Annals of Mathematics **174** (2011), 519–541
doi: 10.4007/annals.2011.174.1.15

Wiener's 'closure of translates' problem and Piatetski-Shapiro's uniqueness phenomenon

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Abstract

N. Wiener characterized the cyclic vectors (with respect to translations) in $\ell^p(\mathbb{Z})$ and $L^p(\mathbb{R})$, $p = 1, 2$, in terms of the zero set of the Fourier transform. He conjectured that a similar characterization should be true for $1 < p < 2$. Our main result contradicts this conjecture.

1. Introduction

1.1. Let G be a locally-compact abelian group, and $1 \leqslant p < \infty$. A function $f \in L^p(G)$ is called a *cyclic vector* (with respect to translations) if the linear span of its translates is dense in the space. It is well known that $f \in L^p(\mathbb{T})$ (where $\mathbb{T}$ is the circle group) is a cyclic vector if and only if all the Fourier coefficients of f are nonzero. The same is true for general compact groups (see [23]).

In the noncompact case the situation is more complicated. N. Wiener [24] characterized the cyclic vectors in $L^p(\mathbb{R})$ (or $\ell^p(\mathbb{Z})$) only for $p = 1$ and 2. We formulate the result for $\ell^p(\mathbb{Z})$, the $L^p(\mathbb{R})$ case is similar.

Theorem A (Wiener). *Let* $\mathbf{c} = \{c_n\}$, $n \in \mathbb{Z}$.

(i) $\mathbf{c}$ *is a cyclic vector in* $\ell^2(\mathbb{Z})$ *if and only if the Fourier transform*

$$\widehat{\mathbf{c}}(t) := \sum_{n\in\mathbb{Z}} c_n e^{int} \tag{1}$$

is nonzero almost everywhere.

(ii) $\mathbf{c}$ *is cyclic in* $\ell^1(\mathbb{Z})$ *if and only if* $\widehat{\mathbf{c}}(t)$ *has no zeros.*

Part (i) is a consequence of the unitarity of the Fourier transform. Part (ii) is more delicate; the proof is based on the fact that the space $\ell^1(\mathbb{Z})$ is a convolution algebra.

Research supported in part by the Israel Science Foundation.

In both cases the result can be stated as follows: $\mathbf{c}$ is a cyclic vector if and only if the set

$$Z_{\hat{\mathbf{c}}} := \{t \in \mathbb{T} : \widehat{\mathbf{c}}(t) = 0\}$$

of the zeros of the Fourier transform (1) is "small" in a certain sense. Wiener conjectured (see [24, p. 93]) that a similar result should be true for ℓ^p spaces, at least for $1 < p < 2$. This problem has been studied by Segal [22], Beurling [5], Pollard [19], Herz [7], Newman [18] and other authors (see [6], [13], [21]).

First of all, one should define precisely how to understand the zero set. The answer is obvious if the vector $\mathbf{c}$ is assumed to be in $\ell^1(\mathbb{Z})$ (or $L^1(\mathbb{R})$). The above mentioned authors have studied the problem under this assumption. We shall keep it as well.

A more serious question is — what kind of "smallness" should one consider? One can show (see [22]) that if $1 < p < 2$, then the condition in part (i) of Theorem A is not sufficient, and the condition in part (ii) is not necessary for cyclicity in ℓ^p. So one should look for an "intermediate measurement" of smallness.

A. Beurling proved in [5] that if the Hausdorff dimension of $Z_{\hat{\mathbf{c}}}$ is less than $2(p-1)/p$, then $\mathbf{c}$ is a cyclic vector in ℓ^p $(1 < p < 2)$. This condition is sharp, but it is not necessary for the cyclicity (see [18]).

On the other hand, it is well known that not only metrical but also arithmetical "thinness" properties may play an important role in problems of harmonic analysis. In the above cited papers, various metrical and nonmetrical properties of the zero set of cyclic vectors in $L^p(\mathbb{R})$ $(\ell^p(\mathbb{Z}))$ have been studied, and a number of interesting results were obtained. In particular, for $p > 2$ the cyclic vectors were indeed characterized by a condition in terms of the zero set. This condition (see Section 2.2 below) is not easy to check, but anyway it supports Wiener's conjecture that cyclicity depends on the set $Z_{\hat{\mathbf{c}}}$ only.

Our main result shows that this is not the case for $1 < p < 2$.

Theorem 1. *Let $1 < p < 2$. Then there is a compact set K on the circle $\mathbb{T}$ with the following properties*:

(a) *If a vector $\mathbf{c}$ has fast decreasing coordinates, say $\sum_{n\in\mathbb{Z}} |c_n|\,|n|^{\varepsilon} < \infty$ for some $\varepsilon > 0$, and $\widehat{\mathbf{c}}$ vanishes on K, then $\mathbf{c}$ is not cyclic in $\ell^p(\mathbb{Z})$.*

(b) *There exists $\mathbf{c} \in \ell^1(\mathbb{Z})$, such that $\widehat{\mathbf{c}}$ vanishes on K, and $\mathbf{c}$ is a cyclic vector in $\ell^p(\mathbb{Z})$.*

It follows that no characterization of the cyclic vectors exists in terms of the zeros of the Fourier transform:

Corollary 1. *Given any p, $1 < p < 2$, one can find two vectors in $\ell^1(\mathbb{Z})$ such that one is cyclic in $\ell^p(\mathbb{Z})$ and the other is not, but their Fourier transforms have an identical set of zeros.*

A similar result is true for $L^p(\mathbb{R})$ (see Section 6 below).

1.2. Our approach to the problem is based on its relation to the uniqueness problem in Fourier analysis, or more specifically, to an aspect of it which we call the "Piatetski-Shapiro phenomenon".

Recall that a set $K \subset \mathbb{T}$ is called a *set of uniqueness* (U-set) if whenever a trigonometric series

$$\sum_{n \in \mathbb{Z}} c_n e^{int}$$

converges to zero at every point $t \notin K$, then all the coefficients c_n must be zero. Otherwise, K is called a *set of multiplicity* (M-set). Classical Riemannian theory allows one to characterize the compact M-sets as the compacts which support a nonzero distribution S with Fourier transform $\widehat{S}(n)$ tending to zero as $|n| \to \infty$ (see [10]).

It was D. E. Menshov who discovered (1916) that a set of Lebesgue measure zero can be an M-set. In fact, Menshov constructed a compact set K of Lebesgue measure zero, which supports a *measure* whose Fourier transform vanishes at infinity (see [3]). It was believed for a long time that every compact M-set must support such a measure. This was disproved in 1954 by I. Piatetski-Shapiro [20], who constructed a compact M-set which does not support such a measure.

This striking result was further developed by T. Körner [14] and R. Kaufman [12], who presented different examples of compact M-sets K which are Helson sets. The latter means that every measure μ supported by K satisfies the condition

$$\limsup_{|n| \to \infty} |\widehat{\mu}(n)| \geqslant \delta(K) \int |d\mu|, \quad \delta(K) > 0.$$

As Kaufman mentioned, his construction was inspired by Piatetski-Shapiro's original ideas. An additional improvement of this technique was done by J.-P. Kahane [10, pp. 213–216] in his presentation of Kaufman's paper. This presentation was the starting point for our approach.

1.3. Let X be a Banach space of sequences (with a norm weaker than ℓ^2). We say that *Piatetski-Shapiro's phenomenon exists for the space* X if there is a compact set $K \subset \mathbb{T}$, which supports a (nonzero) distribution S with Fourier transform $\widehat{S} \in X$, but which does not support such a measure. The result from [20] means that the phenomenon exists for the space c_o. On the other hand, potential theory (see [4], [10]) provides an important example of spaces for which the phenomenon does not exist: the weighted spaces $\ell^2(\mathbb{Z}, w)$ with e.g. the power weight $w(n) = (1 + |n|)^{-\alpha}$, $0 < \alpha \leqslant 1$.

Our concern, inspired by Wiener's problem, was:

Does Piatetski-Shapiro's phenomenon exist for ℓ^q spaces, $q > 2$?

