Annals of Mathematics **173** (2011), 1779–1840
doi: 10.4007/annals.2011.173.3.11

O-minimality and the André-Oort conjecture for $\mathbb{C}^n$

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Abstract

We give an *unconditional* proof of the André-Oort conjecture for arbitrary products of modular curves. We establish two generalizations. The first includes the Manin-Mumford conjecture for arbitrary products of elliptic curves defined over $\overline{\mathbb{Q}}$ as well as Lang's conjecture for torsion points in powers of the multiplicative group. The second includes the Manin-Mumford conjecture for abelian varieties defined over $\overline{\mathbb{Q}}$. Our approach uses the theory of o-minimal structures, a part of Model Theory, and follows a strategy proposed by Zannier and implemented in three recent papers: a new proof of the Manin-Mumford conjecture by Pila-Zannier; a proof of a special (but new) case of Pink's relative Manin-Mumford conjecture by Masser-Zannier; and new proofs of certain known results of André-Oort-Manin-Mumford type by Pila.

1. **Introduction**

In this paper we give an unconditional proof of the André-Oort conjecture for arbitrary products of modular curves. Under the Generalized Riemann Hypothesis for imaginary quadratic fields this result is due to Edixhoven [32], [34]; for $n = 2$ it is an unconditional result of André [3]. Our approach uses the theory of *o-minimal structures*, a part of Model Theory. It leads naturally to a more general result that is an "André-Oort-Manin-Mumford-Lang" statement for varieties of the form

$$X = Y_1 \times \cdots \times Y_n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell,$$

where n, m, ℓ are nonnegative integers, $Y_1 = \Gamma_1\backslash\mathbb{H}, \ldots, Y_n = \Gamma_n\backslash\mathbb{H}$ are modular curves corresponding to the quotient of the upper half-plane $\mathbb{H}$ by congruence subgroups Γ_i of $\mathrm{SL}_2(\mathbb{Z})$, $E_1, \ldots, E_m$ are elliptic curves defined over $\overline{\mathbb{Q}}$, and $\mathbb{G} = \mathbb{G}_{\mathrm{m}}(\mathbb{C})$ is the multiplicative group of nonzero complex numbers. (In this paper complex algebraic varieties will be identified with their sets of complex-valued points.) Combining the methods of this paper with those of Pila and Zannier [71] we prove an "André-Oort-Manin-Mumford" statement for varieties of the form

$$X = Y_1 \times \cdots \times Y_n \times A$$

where Y_i are modular curves as above and A is an abelian variety defined over $\overline{\mathbb{Q}}$.

It is well known ([33], [35]) that level structure is inessential for the André-Oort conjecture. Here too the case in which each $\Gamma_i = \mathrm{SL}_2(\mathbb{Z})$, so that $Y_i = \mathbb{C}$, exhibits all the essential features, and we restrict to this case for the latter part of the introduction. In particular, the definitions of "special point" and "special subvariety" are given for X of this special form in Definitions 1.2 and 1.3 below. The definitions in the general case are given in Definition 6.7. One observes that a "special point" is the same as a "special subvariety of dimension 0", and that special subvarieties of positive dimension contain infinitely many — even a Zariski dense set of — special points (see Aside 1.4). Thus if $V \subset X$ contains a special subvariety of positive dimension then V will contain infinitely many special points.

A weak version of the "André-Oort-Manin-Mumford-Lang" statement for a variety X is the converse of the above statement:

> *If $V \subset X$ contains infinitely many special points, then it contains a special subvariety of positive dimension.*

When such a result is known it is generally known in a more refined version asserting:

> *A subvariety $V \subset X$ contains a finite number of special subvarieties of X (of dimension 0 or greater) that contain all the special points of X lying in V.*

We establish our result in this stronger form, and since any special subvariety contained in V is contained in some *maximal* special subvariety contained in V we can state our main result as follows.

1.1. Theorem. *Let*

$$X = Y_1 \times \cdots \times Y_n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell,$$

where $n, m, \ell \geq 0$ and $Y_i = \Gamma_i \backslash \mathbb{H}$ are modular curves corresponding to congruence subgroups Γ_i of $\mathrm{SL}_2(\mathbb{Z})$ and E_j are elliptic curves defined over $\overline{\mathbb{Q}}$. Suppose V is a subvariety of X. Then V contains only a finite number of maximal special subvarieties.

Note that the subvariety V need not be irreducible, nor need it be defined over $\overline{\mathbb{Q}}$, but since special points are algebraic the proof reduces immediately to this case.

Another way of stating this result concerns the Zariski closure of an arbitrary set Σ of special points of X. Let V_Σ be the Zariski closure of Σ. By

Theorem 1.1, V_Σ contains finitely many maximal special subvarieties. Then it coincides with their union. As special subvarieties are irreducible (see Definition 1.3; this holds also generally Shimura varieties — but note that varieties as in Theorem 1.1 are not in general Shimura varieties, even mixed ones), one concludes the following.

1.1*. Theorem. *Suppose X is as in Theorem* 1.1. *Let Σ be an arbitrary set of special points of X with Zariski closure V_Σ. Then the irreducible components of V_Σ are special subvarieties.*

In fact this second version is equivalent to the first. For suppose V is a subvariety of X. We may apply the second version to the set Σ of special points of X contained in V. The Zariski closure of Σ then comprises a finite union of irreducible components, which (as special points are Zariski dense in a special subvariety) are just the maximal special subvarieties contained in V. In Section 12 we prove the assertion of Theorem 1.1 for subvarieties $V \subset X$ for $X = Y_1 \times \cdots \times Y_n \times A$, where A is an abelian variety of arbitrary dimension defined over $\overline{\mathbb{Q}}$. This is again equivalent to the assertion of Theorem 1.1* for arbitrary sets Σ of special points of X.

For fixed X, the number and "complexity" of maximal special subvarieties contained in V is bounded uniformly for subvarieties V of given degree and degree over $\mathbb{Q}$ of field of definition. A precise statement is formulated in Section 13. Theorem 1.1 is ineffective in the j aspects due to its reliance on lower bounds for class numbers. Siegel's well-known result [84], which is nearly as good as would follow from Generalized Riemann Hypothesis (GRH), is in fact stronger than we need. Landau's weaker lower bound [47] suffices (see Remark 5.9.1) though it too is ineffective, as are all known bounds of the requisite form. Unlike the proofs in [34], [92], which depend on the existence of a small split prime, and so rely on GRH, a suitable lower bound for class numbers is all we require. In Section 13 we explicate what would be required to make the rest of our argument effective, and give a further statement that would follow.

The André-Oort conjecture (AO) is the assertion in Theorem 1.1 (more usually stated in the form Theorem 1.1*) for an arbitrary Shimura variety X (see e.g. [62], [97]). It is trivial if $\dim X = 1$, since X is irreducible as a variety and so a proper $V \subset X$ reduces to a finite set of points. AO is the compositum of a conjecture of Oort [63] (AO for subvarieties of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties) and one of André [2] (AO for curves in an arbitrary Shimura variety). As already mentioned, André [3] proved AO unconditionally for a product of two modular curves. Independently, Edixhoven [32] proved the same under GRH for imaginary quadratic fields, and later, under the same GRH assumptions, for an arbitrary product of modular curves [34] (see also [92]). Under GRH for suitable CM fields, Yafaev [95] affirms AO for products of two Shimura curves, Edixhoven [33] for Hilbert modular surfaces, and Yafaev [96] for curves in an arbitrary Shimura variety. In the subsequent work, equidistribution results (see e.g. [23], [90]) have played a major role. By combining the Galois- and equidistribution-theoretic methods, a proof of the André-Oort conjecture in full, under GRH for CM fields, has been announced in work of Klingler, Ullmo, and Yafaev [46], [91].

Unconditional results have been obtained for certain X and V under additional hypotheses on the special points Σ. In particular if the points in Σ lie in one Hecke orbit, then Theorem 1.1* is affirmed in [33] for Hilbert modular surfaces, in [34] for products of modular curves, and these results are further strengthened and generalized in [35], [96], [46]; see also Zhang [99]. Moonen [60] affirms Theorem 1.1* for $\mathcal{A}_g$ under different conditions on the points in Σ.

Theorem 1.1 affirms AO in the case of a product $X = Y_1 \times \cdots \times Y_n$ of modular curves (the more general X we consider in Theorem 1.1 are not Shimura varieties). To the knowledge of the author, these are the only Shimura varieties X (with $\dim X \geq 2$) for which AO is known unconditionally. (For mixed Shimura varieties one has also the result of André [4] on elliptic pencils, for which a proof along the present lines is given in [68].)

For $X = E_1 \times \cdots \times E_m$, Theorem 1.1 is a special case of the Manin-Mumford conjecture (MM) for subvarieties of abelian varieties and our proof is a variant of the one in [71] for abelian varieties over $\overline{\mathbb{Q}}$. (The Manin-Mumford conjecture was originally proved by Raynaud [76], [77]. For a survey see [88]). For $X = \mathbb{G}^\ell$ the result is a special case of a theorem of Laurent [50] (see also Sarnak-Adams [82]), generalizing earlier cases due to Liardet [52] to affirm for $\mathbb{G}^\ell$ a conjecture of Lang on the intersection of a subvariety of a semi-abelian variety with the *division points* of a finitely generated subgroup. For torsion points (i.e. division points of the trivial subgroup) weak forms of the conjecture go back to Chabauty (see Lang [49]), while proofs of it in the simplest case of plane curves, due to Ihara-Serre-Tate, are given in [48]. For $X = E_1 \times \cdots \times E_m \times \mathbb{G}^\ell$ our result is a special case of Hindry's theorem [42] affirming the torsion point case of Lang's conjecture for subvarieties of commutative group varieties. MM became part of the Mordell-Lang conjecture (ML, proved by Faltings, Vojta,...) on the one hand, and a special case of the Bogomolov conjecture on the other. It has a great variety of proofs, some in conjunction with these other problems, including a proof by Hrushovski [43] using the model theory of difference fields (quite a different flavour of Model Theory to that employed here).

For $X = \mathbb{C} \times A$ a much stronger result than ours, allowing finite generation and points of small height (as in the Bogomolov conjecture) in the abelian variety A, is due to Buium-Poonen [21] (see also [22]), who further allow the modular curve $\mathbb{C}$ to be replaced by a Shimura curve. An earlier result along these lines is due to Nekovar-Schappacher [61], and a proof in the case of $\mathbb{C} \times E$ is in [68].

In making their conjectures, André and Oort were mindful of the analogy between AO and MM, and there has been an interplay of methods used for AO and MM and related problems. Notably, equidistribution played a key role in the proof by Ullmo [89] and Zhang [98] of the Bogomolov conjecture. In the other direction, Ratazzi-Ullmo [75] give a proof of MM using methods developed for AO. A conjecture of Pink [73], [74] combines AO, MM and ML in a far-reaching generalization. A related conjecture in the semi-abelian setting (encompassing MM and ML but not AO) had been earlier proposed by Zilber [100] and, independently, Bombieri-Masser-Zannier [14] proposed a similar conjecture for $\mathbb{G}^\ell$. In [15] it is shown that, for $\mathbb{G}^\ell$, all the formulations are equivalent if taken in sufficient generality. The aforementioned result of André on elliptic pencils is contained in Pink's conjecture, as is the result of Masser-Zannier [56]. Theorem 1.1 (Theorem 12.1) combines AO for products of modular curves with MM for products of elliptic curves and linear tori (abelian varieties), treating the various factors in a uniform manner, although we also exploit incompatibilities in the underlying geometries.

For the rest of the introduction we restrict consideration to varieties

$$X = \mathbb{C}^n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell,$$

where $n, m, \ell \geq 0$ and E_i are elliptic curves defined over $\overline{\mathbb{Q}}$, except that, for convenience (and brevity), the following definitions are given with $E_1 \times \cdots \times E_m$ replaced by an arbitrary abelian variety.

1.2. *Definition.* 1. Let $n \geq 0$. A *special point* of $\mathbb{C}^n$ is a point $c = (c_1, \ldots, c_n)$ such that each c_i is the j-invariant of an elliptic curve with complex multiplication. By convention the point $\mathbb{C}^0$ is special.

2. Let A be an abelian variety of dimension $m \geq 0$. A *special point* of A is a point $a \in A$ of finite order, i.e. a torsion point. So if $m = 0$, then A consists of a single point, which is special.

3. Let $\ell \geq 0$. A *special point* of $\mathbb{G}^\ell$ is a point $g = (g_1, \ldots, g_\ell) \in \mathbb{G}^\ell$ of finite order, i.e., such that each g_i is a root of unity. By convention the point $\mathbb{G}^0$ is special.

4. Let

$$X = \mathbb{C}^n \times A \times \mathbb{G}^\ell,$$

where $n, \ell \geq 0$ and A is an abelian variety of dimension $m \geq 0$. A *special point* of X is a point $(c, a, g) \in X$ such that c is a special point of $\mathbb{C}^n$, a is a special point of A, and g is a special point of $\mathbb{G}^\ell$.

1.3. *Definition.* 1. A special subvariety in $\mathbb{C}^n$ is an irreducible component of a cartesian product of fibred products of modular curves and special points, which we detail more precisely as follows: For $N \geq 1$ let $\Phi_N \in \mathbb{Z}[x, y]$ denote the classical modular polynomial (see e.g. [58]; Φ_N is symmetric for $N \geq 2$, e.g. $\Phi_2 = x^3 + y^3 - x^2y^2 + 1488xy(x + y) - 162000(x^2 + y^2) + 40773375xy + 8748000000(x + y) - 157464000000000$, and we take $\Phi_1(x, y) = x - y$). Let $n \geq 0$. Let $S_0 \cup S_1 \cup \cdots \cup S_w$ be a disjoint partition of $\{1, \ldots, n\}$ with $w \geq 0$ and S_0 only permitted to be empty. Let j_i be a special point of $\mathbb{C}$ for each $i \in S_0$. Let s_i be the smallest element of S_i for each $i > 0$ and for each $j \in S_i, j \neq s_i$ choose a positive integer N_{ij}. A *special subvariety* of $\mathbb{C}^n$ is an irreducible component Y of a subvariety of the form

$$\{(c_1, \ldots, c_n) \in \mathbb{C}^n : c_i = j_i,\ i \in S_0,\ \Phi_{N_{ij}}(c_{s_i}, c_j) = 0,\ j \in S_i,\ j \neq s_i,\ i = 1, \ldots, w\}$$

associated to some choice of data S_i, j_i, N_{ij} as indicated. The dimension of the special subvariety is equal to w. Note that for $n = 0$ one must have $w = 0$.

2. Let A be an abelian variety of dimension $m \geq 0$. A *special subvariety* of A is a subvariety of the form

$$a + B,$$

where B is an abelian subvariety of A (possibly trivial) and a is a torsion point.

3. Let $\ell \geq 0$. A *special subvariety* of $\mathbb{G}^\ell$ is a subvariety of the form

$$gH,$$

where H is an irreducible algebraic subgroup (which may be trivial) and $g = (g_1, \ldots, g_\ell)$ is a torsion point (i.e. g_i are roots of unity).

4. Let $X = \mathbb{C}^n \times A \times \mathbb{G}^\ell$, where $n, \ell \geq 0$ and A is an abelian variety of dimension $m \geq 0$. A *special subvariety* of X is a subvariety of the form

$$Y \times (a + B) \times gH,$$

where Y is a special subvariety of $\mathbb{C}^n$, $a + B$ is a special subvariety of A, and gH is a special subvariety of $\mathbb{G}^\ell$.

1.4. *Aside.* We give a brief indication of the assertion that special points are Zariski dense in special subvarieties. In an abelian variety, the density of torsion points in the analytic topology is evident when A is viewed as a complex torus, the torsion points being the division points of the lattice. This is clearly preserved for torsion cosets of abelian subvarieties. In $\mathbb{G}$, torsion points are Zariski dense because there are infinitely many of them. As an irreducible algebraic subgroup of $\mathbb{G}^\ell$ is isomorphic to $\mathbb{G}^\lambda$ for some $\lambda \leq \ell$, one gets Zariski denisty in an irreducible algebraic subgroup, and thence in any torsion coset. In $\mathbb{C}$ as a modular variety the Zariski density again follows from there being infinitely many special points, but here they are also dense in the analytic topology being the images of quadratic points under the uniformisation $j : \mathbb{H} \to \mathbb{C}$ by the elliptic modular function. If an elliptic curve has CM, then so does any isogenous elliptic curve. Thus if $\Phi_N(x, y) = 0$ and x is special, then y is also special. This gives the (analytic) density of special points in modular curves, and density in all special subvarieties of $\mathbb{C}^n$ follows.

Our method of proof of Theorem 1.1 follows the same basic strategy originally proposed by Zannier and worked out by the present author and Zannier to give a new proof [71] of the Manin-Mumford conjecture. The same strategy has been exploited in two further papers. Masser and Zannier [56] prove a special case of Pink's relative Manin-Mumford conjecture, and Pila [68] gives new proofs of some simple results of André-Oort-Manin-Mumford type (including the $X = \mathbb{C}^2$ and $X = \mathbb{C} \times E$ cases of Theorem 1.1). It relies on results from the theory of *o-minimal structures* over $\mathbb{R}$, a part of Model Theory. O-minimality is used at three distinct junctures in our argument. The definition of an o-minimal structure over $\mathbb{R}$, and the key examples, are set out in Section 2. For some remarks on further prospects for this approach see Remark 11.4.2.

With $X = \mathbb{C}^n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell$, let $\mathbb{H}$ denote the upper half-plane and, for $j = 1, \ldots, m$, let $\Lambda_j \subset \mathbb{C}$ be a lattice such that $\mathbb{C}/\Lambda_j$ is complex analytically isomorphic to E_j by means of the Weierstrass $\wp$-function $\wp_j$ corresponding to Λ_j and its derivative $\wp_j'$. Let

$$U = \mathbb{H}^n \times \mathbb{C}^m \times \mathbb{C}^\ell.$$

The starting point for this strategy is the transcendental uniformization

$$\pi : U \to X,$$

where π is given by applying the j function on the factors of $\mathbb{H}^n$, the functions $\wp_j, \wp_j', j = 1, \ldots, m$ on the factors of $\mathbb{C}^m$, and the exponential function on the factors of $\mathbb{C}^\ell$. The map π is invariant under a discrete group Γ of isometries of U, generated by the action of $\mathrm{SL}_2(\mathbb{Z})^n$ on $\mathbb{H}^n$, translation by the lattices Λ_j on the respective factors of $\mathbb{C}^m$, and translation by $2\pi i\mathbb{Z}$ on the factors of $\mathbb{C}^\ell$. The discrete group Γ is a subgroup of a suitable algebraic group G of isometries of U (G is a product of copies of $\mathrm{SL}_2(\mathbb{R}), \mathbb{R}^2$, and $\mathbb{R}$). Let $\mathbb{F}$ be a standard fundamental domain for the action of Γ on U. (In §4, these definitions are set out formally and more generally for X as in Theorem 1.1. The notation $X, V, U, G, \pi, \Gamma, \mathbb{F}$ above and $\mathcal{Z}, Z$ below remain fixed from §4 onwards, except that we take X to be of more or less restricted form at various places).

Call the pre-images in U of special points of X *pre-special points*. They have certain rationality properties. Specifically, if $\pi(\tau_1, \ldots, \tau_n,\ z_1, \ldots, z_m,\ \zeta_1, \ldots, \zeta_\ell)$ is special, then the τ_i are quadratic algebraic points in $\mathbb{H}$, the z_j are division points with respect to the lattices Λ_j, and the ζ_j are rational multiples of $2\pi i$. Let

$$\mathcal{Z} = \pi^{-1}(V) \quad \text{and} \quad Z = \mathcal{Z} \cap \mathbb{F}.$$

To count special points in V we may count instead their pre-images in Z.

We consider

$$U \subset \mathbb{R}^N, \quad X \subset \mathbb{R}^N, \quad N = 2(n + m + \ell)$$

in suitable real coordinates so that pre-special points are algebraic of bounded degree. All the sets being considered may then be viewed as subsets of $\mathbb{R}^N$. By work of Gabrielov [38] and van den Dries [26] on projections of semi-analytic sets, Wilkie [93] on the exponential function, Peterzil-Starchenko [64] on the Weierstrass $\wp$-function, and others, the set

$$Z \subset \mathbb{R}^N$$

is a *definable set* in a suitable o-minimal structure over $\mathbb{R}$ (a "definable set"; see §2). (In contrast $\mathcal{Z}$ is generally not so definable, due to the Γ-periodicity.) We apply a result of Pila-Wilkie [70] concerning the distribution of rational points on definable sets in $\mathbb{R}^\nu$ (more precisely a refinement [67] of it applicable to algebraic points of bounded degree stated as Theorem 3.2 below). This gives an *upper bound* for the number of pre-special points in Z up to a given height that do not lie on some connected semialgebraic subset of Z of positive dimension.

