satisfying (16) and (17) we resort to our hypothesis, condition (iii) in the theorem, which says that  $T = R \circ U$ , where  $R$  is a higher order Riesz transform,  $U$  is an invertible operator in the algebra  $A$  and the polynomial  $P$  which determines  $R$  divides  $P_{2j}$ ,  $1 \leq j \leq N$ , in the ring of polynomials in  $n$  variables with real coefficients. The construction of  $\beta$  is performed in two steps.

The first step consists in proving that there exists a function  $\beta_1$  in  $L^\infty(B)$ , satisfying a Lipschitz condition of order 1 on  $B$ ,  $\int \beta_1(x) dx = 0$  and such that

$$(19) \quad S(x)\chi_B(x) = R(\beta_1)(x).$$

Later, it will become clear how the Lipschitz condition on  $\beta_1$  is used. To prove (19) we need an explicit formula for  $S(x)$ , and for that we will make use of the following formula of Lyons and Zumbrun [LZ94].

LEMMA 3. *Let  $L$  be a homogeneous polynomial of degree  $l$ , and let  $f$  be a smooth function of one variable. Then*

$$L(\partial)f(r) = \sum_{\nu \geq 0} \frac{1}{2^\nu \nu!} \Delta^\nu L(x) \left(\frac{1}{r} \frac{\partial}{\partial r}\right)^{l-\nu} f(r), \quad r = |x|.$$

An immediate consequence of Lemma 3 is:

LEMMA 4. *Let  $P_{2j}$  a homogeneous harmonic polynomial of degree  $2j$ , and let  $k$  be a nonnegative integer. Then*

$$P_{2j}(\partial)(|x|^{2k}) = 2^{2j} \frac{k!}{(k-2j)!} P_{2j}(x) |x|^{2(k-2j)} \quad \text{if } 2j \leq k,$$

and

$$P_{2j}(\partial)(|x|^{2k}) = 0, \quad \text{if } 2j > k.$$

On the other hand, a routine computation gives

$$(20) \quad \Delta^j(|x|^{2k}) = 4^j \frac{j! k!}{(k-j)!} \binom{\frac{n}{2} + k - 1}{j} |x|^{2(k-j)}, \quad k \geq j,$$

and

$$(21) \quad \Delta^j(|x|^{2k}) = 0, \quad k < j.$$

By Lemma 4, (20) and (21) we get that for some constants  $c_{jk}$ , in view of the definitions of  $Q(x)$  and  $S(x)$ , one has

$$(22) \quad S(x) = \sum_{j=1}^{N-1} \sum_{k=j}^{N-1} c_{jk} P_{2j}(x) |x|^{2(k-j)}.$$

Therefore it suffices to prove (19) with  $S(x)$  replaced by  $P_{2j}(x) |x|^{2k}$  for  $1 \leq j \leq N$  and each nonnegative integer  $k$ . The idea is to look for an appropriate

function  $\psi$  such that

$$(23) \quad P(\partial)\psi(x) = P_{2j}(x) |x|^{2k} \chi_B(x).$$

Indeed, if (23) holds and  $2d$  is the degree of  $P$ , then

$$\psi = E \star \Delta^d \psi,$$

provided  $E$  is the fundamental solution of  $\Delta^d$  and  $\psi$  is good enough. Hence

$$P(\partial)\psi = P(\partial)E \star \Delta^d \psi = c P.V. \frac{P(x)}{|x|^{n+2d}} \star \Delta^d \psi = R(\beta_1)$$

if  $\beta_1 = c \Delta^d \psi$ . The conclusion is that we have to solve (23) in such a way that  $\Delta^d \psi$  is supported on  $B$ , is a Lipschitz function on  $B$  and has zero integral.

Taking Fourier transforms in (23), we get

$$(24) \quad (-1)^d P(\xi) \widehat{\psi}(\xi) = (-1)^{j+k} P_{2j}(\partial) \Delta^k (\widehat{\chi_B}(\xi)).$$

Recall that for  $m = n/2$ , one has [Gra04, A-10]

$$\widehat{\chi_B}(\xi) = \frac{J_m(\xi)}{|\xi|^m}, \quad \xi \in \mathbb{R}^n,$$

where  $J_m$  is the Bessel function of order  $m$ . Set

$$G_\lambda(\xi) = \frac{J_\lambda(\xi)}{|\xi|^\lambda}, \quad \xi \in \mathbb{R}^n, \quad \lambda > 0.$$

In computing the right-hand side of (24), we apply Lemma 3 to  $L(x) = P_{2j}(x) |x|^{2k}$  and  $f(r) = G_m(r)$ , and we get

$$P(\xi) \widehat{\psi}(\xi) = (-1)^{j+k+d} \sum_{\nu \geq 0} \frac{(-1)^\nu}{2^\nu \nu!} \Delta^\nu (P_{2j}(\xi) |\xi|^{2k}) G_{m+2j+2k-\nu}(\xi),$$

owing to the well-known formula, e.g., [Gra04, A-6],

$$\frac{1}{r} \frac{d}{dr} G_\lambda(r) = -G_{\lambda+1}(r), \quad r > 0, \quad \lambda > 0.$$

Since  $P_{2j}(\xi)$  is homogeneous of degree  $2j$ ,  $\nabla P_{2j}(\xi) \cdot \xi = 2j P_{2j}(\xi)$ , and hence one may readily show by an inductive argument that

$$\Delta^\nu (P_{2j}(\xi) |\xi|^{2k}) = a_{jk\nu} P_{2j}(\xi) |\xi|^{2(k-\nu)}$$

for some constants  $a_{jk\nu}$ . Thus, for some other constants  $a_{jk\nu}$ , we get

$$(25) \quad P(\xi) \widehat{\psi}(\xi) = \sum_{\nu \geq 0} a_{jk\nu} P_{2j}(\xi) |\xi|^{2(k-\nu)} G_{m+2j+2k-\nu}(\xi).$$

By hypothesis,  $P$  divides  $P_{2j}$  in the ring of polynomials in  $n$  variables, and so

$$P_{2j}(\xi) = P(\xi) Q_{2j-2d}(\xi)$$

for some homogeneous polynomial  $Q_{2j-2d}$  of degree  $2j - 2d$ . Cancelling out the factor  $P(\xi)$  in (25), we conclude that

$$\widehat{\psi}(\xi) = Q_{2j-2d}(\xi) \sum_{\nu=0}^k a_{jk\nu} |\xi|^{2(k-\nu)} G_{m+2j+2k-\nu}(\xi).$$

Since [Gra04, A-10]

$$((1 - |x|^2)^\lambda \chi_B(x))(\xi) = c_\lambda G_{m+\lambda}(\xi),$$

we finally obtain

$$\psi(x) = Q_{2j-2d}(\partial) \sum_{\nu=0}^k a_{jk\nu} \Delta^{k-\nu} ((1 - |x|^2)^{2j+2k-\nu} \chi_B(x)).$$

Observe that  $\psi$  restricted to  $B$  is a polynomial which vanishes on  $\partial B$  up to order  $2d$  and  $\psi$  is zero off  $B$ . Therefore  $\Delta^d \psi$  is supported on  $B$  and its restriction to  $B$  is a polynomial with zero integral. This completes the first step of the construction of  $\beta$ .

The second step proceeds as follows. Since by hypothesis  $T = R \circ U$ , with  $U$  invertible in the algebra  $A$ , we have

$$R(\beta_1) = T(U^{-1}\beta_1).$$

Setting

$$(26) \quad \beta = U^{-1}\beta_1,$$

we are only left with the task of showing that

$$(27) \quad \beta \in L^\infty(\mathbb{R}^n)$$

and that, for some positive constant  $C$ ,

$$(28) \quad |\beta(x)| \leq \frac{C}{|x|^{n+1}}, \quad |x| \geq 2.$$

Since  $U^{-1} \in A$ , for some real number  $\lambda$  and some smooth homogeneous Calderón-Zygmund operator  $V$ ,

$$U^{-1} = \lambda I + V.$$

Thus

$$\beta = \lambda \beta_1 + V(\beta_1).$$

Now  $\beta_1$  is supported on  $B$  and has zero integral on  $B$ . This is enough to insure the decay estimate (28). Indeed, let  $L(x)$  be the kernel of  $V$  and assume that  $|x| \geq 2$ . Then

$$(29) \quad \begin{aligned} V(\beta_1)(x) &= \int L(x-y) \beta_1(y) dy \\ &= \int (L(x-y) - L(x)) \beta_1(y) dy, \end{aligned}$$

and so

$$\begin{aligned}
 (30) \quad |V(\beta_1)(x)| &\leq \int |(L(x-y) - L(x))| |\beta_1(y)| dy, \\
 &\leq C \int \frac{|y|}{|x|^{n+1}} |\beta_1(y)| dy, \\
 &= \frac{C}{|x|^{n+1}}.
 \end{aligned}$$

The boundedness of  $\beta$  is a more delicate issue. It follows immediately from the next lemma applied to the operator  $V$  and the function  $\beta_1$ . This is precisely where we use the fact that  $\beta_1$  satisfies a Lipschitz condition.

The constant of the kernel  $K(x) = \Omega(x)/|x|^n$  of the smooth homogeneous Calderón-Zygmund operator  $T$  is

$$(31) \quad \|T\|_{CZ} \equiv \|K\|_{CZ} = \|\Omega\|_\infty + \||x| \nabla \Omega(x)\|_\infty.$$

We adopt the standard notation for the minimal Lipschitz constant of a Lipschitz function  $f$  on  $B$ , namely

$$\|f\|_{\text{Lip}(1,B)} = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in B, x \neq y \right\} < \infty.$$

LEMMA 5. *Let  $T$  be the homogeneous singular integral operator with kernel  $K(x) = \frac{\Omega(x)}{|x|^n}$ , where  $\Omega$  is an even homogeneous function of degree 0, continuously differentiable and with zero integral on the unit sphere. Then*

$$\|T(f \chi_B)\|_{L^\infty(\mathbb{R}^n)} \leq C \|K\|_{CZ} (\|f\|_{L^\infty(B)} + \|f\|_{\text{Lip}(1,B)}),$$

where  $C$  is a positive constant which depends only on  $n$ .

*Proof.* We start by examining the behaviour of  $T(f \chi_B)$  on the unit sphere. We claim that

$$|T_\epsilon(f \chi_B)(a)| \leq C \|K\|_{CZ} (\|f\|_{L^\infty(B)} + \|f\|_{\text{Lip}(1,B)}), \quad |a| = 1, \quad \epsilon > 0.$$

Indeed, if one follows in detail the proof of the claim, which we discuss below, one will realize that the principal value integral  $T(f \chi_B)(a)$  exists for all  $a$  in the sphere and satisfies the desired estimate.

We have

$$\begin{aligned}
 T_\epsilon(f \chi_B)(a) &= \int_{\epsilon < |x-a| < 1/2} \chi_B(x) f(x) K(a-x) dx + \int_{1/2 < |x-a|} \dots \\
 &= \text{I}_\epsilon + \text{II}.
 \end{aligned}$$

Clearly,

$$|\text{II}| \leq \int_{1/2 < |x-a|} \chi_B(x) |f(x)| \frac{|\Omega(x-a)|}{|x-a|^n} dx \leq 2^n |B| \|\Omega\|_\infty \|f\|_{L^\infty(B)}.$$

To deal with the term  $I_\epsilon$ , we write

$$\begin{aligned} I_\epsilon &= \int_{\epsilon < |x-a| < 1/2} \chi_B(x) (f(x) - f(a)) K(a-x) dx \\ &\quad + f(a) \int_{\epsilon < |x-a| < 1/2} \chi_B(x) K(a-x) dx \\ &= III_\epsilon + f(a) IV_\epsilon, \end{aligned}$$

and we remark that  $III_\epsilon$  can easily be estimated as follows:

$$|III_\epsilon| \leq \|f\|_{\text{Lip}(1,B)} \int_B |x-a| |K(a-x)| dx \leq C \|\Omega\|_\infty \|f\|_{\text{Lip}(1,B)}.$$

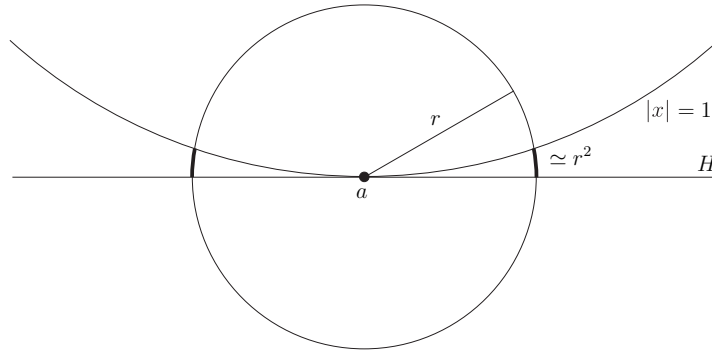
Taking care of  $IV_\epsilon$  is not so easy. Take spherical coordinates centered at the point  $a$ ,  $x = a + r\omega$  with  $0 \leq r$  and  $|\omega| = 1$ . Then

$$(32) \quad IV_\epsilon = \int_\epsilon^{1/2} \left( \int_{A(r)} \Omega(\omega) d\sigma(\omega) \right) \frac{dr}{r},$$

where

$$A(r) = \{\omega : |\omega| = 1 \text{ and } |a + r\omega| < 1\}.$$

Let  $H$  be the tangent hiperplane to  $S = \{x : |x| = 1\}$  at the point  $a$ . Call  $V$  the half space with boundary  $H$  containing the origin. Set  $\Sigma = \{\omega : |\omega| = 1\}$ , and think of  $\Sigma$  as the sphere of radius 1 centered at the point  $a$  expressed in polar coordinates centered at  $a$ . Then  $A(r) \subset \Sigma \cap V$ . See the figure below.



Since  $\Omega$  is even,

$$0 = \int_\Sigma \Omega(\omega) d\sigma(\omega) = 2 \int_{\Sigma \cap V} \Omega(\omega) d\sigma(\omega).$$

Thus

$$\int_{A(r)} \Omega(\omega) d\sigma(\omega) = - \int_{(\Sigma \cap V) \setminus A(r)} \Omega(\omega) d\sigma(\omega),$$

and so

$$\left| \int_{A(r)} \Omega(\omega) d\sigma(\omega) \right| \leq \|\Omega\|_\infty \sigma((\Sigma \cap V) \setminus A(r)).$$

Since  $H$  is tangent to  $S$  at the point  $a$ , we obtain

$$\sigma((\Sigma \cap V) \setminus A(r)) \leq C r,$$

which yields, by (32),

$$|IV_\epsilon| \leq C \|\Omega\|_\infty.$$

Assume now that  $|a| < 1$ . Proceeding as before we estimate in the same way the terms II and III $_\epsilon$ , so that we are again left with IV $_\epsilon$ . Let  $\epsilon_0$  stand for the distance from  $a$  to the boundary of  $B$ . In estimating IV $_\epsilon$  we can assume, without loss of generality, that  $\epsilon_0 \leq 1/4$ . Set  $a_0 = a/|a|$ ,

$$A = \{x \in B : \epsilon_0 < |x - a| < 1/2\}$$

and

$$A_0 = \{x \in B : \epsilon_0 < |x - a_0| < 1/2\}.$$

We compare IV $_\epsilon$  to the expression we get replacing  $a$  by  $a_0$  and  $\epsilon$  by  $\epsilon_0$  in the definition of IV $_\epsilon$ . For  $\epsilon \leq \epsilon_0$ , we have

$$\int_{\epsilon < |x-a| < 1/2} \chi_B(x) K(a-x) dx = \int_{\epsilon_0 < |x-a| < 1/2} \chi_B(x) K(a-x) dx,$$

and then

$$\begin{aligned} & \left| \int_{\epsilon < |x-a| < 1/2} \chi_B(x) K(a-x) dx - \int_{\epsilon_0 < |x-a_0| < 1/2} \chi_B(x) K(a_0-x) dx \right| \\ &= \left| \int_A K(a-x) dx - \int_{A_0} K(a_0-x) dx \right| \\ &\leq \int_{A \cap A_0} |K(a-x) - K(a_0-x)| dx \\ &\quad + \left| \int_{A \setminus A_0} \chi_B(x) K(a-x) dx \right| + \left| \int_{A_0 \setminus A} \chi_B(x) K(a_0-x) dx \right| \\ &= J_1 + J_2 + J_3. \end{aligned}$$

If  $x \in A \cap A_0$ , then

$$|K(a-x) - K(a_0-x)| \leq C \|K\|_{CZ} \frac{|a-a_0|}{|x-a|^{n+1}}.$$

Hence

$$J_1 \leq C \|K\|_{CZ} |a-a_0| \int_{|x-a| > \epsilon_0} \frac{dx}{|x-a|^{n+1}} \leq C \|K\|_{CZ}.$$

To estimate  $J_2$ , observe that

$$A \setminus A_0 = (A \cap B(a_0, \epsilon_0)) \cup (A \cap (\mathbb{R}^n \setminus B(a_0, 1/2))).$$

Now, it is obvious that if  $|x - a_0| \geq 1/2$ , then  $|x - a| \geq 1/4$ , and so

$$J_2 \leq \|\Omega\|_\infty \left( \int_{|x-a_0| < \epsilon_0} \frac{dx}{\epsilon_0^n} + \int_B 4^n dx \right) \leq C \|\Omega\|_\infty.$$

A similar argument does the job for  $J_3$ .

