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# Uniform approximation on manifolds

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*Dedicated to John Wermer on the occasion of his* 80*th birthday*

## Abstract

It is shown that if $A$ is a uniform algebra generated by a family $\Phi$ of complex-valued $C^1$ functions on a compact $C^1$ manifold-with-boundary $M$, the maximal ideal space of $A$ is $M$, and $E$ is the set of points where the differentials of the functions in $\Phi$ fail to span the complexified cotangent space to $M$, then $A$ contains every continuous function on $M$ that vanishes on $E$. This answers a 45-year-old question of Michael Freeman who proved the special case in which the manifold $M$ is two-dimensional. More general forms of the theorem are also established. The results presented strengthen results due to several mathematicians.

## 1. Introduction

In 1965 John Wermer [24] showed that if $f$ is a complex-valued continuously differentiable function on the closed unit disc $\overline{D}$ such that the graph of $f$ is polynomially convex and $E$ is the zero set of $\partial f/\partial \bar{z}$, then the uniformly closed algebra generated by $z$ and $f$ contains every continuous function on $\overline{D}$ that vanishes on $E$. The following year, Michael Freeman [8] generalized this result to the context of uniform algebras on two-dimensional manifolds by proving that if $A$ is a uniform algebra generated by a family $\Phi$ of complex-valued $C^1$ functions on a compact two-dimensional real $C^1$ manifold-with-boundary $M$, the maximal ideal space of $A$ is $M$, and $E = \{p \in M : df_1 \wedge df_2(p) = 0 \text{ for all } f_1, f_2 \in \Phi\}$, then $A$ contains every continuous function on $M$ that vanishes on $E$, or equivalently that $A = \{g \in C(M) : g|E \in A|E\}$. (Here $A|E$ denotes the collection of functions obtained by restricting the functions in $A$ to $E$.) Freeman then asked whether this theorem continues to hold if $M$ is taken to be an $m$-dimensional manifold and $E = \{p \in M : df_1 \wedge \cdots \wedge df_m(p) = 0 \text{ for all } f_1, \ldots, f_m \in \Phi\}$. In this paper we will prove that the answer to Freeman's question is affirmative.

In the case when $M$ is a submanifold of $\mathbf{C}^n$ and $\Phi = \{z_1, \ldots, z_n\}$, and hence $A = P(M)$ (the uniform closure on $M$ of the polynomials in $z_1, \ldots, z_n$),

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the question has been studied by many mathematicians and has been settled for some time. In this setting, it is easily seen that $E$ is exactly the set of points where $M$ has a complex tangent. (This is discussed in [1, p. 190] for instance.) In addition, it is well known that the maximal ideal space of $P(M)$ can be naturally identified with the polynomially convex hull $\widehat{M}$ of $M$, defined by

$$\widehat{M} = \{z \in \mathbf{C}^n : |p(z)| \leq \sup_{x \in M} |p(x)| \text{ for all polynomials } p\},$$

and hence the condition that the maximal ideal space of $A = P(M)$ is $M$ is precisely the condition that $M$ is polynomially convex (that is, that $\widehat{M} = M$). The following theorem is well known.

Theorem 1.0. *Let $M$ be a smooth submanifold of $\mathbf{C}^n$, and let $X$ be a compact subset of $M$ that is polynomially convex. Let $E$ be the set of points of $X$ where $M$ has a complex tangent. Then $P(X) = \{g \in C(X) : g|E \in P(X)|E\}$.*

Note that the conclusion is equivalent to the statement that $P(X)$ contains every continuous function on $X$ that vanishes on $E$. Note also that when $E$ is empty, the conclusion is that $P(X) = C(X)$.

Under various degrees of smoothness (and other conditions) the above theorem is due to several different mathematicians. The case when $M$ is of class $C^\infty$ and $E$ is empty is in papers by Nirenberg and Wells [17], [18]. The case when $M$ is an $m$-dimensional manifold of class $C^r$ with $r \geq (m/2) + 1$ (and $E$ is arbitrary) is in a paper of Hörmander and Wermer [12]. The case when $M$ is just of class $C^1$, and $E$ is empty, can be found in papers by Čirka [6], Harvey and Wells [11], and Berndtsson [5]. The case when $M$ is a $C^1$ graph (and $E$ is arbitrary) is in a paper of Weinstock [22]. Finally, the full theorem is in a paper of O'Farrell, Preskenis, and Walsh [19]. In fact, the theorem there is more general than what is stated above.

The original form of Freeman's question (in which $A$ is a uniform algebra on an abstract manifold) has also been studied. Freeman himself gave an affirmative answer in the case when the manifold $M$ and the functions in $\Phi$ are real-analytic [9]. Fornæss [7] showed that the answer remains affirmative under the weaker condition that $M$ and the functions in $\Phi$ are just of class $C^r$ with $r \geq (m/2) + 1$, where $m$ is the dimension of $M$. In the present paper we will show that class $C^1$ is enough, thus fully answering Freeman's question. Specifically, we have the following theorem.

Theorem 1.1. *Let $M$ be an $m$-dimensional $C^1$ manifold-with-boundary, and let $X$ be a compact subset of $M$. Suppose $A$ is a uniform algebra on $X$ generated by a family $\Phi$ of complex-valued functions $C^1$ on $M$, the maximal ideal space of $A$ is $X$, and $E = \{p \in X : df_1 \wedge \cdots \wedge df_m(p) = 0 \text{ for all } f_1, \ldots, f_m \in \Phi\}$. Then $A = \{g \in C(X) : g|E \in A|E\}$.*

Although this settles Freeman's question, it is sometimes desirable to have a theorem along these lines in which the space $X$ is not required to be a subset of a manifold. Also the theorem of O'Farrell, Preskenis, and Walsh mentioned earlier deals with algebras on spaces more general than subsets of a manifold. In order to give a single theorem that encompasses all of the various cases the author is aware of and that is as widely applicable as possible, we formulate our main theorem as follows.

THEOREM 1.2. *Let* $A$ *be a uniform algebra on a compact Hausdorff space* $X$*. Suppose the maximal ideal space of* $A$ *is* $X$*. Suppose also that* $E$ *is a closed subset of* $X$ *such that each point* $p \in X \backslash E$ *has a neighborhood* $U_p$ *imbeddable in a* $C^1$ *manifold* $M_p$ *of dimension* $m = m(p)$ *such that*

(i) *there exist functions* $f_1, \ldots, f_m \in A$ *whose restrictions to* $U_p$ *extend to be* $C^1$ *on* $M_p$ *so that* $df_1 \wedge \cdots \wedge df_m(p) \neq 0$, *and*

(ii) *the functions in* $A$ *whose restrictions to* $U_p$ *extend to be* $C^1$ *on* $M_p$ *separate points on* $X$*.*

*Then* $A = \{g \in C(X) : g|E \in A|E\}$.

Theorem 1.1 is obviously a special case of Theorem 1.2. The following special case of Theorem 1.2 is also worth noting.

THEOREM 1.3. *Let* $A$ *be a uniform algebra on a compact Hausdorff space* $X$*. Suppose the maximal ideal space of* $A$ *is* $X$*. Suppose also that* $E$ *is a closed subset of* $X$ *such that* $X \backslash E$ *is an* $m$*-dimensional manifold and such that*

(i) *for each point* $p \in X \backslash E$ *there are functions* $f_1, \ldots, f_m \in A$ *that are* $C^1$ *on* $X \backslash E$ *and satisfy* $df_1 \wedge \cdots \wedge df_m(p) \neq 0$, *and*

(ii) *the functions in* $A$ *that are* $C^1$ *on* $X \backslash E$ *separate points on* $X$*.*

*Then* $A = \{g \in C(X) : g|E \in A|E\}$.

In proving Theorem 1.2, we will not use the full strength of condition (ii). All we will use is that the functions in condition (ii) separate every pair of points of $X$ at least one of which lies in $X \backslash E$. It of course follows that condition (ii) of Theorem 1.3 can also be weakened in the analogous way.