The answer is *yes*:

THEOREM 2 ([15]). *Given any* $q>2$ *there is a compact set* $K\subset\mathbb{T}$ *such that*

(a′) K *supports a nonzero distribution* S *such that* $\widehat{S}\in\ell^q$;

(b′) K *does not support any nonzero measure* μ *such that* $\widehat{\mu}\in\ell^q$.

The role of Theorem 2 in the cyclicity problem is clarified by the following observation (see Section 2.2 below):

Condition (*a*′) *is equivalent to part* (*a*) *of Theorem* 1, *while condition* (*b*′) *is necessary for part* (*b*), *with* $q=p/(p-1)$.

So Theorem 2 provides a chance for (although it does not imply) the existence of a counterexample to Wiener's conjecture. Such a counterexample — in a weaker form than Corollary 1 above — was sketched in [16]. The present paper contains full proofs and extensions of the results obtained in [15], [16]. It is organized as follows.

In Section 2 we give some preliminary background and auxiliary lemmas.

Section 3 is the key one in the paper. Our main tools are special measures on the circle, defined by Riesz-type products, and a version of Bernstein's stochastic exponential estimate.

In Section 4 we construct a Helson set with the property (a′) above. In Section 5 we prove that every Helson set admits a vector with property (b). So Theorem 1 follows.

The nonperiodic version is considered in Section 6. Section 7 contains some additional remarks. In particular, we discuss there the relation of Theorem 1 to P. Malliavin's celebrated "non-synthesis" phenomenon, and mention some open problems.

2. **Preliminaries. Lemmas.**

2.1. *Notation.* In what follows $\mathbb{T}$ is the circle group $\mathbb{R}/2\pi\mathbb{Z}$. As usual, $C(\mathbb{T})$ is the space of continuous complex functions on $\mathbb{T}$, with the norm $\|f\|_\infty:=\sup|f(t)|$, $t\in\mathbb{T}$. By a "measure" on $\mathbb{T}$ we always mean an element of the dual space $M(\mathbb{T})$, that is, a finite complex Borel measure.

We denote by $\{\widehat{S}(n)\}$, $n\in\mathbb{Z}$, the Fourier coefficients of a Schwartz distribution S on $\mathbb{T}$. It will also be convenient to keep the notation $\widehat{\mathbf{c}}$ for the Fourier transform of a vector $\mathbf{c}\in\ell^1(\mathbb{Z})$ as defined in (1).

Let $A_p(\mathbb{T})$, $1\leqslant p\leqslant\infty$, denote the Banach space of distributions S on $\mathbb{T}$ with Fourier coefficients belonging to $\ell^p(\mathbb{Z})$, endowed with the norm

$$\|S\|_{A_p}:=\|\widehat{S}\|_{\ell^p}.$$

For $p=1$ this is the Wiener algebra $A(\mathbb{T})$ of absolutely convergent Fourier series (see [9]). Throughout we will use the following standard properties:

$$\|f\|_\infty\leqslant\|f\|_A,\quad f\in A(\mathbb{T}),$$
$$\|f\cdot g\|_{A_p}\leqslant\|f\|_A\,\|g\|_{A_p},\quad f\in A(\mathbb{T}),\ g\in A_p(\mathbb{T}).$$

2.2. *Cyclic vectors.* In this section we refer to some basic results about cyclicity. The results go back to Segal [23], Beurling [5], Pollard [19], Herz [7] and Newman [18]. Actually the first four authors considered Wiener's problem in $L^p(\mathbb{R})$ rather than $\ell^p(\mathbb{Z})$, but, as the last author mentioned, "the distinction is not vital". See also Kahane-Salem [10, pp. 111–112 and 122–123].

First it would be convenient to reformulate the concept of cyclicity in an equivalent way, using the following:

Definition. An element $f \in A_p(\mathbb{T})$, $1 \leqslant p < \infty$, is called a *cyclic vector* (with respect to multiplication by exponentials) if the set $\{P(t)f(t)\}$, where P goes through all trigonometric polynomials, is dense in $A_p(\mathbb{T})$.

Clearly, a vector $\mathbf{c}$ is cyclic (with respect to translations) in $\ell^p(\mathbb{Z})$ if and only if its Fourier transform $f := \widehat{\mathbf{c}}$ is cyclic in $A_p(\mathbb{T})$ in the sense just defined.

In what follows f is assumed to belong to the Wiener algebra $A(\mathbb{T})$, Z_f denotes the set of the zeros of f, and $q = p/(p-1)$.

(i) *f is cyclic in $A_p(\mathbb{T})$ if and only if there is a sequence of trigonometric polynomials P_n such that*

$$\lim_{n\to\infty} \|1 - P_n \cdot f\|_{A_p} = 0.$$

(ii) *If Z_f is finite, then f is cyclic in $A_p(\mathbb{T})$ for every $p > 1$.*

(iii) *If f is a noncyclic vector in $A_p(\mathbb{T})$, then there is a nonzero distribution $S \in A_q(\mathbb{T})$, supported by Z_f.*

(iv) *If there is a nonzero measure $\mu \in A_q(\mathbb{T})$, supported by Z_f, then f is a noncyclic vector in $A_p(\mathbb{T})$.*

(v) *If f is continuously differentiable and there is a nonzero distribution $S \in A_q(\mathbb{T})$ supported by Z_f, then f is a noncyclic vector in $A_p(\mathbb{T})$.*

Actually (see [7]) the smoothness condition in (v) can be reduced up to $f \in \operatorname{Lip}\varepsilon$ for some $\varepsilon > 0$, or, in terms of the Fourier coefficients of f, up to

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|\, |n|^{\varepsilon} < \infty \quad \text{for some } \varepsilon > 0.$$

Observe that if $p > 2$, then $A_q(\mathbb{T})$ is a functional space (embedded in $L^2(\mathbb{T})$). So conditions (iii)–(iv) imply:

A function $f \in A(\mathbb{T})$ is a cyclic vector in $A_p(\mathbb{T})$, $p > 2$, if and only if its zero set Z_f does not support any nonzero function $g \in A_q(\mathbb{T})$.

This condition is not very effective, but it shows that the cyclicity of a vector $c \in \ell^1(\mathbb{Z})$ in the space $\ell^p(\mathbb{Z})$, $p > 2$, admits characterization in terms of the zero set of the Fourier transform (as Wiener thought). Here, by the way, the condition that Z_f has Lebesgue measure zero obviously implies the cyclicity, but not vice versa [18] (see also [11, pp. 101–102]).

The case $p = \infty$ was also considered. D. Newman [18] proved that $\mathbf{c} \in \ell^1(\mathbb{Z})$ is a cyclic vector in the space $c_0(\mathbb{Z})$ if and only if $Z_{\hat{\mathbf{c}}}$ is a nowhere dense set in $\mathbb{T}$.

Now let $1 < p < 2$ (the case where Wiener's conjecture was most certain). Then $A_q(\mathbb{T})$ is not a functional space, and (iii)–(v) only imply the following:

Let $f \in A(\mathbb{T})$ *be smooth* ($f \in \operatorname{Lip} \varepsilon$, $\varepsilon > 0$). *Then it is cyclic in* $A_p(\mathbb{T})$, $1 < p < 2$, *if and only if* Z_f *does not support any nonzero distribution* $S \in A_q(\mathbb{T})$.

However we will see that without the smoothness condition, the zero set Z_f does not provide a characterization of the cyclic vectors.

2.3. *Auxiliary polynomials.* We shall use trigonometric polynomials with the following properties.

LEMMA 1. *Given any* $q > 2$ *and* $\gamma > 0$ *there is a real trigonometric polynomial* $\varphi = \varphi_{q,\gamma}$ *such that*

$$\widehat{\varphi}(0) = 0, \quad \|\varphi\|_\infty \leqslant 1, \quad \|\varphi\|_{L^2} = \tfrac{1}{2}, \quad \|\varphi\|_{A_q} < \gamma. \tag{2}$$

Here and below, $\|\cdot\|_{L^2}$ denotes the L^2 norm on $\mathbb{T}$ with respect to the normalized Lebesgue measure.