The case  $|a| > 1$  is treated in a completely analogous way. □



The construction of  $\beta$  is then completed and the theorem is proved for polynomial operators.

We remark that a variant of Lemma 5 holds, with the same proof, replacing  $B$  by  $\mathbb{R}^n \setminus \overline{B}$ . To control the term II we have to assume, in addition to the hypothesis of Lemma 5, that  $f$  satisfies a decay inequality of the type

$$|f(x)| \leq \frac{\|f\|_{L^\infty(\mathbb{R}^n \setminus \overline{B})}}{|x|^\eta}, \quad |x| \geq 1.$$

Then we conclude that

$$\|T(f \chi_{\mathbb{R}^n \setminus \overline{B}})\|_{L^\infty(\mathbb{R}^n)} \leq C \|K\|_{CZ} \left( \|f\|_{L^\infty(\mathbb{R}^n \setminus \overline{B})} + \|f\|_{\text{Lip}(1, \mathbb{R}^n \setminus \overline{B})} \right),$$

where  $C$  depends on  $n$  and  $\eta$ . Later, we will use this variant of Lemma 5 with  $f(x) = K(x)$  on  $\mathbb{R}^n \setminus \overline{B}$ , so that  $\eta = n$  and the constant  $C$  will depend only on  $n$ .

We mention another straightforward extension of Lemma 5 that will not be used in this paper. The function  $f$  may be assumed to be in  $\text{Lip}(\alpha, B)$   $0 < \alpha \leq 1$ , and the unit ball may be replaced by a domain with boundary of class  $C^{1+\epsilon}$ .

After the paper was completed, we learned from Stephen Semmes that Lemma 5 is known in dimension 2 [Che95, p.52] or [MB02, p.348]. This was used to prove global regularity of vortex patches for incompressible perfect fluids.

#### 4. Proof of the sufficient condition: the general case

We start this section by clarifying several facts about the convergence of the series (6). Let us then assume that  $\Omega$  is a function in  $C^\infty(S^{n-1})$  with zero integral. Then  $\Omega$  has an expansion (6) in spherical harmonics. For each positive integer  $r$ , one has the identity [Ste70, p. 70]

$$(33) \quad \sum_{j \geq 1} (j(j+n-2))^r \|P_j\|_2^2 = (-1)^r \int_{S^{n-1}} \Delta_S^r \Omega \Omega \, d\sigma,$$

where  $\Delta_S$  stands for the spherical Laplacean. Then

$$\sum_{j \geq 1} (j(j+n-2))^r \|P_j\|_2^2 \leq \|\Delta_S^r \Omega\|_2 \|\Omega\|_2,$$

where the  $L^2$  norm is taken with respect to  $d\sigma$ . Thus, by Schwarz's inequality, for each positive integer  $M$ ,

$$(34) \quad \sum_{j \geq 1} j^M \|P_j\|_2 < \infty.$$

We want to see that we also have

$$(35) \quad \sum_{j \geq 1} j^M \|P_j\|_\infty < \infty,$$

where the supremum norm is taken on  $S^{n-1}$ . This follows immediately from the next lemma, whose proof was indicated to us by Fulvio Ricci.

LEMMA 6. *For all homogeneous polynomials of degree  $q$ ,*

$$\|Q\|_\infty \leq C q^{\frac{n-1}{2}} \|Q\|_2,$$

where  $C$  is a positive constant which depends only on  $n$ .

*Proof.* Take an orthonormal basis  $Q_1, \dots, Q_d$ ,  $d = d_q$ , of the subspace of  $L^2(d\sigma)$  consisting of the restrictions to  $S^{n-1}$  of all homogeneous polynomials of degree  $q$ . Consider the function

$$S(x) = \sum_{j=1}^d Q_j(x)^2, \quad x \in S^{n-1}.$$

We claim that  $S$  is rotation invariant, and, hence, constant. Since  $d\sigma$  is a probability measure, this constant must be  $\sum_{j=1}^d \int Q_j(x)^2 d\sigma(x) = d$ . Now let  $Q$  be a homogeneous polynomial of degree  $q$  and set  $Q = \sum_{j=1}^d \lambda_j Q_j$ . Then

$$|Q(x)| \leq \left( \sum_{j=1}^d \lambda_j^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^d Q_j(x)^2 \right)^{\frac{1}{2}} = \|Q\|_2 d^{\frac{1}{2}}, \quad x \in S^{n-1},$$

which proves the lemma because  $d = \binom{n+q-1}{q} \simeq q^{n-1}$  ([SW71, p. 139]).

To show the claim, take a rotation  $\rho$ . Then we have

$$Q_j(\rho(x)) = \sum_{k=1}^d a_{jk} Q_k(x)$$

for some matrix  $(a_{jk})$  which is orthogonal, because the polynomials  $Q_j(\rho(x))$  form also an orthonormal basis due to the rotation invariance of  $\sigma$ . Hence

$$\sum_{j=1}^d Q_j(\rho(x))^2 = \sum_{j=1}^d Q_j(x)^2, \quad x \in S^{n-1}. \quad \square$$

Let us return now to the context of the theorem. Thus  $T$  is an even smooth homogeneous Calderón-Zygmund operator with kernel  $K(x) = \Omega(x)/|x|^n$ , and the expansion of  $\Omega$  in spherical harmonics is

$$(36) \quad \Omega(x) = \sum_{j \geq 1}^\infty P_{2j}(x).$$

By hypothesis there is a homogeneous harmonic polynomial  $P$  of degree  $2d$  which divides each  $P_{2j}$ . In other words,  $P_{2j} = P Q_{2j-2d}$ , where  $Q_{2j-2d}$  is a homogeneous polynomial of degree  $2j - 2d$ . We want to show that the series

$\sum_j Q_{2j-2d}(x)$  is convergent in  $C^\infty(S^{n-1})$ ; that is, for each positive integer  $M$ ,

$$(37) \quad \sum_{j \geq d} j^M \|Q_{2j-2d}\|_\infty < \infty.$$

The next lemma states that when one divides two homogeneous polynomials, then the supremum norm (on  $S^{n-1}$ ) of the quotient is controlled by the supremum norm of the dividend.

LEMMA 7. *Let  $P$  be a homogeneous polynomial nonidentically zero. Then there exists a positive  $\epsilon$  and a positive constant  $C = C(n, P)$  such that*

$$\|Q\|_\infty \leq C q^{2(n-1)/\epsilon} \|PQ\|_\infty$$

for each homogeneous polynomial  $Q$  of degree  $q$ .

*Proof.* Assume we can prove that for some positive  $\epsilon$ ,

$$(38) \quad \int_{|x|=1} \frac{1}{|P(x)|^\epsilon} d\sigma(x) < \infty.$$

Then, by Lemma 6 and Schwarz's inequality,

$$\begin{aligned} \|Q\|_\infty &\leq C q^{(n-1)/2} \|Q\|_2 \\ &\leq C q^{(n-1)/2} \left( \int_{|x|=1} \frac{1}{|P(x)|^\epsilon} d\sigma(x) \right)^{1/4} \left( \int_{|x|=1} |P(x)|^\epsilon |Q|^4 d\sigma(x) \right)^{1/4} \\ &\leq C q^{(n-1)/2} \|PQ\|_\infty^{\epsilon/4} \left( \int_{|x|=1} |Q|^{4-\epsilon} d\sigma(x) \right)^{1/4} \\ &\leq C q^{(n-1)/2} \|PQ\|_\infty^{\epsilon/4} \|Q\|_\infty^{1-\epsilon/4}, \end{aligned}$$

which completes the proof of the lemma.

Let us prove (38). Let  $d$  be the degree of  $P$ . By a well-known result of Ricci and Stein [RS87],  $|P(x)|$  is a weight in the class  $A^\infty$ . Indeed, if  $\epsilon d < 1$ , then

$$\int_{|x|<1} \frac{1}{|P(x)|^\epsilon} dx \leq C(\epsilon, d) \left( \int_{|x|<1} |P(x)| dx \right)^{-\epsilon} < \infty.$$

Since  $P$  is an homogeneous polynomial, (38) follows by changing to spherical coordinates.  $\square$

Now (37) may be proved readily from Lemma 7 and (35). Indeed, setting  $M_0 = 2(n-1)/\epsilon$ , we have

$$\|Q_{2j-2d}\|_\infty \leq C(n, P) (2j)^{M_0} \|P_{2j}\|_\infty$$

and

$$\sum_{j \geq d} j^M \|Q_{2j-2d}\|_\infty \leq C(n, P) \sum_{j \geq 1} (2j)^{M+M_0} \|P_{2j}\|_\infty < \infty.$$

The scheme for the proof of the sufficient condition in the general case is as follows. Taking a large partial sum of the series (36), we pass to a polynomial operator  $T_N$  (associated to a polynomial of degree  $2N$ ), which still satisfies the hypothesis (iii) of the theorem. Then we may apply the construction of Section 3 to  $T_N$  and get functions  $b_N$  and  $\beta_N$ . Unfortunately what was done in Section 3 does not give any uniform estimate in  $N$ , which is precisely what we need to try a compactness argument. The rest of the section is devoted to get the appropriate uniform estimates and to describe the final compactness argument.

By hypothesis,  $T = R \circ U$ , where  $R$  is the higher order Riesz transform associated with the harmonic polynomial  $P$  of degree  $2d$  that divides all  $P_{2j}$ , and  $U$  is invertible in the algebra  $A$ . The Fourier multiplier of  $T$  is

$$\sum_{j=d}^{\infty} \gamma_{2j} \frac{P_{2j}(\xi)}{|\xi|^{2j}} = \gamma_{2d} \frac{P(\xi)}{|\xi|^{2d}} \sum_{j \geq d} \frac{\gamma_{2j}}{\gamma_{2d}} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Therefore the Fourier multiplier of  $U$  is

$$(39) \quad \mu(\xi) = \gamma_{2d}^{-1} \sum_{j \geq d} \gamma_{2j} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}},$$

and the series is convergent in  $C^\infty(S^{n-1})$  because  $\gamma_{2j} \simeq (2j)^{-n/2}$  [SW71, p. 226]. Set, for  $N \geq d$ ,

$$(40) \quad \mu_N(\xi) = \gamma_{2d}^{-1} \sum_{j=d}^N \gamma_{2j} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

If

$$K_N(x) = \sum_{j=d}^N \frac{P_{2j}(x)}{|x|^{2j+n}}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

and  $T_N$  is the polynomial operator with kernel  $K_N$ , then  $T_N = R \circ U_N$ , where  $U_N$  is the operator in the algebra  $A$  with Fourier multiplier  $\mu_N(\xi)$ . From now on  $N$  is assumed to be big enough so that  $\mu_N(\xi)$  does not vanish on  $S^{n-1}$ . In fact, later we will need the inequality

$$(41) \quad |\partial^\alpha \mu_N^{-1}(\xi)| \leq C, \quad |\xi| = 1, \quad 0 \leq |\alpha| \leq 2(n+3),$$

which may be taken for granted owing to the convergence in  $C^\infty(S^{n-1})$  of the series (39). In (41),  $C$  is a positive constant depending only on the dimension  $n$  and  $\mu$ .

Notice that  $T_N$  satisfies condition (iii) in the theorem (with  $T$  replaced by  $T_N$ ), because  $\mu_N(\xi) \neq 0$ ,  $|\xi| = 1$ , and so we can apply the results of Section 3. In particular,

$$K_N(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T_N(b_N)(x) + T_N(\beta_N)(x),$$

where  $b_N$  and  $\beta_N$  are, respectively, the functions  $b$  and  $\beta$  defined in (18). It is important to remark that  $b_N$  does not depend on  $T$ . As (12) shows, the function  $b_N$  depends on  $N$  only through the fundamental solution of the operator  $\Delta^N$ . The uniform estimate we need on  $b_N$  is given by part (i) of the next lemma. The polynomial estimates in  $N$  of (ii) and (iii) are also basic for the compactness argument we are looking for.

LEMMA 8. *There exist a constant  $C$  depending only on  $n$  such that*

- (i)  $|\widehat{b_N}(\xi)| \leq C, \quad \xi \in \mathbb{R}^n;$
- (ii)  $\|b_N\|_{L^\infty(B)} \leq C (2N)^{2n+2};$  and
- (iii)  $\|\nabla b_N\|_{L^\infty(B)} \leq C (2N)^{2n+4}.$

*Proof.* We first prove (i). Let  $h_1, \dots, h_d$  be an orthonormal basis of the subspace of  $L^2(d\sigma)$  consisting of all homogeneous harmonic polynomials of degree  $2N$ . As in the proof of Lemma 6, we have  $h_1^2 + \dots + h_d^2 = d$ , on  $S^{n-1}$ . Set

$$H_j(x) = \frac{1}{\gamma_{2N}\sqrt{d}} h_j(x), \quad x \in \mathbb{R}^n.$$

Let  $S_j$  be the higher order Riesz transform with kernel  $K_j(x) = H_j(x)/|x|^{2N+n}$ . The Fourier multiplier of  $S_j^2$  is

$$\frac{1}{d} \frac{h_j(\xi)^2}{|\xi|^{4N}}, \quad 0 \neq \xi \in \mathbb{R}^n,$$

and thus

$$\sum_{j=1}^d S_j^2 = I.$$

By (10), we get

$$K_j(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x) = S_j(b_N)(x), \quad x \in \mathbb{R}^n, \quad 1 \leq j \leq d,$$

and so

$$(42) \quad b_N = \sum_{j=1}^d S_j \left( K_j(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x) \right).$$

We now appeal to a lemma of Calderón and Zygmund ([CZ55]; see [LS55] for a simpler proof), which can be stated as follows.