The results of this paper can be used to extend the peak point theorems of Anderson, Wermer, and the author for certain uniform algebras on subsets of complex euclidean space [3], [4] to an abstract uniform algebra setting. This is begun in [2] and will be continued in a subsequent paper. Applications to uniform algebras invariant under group actions are given in [14]. (See also [15].) The results of this paper can also be applied to the problems concerning approximation in the plane treated in [13]. In fact [13, Th. 5.4] is the two-dimensional case of Theorem 1.3 above with condition (ii) replaced by the stronger condition that the functions in $A$ that are $C^1$ on $X \backslash E$ are uniformly dense in $A$. Thus by the preceding paragraph it is actually enough to assume in [13, Th. 5.4] that the functions in $A$ that are $C^1$ on $X\backslash E$ separate every pair of points of $X$ at least one of which lies in $X\backslash E$. It follows that condition (iii) can be omitted from [13, Th. 5.1]. Theorems 3.1 and 5.3 in [13] do not seem to follow from the results of the present paper but do follow under suitably strengthened hypotheses. These theorems contain the condition that for almost every point $a$ in a certain planar set $\Omega$ there exists a function $f$ in $A$ that is differentiable at $a$ and such that $(\partial f/\partial \bar{z})(a) \neq 0$. If we make the further requirement that this function $f$ is $C^1$ on $\Omega$, then Theorem 1.3 above can be applied to prove the theorems by an argument similar to how [13, Th. 5.1] is obtained from [13, Th. 5.4]. In the applications given in [13], this further requirement is satisfied.

The present paper owes a great deal to the work of Barnet Weinstock. In particular, the paper relies heavily on ideas from [22], and I would like to thank Weinstock for sending me a copy of that paper without which I certainly would never have found the proof presented here.

It is a pleasure to dedicate this paper to John Wermer on the occasion of his 80th birthday. As discussed above, the long line of research continued here was initiated by him in his papers [23], [24]. In addition, the general areas of uniform algebras and several complex variables owe a great deal to the work of Wermer. On a personal level, Wermer has been a great inspiration to me, and I feel tremendously privileged and honored to have had opportunities to work with him. It is a pleasure to express my thanks for all he has done for me over the years.

## 2. **Preliminaries**

The proof of Theorem 1.2 is based on Weinstock's proof of the following theorem alluded to in the introduction.

THEOREM 2.0 (Weinstock [22]). *Let $X$ be a compact set in $\mathbf{C}^n$, and let $f_1, \ldots, f_k$ be complex-valued $C^1$ functions on a neighborhood of $X$. Let $E = \{z \in X : \operatorname{rank}(\partial f_i/\partial \bar{z}_j) < n\}$, and let $\widetilde{X} = \{\big(z, f_1(z), \ldots, f_k(z)\big) \in \mathbf{C}^{n+k} : z \in X\}$. Assume $\widetilde{X}$ is polynomially convex. Let $A$ be the algebra of functions on $X$ that can be approximated uniformly by polynomials in the functions $z_1, \ldots, z_n, f_1, \ldots, f_k$. Then $A = \{f \in C(X) : f|E \in A|E\}$.*

This theorem of Weinstock can be reformulated as a theorem about approximation on a graph: If $U$ denotes the neighborhood of $X$ on which $f_1, \ldots, f_k$ are defined, $M = \{\big(z, f_1(z), \ldots, f_k(z)\big) : z \in U\}$, and $\widetilde{E} = \{\big(z, f_1(z), \ldots, f_k(z)\big) : z \in E\}$, then $M$ is a smooth graph in $\mathbf{C}^{n+k}$, the set $\widetilde{X}$ is a polynomially convex subset of $M$, and the conclusion of the theorem becomes $P(\widetilde{X}) = \{g \in C(\widetilde{X}) : g|\widetilde{E} \in P(\widetilde{X})|\widetilde{E}\}$. It turns out that the more general case of polynomial approximation on a polynominally convex subset of a smooth manifold in $\mathbf{C}^N$ that is *not* assumed to be a graph *follows* from the graph case. The trick is *not* to attempt to apply a local to a global argument (as would seem natural) but instead to take the whole manifold and imbed it in a higher-dimensional complex euclidean space so as to make it into a graph. Since this trick is one of the main ingredients in the proof of Theorem 1.2, as motivation for the proof of Theorem 1.2 we demonstrate how it is used to obtain approximation on general submanifolds of $\mathbf{C}^N$ from the graph case. Here is the precise statement of the result.

THEOREM 2.1. *Let $M$ be a $C^1$ submanifold of $\mathbf{C}^n$, and let $X$ be a compact subset of $M$ that is polynomially convex. Let $E$ be the set of points of $X$ where $M$ has a complex tangent. Then $P(X) = \{g \in C(X) : g|E \in P(X)|E\}$.*

Before showing that this follows from Weinstock's result quoted above, we establish a lemma that will be useful in the proof of Theorem 1.2.

LEMMA 2.2. *Let $Y$ be a subset of $\mathbf{C}^n$. Suppose $p$ is a point of $Y$ and there is a neighborhood $N$ of $p$ in $\mathbf{C}^n$ such that $Y \cap N$ is a $C^1$ submanifold of $\mathbf{C}^n$ with no complex tangents. Then there exist real-valued $C^1$ functions $h_1, \ldots, h_n$ on $\mathbf{C}^n$ that vanish on $Y$ such that the matrix $((\partial h_i/\partial \bar{z}_j)(p))$ is nonsingular.*

*Proof.* Clearly it suffices to show that we can find real-valued $C^1$ functions $h_1, \ldots, h_k$ (with $k \geq n$) on $\mathbf{C}^n$ that vanish on $Y$ such that the matrix $\big((\partial h_i/\partial \bar{z}_j)(p)\big)$ has rank $n$. For this, choose real-valued $C^1$ functions $h_1, \ldots, h_k$ on a neighborhood $V$ of $p$ with linearly independent differentials at $p$ such that the common zero set of $h_1, \ldots, h_k$ in $V$ is exactly $Y \cap V$. Since we can multiply $h_1, \ldots, h_k$ by a real-valued $C^1$ function that is identically 1 on a neighborhood of $p$ and has support in $V$, we can assume that $h_1, \ldots, h_k$ are defined on all of $\mathbf{C}^n$ and vanish on $Y$. A vector $v$ is then tangent to the submanifold $Y \cap N$ at $p$ if and only if $dh_j(v) = 0$ for every $j$. Consequently, $v$ is a complex tangent to $Y \cap N$ at $p$ if and only if $dh_j(v) = dh_j(iv) = 0$ for every $j$. Since $h_j$ is real-valued, we have $dh_j = \partial h_j + \bar{\partial} h_j = 2 \operatorname{Re} \partial h_j$, and since $\partial h_j$ is complex-linear, we have $\operatorname{Re}[\partial h_j(iv)] = \operatorname{Re}[i \partial h_j(v)] = -\operatorname{Im}[\partial h_j(v)]$. So a vector $v$ is a complex tangent to $Y \cap N$ at $p$ if and only if $\partial h_j(v) = 0$ for every $j$. Since $Y \cap N$ has no complex tangents, we conclude that $\partial h_1, \ldots, \partial h_k$ span an $n$-dimensional space. Since $h_1, \ldots, h_k$ are real-valued, the same conclusion holds then also for $\bar{\partial} h_1, \ldots, \bar{\partial} h_k$, and so the matrix $\big((\partial h_i/\partial \bar{z}_j)(p)\big)$ has rank $n$. ☐

*Proof of Theorem* 2.1. Suppose $f_1, \ldots, f_k$ are $C^1$ functions on a neighborhood of $X$ that vanish on $X$. Then of course the uniform algebra on $X$ generated by $z_1, \ldots, z_n, f_1, \ldots, f_k$ is just $P(X)$, and setting

$$\widetilde{X} = \big\{\big(z, f_1(z), \ldots, f_k(z)\big) \in \mathbf{C}^{n+k} : z \in X\big\},$$

we have $\widetilde{X} = X \times \{0\}^k$, so that $\widetilde{X}$ is polynomially convex. Thus Weinstock's result (quoted as Theorem 2.0 above) will apply to give us the desired conclusion provided we can find functions $f_1, \ldots, f_k$ as above with rank $(\partial f_i / \partial \overline{z}_j) = n$ everywhere on $X \backslash E$.