There are several ways to get Lemma 1. In particular, one may use the Shapiro-Rudin polynomials (see [9, p. 52]). Namely, for an appropriate choice of signs $\varepsilon_n = \pm 1$ ($n = 1, 2, \dots$) the trigonometric polynomial

$$Q_k(t) = \sum_{n=1}^{2^k} \varepsilon_n \cos nt$$

satisfies $\|Q_k\|_\infty \leqslant 2^{(k+1)/2}$. It follows that if $k = k(q, \gamma)$ is sufficiently large, then

$$\varphi(t) := 2^{-(k+1)/2}\, Q_k(t)$$

is a real trigonometric polynomial with properties (2).

2.4. *Kahane's lemma.* One of the key arguments in [20] and [12] is based on the uniqueness theorem for power series. In Kahane's presentation of Kaufman's paper (see [10, pp. 213–216]) this point was performed as follows:

Given any $\delta > 0$ *there is a real, signed measure* ρ, *supported by a finite subset of the interval* $(1 - \delta, 1)$, *such that*

$$\int d\rho = 1 \qquad \text{and} \qquad \left| \int s^k \, d\rho(s) \right| < \delta \quad (k = 1, 2, \dots). \tag{3}$$

This lemma was proved in [10, p. 214] based on the Hahn-Banach theorem. Here we shall need a quantitative version, with an estimate on the total variation of the measure.

LEMMA 2. *Let an interval $I = (a, b)$, $0 < a < b < \frac{1}{2}$, and $0 < \delta < 1$ be given. Then there is a real, signed measure ρ, supported by a finite subset of I, such that* (3) *holds, and such that*

$$\int |d\rho| < \delta^{-c(I)}, \tag{4}$$

where $c(I) > 0$ is a constant which depends only on I.

We proceed to the proof of Lemma 2.

2.4.1. *The measure.* Given n distinct points $s_1, \dots, s_n \in I$, consider a measure ρ supported by these points and defined uniquely by the condition

$$\int p(s)\, d\rho(s) = p(0), \quad \text{for every algebraic polynomial } p \text{ of degree} \leqslant n-1.$$

In particular,

$$\int d\rho = 1, \qquad \int s^k\, d\rho(s) = 0 \quad (k = 1, 2, \dots, n-1). \tag{5}$$

Given any function $f(s)$ one has $\int f(s)\, d\rho(s) = p(0)$, where p is the unique polynomial of degree $\leqslant n-1$ which interpolates f at the nodes $s_1, \dots, s_n$. It is well known (see for example [1, pp. 134–135]) that if $f(s)$ is real-valued and sufficiently smooth, then there is $0 \leqslant \xi < b$ such that

$$f(0) = p(0) + \frac{f^{(n)}(\xi)}{n!} \prod_{j=1}^{n} (0 - s_j).$$

Applying this with $f(s) = s^k$, $k \geqslant n$, gives

$$\int s^k\, d\rho(s) = (-1)^{n-1} \binom{k}{n} \xi^{k-n} \prod_{j=1}^{n} s_j\,,$$

and consequently the moments of ρ satisfy the estimate

$$\left| \int s^k\, d\rho(s) \right| < 2^k \cdot b^{k-n} \cdot b^n = (2b)^k \leqslant (2b)^n \quad (k \geqslant n). \tag{6}$$

2.4.2. *The total variation.* Using the Lagrange polynomials

$$l_j(s) = \prod_{i \neq j} \frac{s - s_i}{s_j - s_i} \quad (1 \leqslant j \leqslant n),$$

one can calculate the masses

$$\rho(\{s_j\}) = \int l_j(s)\, d\rho(s) = l_j(0) = \prod_{i \neq j} \frac{s_i}{s_i - s_j}\,.$$

We choose the points $s_1, \dots, s_n$ as equally spaced nodes, $s_j = a + (j - \frac{1}{2})h$ where $h = (b-a)/n$. Then

$$\left| \rho(\{s_j\}) \right| = \frac{1}{h^{n-1}\,(j-1)!\,(n-j)!} \prod_{i \neq j} s_i\,,$$

and so we have

$$\int |d\rho| \leqslant \Big(\frac{b}{h}\Big)^{n-1} \sum_{j=1}^{n} \frac{1}{(j-1)!\,(n-j)!} = \frac{1}{n!}\Big(\frac{nb}{b-a}\Big)^{n-1} \sum_{j=1}^{n} j\binom{n}{j} = \frac{n^n}{n!}\Big(\frac{2b}{b-a}\Big)^{n-1} \leqslant \Big(\frac{2eb}{b-a}\Big)^{n-1}. \tag{7}$$

Finally, choose n to be the least integer $\geqslant \frac{\log(1/\delta)}{\log(1/2b)}$. It follows from (5), (6) and (7) that ρ satisfies both (3) and (4). This proves the lemma. □

Remark. One can show that a power estimate (4) in Lemma 2 is sharp.

2.5. *Bernstein inequality.* Bernstein exponential estimates for sums of independent random variables are classical. Different versions, adopted for sums of “almost” independent variables, in various senses, are also well known.

In particular, Azuma [2] considered the so-called multiplicatively orthogonal systems and obtained Bernstein-type exponential estimates for them.

It will be convenient for us to consider a similar version, suitable for an “almost multiplicative” system of random variables, in the following sense:

LEMMA 3. *Let $X_1, \dots, X_N$ be random variables on a probability space (Ω, P) such that $-1 \leqslant X_j \leqslant 1$ $(j = 1, 2, \dots, N)$. Suppose that*

$$\mathbb{E}(X_1) = \cdots = \mathbb{E}(X_N) = \mu > 0 \tag{8}$$

and that there is $0 < \varepsilon < 1$ such that

$$(1-\varepsilon)\,\mu^{|A|} \leqslant \mathbb{E}\Big\{\prod_{j\in A} X_j\Big\} \leqslant (1+\varepsilon)\,\mu^{|A|} \tag{9}$$

for every nonempty subset $A \subset \{1, 2, \dots, N\}$, where $|A|$ denotes the number of elements in A. Define

$$X = \frac{1}{N}\sum_{j=1}^{N} X_j\,.$$

Then for any $\alpha > 0$,

$$P\big\{X < \mathbb{E}(X) - \alpha\big\} \leqslant \exp\big(-\tfrac{1}{8}\alpha^2 N\big) + \varepsilon \exp\big(\tfrac{1}{4}N\big). \tag{10}$$

Proof. Fix $\lambda > 0$. By the classical Bernstein method we can estimate the probability on the left-hand side of (10) as follows:

$$P\big\{X < \mu - \alpha\big\} = P\Big\{\prod_{j=1}^{N} e^{\lambda(\mu - X_j)} > e^{\alpha\lambda N}\Big\} \leqslant e^{-\alpha\lambda N}\,\mathbb{E}\prod_{j=1}^{N} e^{\lambda(\mu - X_j)}. \tag{11}$$

To estimate the expectation on the right-hand side we adopt the approach of [2]. Since $|\mu - X_j| \leqslant 2$ and by the convexity of the exponential function, we have

$$e^{\lambda(\mu - X_j)} \leqslant \cosh(2\lambda) + ((\mu - X_j)/2)\,\sinh(2\lambda) = b - aX_j,$$

where

$$a := (1/2)\,\sinh(2\lambda) \quad \text{and} \quad b := \cosh(2\lambda) + (\mu/2)\,\sinh(2\lambda).$$

It follows that

$$\mathbb{E}\prod_{j=1}^{N} e^{\lambda(\mu - X_j)} \leqslant \mathbb{E}\prod_{j=1}^{N}(b - aX_j) = \sum_{A\subset\{1,\dots,N\}} (-a)^{|A|}\, b^{N-|A|}\, \mathbb{E}\prod_{j\in A} X_j. \tag{12}$$

Now we invoke the assumption (9), which (together with the fact that a, b are positive numbers) implies that the right-hand side of (12) is not larger than

$$\sum (-a)^{|A|}\, b^{N-|A|} \Big\{1 + (-1)^{|A|}\,\varepsilon\Big\}\, \mu^{|A|} = (b - a\mu)^N + \varepsilon(b + a\mu)^N.$$