LEMMA (Calderón and Zygmund). *If  $K$  is the kernel of a higher order Riesz transform, then, for some constant  $C$  depending only on  $n$ ,*

$$|(\widehat{K(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x)})(\xi)| \leq C |(\widehat{P.V. K(x)})(\xi)|, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

By (42) and the preceding lemma, we get

$$\begin{aligned} |\widehat{b}_N(\xi)| &\leq \sum_{j=1}^d |\widehat{P.V. K_j(x)}(\xi)| |(K_j(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x))(\xi)| \\ &\leq C \sum_{j=1}^d |\widehat{P.V. K_j(x)}(\xi)|^2 \\ &= C. \end{aligned}$$

We now turn to the proof of (ii) in Lemma 8. In view of expression (42) for  $b_N$ , we apply Lemma 5 to the operators  $S_j$  and the functions  $K_j(x)$ , which satisfy a Lipschitz condition on  $\mathbb{R}^n \setminus \overline{B}$ . We obtain

$$(43) \quad \|b_N\|_\infty \leq C d \max_{1 \leq j \leq d} \|K_j\|_{CZ} (\|K_j\|_{L^\infty(\mathbb{R}^n \setminus \overline{B})} + \|K_j\|_{\text{Lip}(1, \mathbb{R}^n \setminus \overline{B})}).$$

As is well known,  $d \simeq (2N)^{n-2}$  [SW71, p. 140]. On the other hand,

$$\|K_j\|_{CZ} \leq \|H_j\|_\infty + \|\nabla H_j\|_\infty,$$

where the supremum norms are taken on  $S^{n-1}$ . Clearly,

$$\|H_j\|_\infty = \frac{1}{\gamma_{2N}} \left\| \frac{h_j}{\sqrt{d}} \right\|_\infty \leq \frac{1}{\gamma_{2N}} \simeq (2N)^{n/2}.$$

For the estimate of the gradient of  $H_j$ , we use the inequality [Ste70, p. 276]

$$(44) \quad \|\nabla H_j\|_\infty \leq C (2N)^{n/2+1} \|H_j\|_2,$$

where the  $L^2$  norm is taken with respect to  $d\sigma$ . Since the  $h_j$  are an orthonormal system,

$$\|H_j\|_2 = \frac{1}{\sqrt{d} \gamma_{2N}} \simeq \frac{(2N)^{n/2}}{(2N)^{(n-2)/2}} \simeq 2N.$$

Gathering the above inequalities, we get

$$\|K_j\|_{CZ} \leq C (2N)^{n/2+2}.$$

On the other hand, a straightforward computation yields

$$\|K_j\|_{L^\infty(\mathbb{R}^n \setminus \overline{B})} + \|K_j\|_{\text{Lip}(1, \mathbb{R}^n \setminus \overline{B})} \leq C N \|H_j\|_\infty + \|\nabla H_j\|_\infty \leq C (2N)^{n/2+2},$$

and therefore

$$\|b_N\|_{L^\infty(B)} \leq C (2N)^{n-2} (2N)^{n/2+2} (2N)^{n/2+2} = C (2N)^{2n+2}.$$

We are only left with the proof of (iii) in Lemma 8. Recalling the definition of  $b$  in (12), we see that  $b_N$  has the form

$$b_N(x) = \alpha_0 + \alpha_1 |x|^2 + \dots + \alpha_{N-1} |x|^{2N-2}, \quad |x| < 1$$

for some real coefficients  $\alpha_j$ ,  $0 \leq j \leq N - 1$ . Define the polynomial  $p(t)$  of the real variable  $t$  as

$$p(t) = \alpha_0 + \alpha_1 t^2 + \dots + \alpha_{N-1} t^{2N-2}$$

so that  $b_N(x) = p(|x|)$ ,  $|x| < 1$ . By part (ii) of the lemma,

$$\sup_{0 \leq t \leq 1} |p(t)| \leq C (2N)^{2n+2},$$

and thus, appealing to Markov's inequality [Lor86, p. 40],

$$\sup_{0 \leq t \leq 1} |p'(t)| \leq (2N - 2)^2 \sup_{0 \leq t \leq 1} |p(t)| \leq C (2N)^{2n+4}.$$

Now (iii) follows from the obvious identity  $\frac{\partial b_N}{\partial x_j} = p'(|x|) \frac{\partial |x|}{\partial x_j}$ , which gives  $|\nabla b_N(x)| \leq p'(|x|)$ ,  $|x| < 1$ . □

Our goal is now to show that under condition (iii) of the theorem, we can find a function  $\gamma$  in  $L^\infty(\mathbb{R}^n)$  such that

$$(45) \quad K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T(\gamma)(x), \quad x \in \mathbb{R}^n.$$

This was proven in the preceding section for the case in which  $T$  is a polynomial operator (see (18)). However, the bound we got there for  $\|\gamma\|_{L^\infty(\mathbb{R}^n)}$  depends on the degree of the polynomial defining the operator, and thus the proof does not cover the general case we are now tackling.

Since  $\Omega$  has the expansion (36) in spherical harmonics, we have

$$\begin{aligned} K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) &= \sum_{j \geq 1} \frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\mathbb{R}^n \setminus \overline{B}}(x) \\ &= \sum_{j \geq 1} T_j(b_j)(x), \end{aligned}$$

where  $T_j$  is the higher order Riesz transform with kernel  $P_{2j}(x)/|x|^{2j+n}$  and  $b_j$  is the function constructed in Section 2 (see (10) and (12)). The Fourier multiplier of  $T_j$  is

$$\gamma_{2j} \frac{P_{2j}(\xi)}{|\xi|^{2j}} = \gamma_{2d} \frac{P(\xi)}{|\xi|^{2d}} \frac{\gamma_{2j}}{\gamma_{2d}} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Let  $S_j$  be the operator whose Fourier multiplier is

$$(46) \quad \frac{\gamma_{2j}}{\gamma_{2d}} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n \setminus \{0\},$$

so that  $T_j = R \circ S_j$ . Then

$$\begin{aligned} K(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) &= \sum_{j \geq d} (R \circ S_j)(b_j) \\ &= \sum_{j \geq d} T((U^{-1} \circ S_j)(b_j)) \\ &= T\left(\sum_{j \geq d} (U^{-1} \circ S_j)(b_j)\right). \end{aligned}$$

The latest identity is justified by the absolute convergence of the series  $\sum_{j \geq d} (U^{-1} \circ S_j)(b_j)$  in  $L^2(\mathbb{R}^n)$ , which follows from the estimate

$$\begin{aligned} \sum_{j \geq d} \|(U^{-1} \circ S_j)(b_j)\|_2 &\leq C \sum_{j \geq d} \|Q_{2j-2d}\|_\infty \|b_j\|_{L^2(\mathbb{R}^n)} \\ &\leq C \sum_{j \geq d} \|Q_{2j-2d}\|_\infty \|b_j\|_{L^\infty(B)} \\ &\leq C \sum_{j \geq d} (2N)^{2n+2} \|Q_{2j-2d}\|_\infty < \infty. \end{aligned}$$

We claim now that the series  $\sum_{j \geq d} (U^{-1} \circ S_j)(b_j)$  converges uniformly on  $\mathbb{R}^n$  to a function  $\gamma$ , which will prove (45). Observe that the operator  $U^{-1} \circ S_j \in A$  is not necessarily a Calderón-Zygmund operator because the integral on the sphere of its multiplier does not need to vanish. However, it can be written as  $U^{-1} \circ S_j = c_j I + V_j$ , where

$$c_j = \frac{\gamma_{2j}}{\gamma_{2d}} \int_{S^{n-1}} \mu(\xi)^{-1} Q_{2j-2d}(\xi) d\sigma(\xi),$$

and  $V_j$  is the Calderón-Zygmund operator with multiplier

$$(47) \quad \mu(\xi)^{-1} \frac{\gamma_{2j}}{\gamma_{2d}} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}} - c_j.$$

Now

$$\sum_{j \geq d} (U^{-1} \circ S_j)(b_j) = \sum_{j \geq d} c_j b_j + \sum_{j \geq d} V_j(b_j),$$

and the first series offers no difficulties because, by Lemma 8(ii) and (37),

$$\sum_{j \geq d} |c_j| \|b_j\|_{L^\infty(B)} \leq C \sum_{j \geq d} (2j)^{-n/2} (2j)^{2n+2} \|Q_{2j-2d}\|_\infty < \infty.$$

The second series is more difficult to treat. By Lemma 5 and Lemma 8(ii) and (iii),

$$\begin{aligned} \|V_j(b_j)\|_{L^\infty(\mathbb{R}^n)} &\leq C \|V_j\|_{CZ} (\|b_j\|_{L^\infty(B)} + \|\nabla b_j\|_{L^\infty(B)}) \\ &\leq C (2j)^{2n+4} \|V_j\|_{CZ}. \end{aligned}$$

Estimating the Calderón-Zygmund constant of the kernel of the operator  $V_j$  is not an easy task, because we do not have an explicit expression for the kernel. We do know, however, the multiplier (47) of  $V_j$ . We need a way of estimating the constant of the kernel in terms of the multiplier, and this is what the next lemma supplies.

LEMMA 9. *Let  $V$  be a smooth homogeneous Calderón-Zygmund operator with Fourier multiplier  $m$ . Then for some constant  $C$  depending only on  $n$ ,*

$$\|V\|_{CZ} \leq C \|\Delta_S^{n+3} m\|_2^{1/2} \|m\|_2^{1/2},$$



where  $\Delta_S$  is the spherical Laplacean and the  $L^2$  norm is taken with respect to  $d\sigma$ .

*Proof.* Let  $\omega(x)/|x|^n$  be the kernel of  $V$ , so that  $\omega$  is a homogeneous function of degree zero, of class  $C^\infty(S^{n-1})$  and with zero integral on the sphere. Consider the expansion of  $\omega$  in spherical harmonics  $\omega(x) = \sum_{j \geq 1} p_j(x)$ ,  $|x| = 1$ , so that the kernel of  $V$  is  $\sum_{j \geq 1} p_j(x)/|x|^{j+n}$ ,  $x \in \mathbb{R}^n \setminus \{0\}$ , and its Fourier multiplier is  $m(\xi) = \sum_{j \geq 1} \gamma_j p_j(\xi)$ ,  $|\xi| = 1$ . By definition (31) of the constant of the kernel of a Calderón-Zygmund operator, we have

$$\|V\|_{CZ} \leq C \sum_{j \geq 1} (j \|p_j\|_\infty + \|\nabla p_j\|_\infty),$$

where the supremum is taken on  $S^{n-1}$ . By (44) with  $H_j$  replaced by  $p_j$ , and Lemma 6,

$$\begin{aligned} \|V\|_{CZ} &\leq C \sum_{j \geq 1} (j^{1+(n-1)/2} \|p_j\|_2 + j^{n/2+1} \|p_j\|_2) \\ &\leq C \sum_{j \geq 1} (j^{n/2+2} \|p_j\|_2). \end{aligned}$$

Since  $\gamma_j \simeq j^{-n/2}$ , using Schwarz's inequality and (33) with  $\Omega$  replaced by  $m$  and  $P_j$  by  $\gamma_j p_j$ , the above sum can be estimated by

$$\begin{aligned} \sum_{j \geq 1} j^{n+2} \|\gamma_j p_j\|_2 &\leq C \left( \sum_{j \geq 1} j^{2n+6} \|\gamma_j p_j\|_2^2 \right)^{1/2} \\ &\leq C \left( \sum_{j \geq 1} (j(j+n-2))^{n+3} \|\gamma_j p_j\|_2^2 \right)^{1/2} \\ &= C \left( (-1)^n \int_{S^{n-1}} \Delta_S^{n+3} m m d\sigma \right)^{1/2} \\ &\leq C \|\Delta_S^{n+3} m\|_2^{1/2} \|m\|_2^{1/2}. \quad \square \end{aligned}$$

Since the multiplier of  $V_j$  is given by (47) and  $\mu^{-1}$  is in  $C^\infty(S^{n-1})$ , Lemma 9 reduces the estimate of  $\|V_j\|_{CZ}$  to the estimate of the  $L^2(d\sigma)$  norm of  $\nabla^k Q_{2j-2d}$  for  $0 \leq k \leq 2(n+3)$ . Let us consider first the case  $k = 1$ .

Since  $P_{2j} = P Q_{2j-2d}$ , we have

$$\nabla P_{2j} = \nabla P Q_{2j-2d} + P \nabla Q_{2j-2d},$$

and so, by Lemma 7 and (44) with  $H_j$  replaced by  $P_{2j}$ , there is a large positive integer  $M = M(n, P)$  such that

$$\begin{aligned} \|\nabla Q_{2j-2d}\|_\infty &\leq C j^M \|P \nabla Q_{2j-2d}\|_\infty \\ &\leq C j^M (\|\nabla P_{2j}\|_\infty + \|Q_{2j-2d}\|_\infty) \\ &\leq C j^M (C j^{n/2+1} \|P_{2j}\|_2 + C j^M \|P_{2j}\|_\infty) \\ &\leq C j^M \|P_{2j}\|_2, \end{aligned}$$

where in the latest inequality  $M$  has been increased without changing the notation.

By induction, for some large integer  $M = M(n, P)$ , we get

$$\|\nabla^k Q_{2j-2d}\|_\infty \leq C j^M \|P_{2j}\|_2, \quad 0 \leq k \leq 2(n + 3).$$

Therefore, the estimate we finally obtain for the constant of the kernel of  $V_j$  is

$$\|V_j\|_{CZ} \leq C j^M \|P_{2j}\|_2,$$

and thus

$$\|V_j(b_j)\|_{L^\infty(\mathbb{R}^n)} \leq C j^M \|P_{2j}\|_2,$$

where again  $M = M(n, P)$  is a positive integer. Hence the series

$$\sum_{j \geq d} (U^{-1} \circ S_j)(b_j)$$

converges uniformly on  $\mathbb{R}^n$  and the proof of (45) is complete.

We are now ready for the discussion of the final compactness argument that will complete the proof of the sufficient condition. The reader is invited to review the definitions of the operators  $T_N$  (with kernel  $K_N$ ) and  $U_N$  given in this section just before Lemma 8. We know from Section 3 (see (18)) that

$$(48) \quad K_N(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T_N(b_N)(x) + T_N(\beta_N)(x).$$

On the other hand, by the construction of the function  $\gamma$  we have just described, we also have

$$(49) \quad K_N(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T_N(\gamma_N)(x), \quad \gamma_N = \sum_{j \geq d}^N (U_N^{-1} \circ S_j)(b_j).$$

Notice that (41) guarantees that the estimate of the supremum norm of  $\gamma$  on the whole of  $\mathbb{R}^n$  is applicable to the operator  $T_N$ , and thus we get an estimate for  $\|\gamma_N\|_{L^\infty(\mathbb{R}^n)}$  which is uniform in  $N$ . Since  $T_N$  is injective, (48) and (49) imply

$$(50) \quad b_N + \beta_N = \gamma_N,$$

and, in particular, we conclude that the functions  $b_N + \beta_N$  are uniformly bounded in  $L^\infty(\mathbb{R}^n)$ , a fact that cannot be derived from the work done in Section 3. It is worth mentioning that numerical computations indicate that

$b_N$ , and thus  $\beta_N$ , are not uniformly bounded. On the other hand, Section 3 tells us that  $\gamma_N$  satisfies the decay estimate (16) with  $\beta$  replaced by  $\gamma_N$ , which we cannot infer from the preceding construction of  $\gamma$ . The advantages of both approaches will be combined now to get both the boundedness and decay property for  $\gamma$ .