On the other hand, special points of V are algebraic, so their suitable Galois conjugates lie again on V, and are also special points. Siegel's lower bound for class numbers of imaginary orders gives, via the theory of complex multiplication, a lower bound for the degree of a special point of $\mathbb{C}$ in terms of the size of the discriminant of the corresponding order. Masser [54] gives a lower bound for the degree of a torsion point of an abelian variety (i.e. the degree over $\mathbb{Q}$ of a field of definition for the point) in terms of its order of torsion. The degree $\phi(n)$ of a primitive nth root of unity has elementary lower bounds. In combination these give (as in [71], [68]) a *lower* bound for the number of conjugates of a special point, and hence for the number of pre-special points in Z in terms of the"complexity" (size of discriminant of corresponding quadratic order, minimal order of torsion, or maximum of these; see §5) of one such point in V. It is elementary to bound the height of a pre-special point in $\mathbb{F}$ in terms of its complexity.

The crux of the strategy is the incompatibility of the upper and lower bounds once the complexity of the pre-special point is too large, unless Z contains semi-algebraic subsets of positive dimension. Looking back one finds an antecedent of this strategy of opposing Galois lower bounds to archimedean upper bounds used by Sarnak in an unpublished manuscript [81] to reprove Lang's conjecture (on torsion points) for subvarieties of $\mathbb{G}^\ell$. In the published version [82] this proof is replaced by a slicker argument. It was in fact this manuscript [81] that raised the questions about diophantine properties of smooth and analytic curves that led to the paper [16], whose ideas developed ultimately into [70].

To conclude the proof of Theorem 1.1 we must identify the possible semi-algebraic subsets of $\mathcal{Z} = \pi^{-1}(V)$. It turns out that they correspond (in the complex coordinates) almost exactly to components of pre-images of special subvarieties of V (the "almost" is explained in Section 6: there are some additional possible components but they contain no pre-special points). This identification amounts to proving the algebraic independence of certain functions, namely the composition of the component functions of π with algebraic functions, under suitable hypotheses. The main work and the main novelty in this paper occur at this juncture. We make further use of the conjunction of definability and diophantine properties with a second and independent application of the Pila-Wilkie result (with a further slight refinement established here as Theorem 3.6 below)

For $X = \mathbb{G}^\ell$ the result we need is the following. Let $W \subset \mathbb{C}^\ell$ be an irreducible algebraic variety, and $\overline{\zeta_1}, \ldots, \overline{\zeta_\ell}$ the restrictions of the coordinate function on $\mathbb{C}^\ell$ to W, i.e., their images in the function field $\mathbb{C}(W)$. Then the functions

$$\exp(\overline{\zeta_1}), \ldots, \exp(\overline{\zeta_\ell}),$$

mapping $W \to \mathbb{C}$, are algebraically independent (over $\mathbb{C}$) provided that the $\overline{\zeta_i}$ are linearly independent over $\mathbb{Q}$ modulo constants, i.e. provided there do not exist $q_i \in \mathbb{Q}$, not all zero, such that $\sum q_i\overline{\zeta_i} = c \in \mathbb{C}$. (Note that if this condition fails, then the $\exp(\overline{\zeta_i})$ are indeed algebraically dependent over $\mathbb{C}$.) This follows from the results of Ax [5] establishing power series and differential field versions of Schanuel's conjecture, the results known as "Ax-Schanuel". More precisely it is the part of Ax-Schanuel corresponding to the Lindemann-Weierstrass Theorem (which asserts the algebraic independence of the exponentials of algebraic numbers that are linearly independent over $\mathbb{Q}$), and accordingly we call this "Ax-Lindemann-Weierstrass" (ALW). It was remarked in [71] that the method of proof there should give a new proof of Lang's conjecture on torsion points on subvarieties of $\mathbb{G}^\ell$ using Ax-Schanuel. Such a proof is included in the proof of Theorem 1.1, but our method introduced here for studying algebraic subsets of Z also reproves the required ALW part of Ax-Schanuel.

For the case $X = \mathbb{C}^n$ the result we must prove is equivalent (as we show) to the analogue of ALW for the j function. Namely, suppose $W \subset \mathbb{C}^n$ is an irreducible algebraic variety having a nonempty intersection with $\mathbb{H}^n$ (so $W \cap \mathbb{H}^n$ is Zariski dense in W) and that $\overline{\tau_1}, \ldots, \overline{\tau_n}$ are the images in the algebraic function field $\mathbb{C}(W)$ of the coordinate functions on $\mathbb{C}^n$. Let $P \in W \cap \mathbb{H}^n$. Then the functions

$$j(\overline{\tau_1}), \ldots, j(\overline{\tau_n}),$$

mapping $W \cap \Delta \to \mathbb{C}$ for some open neighbourhood Δ of P, are algebraically independent over $\mathbb{C}$, unless some $\overline{\tau_i}$ is constant or there is a relation of the form $\overline{\tau}_a = g\overline{\tau}_b$ where $a \neq b$ and $g \in \mathrm{GL}_2(\mathbb{Q})^+$ (where "$^+$" indicates positive determinant and $\mathrm{GL}_2(\mathbb{Q})^+$ acts on $\mathbb{H}$ by fractional linear transformations). Note again that if the condition on the $\overline{\tau_i}$ fails, then the $j(\overline{\tau_i})$ are algebraically dependent over $\mathbb{C}$, by a suitable modular relation $\Phi_N(j(\overline{\tau_a}), j(\overline{\tau_b})) = 0$ if $\overline{\tau_a} = g\overline{\tau_b}$ with $g \in \mathrm{GL}_2(\mathbb{Q})^+$. This result appears to be new (cf. the very special cases treated by Amou [1]). (For a generalization of Schanuel's conjecture encompassing the j-function, the exponential function and more see [8].)

For a product $X = E_1 \times \cdots \times E_m$ of elliptic curves (defined over $\mathbb{C}$), the corresponding ALW result required is for the composition of Weierstrass $\wp$-functions with algebraic functions

$$\wp_1(\overline{z_1}), \ldots, \wp_m(\overline{z_m}),$$

where $\overline{z_i}$ are the images of the coordinate functions in some algebraic function field, under suitable (and necessary) "linear independence" conditions (see Definition 1.5.2). This follows (even for $X = E_1 \times \cdots \times E_m \times \mathbb{G}^\ell$) from the "Ax-Schanuel" results for Weierstrass functions of Brownawell-Kubota [20], though again we reprove the ALW part directly by our methods.

For $X = A$ an abelian variety (over $\mathbb{C}$) the corresponding characterization of the "algebraic part" proved in [71] is likewise equivalent (by the argument given here) to an ALW-type result. An Ax-Schanuel result for abelian and indeed semi-abelian varieties is established in work of Ax [6] and Kirby [45], (see also [9], [10]), which thus includes all of the results discussed above except the one concerning the j-function. Of course ALW for the j function is the crucial ingredient required to admit products of modular curves in Theorem 1.1.

To prove Theorem 1.1 we must establish a functional algebraic independence result encompassing all the Ax-Lindemann-Weierstrass results mentioned, which we now frame.

1.5. *Definition.* Let n, m, ℓ be nonnegative integers. Let $X = \mathbb{C}^n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell$, where E_i are elliptic curves over $\mathbb{C}$ corresponding to lattices $\Lambda_i \subset \mathbb{C}$ with Weierstrass $\wp$-functions $\wp_i$. Let $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_m \subset \mathbb{C}^m$. Let $U = U_X$. Let $W \subset \mathbb{C}^{n+m+\ell}$ be an irreducible algebraic variety, closed in X, having a nonempty intersection with U. Let

$$\tau_1, \ldots, \tau_n, \quad z_1, \ldots, z_m, \quad \zeta_1, \ldots, \zeta_\ell$$

be the coordinate functions on $\mathbb{C}^{n+m+\ell}$ and

$$\overline{\tau_1}, \ldots, \overline{\tau_n}, \quad \overline{z_1}, \ldots, \overline{z_m}, \quad \overline{\zeta_1}, \ldots, \overline{\zeta_\ell}$$

their images in $\mathbb{C}(W)$. A subset of these, which for simplicity we take to be

$$\overline{\tau_1}, \ldots, \overline{\tau_\nu}, \quad \overline{z_1}, \ldots, \overline{z_\mu}, \quad \overline{\zeta_1}, \ldots, \overline{\zeta_\lambda},$$

where $0 \le \nu \le n, 0 \le \mu \le m, 0 \le \lambda \le \ell$, will be called *geodesically independent* if all of the following conditions hold.

1. The functions $\overline{\tau_1}, \dots, \overline{\tau_\nu}$, are nonconstant and there are no relations of the form $\overline{\tau_a} = g\overline{\tau_b}$ where $a \neq b$ and $g \in \mathrm{GL}_2(\mathbb{Q})^+$. If $\nu = 0$, then we consider this condition to be satisfied.

2. The functions $\overline{z_1}, \dots, \overline{z_\mu}$ do not satisfy any system of $\mu - h$ linearly independent equations $\sum_{j=1}^{\mu} \alpha_{ij} z_j = c_i, i = 1, \dots, \mu - h$, $h < \mu$, where $c_i \in \mathbb{C}$ and the h-dimensional linear subspace L defined by $\sum_{j=1}^{\mu} \alpha_{ij} z_j = 0, i = 1, \dots, \mu - h$ contains $L \cap \Lambda$ as a lattice (i.e. of full rank $2h$). That is, the locus $(\overline{z_1}, \dots, \overline{z_\mu})$ is not a coset of a proper subtorus of $\mathbb{C}/\Lambda$. If $\mu = 0$, then we consider this condition to be satisfied.

3. The functions $\overline{\zeta_1}, \dots, \overline{\zeta_\lambda}$ are $\mathbb{Q}$-linearly independent modulo constants; i.e., there do not exist $q_1, \dots, q_\lambda \in \mathbb{Q}$, not all zero, such that $\sum_{i=1}^{\lambda} q_i \overline{\zeta_i} \in \mathbb{C}$. If $\lambda = 0$, we consider this condition to be satisfied.

The term "geodesic" here is suggested by the notion of a *totally geodesic subvariety* studied by Moonen [59]; see Remark 6.4 below. As observed, the geodesic independence of the arguments is a necessary condition for the compositions with the respective $j, \wp_i, \exp$ to be algebraically independent over $\mathbb{C}$. The required result is the sufficiency of this condition.

1.6. Theorem. *Let the notation* (*and assumption* $W \cap U_X \neq \emptyset$) *be as in Definition* 1.5. *If the functions*

$$\overline{\tau_1}, \dots, \overline{\tau_\nu}, \quad \overline{z_1}, \dots, \overline{z_\mu}, \quad \overline{\zeta_1}, \dots, \overline{\zeta_\lambda}$$

in $\mathbb{C}(W)$ *are geodesically independent*, *then the functions*

$$j(\overline{\tau_1}), \dots, j(\overline{\tau_\nu}), \quad \wp_1(\overline{z_1}), \dots, \wp_\mu(\overline{z_\mu}), \quad \exp(\overline{\zeta_1}), \dots, \exp(\overline{\zeta_\lambda}),$$

defined locally on $W \cap U_X$, *are algebraically independent over* $\mathbb{C}$.

This result is equivalent to the characterization of semi-algebraic subsets of $\mathcal{Z}$ required to prove Theorem 1.1 (see Theorems 9.1 and 9.2). Another way of stating the conclusion of Theorem 1.6 is that under the associated map $\pi : U \to X$, where $U = \mathbb{H}^\nu \times \mathbb{C}^\mu \times \mathbb{C}^\lambda$ and $X = \mathbb{C}^\nu \times E_1 \times \cdots \times E_\mu \times \mathbb{G}^\lambda$, the image $\pi(W)$ is Zariski dense in X. In fact we can prove a stronger version, namely that under the same conditions these functions are algebraically independent over $\mathbb{C}(W)$ (see Theorem 9.6 et seq.). One can rephrase the statement of Theorem 1.6 to consider arbitrary elements $\overline{a_1}, \dots, \overline{a_\nu}, \quad \overline{b_1}, \dots, \overline{b_\mu}, \quad \overline{c_1}, \dots, \overline{c_\lambda}$ in an algebraic function field $\mathbb{C}(W)$. The conclusion of Theorem 1.6 then holds provided that these functions are geodesically independent as in Definition 1.5, and there is a point $P \in W$ such that $(\overline{a_1}, \dots, \overline{a_\nu}, \overline{b_1}, \dots, \overline{b_\mu}, \overline{c_1}, \dots, \overline{c_\lambda})(P) \in U$, so that the required compositions are all defined locally on W.

With the identification of the maximal algebraic subsets of $\mathcal{Z}$ and the upper and lower bounds for prespecial points in Z we can establish the AOMML statement in its weak form: $V \subset X$ contains only finitely many special points unless it contains a special subvariety of positive dimension. The deduction of the stronger form enunciated in Theorem 1.1 is by an induction that requires knowing that only finitely many different (up to "translation") maximal special subvarieties occur. Here we make a third, though quite elementary, use of o-minimality properties in conjunction with rationality. Essentially, we use the fact that *a definable set consisting* only *of rational points is finite.* Probably this step could be effected by elementary means, as is the corresponding result in the case of abelian varieties (see e.g. the corresponding deduction in [71] recalling arguments from [17]), however the argument using o-minimality is quite transparent.

The paper is organized as follows. The definition and key examples of o-minimal structures over $\mathbb{R}$ are recalled in Section 2. The upper bound result for the height density of algebraic points of bounded degree on definable sets is given in Section 3. In Section 4 we set up some notation with respect to the uniformization $\pi : U \to X$ and the discrete group Γ for which π is invariant. We specify the real coordinates that we will use on U, and observe the definability of the key sets. In Section 5 we introduce some height-like quantities, including the "complexity" of a pre-special point alluded to above. Sections 6, 7, and 8 are devoted to proving Theorem 6.8, which characterizes the algebraic part of $\mathcal{Z} = \pi^{-1}(V)$ when X is of the form $\mathbb{C}^n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell$, and are the heart of the paper. In Section 9 we show that Theorem 1.6 is equivalent to Theorem 6.8, and deduce more general forms of both statements. After some further preparations in Section 10 relating to the maximal algebraic components of $\mathcal{Z}$, the proof of Theorem 1.1 is given in Section 11. In Section 12 we show how to combine the present methods with the results of [71] to establish the "André-Oort-Manin-Mumford" statement for varieties $Y_1 \times \cdots \times Y_n \times A$, where A is an abelian variety over $\overline{\mathbb{Q}}$. Finally, Section 13 addresses uniformity and effectivity.

Acknowledgments. It is a pleasure to thank Daniel Bertrand, Jonathan Kirby, Ehud Hrushovski, David Masser, Anand Pillay, Peter Sarnak, Alex Wilkie, and Umberto Zannier for enlightening conversations and communications. The referees offered many valuable suggestions and corrections, I am very appreciative of the careful attention they gave to this paper. My thanks also go to Roger Heath-Brown and the Mathematical Institute, Oxford, for extending gracious hospitality to me as an Academic Visitor in the period of researching and writing this paper, and I am grateful to the Leverhulme Trust for supporting my work through a Research Fellowship.

2. **o-minimal structures over** $\mathbb{R}$

An o-minimal structure over $\mathbb{R}$ is a collection of subsets of $\mathbb{R}^\nu$, $\nu = 1, 2, \ldots$ that is closed under some basic operations corresponding to definability in a suitable first-order language (i.e. a "structure" in the sense of first-order Model

Theory), but which also enjoys strong finiteness properties. The notion grew out of work of van den Dries [25], [26] on Tarski's problem concerning the decidability of the real ordered field with the exponential function, and was studied in the more general context of linearly ordered structures by Pillay and Steinhorn [72], to whom the term "o-minimal" ("order-minimal") is due.

2.1. *Definition.* A *pre-structure* is a sequence $\mathcal{S} = (S_\nu : \nu \geq 1)$ where each $\mathcal{S}_\nu$ is a collection of subsets of $\mathbb{R}^\nu$. A pre-structure $\mathcal{S}$ is called a *structure* (*over the real field*) if, for all $\nu, \mu \geq 1$, the following conditions are satisfied:

(1) $\mathcal{S}_\nu$ is a boolean algebra (under the usual set-theoretic operations);
(2) $\mathcal{S}_\nu$ contains every semi-algebraic subset of $\mathbb{R}^\nu$;
(3) if $A \in \mathcal{S}_\nu$ and $B \in \mathcal{S}_\mu$, then $A \times B \in \mathcal{S}_{\nu+\mu}$;
(4) if $\mu \geq \nu$ and $A \in \mathcal{S}_\mu$, then $\pi(A) \in \mathcal{S}_\nu$, where $\pi : \mathbb{R}^\mu \to \mathbb{R}^\nu$ is projection onto the first ν coordinates.

If $\mathcal{S}$ is a structure, and, in addition,

(5) the boundary of every set in $\mathcal{S}_1$ is finite,

then $\mathcal{S}$ is called an *o-minimal* structure (over the real field).

If $\mathcal{S}$ is a structure and $Z \subset \mathbb{R}^\nu$, then we say Z is *definable in* $\mathcal{S}$ if $Z \in \mathcal{S}_\nu$. A function $f : A \to B$ is *definable in* $\mathcal{S}$ if its graph is definable, in which case the domain A of f and image $f(A)$ will also be definable by the definitions.

Sets that are definable in an o-minimal structure are well-behaved. For example, they have finitely many connected components and admit cell decomposition. Indeed, o-minimal structures over $\mathbb{R}$ can be considered as candidates for Grothendieck's idea of "topologie modérée" [40], [27], [79]. For the theory of o-minimal structures we refer to [27], [31], which we reference as needed. We now describe the key examples.

The collection of all *semi-algebraic subsets* of $\mathbb{R}^\nu, \nu = 1, 2, \ldots$ is a structure, and is o-minimal. Here a *semi-algebraic* set in $\mathbb{R}^\nu$ is a the set of solutions to a finite collection of equations and inequalities ($<, \leq$) involving polynomials in $\mathbb{R}[X_1, \ldots, X_\nu]$. Equivalently, it is the collection of subsets of $\mathbb{R}^\nu, \nu = 1, 2, \ldots$ definable with parameters in the language of ordered fields. Conditions (1), (2), and (3) are evidently satisfied, while (4) follows from the Tarski-Seidenberg Theorem. The collection $\mathbb{R}_{\rm an}$ of *globally subanalytic* sets in $\mathbb{R}^\nu$, $\nu = 1, 2, \ldots$ is an o-minimal structure. These are the subsets of $\mathbb{R}^\nu$ that are subanalytic when considered as subsets of $\mathbb{P}^\nu(\mathbb{R})$; the o-minimality follows from Gabrielov's Theorem [38], as observed by van den Dries [26]. The collection $\mathbb{R}_{\rm exp}$ of subsets of $\mathbb{R}^\nu, \nu = 1, 2, \ldots$ that are *definable using the exponential function* (or, alternatively, the smallest structure containing the graph

$$\Gamma_{\rm exp} = \{(x, y) \in \mathbb{R}^2 : y = e^x, x \in \mathbb{R}\}$$

of the exponential function) is o-minimal. This follows from the work of Wilkie [93] in conjunction with Khovanskii's finiteness results [44]. Neither of the structures $\mathbb{R}_{\mathrm{an}}, \mathbb{R}_{\exp}$ contains the other. For example (see e.g. [30]) the set $\Gamma_{\exp}$ is not subanalytic at infinity, so is not contained in $\mathbb{R}_{\mathrm{an}}$, while $\mathbb{R}_{\mathrm{an}}$ contains the graphs of restricted analytic functions such as $\{(x,y) \in \mathbb{R}^2 : y = \sin(x), x \in [-1,1]\}$ that are not definable in $\mathbb{R}_{\exp}$ (see [12]). However the structure $\mathbb{R}_{\mathrm{an,exp}}$ generated by the union of $\mathbb{R}_{\mathrm{an}}$ and $\mathbb{R}_{\exp}$ is o-minimal (van den Dries and Miller [31]; see also [28]).

Further examples may be found described in [80], [86], [79]. The latter surveys methods of constructing o-minimal structures and discusses the connections with "topologie modérée". In particular [80], there exist pairs of o-minimal structures that are incompatible in that their union is not contained in any o-minimal structure, and consequently there does not exist a "largest" o-minimal structure over $\mathbb{R}$. Examples are given in [29] of natural functions that are not definable in $\mathbb{R}_{\mathrm{an,exp}}$. For example the error function $\int_0^x \exp(-t^2)dt$ and the logarithmic integral $\int_x^\infty \exp(-t)dt/t$ on $(0,\infty)$ are not definable in $\mathbb{R}_{\mathrm{an,exp}}$, though their restrictions to any compact subinterval are, and they are definable in the o-minimal structure $\mathbb{R}_{\mathrm{Pfaff}}$ generated by Pfaffian functions (see e.g. [94], [79]).

However, $\mathbb{R}_{\mathrm{an,exp}}$ contains all the sets that are required in this paper. *Therefore, from Section* 4 *onwards, "definable" will mean "definable in* $\mathbb{R}_{\mathrm{an,exp}}$". The reader who is unfamiliar with these notions need only be content to accept that certain sets are definable in the structure $\mathbb{R}_{\mathrm{an,exp}}$ and that, as a consequence of this and the o-minimality of the structure, various properties, notably the diophantine properties set out in Theorem 3.6, hold for those sets.