It is easy to see that $b - a\mu \leqslant \exp(2\lambda^2)$ and $b + a\mu \leqslant \exp(2\lambda)$, so it follows that

$$\mathbb{E}\prod_{j=1}^{N} e^{\lambda(\mu - X_j)} \leqslant \exp(2\lambda^2 N) + \varepsilon\, \exp(2\lambda N). \tag{13}$$

Finally, a combination of (11) and (13), with $\lambda = \alpha/4$, gives

$$P\Big\{X < \mu - \alpha\Big\} \leqslant \exp\Big(-\tfrac{1}{8}\alpha^2 N\Big) + \varepsilon \exp\Big(\tfrac{1}{4}\alpha(2-\alpha)N\Big).$$

However $\alpha(2 - \alpha) \leqslant 1$, so the estimate (10) follows. □

3. **Riesz-type products and exponential estimates**

This section contains the central part of our approach. We prove here the following main lemma:

LEMMA 4. *Suppose that we are given numbers $q > 2$, $\varepsilon > 0$, and a real trigonometric polynomial u, not identically zero. Then we can find a compact set K (a finite union of segments), an infinitely smooth function f and a real trigonometric polynomial P such that*

(i) *f is supported by K,* $\|1 - f\|_{A_q} < \varepsilon$;

(ii) $\inf\limits_{t\in K} |P(t)| > 1$, $P(t)u(t) > 0$ *on K,* $\|P\|_A \leqslant C(q)$,

where $C(q)$ is a constant which depends only on q.

The proof involves several steps.

3.1. *Multiplicativity.* We start with the following simple property.

LEMMA 5. *Let ν be a positive integer, and suppose that P_j are trigonometric polynomials,* $\deg P_j < \nu$ $(j = 0, 1, \dots, N)$. *Then*

$$\int_{\mathbb{T}} \Big\{ \prod_{j=0}^{N} P_j(\nu^j t)\Big\} \frac{dt}{2\pi} = \prod_{j=0}^{N} \Big\{ \int_{\mathbb{T}} P_j(t)\, \frac{dt}{2\pi}\Big\}.$$

Proof. By Fourier expansion, the left-hand side is equal to

$$\sum_{\mathbf{k}} \Big\{ \prod_{j=0}^{N} \widehat{P}_j(k_j) \Big\} \int_{\mathbb{T}} e^{i(k_0+k_1\nu+k_2\nu^2+\cdots+k_N\nu^N)t} \, \frac{dt}{2\pi} \,,$$

where the sum goes through all integer vectors $\mathbf{k} = (k_0, k_1, \dots, k_N)$ such that $|k_j| \leqslant \deg P_j$. However it is easy to check that the only solution of the equation

$$k_0 + k_1\nu + k_2\nu^2 + \cdots + k_N\nu^N = 0$$

with $\mathbf{k}$ as above, is $\mathbf{k} = (0, 0, \dots, 0)$. This implies the result. □

3.2. *Riesz-type measures.* Suppose that we are given a positive integer N, a real trigonometric polynomial φ with the properties

$$\widehat{\varphi}(0) = 0, \quad \|\varphi\|_\infty \leqslant 1 \tag{14}$$

and also a real trigonometric polynomial w such that

$$\|w\|_\infty \leqslant 1. \tag{15}$$

Choose a large integer ν, satisfying the condition

$$\nu > 2 \max\{\deg \varphi,\ N \deg w\}, \tag{16}$$

and define a "Riesz-type product"

$$\lambda_s(t) = \prod_{j=1}^{N} \Big(1 + s\, w(t)\, \varphi(\nu^j t)\Big), \quad 0 < s < 1. \tag{17}$$

Introduce a measure μ_s on the circle $\mathbb{T}$:

$$d\mu_s(t) = \lambda_s(t)\, \frac{dt}{2\pi} \,. \tag{18}$$

Observe first that it is a probability measure on $\mathbb{T}$. Indeed, it is clear from the properties above that λ_s is everywhere positive. Now expand the product (17) into the form

$$\lambda_s(t) = 1 + \sum_{B} \Big(s\, w(t)\Big)^{|B|} \prod_{j\in B} \varphi(\nu^j t), \tag{19}$$

where the sum goes through all nonempty subsets $B \subset \{1, \dots, N\}$. Condition (16) allows one to use Lemma 5, which implies that

$$\int_{\mathbb{T}} \lambda_s(t)\, \frac{dt}{2\pi} = 1 + \sum_{B} \Big\{ \int_{\mathbb{T}} w(t)^{|B|}\, \frac{dt}{2\pi} \Big\} \Big\{ s \int_{\mathbb{T}} \varphi(t)\, \frac{dt}{2\pi} \Big\}^{|B|}.$$

However all terms in the above sum are zero, since $\widehat{\varphi}(0) = 0$. So it follows that

$$\int_{\mathbb{T}} \lambda_s(t)\, \frac{dt}{2\pi} = 1,$$

and this proves the claim.

3.3. *Random variables.* Consider random variables defined by

$$X_j(t) = w(t)\,\varphi(\nu^j t), \quad 1 \leqslant j \leqslant N, \tag{20}$$

on the probability space $(\mathbb{T}, \mu_s)$.

It is well known that these variables are "almost independent" with respect to the Lebesgue measure on $\mathbb{T}$. However, we will see that (under some additional condition) they are "almost independent" also with respect to μ_s, which is going to be essentially "singular" with respect to the Lebesgue measure.

To establish such a property we first compute the "multiplicative moments".

Lemma 6. *Let A be a nonempty subset of $\{1, 2, \dots, N\}$. Then*

$$\mathbb{E}\Big\{\prod_{j\in A} X_j\Big\} = \Big\{s\int_{\mathbb{T}} \varphi(t)^2\,\frac{dt}{2\pi}\Big\}^{|A|}\Big\{\int_{\mathbb{T}} w(t)^{2|A|}\,\frac{dt}{2\pi}\Big\}. \tag{21}$$

Proof. By (18) and (20), the left-hand side of (21) is equal to

$$\int_{\mathbb{T}} \Big\{w(t)^{|A|}\prod_{j\in A}\varphi(\nu^j t)\Big\}\,\lambda_s(t)\,\frac{dt}{2\pi}\,. \tag{22}$$

Let us again consider the expansion (19) for λ_s; however, this time we do not distinguish the constant term as before, but rather write

$$\lambda_s(t) = \big(s\,w(t)\big)^{|A|}\prod_{j\in A}\varphi(\nu^j t), + \cdots$$

where the implicit terms correspond to all subsets $B \subset \{1, \dots, N\}$ which are different from A. Inserting this expression into (22) one can see that the integration of the explicit term gives

$$s^{|A|}\int_{\mathbb{T}} \Big\{w(t)^{2|A|}\prod_{j\in A}\varphi^2(\nu^j t)\Big\}\frac{dt}{2\pi}$$

which, by condition (16) and Lemma 5, provides the right-hand side of (21). So to conclude the proof it is enough to show that the integrals of the other terms in the sum are all zero.

Indeed, if B is any subset $\neq A$, then the corresponding term is

$$s^{|B|}\int_{\mathbb{T}} \Big\{w(t)^{|A|+|B|}\prod_{j\in A\triangle B}\varphi(\nu^j t)\prod_{j\in A\cap B}\varphi^2(\nu^j t)\Big\}\frac{dt}{2\pi}$$

which, again by (16) and Lemma 5, is equal to

$$s^{|B|}\Big\{\int_{\mathbb{T}} w(t)^{|A|+|B|}\,\frac{dt}{2\pi}\Big\}\Big\{\int_{\mathbb{T}}\varphi(t)\,\frac{dt}{2\pi}\Big\}^{|A\triangle B|}\Big\{\int_{\mathbb{T}}\varphi^2(t)\,\frac{dt}{2\pi}\Big\}^{|A\cap B|}.$$

However this is zero, because $\widehat{\varphi}(0) = 0$, so the lemma is proved. ☐

One can see that if the trigonometric polynomial w is mostly close to 1 in modulus, then the integrals of the even powers of w which appear in (21) are almost equal to 1. We will see that in such a case the $X_1, \dots, X_N$ form an "almost multiplicative" system of random variables (in the sense of Lemma 3) with respect to the measure μ_s.