In view of (49) and the expressions of the multipliers of  $U_N$  and  $S_j$  (see (46)),

$$\widehat{\gamma}_N(\xi) = \sum_{j=d}^N \frac{1}{\mu_N(\xi)} \frac{\gamma_{2j}}{\gamma_{2d}} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}} \widehat{b}_j(\xi),$$

which yields, by Lemma 8 and (37) for  $M = 0$ ,

$$\begin{aligned} (51) \quad \|\widehat{\gamma}_N\|_{L^\infty(\mathbb{R}^n)} &\leq C \sum_{j=d}^N \|Q_{2j-2d}\|_\infty \\ &\leq C \sum_{j=d}^{\infty} \|Q_{2j-2d}\|_\infty \\ &\leq C, \end{aligned}$$

where  $C$  does not depend on  $N$ . Recall that, from (26) in Section 3, we have

$$\beta_N = U_N^{-1}(\beta_{1,N}),$$

with  $\beta_{1,N}$  a bounded function supported on  $B$  satisfying  $\int \beta_{1,N}(x) dx = 0$ . Since

$$\widehat{\beta}_{1,N} = \mu_N \widehat{\beta}_N = \mu_N (\widehat{\gamma}_N - \widehat{b}_N),$$

we have, again by Lemma 8,

$$\|\widehat{\beta}_{1,N}\|_{L^\infty(\mathbb{R}^n)} \leq C.$$

Therefore, passing to a subsequence, we may assume that, as  $N$  goes to  $\infty$ ,

$$\widehat{b}_N \longrightarrow a_0 \quad \text{and} \quad \widehat{\beta}_{1,N} \longrightarrow a_1,$$

weak  $\star$  in  $L^\infty(\mathbb{R}^n)$ . Hence

$$b_N \longrightarrow \Phi_0 = \mathcal{F}^{-1}a_0 \quad \text{and} \quad \beta_{1,N} \longrightarrow \Phi_1 = \mathcal{F}^{-1}a_1$$

in the weak  $\star$  topology of tempered distributions, where  $\mathcal{F}^{-1}$  is the inverse Fourier transform. In particular,  $\Phi_0$  and  $\Phi_1$  are distributions supported on  $\overline{B}$  and

$$(52) \quad \langle \Phi_1, 1 \rangle = \lim_{N \rightarrow \infty} \int \beta_{1,N}(x) dx = 0.$$

Now we would like to understand the convergence properties of the sequence of the  $\beta_N$ 's. Since

$$\widehat{\beta}_N(\xi) = \mu_N^{-1}(\xi) \widehat{\beta}_{1,N}(\xi),$$

and we have pointwise bounded convergence of  $\mu_N^{-1}(\xi)$  towards  $\mu^{-1}(\xi)$  on  $\mathbb{R}^n \setminus \{0\}$ , we get that  $\widehat{\beta}_N \rightarrow \mu^{-1} a_1$  in the weak  $\star$  topology of  $L^\infty(\mathbb{R}^n)$ . Thus  $\beta_N \rightarrow U^{-1}(\Phi_1)$  in the weak  $\star$  topology of tempered distributions. Letting  $N \rightarrow \infty$  in (50), we obtain

$$\Phi_0 + U^{-1}(\Phi_1) = \gamma.$$

Now we come to the last key point of the proof, namely, that one has the decay estimate

$$(53) \quad |\gamma(x)| \leq \frac{C}{|x|^{n+1}}, \quad |x| \geq 2.$$

Since  $\Phi_0$  and  $\Phi_1$  are supported on  $\overline{B}$  and  $U^{-1}(\Phi_1) = \lambda \Phi_1 + V(\Phi_1)$ , where  $\lambda$  is a real number and  $V$  a smooth homogeneous Calderón-Zygmund operator, it is enough to show that  $V(\Phi_1)$  has the appropriate behavior off the ball  $B(0, 2)$ . Let  $L$  be the kernel of  $V$ . Regularizing  $\Phi_1$ , one checks that, for a fixed  $x$  satisfying  $|x| \geq 2$ ,

$$(54) \quad \begin{aligned} V(\Phi_1)(x) &= \langle \Phi_1, L(x - y) \rangle \\ &= \langle \Phi_1, L(x - y) - L(x) \rangle, \end{aligned}$$

where the latest identity follows from (52). Since  $\Phi_1$  is a distribution supported on  $\overline{B}$ , there exists a positive integer  $\nu$  and a constant  $C$  such that

$$(55) \quad |\langle \Phi_1, \varphi \rangle| \leq C \sup_{|\alpha| \leq \nu} \sup_{|y| \leq 3/2} |\partial^\alpha \varphi(y)|$$

for each infinitely differentiable function  $\varphi$  on  $\mathbb{R}^n$ . The kernel  $L$  satisfies

$$\left| \frac{\partial^\alpha}{\partial y^\alpha} (L(x - y) - L(x)) \right| \leq \frac{C_\alpha}{|x|^{n+1+|\alpha|}}, \quad |y| \leq 3/2,$$

and hence by (54) and (55),

$$|V(\Phi_1)(x)| \leq \frac{C}{|x|^{n+1}}, \quad |x| \geq 2,$$

which proves (53) and then completes the proof of the sufficient condition in the general case.

### 5. Proof of the necessary condition: the polynomial case

We assume in this section that  $T$  is a polynomial operator with kernel

$$K(x) = \frac{\Omega(x)}{|x|^n} = \frac{P_2(x)}{|x|^{2+n}} + \frac{P_4(x)}{|x|^{4+n}} + \cdots + \frac{P_{2N}(x)}{|x|^{2N+n}}, \quad x \neq 0,$$

where  $P_{2j}$  is a homogeneous harmonic polynomials of degree  $2j$ . Let  $Q$  be the homogeneous polynomial of degree  $2N$  defined by

$$Q(x) = \gamma_2 P_2(x)|x|^{2N-2} + \cdots + \gamma_{2j} P_{2j}(x)|x|^{2N-2j} + \cdots + \gamma_{2N} P_{2N}(x).$$

Then

$$\widehat{P.V.K}(\xi) = \frac{Q(\xi)}{|\xi|^{2N}}, \quad \xi \neq 0.$$

Our assumption is now the  $L^2$  estimate between  $T^*$  and  $T$  (see (ii) in the statement of the theorem). Since the truncated operator  $T^1$  at level 1 is obviously dominated by  $T^*$ , we have

$$\int (T^1 f)^2(x) dx \leq \int (T^* f)^2(x) dx \leq C \int (Tf)^2(x) dx.$$

The kernel of  $T^1$  is (see (15))

$$(56) \quad K(x) \chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T(b)(x) + S(x) \chi_B(x),$$

where  $b$  is given in equation (12), and

$$-S(x) = Q(\partial)(A_0 + A_1 |x|^2 + \cdots + A_{2N-1} |x|^{4N-2})(x), \quad x \in \mathbb{R}^n.$$

The reader may consult the beginning of Section 3 to review the context of the definition of  $S$ . In view of (56), for each  $f \in L^2(\mathbb{R}^n)$ , we have

$$\begin{aligned} \|S \chi_B \star f\|_2 &\leq C \|T^1 f\|_2 + \|b \star Tf\|_2 \\ &\leq C (\|Tf\|_2 + \|\widehat{b}\|_\infty \|Tf\|_2) \\ &= C \|Tf\|_2. \end{aligned}$$

By Plancherel, the above  $L^2$  inequality translates into a pointwise inequality between the Fourier multipliers, namely,

$$(57) \quad |\widehat{S \chi_B}(\xi)| \leq C |\widehat{P.V.K}(\xi)| = C \frac{|Q(\xi)|}{|\xi|^{2N}}.$$

Our next goal is to show that (57) provides interesting relations between the zero sets of  $Q$  and the  $P_{2j}$ . For each function  $f$  on  $\mathbb{R}^n$ , set  $Z(f) = \{x \in \mathbb{R}^n : f(x) = 0\}$ .

LEMMA (Zero Sets Lemma).

$$Z(Q) \subset Z(P_{2j}), \quad 1 \leq j \leq N.$$

*Proof.* We know that  $S$  has an expression of the form (see (22))

$$(58) \quad S(x) = \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} c_{lj} P_{2j}(x) |x|^{2(l-N-j)}.$$

Since  $\widehat{\chi}_B = G_m$ ,  $m = n/2$ , Lemma 3 yields

$$\begin{aligned}
 (59) \quad \widehat{S\chi}_B(\xi) &= S(i\partial)\widehat{\chi}_B(\xi) \\
 &= \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} c_{lj} (-1)^{l-N} P_{2j}(\partial) \Delta^{l-N-j} G_m(\xi) \\
 &= \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} \sum_{k=0}^{l-N-j} c_{l,j,k} P_{2j}(\xi) |\xi|^{2(l-N-j-k)} G_{m+2l-2N-k}(\xi).
 \end{aligned}$$

The function  $G_p(\xi)$  is, for each  $p \geq 0$ , a radial function which is the restriction to the real positive axis of an entire function [Gra04, A-8]. Set  $\xi = r\xi_0$ ,  $|\xi_0| = 1$ ,  $r \geq 0$ . Then

$$(60) \quad \widehat{S\chi}_B(r\xi_0) = \sum_{p=1}^{\infty} a_{2p}(\xi_0) r^{2p},$$

and the power series has infinite radius of convergence for each  $\xi_0$ . Assume now that  $Q(\xi_0) = 0$ . Then, by (57),  $\widehat{S\chi}_B(r\xi_0) = 0$  for each  $r \geq 0$ , and hence  $a_{2p}(\xi_0) = 0$  for each  $p \geq 1$ . For  $p = 1$ , one has  $a_2(\xi_0) = P_2(\xi_0) C_2$ , where

$$C_2 = \sum_{l=N+1}^{2N-1} c_{l,1,l-N-1} G_{m+l-N+1}(0).$$

It will be shown later that  $C_2 \neq 0$ , and then we get  $P_2(\xi_0) = 0$ . Let us make the inductive hypothesis that  $P_2(\xi_0) = \dots = P_{2(j-1)}(\xi_0) = 0$ . Then we obtain, if  $j \leq N - 1$ ,  $a_{2j}(\xi_0) = P_{2j}(\xi_0) C_{2j}$ , where

$$(61) \quad C_{2j} = \sum_{l=N+j}^{2N-1} c_{l,j,l-N-j} G_{m+l-N+j}(0).$$

Since we will show that  $C_{2j} \neq 0$ ,  $P_{2j}(\xi_0) = 0$ ,  $1 \leq j \leq N - 1$ . We have

$$0 = Q(\xi_0) = \sum_{j=1}^N \gamma_{2j} P_{2j}(\xi_0),$$

and so we also get  $P_{2N}(\xi_0) = 0$ . Therefore the Zero Sets Lemma is completely proved provided we have at our disposition the following:

LEMMA 10.

$$C_{2j} = \frac{-\pi^{\frac{n}{2}}}{V_n 2^{\frac{n}{2}} \Gamma(\frac{n}{2} + 1)} \frac{(-1)^j}{j 4^j \Gamma(2j + \frac{n}{2})}, \quad 1 \leq j \leq N - 1.$$

The proof of Lemma 10 is lengthy and rather complicated from the computational point of view, and so we postpone it to Section 7. □

Notice that, although the constants  $C_{2j}$  are nonzero, they become rapidly small as the index  $j$  increases.

The reason why Lemma 10 is involved is that one has to trace back the exact values of the constants  $C_{2j}$  from the very beginning of our proof of (56). This forces us to take into account the exact values of various other constants; for instance, those which appear in the expression of the fundamental solution of  $\Delta^N$  and the constants  $A_0, A_1, \dots, A_{2N-1}$  in formula (11). Finally, we need to prove some new identities involving a triple sum of combinatorial numbers, in spirit of those that can be found in the book of R. Graham, D. Knuth and O. Patashnik [GKP94].

The following is elementary folklore, but is proved here for the reader's sake.

**LEMMA (Dimension Lemma).** *If  $f$  is a real-valued, continuous function on  $\mathbb{R}^n$  which changes sign, then  $H^{n-1}(Z(f)) > 0$ ,  $H^{n-1}$  being the Hausdorff measure in dimension  $n - 1$ . In particular, the Hausdorff dimension of  $Z(f)$  is at least  $n - 1$ .*

*Proof.* Assume, without loss of generality, that  $f(0) > 0$  and  $f(p) < 0$ , where  $p = (0, \dots, 0, 1)$ . For  $\epsilon > 0$  small enough we have  $f(x) > 0$  if  $|x| < \epsilon$  and  $f(x) < 0$  if  $|x - p| < \epsilon$ . Set  $B = \{x \in \mathbb{R}^n : x_n = 0 \text{ and } |x| < \epsilon\}$ . Bolzano's theorem tells us that, for each  $x \in B$ ,  $f$  vanishes at some point of the segment  $(x_1, \dots, x_{n-1}, t)$ ,  $0 \leq t \leq 1$ . Hence the orthogonal projection of the set  $Z(f)$  onto the hyperplane  $\{x : x_n = 0\}$  contains  $B$ , and so  $H^{n-1}(Z(f)) \geq H^{n-1}(B) > 0$ .  $\square$

We now turn our attention to an algebraic lemma which plays a key role in obtaining the necessary condition we are looking for.

**LEMMA (Division Lemma).** *Let  $F$  and  $G$  be polynomials in  $\mathbb{R}[x_1, \dots, x_n]$ . Assume that  $G$  is irreducible and that  $H^{n-1}(Z(F) \cap Z(G)) > 0$ . Then there exists a polynomial  $H$  in  $\mathbb{R}[x_1, \dots, x_n]$  such that  $F = GH$ .*

*Proof.* Denote by  $V(P)$  the complex hyper-surface  $\{z \in \mathbb{C}^n : P(z) = 0\}$  of a polynomial  $P$ . By hypothesis,  $V(F) \cap V(G)$  is not empty. If  $V(G)$  is not contained in  $V(F)$ , then the complex dimension of  $V(G) \cap V(F)$  is not greater than  $n - 2$  [Kun85, 3.2 p. 131]. Since the real dimension of a variety is less than or equal to the complex dimension, we conclude that  $Z(G) \cap Z(F)$  has real dimension not greater than  $n - 2$ , which contradicts the fact that it has positive  $n - 1$ -dimensional Hausdorff measure. Thus  $V(G) \subset V(F)$ , and therefore  $F = GH$  for some polynomial  $H$  in  $\mathbb{C}[x_1, \dots, x_n]$ . Since  $F$  and  $G$  have real coefficients, the same happens to  $H$ .  $\square$

We proceed to the proof of the necessary condition.

Let  $j_0$  be the first positive index such that  $P_{2j_0}$  does not vanish identically. We want to show that  $P_{2j_0}$  divides  $P_{2j}$  for  $j_0 \leq j \leq N$ .

Since  $\mathbb{R}[x_1, \dots, x_n]$  is a unique factorization domain, we can express  $P_{2j_0}$  as a product of irreducible factors, say  $R_k$ ,  $1 \leq k \leq M$ , which are also homogeneous. Clearly,  $Z(P_{2j_0}) = \cup_k Z(R_k)$ , and so

$$Z(Q) = \bigcup_k (Z(Q) \cap Z(R_k)).$$

Since the integral of  $Q$  on the sphere is 0,  $Q$  changes sign. Thus, by the Dimension Lemma, there is at least a  $k$  such that  $H^{n-1}(Z(Q) \cap Z(R_k)) > 0$ . Change notation if necessary so that  $k = 1$ . Then  $R_1$  divides  $Q$ , by the Division Lemma. We may also apply the Division Lemma to  $R_1$  and  $P_{2j}$  for each  $j_0 \leq j \leq N$ , because  $Z(Q) \cap Z(R_1) \subset Z(P_{2j}) \cap Z(R_1)$  by the Zero Sets Lemma. Hence  $R_1$  also divides  $P_{2j}$  for  $j_0 \leq j \leq N$ . Set

$$Q = R_1 Q_1 \quad \text{and} \quad P_{2j} = R_1 P_{2j,1}, \quad j_0 \leq j \leq N$$

for certain homogeneous polynomials  $Q_1$  and  $P_{2j,1}$ .

If  $M = 1$ , we are done. Otherwise our intention is to repeat as many times as we can the above division process. With this in mind, we use (60) to rewrite inequality (57) in the form

$$(62) \quad \left| \sum_{p=1}^{\infty} a_{2p}(\xi_0) r^{2p} \right| \leq C |Q(\xi_0)|, \quad 0 < r.$$

The definition of the coefficients  $a_{2p}$  and (59) show that there exist real numbers  $\mu_j(p)$  such that

$$a_{2p}(\xi_0) = \sum_{j=j_0}^{N-1} \mu_j(p) P_{2j}(\xi_0),$$

and so

$$(63) \quad \begin{aligned} a_{2p}(\xi_0) &= R_1(\xi_0) \sum_{j=j_0}^{N-1} \mu_j(p) P_{2j,1}(\xi_0) \\ &= R_1(\xi_0) a_{2p,1}(\xi_0), \end{aligned}$$

where the last identity provides the definition of the numbers  $a_{2p,1}(\xi_0)$ . We can simplify the common factor  $R_1(\xi_0)$  in (62) and get

$$(64) \quad \left| \sum_{p=1}^{\infty} a_{2p,1}(\xi_0) r^{2p} \right| \leq C |Q_1(\xi_0)|, \quad 0 < r.$$

Equipped with (64), we are ready to begin the second step in the division process. If  $Q_1(\xi_0) = 0$ , then  $a_{2p,1}(\xi_0) = 0$  for each  $p \geq 1$ . Hence, as in the proof of the Zero Sets Lemma,

$$Z(Q_1) \subset Z(P_{2j,1}), \quad j_0 \leq j \leq N.$$



To apply the Division Lemma we need to ascertain that the zero set of  $Q_1$  is big enough. For that it suffices to show, by the Dimension Lemma, that  $Q_1$  changes sign. As we are assuming that  $M$  is greater than 1, the degree of  $R_1$  is less than  $2j_0$ . Considering the expansions of  $R_1$  and  $Q$  in spherical harmonics, we see that they are orthogonal in  $L^2(d\sigma)$  [Ste70, p. 69]. Hence

$$0 = \int R_1(\xi) Q(\xi) d\sigma(\xi) = \int R_1^2(\xi) Q_1(\xi) d\sigma(\xi),$$

which tells us that  $Q_1$  changes sign.