3. **Rational (and algebraic) points of definable sets**

Let $\mathcal{S}$ be an o-minimal structure over $\mathbb{R}$, fixed for this section, so that "definable" will, in this section, mean "definable in $\mathcal{S}$". The distribution of rational points on definable sets is studied in [70], with some refinement to deal with algebraic points of bounded degree in [67].

We first state the basic result to the effect that, if $Z \subset \mathbb{R}^\nu$ is definable in an o-minimal structure over $\mathbb{R}$, then Z contains only "few" rational (or algebraic of bounded degree) points of height $\leq T$, in a suitable sense, as $T \to \infty$, unless Z contains a semi-algebraic subset of positive dimension. More precisely, we consider the distribution of rational (or algebraic of bounded degree) points that lie outside the *algebraic part* of a set $Z \subset \mathbb{R}^\nu$, defined as follows.

3.1. *Definition.* Let $Z \subset \mathbb{R}^\nu$. The *algebraic part* of Z, which we denote Z^{alg}, is the union of all connected positive-dimensional semi-algebraic subsets of Z.

For a set $Z \subset \mathbb{R}^\nu$, an integer $k \geq 1$ and a real number $T \geq 1$, we set

$$Z(k,T) = \{z = (z_1, \ldots, z_\nu) \in Z : \max_i [\mathbb{Q}(z_i) : \mathbb{Q}] \leq k, \max_i H(z_i) \leq T\},$$

where $H(\alpha)$ is the absolute multiplicative height of an algebraic number, as defined in e.g. [13], and

$$N_k(Z,T) = \#Z(k,T).$$

3.2. THEOREM ([70] for $k = 1$ and [67] in general). *Let $Z \subset \mathbb{R}^\nu$ be definable, let $k \geq 1$ and $\epsilon > 0$. There is a constant $c(Z,k,\epsilon)$ such that, for all $T \geq 1$,*

$$N_k(Z - Z^{\mathrm{alg}}, T) \leq c(Z,k,\epsilon) T^\epsilon.$$

This statement suffices for our first application to the algebraic points of Z. In fact Theorem 3.2 is proved in [70], [67] in a more elaborate form; in particular, it is proved for *definable families* of sets (see below), which is the source of the uniformity mentioned for Theorem 1.1, and using a variant height. In considering the semi-algebraic subsets of $\mathcal{Z}$ we need a more refined version.

Let us note for definiteness that, for a rational number $q = a/b$ in lowest terms ($(a,b) = 1$)) we have $H(q) = \max\{|a|, |b|\}$. For a ν-tuple $q = (q_1, \ldots, q_\nu) \in \mathbb{Q}^\nu$ we will adopt a coordinate-wise height (rather than projective height) setting $H(q) = \max_i H(q_i)$.

3.3. *Definition.* Let k be a positive integer. The *polynomial height* (*of degree k*), denoted $H_k^{\mathrm{poly}}(\alpha)$ of a real number α is given by

$$H_k^{\mathrm{poly}}(\alpha) = \min\{H(q) : q = (q_0, \ldots, q_k) \in \mathbb{Q}^{k+1} - \{(0, \ldots, 0)\}, \sum_{i=0}^{k} q_i \alpha^i = 0\}$$

if $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq k$. Otherwise we take $H_k^{\mathrm{poly}}(\alpha) = \infty$. For a ν-tuple $z = (z_1, \ldots, z_\nu)$ we set $H_k^{\mathrm{poly}}(z) = \max_i H^{\mathrm{poly}}(z_i)$. The relation between absolute height and Mahler measure ([13, 1.6.5, 1.6.6]) implies that, when $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq k$,

$$H_k^{\mathrm{poly}}(\alpha) \leq 2^k H(\alpha)^k.$$

Let us put, for a set $Z \subset \mathbb{R}^\nu$,

$$Z^{\mathrm{poly}}(k,T) = \{z \in Z : H_k^{\mathrm{poly}}(z) \leq T\}, \quad N_k^{\mathrm{poly}}(Z,T) = \#Z^{\mathrm{poly}}(k,T).$$

Then Theorem 3.2 may be proved using H^{poly} rather than H. That is, there is a constant $c^{\mathrm{poly}}(Z,k,\epsilon)$ such that, for $T \geq 1$,

$$N_k^{\mathrm{poly}}(Z - Z^{\mathrm{alg}}, T) \leq c^{\mathrm{poly}}(Z,k,\epsilon) T^\epsilon.$$

This version evidently implies Theorem 3.2 in view of the above exhibited relation between the two heights.

By a *definable family* of sets we mean a definable set in $\mathbb{R}^\nu \times \mathbb{R}^\mu$, considered as the family of fibres

$$Z_y = \{x \in \mathbb{R}^\nu : (x, y) \in Z\}, \quad y \in \mathbb{R}^\mu.$$

The set $Y = \{y \in \mathbb{R}^\mu : Z_y \neq \emptyset\}$ is then definable, so it will be immaterial whether we consider quantifications over Y or $\mathbb{R}^\mu$. Note that we consider the fibre Z_y to be a subset of $\mathbb{R}^\nu$, so any rationality considerations relate to the $\mathbb{R}^\nu$-coordinates and not to the coordinates of the parameter $y \in \mathbb{R}^\mu$.

For a definable set Z and each pair $\kappa, p \in \mathbb{N} = \{0, 1, 2, \ldots\}$ we define the *p-regular points of Z of dimension κ*, denoted $\mathrm{reg}^p_\kappa(Z)$, to be the set of $x \in Z$ such that there is an open neighbourhood U of x with $U \cap Z$ a C^p embedded submanifold of $\mathbb{R}^\nu$ of dimension κ. Then each $\mathrm{reg}^p_\kappa(Z)$ is definable, and indeed this is true over a family, i.e. for a definable family Z the set

$$\{z = (x, y) \in Z : x \in \mathrm{reg}^p_\kappa(Z_y)\}$$

is definable ([30, B.10]). A *regular point of dimension κ* will mean a 1-regular point of dimension κ. The *dimension* of a definable set Z is the maximum κ such that Z has a regular point of dimension κ. Therefore, if Z has dimension κ, then $Z - \mathrm{reg}^1_\kappa(Z)$ has dimension $\leq \kappa - 1$. A *regular* point of a definable set of dimension κ will mean a regular point of dimension κ.

The term “definable block” which we now introduce was termed a “basic block” in [67]. However, our purpose here is to eliminate the need for what was termed a “block” in [67], so here we will just use the term “definable block”. We also explicate the degree in our definitions.

3.4. *Definition.* 1. A *definable semialgebraic block* or *definable block* of dimension w and degree d in $\mathbb{R}^\nu$ is a connected definable set $W \subset \mathbb{R}^\nu$ of dimension w, regular at every point, such that there is a semialgebraic set $A \subset \mathbb{R}^\nu$, of dimension w and degree $\leq d$, regular at every point, with $W \subset A$.

2. A *definable semialgebraic block family* or *definable block family* of dimension w and degree d is a definable family $W \subset \mathbb{R}^\nu \times \mathbb{R}^\mu$ such that every nonempty fibre $W_y, y \in \mathbb{R}^\mu$ is a block of dimension w and degree $\leq d$.

Note that dimension 0 is allowed: a point is a definable block. Further, a definable block of positive dimension is a union of connected semi-algebraic sets of positive dimension (the intersection of the definable block with small neighbourhoods of each point), and so if such a definable block is contained in a set Z, it is contained in Z^{alg}. In the following lemma, a *semi-algebraic map* means a definable function in the structure of semi-algebraic sets, i.e. $f : B \to \mathbb{R}^\mu$, where $B \subset \mathbb{R}^\nu$ and $\{(x, f(x)) \in \mathbb{R}^{\nu+\mu} : x \in B\}$ are semi-agebraic sets. If $W \subset \mathbb{R}^\nu$, then $f(W)$ will mean $f(W \cap B)$.

3.5. Lemma. *Suppose $B \subset \mathbb{R}^\kappa$ is a semi-algebraic set and $\phi : B \to \mathbb{R}^\nu$ is a semialgebraic map.*

1. *If $W \subset \mathbb{R}^\kappa$ is a definable block, then $\phi(W)$ is a finite union of definable blocks.*
2. *If $W \subset \mathbb{R}^\kappa \times \mathbb{R}^\mu$ is a definable block family, then $\phi(W)$ is a finite union of definable block families in $\mathbb{R}^\nu \times \mathbb{R}^\mu$.*

Proof. 0. Suppose $W \subset \mathbb{R}^\kappa$ is a definable block with respect to a semi-algebraic set $A \subset \mathbb{R}^\kappa$, and $B \subset \mathbb{R}^\kappa$ is a semi-algebraic set. Let A' be the set of regular points of $A \cap B$ and W' a connected component of $W \cap A'$. Then W' is a definable block with respect to A', since locally at each point of W it coincides with A. Since $W \cap A'$ is definable it has finitely many connected components (see the remarks following Definition 2.1), it suffices to consider the intersection of W' with the set of singular points of $A \cap B$. This set has lower dimension. Thus, by induction on dimension, $W \cap B$ is a finite union of definable blocks.

1. By the above, $\phi(W) = \phi(W \cap B)$, and $W \cap B$ is a finite union of definable blocks. So we reduce to the case that ϕ is defined on A, and by the same argument we reduce further to the case that ϕ is continuous on A. Now we look at the image. The image $\phi(A)$ is semi-algebraic, and has some degree d' and dimension w'. There is a semialgebraic set $S \subset A$, closed in A and of lower dimension, such that, on $A - S$, the image of ϕ is a regular point of dimension w'. The set $A - S$ consists of finitely many connected components, as does $W \cap (A - S)$, and we can reduce to the case that $A - S$ and $W \cap (A - S)$ are connected. Then $W' = W \cap (A - S)$ is a definable block with respect to $A - S$ and $\phi(W')$ is a definable block with respect to $\phi(A - S)$ of dimension w' and degree d'. Further $W \cap S$ is a finite union of definable blocks, and the proof of assertion 1 is completed by induction.

2. We need only observe that all the steps above can be carried out in definable families. $\square$

We can now state our refinement of Theorem 3.2, incorporating the refinements of the versions in [70], [67].

3.6. Theorem. *Let $Z \subset \mathbb{R}^\nu \times \mathbb{R}^\mu$ be a definable family, $k \geq 1$ and $\epsilon > 0$. There is a finite number $J = J(Z, k, \epsilon)$ of definable block families*

$$W^{(j)} \subset \mathbb{R}^\nu \times (\mathbb{R}^\mu \times \mathbb{R}^\lambda), \quad j = 1, \ldots, J,$$

of dimension w_j and degree d_j, and a constant $c^{\mathrm{poly}}(Z, k, \epsilon)$ with the following properties:

1. *For all $(y, \eta) \in \mathbb{R}^\mu \times \mathbb{R}^\lambda$, we have $W^{(j)}_{(y,\eta)} \subset Z_y$.*

2. *For all* $y \in \mathbb{R}^\mu$ *and* $T \geq 1$, $Z_y^{\text{poly}}(k,T)$ *is contained in the union of at most* $c^{\text{poly}}(Z,k,\epsilon)T^\epsilon$ *definable blocks of the form* $W^{(j)}_{(y,\eta)}$ *for some* $j = 1, \ldots, J$ *and* $\eta \in \mathbb{R}^\lambda$.

3.7. *Remarks.* 1. Since each definable block in Z_y of positive dimension is contained in Z_y^{alg}, Theorem 3.6 implies that

$$N_k^{\text{poly}}(Z_y - Z_y^{\text{alg}}, T) \leq c^{\text{poly}}(Z,k,\epsilon)\ T^\epsilon$$

for all $y \in \mathbb{R}^\nu$ and $T \geq 1$, thus giving a uniform version of Theorem 3.2.

2. However, the main point of this version is that not just the number of points outside the algebraic part is T^ϵ bounded, but that the "connected semi-algebraic pieces", i.e., definable blocks required to contain the rational points are similarly controlled in number and come from finitely many definable block families. Further all the points of all the definable blocks are regular.

3. Let $W^{Z,k,\epsilon} \subset \mathbb{R}^\nu \times \mathbb{R}^\mu$ be the family whose fibre at $y \in \mathbb{R}^\nu$ is the union over all $j = 1, \ldots, J(Z,k,\epsilon)$ and $\eta \in \mathbb{R}^\lambda$ of the fibres of $W^{(j)}_{(y,\eta)}$ of positive dimension. Then $W^{Z,k,\epsilon}$ is definable,

$$W_y^{Z,k,\epsilon} \subset Z_y^{\text{alg}},$$

and

$$N_k^{\text{poly}}(Z_y - W_y^{Z,k,\epsilon}, T) \leq c^{\text{poly}}(Z,k,\epsilon)T^\epsilon.$$

Since the algebraic part of a definable set may not be definable, this shows that the T^ϵ bound may be achieved by removing a *definable* subset of the algebraic part, and this may be done uniformly over families. Corresponding assertions appeared in [70], [67]. Indeed, the degrees d_j are bounded by some $d(n,k,\epsilon)$, independent of Z (but not so the $J(Z,k,\epsilon)$).

4. The result for H^{poly} implies the same result using H (with possibly different constants J, c and fibres W).

Proof. We need to elaborate the proof of [67, Th. 5.3], which gives the conclusion for a finite number of families of semi-algebraic images of definable block families. Here we just need to apply Lemma 3.5 above at a suitable juncture to get the additional refinement of the conclusion required. For $k = 1$ (i.e. for rational points), however, the required conclusion is established in , in which a "basic block family" is precisely our present "definable block family". We did not explicate there that the definable blocks have degrees, but this follows from the proof. For $k > 1$, a slightly weaker result is established in [67, Th. 3.5]. In the course of the proof of [67, Th. 3.5], a definable family

$$Y \subset \mathbb{R}^{(k+1)\nu} \times \mathbb{R}^\mu$$

(depending on Z, k) is constructed, together with a finite number of definable maps $Y \to Z$, preserving the fibres, such that the algebraic points of the fibres Z_y are images of rational points on the corresponding fibre Y_y. Moreover, the definable maps alluded to are the restrictions of semi-algebraic maps ϕ_i, defined and continuous on semi-algebraic subsets $B_i \subset \mathbb{R}^{(k+1)\nu}$ (depending only on ν, k) such that:

1. For all $y \in \mathbb{R}^\mu$, Y_y is the union of pre-images of Z_y under the maps ϕ_i.
2. If $x \in \mathbb{R}^\nu$ with $H_k^{\text{poly}}(x) \le T$, then there is an index i and a preimage ξ of x under ϕ_i with $H(\xi) \le T$.

Now [67, Th. 3.5] establishes the conclusion of the theorem for the rational points on the fibres of Y: we have a finite number of definable block families $V^{(j)} \subset \mathbb{R}^{(k+1)\nu} \times (\mathbb{R}^\mu \times \mathbb{R}^\lambda)$ satisfying the desired conclusions for Y. We have only to show that these conclusions regarding block families are preserved under the semialgebraic maps ϕ_i. This is afforded by the present Lemma 3.5: the ϕ_i images of the $V^{(j)}$ can be decomposed into a finite number of definable block families satisfying the required conclusions for Z. □

4. Uniformization, group actions, fundamental domains, real coordinates, and definability

In this section we give formally the definition of the uniformizing space U associated to the variety X, an associated real algebraic group G of isometries of U and a discrete subgroup Γ such that the map $\pi : U \to X$ is Γ-invariant. We normalize the definition in such a way that $\Gamma < G(\mathbb{Z})$ in each case. We specify a fundamental domain. We specify real coordinates on U, and observe the definability properties that will be crucial to the application of Theorem 3.6. The variety X is specified in the notation as it determines U and Γ (while U does not determine Γ). However it will be omitted when the intended variety X is clear from the context.

4.1. *Notation.* 1. Let X be a modular curve $\Gamma\backslash\mathbb{H}$. Then $U_X = \mathbb{H}$ and $\pi_X : U \to X$ is an embedding of the quotient as a quasiprojective curve given by a suitable choice of modular functions for Γ. The group $G_X = \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformations and $\Gamma_X = \Gamma$. The fundamental domain $\mathbb{F}_X$ is taken to be a suitable finite union of $\mathrm{SL}_2(\mathbb{Z})$ translates of the standard fundamental domain for the modular group (see e.g. Serre [83]).

2. Let $X = A$ be an abelian variety of dimension m. Let Λ be a lattice in $\mathbb{C}^m$ such that $\mathbb{C}^m/\Lambda$ is complex analytically isomorphic to A. (For definiteness we could specify that Λ corresponds a point in some chosen fundamental domain of moduli, so e.g. for an elliptic curve that Λ has generators $1, \tau$, where τ is in the usual fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, but this is not

necessary.) Then $U_X = \mathbb{C}^m$ and $\pi_X : \mathbb{C}^m \to A$ is the composition of the quotient map $\mathbb{C}^m \to \mathbb{C}^m/\Lambda$ with the isomorphism $\mathbb{C}^m/\Lambda \to A$. Let $\{\lambda_1, \ldots, \lambda_{2m}\}$ be a $\mathbb{Z}$-basis for Λ. We take $G_A = \mathbb{R}^{2m}$, acting as translations of $\mathbb{C}^m$ as follows. If $g = (r_1, \ldots, r_{2m}) \in G_A$, then $g(z) = z + t$ for $z \in \mathbb{C}^m$ where $t = \sum_{i=1}^{2m} r_i\lambda_i$. Then $\Gamma_A = G_A(\mathbb{Z}) = \mathbb{Z}^{2m}$ corresponds to translations by elements of Λ. We take $\mathbb{F}_A$ to be the fundamental parallelogram for Λ given by $\{\sum t_i\lambda_i : 0 \le t_i < 1, i = 1, \ldots, 2m\}$.

3. Let $X = \mathbb{G}$. Then $U_X = \mathbb{C}$ and $\pi_X : U \to X$ is the exponential function. We take $G_X = \mathbb{R}$ acting as translations in the imaginary direction, where $g = r \in \mathbb{R}$ acts by $g(z) = z + 2\pi i r$, and $\Gamma_X = G(\mathbb{Z}) = \mathbb{Z}$ acts as translations by $2\pi i\mathbb{Z}$. We take $\mathbb{F}_X = \{z \in \mathbb{C} : 0 \le \mathrm{Im}(z) < 2\pi\}$.

4. For a cartesian product $X = Y_1 \times \cdots \times Y_n \times A \times \mathbb{G}^\ell$, where $n, \ell \ge 0$, $Y_i = \Gamma_i\backslash\mathbb{H}$ are modular curves with uniformisations $\pi_i : \mathbb{H} = U_i \to Y_i$ and fundamental domains $\mathbb{F}_i$, and A is an abelian variety of dimension $m \ge 0$, we take the cartesian product of the uniformisations, groups, and fundamental domains. Thus $U_X = U_1 \times \cdots \times U_n \times U_A \times (U_{\mathbb{G}})^\ell$ and $\pi_X : U \to X$, $\pi_X = \pi_1 \times \cdots \times \pi_n \times \pi_A \times (\pi_{\mathbb{G}})^\ell$. We take $G_X = \mathrm{SL}_2(\mathbb{R})^n \times G_A \times (G_{\mathbb{G}})^\ell$, $\Gamma_X = \Gamma_1 \times \cdots \times \Gamma_n \times \Gamma_A \times \Gamma_{\mathbb{G}}^\ell$, and $\mathbb{F}_X = \mathbb{F}_1 \times \cdots \times \mathbb{F}_n \times \mathbb{F}_A \times \mathbb{F}_{\mathbb{G}}^\ell$.

We adopt real coordinates on the spaces U_X in such a way that pre-special points have suitable algebraicity properties. By giving real coordinates for an open domain $U \subset \mathbb{C}^k$ we mean giving functions $x_1, \ldots, x_{2k} : U \to \mathbb{R}$ such that the assignment $z \mapsto x(z) = (x_1(z), \ldots, x_{2k}(z))$ gives a bijection of U with an open domain in $\mathbb{R}^{2k}$. We identify subsets of U (including U itself) with their images in $\mathbb{R}^{2k}$.

4.2. *Real coordinates.* 1. For $X = \Gamma\backslash\mathbb{H}$ we put real coordinates on $U_X = \mathbb{H}$ using the real and imaginary parts. If we write $\tau = u + iv$, then pre-special points; i.e., quadratic $\tau \in \mathbb{H}$ are then certain points (u, v) with $u \in \mathbb{Q}$ and v of degree ≤ 2.

2. For $X = A$, an abelian variety, we put real coordinates on $U_X = \mathbb{C}^m$ using a basis of Λ. Then the pre-special points are rational points. If $\pi_A(z) = P \in A$ is special (i.e. torsion), then the minimal order of P is equal to the minimal denominator z.

3. For $X = \mathbb{G}$ we put real coordinates on $U_X = \mathbb{C}$ by using

$\mathrm{Re}(z)$ and $\mathrm{Im}(z)/2\pi$. Then the set of pre-special points is $\{(0, q) : q \in \mathbb{Q}\}$.

4. For $X = Y_1 \times \cdots \times Y_n \times A \times \mathbb{G}^\ell$, where $n, \ell \ge 0$, $Y_i = \Gamma_i\backslash\mathbb{H}$ and A is an abelian variety of dimension $m \ge 0$, we put real coordinates on U_X using the real coordinates on the cartesian factors.