Precisely, Lemma 6 allows one to find the expectations

$$\mathbb{E}(X_j) = s\,\|\varphi\|_{L^2}^2\,\|w\|_{L^2}^2 \quad (1 \leqslant j \leqslant N). \tag{23}$$

In particular all the X_j have the same expectation, as in (8). Now suppose that the trigonometric polynomial w satisfies, in addition to property (15), also the condition

$$\Big\{\int_{\mathbb{T}} w(t)^2\,\frac{dt}{2\pi}\Big\}^N > \frac{1}{1+\varepsilon} \qquad \text{for some } 0<\varepsilon<1. \tag{24}$$

Then, given any nonempty $A \subset \{1,2,\dots,N\}$, by (21) and Jensen's inequality

$$\mathbb{E}\Big\{\prod_{j\in A} X_j\Big\} \geqslant \Big\{s\int_{\mathbb{T}} \varphi(t)^2\,\frac{dt}{2\pi}\Big\}^{|A|}\Big\{\int_{\mathbb{T}} w(t)^2\,\frac{dt}{2\pi}\Big\}^{|A|} = \prod_{j\in A}\mathbb{E}(X_j).$$

On the other hand (15), (21) and (24) imply that

$$\mathbb{E}\Big\{\prod_{j\in A} X_j\Big\} \leqslant \Big\{s\int_{\mathbb{T}} \varphi(t)^2\,\frac{dt}{2\pi}\Big\}^{|A|} \leqslant (1+\varepsilon)\prod_{j\in A}\mathbb{E}(X_j).$$

This shows that the "almost multiplicativity" condition (9) is satisfied.

3.4. *Concentration.* Define a trigonometric polynomial

$$X(t) = \frac{1}{N}\sum_{j=1}^{N} X_j(t) = w(t)\cdot\frac{1}{N}\sum_{j=1}^{N}\varphi(\nu^j t). \tag{25}$$

"Almost independence" suggests that this average is strongly concentrated (with respect to the measure μ_s) near its expectation, and the rate of concentration is governed by the classical exponential estimates.

Indeed, assuming (14), (15) and (24) one may use Lemma 3, which implies

$$\mu_s\big\{t : X(t) < \mathbb{E}(X) - \alpha\big\} \leqslant \exp\big(-\tfrac{1}{8}\alpha^2 N\big) + \varepsilon\exp\big(\tfrac{1}{4}N\big), \quad \alpha>0. \tag{26}$$

We use this to prove the following L^2-concentration estimate.

Lemma 7. *Suppose that* (14) *and* (15) *hold, and furthermore suppose that*

$$\|\varphi\|_{L^2} \geqslant \tfrac{1}{2} \tag{27}$$

and

$$\Big\{\int_{\mathbb{T}} w(t)^2\,\frac{dt}{2\pi}\Big\}^N > \frac{1}{1+e^{-N}}\,. \tag{28}$$

Then for every

$$s \in I_0 := \big(\tfrac{1}{4}, \tfrac{1}{3}\big), \tag{29}$$

one has

$$\int_{\{t\colon X(t)<c_1\}} \lambda_s^2(t)\,\frac{dt}{2\pi} \;<\; 2\,e^{-c_2 N}$$

for some absolute positive constants c_1, c_2.

Proof. It follows from (23), (27), (28) and (29) that

$$\mathbb{E}(X) = s\,\|\varphi\|_{L^2}^2\,\|w\|_{L^2}^2 > \tfrac{1}{100} \quad (s \in I_0).$$

So the estimate (26) with $\varepsilon = e^{-N}$ implies that

$$\mu_s\Big\{t : X(t) < c_1\Big\} < 2\exp\Big(-\tfrac{1}{8}(\tfrac{1}{100} - c_1)^2 N\Big), \quad 0 < c_1 < \tfrac{1}{100}. \tag{30}$$

Using (17) we also obtain the estimate

$$\lambda_s(t) \leqslant \exp\Big(s\,w(t)\sum_{j=1}^{N}\varphi(\nu^j t)\Big) = \exp\big(sNX(t)\big). \tag{31}$$

A combination of (30) and (31) gives, for every $s \in I_0$,

$$\int_{\{t\,:\,X(t)<c_1\}} \lambda_s^2(t)\,\frac{dt}{2\pi} \leqslant \bigg(\int_{\{t\,:\,X(t)<c_1\}} \lambda_s(t)\,\frac{dt}{2\pi}\bigg)\bigg(\sup_{\{t\,:\,X(t)<c_1\}} \lambda_s(t)\bigg)$$
$$< 2\exp\Big(-\tfrac{1}{8}(\tfrac{1}{100} - c_1)^2 N\Big)\exp\Big(\tfrac{1}{3}c_1 N\Big) = 2\,e^{-c_2 N}$$

for appropriate absolute positive constants c_1, c_2. □

Below we continue to denote by c_1, c_2 the constants from Lemma 7, and let $c_3, c_4, \ldots$ denote other absolute positive constants.

3.5. *Proof of Lemma* 4. Let the numbers $q > 2$ and $\varepsilon > 0$, and the real trigonometric polynomial u (not identically zero) be given. Let $N = N(\varepsilon)$ be a sufficiently large integer, which will be chosen later. Denote by $\varphi = \varphi_{q,\gamma}$ the trigonometric polynomial from Lemma 1. Also let $w = w_{N,u}$ be a real trigonometric polynomial, satisfying (15) and (28), which has the following additional property:

(32) for every $t \in \mathbb{T}$ either $w(t)u(t) > 0$ or otherwise $|w(t)| < c_1/2$.

Such a w can be found easily by taking an approximation of the function $\operatorname{sign}(u)$.

Given $0 < \delta < 1$ we use Lemma 2 to find a measure ρ, supported by the interval $I_0 = \big(\frac{1}{4}, \frac{1}{3}\big)$, satisfying (3) and such that

$$\int |d\rho| < \delta^{-c_3}, \quad \text{where } c_3 := c(I_0). \tag{33}$$

Define

$$\lambda(t) = \int \lambda_s(t)\,d\rho(s).$$

One can expand the product (17) using the Fourier representation of the trigonometric polynomial φ. This yields the expression

$$\lambda(t) = 1 + \sum_{\mathbf{k}} \Big\{ \int s^{l(\mathbf{k})} d\rho(s) \Big\} \Big\{ \prod_{k_j \neq 0} \widehat{\varphi}(k_j) \Big\} \; w(t)^{l(\mathbf{k})} \; e^{i(k_1\nu + k_2\nu^2 + \cdots + k_N \nu^N)t},$$

where the sum goes through all nonzero vectors

$$\mathbf{k} = (k_1, \dots, k_N) \in \mathbb{Z}^N, \quad |k_j| \leqslant \deg \varphi,$$

and $l(\mathbf{k}) > 0$ denotes the number of nonzero coordinates of $\mathbf{k}$. Note that each polynomial $w(t)^{l(\mathbf{k})}$ has degree $\leqslant N \deg w$. So condition (16) ensures that the summands in the above sum have disjoint spectra. Taking advantage of the fact that $\|w(t)^{l(\mathbf{k})}\|_{A_q} \leqslant 1$ (which follows from (15)) we deduce that

$$\|1 - \lambda\|_{A_q}^q < \delta^q \sum_{\mathbf{k}} \prod_{k_j \neq 0} |\widehat{\varphi}(k_j)|^q < \delta^q \, (1 + \|\varphi\|_{A_q}^q)^N < \delta^q \, \exp(N \|\varphi\|_{A_q}^q).$$

Using (2) this implies

$$\|1 - \lambda\|_{A_q} < \delta \, \exp\Big(\tfrac{1}{q} \gamma^q N\Big). \tag{34}$$

Now consider the trigonometric polynomial X defined in (25). Set

$$E := \{t \in \mathbb{T} : X(t) \geqslant c_1\} \quad \text{and} \quad h := \lambda \cdot \mathbb{1}_E \, ;$$

then

$$\|\lambda - h\|_{A_q} \leqslant \|\lambda - h\|_{L^2(\mathbb{T})} = \|\lambda\|_{L^2(\mathbb{T} \setminus E)} \leqslant \int \|\lambda_s\|_{L^2(\mathbb{T} \setminus E)} \, |d\rho(s)|.$$