Since  $P_{2j_0,1} = \prod_{k=2}^M R_k$ , we conclude that one of the  $R_k$ , say  $R_2$ , divides the  $P_{2j,1}$ ,  $j_0 \leq j \leq N$ . An inductive argument gives that  $P_{2j_0}$  divides the  $P_{2j}$ ,  $j_0 \leq j \leq N$ . At the  $k$ -th step, one should observe that  $Q = \prod_{l=1}^k R_l Q_k$  and

$$0 = \int \prod_{l=1}^k R_l(\xi) Q(\xi) d\sigma(\xi) = \int \prod_{l=1}^k R_l^2(\xi) Q_k(\xi) d\sigma(\xi),$$

so that  $Q_k$  changes sign. It is also important to remark that

$$a_{2p,k}(\xi_0) = \sum_{j=j_0}^{N-1} \mu_j(p) P_{2j,k}(\xi_0), \quad 1 \leq k \leq M.$$

Thus, at the  $M$ -th step, we get

$$(65) \quad a_{2j_0,M}(\xi_0) = C_{2j_0} \neq 0$$

for  $p = j_0$ . We have  $P_{2j} = P_{2j_0} Q_{2j-2j_0}$  for some homogeneous polynomials  $Q_{2j-2j_0}$  of degree  $2j - 2j_0$ , and so

$$\begin{aligned} Q(\xi) &= \sum_{j=j_0}^N \gamma_{2j} P_{2j}(\xi) |\xi|^{2N-2j} \\ &= P_{2j_0}(\xi) \sum_{j=j_0}^N \gamma_{2j} Q_{2j-2j_0}(\xi) |\xi|^{2N-2j}. \end{aligned}$$

By (62) and the definition of the coefficients  $a_{2p,M}(\xi_0)$ , for  $|\xi_0| = 1$  and  $0 < r$ , we get

$$\left| \sum_{p=j_0}^{\infty} a_{2p,M}(\xi_0) r^{2p} \right| \leq C \left| \sum_{j=j_0}^N \gamma_{2j} Q_{2j-2j_0}(\xi_0) \right|, \quad 0 < r.$$

Taking (65) into account, we conclude that

$$\sum_{j=j_0}^N \gamma_{2j} Q_{2j-2j_0}(\xi_0) \neq 0, \quad |\xi_0| = 1,$$

which completes the proof of the necessary condition in the polynomial case.

**6. Proof of the necessary condition: the general case**

In this section the kernel of our operator has the general form  $K(x) = \Omega(x)/|x|^n$  with  $\Omega$  a homogeneous function of degree 0, with vanishing integral on the sphere and of class  $C^\infty(S^{n-1})$ . Then  $\Omega(x) = \sum_{j \geq 1}^\infty P_{2j}(x)/|x|^{2j}$  with  $P_{2j}$  a homogeneous, harmonic polynomial of degree  $2j$ . The strategy consists in passing to the polynomial case by looking at a partial sum of the series above. Set, for each  $N \geq 1$ ,  $K_N(x) = \Omega_N(x)/|x|^n$ , where  $\Omega_N(x) = \sum_{j=1}^N P_{2j}(x)/|x|^{2j}$ , and let  $T_N$  be the operator with kernel  $K_N$ . The difficulty is now that there is no obvious way of obtaining the inequality

$$(66) \quad \|T_N^* f\|_2 \leq C \|T_N f\|_2, \quad f \in L^2(\mathbb{R}^n),$$

from our hypothesis; namely,

$$\|T^* f\|_2 \leq C \|T f\|_2, \quad f \in L^2(\mathbb{R}^n).$$

Instead we try to get (66) with  $\|T_N f\|_2$  replaced by  $\|T f\|_2$  in the right-hand side plus an additional term which becomes small as  $N$  tends to  $\infty$ . We start as follows:

$$\begin{aligned} \|T_N^1 f\|_2 &\leq \|T^1 f\|_2 + \|T^1 f - T_N^1 f\|_2 \\ &\leq C \|T f\|_2 + \left\| \sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\mathbb{R}^n \setminus \bar{B}} * f \right\|_2. \end{aligned}$$

By (10), there exists a bounded function  $b_j$  supported on  $B$  such that

$$\frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\mathbb{R}^n \setminus \bar{B}}(x) = P.V. \frac{P_{2j}(x)}{|x|^{2j+n}} * b_j.$$

By Lemma 8(i),  $\|\widehat{b_j}\|_{L^\infty(\mathbb{R}^n)}$  is bounded uniformly in  $j$ , and then an application of Plancherel yields

$$\begin{aligned} \left\| \sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\mathbb{R}^n \setminus \bar{B}} * f \right\|_2 &= \left\| \sum_{j>N} P.V. \frac{P_{2j}(x)}{|x|^{2j+n}} * b_j * f \right\|_2 \\ &\leq C \left( \sum_{j>N} \|P_{2j}\|_\infty \right) \|f\|_2, \end{aligned}$$

where the supremum norm is taken on the sphere. By (56) applied to  $T_N$ ,

$$T_N^1 f = K_N \chi_{\mathbb{R}^n \setminus \bar{B}} * f = K_N * b_N * f + S_N \chi_B * f.$$

Hence, for each  $f$  in  $L^2(\mathbb{R}^n)$ ,

$$\begin{aligned} \|S_{N\chi_B} * f\|_2 &\leq \|T_N^1 f\|_2 + \|K_N * f * b_N\|_2 \\ &\leq C\|Tf\|_2 + C \left( \sum_{j>N} \|P_{2j}\|_\infty \right) \|f\|_2 + \|K_N * f * b_N\|_2 \\ &\leq C\|Tf\|_2 + \|Tf * b_N\|_2 + C \left( \sum_{j>N} \|P_{2j}\|_\infty \right) \|f\|_2 \\ &\leq C\|Tf\|_2 + C \left( \sum_{j>N} \|P_{2j}\|_\infty \right) \|f\|_2, \end{aligned}$$

where in the latest inequality, Lemma 8(i) was used. The above  $L^2$  inequality translates, via Plancherel, into the pointwise estimate

$$(67) \quad |\widehat{S_{N\chi_B}}(\xi)| \leq C|\widehat{P.V.K}(\xi)| + C \left( \sum_{j>N} \|P_{2j}\|_\infty \right), \quad \xi \neq 0.$$

The idea is now to take limits, as  $N$  goes to  $\infty$ , in the preceding inequality. The remainder of the convergent series will disappear and we will get a useful analog of (57). The first task is to clarify how the left-hand side converges.

Set  $\xi = r\xi_0$ , with  $|\xi_0| = 1$  and  $r > 0$ . Rewrite (60) with  $S$  replaced by  $S_N$  and  $a_{2p}$  by  $a_{2p}^N$ :

$$\widehat{S_{N\chi_B}}(r\xi_0) = \sum_{p=1}^\infty a_{2p}^N(\xi_0) r^{2p}.$$

It is a remarkable key fact that, for a fixed  $p$ , the sequence of the  $a_{2p}^N$  stabilizes for  $N$  large. This fact depends on a laborious computation of various constants and will be proved in Section 7 in the following form.

LEMMA 11. *If  $p + 1 \leq N$ , then  $a_{2p}^N = a_{2p}^{p+1}$ .*

If  $p \geq 1$  and  $p + 1 \leq N$ , we set  $a_{2p} = a_{2p}^N$ . We need an estimate for the  $a_{2p}^N$ , which will be proved as well in Section 7.

LEMMA 12. *We have, for a constant  $C$  depending only on  $n$ ,*

$$(68) \quad |a_{2p}| \leq \frac{C}{(p-1)! 4^p} \sum_{j=1}^p \|P_{2j}\|_\infty, \quad 1 \leq p \leq N-1$$

and

$$(69) \quad |a_{2p}^N| \leq \frac{C}{4^p} \binom{\frac{n}{2} + N - 1}{N - 1} \sum_{j=1}^{N-1} \|P_{2j}\|_\infty, \quad 1 < N \leq p.$$

Let us prove that for each  $\xi_0$  in the sphere, the sequence  $S_N \chi_B(r \xi_0)$  converges uniformly on  $0 \leq r \leq 1$ . For  $1 \leq N \leq M$ ,

$$\begin{aligned} & \left| \widehat{S_N \chi_B}(r \xi_0) - \widehat{S_M \chi_B}(r \xi_0) \right| \\ & \leq \sum_{p \geq N} |a_{2p}^N| r^{2p} + \sum_{p=N}^{M-1} |a_{2p}| r^{2p} + \sum_{p \geq M} |a_{2p}^M| r^{2p} \\ & \leq C \left( \frac{1}{4^N} \binom{\frac{n}{2} + N - 1}{N - 1} + \sum_{p \geq N} \frac{1}{(p - 1)! 4^p} \right) \sum_{j=1}^{\infty} \|P_{2j}\|_{\infty}, \end{aligned}$$

which clearly tends to 0 as  $N$  goes to  $\infty$ . Letting  $N$  go to  $\infty$  in (67), we get

$$(70) \quad \left| \sum_{p=1}^{\infty} a_{2p}(\xi_0) r^{2p} \right| \leq C |\widehat{P.V.K}(\xi_0)|, \quad 0 \leq r \leq 1, \quad |\xi_0| = 1.$$

At this point we may repeat almost verbatim the proof we presented in the previous section, because the coefficients  $a_{2p}^N$  stabilize. This allows us to argue as in the polynomial case. The only difference lays in the fact that now we are dealing with infinite sums. However, no convergence problems will really arise because of (35).

### 7. Proof of the combinatorial lemmata

This section will be devoted to prove Lemmas 10, 11 and 12 stated and used in the preceding sections.

For the proof of Lemma 10 (see §5) we need to carefully trace back the path that led us to the constants  $C_{2j}$ . To begin with, we need a formula for the coefficients  $A_l$  in (11), and for that it is essential to have the expression for the fundamental solution  $E_N = E_N^n$  of  $\Delta^N$  in  $\mathbb{R}^n$ . One has [ACL83]

$$E_N(x) = \frac{1}{|x|^{n-2N}} (\alpha(n, N) + \beta(n, N) \log |x|^2),$$

where  $\alpha$  and  $\beta$  are constants that depend on  $n$  and  $N$ . To write in close form  $\alpha$  and  $\beta$ , we consider different cases. Let  $\omega_n$  be the surface measure of  $S^{n-1}$ .

*Case 1:  $n$  is odd.* Then

$$\alpha(n, N) = \frac{\Gamma(2 - \frac{n}{2})}{4^{N-1} (N - 1)! \Gamma(N + 1 - \frac{n}{2}) (2 - n) \omega_n}$$

and

$$\beta(n, N) = 0.$$

Case 2:  $n$  is even,  $n \neq 2$  and  $N \leq \frac{n}{2} - 1$ . Then

$$\alpha(n, N) = \frac{(-1)^{N-1} (\frac{n}{2} - N - 1)!}{4^{N-1} (N - 1)! (\frac{n}{2} - 2)! (2 - n) \omega_n}$$

and

$$\beta(n, N) = 0.$$

Case 3:  $n$  is even,  $n \neq 2$  and  $N \geq \frac{n}{2}$ . Then

$$\beta(n, N) = \frac{1}{(-1)^{\frac{n}{2}+1} (N - \frac{n}{2})! 4^{N-1} (N - 1)! (\frac{n}{2} - 2)! (2 - n) \omega_n}$$

and

$$\alpha(n, N) = 2 \beta(n, N) S_{N-\frac{n}{2}},$$

where  $S_0 = 0$  and

$$S_L = \sum_{k=1}^L \frac{1}{2^k} + \sum_{k=\frac{n}{2}}^{L+\frac{n}{2}-1} \frac{1}{2^k}, \quad 1 \leq L.$$

Case 4:  $n = 2$ . Then

$$\beta(2, N) = \frac{1}{2} \frac{1}{4^{N-1} (N - 1)!^2 \omega_2}$$

and

$$\alpha(2, N) = 2 \beta(2, N) S_{N-1}.$$

Recall that the constants  $A_0, A_1, \dots, A_{2N-1}$  are chosen so that the function (see (11))

$$(71) \quad \varphi(x) = E(x) \chi_{\mathbb{R}^n \setminus \bar{B}}(x) + (A_0 + A_1 |x|^2 + \dots + A_{2N-1} |x|^{4N-2}) \chi_B(x),$$

and all its partial derivatives of order not greater than  $2N - 1$  extend continuously up to  $\partial B$ .

LEMMA 13. For  $L = N + 1, \dots, 2N - 1$ , we have

$$A_L = \frac{(-1)^{N+L} \binom{N+\frac{n}{2}-1}{N-1}}{V_n 4^N (L + \frac{n}{2} - N) (2N - L - 1)! L!},$$

where  $V_n$  is the volume of the unit ball.

Proof. Set  $t = |x|^2$ , so that

$$(72) \quad E_N^n(x) \equiv E(t) = t^{N-\frac{n}{2}} (\alpha + \beta \log(t)).$$

Let  $P(t)$  be the polynomial  $\sum_{L=0}^{2N-1} A_L t^L$ . By Corollary 2 in Section 2, we need that

$$P^k(1) = E^k(1), \quad 0 \leq k \leq 2N - 1.$$

By Taylor’s expansion, we have  $P(t) = \sum_{i=0}^{2N-1} \frac{E^i(1)}{i!} (t - 1)^i$ , and hence, by the binomial formula applied to  $(t - 1)^i$ ,

$$A_L = \sum_{i=L}^{2N-1} \frac{E^i(1)}{i!} (-1)^{i-L} \binom{i}{L}, \quad 0 \leq L \leq 2N - 1.$$

Now we want to compute  $E^i(1)$ . Clearly

$$\left(\frac{d}{dt}\right)^i (t^{N-\frac{n}{2}}) = \left(N - \frac{n}{2}\right) \cdots \left(N - \frac{n}{2} - i + 1\right) t^{N-\frac{n}{2}-i}.$$

Notice that the logarithmic term in (72) only appears when the dimension  $n$  is even. In this case, for each  $i \geq N + 1$ ,

$$\left(\frac{d}{dt}\right)^i (t^{N-\frac{n}{2}} \log t) = \left(N - \frac{n}{2}\right)! (-1)^{i-N+\frac{n}{2}-1} \left(i - N + \frac{n}{2} - 1\right)! t^{-i+N-\frac{n}{2}}.$$

Hence, for  $i \geq N + 1$ , we obtain

$$\begin{aligned} E^i(1) &= \alpha(n, N) \left(N - \frac{n}{2}\right) \cdots \left(n - \frac{n}{2} - i + 1\right) \\ &\quad + \beta(n, N) \left(N - \frac{n}{2}\right)! (-1)^{i-N+\frac{n}{2}-1} \left(i - N + \frac{n}{2} - 1\right)! \end{aligned}$$

Consequently,

$$\begin{aligned} (73) \quad A_L &= (-1)^L \alpha(n, N) \sum_{i=L}^{2N-1} \left(N - \frac{n}{2}\right) \cdots \left(N - \frac{n}{2} - i + 1\right) \frac{(-1)^i}{i!} \binom{i}{L} \\ &\quad + (-1)^{L-N+\frac{n}{2}-1} \beta(n, N) \left(N - \frac{n}{2}\right)! \sum_{i=L}^{2N-1} \left(i - N + \frac{n}{2} - 1\right)! \frac{\binom{i}{L}}{i!}. \end{aligned}$$

Let us remark that for the cases  $n = 2$  or  $n$  even and  $N \geq \frac{n}{2}$ , the first term in (73) is zero, while for the cases  $n$  odd or  $n$  even and  $N \leq \frac{n}{2} - 1$ , the second term is zero because  $\beta(n, N) = 0$ . This explains why we compute below the two terms separately.