We observe that, with the real coordinates we have adopted, the restriction $\pi_X : \mathbb{F}_X \to X$ is definable (in $\mathbb{R}_{\mathrm{an,exp}}$).

4.3. Proposition. *Let* $X = Y_1 \times \cdots \times Y_n \times A \times \mathbb{G}^\ell$, *where* $n, \ell \geq 0$, $Y_i = \Gamma_i\backslash\mathbb{H}$ *are modular curves, and* A *is an abelian variety of dimension* $m \geq 0$. *Then the restriction of* π_X *to* $\mathbb{F}_X$ *is definable in* $\mathbb{R}_{\mathrm{an,exp}}$.

Proof. For $X = \mathbb{C}$, the restriction of $\pi_{\mathbb{C}} = j$ to $\mathbb{F}_{\mathbb{C}}$ is definable by the results of Peterzil-Starchenko [64]. Hence it is definable on any other fixed $\mathrm{SL}_2(\mathbb{Z})$ translate of $\mathbb{F}_{\mathbb{C}}$, and on any finite union of such domains. Then for a modular curve $X = \Gamma\backslash\mathbb{H}$ definability follows since j is definable on the fundamental domain, and so any algebraic function of j is too. For $X = A$, the restriction of π_A to $\mathbb{F}$ is definable in $\mathbb{R}_{\mathrm{an}}$, since, in the real coordinates, the map is real analytic on (a neighbourhood of the closure of) the bounded semi-algebraic set $\mathbb{F}_A$. For $X = \mathbb{G}$, the restriction of $\pi_{\mathbb{G}} = \exp$ to $\mathbb{F}_{\mathbb{G}}$ is given by a polynomials in the real exponential function and the restrictions of the sine and cosine function to $[0, 2\pi)$. The former is definable in $\mathbb{R}_{\exp}$, the latter in $\mathbb{R}_{\mathrm{an}}$, so π_X on $\mathbb{F}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$. For the cartesian product $X = Y_1 \times \cdots \times Y_n \times A \times \mathbb{G}^\ell$, the restriction of π_X to $\mathbb{F}_X$ is the cartesian product of the corresponding maps on the factors, and so is definable in $\mathbb{R}_{\mathrm{an,exp}}$ by the basic properties of structures. □

4.4. *Remark.* Peterzil and Starchenko [64] establish a definability result for $\wp(\tau, z)$ as a function of both variables. Here only the definability j on $\mathbb{F}_{\mathbb{C}}$ is required, which follows easily from the q-expansion.

5. **Intricacy and complexity**

We introduce a notion of *intricacy* for the points of U, and of *complexity* for pre-special points in U. The former will be used in the arguments in Section 8 characterizing the maximal algebraic subsets of $\pi^{-1}(V)$. The latter is the natural quantity to which we relate the lower bound for the number of conjugates of a special point, and the height of a corresponding pre-special point lying in $\mathbb{F}_X$.

5.1. *Definition.* Let $X = Y_1 \times \cdots \times Y_n \times A \times \mathbb{G}^\ell$, where $n, \ell \geq 0$, Y_i are modular curves, and A is an abelian variety of dimension $m \geq 0$. Let $\mathbb{F}$ be a fundamental domain for the action of Γ_X on U_X and $u \in U_X$. We define the Γ_X*-intricacy of* u *with respect to* $\mathbb{F}$, denoted $I_{\mathbb{F}}^X(u)$, by

$$I_{\mathbb{F}}^X(u) = H(g),$$

where $g \in \Gamma_X$ is the unique element such that $g(u) \in \mathbb{F}$.

5.2. Proposition. *Let* $X = \mathbb{C}$, *with* $\mathbb{F} = \mathbb{F}_{\mathbb{C}}$ *and* $\mathbb{D}$ *any fundamental domain of the form* $g\mathbb{F}, g \in \Gamma$. *Let* $\tau \in \mathbb{H}$. *Then there is a bivariate polynomial* $P = P_{\mathbb{D}}$ *with positive real coefficients such that*

$$I_{\mathbb{D}}(\tau) \leq P\left(|\tau|, \frac{1}{\mathrm{Im}(\tau)}\right).$$

Proof. For $\mathbb{D} = \mathbb{F}$ we observe the quantitative statement we need from the proof that $\mathbb{F}$ is a fundamental domain given e.g. in Serre [83]. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$\mathrm{Im}(g\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}.$$

Therefore, $\mathrm{Im}(gz)$ has a maximum as g varies over Γ, and it is attained for some g with

$$|c| \leq \frac{1}{\mathrm{Im}(\tau)};$$

otherwise we could take $c = 0, d = 1$, and

$$|d| \leq |c||\mathrm{Re}(\tau)| \leq \frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} \leq \frac{|\tau|}{\mathrm{Im}(\tau)}.$$

Then a, b can be chosen with

$$|a|, |b| \leq \frac{|\tau|}{\mathrm{Im}(\tau)^2}.$$

We next take a translation $h = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ such that $hg\tau$ has real part between -1/2 and 1/2. As shown in [83], $hg\tau \in \mathbb{F}$, so that

$$I_{\mathbb{F}}(\tau) = H(hg).$$

We estimate the height of h and then of hg. If $c \neq 0$, then

$$|n| \leq |g\tau| \leq \frac{|a\tau + b|}{|c||\tau + d/c|} \leq \frac{|\tau|(|\tau| + 1)}{\mathrm{Im}(\tau)^3},$$

while if $c = 0$ we have $d \neq 0$ and

$$|n| \leq |g\tau| \leq \frac{|a\tau + b|}{|d|} \leq \frac{|\tau|(|\tau| + 1)}{\mathrm{Im}(\tau)^2}.$$

Then

$$H(hg) = H(a + nc, b + nd, c, d) \leq |\tau|(|\tau| + 1)^2 \left(\frac{1}{\mathrm{Im}(\tau)} + \frac{2}{\mathrm{Im}(\tau)^4} \right),$$

which gives what we need for $\mathbb{F}$. For general $\mathbb{D}$ we need only observe that there is a fixed element $g_0 \in \Gamma$ with $\mathbb{D} = g_0\mathbb{F}$, so that

$$I_{\mathbb{D}}(\tau) = H(g_0hg) \leq CH(hg)$$

for some constant C depending only on $\mathbb{D}$. □

The result we need is that the intricacy of a point is not increased too much by application of an algebraic function, under suitable conditions. By a (*complex*) *algebraic function* on $\mathbb{C}$ we will mean a function $\phi(x)$ defined and univalent on some connected open domain in $\mathbb{C}$ formed by removing some branch points and cuts (which we can always assume are line segements between branch points or rays joining a branch point to ∞) that satisfies an algebraic relation $P(x, \phi(x)) = 0$, where $P \in \mathbb{C}[X][Y]$ is nonconstant in Y and absolutely irreducible over $\mathbb{C}(X)$.

5.3. PROPOSITION. *Let $X = \Gamma\backslash\mathbb{H}$. Suppose that ϕ is an algebraic function on $\mathbb{C}$, real-valued on $\mathbb{R}$. Let $P \in \mathbb{R}$, and B an open disk centred at P. Suppose that the closure of B is at positive distance from any components of $\{\tau \in \mathbb{C} : \phi(\tau) \in \mathbb{R}\}$ other than $\mathbb{R}$, and from the poles of ϕ. Suppose that $\phi(B \cap \mathbb{H}) \subset \mathbb{H}$. Suppose that $\mathbb{D}$ is a fundamental domain for Γ_X of the form $g\mathbb{F}_X, g \in \Gamma$, and that $\tau \in B$. There is a univariate polynomial $P = P_{X,B,\tau,\phi}$ with positive real coefficients with the following property. If $g \in \Gamma_X$ is such that $g\tau \subset B$, then*

$$I_{\mathbb{D}}(\phi(g\tau)) \leq P_{X,B,\tau,\phi}(H(g)).$$

Proof. Since $\mathbb{F}_X$ is a finite union of $g\mathbb{F}_{\mathbb{C}}$ for some fixed $g \in \Gamma_{\mathbb{C}}$, and the conclusion is easily seen to be true for $g\mathbb{F}$, for a fixed g, if it is true for $\mathbb{F}$, it suffices to assume that $X = \mathbb{C}$ and $\mathbb{D} = \mathbb{F} = \mathbb{F}_{\mathbb{C}}$. We have then

$$I_{\mathbb{F}}(\phi(g\tau)) \leq P_{\mathbb{F}}\left(|\phi(g\tau)|, \frac{1}{\mathrm{Im}(\phi(g\tau))}\right).$$

Since B is away from the poles of ϕ, we see that ϕ is bounded on B, and so $\phi(g\tau)$ is bounded by a quantity depending on B and ϕ under our assumptions. Since B is away from all loci apart from $\mathbb{R}$ where ϕ is real, we have that the zero-set of $\mathrm{Im}(\phi(z))$ in the closure $\overline{B}$ is contained in the zero set of $\mathrm{Im}(z)$ on $\overline{B}$. Both functions $\mathrm{Im}(\phi(z))$ and $\mathrm{Im}(z)$ are continuous on $\overline{B}$. Since the structure of semi-algebraic sets is polynomially bounded, we can apply the Lojasiewicz inequality [30, 4.14(2)] to get positive constants $C(B, \phi), c(B, \phi)$, that

$$\mathrm{Im}(\phi(z)) \geq C\Big(\mathrm{Im}(z)\Big)^c$$

for $z \in \overline{B}$.

If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we see that

$$\frac{1}{\mathrm{Im}(\phi(g\tau))} \leq \frac{1}{C}\left(\frac{|c\tau + d|}{\mathrm{Im}(\tau)}\right)^c \leq C' H(g)^c$$

which gives the required form of dependence on $H(g)$. □

The corresponding results when X is an elliptic curve or $X = \mathbb{G}$ are more trivial, but we give the statements we will use later.

5.4. PROPOSITION. *Let $X = E$ be an elliptic curve, $U = U_E = \mathbb{C}$, and $\mathbb{D}$ a fundamental domain for Γ_E of the form $g\mathbb{F}_E, g \in \Gamma_E$.*

1. *There is a (linear) polynomial $P = P_{E,\mathbb{D}}$ with positive real coefficients such that, for $z \in \mathbb{C}$,*

$$I_{\mathbb{D}}(z) \leq P(|z|).$$

2. *Suppose* Λ *is a lattice with* $E = \mathbb{C}/\Lambda$, *with a chosen basis, and that* $\lambda \in \Lambda$. *Let* ϕ *be an algebraic function,* $z \in \mathbb{C}$. *There is a polynomial* $P = P_{E,\mathbb{D},\lambda,z,\phi}$ *such that, for sufficiently large* $|t|$ (*depending on* Λ (*with its basis*), $\mathbb{D}, \phi, z$),

$$I_{\mathbb{D}}(\phi(z + t\lambda)) \leq P(|t|).$$

Proof. 1. $\Gamma_E = \mathbb{Z}^2$ acts by translations by Λ, the identification being provided by the chosen basis. The size of the element of $\mathbb{Z}^2$ required to translate a given z into the $\mathbb{D}$ is evidently bounded by $C \max\{1, |z|\}$ for some suitable $C = C(\Lambda)$, where this dependence assumes a choice of basis.

2. By the first part of the proof we have

$$I_{\mathbb{D}}(\phi(z + t\lambda)) \leq C \max\{1, |\phi(z + t\lambda)|\}.$$

But $|\phi(z+\lambda t)|$ grows polynomially (depending on ϕ, λ, z) in $|t|$ for large $|t|$. □

5.5. Proposition. *Let* $X = \mathbb{G}$, $U = U_{\mathbb{G}} = \mathbb{C}$ *and* $\mathbb{D}$ *a fundamental domain for* $\Gamma_{\mathbb{G}}$ *of the form* $g\mathbb{F}_{\mathbb{G}}, g \in \Gamma_{\mathbb{G}}$.

1. *There is a* (*linear*) *polynomial* $P = P_{\mathbb{G},\mathbb{D}}$ *with real coefficients such that, for* $\zeta \in \mathbb{C}$,
$$I_{\mathbb{D}}(\zeta) \leq P(|\mathrm{Im}(\zeta)|).$$
2. *Let* ϕ *be an algebraic function and* $\zeta \in \mathbb{C}$. *There is a polynomial* $P = P_{\mathbb{G},\mathbb{D},\phi,\zeta}$ *such that, for sufficiently large* t (*depending on* $\mathbb{G}, \mathbb{D}, \phi, \zeta$),
$$I_{\mathbb{D}}(\phi(\zeta + 2\pi it)) \leq P(|t|).$$

Proof. 1. Now $\Gamma_{\mathbb{G}} = \mathbb{Z}$ acting as translations, with $1 \in \mathbb{Z}$ acting as translation by $2\pi i$. For $\mathbb{D} = \mathbb{F}_{\mathbb{G}}$ we then have $I_{\mathbb{D}}(\zeta) \leq \max\{1, \mathrm{Im}(\zeta)/2\pi i\}$. For general $\mathbb{D}$ we need only add a bounded quantity to the height.

2. Combine part 1 with the polynomial growth (for sufficiently large $|t|$) of the imaginary part of $\phi(\zeta + 2\pi it)$. □

We next formalize our notion of "complexity" of a pre-special point. For a complex quadratic $\tau \in \mathbb{H}$ we have that τ is the root of a unique polynomial $a\tau^2 + b\tau + c$ with $a, b, c \in \mathbb{Z}, (a, b, c) = 1, a > 0$. The discriminant $D(\tau)$ of τ is then the discriminant $b^2 - 4ac$ of this polynomial.

5.6. *Definition.* Let $X = Y_1 \times \cdots \times Y_n \times E_1 \times \cdots E_m \times \mathbb{G}^\ell$, where $n, m, \ell \geq 0$, $Y_i = \Gamma_i \backslash \mathbb{H}$, and E_i are elliptic curves. Let $u = (\tau_1, \ldots, \tau_n, z_1, \ldots, z_m, \zeta_1, \ldots, \zeta_\ell) \in U_X$ be a pre-special point. Let D_i be the discriminant of $\tau_i, i = 1, \ldots, n$, let T be the order of the image of $(z_1, \ldots, z_m)$ in $\mathbb{C}^m/\Lambda_1 \oplus \cdots \oplus \Lambda_m$ and let N be the order of the image of $(\zeta_1, \ldots, \zeta_\ell)$ in $(\mathbb{C}/2\pi i\mathbb{Z})^\ell$. We define the *complexity* of u to be

$$\Delta(u) = \max(|D_1|, \ldots, |D_n|, T, N).$$

Observe that, given X and a positive B, there are only finitely many special points of X corresponding to pre-special points u with $\Delta(u) \leq B$.

5.7. PROPOSITION. *Let* $X = Y_1 \times \cdots \times Y_n \times E_1 \times \cdots E_m \times \mathbb{G}^\ell$. *There is a positive constant* $c_{\text{height}}(X)$ *such that if* $u = (\tau_1, \ldots, \tau_n, z_1, \ldots, z_m, \zeta_1, \ldots, \zeta_\ell) \in \mathbb{F}_X$ *be a pre-special point. Then*

$$H(u) \leq c_{\text{height}}(X)\Delta(u).$$

Proof. Write $\tau_j = u_j + iv_j, j = 1, \ldots, n$. Consider some τ_j, the root of a quadratic polynomial $a\tau^2 + b\tau + c = 0$ as above. Since $u \in \mathbb{F}_X$, we have τ_j belongs to one of finitely many $g\mathbb{F}_\mathbb{C}$, $g \in \mathrm{SL}_2(\mathbb{Z})$. Suppose that $\tau_j \in \mathbb{F}_\mathbb{C}$, which is equivalent to the triple (a, b, c) being *reduced*, namely $|b| \leq a \leq c$ and $b \geq 0$ if $a = |b|$ or $a = c$. Then

$$4ac = b^2 - D(\tau_j) \leq ac - D(\tau_j),$$

whence

$$3ac \leq |D(\tau_j)|.$$

We have

$$u_j = \frac{-b}{2a}, \quad v_j = \frac{\sqrt{|D(\tau_j)|}}{2a}$$

so that v is a root of the polynomial $4a^2v^2 - |D|$. Then, using [13, 1.6.5, 1.6.6],

$$\begin{aligned} H(u) &\leq \max\{b, 2a\} \leq 2a \leq |D(\tau_j)| \leq \Delta(u),\\ H(v) &\leq \max\{4a^2, |D(\tau_j)|\} \leq 4|D(\tau_j)|/3 \leq 2\Delta(u). \end{aligned}$$

In general, these inequalities hold for some $g\tau$, where $g \in \mathrm{SL}_2(\mathbb{Z})$ are from a finite set. If $g\tau$ satisfies $a\tau^2 + b\tau + c = 0$, then $g^{-1}\tau$ satisfies $A\tau^2 + B\tau + C = 0$ where A, B, C are bounded by some fixed constant multiple (depending on g) of $\max(|a|, |b|, |c|)$. Then the height of τ as a real point is at most some constant multiple of the height of $g\tau$, and we conclude

$$H(u), H(v) \leq c_{\text{height}}(X)\Delta(u).$$

Since $z_j \in \mathbb{F}_E$ is pre-special we have that the real coordinates of z_j are rational fractions ≤ 1 with denominator T, and hence of height $\leq T \leq \Delta(u)$. Similarly for $\zeta_j \in \mathbb{F}_\mathbb{G}$ and pre-special, the corresponding real point is rational with height $\leq N \leq \Delta(u)$. $\square$

It is convenient to record here the results we will use for the lower bound on the number of conjugates of a special point. This combines results that are rather deep for the cases $X = \mathbb{C}$ and $X = E$, with the elementary one required for the case $X = \mathbb{G}$.

5.8. Proposition. *Let* $X = Y_1 \times \cdots \times Y_n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell$, *where* $n, m, \ell \geq 0$, $Y_i = \Gamma_i \backslash \mathbb{H}$, *and* $E_1, \dots, E_m$ *are elliptic curves defined over* $\overline{\mathbb{Q}}$. *There is a positive constant* $c_{\text{degree}}(X)$ *such that if* $u \in U_X$ *be a pre-special point, then*

$$[\mathbb{Q}(\pi(u)) : \mathbb{Q}] \geq c_{\text{degree}}(X)\Delta(u)^{1/7}.$$

Proof. Write

$$u = (\tau_1, \dots, \tau_n, z_1, \dots, z_m, \zeta_1, \dots, \zeta_\ell).$$

By the theory of Complex Multiplication (see e.g. [18]) for the equality and Siegel (see e.g. [39] for the statement for maximal orders, [51] for the general version) for the inequality we have, if $\nu > 0$,

$$[\mathbb{Q}(j(\tau_i)) : \mathbb{Q}] = h(D(\tau_i)) \geq c_{\text{Siegel}}(\nu)|D(\tau_i)|^{1/2-\nu}.$$

The modular curve Y_i is some finite cover of $\mathbb{C}$, and we get a similar lower bound up some constant depending on Y_i.

By the results of Masser [54] we have (effectively), if $P_i \in E_i$ is the image of z_i, that

$$[\mathbb{Q}(P_i) : \mathbb{Q}] \geq c(E_i)T^{1/7}.$$

Finally, according to [41, Th. 327] we have (effectively)

$$\frac{\phi(n)}{n^{1-\nu}} \to \infty$$

for every positive ν. □

5.9. *Remarks.* 1. In the proof of Proposition 5.8 we have appealed to Siegel's lower bounds [84] for class numbers of imaginary quadratic fields. In fact any bound of the form $h(D) \geq c|D|^\delta$ with $c, \delta > 0$ would suffice for the eventual proof of Theorem 1.1. In particular at the cost of replacing the exponent 1/7 by 1/8 we could use Landau's [47] bound $h(D) \geq c|D|^{1/8}$. This highlights the fact that the present proof requires only rather weaker bounds than are afforded by GRH. I thank Peter Sarnak for this observation and the reference to Landau. Of course Landau's result is ineffective, and the known effective lower bounds for $h(D)$ due to Goldfeld-Gross-Zagier (see [39]) are of the form $h(D) \geq C(\log|D|)^c$ while our argument requires a lower bound by a positive power of $|D|$.

2. The results of Masser [54] appealed to in 5.8 hold for abelian varieties. For elliptic curves they have been improved subsequently by Masser [55] and David [24], and there are alternative bounds available. Masser [54] mentions results of P. B. Cohen. Ineffectively one has even better results from Serre's open image theorem, in the non-CM case, while for CM elliptic curves one has results of Silverberg [85]. However, for us it suffices to have any positive exponent of $\Delta(u)$ (even one depending on X would suffice), and since the

constant is anyway ineffective due to the Landau/Siegel bound, there seems little point optimizing the exponent at this juncture.

3. Apart from the lower bound for class numbers, the other ingredients of the lower bound are effective.

6. **The algebraic part: preliminaries**

In this and the subsequent sections we characterize maximal algebraic subsets of

$$\mathcal{Z} = \pi^{-1}(V) \subset U = U_X.$$

It is convenient to do this first (in §§6–8) for X of the special form $X = \mathbb{C}^n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell$. The same result for X of the more general form required in Theorem 1.1 is deduced in Section 9.