For each point $p$ in $M$, choose real-valued $C^1$ functions $g_1, \ldots, g_r$ on $\mathbf{C}^n$ vanishing on $M$ such that the differentials $dg_1, \ldots, dg_r$ span the annihilator of the tangent space $T_pM$ to $M$ at $p$ in the cotangent space $(T_p\mathbf{C}^n)^*$ to $\mathbf{C}^n$. Then $dg_1, \ldots, dg_r$ span the annihilator of $T_qM$ in $(T_q\mathbf{C}^n)^*$ at all points $q$ in some neighborhood of $p$. Hence by a compactness argument, we can obtain real-valued $C^1$ functions $f_1, \ldots, f_k$ on $\mathbf{C}^n$ that vanish on $M$ such that $df_1, \ldots, df_k$ span the annihilator of $T_pM$ in $(T_p\mathbf{C}^n)^*$ at every point $p$ in $X$. As in the proof of Lemma 2.2, we conclude that at each point of $X \backslash E$, the forms $\overline{\partial} f_1, \ldots, \overline{\partial} f_k$ span an $n$-dimensional space, and hence the matrix $\big((\partial f_i / \partial \overline{z}_j)(p)\big)$ has rank $n$ everywhere on $X \backslash E$. $\square$

Theorem 1.2 is proved in the next section by a duality argument; we start with a measure $\mu$ on $X$ that annihilates $A$ and seek to show that $\mu = 0$ on $X \backslash E$. To this end we fix a point $p$ in $X \backslash E$ and seek to show that $\mu = 0$ on a neighborhood of $p$. In order to apply results from several complex variables, we need to map $X$ into $\mathbf{C}^n$ by a map $F$ having certain properties (see Step 1 of the proof). Then to show that $\mu = 0$ on a neighborhood of $p$, it is enough to show that the push forward measure $F_*(\mu)$ is 0 on a neighborhood of $F(p)$. Next we apply the trick discussed above to, in effect, reduce to the case where $F(X)$ lies on a graph. Then we imitate Weinstock's proof. One of the difficulties that arises in trying to carry over Weinstock's argument is that we do not know that $F(X)$ is polynomially convex. This difficulty is handled by applying the Arens-Calderon lemma. This involves the introduction of certain auxiliary functions and leads to additional complications. In particular, it seems that it is no longer possible to apply the Cauchy-Fantappié formula used by Weinstock. To overcome this obstacle, we generalize this Cauchy-Fantappié formula to include dependence on parameters. Specifically, the result we will need is the following:

THEOREM 2.3. *Let $U$, $V$, and $M$ be open sets in $\mathbf{C}^n$, $\mathbf{C}^k$, and $\mathbf{C}^{n+k}$ respectively with $M \cap (U \times \mathbf{C}^k) = U \times V$. Write points in $M$ as $(z, u)$ with $z \in \mathbf{C}^n$ and $u \in \mathbf{C}^k$. Let $G_1, \ldots, G_n \in C^1(U \times M)$ and define $G$ on $U \times M$ by*

$$G(\zeta, z, u) = \sum_{j=1}^{n} (\zeta_j - z_j) G_j(\zeta, z, u).$$

*Suppose that*

(i) *$G(\zeta, z, u)$ vanishes only when $\zeta = z$, and*

(ii) *for each $j$, the function $\zeta \mapsto G_j(\zeta, z, u) G(\zeta, z, u)^{-n}$ belongs to $L^1_{\mathrm{loc}}$ uniformly for $(z, u)$ in compact subsets of $M$.*

*Define* $\Omega(\zeta, z, u)$ *by*

$$\Omega(\zeta, z, u)$$
$$= (n-1)!(2\pi i)^{-n}(-1)^{n(n-1)/2}G(\zeta, z, u)^{-n}\sum_{j=1}^{n}\Big[(-1)^j G_j(\zeta, z, u)\bigwedge_{r\neq j}\overline{\partial}_\zeta G_r \wedge \alpha\Big],$$

*where* $\alpha = d\zeta_1 \wedge \cdots \wedge d\zeta_n$. *Then every* $\phi \in C^\infty(\mathbf{C}^{n+k})$ *whose support lies in a set of the form* $K \times \mathbf{C}^k$ *with* $K$ *a compact subset of* $U$ *admits the representation*

$$(*) \qquad \phi(z, u) = \int_{\zeta\in U}\Omega(\zeta, z, u)\wedge\overline{\partial}_\zeta\phi(\zeta, u)$$

*with equality for all* $(z, u) \in M$. (*In particular, both sides vanish whenever* $(z, u) \in M\backslash(U \times V)$.)

*Proof.* We verify $(*)$ separately at points in $U\times V$ and not in $U\times V$. First consider an arbitrary point $(z^0, u^0)$ in $U \times V$. Define $G'_1, \dots, G'_n$, $G'$, and $\Omega'$ on $U \times U$ by setting the value of each of these at a point $(\zeta, z)$ to be the value of the corresponding unprimed object at the point $(\zeta, z, u^0)$. It follows from (i) and (ii) that

(i′) $G'(\zeta, z)$ vanishes only when $\zeta = z$, and
(ii′) for each $j$, the function $G'_j(\cdot, z)G(\cdot, z)^{-n}$ belongs to $L^1_{\text{loc}}$ uniformly for $z$ in compact subsets of $U$.

Thus if we define $\phi' \in C^\infty_c(U)$ by $\phi'(z) = \phi(z, u^0)$, we get from the usual Cauchy-Fantappié formula as given in [22] that

$$\phi'(z) = \int_{\zeta\in U}\Omega'(\zeta, z)\wedge\overline{\partial}\phi'(\zeta).$$

(A factor $(-1)^{n(n-1)/2}$ is missing in [22].) Substituting $z^0$ for $z$ and going back to the unprimed objects we see that

$$\phi(z^0, u^0) = \int_{\zeta\in U}\Omega(\zeta, z^0, u^0)\wedge\overline{\partial}_\zeta\phi(\zeta, u^0).$$

Thus $(*)$ holds for points in $U \times V$.

Now let $(z^0, u^0)$ be an arbitrary point in $M\backslash(U \times V)$. Then $\Omega(\cdot, z^0, u^0) \wedge \phi(\cdot, u^0)$ is a smooth form on $U$. Set $w_j(\zeta) = G_j(\zeta, z^0, u^0)/G(\zeta, z^0, u^0)$. Then in the notation of [20, Lemma IV.3.1] we have $\Omega(\zeta, z^0, u^0) = \Omega_0(W)(\zeta)$. Thus $\overline{\partial}_\zeta\Omega(\zeta, z^0, u^0) = 0$ by [20, Lemma IV.3.1 and its addendum]. Consequently,

$$d_\zeta\big(\Omega(\zeta, z^0, u^0)\wedge\phi(\zeta, u^0)\big) = -\Omega(\zeta, z^0, u^0)\wedge\overline{\partial}_\zeta\phi(\zeta, u^0).$$

(Note that $\partial_\zeta\big(\Omega(\zeta, z^0, u^0)\wedge\phi(\zeta, u^0)\big) = 0$ because the form $\Omega(\cdot, z^0, u^0)\wedge\phi(\cdot, u^0)$ is of bidegree $(n, n-1)$.) Thus since $\phi(\cdot, u^0)$ has compact support in $U$, Stokes' theorem gives that

$$\int_{\zeta\in U}\Omega(\zeta, z^0, u^0)\wedge\overline{\partial}_\zeta\phi(\zeta, u^0) = 0.$$

Thus $(*)$ holds for points in $M\backslash(U \times V)$. □

The next lemma is essentially [24, Lemma 3], and the simple proof we give follows [25]. (Here $\arg z$ denotes the argument of $z$.)