For the first term we show that

$$\begin{aligned} (74) \quad \sum_{i=L}^{2N-1} \left(N - \frac{n}{2}\right) \cdots \left(N - \frac{n}{2} - i + 1\right) \frac{(-1)^i}{i!} \binom{i}{L} \\ = (-1)^L \binom{N - \frac{n}{2}}{L} \binom{\frac{n}{2} + N - 1}{2N - 1 - L}. \end{aligned}$$

Indeed, the left-hand side of (74) is, setting  $k = i - L$ ,

$$\begin{aligned} & \frac{1}{L!} \sum_{k=0}^{2N-1-L} \left(N - \frac{n}{2}\right) \cdots \left(N - \frac{n}{2} - L - k + 1\right) \frac{(-1)^{L+k}}{k!} \\ &= (-1)^L \binom{N - \frac{n}{2}}{L} \sum_{k=0}^{2N-1-L} \binom{\frac{n}{2} + L - N + k - 1}{k} \\ &= (-1)^L \binom{N - \frac{n}{2}}{L} \binom{N + \frac{n}{2} - 1}{2N - 1 - L}, \end{aligned}$$

where the last identity comes from ([GKP94, (5.9), p. 159]).

To compute the second term we first show that

$$(75) \quad \sum_{i=L}^{2N-1} \left(i - N + \frac{n}{2} - 1\right)! \frac{1}{i!} \binom{i}{L} = \frac{(L - N + \frac{n}{2} - 1)!}{L!} \binom{N + \frac{n}{2} - 1}{2N - 1 - L}.$$

As before, setting  $k = i - L$  and applying [GKP94, (5.9), p. 159], we see that the left-hand side of (75) is

$$\begin{aligned} & \frac{1}{L!} \sum_{k=0}^{2N-1-L} \left(L + k - N + \frac{n}{2} - 1\right)! \frac{1}{k!} \\ &= \left(L - N + \frac{n}{2} - 1\right)! \sum_{k=0}^{2N-1-L} \binom{\frac{n}{2} + L - N + k - 1}{k} \\ &= \frac{(L - N + \frac{n}{2} - 1)!}{L!} \binom{N + \frac{n}{2} - 1}{2N - 1 - L}. \end{aligned}$$

We are now ready to complete the proof of the lemma distinguishing four cases.

*Case 1:  $n$  odd.* Since  $\beta(n, N) = 0$ , replacing in (73)  $\alpha(n, N)$  by its value and using (74) we get, by elementary arithmetics and the identity  $n V_n = \omega_n$ ,

$$\begin{aligned} A_L &= \frac{(-1)^L \Gamma(2 - \frac{n}{2})}{4^{N-1} (N-1)! \Gamma(N+1 - \frac{n}{2}) (2-n) \omega_n} (-1)^L \binom{N - \frac{n}{2}}{L} \binom{\frac{n}{2} + N - 1}{2N - 1 - L} \\ &= \frac{(-1)^{N+L} (N + \frac{n}{2} - 1) \cdots (\frac{n}{2} + 1)}{4^N V_n L! (2N - 1 - L)! (L + \frac{n}{2} - N) (N-1)!} \\ &= \frac{(-1)^{N+L} \binom{N + \frac{n}{2} - 1}{N-1}}{4^N V_n L! (2N - 1 - L)! (L + \frac{n}{2} - N)}. \end{aligned}$$

Case 2:  $n$  even,  $n \neq 2$  and  $N \leq \frac{n}{2} - 1$ . As in Case 1,  $\beta(n, N) = 0$ . We proceed similarly using (74) to obtain

$$\begin{aligned} A_L &= \frac{(-1)^L (-1)^{N-1} (\frac{n}{2} - N - 1)!}{4^N (N-1)! (\frac{n}{2} - 2)! (2-n)\omega_n} (-1)^L \binom{N-\frac{n}{2}}{L} \binom{N+\frac{n}{2}-1}{2N-1-L} \\ &= \frac{(-1)^{N+1} \binom{N+\frac{n}{2}-1}{N-1} \frac{n! (\frac{n}{2} - N - 1)! \{(N - \frac{n}{2}) \cdots (N - \frac{n}{2} - L + 1)\}}{(2N-1-L)! L! 4^{N-1} (2-n)\omega_n (\frac{n}{2} - 2)! (\frac{n}{2} - N + L)!}}{4^N V_n L! (2N-1-L)! (\frac{n}{2} - N + L)!} \\ &= \frac{(-1)^{N+L} \binom{N+\frac{n}{2}-1}{N-1}}{4^N V_n L! (2N-1-L)! (L + \frac{n}{2} - N)}. \end{aligned}$$

Case 3:  $n$  even,  $n \neq 2$  and  $N \geq \frac{n}{2}$ . Replacing in (73)  $\alpha(n, N)$  and  $\beta(n, N)$  by their values and using (75), by elementary arithmetics and the identity  $n V_n = \omega_n$ , we get

$$\begin{aligned} A_L &= \beta(n, N) (N - \frac{n}{2})! (-1)^{N+\frac{n}{2}-1+L} \sum_{i=L}^{2N-1} (i - N + \frac{n}{2} - 1)! \frac{1}{i!} \binom{i}{L} \\ &= \frac{(-1)^{N+L} (L - N + \frac{n}{2} - 1)!}{L! 4^{N-1} (\frac{n}{2} - 2)! (N-1)! (2-n)\omega_n} \binom{N+\frac{n}{2}-1}{2N-1-L} \\ &= \frac{(-1)^{L+N} \frac{n}{2} (\frac{n}{2} - 1)}{4^{N-1} (2-n)\omega_n L! (2N-1-L)! (\frac{n}{2} - N + L)} \binom{N+\frac{n}{2}-1}{N-1} \\ &= \frac{(-1)^{L+N} \binom{N+\frac{n}{2}-1}{N-1}}{4^N L! V_n (2N-1-L)! (\frac{n}{2} - N + L)}. \end{aligned}$$

Case 4:  $n = 2$ . Proceeding as in Case 3, we obtain

$$\begin{aligned} A_L &= \beta(n, N) (N - \frac{n}{2})! (-1)^{N+\frac{n}{2}-1+L} \sum_{i=L}^{2N-1} (i - N + \frac{n}{2} - 1)! \frac{1}{i!} \binom{i}{L} \\ &= \frac{(-1)^{N+L} N!}{2\omega_2 4^{N-1} L! (N-1)! (2N-1-L)! (L+1-N)} \\ &= \frac{(-1)^{L+N} \binom{N}{N-1}}{V_2 4^N L! (2n-1-L)! (L+1-N)}. \end{aligned}$$

□

Proof of Lemma 10. Recall that (see (61))

$$C_{2j} = \sum_{l=N+j}^{2N-1} c_{l,j,l-N-j} G_{\frac{n}{2}+l-N+j}(0).$$



Thus, we have to compute the constants  $c_{l,j,k}$  appearing in expression (59) for  $\widehat{S\chi_B}(\xi)$ . For that we need the constants  $c_{l,j}$  appearing in formula (22) for  $S(x)$ . We start by computing  $P_{2j}(\partial)\Delta^{N-j}(|x|^{2l})$ . Using (20) and Lemma 4, one gets

$$P_{2j}(\partial)\Delta^{N-j}(|x|^{2l}) = \frac{4^N l!(N-j)!}{(l-N-j)!} \binom{l-1+\frac{n}{2}}{N-j} P_{2j}(x)|x|^{2(l-N-j)}$$

if  $l-N-j \geq 0$  (and  $= 0$  if  $l-N-j < 0$ ).

As in (25), we want to compute  $P_{2j}(\partial)\Delta^{l-N-j}G_{\frac{n}{2}}(\xi)$  by using Lemma 3 applied to  $f(r) = G_{\frac{n}{2}}(r)$  and the homogeneous polynomial

$$L(x) = P_{2j}(x)|x|^{2(l-N-j)}.$$

We obtain

$$\begin{aligned} P_{2j}(\partial)\Delta^{l-N-j}G_{\frac{n}{2}}(\xi) &= (-1)^{2(l-N)} \sum_{k \geq 0} \frac{(-1)^k}{2^k k!} \Delta^k (P_{2j}(x)|x|^{2(l-N-j)}) G_{\frac{n}{2}+2(l-N)-k}(\xi) \\ &= \sum_{k=0}^{l-N-j} \frac{(-1)^k}{2^k k!} 4^k \frac{(l-N-j)!}{(l-N-j-k)!} k! \binom{\frac{n}{2}+j+l-N-1}{k} \\ &\quad \times P_{2j}(\xi)|\xi|^{2(l-N-j-k)} G_{\frac{n}{2}+2(l-N)-k}(\xi). \end{aligned}$$

In view of the definitions of  $Q(x)$  and  $S(x)$ ,

$$\begin{aligned} S(x) &= -Q(\partial) \left( \sum_{l=0}^{2N-1} A_l |x|^{2l} \right) = - \sum_{l=0}^{2N-1} A_l \sum_{j=1}^N \gamma_{2j} P_{2j}(\partial) \Delta^{N-j}(|x|^{2l}) \\ &= - \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} A_l \gamma_{2j} \frac{4^N l!(N-j)!}{(l-N-j)!} \binom{l-1+\frac{n}{2}}{N-j} P_{2j}(x)|x|^{2(l-N-j)} \\ &= \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} c_{l,j} P_{2j}(x)|x|^{2(l-N-j)}, \end{aligned}$$

where the last identity defines the  $c_{l,j}$ . In (59) we set

$$\begin{aligned} \widehat{S\chi_B}(\xi) &= S(\iota\partial) \widehat{\chi_B}(\xi) \\ &= \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} c_{l,j} (-1)^{l-N} P_{2j}(\partial) \Delta^{l-N-j} G_{\frac{n}{2}}(\xi) \\ &= \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} \sum_{k=0}^{l-N-j} c_{l,j,k} P_{2j}(\xi) |\xi|^{2(l-N-j-k)} G_{\frac{n}{2}+2l-2N-k}(\xi). \end{aligned}$$

Consequently,

$$\begin{aligned} c_{l,j,k} &= c_{l,j}(-1)^{l-N} \frac{(-1)^k}{2^k} 4^k \frac{(l-N-j)!}{(l-N-j-k)!} \binom{\frac{n}{2} + j + l - N - 1}{k} \\ &= -(-1)^{l+k+N} A_l \gamma_{2j} \frac{4^N l! (N-j)!}{(l-N-j)!} \binom{l-1+\frac{n}{2}}{N-j} \\ &\quad \times 2^k \frac{(l-N-j)!}{(l-N-j-k)!} \binom{\frac{n}{2} + j + l - N - 1}{k}. \end{aligned}$$

Replacing  $A_l$  by the formula given in Lemma 13 and performing some easy arithmetics, we get

$$(76) \quad c_{l,j,k} = -\frac{(-1)^k \gamma_{2j} \binom{N+\frac{n}{2}-1}{N-1} 2^k (N-j)! \binom{l-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+l-N-1}{k}}{V_n (l+\frac{n}{2}-N)(2N-l-1)!(l-N-j-k)!}.$$

The final computation of the  $C_{2j}$  is as follows:

$$\begin{aligned} C_{2j} &= \sum_{l=N+j}^{2N-1} c_{l,j,l-N-j} G_{\frac{n}{2}+l-N+j}(0) \\ &= \quad \text{[by the explicit value of } G_p(0) \text{ given in (78) below]} \\ &= \sum_{l=N+j}^{2N-1} c_{l,j,l-N-j} \frac{1}{2^{\frac{n}{2}+l-N+j} \Gamma(\frac{n}{2} + l - N + j + 1)} \\ &= \quad \text{[by (76)]} \\ &= -\sum_{l=N+j}^{2N-1} \frac{(-1)^{l-N-j} \gamma_{2j} \binom{N+\frac{n}{2}-1}{N-1} 2^{l-N-j} (N-j)! \binom{l-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+l-N-1}{l-N-j}}{V_n (l+\frac{n}{2}-N)(2N-l-1)! 2^{\frac{n}{2}+l-N+j} \Gamma(\frac{n}{2} + l - N + j + 1)} \\ &= \quad \text{[substituting the value given in (7) in } \gamma_{2j}] \\ &= -\frac{\pi^{\frac{n}{2}} \Gamma(j) \binom{N+\frac{n}{2}-1}{N-1} (N-j)!}{V_n \Gamma(j + \frac{n}{2}) 2^{\frac{n}{2}+2j}} \\ &\quad \times \sum_{l=N+1}^{2N-1} \frac{(-1)^{l+N} \binom{l-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+l-N-1}{l-N-j}}{(l+\frac{n}{2}-N)(2N-l-1)! \Gamma(\frac{n}{2} + l - N + j + 1)} \\ &= \quad \text{[setting } l = i + N + j] \\ &= -\frac{\pi^{\frac{n}{2}} \Gamma(j) \binom{N+\frac{n}{2}-1}{N-1} (N-j)!}{V_n \Gamma(j + \frac{n}{2}) 2^{\frac{n}{2}+2j}} \\ &\quad \times \sum_{i=0}^{N-1-j} \frac{(-1)^{i+j} \binom{i+N+j-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+2j+i-1}{i}}{(i+j+\frac{n}{2})(N-i-j-1)! \Gamma(\frac{n}{2} + i + 2j + 1)} \end{aligned}$$

$$\begin{aligned}
 &= \quad \left[ \text{because } \Gamma\left(\frac{n}{2} + i + 2j + 1\right) = \Gamma\left(2j + \frac{n}{2}\right) \prod_{k=0}^i \left(\frac{n}{2} + 2j + i - k\right) \right] \\
 &= - \frac{(-1)^j \pi^{\frac{n}{2}} \Gamma(j) \binom{N+\frac{n}{2}-1}{N-1} (N-j)!}{V_n \Gamma\left(j + \frac{n}{2}\right) 2^{\frac{n}{2}+2j} \Gamma\left(2j + \frac{n}{2}\right)} \\
 &\quad \times \sum_{i=0}^{N-1-j} \frac{(-1)^i \binom{i+N+j-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+2j+i-1}{i}}{(i+j+\frac{n}{2})(N-i-j-1)! \prod_{k=0}^i \left(\frac{n}{2} + 2j + i - k\right)} \\
 &= \quad \left[ \text{using Lemma 14 below} \right] \\
 &= - \frac{(-1)^j \pi^{\frac{n}{2}} \Gamma(j) \binom{N+\frac{n}{2}-1}{N-1} (N-j)!}{V_n \Gamma\left(j + \frac{n}{2}\right) 2^{\frac{n}{2}+2j} \Gamma\left(2j + \frac{n}{2}\right)} \binom{N-1}{j-1} \frac{\Gamma\left(j + \frac{n}{2}\right)}{j \Gamma\left(N + \frac{n}{2}\right)} \\
 &= \frac{-\pi^{\frac{n}{2}}}{V_n 2^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right)} \frac{(-1)^j}{j 4^j \Gamma\left(2j + \frac{n}{2}\right)}. \quad \square
 \end{aligned}$$