We have defined the map $\pi : U \to X$ by means of Weierstrass $\wp$-functions for the elliptic curve factors. These are meromorphic, but the maps may be alternatively given by entire (theta-)functions. Then $\mathcal{Z} \subset U$ is a complex analytic subset of U (i.e. it is defined in a neighbourhood of each point $P \in U$ by the vanishing of a finite number of regular functions depending on P), indeed it is defined by the vanishing of finitely many polynomials in the coordinate functions of π, which may be taken to be regular on U (the j-function has a natural boundary on the real line).

First we will observe that, in studying $\mathcal{Z}^{\text{alg}}$, we may reduce to considering *complex* algebraic sets rather than *real semi-algebraic* subsets. Suppose that W is an irreducible complex algebraic set in $\mathbb{C}^{n+m+\ell}$. Then $W \cap U$ consists, as a complex analytic set, of finitely many connected components (since $W \cap U$ is semi-algebraic as a real set), and (since U is open in $\mathbb{C}^{n+m+\ell}$) these components are then complex analytic subsets of U all having the same dimension as W. If Y is such a component, and $\mathcal{Z}$ contains the intersection of Y with any open disc, then, by analytic continuation, $Y \subset \mathcal{Z}$. The union of such components we call the *complex algebraic part* of $\mathcal{Z}$.

6.1. *Definition.* Let U be an open domain in $\mathbb{C}^M$ that is semi-algebraic considered as a subset of $\mathbb{R}^{2M}$, and let $\mathcal{Z} \subset U$ be a complex analytic subset. We define a *complex algebraic component* of $\mathcal{Z}$ to be a connected component Y of positive dimension of $W \cap U$ with $Y \subset \mathcal{Z}$, where W is an irreducible closed complex algebraic set $W \subset \mathbb{C}^M$. The *complex algebraic part of* $\mathcal{Z}$, denoted $\mathcal{Z}^{\text{ca}}$, is the union of complex algebraic components of $\mathcal{Z}$.

Let again $\mathcal{Z} = \pi^{-1}(V) \subset U = U_X$, where X is as above. With the real coordinates described in Section 4 we have $\mathcal{Z} \subset U \subset \mathbb{R}^N$, where $N = 2(n + m + \ell)$ and we have then the algebraic part $\mathcal{Z}^{\text{alg}}$ as defined in Section 3.

6.2. PROPOSITION. *Let* $X = \mathbb{C}^n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell$, $V \subset X$ *a subvariety, and* $\mathcal{Z} = \pi^{-1}(V) \subset U = U_X$. *Then* $\mathcal{Z}^{\rm alg} = \mathcal{Z}^{\rm ca}$.

Proof. This follows from Lemma 2.1 of [68] as the complex coordinates are polynomial functions of the real coordinates on U. □

To study $\mathcal{Z}^{\rm alg}$ for $\mathcal{Z}$ as in Proposition 6.2 we may thus study its complex algebraic components. We will call such a component Y *maximal* if it is not contained in a complex algebraic component Y' of larger dimension. Every constituent component Y of $\mathcal{Z}^{\rm ca}$ is contained in some maximal component. The main result of this and the following two sections describes the possible form of such maximal algebraic components of $\mathcal{Z} = \pi^{-1}(V)$: They are components of the inverse image of a subvariety of $V \subset X$ that is *almost* special.

6.3. *Definition.* 1. A *quasi-special subvariety* of $X = \mathbb{C}^n$ is a subvariety as set out in Definition 1.2.1 except that the points j_i for $i \in S_0$ need not be special.

2. A *quasi-special subvariety* of $X = A$, an abelian variety, is a translate of an abelian subvariety (i.e. by a not-necessarily special point).

3. A *quasi-special subvariety* of $X = \mathbb{G}^\ell$ is a translate of an absolutely irreducible algebraic subgroup.

4. A *quasi-special subvariety* of $X = \mathbb{C}^n \times A \times \mathbb{G}^\ell$ is a subvariety of the form $Y \times (a+B) \times gH$, where Y is a special subvariety of $\mathbb{C}^n$, $a+B$ is a special subvariety of A, and gH is a special subvariety of G.

6.4. *Remark.* For $X = \mathbb{C}^n$ the notion of quasi-special subvariety coincides with the notion of *totally geodesic subvariety* studied for general Shimura varieties by Moonen [59].

The following definitions are given for more general X than those under consideration in the present section, so that we have them in hand when considering more general X in Sections 9–13.

6.5. *Definition.* (The use of the word “basic” here adapts the usage in [100].)

1. Let $n \geq 0$. Let $S_0 \cup S_1 \cup \cdots \cup S_k$ be a disjoint partition of $\{1, \ldots, n\}$ with $k \geq 0$ and S_0 only permitted to be empty. Let $h_i \in \mathbb{H}$ for each $i \in S_0$ be an arbitrary point. Let s_i be the smallest element of S_i for each $i \geq 1$ and for each $j \in S_i, j \neq s_i$, choose an element $g_{ij} \in {\rm GL}_2(\mathbb{Q})^+$. A *quasi-pre-special subvariety* of $\mathbb{H}^n$ is a subvariety

$$N = \{(\tau_1, \ldots, \tau_n) \in \mathbb{H}^n : \tau_i = h_i, i \in S_0, \tau_j = g_{ij}(\tau_{s_i}), i = 1, \ldots, k, j \in S_i, j \neq s_i\}$$

for some choice of data S_i, h_i, g_{ij} as indicated. If S_0 is empty, then we will call the corresponding quasi-special subvariety *basic*. The data $\{1, \ldots, m\} - S_0, g_{ij}$

determine a basic quasi-special subvariety of the product of upper half-planes in the variables indexed by $\{1, \ldots, n\} - S_0$, and we will say that the quasi-special subvariety N with data S_i, h_i, g_{ij} is the *translate by* $h_i, i \in S_0$ of the basic quasi-special subvariety N_0 (in the smaller set of coordinates) specified by $\{1, \ldots, m\} - S_0, g_{ij}$.

2. Let Λ be a lattice in $\mathbb{C}^m$ satisfying the Riemann relations, so that $\mathbb{C}^m/\Lambda$ is an abelian variety. A *quasi-pre-special subvariety of* $\mathbb{C}^m$ (*with respect to* Λ) is a subvariety of the form $b + L$, where L is a linear subspace of $\mathbb{C}^m$ (i.e. through the origin) in which $L \cap \Lambda$ is a lattice (i.e. of maximal rank $2 \dim_{\mathbb{C}} L$), and $b = (b_1, \ldots, b_m) \in \mathbb{C}^m$. Thus $L/(L \cap \Lambda)$ is an abelian subvariety of $\mathbb{C}^m/\Lambda$, and $b + L$ is its translate by the (arbitrary) point b. If $b + L = L$, then we call the corresponding quasi-pre-special subvariety *basic*, and we will refer to an arbitrary quasi-pre-special subvariety $b + L$ as the *translate by* b of the basic quasi-pre-special subvariety L.

3. Let $\ell \geq 0$. A *quasi-pre-special subvariety in* $\mathbb{C}^\ell$ (*with respect to* exp) is a subvariety of the form

$$b + L,$$

where L is a linear subspace defined over $\mathbb{Q}$, and $b \in \mathbb{C}^\ell$ is arbitrary. If $b + L = L$, then we call the corresponding quasi-pre-special subvariety *basic*, and we refer to a quasi-pre-special subvariety $b + L$ as the *translate by* b of the basic quasi-pre-special subvariety$L$.

4. Let $n, \ell \geq 0$ and A an abelian variety of dimension $m \geq 0$. Let $X = Y_1 \times \cdots \times Y_n \times A \times \mathbb{G}^\ell$, where $Y_i = \Gamma_i \backslash \mathbb{H}$. A *quasi-pre-special subvariety for* X in $\mathbb{H}^n \times \mathbb{C}^m \times \mathbb{C}^\ell$ is a subvariety of the form

$$N \times (b + L) \times (c + M),$$

where $N, b+L, c+M$ are quasi-pre-special subvarieties of $\mathbb{H}^n, \mathbb{C}^m$, (with respect to Λ), and $\mathbb{C}^\ell$ (with respect to exp) respectively. If N is the translate by $h_i, i \in S_0$ of the basic quasi-pre-special subvariety N_0, then we will refer to $N \times (b + L) \times (c + M)$ as the *translate by* $(h_i, i \in S_0, b, c)$ of the *basic* quasi-pre-special subvariety $N_0 \times L \times M$.

6.6. *Definition.* With the same conditions as in Definition 6.5, if the translation data $h_i, i \in S_0$ in 6.5.1 (or if S_0 is empty), a in 6.5.2, b in 6.5.3 and all these in 6.5.4 are pre-special points, we call the subvariety *pre-special*. (So a basic quasi-pre-special subvariety is always pre-special.)

6.7. *Definition.* Let $n, \ell \geq 0$ and A an abelian variety of dimension $m \geq 0$. Let $X = Y_1 \times \cdots \times Y_n \times A \times \mathbb{G}^\ell$, where $Y_i = \Gamma_i \backslash \mathbb{H}$. A *special subvariety* of X is the image under $\pi : U_X \to X$ of a pre-special subvariety of U_X.

According to [34, 3.1], this definition of a special subvariety coincides with the one given in Definition 1.3 when $X = \mathbb{C}^n \times A \times \mathbb{G}^\ell$.

If $\mathcal{Z}$ contains an algebraic component Y, then, by Γ-periodicity, it contains all its translates gY under Γ. The union $\cup_g gY$ is not generally algebraic as it has, generally, infinitely many components (the exception is if $Y = U$) and we will refer to it as a *locus*. Thus the inverse image of a quasi-special subvariety of X is a *quasi-pre-special locus*, and this in turn is the union of translates under Γ of a quasi-pre-special subvariety as above. (In my earlier paper [68], I called such subvarieties "quasi-special" but here I prefer to include the "-pre-" for the objects in U corresponding to objects in X.) The preimage in U of a special subvariety in X is a pre-special locus.

The following is our key result identifying the possible maximal algebraic components of $\mathcal{Z}$.

6.8. Theorem. *Let $Y \subset \mathcal{Z}$ be a maximal complex algebraic component. Then Y is quasi-pre-special.*

The proof of this theorem is carried out over the next two sections. As it is somewhat involved in detail, we sketch the main idea to highlight our second use of the Pila-Wilkie result (in the form of Theorem 3.6).

Suppose that Y is a complex algebraic component of $\mathcal{Z}$. Since $\mathcal{Z}$ is Γ-invariant, we see that

$$gY \subset \mathcal{Z}$$

for any $g \in \Gamma$, and therefore

$$gY \cap \mathbb{F} \subset Z,$$

though $gY \cap \mathbb{F}$ will be empty for "most" g. Now Γ is a discrete arithmetic subgroup of some real algebraic group G and, with the normalization we have adopted, such g are *integer* points of a semi-algebraic (hence definable) set G. Let Y be a maximal complex algebraic component of $\mathcal{Z}$. The proof proceeds by considering the set of $g \in G$ such that $gY \cap Z$ has the full dimension of Y. (Actually we will consider $g \in H$ for certain subsets H of G.) This is a definable set, and we show that, as a consequence of the results on intricacy in Section 5, it contains "many" rational points — specifically it contains at least $\gg T^\delta$ integer points up to height T for some fixed $\delta > 0$ and implied constant. Therefore, by Theorem 3.6, it contains a positive dimensional semi-algebraic subset. Such an algebraic set of translates of Y contained in $\mathcal{Z}$ gives one of two possible outcomes. If Y is not invariant as a set under these translations, then we get an algebraic subset of $\mathcal{Z}$ containing Y but of strictly larger dimension, contradicting the assumption that Y is maximal. Otherwise Y is invariant as a set under an algebraic family of translations in G, which results in suitable identities being satisfied by the algebraic functions parametrizing Y. Enough such identities entail Y being of the sought form.

In the next section we isolate some technical results that we require to carry out the plan sketched above. Theorem 6.8 is then proved in Section 8.

6.9. *Remark.* In [71], the corresponding result ([71, Th. 2.1]) is also proved using o-minimality in the form of Gabrielov's theorem for subanalytic sets (appealed to in [71, Lemma 2.2]). In [56] and [68], which use upper bounds on rational points from o-minimality in the same strategy of opposing them with Galois lower bounds, o-minimality is not used in obtaining the analogous results characterizing the algebraic part. In [68] these are obtained by elementary arguments and in [56] by monodromy. As said in Section 1, Theorem 6.8 is equivalent to a suitable Ax-Lindemann-Weierstrass result which, apart from the modular curve aspects, follows from known Ax-Schanuel results [5], [20], [45], proved by differential-algebraic methods.

7. The algebraic part: technicalities

Families containing maximal algebraic components. Let $X = \mathbb{C}^n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell$ with $U = U_X = \mathbb{H}^n \times \mathbb{C}^m \times \mathbb{C}^\ell$ and $G = G_X$ as previously defined in 4.1. Suppose $V \subset X$ and $\mathcal{Z} = \pi^{-1}(V) \subset U$, and W an irreducible algebraic set in $\mathbb{C}^{n+m+\ell}$ of dimension w. We wish to study maximal complex algebraic components of $\mathcal{Z}$. We begin with some observations. If Y is a component of $W \cap U$, then, for any $g \in G$, gY is a component of $gW \cap U$. If Y is a complex algebraic component of $\mathcal{Z}$, then, as already noted, so is gY for any $g \in \Gamma$. Moreover, if Y is maximal, then gY is also maximal for any $g \in \Gamma$.

In our proof of Theorem 6.8, we will assume that Y is a maximal complex algebraic component and we will show that a translate gY, for some $g \in \Gamma$, lies in a semi-algebraic family of complex algebraic components of $\mathcal{Z}$. We will have need of the following result showing that a maximal algebraic component cannot be a fibre in a nonconstant family of components.

We keep all the above notation, but one may observe that Proposition 7.1 holds under the weaker assumption that $\mathcal{Z}$ is a complex analytic subset of U, which need not be of the form $\pi^{-1}(V)$ or even Γ-invariant.

7.1. Proposition. *Suppose that W is an irreducible closed algebraic subset of $\mathbb{C}^{n+m+\ell}$ of dimension w and that Y is a component of $W \cap U$. Let $g : (-1,1) \to G$ be a semialgebraic map which is regular (analytic) for $t \in (-1,1)$. Suppose that $g(t)Y \subset \mathcal{Z}$ for all $t \in (-1,1)$ and that $g(0)Y$ is a maximal complex algebraic component of $\mathcal{Z}$. Then $g(t)Y = Y$ for all $t \in (-1,1)$.*

Proof. Suppose $P \in Y$. Then $g(t)P \in g(t)Y \subset \mathcal{Z}$ for all $t \in (-1,1)$. The map $t \mapsto g(t)P \in U$ extends to a complex algebraic map on some complex neighbourhood of 0, and since $\mathcal{Z}$ is analytic we have that $g(t)P \in \mathcal{Z}$ for such complex t. If we do not have $g(t)Y = g(0)Y$ for all $t \in (-1,1)$, then there is a point $P \in Y$ and some $t \in (-1,1)$ such that $g(t)P$ does not belong to $g(0)Y$, and hence (by analyticity) $g(t)P \in g(0)Y$ for only finitely many t in some complex neighbourhood of 0. We can take a suitable complex neighbourhood of $t = 0$ so that $g(t)P$ does not belong to $g(0)Y$ except for $t = 0$. Then for some equation $F = 0$ defining Y we have that $F(g(t)P)/t^p$ is nonzero in a complex neighbourhood of $t = 0$ for some positive integer p, and hence there is a neighbourhood D of P and a complex neighbourhood of $t = 0$ such that, for all $Q \in D \cap Y$, and t in the neighbourhood, $g(t)Q$ is not in $g(0)Y$. Therefore the union of $g(t)(D \cap Y)$ contains a complex algebraic set of dimension $w + 1$ contained in $\mathcal{Z}$. Then $\mathcal{Z}$ contains a complex algebraic component Y' of dimension $w + 1$ containing Y, contradicting the hypothesis that Y was maximal. □

7.2. Proposition. *Retaining all the hypotheses of Proposition* 7.1, *suppose that* $x_1, \ldots, x_w$ *is a subset of the variables*

$$\{\tau_1, \ldots, \tau_n, z_1, \ldots, z_m, \zeta_1, \ldots, \zeta_\ell\}$$

consisting of w *distinct elements, with* $y_1, \ldots, y_{n+m+\ell-w}$ *the complementary set of variables. Let us write, for* $t \in (-1, 1)$,

$$g(t) = (g_1(t), \ldots, g_w(t), h_1(t), \ldots, h_{n+m+\ell-w}(t)) \in G$$

with respect to the variables $(x_1, \ldots, x_w, y_1, \ldots, y_{n+m+\ell-w})$ *so that each* $g_i(t)$, $h_i(t)$ *is an element of* $\mathrm{SL}_2(\mathbb{R}), \mathbb{R}^2$, *or* $\mathbb{R}$ *according as* x_i, y_j *is a* τ*-variable or a* z*-variable, or a* ζ*-variable. Suppose that* $g(t)Y$ *contains the graph*

$$y = \phi(x), \quad x = (x_1, \ldots, x_w), \quad y = (y_1, \ldots, y_{n+m+\ell})$$

given by

$$y_j = h_j(t)\phi_j(g_1^{-1}(t)x_1, \ldots, g_w^{-1}(t)x_w), \quad j = 1, \ldots, n + m + \ell,$$

where ϕ_j *are algebraic functions, for* $(x_1, \ldots, x_w) \in D$, *where* D *is the product of some open disk in each variable. Then each of the functions*

$$h_j(t)\phi_j(g_1(t)x_1, \ldots, g_w(t)x_w)$$

is independent of t.

Proof. Under the hypotheses of Proposition 7.1, the set $g(t)Y = g(0)Y$ for all t. For a given choice of $x_1, \ldots, x_w$ there are some finite number of points $(x_1, \ldots, x_w, y_1, \ldots, y_{n+m+\ell-w})$ belonging to $g(0)Y$, and so as t varies the point $(x_1, \ldots, x_w, y_1, \ldots, y_{n+m+\ell-w}, z)$ with

$$y_j = h_j(t)\phi_j(g_1(t)x_1, \ldots, g_w(t)x_w)$$

varies over a finite set. Since $t \mapsto g(t)$ is smooth, it is continuous. Away from some lower-dimensional set where the algebraic functions ϕ_j may be discontinuous, the y_j are constant as t varies and equal the value they take at $t = 0$.

So the functions are locally constant near $t = 0$ on some dense set, and hence, being algebraic, are constant identically. $\square$

Algebraic functions satisfying identities. Recall our convention on complex algebraic functions above Proposition 5.3.

7.3. PROPOSITION. *Let $g, h \in \mathrm{SL}_2(\mathbb{R})$ and suppose $x_0, x_1 \in \mathbb{C}$ with $g(x_0) = x_0$ and $x_1 \neq x_0$. Let ϕ be an algebraic function of degree $\leq k$, with $\phi(gx) = h\phi(x)$ on some nonempty connected open domain $D \subset \mathbb{C}$. Suppose ϕ is not branched at $x_0, g^{-1}(x_1), x_1, g(x_1), g^2(x_1), \dots g^k(x_1)$, and that $\phi(x_1) = \phi(x_0)$. Then either ϕ is constant or x_1 is preperiodic under g (with orbit of length $\leq k$).*

Proof. We can connect x_0, x_1 by a path avoiding the branch points. By changing the domain on which ϕ is defined (introducing suitable branch cuts that avoid the path connecting x_0, x_1), we can assume that $\phi(x)$ and $\phi(gx)$ are single valued on a domain containing $x_0, x_1, g(x_1), \dots, g^k(x_1)$, that $\phi(x_0) = \phi(x_1)$, and the relation $\phi(gx) = h\phi(x)$ holds. Then

$$\phi(gx_1) = h\phi(x_1) = h\phi(x_0) = \phi(gx_0) = \phi(x_0),$$

and so inductively $\phi(g^n x_1) = \phi(x_0)$ for $n = 1, 2, \dots, k$. If ϕ is nonconstant, then it is at most k-to-one, and so the points $x_1, g(x_1) \cdots, g^k(x_1)$ cannot be distinct. $\square$

7.4. PROPOSITION. *Let ϕ be an algebraic function. Let g_n, h_n be elements of $\mathrm{SL}_2(\mathbb{R})$ for $n = 1, 2, \dots$ such that $\phi(g_n x) = h_n\phi(x)$ on some nonempty connected open domain $D_n \subset \mathbb{C}$. Suppose the g_n are all parabolic with distinct fixed points. Then ϕ is constant or one-to-one.*

Proof. Suppose that ϕ is nonconstant and generically k-to-one for some $k \geq 2$. Let $b_1, \dots, b_K$ be the branch points, including any points where ϕ is not k-to-one.