Using Lemma 7 and (33) this implies

$$\|\lambda - h\|_{A_q} \leqslant \sqrt{2} \, e^{-\frac{1}{2} c_2 N} \, \delta^{-c_3}. \tag{35}$$

Let $c_4 > 0$ be an absolute constant so small that, setting $\delta := e^{-c_4 N}$, the right-hand side of (35) will tend to zero as $N \to \infty$. Next, let the number $\gamma > 0$ be an absolute constant, so small that also the right-hand side of (34) will tend to zero as $N \to \infty$. Now we fix $N = N(\varepsilon)$ so large, such that the right-hand sides of both (34) and (35) will be smaller than $\varepsilon/2$. Having fixed N, the functions w, λ, X and h are also fixed, and it follows that

$$\|1 - h\|_{A_q} \leqslant \|1 - \lambda\|_{A_q} + \|\lambda - h\|_{A_q} < \varepsilon.$$

Finally we will define the compact K, the function f and the trigonometric polynomial P with properties (i) and (ii). Let χ be a nonnegative, infinitely smooth function, with integral $= 1$. Set $f := h * \chi$; then $\|1 - f\|_{A_q} < \varepsilon$. By choosing χ supported on a sufficiently small neighborhood of zero, we may assume that f is supported by a compact K (a finite union of segments) such that $X(t) > c_1/2$ on K. Thus (i) is satisfied. Now we set

$$P(t) := (2/c_1) \cdot \frac{1}{N} \sum_{j=1}^{N} \varphi(\nu^j t)$$

and check that (ii) is satisfied. First, due to (15),

$$|P(t)| \geqslant P(t)\, w(t) = (2/c_1)\, X(t) > 1, \quad t \in K.$$

Secondly, since $\|\varphi\|_\infty \leqslant 1$, for every $t \in K$ we have $|w(t)| \geqslant X(t) > c_1/2$, and (32) implies that $w(t)u(t) > 0$. Hence $P(t)u(t) > 0$ on K. Lastly,

$$\|P\|_A \leqslant (2/c_1)\, \|\varphi\|_A = C(q),$$

and our main lemma is proved. □

4. **Helson sets and distributions**

4.1. Recall the main two properties of Piatetski-Shapiro's compact K:

(I) K supports a nonzero distribution S with $\widehat{S}(n) \to 0$ as $|n| \to \infty$.

(II) For every nonzero measure μ supported by K,

$$\limsup_{|n|\to\infty} |\widehat{\mu}(n)| > 0.$$

In a way, the existence of such a compact reveals a "compromise" between certain "thickness" and "thinness" conditions of a set (understood not in a metrical but rather an arithmetical sense). We will see that this compromise can be achieved under stronger conditions, in both directions.

Definition (see for example [9, Chap. IV]). A compact set K is called a *Helson set* if it satisfies any one of the following equivalent conditions:

(i) Every continuous function on K admits extension to a function in $A(\mathbb{T})$.

(ii) There is $\delta_1(K) > 0$ such that, for every measure μ supported by K,

$$\sup_{n\in\mathbb{Z}} |\widehat{\mu}(n)| \geqslant \delta_1(K) \int |d\mu|. \tag{36}$$

(iii) There is $\delta_2(K) > 0$ such that, for every measure μ supported by K,

$$\limsup_{|n|\to\infty} |\widehat{\mu}(n)| \geqslant \delta_2(K) \int |d\mu|. \tag{37}$$

Körner [14] and Kaufman [12] generalized Piatetski-Shapiro's result by constructing Helson sets with property (I) above (that is, Helson M-sets).

We will prove the following stronger theorem:

Theorem 3. *For any $q > 2$ there is a Helson set K on the circle $\mathbb{T}$, which supports a nonzero distribution S such that $\widehat{S} \in \ell^q$.*

Clearly this also implies Theorem 2.

4.2. For the proof of Theorem 3 we need the following:

LEMMA 8. *Let K be a totally disconnected compact set on $\mathbb{T}$. Suppose that there is a constant $C > 0$ such that the following is true: given any real-valued function $h \in C(\mathbb{T})$ with no zeros in K, one can find a real trigonometric polynomial $P(t)$ such that*

$$\inf_{t \in K} |P(t)| > 1, \quad P(t)h(t) > 0 \text{ on } K, \quad \|P\|_A \leqslant C. \tag{38}$$

Then K is a Helson set.

Proof. It would be enough to show that there is $\delta_1(K) > 0$, such that (36) is satisfied by every measure μ supported by K. In fact, it is enough to prove (36) only for real, signed measures μ, as one can check easily by decomposing a complex measure into its real and imaginary parts.

Let therefore μ be a real, signed measure supported by K, and suppose that $\int |d\mu| = 1$. Since K is totally disconnected, given $\varepsilon > 0$ there is a real-valued function $h \in C(\mathbb{T})$ such that $h(t) = \pm 1$ on K, and $\int h\, d\mu > 1 - \varepsilon$. Let $P(t)$ be a real trigonometric polynomial satisfying (38). Then

$$\int_K P\, d\mu = \int_K P\, h\, |d\mu| - \int_K P\, h\, (|d\mu| - h\, d\mu) > 1 - C\varepsilon.$$

On the other hand,

$$\int_K P\, d\mu = \int_{\mathbb{T}} P\, d\mu = \sum_{n \in \mathbb{Z}} \widehat{P}(-n)\, \widehat{\mu}(n) \leqslant C \sup_{n \in \mathbb{Z}} |\widehat{\mu}(n)|.$$

Since ε was arbitrary, this shows that (36) is true with $\delta_1(K) = C^{-1}$. □

Remark. One can show that the condition in Lemma 8 is also necessary for Helson sets. For comparison, we mention another necessary and sufficient condition in a similar spirit: a compact K is a Helson set if and only if it is totally disconnected, and every $\{0, 1\}$-valued continuous function on K admits an extension to $\mathbb{T}$ with bounded $A(\mathbb{T})$ norm (see [9, p. 52]).

4.3. *Proof of Theorem* 3. Fix $q > 2$. Choose a sequence u_j of real, nonzero trigonometric polynomials, which is dense in the metric space of real-valued continuous functions on $\mathbb{T}$. For a sequence ε_j use Lemma 4 with $\varepsilon = \varepsilon_j$ and $u = u_j$ to choose K_j, f_j and P_j. We choose the ε_j by induction, such that

$$\varepsilon_1 < 2^{-2} \qquad \text{and} \qquad \|f_1 \cdot f_2 \cdots f_j\|_A\, \varepsilon_{j+1} < 2^{-2-j} \quad (j = 1, 2, \dots).$$

This condition allows us to define a distribution $S \in A_q(\mathbb{T})$ by the infinite product $\prod_{j=1}^{\infty} f_j$. Indeed, the partial products $S_j = f_1 \cdot f_2 \cdots f_j$ satisfy

$$\|S_{j+1} - S_j\|_{A_q} = \|f_1 \cdots f_j \cdot (f_{j+1} - 1)\|_{A_q} \leqslant \|f_1 \cdots f_j\|_A\, \varepsilon_{j+1} < 2^{-2-j},$$

hence the S_j converge in $A_q(\mathbb{T})$ to a limit S. Observe that S is nonzero, since

$$\|S-1\|_{A_q} \leqslant \sum_{j=0}^{\infty} \|S_{j+1}-S_j\|_{A_q} < \sum_{j=0}^{\infty} 2^{-2-j} < 1$$

and that S is supported by the compact $K := \bigcap_{j=1}^{\infty} K_j$.