LEMMA 14. For each  $j = 1, \dots, N - 1$ ,

$$\sum_{i=0}^{N-1-j} \frac{(-1)^i \binom{i+N+j-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+2j+i-1}{i}}{(i+j+\frac{n}{2})(N-i-j-1)! \prod_{k=0}^i \left(\frac{n}{2} + 2j + i - k\right)} = \binom{N-1}{j-1} \frac{\Gamma\left(j + \frac{n}{2}\right)}{j \Gamma\left(N + \frac{n}{2}\right)}.$$

*Proof.* Divide the left-hand side by the right-hand side and denote the quotient by  $A$ . We have to prove that  $A = 1$ . Using elementary arithmetics,

$$\begin{aligned}
 &\frac{\binom{i+N+j-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+2j+i-1}{i} \Gamma\left(N + \frac{n}{2}\right)}{(i+j+\frac{n}{2}) \prod_{k=0}^i \left(\frac{n}{2} + 2j + i - k\right) \Gamma\left(j + \frac{n}{2}\right)} \\
 &= \binom{N+i+j+\frac{n}{2}-1}{N-j-1} \binom{N+\frac{n}{2}-1}{N-i-j-1} \frac{(N-i-j-1)!}{N-j} \binom{\frac{n}{2}+i+j-1}{i},
 \end{aligned}$$

and so

$$\begin{aligned}
 A &= \frac{j}{\binom{N-1}{j-1} (N-j)} \sum_{i=0}^{N-1-j} (-1)^i \binom{N+i+j+\frac{n}{2}-1}{N-j-1} \binom{N+\frac{n}{2}-1}{N-i-j-1} \binom{\frac{n}{2}+i+j-1}{i} \\
 &= \left[ \text{because } \binom{a+i}{i} = (-1)^i \binom{-a-1}{i} \right] \\
 &= \frac{j}{\binom{N-1}{j-1} (N-j)} \sum_{i=0}^{N-1-j} \binom{N+i+j+\frac{n}{2}-1}{N-j-1} \binom{N+\frac{n}{2}-1}{N-i-j-1} \binom{-\frac{n}{2}-j}{i} \\
 &= \left[ \text{by the triple-binomial identity (5.28) of ([GKP94, p. 171]); see below} \right] \\
 &= \frac{j}{\binom{N-1}{j-1} (N-j)} \binom{\frac{n}{2}+N+j-1}{0} \binom{N-1}{N-j-1} = 1.
 \end{aligned}$$

For the reader's convenience and later reference we state the triple-binomial identity [GKP94, (5.28), p. 171]:

$$(77) \quad \sum_{k=0}^n \binom{m-r+s}{k} \binom{n+r-s}{n-k} \binom{r+k}{m+n} = \binom{r}{m} \binom{s}{n},$$

where  $m$  and  $n$  are nonnegative integers.  $\square$

Our next task is to prove Lemmas 11 and 12. Setting  $\xi = r \xi_0$ ,  $|\xi_0| = 1$ , in (59) we obtain

$$\begin{aligned} \widehat{S_N \chi_B}(r\xi_0) &= \sum_{l=N+1}^{2N-1} \sum_{j=1}^{l-N} \sum_{k=0}^{l-N-j} c_{l,j,k} P_{2j}(r\xi_0) |r\xi_0|^{2(l-N-j-k)} G_{\frac{n}{2}+2l-2N-k}(r\xi_0) \\ &\quad [\text{make the change of indexes } l = N + s] \\ &= \sum_{s=1}^{N-1} \sum_{j=1}^s \sum_{k=0}^{s-j} c_{N+s,j,k} P_{2j}(\xi_0) r^{2(s-k)} G_{\frac{n}{2}+2s-k}(r) \\ &= \sum_{j=1}^{N-1} \sum_{s=j}^{N-1} \sum_{k=0}^{s-j} c_{N+s,j,k} P_{2j}(\xi_0) r^{2(s-k)} G_{\frac{n}{2}+2s-k}(r) \\ &:= \sum_{p=1}^{\infty} a_{2p}^N(\xi_0) r^{2p}. \end{aligned}$$

In order to compute the coefficients  $a_{2p}^N(\xi_0)$ , we substitute the power series expansion of  $G_q(r)$  [Gra04, A-8]; namely,

$$(78) \quad G_q(r) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i! \Gamma(q+i+1)} \frac{r^{2i}}{2^{2i+q}}$$

in the last triple sum above.

*Proof of Lemma 11.* We are assuming that  $1 \leq p \leq N-1$ . It is important to remark that, for this range of  $p$ , after introducing (78) in the triple sum above, only the values of the index  $j$  satisfying  $1 \leq j \leq p$  are involved in the expression for  $a_{2p}^N$ . Once (78) has been introduced in the triple sum, one should sum, in principle, on the four indexes  $i$ ,  $j$ ,  $s$  and  $k$ . But since we are looking at the coefficient of  $r^{2p}$ , we have the relation  $2(s-k) + 2i = 2p$ , which actually leaves us with three indexes. The range of each of these indexes is easy to determine and one gets

$$a_{2p}^N = \sum_{j=1}^p P_{2j}(\xi_0) \sum_{i=0}^{p-j} \sum_{s=p-i}^{N-1} c_{N+s,j,s-(p-i)} \times \text{coefficient of } r^{2i} \text{ from } G_{\frac{n}{2}+s+p-i}(r).$$

In view of (78),

$$\begin{aligned}
 a_{2p}^N &= \sum_{j=1}^p P_{2j}(\xi_0) \sum_{i=0}^{p-j} \sum_{s=p-i}^{N-1} c_{N+s,j,s-(p-i)} \frac{(-1)^i}{i! 2^{i+\frac{n}{2}+s+p} \Gamma(\frac{n}{2} + s + p + 1)} \\
 &= \text{[by the expression (76) for } c_{l,j,k}\text{]} \\
 &= \sum_{j=1}^p P_{2j}(\xi_0) \sum_{i=0}^{p-j} \sum_{s=p-i}^{N-1} \frac{(-1)^{i+1}}{i! 2^{i+\frac{n}{2}+s+p} \Gamma(\frac{n}{2} + s + p + 1)} \frac{(-1)^{s-(p-i)}}{V_n} \\
 &\quad \times \frac{\gamma_{2j} \binom{N+\frac{n}{2}-1}{N-1} 2^{s-(p-i)} (N-j)! \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{s-(p-i)}}{(s+\frac{n}{2})(N-s-1)!(p-i-j)!} \\
 &= (-1)^{p+1} \sum_{j=1}^p P_{2j}(\xi_0) \frac{(-1)^j \pi^{\frac{n}{2}} \Gamma(j)(N-j)!}{V_n \Gamma(\frac{n}{2} + j) 2^{\frac{n}{2}+2p}} \sum_{i=0}^{p-j} \frac{\binom{N+\frac{n}{2}-1}{N-1}}{i!(p-i-j)!} \\
 &\quad \times \sum_{s=p-i}^{N-1} \frac{(-1)^s \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{s-(p-i)}}{(s+\frac{n}{2})(N-s-1)! \Gamma(\frac{n}{2} + s + p + 1)}.
 \end{aligned}$$

In Lemma 15 below we give a useful compact form for the last sum. Using it, we obtain

$$\begin{aligned}
 a_{2p}^N &= (-1)^{p+1} \sum_{j=1}^p P_{2j}(\xi_0) \frac{(-1)^j \pi^{\frac{n}{2}} \Gamma(j)(N-j)!}{V_n \Gamma(\frac{n}{2} + j) 2^{\frac{n}{2}+2p}} \sum_{i=0}^{p-j} \frac{\binom{N+\frac{n}{2}-1}{N-1}}{i!(p-i-j)!} \\
 &\quad \times \frac{(-1)^{p-i} (N-p-1)! \binom{N-1}{p}}{(N-j)! \Gamma(N+\frac{n}{2}) j! \binom{\frac{n}{2}+p-i+j-1}{j}}.
 \end{aligned}$$

Easy arithmetics with binomial coefficients gives

$$\binom{N+\frac{n}{2}-1}{N-1} (N-p-1)! \binom{N-1}{p} \frac{1}{\Gamma(N+\frac{n}{2})} = \frac{1}{p! \Gamma(\frac{n}{2} + 1)}.$$

We finally get the extremely surprising identity

$$(79) \quad a_{2p}^N = \frac{-\pi^{\frac{n}{2}}}{V_n 2^{\frac{n}{2}+2p} p! \Gamma(\frac{n}{2} + 1)} \sum_{j=1}^p \frac{(-1)^j \Gamma(j) P_{2j}(\xi_0)}{\Gamma(\frac{n}{2} + j)} \sum_{i=0}^{p-j} \frac{(-1)^i}{i!(p-i-j)! j! \binom{\frac{n}{2}+p-i+j-1}{j}}$$

in which  $N$  has miraculously disappeared. Thus Lemma 11 is proved.  $\square$

*Proof of Lemma 12.* We start by proving the inequality (68), so that  $1 \leq p \leq N - 1$ . We roughly estimate  $a_{2p} = a_{2p}^N$  by putting the absolute value inside the sums in (79). The absolute value of each term in the innermost sum in (79) is obviously not greater than 1, and there are at most  $p$  terms. The factor

in front of  $P_{2j}(\xi_0)$  is again not greater than 1 in absolute value. Denoting by  $C$  the terms that depend only on  $n$ , we obtain the desired inequality (68).

We turn now to the proof of inequality (69). Recall that

$$\widehat{S_N \chi_B}(r\xi_0) = \sum_{p=1}^{\infty} a_{2p}^N(\xi_0)r^{2p} = \sum_{j=1}^{N-1} \sum_{s=j}^{N-1} \sum_{k=0}^{s-j} c_{N+s,j,k} P_{2j}(\xi_0) r^{2(s-k)} G_{\frac{n}{2}+2s-k}(r).$$

Replacing  $G_{\frac{n}{2}+2s-k}(r)$  by the expression given by (78), we obtain, as before, a sum with four indexes. Now we eliminate the index  $i$  of (78) using  $s - k + i = p$ . Hence

$$\begin{aligned} a_{2p}^N &= \sum_{j=1}^{N-1} P_{2j}(\xi_0) \sum_{s=j}^{N-1} \sum_{k=0}^{s-j} c_{N+s,j,k} \times \text{coefficient of } r^{2(p-s+k)} \text{ from } G_{\frac{n}{2}+2s-k}(r) \\ &= \sum_{j=1}^{N-1} P_{2j}(\xi_0) \sum_{s=j}^{N-1} \sum_{k=0}^{s-j} c_{N+s,j,k} \frac{(-1)^{p-s+k}}{(p-s+k)! \Gamma(\frac{n}{2} + p + s - 1) 2^{2p+\frac{n}{2}+k}} \\ &= \sum_{j=1}^{N-1} P_{2j}(\xi_0) \sum_{k=0}^{N-1-j} \sum_{s=j+k}^{N-1} c_{N+s,j,k} \frac{(-1)^{p-s+k}}{(p-s+k)! \Gamma(\frac{n}{2} + p + s - 1) 2^{2p+\frac{n}{2}+k}} \\ &= \frac{(-1)^p}{V_n 4^p 2^{\frac{n}{2}}} \binom{N + \frac{n}{2} - 1}{N - 1} \sum_{j=1}^{N-1} \gamma_{2j} P_{2j}(\xi_0) \sum_{k=0}^{N-1-j} \sum_{s=j+k}^{N-1} \\ &\quad \times \frac{(-1)^s (N-j)! \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{k}}{(p-s+k)! \Gamma(\frac{n}{2} + p + s + 1) (s + \frac{n}{2}) (N-s-1)! (s-j-k)!}. \end{aligned}$$

The second identity is just (78). The third is a change of the order of summation and the latest follows from the formula (76) for the constants  $c_{l,j,k}$ .

In view of the elementary fact that

$$(N-j)! \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{k} = \frac{\Gamma(s + \frac{n}{2} + N)}{k! \Gamma(s + \frac{n}{2} + j - k)},$$

we get

$$\begin{aligned} &\left| \sum_{k=0}^{N-1-j} \sum_{s=j+k}^{N-1} \frac{(-1)^s (N-j)! \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{k}}{(p-s+k)! \Gamma(\frac{n}{2} + p + s + 1) (s + \frac{n}{2}) (N-s-1)! (s-j-k)!} \right| \\ &\leq \sum_{k=0}^{N-1-j} \frac{1}{k!} \\ &\quad \times \sum_{s=j+k}^{N-1} \frac{1}{\Gamma(s + \frac{n}{2} + j - k) (p-s+k)! (\frac{n}{2} + p + s) (s + \frac{n}{2}) (N-s-1)! (s-j-k)!} \\ &\leq \sum_{k=0}^{N-1-j} \frac{1}{k!} \sum_{s=j+k}^{N-1} \frac{1}{(s-j-k)!} \leq e^2, \end{aligned}$$

where in the first inequality we used that, since  $N \leq p$ ,

$$\frac{\Gamma(s + \frac{n}{2} + N)}{\Gamma(\frac{n}{2} + p + s + 1)} \leq \frac{1}{\frac{n}{2} + p + s}.$$

The proof of (69) is complete. □

LEMMA 15. *Let  $N - 1 \geq L \geq j \geq k \geq 0$  be integers. Then*

$$\begin{aligned} \sum_{s=j}^{N-1} \frac{(-1)^s \binom{\frac{n}{2}+N+s-1}{N-k} \binom{\frac{n}{2}+k+s-1}{s-j}}{(s + \frac{n}{2})(N - s - 1)! \Gamma(\frac{n}{2} + s + L + 1)} \\ = (-1)^j \frac{(N - L - 1)!}{(N - k)!} \binom{N - 1}{L} \frac{1}{k! \Gamma(\frac{n}{2} + N)} \frac{1}{\binom{\frac{n}{2}+k+j-1}{k}}. \end{aligned}$$

*Proof.* To simplify notation set

$$A = \sum_{s=j}^{N-1} \frac{(-1)^s \binom{\frac{n}{2}+N+s-1}{N-k} \binom{\frac{n}{2}+k+s-1}{s-j}}{(s + \frac{n}{2})(N - s - 1)! \Gamma(\frac{n}{2} + s + L + 1)}.$$

Making the change of index of summation  $i = s - j$  and using repeatedly the identity  $\Gamma(x + 1) = x \Gamma(x)$ , we have

$$\begin{aligned} A &= \frac{(-1)^j}{\Gamma(\frac{n}{2} + j + L)} \sum_{i=0}^{N-1-j} \frac{(-1)^i \binom{\frac{n}{2}+N+i+j-1}{N-k} \binom{\frac{n}{2}+k+i+j-1}{i}}{(i+j + \frac{n}{2})(N - i - j - 1)! \{\prod_{p=0}^i (\frac{n}{2} + i + j + L - p)\}} \\ &= \frac{(-1)^j}{\Gamma(\frac{n}{2} + j + L)} B, \end{aligned}$$

where the last identity defines  $B$ . To compute  $B$  we need to rewrite the terms in a more convenient way so that we may apply well-known equalities, among which the triple-binomial identity (77). Notice that

$$\begin{aligned} \frac{\binom{\frac{n}{2}+N+i+j-1}{N-k} \binom{\frac{n}{2}+k+i+j-1}{i}}{\{\prod_{p=0}^i (\frac{n}{2} + i + j + L - p)\}} \\ = \frac{(N - L - 1)!(L - k)!}{i!(N - k)!} \binom{\frac{n}{2} + N + i + j - 1}{N - L - 1} \binom{\frac{n}{2} + j + L - 1}{L - k}. \end{aligned}$$

Hence

$$B = \sum_{i=0}^{N-1-j} \frac{(-1)^i (N - L - 1)!(L - k)!}{i!(N - k)!(N - i - j - 1)!(i + j + \frac{n}{2})} \binom{\frac{n}{2} + N + i + j - 1}{N - L - 1} \binom{\frac{n}{2} + j + L - 1}{L - k}.$$

Since clearly

$$\frac{\Gamma(\frac{n}{2} + N)}{\Gamma(\frac{n}{2} + j)} \frac{1}{(i + j + \frac{n}{2})(N - i - j - 1)!} = \binom{\frac{n}{2} + N - 1}{N - j - i - 1} \binom{\frac{n}{2} + j + i - 1}{i} i!,$$

$B$  can be written as

$$\begin{aligned} B &= \frac{\Gamma(\frac{n}{2} + j)(N - L - 1)!(L - k)!}{\Gamma(\frac{n}{2} + N)(N - k)!} \binom{\frac{n}{2} + j + L - 1}{L - k} \\ &\quad \times \sum_{i=0}^{N-1-j} (-1)^i \binom{\frac{n}{2} + N - 1}{N - j - i - 1} \binom{\frac{n}{2} + j + i - 1}{i} \binom{\frac{n}{2} + N + i + j - 1}{N - L - 1} \\ &= \frac{\Gamma(\frac{n}{2} + j)(N - L - 1)!(L - k)!}{\Gamma(\frac{n}{2} + N)(N - k)!} \binom{\frac{n}{2} + j + L - 1}{L - k} C, \end{aligned}$$

where the last identity defines  $C$ .