The function ϕ satisfies some irreducible algebraic relation $P(x, \phi(x)) = 0$. Let us call an algebraic function ψ satisfying the same algebraic relation as ϕ but on a possibly different domain a *re-definition* of ϕ. Any such ψ will be nonconstant and generically k-to-one. There are only finitely many points in $\mathbb{C}$ where ϕ or any re-definition of it takes the same value as one of $\phi(g^i(b_j))$ for $i = 1, \dots, k, j = 1, \dots, K$. If x_0 is not one of those points, any point x_1 with $\phi(x_1) = \phi(x_0)$ is also not one of those points.

Since we have infinitely many g_n with distinct fixed points, we can find $g = g_n$ with fixed point x_0 and $x_1 \in \mathbb{C}$ such that the hypotheses of Proposition 7.3 are satisfied. This leads to a contradiction as g has no preperiodic points other than its fixed points. $\square$

8. **The algebraic part: conclusion**

8.1. *Proof of Theorem* 6.8. Suppose that Y is maximal complex algebraic component of $\mathcal{Z} = \pi^{-1}(V)$, so that Y is a connected component $W \cap U$ for some irreducible algebraic $W \subset \mathbb{C}^{n+m+\ell}$.

The proof is in several stages, which we separate for the convenience of the reader. We choose suitable variables to give a parametrization of Y in which the dependencies between variables of different type occur only in specified ways. We restrict to a suitable subdomain where the parametrizing functions are well behaved. We then produce definable subsets R of G such that the corresponding translates of Y intersect a definable subset of $\mathcal{Z}$ in their full dimension (i.e. of Y). Due to the periodicity of $\mathcal{Z}$ these sets R contain "many" Γ-translates . Then Theorem 3.6 gives positive dimensional semi-algebraic families of G-translates of Y contained in $\mathcal{Z}$. Since Y is maximal we derive identities for the parametrizing functions. These identities force Y to have the required form.

Choosing suitable variables to parametrize Y. We take variables $(\tau_1, \ldots, \tau_n)$ for $\mathbb{H}^n$, $(z_1, \ldots, z_m)$ for $\mathbb{C}^m$, and $(\zeta_1, \ldots, \zeta_\ell)$ for $\mathbb{C}^\ell$. Suppose W is w-dimensional. We can choose some w variables

$$(\tau_{f,i}, z_{f,j}, \zeta_{f,k}), \quad i = 1, \ldots,\ n',\ j = 1, \ldots, m',\ k = 1, \ldots, \ell'$$

(the subscript f stands for "free") to parametrize Y locally by means of some algebraic functions

$$\tau_{d,a} = \phi_a(\tau_{f,i}, z_{f,j}, \zeta_{f,k}), \quad z_{d,b} = \theta_b(\tau_{f,i}, z_{f,j}, \zeta_{f,k}), \quad \zeta_{d,c} = \psi_c(\tau_{f,i}, z_{f,j}, \zeta_{f,k})$$

(the subscript d stands for "dependent"), defined on some connected open subset of

$$U' = \mathbb{H}^{n'} \times \mathbb{C}^{m'} \times \mathbb{C}^{\ell'}$$

on the "free" coordinates. I.e. the functions on W induced by these "free" variables are a transcendence basis for the function field $\mathbb{C}(W)$.

The variables $\tau \ldots, z \ldots, \zeta \ldots$ play different roles with respect to the map $\pi : U \to X$, but from the point of view of parametrizing Y any choice of w algebraically independent variables will do. We show that we can make a choice of independent variables such that certain dependencies are avoided.

Some of the dependant variables may be constant. We exchange some of the nonconstant dependent variables and free variables using the Steinitz exchange property (see e.g. [36, Th. A1.1, et seq.]). Suppose $u, v, w_1, \ldots, w_k, y$ are elements of some field L containing $\mathbb{C}$. We will say that v *depends on* u *over* $w_1, \ldots, w_k$ if v is a nonconstant algebraic element over $K(u)$, where $K = \mathbb{C}(w_1, \ldots, w_k)$. In that case, u depends on v over $w_1, \ldots, w_k$. Further, if y is algebraic over $\mathbb{C}(w_1, \ldots, w_k, u)$, then it will be algebraic over

$\mathbb{C}(w_1, \dots, w_k, v)$. In particular if $w_1, \dots, w_k, u$ are a transcendence basis of L, then so are $w_1, \dots, w_k, v$.

We use this property to exchange elements in our transcendence basis of $\mathbb{C}(W)$ given by the "free" variables above. First, if some $\tau_{d,a}$ depends on some $z_{f,j}$, we interchange them. So we may assume that any dependent variables $\tau_{d,a}$ are independent of any free variables $z_{f,j}$. Next, we do the same for dependent $\tau_{d,a}$ and free $\zeta_{f,k}$, and finally we do the same for dependant $z_{d,b}$ and free $\zeta_{f,k}$.

After these interchanges we have Y parametrized, locally on some open region in the free variables, by algebraic functions

$$\tau_{d,a} = \phi_a(\tau_{f,i}), \quad z_{d,b} = \theta_b(\tau_{f,i}, z_{f,j}), \quad \zeta_{d,c} = \psi_c(\tau_{f,i}, z_{f,j}, \zeta_{f,k}).$$

A suitable subregion of U'. We can analytically continue these functions (perhaps with some branching) through a subregion U'' of U' bounded by the following loci corresponding to the boundary of U:

$$L_{f,i} = \{(\tau_{f,i}, z_{f,j}, \zeta_{f,k}) : \mathrm{Im}(\tau_{f,i}) = 0\}, \quad L_{d,a} = \{(\tau_{f,i}, z_{f,j}, \zeta_{f,k}) : \mathrm{Im}(\tau_{d,a}) = 0\}.$$

If the region U'' is bounded only by loci $L_{d,a}$, then some $\tau_{d,a}$ depends on some $\tau_{f,i}$ (i.e. "over" the others, as before) and we can interchange them (and renumber the $\tau_{f,i}$) to get that U'' has some nontrivial boundary in $L_{f,1}$. Some of the loci $L_{d,a}$ may contain $L_{f,1}$, and we will denote by $\tau_{d,\alpha}$ the variables whose loci $L_{d,\alpha}$ contain $L_{d,a}$, but other loci $L_{d,a}$ may not contain $L_{f,1}$, and we will denote those variables $\tau_{d,\beta}$ with corresponding loci $L_{d,\beta}$. These $L_{d,\beta}$ intersect $L_{f,1}$ in some lower dimensional set, and so a suitable product of open disks in each variable inside U'' can be taken having some boundary in $L_{f,1}$ while being at positive distance from all other boundary components. That is, we can take a point $P \in L_{f,1}$ and a product of disks centred at the coordinates of P such that if $U^{\tau}_{f,1}$ is the open half-disk in the $\tau_{f,1}$ variable lying in its upper half-plane, and $U^{\tau}_{f,i}, i \neq 1, U^{z}_{f,j}, U^{\zeta}_{f,k}$ are the disks in the other variables, then

$$U^* = \prod_i U^{\tau}_{f,i} \times \prod_j U^{z}_{f,j} \times \prod_k U^{\zeta}_{f,k} \subset U''$$

has a part

$$\partial U^{\tau}_{f,1} \cap \{\tau_{f,1} : \mathrm{Im}(\tau_{f,1}) = 0\} \times \prod_{i \neq 1} U^{\tau}_{f,i} \times \prod_j U^{z}_{f,j} \times \prod_k U^{\zeta}_{f,k}$$

of its boundary that is contained in $L_{f,1}$, while all of U^* is at positive distance from all the other $L_{f,i}, L_{d,\beta}$ and components of any $L_{d,\alpha}$ besides $L_{f,1}$.

We may further assume (taking smaller discs if need be) that the algebraic functions ϕ_a, θ_b, ψ_c are all bounded and univalent on U^*, and we denote the image regions

$$V^{\tau}_{d,a} = \phi_a(U^*), \quad V^{z}_{d,b} = \theta_b(U^*), \quad V^{z}_{d,c} = \psi_c(U^*).$$

Let us write

$$\Psi = (\phi_a,\ a = 1, \ldots, \theta_b, \quad b = 1, \ldots, \psi_c,\ \ c = 1, \ldots)$$

for the tuple of functions parametrizing Y, and write

$$Y^* = \{(u, \Psi(u)) : u \in U^*\} \subset Y$$

for the graph of the parametrization on the set U^*. The set Y^* will play a key role.

A definable set. We can now take fundamental domains (or *finite* unions thereof)

$$\mathbb{D}^{\tau}_{f,1} \subset U^{\tau}_{f,1},$$
$$\mathbb{D}^{\tau}_{f,i} \supset U^{\tau}_{f,i}, i \neq 1, \quad \mathbb{D}^{z}_{f,j} \supset U^{z}_{f,j}, \quad \mathbb{D}^{\zeta}_{f,j} \supset U^{\zeta}_{f,k}.$$

We have arranged that the algebraic functions parametrizing Y are bounded on U^*. The $\tau_{d,\alpha}$ become real on the part of the boundary of U^* described above corresponding to $\mathrm{Im}(\tau_{f,1}) = 0$ while no other $\tau_{f,i}$ or $\tau_{d,\beta}$ do. So for the image domains we can choose fundamental domains (or finite unions thereof) with

$$\mathbb{D}^{\tau}_{d,\alpha} \subset V^{\tau}_{d,\alpha},$$
$$\mathbb{D}^{\tau}_{d,\beta} \supset V^{\tau}_{d,\beta}, \quad \mathbb{D}^{z}_{d,b} \supset V^{z}_{d,b}, \quad \mathbb{D}^{\zeta}_{d,k} \supset V^{\zeta}_{d,c}.$$

Then

$$\mathbb{D}^* = \prod_i \mathbb{D}^{\tau}_{f,i} \times \prod_j \mathbb{D}^{z}_{f,j} \times \prod_k \mathbb{D}^{\zeta}_{f,k} \times \prod_\alpha \mathbb{D}^{\tau}_{d,\alpha} \times \prod_\beta \mathbb{D}^{\tau}_{d,\beta} \times \prod_b \mathbb{D}^{z}_{d,b} \times \prod_c \mathbb{D}^{\zeta}_{d,c}$$

is a finite union of fundamental domains for the action of Γ on U; whence

$$Z^* = \mathcal{Z} \cap \mathbb{D}^*$$

is definable.

Definable sets of G translates of Y. The set Y is semi-algebraic, hence definable. The set U^* is semi-algebraic, hence definable. Thus Y^* is definable as a graph over the region U^* in which, crucially, there are infinitely many fundamental domains for the variable $\tau_{f,1}$. (So we restricted further to one of them to make Z^* definable.)

Likewise G is definable, and the translations of Y by $g \in G$ is given by a definable subset of $G \times \mathbb{R}^N, N = 2(n + m + \ell)$ (the fibre at $g \in G$ is gY). Let G' be a definable subset of G, Y' a definable subset of Y, Z' a definable subset of $\mathcal{Z}$, and $w' \geq 0$. By the properties in [30, B.10], the set

$$R(G', Y', Z') = \{g \in G' : \dim(gY' \cap Z') = w'\}$$

is a definable set. Note that if $Y' \subset Y$ has dimension w, then $g \in R(G', Y', Z')$ implies that there is a neighbourhood of a regular point of gY contained in $\mathcal{Z}$; whence by analytic continuation we will have $gY \subset \mathcal{Z}$.

While G, Y are definable in their entireties, it is convenient to work with subsets of both: on subsets of Y^* the parametrization is controlled, while restricting to different one parameter subsets of G generates different identities.

We consider translates of Y^* by certain elements $g \in G$ whose elements we denote by

$$g = (g^{\tau}_{f,i}, g^{z}_{f,j}, g^{\zeta}_{f,k}, g^{\tau}_{d,a}, g^{z}_{d,b}, g^{\zeta}_{d,c})$$

acting on the corresponding variables in the obvious way. Put

$$\mathcal{E} = \mathbb{D}^{\tau}_{f,1} \times \prod_{i\neq 1} U^{\tau}_{f,i} \times \prod_{j} U^{\lambda}_{f,j} \times \prod_{k} U^{z}_{f,k}.$$

Since $U^{\tau}_{f,1}$ borders the real $\tau_{f,1}$-axis, it contains infinitely many $\mathrm{SL}_2(\mathbb{Z})$ translates of $\mathbb{D}^{\tau}_{f,1}$. If $g \in \Gamma$ and $g\mathcal{E} \subset U^*$, then the graph

$$Y_{g,\mathcal{E}} = \{(u, \Psi(gu)) : u \in \mathcal{E}\} \subset Y^*$$

is contained in $\mathcal{Z}$, and a suitable Γ-translate of the dependent variables will give a translate (of a part of) Y^* into $\mathbb{D}$, which will then be contained in Z^*.

Fix

$$g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and consider

$$G(g_0) = \Big\{ g \in G : g^{\tau}_{f,1} = \begin{pmatrix} a & b+ta \\ c & d+tc \end{pmatrix},\ \text{some } t \in \mathbb{R},$$
$$g^{\tau}_{f,i} = 1,\ \text{all } i \neq 1, \quad g^{z}_{f,j} = g^{\zeta}_{f,k} = 1,\ \text{all } j,k \Big\},$$

with no restriction on the group elements corresponding to the dependent variables.

We now consider definable sets of the form

$$R(G(g_0), Y^*, Z^*).$$

Rational points on $R(G(g_0), Y^*, Z^*)$. Suppose $a/c \in \partial U^{\tau}_{f,1} \cap \mathbb{R}$. For every sufficiently large positive $t \in \mathbb{Z}$ (depending on g_0) we have

$$g_{f,1}\mathbb{D}^{\tau}_{f,1} \subset U^{\tau}_{f,1}.$$

Then

$$\{(\tau_{f,i}, z_{f,j}, \zeta_{f,k}, \Psi(g_{f,1}\tau_{f,1}, \tau_{f,i}, i \neq 1, z_{f,j}, \zeta_{f,k}) : (\tau_{f,i}, z_{f,j}, \zeta_{f,k}) \in \mathcal{E}\} \subset Z$$

and so also its translation by any element of Γ. We can choose an element of Γ, trivial on all the free variables, to bring some regular point of the translation into $\mathbb{D}^*$ — in fact into the interior of the factor of $\mathbb{D}^*$ corresponding to any nonconstant variable — and moreover by Propositions 5.3, 5.4, and 5.5 we can do so with an element of height

$$\ll t^c$$

for some positive c, where c and the implied constant depend on the choice of g_0, Y^*, Z^*, but is independent of t. Such a translate intersects Z^* in full dimension.

Therefore, for all large T,

$$N(R(G(g_0), Y^*, Z^*), T) \gg T^\delta$$

for some $\delta > 0$ (with a constant depending on g_0, Y^*, Z^*), indeed this holds for integer points. So, by [70], $R(G(g_0), Y^*, Z^*)$ contains semi-algebraic sets of positive dimension. Moreover, by Theorem 3.6, for any choice of $\epsilon > 0$ it contains such sets that contain at least

$$\gg T^{\delta-\epsilon}$$

integer points, all regular. Such sets may have t constant, or variable. We show that there must be such sets with t variable.

Fix $\epsilon = \delta/2$ say. Then the semi-algebraic subsets have bounded degree independent of T. Their intersections with the subvariety of G with $g^\tau_{f,1}$ (i.e. t) constant have bounded degree, and so the number of singular points on them is bounded, say by B. Suppose that, for some fixed t, there is a positive dimensional semi-algebraic subset of translations of the $\tau_{d,a}, z_{d,b}, \zeta_{d,c}$ that brings

$$\{(\tau_{f,i}, z_{f,j}, \zeta_{f,k}, \Psi(g_{f,1}\tau_{f,1}, \tau_{f,i}, i \neq 1, z_{f,j}, \zeta_{f,k}) : (\tau_{f,i}, z_{f,j}, \zeta_{f,k}) \in \mathcal{E}\}$$

into $\mathbb{D}^*$ and contains more than B integer points. Then there exists such containing an integer point as a smooth point of a one-dimensional arc. The integer translate of Y is maximal, and since the arc gives a family that is clearly not constant, we contradict the conclusion of Proposition 7.2.

Therefore we may assume that, for all sufficiently large t (depending on g_0, Y^*, Z^*), there is a connected positive-dimensional semi-algebraic family of translations of Y^* by elements of $G(g_0)$ containing arbitrarily many regular integer points with t varying. The integer (Γ-) translates of Y^* are maximal, so by Proposition 7.2 the corresponding algebraic functions are constant.

Dependent variables $\tau_{d,\beta}, z_{d,b}, \zeta_{d,c}$. Consider then some $\tau_{d,\beta}$. For large t, $g^\tau_{f,1}\mathbb{D}^\tau_{f,1} \subset U^\tau_{f,1}$ and so $\tau_{d,\beta} = \phi_\beta(g_{f,1}\tau_{f,1}, \tau_{f,i})$ remains in $\mathbb{D}^\tau_{d,\beta}$, and there is a fixed finite set of translations on the $\tau_{d,\beta}$ variable that stay inside $\mathbb{D}^\tau_{d,\beta}$. By the constancy of the family, we conclude that

$$\phi_\beta(g_{f,1}\tau_{f,1}, \tau_{f,i})$$

is constant, so that $\tau_{d,\beta}$ is in fact independent of $\tau_{f,1}$. The same argument shows that the $z_{d,b}, \zeta_{d,c}$ are independent of $\tau_{f,1}$.

Dependent variables $\tau_{d,\alpha}$. Consider now some $\tau_{d,\alpha}$. Write $g_{f,1}(t) = g_{f,1}$. As t varies, we have some $h(t) \in \mathrm{SL}_2(\mathbb{R})$ varying semialgebraically in t over an interval I as described above such that

$$h(t)\phi_\alpha(g_{f,1}(t)\tau_{f,1}, \tau_{f,i})$$

belongs to some fixed maximal algebraic component corresponding to some integer $t_0 \in I$, where $g_{f,1}, h$ are smooth. Given a choice of $\tau_{f,i}$ there are only finitely many $\tau_{d,\alpha}$ corresponding to points of this component. So

$$h(t)\phi_\alpha(g_{f,1}(t)\tau_{f,1}, \tau_{f,i}) = h(t_0)\phi_\alpha(g_{f,1}(t_0)\tau_{f,1}, \tau_{f,i})$$

identically for $\tau_{f,1} \in U^\tau$ (where there is no branching of the algebraic functions), and hence identically on some subregion of the $\tau_{f,1}$-plane obtained by removing branch loci.

We now fix the $\tau_{f,i}, i \neq 1$; put

$$g = g_{f,1}(t)g_{f,1}(t_0)^{-1} = \begin{pmatrix} 1 - ac(t-t_0) & a^2(t-t_0) \\ -c^2(t-t_0) & 1 + ac(t-t_0) \end{pmatrix},$$

which is parabolic with fixed point a/c, and $h = h(t)^{-1}h(t_0)$ and we find that ϕ_α satisfies

$$\phi_\alpha(g\tau) = h\phi_\alpha(\tau)$$

locally, and hence this relation holds globally by analytic continuation. We have infinitely many different possible choices for a/c, and so we can apply Proposition 7.4 to conclude that ϕ_α is constant or one-to-one (i.e. under any choice of cuts). However ϕ_α is not constant, therefore it is one-to-one.

Since $\tau_{f,1}$ and $\tau_{d,\alpha}$ depend on each other, we can interchange them, and we find that ϕ_α^{-1} is also one-to-one. Then ϕ_α is a fractional linear transformation, and since $\tau_{d,\alpha}$ is real on the real line for $\tau_{f,1}$, we see that (having fixed the other $\tau_{f,i}$) $\phi_\alpha \in \mathrm{SL}_2(\mathbb{R})$.

Now as the $\tau_{f,i}, i \neq 1$ vary, ϕ_α is an element of $\mathrm{SL}_2(\mathbb{R})$ that depends complex algebraically on $\tau_{f,i}, i \neq 1$. Therefore it is constant (consider e.g. the images of $0, 1, \infty$, which must be real numbers varying algebraically in the complex parameters $\tau_{f,i}, i \neq 1$).

Therefore ϕ_α depends on $\tau_{f,1}$ alone and is an element of $\mathrm{SL}_2(\mathbb{R})$. We conclude that the $\tau_{d,\beta}$ depend on $\tau_{f,1}$ alone, and by elements of $\mathrm{SL}_2(\mathbb{R})$, while the $\tau_{d,\alpha}$ are independent of $\tau_{f,1}$. We may repeat the above arguments to show that all the dependencies among the τ-variables are of this form, and so we may take $U'' = U'$, and for the parametrization of Y we may allow each $\tau_{f,i}$ to range over its upper half-plane.

From $\mathrm{SL}_2(\mathbb{R})$ *to* $\mathrm{GL}_2(\mathbb{Q})^+$. We can repeat our argument above for any nonconstant ϕ_α in the neighbourhood of any point on the real axis to show that

$$\phi_\alpha g \phi_\alpha^{-1} = \lambda h, \quad \lambda \in \mathbb{R}, \quad h \in \mathrm{GL}_2(\mathbb{Q})^+$$

for suitable matrices g as above with integer entries: namely for any $g_0 \in \mathrm{SL}_2(\mathbb{Z})$ (i.e. with any a/c) and for at least $\gg T^{\delta/2}$ choices of $s = t - t_0 \in \mathbb{Z}$ up to T. We show elementarily that this implies that the ϕ_α, up to scaling, are in fact in $\mathrm{GL}_2(\mathbb{Q})^+$, i.e. that the ratio of any two entries of ϕ_α is rational.