Lemma 2.4. *Let* $T = \{z \in \mathbf{C} : 3\pi/4 \leq \arg z \leq 5\pi/4,\ 0 < |z| \leq \varepsilon\}$. *Then there is a sequence* $(\alpha_r)_{r=1}^{\infty}$ *of functions each holomorphic on a neighborhood of* $\mathbf{C}\backslash T$ *and a positive constant* $c$ *such that*

(i) $\alpha_r(z) \to 1/z$ *for* $z \in \mathbf{C}\backslash(T \cup \{0\})$, *and*
(ii) $|\alpha_r(z)| \leq c/|z|$ *for* $z \in \mathbf{C}\backslash T$.

*Proof.* Set $\alpha_r(z) = \frac{1}{z+(1/r)}$. Then for large $r$, we have that $\alpha$ is holomorphic on a neighborhood of $\mathbf{C}\backslash T$. Clearly (i) holds. Also, a little thought shows that there is a positive constant $c_1$ such that for all $r$ large and $z \in \mathbf{C}\backslash T$, we have

$$\left|1 + \tfrac{1}{rz}\right| \geq c_1$$

or equivalently

$$\left|z + \tfrac{1}{r}\right| \geq c_1|z|.$$

Thus setting $c = 1/c_1$ gives (ii). □

We conclude this section with three more lemmas that will be used in the proof of Theorem 1.2.

Lemma 2.5. *Let* $M$ *be a* $C^1$ *manifold and* $f_1, \ldots, f_n$ *be* $C^1$ *complex-valued functions on* $M$. *Let* $F : M \to \mathbf{C}^n$ *be given by* $F(x) = \big(f_1(x), \ldots, f_n(x)\big)$. *If* $df_1, \ldots, df_n$ *span the complexified cotangent space to* $M$ *at a point* $p$, *then the image of some neighborhood of* $p$ *is a submanifold of* $\mathbf{C}^n$ *with no complex tangents.*

*Proof.* Since $df_1, \ldots, df_n$ span the complexified cotangent space to $M$ at $p$, the same is true on a neighborhood $U$ of $p$. Write $f_j = u_j + iv_j$ with $u_j$ and $v_j$ real-valued. Then $du_1, \ldots, du_n, dv_1, \ldots, dv_n$ also span the complexified cotangent space on $U$, and hence span the real cotangent space there. Consequently, the derivative $dF$ is injective on the tangent space to $M$ at all points of $U$. Hence shrinking $U$ if necessary, we have that $F$ is an embedding on $U$, so $F(U)$ is a submanifold of $\mathbf{C}^n$.

For the no-complex tangents condition, fix $q \in U$ and consider the pull back $F^* : T^*_{F(q)}F(U) \to T^*_qU$ on complexified cotangent spaces. Note that $F^*(dz_j) = d(z_j \circ F) = df_j$. Since $df_1, \ldots, df_n$ span $T^*_qU$, and $F^*$ is an isomorphism, we get that $dz_1, \ldots, dz_n$ span $T^*_{F(q)}F(U)$. This gives that $F(U)$ has no complex tangents (see [3, Lemma 2.5] for instance). □

Lemma 2.6. *Let* $Y$ *be a compact polynomially convex set in* $\mathbf{C}^{n+k}$, *let* $p = (p_1, \ldots, p_n) \in \mathbf{C}^n$, *let* $N$ *be a compact subset of* $\mathbf{C}^k$, *and let* $L = \{p\} \times N$. *Then* $(Y \cup L)\widehat{\ } \subset Y \cup \big(\{p\} \times \mathbf{C}^k\big)$.

*Proof.* We must show that if $\zeta \neq p$ and $(\zeta, w) \notin Y$, then there is a polynomial $q$ such that $q(\zeta, w) > \sup_{x \in Y \cup L} |q(x)|$. Choose $j$ such that $\zeta_j \neq p_j$. Then the function $g = (z_j - p_j)/(\zeta_j - p_j)$ is 0 on $L$ and 1 at $(\zeta, w)$. Let $M = \sup_{x \in Y} |g(x)|$. By the polynomial convexity of $Y$, there is a polynomial $f$ such that $f(\zeta, w) = 1$ and $\sup_{x \in Y} |f(x)| < 1$. Replacing $f$ by a sufficiently high power of $f$, we may assume that $\sup_{x \in Y} |f(x)| < 1/M$. Then $f \cdot g$ is a polynomial such that $(f \cdot g)(\zeta, w) = 1$, $(f \cdot g)(L) = 0$, and $|f \cdot g| < 1$ on $Y$. Thus $(f \cdot g)(\zeta, w) > \sup_{x \in Y \cup L} |(f \cdot g)(x)|$. $\square$

LEMMA 2.7. *Let $X$ and $Y$ be regular spaces, let $Z$ be a compact subset of $X \times Y$, and let $N$ be a neighborhood of $Z$ in $X \times Y$. Let $(p, q) \in Z$ be arbitrary. Then there exists a neighborhood $W$ of $Z$ contained in $N$ and neighborhoods $U$ of $p$ in $X$ and $V$ of $q$ in $Y$ such that $W \cap (U \times Y) = U \times V$.*

*Proof. Step* 1: We show there exist neighborhoods $U'$ of $p$ in $X$ and $V$ of $q$ in $Y$ such that

$$Z \cap (U' \times Y) \subset U' \times V \subset N.$$

Let $Z_p = \{y \in Y : (p, y) \in Z\}$. Then $Z_p$ is compact. Hence by the "generalized tube lemma" [16, §26, Ex. 9] there are open sets $U''$ and $V$ in $X$ and $Y$ respectively, such that $Z \cap (\{p\} \times Y) = \{p\} \times Z_p \subset U'' \times V \subset N$. Now it suffices to show that there is a neighborhood $U'''$ of $p$ such that $Z \cap (U''' \times Y) \subset U''' \times V$ for then setting $U' = U'' \cap U'''$ gives $Z \cap (U' \times Y) \subset U' \times V \subset N$, as desired.

Let $Y_1 = \pi_2(Z)$, where $\pi_2 : X \times Y \to Y$ is projection onto the second coordinate. Then $Y_1$ is compact. Let $S = \big[(X \times Y) \backslash Z\big] \cup (X \times V)$. Then $S$ is open in $X \times Y$. Furthermore, $S \supset \{p\} \times Y$. In particular, $S \supset \{p\} \times Y_1$. Hence by the "generalized tube lemma" again, there is a neighborhood $U'''$ of $p$ in $X$ such that $U''' \times Y_1 \subset S$. Then $Z \cap (U''' \times Y) \subset U''' \times V$, as the reader can check.

*Step* 2: We prove the lemma. By the regularity of $X$, we can choose a neighborhood $U$ of $p$ with $\overline{U} \subset U'$. Then $Z \cap (\overline{U} \times Y) \subset \overline{U} \times V \subset N$. Set $W = N \backslash \big[\overline{U} \times (Y \backslash V)\big]$. Then $W$ is a neighborhood of $Z$ contained in $N$ and $W \cap (U \times Y) = U \times V$. $\square$

## 3. **Proof of Theorem 1.2**

It suffices to show that if $\mu$ is a regular complex Borel measure on $X$ such that $\int f d\mu = 0$ for all $f \in A$, then $\mu = 0$ on $X \backslash E$, or equivalently that each point of $X \backslash E$ has a neighborhood on which $\mu = 0$. Throughout the proof we will let $\mu$ be a regular complex Borel measure on $X$ such that $\int f d\mu = 0$ for all $f \in A$ and let $p \in X \backslash E$ be fixed. Our goal is to show that $\mu = 0$ on some neighborhood of $p$. The proof will be divided into several steps.

*Step* 1: We show that there exists a neighborhood $U$ of $p$ in $X$, finitely many functions $f_1, \ldots, f_n$ in $A$, and a neighborhood $N$ of $\big(f_1(p), \ldots, f_n(p)\big)$ in $\mathbf{C}^n$, such that the map $F : X \to \mathbf{C}^n$ given by $F(x) = \big(f_1(x), \ldots, f_n(x)\big)$ maps $U$ one-to-one into a $C^1$ submanifold of $N$ with no complex tangents and maps $X \backslash U$ outside of $N$.