On the other hand, we will show that K is a Helson set. It is enough to check that K satisfies the conditions of Lemma 8. Indeed, for each j we have

$$\inf_{t\in K} |P_j(t)| > 1, \quad P_j(t)u_j(t) > 0 \text{ on } K, \quad \|P_j\|_A \leqslant C(q).$$

In particular, none of the u_j has a zero in K. Since they are dense in the metric space of real-valued continuous functions on $\mathbb{T}$, it follows that K is totally disconnected. Let now $h \in C(\mathbb{T})$ be a real-valued function, with no zeros in K. Choose j such that $u_j(t)h(t) > 0$ on K, then (38) is satisfied with $P = P_j$ and $C = C(q)$. It therefore follows from Lemma 8 that K is a Helson set. □

5. **Helson sets and cyclic vectors**

5.1. The role of Helson sets in our problem is clarified by the following:

Lemma 9. *Let K be a Helson set on $\mathbb{T}$. Then there is a function $g \in A(\mathbb{T})$, vanishing on K, which is a cyclic vector in $A_p(\mathbb{T})$ for every $p > 1$.*

For the proof of Lemma 9 we will need the following property of Helson sets. Denote by $C(K)$ the space of continuous functions on K with the norm

$$\|h\|_{C(K)} = \sup_{t\in K} |h(t)|.$$

Recall that one of the equivalent definitions of a Helson set is that every element of $C(K)$ admits an extension to a function in $A(\mathbb{T})$. The next lemma shows that one can actually find such extensions with arbitrarily small A_p norms.

Lemma 10. *Let K be a Helson set, and suppose that $\varepsilon > 0$, $p > 1$ and $h \in C(K)$ are given. Then one can find $f \in A(\mathbb{T})$ such that*

$$f|_K = h, \quad \|f\|_A \leqslant (1/\delta)\,\|h\|_{C(K)}, \quad \|f\|_{A_p} < \varepsilon,$$

where $\delta = \delta_2(K) > 0$ is the constant from (37).

Proof. Fix $p > 1$ and $\varepsilon > 0$. Introduce a Banach space $B = B_{p,\varepsilon}$ of functions f on the circle $\mathbb{T}$ such that

$$\|f\|_B := \|f\|_A + (1/\varepsilon)\,\|f\|_{A_p} < \infty.$$

In other words, the space B coincides with the space $A(\mathbb{T})$ but is equipped with a different (equivalent) norm. Let also $T : B \to C(K)$ be the restriction operator $f \mapsto f|_K$, and denote by T^* its dual operator.

Given a measure μ supported by K, by (37) we have

$$L(\mu) := \limsup_{|n|\to\infty} |\widehat{\mu}(n)| \geqslant \delta \int |d\mu|.$$

Take a sequence of integers n_j, $|n_1| < |n_2| < \cdots$, and real numbers θ_j such that

$$\lim_{j\to\infty} \widehat{\mu}(n_j)\, e^{-i\theta_j} = L(\mu)$$

and define

$$f_N(t) = \frac{1}{N} \sum_{j=1}^{N} e^{-i(n_j t + \theta_j)}.$$

Then $\|f_N\|_B = 1 + (1/\varepsilon)\, N^{(1/p)-1}$ and

$$\langle f_N, T^*\mu\rangle = \langle Tf_N, \mu\rangle = \int_K f_N(t)\, d\mu(t) = \frac{1}{N}\sum_{j=1}^{N} \widehat{\mu}(n_j)\, e^{-i\theta_j}.$$

It follows that

$$\|T^*\mu\|_{B^*} \geqslant \lim_{N\to\infty} \frac{|\langle f_N, T^*\mu\rangle|}{\|f_N\|_B} = L(\mu) \geqslant \delta \int |d\mu|,$$

for every measure μ supported by K.

By a classical theorem of Banach (see [10, p. 141]) this implies that for every $h \in C(K)$, the equation $Tf = h$ admits a solution $f \in B$ such that $\|f\|_B \leqslant (1/\delta)\, \|h\|_{C(K)}$. This proves the lemma. □

5.2. Using Lemma 10 we can prove Lemma 9 above.

Proof of Lemma 9. It will be convenient to use Baire categories in the proof. Let $I(K)$ denote the set of functions $g \in A(\mathbb{T})$ which vanish on K. This is a complete metric space, with the metric inherited from $A(\mathbb{T})$. We will prove that the set of functions $g \in I(K)$ which are cyclic in $A_p(\mathbb{T})$ for every $p > 1$, is a countable intersection of open, dense sets in the space $I(K)$. By Baire's theorem, this set is therefore nonempty (and in fact is dense in the space).

For $\varepsilon > 0$ and $p > 1$, denote by $G(\varepsilon, p)$ the set of $g \in I(K)$ for which there exists a trigonometric polynomial P such that $\|1 - P\cdot g\|_{A_p} < \varepsilon$. Choose a sequence $\varepsilon_n \to 0$ and a sequence $p_n \to 1$ $(n \to \infty)$, and consider the intersection

$$\bigcap_{n=1}^{\infty} G(\varepsilon_n, p_n).$$

According to condition (i) from Section 2.2, a function $g \in I(K)$ belongs to this intersection if and only if it is cyclic in $A_p(\mathbb{T})$ for every $p > 1$. So to conclude the proof it remains to show that each $G(\varepsilon, p)$ is an open, dense set in $I(K)$.

Let $g_0 \in G(\varepsilon, p)$ be given. Then $\|1 - P \cdot g_0\|_{A_p} < \varepsilon$ for some trigonometric polynomial P. Given $\eta > 0$, suppose that $g \in I(K)$ and $\|g - g_0\|_A < \eta$. Then

$$\|1 - P \cdot g\|_{A_p} \leqslant \|1 - P \cdot g_0\|_{A_p} + \eta \, \|P\|_{A_p}.$$

It η is chosen sufficiently small then the right-hand side is smaller than ε. Hence $G(\varepsilon, p)$ contains the open ball $B(g_0, \eta)$ of radius η centered at g_0, and this shows that $G(\varepsilon, p)$ is open.

Finally we show that $G(\varepsilon, p)$ is dense. Let a ball $B(g_0, \eta)$ in $I(K)$ be given. Choose a trigonometric polynomial h, not identically zero, such that

$$\|h - g_0\|_A < \frac{\delta}{1+\delta} \cdot \eta \,,$$

where $\delta = \delta_2(K) > 0$ is the constant from (37). In particular, this implies that

$$\sup_{t \in K} |h(t)| < \frac{\delta}{1+\delta} \cdot \eta \,.$$

Since h is nonzero, it has finitely many zeros, so by conditions (i) and (ii) from Section 2.2 there is a trigonometric polynomial P such that $\|1 - P \cdot h\|_{A_p} < \varepsilon/2$. Now use Lemma 10 to find $f \in A(\mathbb{T})$ such that

$$f|_K = h|_K, \quad \|f\|_A < \eta/(1+\delta), \quad \|f\|_{A_p} < \frac{\varepsilon}{2\,\|P\|_A},$$

and set $g := h - f$. Then clearly $g \in I(K)$. Moreover

$$\|g - g_0\|_A \leqslant \|h - g_0\|_A + \|f\|_A < \eta;$$

that is, $g \in B(g_0, \eta)$. Also,

$$\|1 - P \cdot g\|_{A_p} \leqslant \|1 - P \cdot h\|_{A_p} + \|P\|_A \|f\|_{A_p} < \varepsilon,$$

and therefore $g \in G(\varepsilon, p)$. This shows that $G(\varepsilon, p)$ is dense. □

5.3. Our main result now follows:

Proof of Theorem 1. By Theorem 3 there is a Helson set K satisfying condition (a′) in Theorem 2. This condition is equivalent to condition (a) in Theorem 1 (see Section 2.2, (iii) and (v)). On the other hand, Lemma 9 implies that K also satisfies condition (b). So Theorem 1 is proved. □

Proof of Corollary 1. Let K be the compact set of Theorem 1. By property (b) there is $g \in A(\mathbb{T})$ vanishing on K, which is cyclic in $A_p(\mathbb{T})$. Choose a smooth (say, twice continuously differentiable) function f on $\mathbb{T}$, such that $Z_f = Z_g$. In particular, f vanishes on K. Since the Fourier coefficients of f decrease sufficiently fast, property (a) implies that f is a noncyclic vector in $A_p(\mathbb{T})$. Thus our corollary is proved. □

Remark. One can see from the proof above that if p remains bounded away from 2, then the two vectors in Corollary 1 may be chosen independently of p.