Write $\phi_\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), AD - BC = 1$. Taking $a = 1, c = 0$ in g we find that $\left(\begin{smallmatrix} 1-sAC & sA^2 \\ -sC^2 & 1+sAC \end{smallmatrix}\right)$ is of the form $\lambda h, \lambda \in \mathbb{R}, h \in \mathrm{GL}_2(\mathbb{Q})^+$ for suitable choices of s as above and so the ratio of any two entries is rational. If $C = 0$, we find that $A^2 \in \mathbb{Q}$ and then $AD = 1$ implies $A/D \in \mathbb{Q}$. Similarly, $A = 0$ leads to $B/C \in \mathbb{Q}$. Otherwise (if $A, C \neq 0$) we see that $A^2/C^2 \in \mathbb{Q}$ and $(1 - sAC)/sC^2 \in \mathbb{Q}$ giving $A/C \in \mathbb{Q}$. Taking $a = 0, c = 1$ in g we find that $\left(\begin{smallmatrix} 1-sBD & sB^2 \\ -sD^2 & 1+sBD \end{smallmatrix}\right)$ is of the form $\lambda h, \lambda \in \mathbb{R}, h \in \mathrm{GL}_2(\mathbb{Q})^+$. Now $B = 0$ leads to $A/D \in \mathbb{Q}$ and $D = 0$ leads to $B/C \in \mathbb{Q}$, otherwise $(B, D \neq 0)$ we get $B/D \in \mathbb{Q}$.

Suppose $C = 0$, so that $A/D \in \mathbb{Q}$. If $B = 0$, then $A/D \in \mathbb{Q}$ and ϕ_α has the required form. Having $D = 0$ is excluded by $AD - BC = 1$, and $B, D \neq 0$ gives $B/D \in \mathbb{Q}$ and ϕ_α is again of the required form. Similarly, if any of $A, B, D = 0$, we conclude that ϕ_α has the required form.

Therefore we may assume $A, B, C, D \neq 0$, so that $C/A = q \in \mathbb{Q}, D/B = r \in \mathbb{Q}$ and $\phi_\alpha = \left(\begin{smallmatrix} 1 & \alpha \\ q & r\alpha \end{smallmatrix}\right)$, up to scaling, with $r \neq q$. Then $\psi = \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ -q & 1 \end{smallmatrix}\right) \phi_\alpha = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & (r-q)\alpha \end{smallmatrix}\right)$ has the same property as ϕ_α (i.e. $\psi g \psi^{-1} = \lambda h, \lambda \in \mathbb{R}, h \in \mathrm{GL}_2(\mathbb{Q})^+$ for the same matrices g), but now $C' = 0$ and we conclude as above that ψ, and hence ϕ_α, has the required form.

This shows that the τ-variables have the required form of dependencies, and we have also shown that the z-variables and ζ-variables do not depend on the τ-variables.

The $z_{f,j}$ dependencies. Next we consider the $z_{f,j}$ variables, considering in particular dependencies on $z_{f,1}$ say. Here we will just consider the intersection of translates of Y itself (definable) with $Z = \mathcal{Z} \cap \mathbb{F}_X$. There being no τ dependencies, we can suppress the τ variables. We take bounded disks

$$U^z_{f,j}, \quad U^\zeta_{f,k}$$

such that the algebraic functions parametrizing Y are univalent and bounded on the product

$$U^\flat = \prod_j U^z_{f,j} \times \prod_k U^\zeta_{f,k}.$$

We consider now translations of Y. Fix an element λ_1 of the period lattice $\Lambda_1 = \mathbb{Z} \oplus \mathbb{Z}\tau_1$ of E_1. So $\Gamma_{E_1} = \mathbb{Z} \oplus \mathbb{Z}$ and let $s_{f,1}$ be element of Γ_{E_1} corresponding to translation by λ_1. Fixing the other free variables $z_{f,j}$, $j \neq 1$, $\zeta_{f,k}$, the algebraic functions can all be defined univalently for $z_{f,1} + \lambda_1 t$ for sufficiently large t. So for large integer t we have

$$\Big\{(z_{f,1}+t\lambda_1, z_{f,j}, \zeta_{f,k}, \theta_{d,b}(z_{f,1}+t\lambda_1, z_{f,j}, \zeta_{f,k}), \\ \psi_{d,c}(z_{f,1}+t\lambda_1, z_{f,j}, \zeta_{f,k}) : (z_{f,1}, z_{f,j}, \zeta_{f,k}) \in U^\flat\Big\} \subset \mathcal{Z}$$

and therefore, by the Γ_X- invariance of $\mathcal{Z}$,

$$\Big\{(z_{f,j}, \zeta_{f,k}, \theta_{d,b}(z_{f,1}+t\lambda_1, z_{f,j}, \lambda_{f,k}) \\ -\lambda_{d,b}, \psi_{d,c}(z_{f,1}+t\lambda_1, z_{f,j}, \lambda_{f,k}) - k_{d,c}2\pi i : (z_{f,j}, \zeta_{f,k}) \in U^\flat\Big\} \subset \mathcal{Z}$$

for any $\lambda_{d,b} \in \Lambda_{d,b}$ and $k_{d,c} \in \mathbb{Z}$. Set

$$G(\lambda_1) = \Big\{g \in G : g^z_{f,1} = ts_{f,1}, g^z_{f,j} = 1, j \neq 1, g^\lambda_{f,c} = 1\Big\}$$

with no restriction on the group elements corresponding to the dependent variables. Put

$$R(G(\lambda_1), Y, Z) = \{g \in G(\lambda_1) : \dim(gY \cap Z) = w\}.$$

Then $R(G(\lambda_1), Y, Z)$ is definable, and by Proposition 5.4 contains an integer point of size at most $\ll t^c$ for every large integer t, where c and the implied constant depend on $\lambda_1, U^\flat$. Therefore,

$$N(R(G(\lambda_1), Y, Z), T) \gg T^\delta$$

for some positive δ, and $R(G(\lambda_1), Y, Z)$ contains connected semi-algebraic subsets of positive dimension. As before, if, for a fixed t, there is a positive-dimensional set of translations of the dependent variables with full dimensional intersection with Z, then there is one-dimensional such family with a smooth integer point, and this contradicts the maximality of Y. So we may assume that the semi-algebraic subsets are curves with varying t, and that, given $\epsilon > 0$, there exist such curves containing $\gg T^{\delta-\epsilon}$ regular integer points for all large T.

Consider some dependent z variable $z_{d,b}$. We then have identities of the form (suppressing the fixed variables)

$$\theta_{d,b}(z_{f,1}+t\lambda_1) - \theta_{d,b}(z_{f,1}+t_0\lambda_1) = \lambda_{d,b}(t, t_0),$$

where $\lambda_{d,b}(t, t_0)$ is a semi-algebraic function, valid for intervals of t containing $\gg T^{\delta-\epsilon}$ integers t, t_0 for which $\lambda_{d,b}(t, t_0)$ is in the period lattice. Taking derivatives with respect to $z_{f,1}$, the algebraic function with a period must be constant, so that

$$\theta_{d,b}(z_{f,1}) = qz_{f,1} + b$$

for some $q, b \in \mathbb{C}$. Further, the existence of integer points t, t_0 for which such an identity holds implies that, for suitable nonzero integer N, $Nq\lambda_1 \in \Lambda_{d,b}$.

Since we can repeat this argument with any $\lambda_1 \in \Lambda_{f,1}$ we see that, for suitable nonzero integer N,

$$Nq\Lambda_{f,1} \subset \Lambda_{d,b}.$$

Now as we vary the other free variables, such q cannot vary continuously, and we see that

$$\theta_{d,b}(z_{f,1}) = qz_{f,1} + b(z_{f,j}, j \neq 1).$$

Repeating the argument with the other variables shows that

$$\theta_{d,b}(z_{f,j}) = \sum_j q_j z_{f,j} + b,$$

where $b \in \mathbb{C}$ is independent of all the variables, and, for suitable nonzero integer N_j,

$$N_j q_j \Lambda_{f,j} \subset \Lambda_{d,b}.$$

Such a locus is then quasi-special.

If we consider now the dependence of one of the $\zeta_{d,c}$ variables on one of the $z_{f,j}$ variables (the others being fixed), then we find that such dependencies must also be linear of the form

$$\psi_{d,c} = qz_{f,j} + b,$$

where $q, b \in \mathbb{C}$ and

$$Nq\Lambda_{f,j} \subset 2\pi i\mathbb{Z}$$

for some nonzero integer N. This is however impossible for nonzero q, and we find that the $\zeta_{d,c}$ are independent of the $z_{f,j}$.

The $\zeta_{f,j}$ dependencies. Finally we consider the dependence of the $\zeta_{d,b}$ on the $\zeta_{f,j}$. These must again, by similar arguments, be linear and of the form

$$\zeta_{d,c} = \sum_k q_k \zeta_{f,k} + b,$$

where $q_k \in \mathbb{Q}$. Thus Y is quasi-pre-special, as required. □

In fact we can prove a more general form of Theorem 6.8. This is not needed for proving Theorem 1.1 but gives a natural extension Theorem 9.6 of Theorem 1.6 which shows that, under the hypotheses of Theorem 1.6, the functions are algebraically independent over the underlying algebraic function field $\mathbb{C}(W)$. We consider varieties

$$X = \mathbb{C}^n \times E_1 \times \cdots \times E_m \times \mathbb{G}^m \times \mathbb{C}^k$$

(with the elliptic curves E_i over $\mathbb{C}$) uniformized by

$$U_X = \mathbb{H}^n \times \mathbb{C}^m \times \mathbb{C}^\ell \times \mathbb{C}^k$$

in which the uniformization $\pi : U_X \to X$ is trivial on the variables $t_1, \ldots, t_k$ of $\mathbb{C}^k$. (So $\Gamma_{\mathbb{C}^k}$ is trivial, the fundamental domain $\mathbb{F}_{\mathbb{C}^k} = \mathbb{C}^k$, and algebraic subvarieties of $\mathbb{F}_{\mathbb{C}^k}$ are definable.) A *quasi-pre-special* component of U_X is now a cartesian product of quasi-pre-special components in the factors where a *quasi-pre-special component* of $\mathbb{C}^k$ is simply an irreducible algebraic subvariety. We then have the following result, which leads to a stronger version of Theorem 1.6 stated as Section 9.3 below.

8.2. THEOREM. *With X and U_X as above, let $V \subset X$ be a subvariety and $\mathcal{Z} = \pi^{-1}(V) \subset U_X$. Suppose that Y is a maximal complex algebraic component of $\mathcal{Z}$. Then Y is quasi-pre-special.*

Proof. We follow the same procedure as in the proof of Theorem 6.8. We parametrize Y by means of some algebraic functions on some choice of free variables. We can rearrange the variables so that no dependant τ_i, z_i, ζ_i depends on any free t_j. The proof now shows that the t_j are in fact independent of all the other variables. □

9. Ax-Lindemann-Weierstrass

We deduce the equivalence of the functional algebraic independence statement Theorem 1.6 to Theorem 6.8, and establish both in more general form. We consider now X of the more general form required for Theorem 1.1, namely

$$X = Y_1 \times \cdots \times Y_n \times E_1 \times \cdots \times E_m \times \mathbb{G}^\ell,$$

where $n, m, \ell \geq 0$, $Y_i = \Gamma_i \backslash \mathbb{H}$, $i = 1, \ldots, n$ are modular curves and E_j are elliptic curves defined over $\overline{\mathbb{C}}$. Let $U = U_X$ and $\pi : U \to X$.

9.1. THEOREM. *Let $V \subset X$ and $\mathcal{Z} = \pi^{-1}(V) \subset U$. Let Y be a maximal complex algebraic component of $\mathcal{Z}$. Then Y is geodesic.*

9.2. THEOREM. *Let W be an irreducible algebraic subvariety of $\mathbb{C}^{n+m+\ell}$ such that $W \cap U \neq \emptyset$. If the (locally defined) functions*

$$\overline{\tau_1}, \ldots, \overline{\tau_\nu}, \quad \overline{z_1}, \ldots, \overline{z_\mu}, \quad \overline{\zeta_1}, \ldots, \overline{\zeta_\lambda}$$

in $\mathbb{C}(W)$ are geodesically independent, then $\pi(W)$ is Zariski-dense in $Y_1 \times \cdots \times Y_\nu \times E_1 \times \cdots \times E_\mu \times \mathbb{G}^\lambda$.

Let us first observe the equivalence of these two statements.

9.3. *Proof of Theorem* 9.2 *from Theorem* 9.1. We prove the contrapositive statement. Suppose that $W \subset \mathbb{C}^{\nu+\mu+\lambda}$ is an irreducible algebraic variety with $W \cap U \neq \emptyset$. Suppose that $\pi(W)$ is not Zariski dense in X. Then it is contained in some algebraic subvariety $V \subset X$, $V \neq X$, where V is defined by some equation on the images of the indicated variables only (i.e. V is a cylinder on these variables). Then $W \cap U$ is contained in some maximal algebraic component Y of $\mathcal{Z} = \pi^{-1}(V)$ with $Y \neq U$. We have that Y is also a cylinder on the indicated variables, and is quasi-pre-special by Theorem 9.1. So Y

is a cylinder on a product of quasi-pre-special subvarieties of $\mathbb{H}^\nu$, $\mathbb{C}^\mu$, $\mathbb{C}^\lambda$, at least one of which is proper. If the quasi-pre-special component of $\mathbb{H}^\nu$ is proper, then we have either some $\overline{\tau_i}$ is constant, or some relation $\tau_a = g\tau_b$ holding with $a \neq b$ and $g \in GL_2(\mathbb{Q})^+$, so that the $\overline{\tau_i}$ are not geodesically independent. If the quasi-pre-special component of $\mathbb{C}^\mu$ is proper, then the functions $\overline{z_i}$ satisfy nontrivial linear relations as in Definition 1.5.2 and are not geodesically independent. If the quasi-pre-special component of $\mathbb{C}^\lambda$ is proper, then the functions $\overline{\zeta_i}$ satisfy nontrivial linear relations as in Definition 1.5.3 and are not geodescially independent . So the functions are not geodesically independent □

9.4. *Proof of Theorem* 9.1 *from Theorem* 9.2. Suppose $V \subset X$, $\mathcal{Z} = \pi^{-1}(V)$ and Y a maximal complex algebraic component. We show, assuming Theorem 9.2, that Y is quasi-pre-special. Let W be the Zariski closure of Y in $\mathbb{C}^{n+m+\ell}$, which is then an irreducible algebraic variety with a nonempty intersection with U. Take a subset of

$$\overline{\tau_1}, \ldots, \overline{\tau_n}, \quad \overline{z_1}, \ldots, \overline{z_m}, \quad \overline{\zeta_1}, \ldots, \overline{\zeta_\ell}$$

maximal with the property that the restriction of π to the factors corresponding to these variables is Zariski dense in the corresponding product of modular curves, elliptic curves and linear tori. By Theorem 9.2, each of the remaining $\overline{\tau_\alpha}$ is either constant or is related by an element of $\mathrm{GL}_2(\mathbb{Q})^+$ to one of the $\overline{\tau_a}$. Likewise, each of the remaining $\overline{z_\beta}, \overline{\zeta_\gamma}$ are dependent on the $\overline{z_b}, \overline{\zeta_c}$ respectively in the manner prescribed in Definition 1.5. Thus V contains the quasi-special variety T defined by these equations on the selected maximally algebraically independent coordinates, and W is contained in a component of $\pi^{-1}(T)$. Since Y is maximal, it coincides with this component . □

9.5. *Proof of Theorems* 9.2 *and* 9.1. In proving Theorem 6.8 we have established Theorem 9.1 in the case where each modular curve is $\mathbb{C} = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. By 9.3 above we conclude then Theorem 9.2 holds in such a situation, i.e., that Theorem 1.6 holds. Now we establish Theorem 9.2 in general simply by field theory, as every modular function is algebraically dependent on j. Suppose

$$X = Y_1 \times \ldots \times Y_\nu \times E_1 \times \ldots \times E_\mu \times \mathbb{G}^\lambda$$

and W as in the hypotheses of Theorem 9.2 with the images of the coordinate functions in $\mathbb{C}(W)$ geodesically independent. Then the corresponding

$$j(\overline{\tau_1}), \ldots, j(\overline{\tau_\nu}), \quad \wp_1(\overline{z_1}), \ldots, \wp_\mu(\overline{z_\mu}), \quad \exp(\overline{\zeta_1}), \ldots, \exp(\overline{\zeta_\lambda})$$

are algebraically independent (over $\mathbb{C}$), and this set of functions has transcendence degree $\nu + \mu + \lambda$ over $\mathbb{C}$. However this set of functions is algebraically dependent (over $\mathbb{C}$) on the coordinate functions of $\pi : U \to X$, which must then have the same transcendence degree (the maximum possible), and since X is irreducible, the image of W in X is Zariski dense. $\square$

Using Theorem 8.2 in place of Theorem 6.8 we get a more general version of Theorem 1.6. With $X = \mathbb{C}^n \times E_1 \times \ldots \times E_m \times \mathbb{G}^\ell \times \mathbb{C}^k$ and U_X as in Theorem 8.2, we take $t_1, \ldots, t_k$ for the variables in $\mathbb{C}^k$ and $\overline{t_i}$ their images in $\mathbb{C}(W)$ for some irreducible subvariety $W \subset \mathbb{C}^{n+m+\ell+k}$. Extending Definition 1.5 we will say that

$$\overline{\tau_1}, \ldots, \overline{\tau_\nu}, \quad \overline{z_1}, \ldots, \overline{z_\mu}, \quad \overline{\zeta_1}, \ldots, \overline{\zeta_\lambda}, \quad \overline{t_1}, \ldots, \overline{t_\kappa}$$

(where $0 \leq \kappa \leq k$) are *geodesically independent* if the $\overline{\tau_i}, \overline{z_i}, \overline{\zeta_i}$ are geodesically independent as previously (i.e. as in Definition 1.5) and, in addition, the $\overline{t_i}$ are algebraically independent over $\mathbb{C}$.

9.6. Theorem. *With the notation as above* (*and* $W \cap U_X \neq \emptyset$), *if*

$$\overline{\tau_1}, \ldots, \overline{\tau_\nu}, \quad \overline{z_1}, \ldots, \overline{z_\mu}, \quad \overline{\zeta_1}, \ldots, \overline{\zeta_\lambda}, \quad \overline{t_1}, \ldots, \overline{t_\kappa}$$

in $\mathbb{C}(W)$ *are geodesically independent then the functions*

$$j(\overline{\tau_1}), \ldots, j(\overline{\tau_\nu}), \quad \wp_1(\overline{z_1}), \ldots, \wp_\mu(\overline{z_\mu}), \quad \exp(\overline{\zeta_1}), \ldots, \exp(\overline{\zeta_\lambda}), \quad \overline{t_1}, \ldots, \overline{t_\kappa}$$

on $W \cap U_X$ *are algebraically independent over* $\mathbb{C}$.

Proof. This follows exactly the deduction of Theorem 1.6 from Theorem 6.8 in Theorem 9.1 above. $\square$

We can take additional variables t_i that are set equal to any of the τ_j , z_j, ζ_j whose images in $\mathbb{C}(W)$ are algebraically independent of $\overline{t_1}, \ldots, \overline{t_\kappa}$ over $\mathbb{C}$, giving (assuming geodesic independence) the algebraic independence of the functions $j(\overline{\tau_a})$, $\wp_b(\overline{z_b})$, $\exp(\overline{\zeta_c})$ over the algebraic function field (over $\mathbb{C}$) generated by their arguments. The corresponding statement can be proved for modular functions. Following 9.4 shows that Theorem 8.2 may also be deduced from Theorem 9.6, and so these statements are essentially equivalent.

Note added in revision. In forthcoming work by the author the conclusion of Theorem 9.6 is strengthened to include the algebraic independence of $j'(\overline{\tau_1}), \ldots, j'(\overline{\tau_\nu})$ and $j''(\overline{\tau_1}), \ldots, j''(\overline{\tau_\nu})$ in addition to the exhibited functions. This is the natural strengthening, given that $j, j', j'' : \mathbb{H} \to \mathbb{C}$ are algebraically independent while $j''' \in \mathbb{Q}(j, j', j'')$; see [53], [11].