Let $U_p$ and $M_p$ be as in the statement of the theorem. We regard $U_p$ as a subset of $M_p$ as well as regarding $U_p$ as a subset of $X$. In addition, when $f$ is a function in $A$ whose restriction to $U_p$ extends to be $C^1$ on $M_p$, we will, for notational simplicity, also use the symbol $f$ to denote a fixed such $C^1$ extension.

Using condition (i) and Lemma 2.5 choose a neighborhood $V_p$ of $p$ in $M_p$ and functions $f_1, \ldots, f_m$ in $A$ that are $C^1$ on $M_p$ and such that $df_1 \wedge \cdots \wedge df_m$ is nowhere zero on $V_p$ and the map $x \mapsto \big(f_1(x), \ldots, f_m(x)\big)$ takes $V_p$ diffeomorphically onto a submanifold of $\mathbf{C}^m$ with no complex tangents. Using condition (ii) we can find functions $f_{m+1}, \ldots, f_n$ in $A$ that are $C^1$ on $M_p$ such that the set $U = \{x \in X : |f_j(x)| < 1 \text{ for all } j = m+1, \ldots, n\}$ satisfies $p \in U \subset U_p \cap V_p$. Set $N = \{z \in \mathbf{C}^n : |z_j| < 1 \text{ for all } j = m+1, \ldots, n\}$.

Now under the map $x \mapsto \big(f_1(x), \ldots, f_n(x)\big)$, the set $V_p$ is taken diffeomorphically onto a submanifold of $\mathbf{C}^n$ with no complex tangents. Obviously, $F(U) \subset N$ and $F(X \backslash U) \subset \mathbf{C}^n \backslash N$, so $F$ has the desired properties.

*Step* 2: Let $\widetilde{p} = F(p)$. Let $Y$ be the union of $F(X)$ and the submanifold of $N$ with no complex tangents containing $F(U)$. Then applying Lemma 2.2 gives the existence of real-valued $C^1$ functions $h_1, \ldots, h_n$ on $\mathbf{C}^n$ that vanish on $F(X)$ such that the matrix $\big((\partial h_i/\partial \overline{z}_j)(\widetilde{p})\big)$ is nonsingular. Let $T(z)$ denote the matrix $\big((\partial h_i/\partial \overline{z}_j)(z)\big)$, and let $S(z)$ denote the matrix $\big((\partial h_i/\partial z_j)(z)\big)$. Also set $h = (h_1, \ldots, h_n)$. Define $g : \mathbf{C}^{3n} \to \mathbf{C}^n$ by

$$g(\zeta, z, w) = \big(T(\widetilde{p})\big)^{-1}\big(h(\zeta) - w - S(\widetilde{p})(\zeta - z)\big).$$

*Step* 3: We show that there is a neighborhood $U_1$ of $\widetilde{p}$ such that if $\zeta, z \in \overline{U}_1$, then $\Big|g(\zeta, z, h(z)) - (\overline{\zeta} - \overline{z})\Big| \leq \frac{3}{4}|\zeta - z|$.

Let $R(\zeta, z)$ be defined by the equation

$$h(\zeta) = h(z) + S(z)(\zeta - z) + T(z)(\overline{\zeta} - \overline{z}) + R(\zeta, z).$$

Let $C = \|T(\widetilde{p})^{-1}\|$. Choose $U_1$ to be an open ball centered at $\widetilde{p}$ small enough that $\|S(z) - S(\widetilde{p})\| \leq (4C)^{-1}$ and $\|T(z) - T(\widetilde{p})\| \leq (4C)^{-1}$ for $z \in \overline{U}_1$ and $|R(\zeta, z)| \leq (4C)^{-1}|\zeta - z|$ for $\zeta, z \in \overline{U}_1$. Then for $\zeta, z \in \overline{U}_1$,

$$\begin{aligned}\Big|g(\zeta, z, h(z)) - (\overline{\zeta} - \overline{z})\Big| &= \Big|T(\widetilde{p})^{-1}\big[h(\zeta) - h(z) - S(\widetilde{p})(\zeta - z)\big] - (\overline{\zeta} - \overline{z})\Big| \\ &= \Big|T(\widetilde{p})^{-1}\Big[S(z)(\zeta - z) + T(z)(\overline{\zeta} - \overline{z}) + R(\zeta, z) - S(\widetilde{p})(\zeta - z)\Big] - (\overline{\zeta} - \overline{z})\Big|\end{aligned}$$

$$= \left| T(\widetilde{p})^{-1}\Big[\big(S(z)-S(\widetilde{p})\big)(\zeta - z) + \big(T(z)-T(\widetilde{p})\big)(\overline{\zeta}-\overline{z}) + R(\zeta,z)\Big]\right|$$
$$\leq \|T(\widetilde{p})^{-1}\|\Big[\|S(z)-S(\widetilde{p})\|\,|\zeta-z| + \|T(z)-T(\widetilde{p})\|\,|\zeta-z| + |R(\zeta,z)|\Big]$$
$$\leq \tfrac{3}{4}|\zeta - z|.$$

*Step* 4: Define $\Gamma : \mathbf{C}^{3n} \to \mathbf{C}$ by $\Gamma(\zeta,z,w) = (\zeta - z)\cdot g(\zeta,z,w)$ where $\alpha\cdot\beta$ denotes the standard bilinear form on $\mathbf{C}^n$. Using the result of Step 3, we may easily verify the following properties of $\Gamma$:

(i) $\Gamma$ is holomorphic in $z$ and $w$ for fixed $\zeta$, and $\Gamma$ is of class $C^1$;
(ii) $|\Gamma(\zeta,z,h(z))| \geq \frac{1}{4}|\zeta-z|^2$ for $\zeta, z \in \overline{U}_1$;
(iii) $\operatorname{Re}\Gamma(\zeta,z,h(z)) > 0$ for $\zeta, z \in \overline{U}_1$ with $\zeta \neq z$;
(iv) $\Big|\Gamma(\zeta,z,h(z))\Big| \leq \frac{7}{4}|\zeta-z|^2$ for $\zeta, z \in \overline{U}_1$.

*Step* 5: We show that there exist finitely many functions $l_1,\dots,l_k$ in $A$, a domain of holomorphy $\widetilde{M}$ in $\mathbb{C}^{2n+k}$ that contains

$$\sigma(f_1,\dots,f_n,h_1\circ F,\dots,h_n\circ F,l_1,\dots,l_k) = \sigma(f_1,\dots,f_n,0,\dots,0,l_1,\dots,l_k)$$

(the joint spectrum of the indicated functions) and also contains $\{\widetilde{p}\}\times\{0\}^n\times(l_1,\dots,l_k)(X)$, a neighborhood $U_2$ of $\widetilde{p}$, and a function $\widetilde{G}$ of class $C^1$ on $U_2\times\widetilde{M}$ such that

(i) for each fixed $\zeta \in U_2$, $\widetilde{G}$ is holomorphic on $\widetilde{M}$;
(ii) for each fixed $\zeta \in U_2$, there exists an $\varepsilon > 0$ such that for $x \in X$, the point $\widetilde{G}\big(\zeta, f_1(x),\dots,f_n(x),0,\dots,0,l_1(x),\dots,l_k(x)\big)$ lies in the sector $\{z\in\mathbf{C} : \frac{3\pi}{4} \leq \arg z \leq \frac{5\pi}{4},\ |z|\leq\varepsilon\}$ only if
$$\widetilde{G}\big(\zeta, f_1(x),\dots,f_n(x),0,\dots,0,l_1(x),\dots,l_k(x)\big) = 0;$$
and
(iii) for each pair of compact sets $E$ and $E'$ in $U_2$ and $\widetilde{M}$ respectively, there exists a $\lambda > 0$ such that $\Big|\widetilde{G}(\zeta,z,h(z),u)\Big| \geq \lambda|\zeta-z|^2$ for $(\zeta,z,h(z),u) \in E\times E'$.