6. Nonperiodic version

Here we extend the results to $L^p(\mathbb{R})$ spaces, $1 < p < 2$. This can be deduced easily from the previous results, so we may be brief. In particular, we skip the formulation of the corresponding version of Theorem 1, and restrict ourselves to

COROLLARY 2. *Given any* $1 < p < 2$ *one can find two functions in* $L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, *such that one is cyclic in* $L^p(\mathbb{R})$ *and the other is not, but their Fourier transforms have the same* (*compact*) *set of zeros.*

Here $C_0(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$ vanishing at infinity.

It will be convenient to denote by $\widehat{\mathbb{R}}$ another copy of the real line. We consider distributions on the Schwartz space $S(\widehat{\mathbb{R}})$. We denote by $A_p(\widehat{\mathbb{R}})$, $1 \leqslant p < \infty$, the space of Fourier transforms of functions in $L^p(\mathbb{R})$, with the corresponding norm. In particular, for $p = 1$ this is the Wiener algebra $A(\widehat{\mathbb{R}})$ of functions with an absolutely convergent Fourier integral.

Recall that, by definition, a function $F(x) \in L^p(\mathbb{R})$ is a cyclic vector if the translates $\{F(x-y)\}$, $y \in \mathbb{R}$, span the whole space. Equivalently, F is cyclic if the set $\{f(t)\,\phi(t)\}$, where $f = \widehat{F}$ and ϕ runs over $S(\widehat{\mathbb{R}})$, is dense in $A_p(\widehat{\mathbb{R}})$.

Proof of Corollary 2. Fix $1 < p < 2$, and take the compact K of Theorem 1. By property (b) there is $h \in A(\mathbb{T})$, vanishing on K, which is cyclic in $A_p(\mathbb{T})$ (with respect to multiplication by trigonometric polynomials).

We may assume that $h(t)$ is positive at some point t, and by rotation, that $h(\pi) > 0$. It follows that there is an interval

$$I := (-\pi + \delta,\ \pi - \delta), \quad \delta > 0,$$

such that $K \subset \{t : \operatorname{Re} h(t) \leqslant 0\} \subset I$. Choose a function $\chi \in S(\widehat{\mathbb{R}})$, $0 \leqslant \chi \leqslant 1$, compactly supported by $(-\pi, \pi)$, and such that $\chi(t) = 1$ on I. Define

$$g(t) := \chi(t)\,h(t) + (1 - \chi(t))\,e^{-t^2}, \quad t \in \widehat{\mathbb{R}}.$$

It is easy to see that:

(i) The zero set Z_g is compact, $K \subset Z_g \subset I$.

(ii) $g = \widehat{G}$ for some $G \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, which implies $g \in A_p(\widehat{\mathbb{R}})$.

Claim 1. The set $\{g(t)\,\phi(t)\}$, $\phi \in S(\widehat{\mathbb{R}})$, is dense in $A_p(\widehat{\mathbb{R}})$.

If not then, by duality, there is a (nonzero) distribution $S \in A_q(\widehat{\mathbb{R}})$, $q = p/(p-1)$, such that $\langle S,\ g \cdot \phi \rangle = 0$ for every $\phi \in S(\widehat{\mathbb{R}})$. It follows that

$$\operatorname{supp}(S) \subset Z_g \subset I, \quad |I| < 2\pi. \tag{39}$$

We have $g(t) = h(t)$ on I, since $\chi(t) = 1$ on I. Hence

$$\langle S,\ h \cdot \phi \rangle = 0 \quad \text{for every } \phi \in S(\widehat{\mathbb{R}}). \tag{40}$$

Condition (39) allows us to regard S also as a distribution on $\mathbb{T}$. It is well known that under this condition the following equivalence holds:

$$S \in A_q(\mathbb{T}) \iff S \in A_q(\widehat{\mathbb{R}}).$$

But h is cyclic in $A_p(\mathbb{T})$, so (40) implies that $S = 0$, which proves the claim.

Now take an arbitrary function $f \in S(\widehat{\mathbb{R}})$ with $Z_f = Z_g$.

Claim 2. The set $\{f(t)\,\phi(t)\}$, $\phi \in S(\widehat{\mathbb{R}})$, is not dense in $A_p(\widehat{\mathbb{R}})$.

Indeed, property (a) from Theorem 1 implies that K supports a (nonzero) distribution $S \in A_q(\mathbb{T})$. As above we can regard it as a distribution on $\widehat{\mathbb{R}}$, belonging to $A_q(\widehat{\mathbb{R}})$. But f is a *smooth* function in $A_p(\widehat{\mathbb{R}})$, and $f|_K = 0$, hence $\langle S,\ f \cdot \phi\rangle = 0$ for every $\phi \in S(\widehat{\mathbb{R}})$.

This means that the inverse Fourier transform of f is a function $F \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, which is noncyclic in $L^p(\mathbb{R})$. Our corollary is thus proved. □

7. Remarks

7.1. Theorem 1 may be put into the context of the theory of translation-invariant subspaces. A linear subspace $M \subset L^p(G)$ is called translation-invariant if whenever f belongs to M, then so do all of the translates of f. Observe that $f \in L^p(G)$ is a cyclic vector if and only if it does not belong to any proper closed translation-invariant subspace of $L^p(G)$.

It is well known that any closed translation-invariant subspace in $\ell^2(\mathbb{Z})$ can be uniquely recovered from the set of the common zeros of the Fourier transforms of its elements.

This is not the case in $\ell^1(\mathbb{Z})$. Malliavin's "non-synthesis" example [17] means that different closed translation-invariant subspaces in this space may have the same set of common zeros. More precisely, for a compact set $K \subset \mathbb{T}$ consider the invariant subspaces

$$I(K) = \{\mathbf{c} \in \ell^1(\mathbb{Z}) \,:\, \widehat{\mathbf{c}} \text{ vanishes on } K\},$$

$$J(K) = \{\mathbf{c} \in \ell^1(\mathbb{Z}) \,:\, \widehat{\mathbf{c}} \text{ vanishes on some open set containing } K\}.$$

Malliavin proved that there is a compact compact K such that the closure of $J(K)$ is strictly smaller than $I(K)$.

Kahane [8] (see also [10, p. 121]) showed that such a result still holds if one takes the closures of $J(K)$ and $I(K)$ in $\ell^p(\mathbb{Z})$, $1 < p < 2$.

Theorem 1 reveals a sharper phenomenon in these spaces, which is not possible in $\ell^1(\mathbb{Z})$. Namely, there is a compact K such that the ℓ^p-closures satisfy

$$\operatorname{Clos} J(K) \subsetneqq \operatorname{Clos} I(K) = \ell^p(\mathbb{Z}).$$

7.2. Strictly speaking, it is not necessary to require that $\mathbf{c} \in \ell^1$ in order to have the zero set $Z_{\widehat{\mathbf{c}}}$ well defined. The continuity of $\widehat{\mathbf{c}}$ is sufficient for that, as appeared in the weaker version of Theorem 1 proved in [16]. However, the advantage of the present version seems to be substantial, since very little is known about the relation between cyclicity in ℓ^p $(1 < p < 2)$ and the zero set unless $\mathbf{c} \in \ell^1$. In particular, we do not know the answer to the following question: let $f \in C(\mathbb{T}) \cap A_p(\mathbb{T})$ have no zeros; does this imply that $\mathbf{c} = \widehat{f}$ is a cyclic vector?

7.3. Let K be a Helson set. Define its 'Helson constant' as the maximal possible $\delta_1(K)$ in (36). Körner's [14] and Kaufman's [12] constructions (see Section 4.1 above) give a Helson constant 1. What can be said about the Helson constant of K in Theorem 3? Specifically, must it tend to zero when $q \to 2$?

7.4. Perhaps the most interesting problem left open is: could one characterize in reasonable terms the cyclic vectors $\mathbf{c}$ in ℓ^p, $1 < p < 2$, under the standard assumption $\mathbf{c} \in \ell^1$ with no extra restrictions?

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(Received: October 19, 2009)

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