To simplify notation set  $L = m + j$ , where  $0 \leq m \leq L - j$ . Using the elementary identity

$$\binom{\frac{n}{2} + N - 1}{N - j - i - 1} \binom{\frac{n}{2} + i + j - 1}{i} = \frac{(N - j)!}{i!(N - i - j - 1)!(\frac{n}{2} + i + j)} \binom{\frac{n}{2} + N - 1}{N - j},$$

we get

$$C = \binom{\frac{n}{2} + N - 1}{N - j} (N - j)! \sum_{i=0}^{N-1-j} \frac{(-1)^i \binom{\frac{n}{2} + N + i + j - 1}{N - j - m - 1}}{i!(N - i - j - 1)!(\frac{n}{2} + i + j)}.$$

The only task left is the computation of the sum

$$D(j, m) = \sum_{i=0}^{N-1-j} \frac{(-1)^i \binom{\frac{n}{2} + N + i + j - 1}{N - j - m - 1}}{i!(N - i - j - 1)!(\frac{n}{2} + i + j)}.$$

The identity

$$\frac{1}{\frac{n}{2} + i + j} = \frac{1}{\frac{n}{2} + j} \left( 1 - \frac{i}{\frac{n}{2} + i + j} \right),$$

yields the expression

$$\begin{aligned} D(j, m) &= \sum_{i=0}^{N-1-j} \frac{(-1)^i}{(\frac{n}{2} + j)i!(N - i - j - 1)!} \binom{\frac{n}{2} + i + j + N - 1}{N - j - m - 1} \\ &\quad - \sum_{i=0}^{N-1-j} \frac{(-1)^i i}{i!(\frac{n}{2} + j)(\frac{n}{2} + j + i)(N - i - j - 1)!} \binom{\frac{n}{2} + i + j + N - 1}{N - j - m - 1}. \end{aligned}$$



The first sum in the above expression for  $D(j, m)$  turns out to vanish for  $m \geq 1$ . This is because

$$\begin{aligned} & \sum_{i=0}^{N-1-j} \frac{(-1)^i}{\left(\frac{n}{2} + j\right)! (N - i - j - 1)!} \binom{\frac{n}{2} + i + j + N - 1}{N - j - m - 1} \\ &= \frac{1}{(N - j - 1)! \left(\frac{n}{2} + j\right)} \sum_{i=0}^{N-1-j} (-1)^i \binom{N - j - 1}{i} \binom{\frac{n}{2} + i + j + N - 1}{N - j - m - 1} \\ &= (-1)^{N-j-1} \binom{\frac{n}{2} + j + N - 1}{-m} \\ &= 0, \end{aligned}$$

where the next to the last equality follows from an identity proven in [GKP94, (5.24), p. 169] and the last equality follows from the fact that  $\binom{s}{k} = 0$  provided  $k$  is a negative integer. Hence, setting  $s = i - 1$ ,

$$\begin{aligned} D(j, m) &= - \sum_{i=0}^{N-1-j} \frac{(-1)^i i}{i! \left(\frac{n}{2} + j\right) \left(\frac{n}{2} + j + i\right) (N - i - j - 1)!} \binom{\frac{n}{2} + i + j + N - 1}{N - j - m - 1} \\ &= \frac{1}{\frac{n}{2} + j} \sum_{s=0}^{N-(j+1)-1} \frac{(-1)^s}{s! \left(\frac{n}{2} + s + (j + 1)\right)} \binom{\frac{n}{2} + (j+1) + N + s - 1}{N - (j+1) - (m-1) - 1} \\ &= \frac{1}{\left(\frac{n}{2} + j\right)} D(j + 1, m - 1). \end{aligned}$$

Repeating the above argument  $m$  times we obtain that

$$D(j, m) = \frac{1}{\left(\frac{n}{2} + j\right) \left(\frac{n}{2} + j + 1\right) \cdots \left(\frac{n}{2} + j + m - 1\right)} D(L, 0).$$

To compute  $D(L, 0)$  we use the elementary identity

$$\frac{1}{\left(\frac{n}{2} + L + s\right)} = \frac{s! (N - L - s - 1)! \Gamma\left(\frac{n}{2} + L\right)}{\Gamma\left(\frac{n}{2} + N\right)} \binom{\frac{n}{2} + L + s - 1}{s} \binom{\frac{n}{2} + N - 1}{N - L - s - 1},$$

from which we get

$$\begin{aligned} D(L, 0) &= \sum_{s=0}^{N-1-L} (-1)^s \frac{\Gamma\left(\frac{n}{2} + L\right)}{\Gamma\left(\frac{n}{2} + N\right)} \\ &\quad \times \binom{\frac{n}{2} + L + s - 1}{s} \binom{\frac{n}{2} + N - 1}{N - L - s - 1} \binom{\frac{n}{2} + L + N + s - 1}{N - L - 1} \\ &= \frac{\Gamma\left(\frac{n}{2} + L\right)}{\Gamma\left(\frac{n}{2} + N\right)} \sum_{s=0}^{N-1-L} \binom{-\frac{n}{2} - L}{s} \binom{\frac{n}{2} + N - 1}{N - L - s - 1} \binom{\frac{n}{2} + L + N + s - 1}{N - L - 1} \\ &= \frac{\Gamma\left(\frac{n}{2} + L\right)}{\Gamma\left(\frac{n}{2} + N\right)} \binom{N - 1}{L}, \end{aligned}$$

where in the second identity we applied [GKP94, (5.14), p. 164] and the latest equality is consequence of the triple-binomial identity [GKP94, (5.28), p. 171] (for  $n = N - L - 1$ ,  $m = 0$ ,  $r = \frac{n}{2} + N + L - 1$  and  $s = N - 1$ ). Consequently,

$$\begin{aligned} D(j, m) &= \frac{\Gamma(\frac{n}{2} + L)}{\Gamma(\frac{n}{2} + N)} \binom{N - 1}{L} \frac{1}{(\frac{n}{2} + j) \cdots (\frac{n}{2} + L - 1)} \\ &= \frac{\Gamma(\frac{n}{2} + j)}{\Gamma(\frac{n}{2} + N)} \binom{N - 1}{L}. \end{aligned}$$

Hence

$$C = \binom{\frac{n}{2} + N - 1}{N - j} (N - j)! \frac{\Gamma(\frac{n}{2} + j)}{\Gamma(\frac{n}{2} + N)} \binom{N - 1}{L},$$

and

$$\begin{aligned} B &= \frac{\Gamma(\frac{n}{2} + j)}{\Gamma(\frac{n}{2} + N)} \frac{(N - L - 1)!(L - k)!}{(N - k)!} \binom{\frac{n}{2} + j + L - 1}{L - k} \binom{\frac{n}{2} + N - 1}{N - j} \\ &\quad \times \binom{\frac{n}{2} + N - 1}{N - j} (N - j)! \frac{\Gamma(\frac{n}{2} + j)}{\Gamma(\frac{n}{2} + N)} \binom{N - 1}{L}. \end{aligned}$$

Finally, after appropriate simplifications,

$$\begin{aligned} A &= \frac{(-1)^j}{\Gamma(\frac{n}{2} + j + L)} B \\ &= (-1)^j \frac{(N - L - 1)!}{(N - k)!} \frac{\binom{N - 1}{L}}{\Gamma(\frac{n}{2} + N) k! \binom{\frac{n}{2} + j + k - 1}{k}}, \end{aligned}$$

which completes the proof of the lemma. □

### 8. Examples and questions

*Example 1.* Consider the polynomial operator  $T = T_\lambda$  in  $\mathbb{R}^2$  determined by

$$\Omega(x, y) = \frac{xy}{|z|^2} + \lambda \frac{x^3y - xy^3}{|z|^4},$$

where  $z = x + iy$ . We claim that the inequality  $\|T_\lambda^* f\|_2 \leq C \|T_\lambda f\|_2$ ,  $f \in L^2(\mathbb{R}^2)$ , holds if and only if  $|\lambda| < 2$ . This follows from the theorem because the multiplier of the operator  $T_\lambda$  is

$$\frac{\xi\eta}{\xi^2 + \eta^2} \left( -\pi + \lambda \frac{\pi}{2} \frac{(\xi^2 - \eta^2)}{\xi^2 + \eta^2} \right),$$

$\frac{\xi\eta}{\xi^2 + \eta^2}$  is the multiplier of a second order Riesz Transform and the function  $-\pi + \lambda \frac{\pi}{2} \frac{(\xi^2 - \eta^2)}{\xi^2 + \eta^2}$  has no zeroes on the unit circle if and only if  $|\lambda| < 2$ .

*Example 2.* Let  $B$  stand for the Beurling transform and consider the operator

$$T_\lambda = B - \lambda B^2, \quad \lambda \in \mathbb{C},$$

which corresponds to

$$\Omega(z) = -\frac{1}{\pi} \left( \frac{\bar{z}^2 |z|^2 + 2\lambda \bar{z}^4}{|z|^4} \right).$$

In this case the multiplier is

$$\frac{\bar{\xi}^3 \xi - \lambda \bar{\xi}^4}{|\xi|^4} = \frac{\bar{\xi}}{\xi} \left( 1 - \lambda \frac{\bar{\xi}}{\xi} \right), \quad \xi \in \mathbb{C},$$

which is the multiplier of  $B$  times a function that vanishes on the unit circle if and only if  $|\lambda| = 1$ . Then the theorem tells us that  $\|T_\lambda^* f\|_2 \leq C \|T_\lambda f\|_2$ ,  $f \in L^2(\mathbb{R}^2)$ , if and only if  $|\lambda| \neq 1$ . Indeed, strictly speaking, the necessary condition of the theorem, i.e., (iii) implies (i), applies only to *real* homogeneous polynomials. However, in the case at hand, the factorization results one needs can be checked by direct inspection.

*Example 3.* We give an example of a polynomial operator  $T$  which is of the form  $T = R \circ U$ , where  $R$  is an even higher order Riesz Transform and  $U$  is invertible, but the  $L^2$  inequality  $\|T^* f\|_2 \leq C \|T f\|_2$ ,  $f \in L^2(\mathbb{R}^n)$ , does not hold. The operator  $T$  is associated with the homogenous polynomial of degree 8

$$P(x, y) = \frac{1}{\gamma_2} P_2(x, y)(x^2 + y^2)^3 + \varepsilon \left( \frac{1}{\gamma_4} P_4(x, y)(x^2 + y^2)^2 - \frac{1}{\gamma_8} P_8(x, y) \right),$$

where

$$P_2(x, y) = xy, \quad P_4(x, y) = x^4 - 6x^2y^2 + y^4,$$

and

$$P_8(x, y) = x^8 + y^8 - 28x^6y^2 - 28x^2y^6 + 70x^4y^4$$

are harmonic polynomials and  $\varepsilon$  is small enough. Notice that  $P_2$  does not divide  $P_4$  or  $P_8$ . Therefore, by the Theorem, there is no control of  $T^*$  in terms of  $T$ . On the other hand,  $P_4$  and  $P_8$  have been chosen so that  $P_4(\xi_1, \xi_2)|\xi|^4 - P_8(\xi_1, \xi_2)$  is divisible by  $P_2$ , and so the multiplier of  $T$  is

$$\frac{P_2(\xi_1, \xi_2)}{|\xi|^2} + \varepsilon \left( \frac{P_4(\xi_1, \xi_2)}{|\xi|^4} - \frac{P_8(\xi_1, \xi_2)}{|\xi|^8} \right) = \frac{P_2(\xi_1, \xi_2)}{|\xi|^2} \left( 1 + \varepsilon \frac{Q(\xi_1, \xi_2)}{|\xi|^6} \right),$$

where  $Q$  is a homogeneous polynomial of degree 6. Define  $R$  as the higher order Riesz transform whose multiplier is  $\frac{P_2(\xi_1, \xi_2)}{|\xi|^2}$  and  $U$  as the operator whose multiplier is  $1 + \varepsilon \frac{Q(\xi_1, \xi_2)}{|\xi|^6}$ . If  $\varepsilon$  is small enough, then  $U$  is invertible.

*Example 4.* We show here that condition (iii) in the theorem is very restrictive. Take  $n = 2$ . Since  $\Omega(e^{i\theta})$  is an even function, it has, modulo a rotation, a Fourier series expansion of the type

$$(80) \quad \Omega(e^{i\theta}) = \sum_{j \geq 1} a_j \sin(2j\theta).$$

Thus the harmonic polynomials  $P_{2j}$  are

$$P_{2j}(z) = a_j \frac{z^{2j} - \bar{z}^{2j}}{2i}.$$

If  $a_j \neq 0$ , then  $P_{2j}$  vanishes exactly on  $2j$  straight lines through the origin uniformly distributed and containing the two axis. Each such line is determined by a pair of opposed  $4j$ -th roots of the unity. Assume now that the operator determined by  $\Omega(e^{i\theta})$  satisfies condition (iii) of the theorem. Let  $j_0$  be the first positive integer with  $a_{j_0} \neq 0$ . Then only the  $a_j$  with  $j$  a multiple of  $j_0$  may be nonzero, owing to the particular structure of the zero set of  $P_{2j}$ . In other words, the first nonzero  $P_{2j}$  determines all the others, modulo the constants  $a_j$ .

*Example 5.* Now, we show a method to construct even kernels in  $\mathbb{R}^3$  that satisfy condition (iii) in the theorem. It can be easily adapted to any dimension. The kernel is determined by the function

$$\Omega(x, y, z) = xy \sum_{j \geq 0} \epsilon_j Q_{2j}(x, y, z),$$

where the sequence  $(\epsilon_j)$  is chosen so that  $\Omega$  is in  $C^\infty(S^2)$  and the  $Q_{2j}$  are defined by

$$Q_{2j}(x, y, z) = \sum_{k=0}^{2j} c_k y^{2k} z^{2j-2k}.$$

The  $c_k$  are determined by a recurrent formula obtained by requiring that  $xy Q_{2j}(x, y, z)$  is harmonic. Computing its Laplacean, we get the recurrent condition

$$c_k = -c_{k-1} \frac{2k(2k+1)}{(2j-2k+1)(2j-2k+2)}, \quad 1 \leq k \leq j,$$

where  $c_0$  may be freely chosen.

We would like to close the paper by asking a couple of questions which we have not been able to answer.

*Question 1.* Since our methods are very much dependent on the Fourier transform, we do not know whether either the weak type inequality

$$\|T^*f\|_{1,\infty} \leq C\|Tf\|_1$$

or the  $L^p$  inequality with  $1 < p < \infty$ ,  $p \neq 2$ ,

$$\|T^*f\|_p \leq C\|Tf\|_p, \quad f \in L^p(\mathbb{R}^n),$$

imply the  $L^2$  inequality

$$\|T^*f\|_2 \leq C\|Tf\|_2, \quad f \in L^2(\mathbb{R}^n).$$

As suggested by Carlos Pérez, this might be related to interpolation results for couples of sub-linear operators.

*Question 2.* How far may the smoothness assumption on  $\Omega$  be weakened? More concretely, does the theorem still hold true for  $\Omega$  of class  $C^m(S^{n-1})$  for some positive integer  $m$ ?

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