Choose a neighborhood $U_2$ of $\widetilde{p}$ with $\overline{U}_2 \subset U_1$ and $\overline{U}_2$ compact. Let $W_1 = U_1 \times \mathbf{C}^n$ and let

$$W_2 = \Big((\mathbf{C}^n\backslash\overline{U}_2)\times\mathbf{C}^n\Big)$$
$$\bigcap\Big(\Big[(\mathbf{C}^n\backslash\overline{U}_1)\times\mathbf{C}^n\Big] \cup \Big\{(z,w)\in\mathbf{C}^{2n} : \operatorname{Re}\Gamma(\zeta,z,w) > 0\ \forall\zeta\in\overline{U}_2\Big\}\Big).$$

Let $\widetilde{X} = \{\big(z,h_1(z),\dots,h_n(z)\big) : z \in F(X)\} = F(X)\times\{0\}^n$. Then

(a) $W_1 \cup W_2 \supset \widetilde{X}$;
(b) if $z \in U_1$, then $\big(z,h(z)\big) \in W_1$;
(c) if $z \in U_2$, then $\big(z,h(z)\big) \notin W_1 \cap W_2$;
(d) $\operatorname{Re}\Gamma(\zeta,z,w) > 0$ on $U_2 \times (W_1\cap W_2)$.

The functions $f_1, \ldots, f_n$ are in $A$, and the functions $h_1 \circ F, \ldots, h_n \circ F$ are identically zero and hence are also in $A$. Since the maximal ideal space of $A$ is $X$, we have $\sigma(f_1, \ldots, f_n, h_1 \circ F, \ldots, h_n \circ F) = \widetilde{X}$. By the Arens-Calderon lemma [10, Lemma III.5.2] there exist functions $l_1, \ldots, l_k$ in $A$ such that $\pi\Big(\widehat{\sigma}(f_1, \ldots, f_n, h_1 \circ F, \ldots, h_n \circ F, l_1, \ldots, l_k)\Big) \subset W_1 \cup W_2$ where $\widehat{\sigma}$ denotes the polynomially convex hull of the joint spectrum and $\pi : \mathbf{C}^{2n+k} \to \mathbf{C}^{2n}$ is projection onto the first $2n$ coordinates. Henceforth we shall write $(f_1, \ldots, l_k)$ for $(f_1, \ldots, f_n, h_1 \circ F, \ldots, h_n \circ F, l_1, \ldots, l_k)$. Let $L(x) = \big(l_1(x), \ldots, l_k(x)\big)$. By Lemma 2.6

$$\Big[\widehat{\sigma}(f_1, \ldots, l_k) \cup \big(\{\widetilde{p}\} \times \{0\}^n \times L(X)\big)\Big]^{\widehat{\ }} \subset \widehat{\sigma}(f_1, \ldots, l_k) \cup \big(\{\widetilde{p}\} \times \mathbf{C}^{n+k}\big).$$

Since $\pi^{-1}(W_1)$ and $\pi^{-1}(W_2)$ cover $\widehat{\sigma}(f_1, \ldots, l_k)$, and $\pi^{-1}(W_1)$ contains $\{\widetilde{p}\} \times \mathbf{C}^{n+k}$, we get

$$\Big[\widehat{\sigma}(f_1, \ldots, l_k) \cup \big(\{\widetilde{p}\} \times \{0\}^n \times L(X)\big)\Big]^{\widehat{\ }} \subset \pi^{-1}(W_1) \cup \pi^{-1}(W_2).$$

Consequently, there is a domain of holomorphy $\widetilde{M}$ in $\mathbf{C}^{2n+k}$ such that

$$\Big[\widehat{\sigma}(f_1, \ldots, l_k) \cup \big(\{\widetilde{p}\} \times \{0\}^n \times L(X)\big)\Big]^{\widehat{\ }} \subset \widetilde{M} \subset \pi^{-1}(W_1) \cup \pi^{-1}(W_2).$$

Extend $\Gamma$ to $\mathbf{C}^{3n} \times \mathbf{C}^k$ by making it independent of the last $k$ coordinates. Obviously $\{\pi^{-1}(W_1) \cap \widetilde{M}, \pi^{-1}(W_2) \cap \widetilde{M}\}$ is an open covering of $\widetilde{M}$. Furthermore, for fixed $\zeta \in U_2$, note that $\log \Gamma$ is holomorphic on $\pi^{-1}(W_1) \cap \pi^{-1}(W_2) \cap \widetilde{M}$. Thus we can apply [21, Prop. 2] to get that there exist $C^1$ functions $P$ on $U_2 \times (\pi^{-1}(W_1) \cap \widetilde{M})$ and $Q$ on $U_2 \times (\pi^{-1}(W_2) \cap \widetilde{M})$ that are holomorphic in $\pi^{-1}(W_1) \cap \widetilde{M}$ and $\pi^{-1}(W_2) \cap \widetilde{M}$ respectively for fixed $\zeta \in U_2$ and satisfy

$$\log \Gamma = Q - P \quad \text{on } U_2 \times (\pi^{-1}(W_1) \cap \pi^{-1}(W_2) \cap \widetilde{M}).$$

If we now define $\widetilde{G}$ by

$$\widetilde{G} = \begin{cases} \Gamma e^P & \text{on } U_2 \times (\pi^{-1}(W_1) \cap \widetilde{M}) \\ e^Q & \text{on } U_2 \times (\pi^{-1}(W_2) \cap \widetilde{M}), \end{cases}$$

then $\widetilde{G}$ is a well-defined $C^1$ function on $U_2 \times \widetilde{M}$. Furthermore, $\widetilde{G}$ is holomorphic on $\widetilde{M}$ for fixed $\zeta \in U_2$, and so (i) holds.

To prove (iii), suppose $E$ and $E'$ are compact sets in $U_2$ and $\widetilde{M}$ respectively. Choose compact sets $E_1$ and $E_2$ such that $E' = E_1 \cup E_2$ and $E_1 \subset \pi^{-1}(W_1)$ and $E_2 \subset \pi^{-1}(W_2)$. We establish (iii) separately for points in $E \times E_1$ and $E \times E_2$.

On $E \times E_1 \subset U_2 \times (\pi^{-1}(W_1) \cap \widetilde{M})$, by definition $\widetilde{G} = \Gamma e^P$. Note that $(z, h(z), u) \in E_1 \subset \pi^{-1}(W_1)$ implies $z \in U_1$. Hence given $(\zeta, z, h(z), u) \in E \times E_1$, Step 4 (ii) gives $\Big|\Gamma(\zeta, z, h(z), u)\Big| \geq \frac{1}{4}|\zeta - z|^2$. Since by compactness $|e^P|$ is bounded below on $E \times E_1$, this gives (iii) on $E \times E_1$.

On $E \times E_2 \subset U_2 \times (\pi^{-1}(W_2) \cap \widetilde{M})$, by definition $\widetilde{G} = e^Q$. Note that $(z, h(z), u) \in E_2 \subset \pi^{-1}(W_2)$ implies $z \notin \overline{U}_2$. Thus for $(\zeta, z, h(z), u) \in E \times E_2$ we have $\zeta \neq z$. Hence $|\widetilde{G}|/|\zeta - z|^2 = |e^Q|/|\zeta - z|^2$ is a continuous function on $E \times E_2$ that is never zero. Consequently, (iii) holds on $E \times E_2$ by compactness.

It remains to establish (ii). We may assume without loss of generality that $P\big(\widetilde{p}, \widetilde{p}, 0, L(p)\big) = 0$. Then for all points $(\zeta, z, w, u)$ in some neighborhood of $\big(\widetilde{p}, \widetilde{p}, 0, L(p)\big)$ we have

$$|e^{P(\zeta,z,w,u)} - 1| < 1/\sqrt{2}.$$

Consequently, if we replace $U_2$ by a sufficiently small neighborhood of $\widetilde{p}$, then there is a neighborhood $U'$ of $p$ such that the inequality

$$|e^{P(\zeta,F(x),0,L(x))} - 1| < 1/\sqrt{2}$$

holds for all $\zeta \in U_2$ and all $x \in U'$. By choosing $U'$ small enough, we may assume that $F(U') \subset U_1$. Since $p$ is the only point mapped by $F$ to $\widetilde{p}$, there is a neighborhood of $\widetilde{p}$ disjoint from $F(X \backslash U')$. Thus by shrinking $U_2$ again if necessary, we may assume $F(X \backslash U')$ is disjoint from $U_2$.

Now fix $\zeta \in U_2$. Consider a point $x \in U'$. Note that if $F(x) = \zeta$, then $\widetilde{G}\big(\zeta, F(x), 0, L(x)\big) = \Gamma\big(\zeta, F(x), 0, L(x)\big) e^{P(\zeta,F(x),0,L(x))} = 0$, while if $F(x) \neq \zeta$, then multiplying the preceding inequality by $\Gamma$ gives

$$\Big|\widetilde{G}\big(\zeta, F(x), 0, L(x)\big) - \Gamma\big(\zeta, F(x), 0, L(x)\big)\Big| < (1/\sqrt{2})\Big|\Gamma\big(\zeta, F(x), 0, L(x)\big)\Big|.$$

This inequality together with Step 4(iii) gives that $\widetilde{G}\big(\zeta, F(x), 0, L(x)\big)$ lies outside the sector $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$.

Now to show there exists an $\varepsilon > 0$ such that (ii) holds, it suffices by the compactness of $X \backslash U'$ to show that $\widetilde{G}\big(\zeta, F(x), 0, L(x)\big)$ is never 0 for $x \in X \backslash U'$. So consider $x \in X \backslash U'$. If $F(x) \in U_1$, then $\widetilde{G}\big(\zeta, F(x), 0, L(x)\big) = \Gamma\big(\zeta, F(x), 0, L(x)\big) e^{P(\zeta,F(x),0,L(x))} \neq 0$ by Step 4(iii). If $F(x) \notin U_1$, then $\widetilde{G}\big(\zeta, F(x), 0, L(x)\big) = e^{Q(\zeta,F(x),0,L(x))} \neq 0$.

*Step* 6: There exist a neighborhood $U_3$ of $\widetilde{p}$ contained in $U_2 \subset \mathbf{C}^n$, a neighborhood $M$ of $(F, L)(X) = \{\big(f_1(x), \ldots, f_n(x), l_1(x), \ldots, l_k(x)\big) : x \in X\}$ in $\mathbf{C}^{n+k}$, and an open set $V$ in $\mathbf{C}^k$ such that

(i) $(z, u) \in M \Rightarrow (z, h(z), u) \in \widetilde{M}$,
(ii) $\zeta \in U_3, (z, u) \in M \Rightarrow (\zeta, h(z), u) \in \widetilde{M}$, and
(iii) $M \cap (U_3 \times \mathbf{C}^k) = U_3 \times V$.

Since $\widetilde{M}$ contains the set $\{\big(F(x), h(F(x)), L(x)\big) : x \in X\}$, there is a neighborhood $M_1$ of $(F, L)(X)$ such that (i) holds with $M_1$ in place of $M$. Since $\widetilde{M}$ also contains the set $\big\{\big(\widetilde{p}, h(F(x)), L(x)\big) : x \in X\big\}$, there is a neighborhood $W$ of $\{\widetilde{p}\} \times (F, L)(X)$ such that $(\zeta, z, u) \in W \Rightarrow (\zeta, h(z), u) \in \widetilde{M}$.

By compactness of $(F,L)(X)$, there exist neighborhoods $U'$ of $\widetilde{p}$ and $M_2$ of $(F,L)(X)$ such that $U' \times M_2 \subset W$. Then with $M_1 \cap M_2$ in place of $M$ and $U'$ in place of $U_3$, both (i) and (ii) hold. Finally, applying Lemma 2.7 we obtain a neighborhood $M \subset M_1 \cap M_2$ of $(F,L)(X)$, a neighborhood $U_3$ of $p$, and an open set $V$ in $\mathbf{C}^k$, such that (iii) holds. We may assume $U_3 \subset U' \cap U_2$, and then all conditions are satisfied.

*Step* 7: Define $G$ on $U_3 \times M$ by

$$G(\zeta,z,u) = \widetilde{G}(\zeta,z,h(z),u)$$

for $\zeta \in U_3$ and $(z,u) \in M$. By Step 5(iii), for each pair of compact sets $E$ and $E''$ in $U_3$ and $M$ respectively, there exists a $\lambda > 0$ such that

$$\Big|G(\zeta,z,u)\Big| \geq \lambda|\zeta - z|^2$$

for $(\zeta,z,u) \in E \times E''$.

*Step* 8: We show that there exist functions $G_1,\ldots,G_n \in C^1(U_3 \times M)$ such that

(i) $G(\zeta,z,u) = \sum_{j=1}^{n}(\zeta_j - z_j)G_j(\zeta,z,u)$;
(ii) for fixed $\zeta \in U_3$, the function $x \mapsto G_j\big(\zeta,F(x),L(x)\big)$ is in $A$, $1 \leq j \leq n$;
(iii) for fixed $\zeta \in U_3$, the function $x \mapsto \frac{\partial}{\partial\zeta_r}G_j\big(\zeta,F(x),L(x)\big)$ is in $A$, $1 \leq j,r \leq n$;
(iv) for each pair of compact sets $E$ and $E''$ in $U_3$ and $M$ respectively, there exists a constant $C$ such that $\Big|G_j(\zeta,z,u)\Big| \leq C|\zeta - z|$ for $(\zeta,z,u) \in E \times E''$; and
(v) for each $j$, the function $\zeta \mapsto G_j(\zeta,z,u)G(\zeta,z,u)^{-n}$ belongs to $L^1_{\mathrm{loc}}$ uniformly for $(z,u)$ in compact subsets of $M$.

By [21, Prop. 4] there exist functions $R_1,\ldots,R_n,S_1,\ldots,S_n,T_1,\ldots,T_k$ of class $C^1$ on $U_2 \times (\widetilde{M} \times \widetilde{M})$, holomorphic on $\widetilde{M} \times \widetilde{M}$ for fixed $\zeta \in U_2$ such that

$$\begin{aligned}(1)\qquad \widetilde{G}(\zeta,z,w,u) - \widetilde{G}(\zeta,z',w',u') &= \sum(z_j - z'_j)R_j(\zeta,z,w,u,z',w',u')\\ &\quad+\sum(w_j - w'_j)S_j(\zeta,z,w,u,z',w',u')\\ &\quad+\sum(u_j - u'_j)T_j(\zeta,z,w,u,z',w',u')\end{aligned}$$

for all $\zeta \in U_2$ and $(z,w,u),(z',w',u') \in \widetilde{M}$. Recall from Step 6 that for $\zeta \in U_3$ and $(z,u) \in M$ we have $(z,h(z),u) \in \widetilde{M}$ and $(\zeta,h(z),u) \in \widetilde{M}$. Thus setting $w = h(z)$, $z' = \zeta$, $w' = h(z)$, and $u' = u$ in (1) and applying the definition of $G$ from Step 7, we get

$$\begin{aligned}G(\zeta,z,u) &= \widetilde{G}(\zeta,z,h(z),u)\\ &= \widetilde{G}(\zeta,\zeta,h(z),u) + \sum(z_j - \zeta_j)R_j(\zeta,z,h(z),u,\zeta,h(z),u)\end{aligned}$$

for all $(\zeta, z, u) \in U_3 \times M$. Now note that $\widetilde{G}(\zeta, \zeta, h(z), u) = 0$ for $\zeta \in U_3$ by the definition of $\widetilde{G}$ and $\Gamma$. Thus setting $G_j(\zeta, z, u) = -R_j(\zeta, z, h(z), u, \zeta, h(z), u)$ we have that (i) holds and $G_1, \dots, G_n \in C^1(U_3 \times M)$.

Condition (ii) that the map

$$x \mapsto G_j\big(\zeta, F(x), L(x)\big) = -R_j\big(\zeta, F(x), h(F(x)), L(x), \zeta, h(F(x)), L(x)\big)$$

is in $A$ follows from the functional calculus in several variables since $R_j$ is holomorphic on $\widetilde{M} \times \widetilde{M}$ for fixed $\zeta \in U_3$, and the components of $F$, $h \circ F$, and $L$ all lie in $A$. It is well known that the conditions that $R_j$ is of class $C^1$ and holomorphic on $\widetilde{M} \times \widetilde{M}$ for fixed $\zeta$ imply that each first partial derivative of $R_j$ is holomorphic on $\widetilde{M} \times \widetilde{M}$ for fixed $\zeta$. Therefore, condition (iii) also follows from the functional calculus in several variables.

With $E$ and $E''$ as in (iv), we see from the definitions of $G$ and $\widetilde{G}$ in Steps 7 and 5, that Step 4(iv) implies that there exists a constant $C'$ such that

$$\Big|G(\zeta, z, u)\Big| \le C'|\zeta - z|^2$$

for all $(\zeta, z, u) \in E \times E''$. In view of (i), it follows that $G_j(\zeta, z, u) = 0$ for $\zeta = z$ and now (iv) follows from the continuous differentiability of $G_j$.

Finally, (iv) and Step 7 give the existence of a constant $C''$ such that

$$\Big|G_j(\zeta, z, u) G(\zeta, z, u)^{-n}\Big| \le C''|\zeta - z|^{1-2n}$$

for all $(\zeta, z, u) \in E \times E^{''}$, and this implies condition (v).

*Step* 9: Define the form $\Omega(\zeta, z, u)$ on $U_3 \times M$ in terms of the functions $G, G_1, \dots, G_n$ by the formula given in Theorem 2.3. Define functions $K_j(\zeta, z, u)$ on $U_3 \times M$ by the equation $\Omega(\zeta, z, u) = \sum_{j=1}^n K_j(\zeta, z, u) \bigwedge_{r \neq j} d\overline{\zeta}_r \wedge \alpha$ where $\alpha = d\zeta_1 \wedge \ldots \wedge d\zeta_n$. We show that

$$\int K_j(\zeta, z, u)\, d(F, L)_*(\mu)(z, u) = 0 \tag{2}$$

for almost all $\zeta \in U_3$. Here $d(F, L)_*(\mu)$ denotes the push forward of $\mu$ under the map $(F, L)$. (Recall from the beginning of the proof that $\mu$ is an annihilating measure for $A$.)

Each of the functions $K_j$ is the product of $G_j G^{-n}$ with some $\overline{\zeta}$-derivatives of the functions $G_r$. Thus Step 8(v) gives, for an arbitrary compact set $E$ in $U_3$, that

$$\sup\nolimits_{(z,u) \in (F,L)(X)} \int_E \Big|K_j(\zeta, z, u)\Big|\, dm(\zeta) < \infty$$

where $m$ denotes Lebesgue measure on $\mathbf{C}^n$. Hence

$$\int_{(F,L)(X)} \int_E \Big|K_j(\zeta, z, u)\Big|\, dm(\zeta)\, d(F, L)_*(\mu)(z, u)$$

is finite, so an application of Fubini's theorem gives that

$$\int_{(F,L)(X)} \Big|K_j(\zeta, z, u)\Big|\, d(F, L)_*(\mu)(z, u) < \infty \tag{3}$$

for almost all $\zeta$ in $U_3$. Thus it suffices to establish (2) for those $\zeta$ satisfying (3). Furthermore, it is easily seen that

$$\Big|(F, L)_*(\mu)\Big|(\{\zeta\} \times \mathbf{C}^k) = 0 \tag{4}$$

for almost all $\zeta \in U_3$, so that we may further restrict our attention to only these $\zeta$. Now fix $\zeta \in U_3$ satisfying (3) and (4).

As previously noted, each of the functions $K_j$ is the product of $G_j G^{-n}$ with some $\overline{\zeta}$-derivatives of the functions $G_r$. Consequently, Step 8(ii) and (iii) give that the function on $X$ given by $x \mapsto K_j\big(\zeta, F(x), L(x)\big) \cdot G^n\big(\zeta, F(x), L(x)\big)$ is in $A$. Let $(\alpha_r)$ be the sequence of holomorphic functions given by Lemma 2.4. Then the map $(z, w, u) \mapsto (\alpha_r \circ \widetilde{G})(\zeta, z, w, u)$ is holomorphic on a neighborhood of $\sigma(f_1, \ldots, f_n, 0, \ldots, 0, l_1, \ldots, l_k)$. Hence the functional calculus in several variables shows that the function

$$x \mapsto \alpha_r\Big(\widetilde{G}\big(\zeta, F(x), 0, L(x)\big)\Big) = \alpha_r\Big(G\big(\zeta, F(x), L(x)\big)\Big)$$

is in $A$. We conclude that regarding $K_j, G,$ etc. as functions of $x \in X$ in the obvious way, we have $K_j G^n(\alpha_r^n \circ G) \in A$.

Now note that for all $(z, u)$ in $(F, L)(X)$ with $z \neq \zeta$ we have

$$K_j(\zeta, z, u) G^n(\zeta, z, u) \alpha_r^n\Big(G(\zeta, z, u)\Big) \to K_j(\zeta, z, u) \text{ as } r \to \infty,$$

and

$$\Big|K_j(\zeta, z, u) G^n(\zeta, z, u) \alpha_r^n(G(\zeta, z, u))\Big| \leq c^n |K_j|,$$

where $c$ is the constant in Lemma 2.4. Thus by (3) and (4) we can apply the Lebesgue dominated convergence theorem to get

$$\begin{aligned} &\int K_j(\zeta, z, u) d(F, L)_*(\mu)(z, u) \\ &\quad = \lim_{r\to\infty} \int K_j(\zeta, z, u) G^n(\zeta, z, u) \alpha_r^n\Big(G(\zeta, z, u)\Big)\, d(F, L)_*(\mu)(z, u) \\ &\quad = \lim_{r\to\infty} \int K_j\big(\zeta, F(x), L(x)\big) G^n\big(\zeta, F(x), L(x)\big) \alpha_r^n\Big(G(\zeta, F(x), L(x)\Big)\, d\mu(x). \end{aligned}$$

Since we showed that the integrand on the last line is in $A$, the expression on the last line is 0. Thus (2) holds.

*Step* 10: We show that $\mu = 0$ on some neighborhood of $p$ in $X$, thus completing the proof.

From Step 1 we have that $F$ maps the neighborhood $U$ of $p$ one-to-one into the neighborhood $N$ of $F(p)$ and maps $X \backslash U$ outside of $N$. Consequently, to show that $\mu = 0$ on a neighborhood of $p$ in $X$, it suffices to show that the push forward measure $F_*(\mu)$ is 0 on a neighborhood of $F(p)$ in $\mathbf{C}^n$. We show that $F_*(\mu) = 0$ on $U_3$ by showing that

$$\int \phi(z)\, dF_*(\mu)(z) = 0 \tag{5}$$

for every $\phi \in C_c^\infty(U_3)$ ($C^\infty$ functions with compact support in $U_3$).

By Steps 6, 7, 8, and 9, the hypotheses of Theorem 2.3 are satisfied (with $U_3$ as $U$), and so regarding $\phi$ as a function on $\mathbf{C}^n \times \mathbf{C}^k$ that is independent of the second variable, the representation $(*)$ given in that theorem holds. Thus by an application of Fubini's theorem we have (with the notation of Step 9)

$$\begin{aligned}\int \phi(z)\, dF_*(\mu)(z) &= \int \phi(z)\, d(F,L)_*(\mu)(z,u)\\ &= \int \left[\int_{\zeta\in U_3} \sum_j K_j(\zeta,z,u) \bigwedge_{r\neq j} d\bar{\zeta}_r \wedge \alpha \wedge \bar{\partial}\phi(\zeta)\right] d(F,L)_*(\mu)(z,u)\\ &= \sum_j \int_{\zeta\in U_3} \left[\int K_j(\zeta,z,u)\, d(F,L)_*(\mu)(z,u)\right] \bigwedge_{r\neq j} d\bar{\zeta}_r \wedge \alpha \wedge \bar{\partial}\phi(\zeta).\end{aligned}$$

By Step 9, the expression in square brackets on the last line is 0 for almost all $\zeta \in U_3$. Hence (5) holds. This completes the proof. □

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