10. Basic pre-special components

Here we show that, given $V \subset X$ as in Theorem 1.1, there are only a finite number of basic quasi-pre-special loci that have a translate that is a maximal quasi-pre-special locus contained in Z. This is evidently implied by Theorem 1.1, and though much weaker it enables an inductive proof of Theorem 1.1. As observed in [71], for abelian varieties this follows by relatively elementary considerations given in [17, Lemma 2]. Our argument is also quite elementary, but uses o-minimality. The most arduous part is spelling out the required new definition. The argument hinges on the simple observation that *a countable definable set is a finite set*, and hence *a definable set all of whose points are rational* (*or even algebraic*) *is finite.*

10.1. *Definition.* 1. Let $n \geq 0$ and let $Y_1 = \Gamma_1\backslash\mathbb{H}, \dots, Y_n = \Gamma_n\backslash\mathbb{H}$ be modular curves. Let $S_0 \cup S_1 \cup \dots \cup S_w$ be a disjoint partition of $\{1, \dots, n\}$ with $w \geq 0$ and S_0 only permitted to be empty. Let $h_i \in \mathbb{H}$ for each $i \in S_0$ be an arbitrary point. Let s_i be the smallest element of S_i for each $i \geq 1$ and for each $j \in S_i, j \neq s_i$, choose an element $g_{ij} \in \mathrm{SL}_2(\mathbb{R})$. A *linear subvariety* of $\mathbb{H}^n$ is a subvariety

$$Y = \{(\tau_1, \dots, \tau_n) \in \mathbb{H}^n : \tau_i = h_i, i \in S_0, \tau_j = g_{ij}(\tau_{s_i}), i = 1, \dots, k, j \in S_i, j \neq s_i\}$$

for some choice of data S_i, h_i, g_{ij} as indicated. The union of gY over $g \in \Gamma_1 \times \dots \times \Gamma_n$ we call a *linear locus for* $\mathbb{H}^n$ (*with respect to* $\Gamma_1 \times \dots \times \Gamma_n$) If S_0 is empty, we will call the corresponding linear subvariety (locus) *basic*. The data $\{1, \dots, m\} - S_0, g_{ij}$ determine a basic linear component of the product of upper half-planes in the variables indexed by $\{1, \dots, m\} - S_0$, and we will say that the linear locus with data S_i, h_i, g_{ij} is the *translate by* $h_i, i \in S_0$ of the basic linear locus (in the reduced set of coordinates) specified by $\{1, \dots, n\} - S_0, g_{ij}$. Note that, in specifying the locus, the h_i, g_{ij} are not uniquely determined.

2. Let Λ be a lattice in $\mathbb{C}^m$. A *linear subvariety* of $\mathbb{C}^k$ is a subvariety of the form

$$a + L,$$

where L is a $\mathbb{C}$-linear subspace of $\mathbb{C}^m$ (i.e. through the origin) and $a \in \mathbb{C}^m$. With the same conditions we call $a + L + \Lambda$ a *linear locus in* $\mathbb{C}^m$ (*with respect to* Λ). If $a + L = L + \Lambda$, we call the corresponding linear locus *basic*, and we will refer to an arbitrary linear locus $a + L + \Lambda$ as the *translate by* a of the basic linear locus $L + \Lambda$. (Note a is only determined up to elements of $L + \Lambda$.)

3. Let $n \geq 0$. A *linear component in* $\mathbb{C}^\ell$ is (just as above) a subvariety of the form

$$b + M,$$

where M is a $\mathbb{C}$-linear subspace, and $b \in \mathbb{C}^\ell$ is arbitrary. With the same conditions, a *linear locus in* $\mathbb{C}^\ell$ (*with respect to* exp) is a locus of the form $b + M + 2\pi i\mathbb{Z}$. If $b + M + 2\pi i\mathbb{Z} = M + 2\pi i\mathbb{Z}$, then we call the corresponding linear locus *basic*, and we refer to a linear locus $b + M + 2\pi i\mathbb{Z}$ as the *translate by* b of the basic linear locus $M + 2\pi i\mathbb{Z}$. (So in specifying a translation $b + M + 2\pi i\mathbb{Z}$ of the basic linear locus $M + 2\pi i\mathbb{Z}$ the translation b is determined only up to $M + 2\pi i\mathbb{Z}$.)

4. Let $n, \ell \geq 0$ and A an abelian variety of dimension $m \geq 0$. Let $X = Y_1 \times \cdots \times Y_n \times A \times \mathbb{G}^\ell$ where $Y_i = \Gamma_i \backslash \mathbb{H}$. A *linear subvariety* in $U_X = \mathbb{H}^n \times \mathbb{C}^m \times \mathbb{C}^\ell$ is a subvariety of the form

$$Y \times (a + L) \times (b + M),$$

where Y is a linear subvariety of $\mathbb{H}^n$, $a + L$ is a linear subvariety in $\mathbb{C}^m$, and $b + M$ is a linear subvariety in $\mathbb{C}^\ell$. With the same conditions we call the union of $g(Y \times (a + L) \times (b + M))$ over $g \in \Gamma_X$ a *linear locus in X* (*with respect to* Γ_X). If the set of variables S_0 in the data for Y is empty, $a + L = L + \Lambda$, and $b + M + 2\pi i\mathbb{Z} = M + 2\pi i\mathbb{Z}$, then we call the linear locus *basic*. We note that the data for a locus are not uniquely determined. We refer to an arbitrary linear locus as the *translate by* $(h_i, i \in S_0, a, b)$ of the corresponding basic linear locus. We note that the h_i are not uniquely determined.

10.2. PROPOSITION. *Let $X = Y_1 \times \cdots \times Y_n \times A \times \mathbb{G}^\ell$, where $n, \ell \geq 0$, $Y_i = \Gamma_i \backslash \mathbb{H}$ are modular curves and A is an abelian variety of dimension $m \geq 0$ defined over $\mathbb{C}$. Let $V \subset X$ be a subvariety and $\mathcal{Z} = \pi^{-1}(V)$. Then there are only finitely many basic quasi-pre-special loci having a translate that is a maximal quasi-pre-special locus contained in $\mathcal{Z}$.*

Proof. Let $Z = \mathcal{Z} \cap \mathbb{F}_X$. If Y is a quasi-pre-special locus contained in $\mathcal{Z}$, consisting of translates of a basic pre-special subvariety B, then it has a component subvariety that intersects Z in full dimension. Further, Y and B are linear. Conversely, if Y is a linear locus contained in $\mathcal{Z}$, a union of translates of a basic linear subvariety B, then it has a component subvariety that intersects Z in full dimension, and B and the components of such Y are algebraic components of $\mathcal{Z}$. Therefore the set of basic pre-special subvarieties that have a translate maximal among quasi-special subvarieties contained in $\mathcal{Z}$ coincides with the set of basic linear subvarieties that have a translate maximal among linear subvarieties contained in $\mathcal{Z}$. The sets of linear subvarieties and basic linear subvarieties are semialgebraic (a product of copies of $\mathrm{SL}_2(\mathbb{R})$ and certain Grassmann varieties), hence definable, and the set M of basic linear subvarieties that have a translate occurring maximally among linear subvarieties contained in $\mathcal{Z}$ is a definable subset (there is always a translate intersecting the definable set Z in full dimension). However, these maximal linear subvarieties are quasi-pre-special, so correspond to algebraic points (the corresponding points in $\mathrm{SL}_2(\mathbb{R})$ are in the image of $\mathrm{GL}_2(\mathbb{Q})$, the points in the Grassmann varieties are rational in suitable coordinates). Thus the definable set M consists entirely of algebraic points, and so is finite. $\square$

10.3. *Remarks.* 1. It follows that, for fixed X, the same conclusion holds over any *definable family* of subvarieties V: there are only finitely many basic pre-special subvarieties that have translates that are contained in $\mathcal{Z}$ for any variety V in the family. This is formally framed in Section 13.

2. Let us briefly compare the method of proof of Proposition 10.2 with the corresponding proof for abelian varieties in [17, Lemma 2]. Both arguments use the fact that the varieties in question (basic pre-special subvarieties/ abelian subvarieties) do not have moduli (in [17, Lemma 1]: only finitely many abelian subvarieties up to a given degree). The argument in [17] leads to a degree bound. Our argument seems to give no information about this (which corresponds to the height of the rational numbers), we get only a bound for the number of them. The argument in [17] looks more likely to yield an effective version. Effectivity would be needed for Proposition 13.5.

11. **Proof of Theorem 1.1**

We begin with an intermediate version of the AOMML statement that assumes that only finitely many maximal special subvarieties of positive dimension are contained in V.

11.1. *Definition.* Let $V \subset X$. The *special set* of V, which we denote V^{sp}, is the union of special subvarieties of positive dimension contained in V.

Special subvarieties are algebraic varieties (irreducible). If V^{sp} consists of a *finite* union of special subvarieties, then it is an algebraic variety (generally reducible). Otherwise it is not, as a variety cannot consist of infinitely many irreducible components. Special subvarieties are defined over $\overline{\mathbb{Q}}$. If X is defined over a numberfield K, then the conjugate over K of a special subvariety is again a special subvariety. If X and V are defined over K, then such a conjugate is again contained in V, so that V^{sp} is defined over K as well.

11.2. Theorem. *Suppose X is as in the hypotheses of Theorem* 1.1 *and that V is a subvariety of X defined over a number field K that contains a field of definition for X. Suppose that V^{sp} is a variety. Then $V - V^{\mathrm{sp}}$ contains only finitely many special points.*

Proof. Let $\mathcal{Z} = \pi^{-1}(V)$. Then $\mathcal{Z}^{\mathrm{alg}}$ consists of $\mathcal{Z}^{\mathrm{ps}} = \pi^{-1}(V^{\mathrm{sp}})$ together with other quasi-pre-special loci that contain no pre-special points. Put $Z^{\mathrm{ps}} = \mathcal{Z}^{\mathrm{ps}} \cap \mathbb{F}$. We have $Z^{\mathrm{alg}} = \mathcal{Z}^{\mathrm{alg}} \cap \mathbb{F}$. If we let

$$N_2^{\mathrm{prespecial}}(W, T)$$

denote the number of pre-special points in a set W up to height T, then, for all $\epsilon > 0$ and $T \geq 1$, we have

$$N_2^{\mathrm{prespecial}}(Z - Z^{\mathrm{ps}}, T) = N_2^{\mathrm{prespecial}}(Z - Z^{\mathrm{alg}}, T) \leq N_2(Z - Z^{\mathrm{alg}}, T) \leq c(Z, 2, \epsilon)T^{\epsilon},$$

where $c(Z, 2, \epsilon)$ is provided by Theorem 3.2. Suppose that $Z - Z^{\mathrm{ps}}$ contains a pre-special point u of complexity $\Delta = \Delta(u)$. Then $x = \pi(u) \in V - V^{\mathrm{sp}}$ is special and has at least

$$[K : \mathbb{Q}]^{-1} c_{\text{degree}}(X)\Delta^{1/7}$$

conjugates x' which also lie in $V - V^{\text{sp}}$. These conjugates have distinct pre-images $u' \in Z - Z^{\text{ps}}$, having complexity

$$\Delta(u') = \Delta(u) = \Delta,$$

and hence

$$H(u') \leq c_{\text{height}}(X)\Delta$$

by Proposition 5.7. Put $T = c_{\text{special}}(X)\Delta$. Then (finally opposing the upper bound from o-minimality with the lower bound from Galois conjugates) we have

$$[K : \mathbb{Q}]^{-1} \frac{c_{\text{special}}(X)}{c_{\text{height}}(X)^{1/7}} T^{1/7} \leq N_2(Z - Z^{\text{ps}}, T) \leq c(Z, 2, \epsilon)T^{\epsilon}$$

and, choosing $\epsilon = 1/8$ (say), the inequalities are untenable once T, and hence Δ, is sufficiently large. Hence $\Delta(u)$ is bounded for a prespecial point in $Z - Z^{\text{ps}}$, and the special points of $V - V^{\text{sp}}$ come from a finite set. $\square$

11.3. *Proof of Theorem* 1.1. There is a subvariety $\overline{V} \subset V$ (not necessarily irreducible), defined over $\overline{\mathbb{Q}}$, that contains all the algebraic points of V. So we may assume that V is defined over $\overline{\mathbb{Q}}$.

We prove the theorem by induction on $\dim X$ as a complex variety. The result clearly holds if $\dim X = 1$, since we have then $V = X$ or V is a finite set of points. We can also argue directly that it holds if $\dim X = 2$, for then V, if proper, has dimension ≤ 1 and can contain only finitely many components of dimension 1, so that V^{sp} is certainly a subvariety and the conclusion holds by Theorem 11.2.

Let then X be of dimension $n \geq 3$, and $V \subset X$. Since the conclusion holds by Theorem 11.2 if V^{sp} is a variety, it suffices to prove this for $V \subset X$ under the inductive assumption that Theorem 1.1 holds for all X' of smaller dimension. So we may assume that V is a proper subvariety of X.

Now there are just finitely many basic special subvarieties whose translates occur as maximal special subvarieties. So it suffices to show that, given a basic special subvariety $Y \times B \times H$ of positive dimension, that there are only finitely many translates of it that occur as maximal *special* subvarieties of V.

Suppose B has dimension $h \leq m$. Then we can choose h of the elliptic curves, which we may assume to be $E_1, \ldots, E_h$, such that B is the image under

$$\mathbb{C}^m \to E_1 \times \cdots \times E_m$$

of a basic pre-special component L of the form

$$L = \Big\{(z_1, \ldots, z_m) : z_j = \sum_{i=1}^{h} q_{ij} z_i, \quad j = h+1, \ldots, m\Big\},$$

where $q_{ij} \in \mathbb{C}$ and there exist nonzero integers N_{ij} are such that $N_{ij}q_{ij}\Lambda_i \subset \Lambda_j$. A translate of B inside $E_1 \times \cdots \times E_m$ is the image of some

$$L + (a_1, \ldots, a_h, a_{h+1}, \ldots, a_m) \subset \mathbb{C}^m,$$

and if $(a_1, \ldots, a_m)$ is a torsion point with respect to $\Lambda_1 \oplus \cdots \oplus \Lambda_m$, then $\phi(a_1, \ldots, a_h) = (a'_{h+1}, \ldots, a'_m)$ is torsion in $\mathbb{C}^{m-h}$ with respect to $\Lambda_{h+1} \oplus \cdots \oplus \Lambda_m$, and so the same translate is given by

$$L + (0, \ldots, 0, a_{h+1} - a'_{h+1}, \ldots, a_m - a'_m).$$

Therefore: the translate of B by a torsion point of $E_1 \times \cdots \times E_m$ is equal to translate of B by a torsion point of the form $(0, \ldots, 0, a_{h+1}, \ldots, a_m)$.

Similarly, suppose H has dimension $p \leq \ell$. We can choose p of the factors of $\mathbb{C}^\ell$, say the first p, such that H is the image of a basic pre-special component of $\mathbb{C}^\ell$ defined as in Definition 6.5. Then we see that a translate gH can be given in the form $g'H$, where $g' = (1, \ldots, 1, g'')$ for some $g'' \in \mathbb{G}^{\ell-p}$, and that if g is torsion, we can take g'' to be torsion.

A translate of Y is given by some element $s \in \mathbb{C}^{\#S_0}$ for the S_0 in the underlying partition.

The variety

$$X' = \mathbb{C}^{\#S_0} \times E_{h+1} \times \cdots \times E_m \times \mathbb{G}^{n-\ell}$$

parametrizes the possible "translations" of the basic special subvariety $Y \times B \times H$, and might be termed the "quotient" of X by $Y \times B \times H$. The set of points

$$V' = \{(s, a, g) \in X' : \text{ the translate of } Y \times B \times H \text{ by } (s, a, g) \text{ is contained in } V\}$$

is an algebraic subvariety $V' \subset X'$, defined over $\overline{\mathbb{Q}}$. The translates of $Y \times B \times H$ which are maximal special subvarieties of V are the special points of $V' - (V')^{\mathrm{sp}}$. However, X' has lower dimension than X, as $Y \times B \times H$ has positive dimension, and so by induction we have that V' has only finitely many such special points.

Therefore only finitely many translates of $Y \times B \times H$ occur as maximal special subvarieties of V, and since there are only finitely many possibilities for $Y \times B \times H$ we see that V^{sp} is a subvariety. But then the conclusion of Theorem 1.1 holds for it by Theorem 11.2, and the proof is complete. $\square$

11.4. *Remarks.* 1. One may observe that Theorem 1.1 holds more generally when the factors $Y_i = \Gamma_i \backslash \mathbb{H}$ are quotients by finite index subgroups Γ_i of $\mathrm{SL}_2(\mathbb{Z})$, where a "special subvariety" means just the image in Y_i of a pre-special subvariety in $\mathbb{H}^n$.

2. In work in progress I affirm AO unconditionally for the product of two Shimura curves associated to indefinite quaternion algebras over $\mathbb{Q}$ (under GRH this is due to Yafaev [95]). In view of this one can reasonably aspire in

the first instance to replace the $\mathbb{C}$ factors in Theorems 1.1 and 12.1 by Shimura curves (one need only provide the Ax-Lindemann-Weierstrass statement). One can of course seek to adapt the present methods much more generally. However, suitable lower bounds for the degree of special points are not presently available in general (apparently even under GRH; see [96]). A conjectural strengthening of Theorem 3.2 for sets definable in $\mathbb{R}_{\exp}$ proposed by Wilkie (in [70]) could, if extended to an o-minimal structure containing the j-function, enable the proofs to go through using substantially weaker lower bounds for the degree of special points. For some discussion of this conjecture, see [69]. Definability results generalizing [64] would also be required in general (though in several interesting cases such as nonmodular Shimura curves they would not be needed).

11.5. *Note added in revision.* Peterzil-Starchenko [66] have generalized their result [64] on the Weierstrass $\wp$-function to show definability in $\mathbb{R}_{\text{an exp}}$ for theta-functions (in both sets of variables) restricted to suitable fundamental domains.

12. AOMM for $\mathbb{C}^n \times A$

12.1. THEOREM. *Let $X = Y_1 \times \cdots \times Y_n \times A$, where $n \geq 0$, $Y_i = \Gamma_i \backslash \mathbb{H}$ are modular curves, and A is an abelian variety of dimension $m \geq 0$ defined over $\overline{\mathbb{Q}}$. Let $V \subset X$ be a subvariety. Then V contains only a finite number of maximal special subvarieties.*

Proof. We may assume that V is defined over $\overline{\mathbb{Q}}$. Let $U = U_X$ and $\mathcal{Z} = \pi^{-1}(V)$. Repeating the start of the proof of Theorem 6.8, we see that for a maximal algebraic component of $\mathcal{Z}$ the z variables are independent of the τ variables, and the dependencies among τ variables are of quasi-pre-special form. By the results of [71], the dependencies among the z variables are also of quasi-pre-special form, so Y is quasi-pre-special.

By Proposition 10.2 there are only finitely many basic quasi-special subvarieties having translates that are maximal among translates of quasi-linear subvarieties contained in Z.

We now repeat the proof procedure of Section 11. First, if V^{sp} is a variety, the result holds by comparing the upper and lower estimates for pre-special points in $Z = \mathcal{Z} \cap \mathbb{F}_X$. Finally we prove inductively that V^{sp} is indeed a subvariety, as we need consider only translates of a finite number of basic special subvarieties, for which the problem reduces to special points on a lower dimensional set of the same form. Here the translates of a basic special subvariety (i.e. abelian subvariety) B of A are parametrized by A/B. $\square$

12.2. *Remarks.* 1. The characterization of maximal algebraic components of $\mathcal{Z}$ in the course of the proof of Theorem 12.1 can also be phrased as an ALW statement relative to a suitable notion of "geodesic independence".

2. Peterzil and Starchenko [65] have extended a (simplification) of the method of [71] to prove MM for semiabelian varieties S over $\overline{\mathbb{Q}}$, in the course of which they reprove in effect the ALW part of Ax-Schanuel for semi-abelian varieties by o-minimal methods. It seems likely that by combining the various approaches one can encompass both Theorems 1.1 and 12.1 in a result for varieties $X = Y_1 \times \cdots \times Y_n \times S$.

13. Uniformity and effectivity issues

Let X be as in Theorem 1.1. We may consider X to be embedded as a quasi-projective variety in some projective space $\mathbb{P}^N$. If V is a subvariety of X, we denote by $d(V)$ the degree of V, meaning the degree of its Zariski closure $\overline{V}$ as a subvariety of $\mathbb{P}^N$, where, for a reducible projective variety W we take $d(W)$ to be the sum of the degrees of its irreducible components. If a subvariety V is defined over $\overline{\mathbb{Q}}$, let $\delta(V)$ denote the minimal degree over $\mathbb{Q}$ of a field of definition for V.

For definability purposes we identify $\mathbb{P}^N$ with a subset of unit length elements in $\mathbb{C}^{N+1}$, which is real semi-algebraic in real coordinates given by real and imaginary parts. By a *definable family of subvarieties of* X we mean a definable family $\mathbf{V}$ whose fibres are relatively closed complex subvarieties of X. We do not insist that the parameter space be complex, though in the cases of interest it will be. Thus such $\mathbf{V} \subset \mathbb{P}^N \times \mathbb{R}^\nu$ with $V = \mathbf{V}_y$ a subvariety of X for each $y \in \mathbb{R}^\mu$. Then the subvarieties $V \subset X$ of given degree form a definable family of subvarieties (their dimension being bounded by $\dim X$).

13.1. PROPOSITION. *Let $X = Y_1 \times \cdots \times Y_n \times A \times \mathbb{G}^\ell$, where $n, \ell \geq 0$, $Y_i = \Gamma_i \backslash \mathbb{H}$ are modular curves and A is an abelian variety of dimension $m \geq 0$ defined over $\mathbb{C}$. Let $\mathbf{V}$ be a definable family subvarieties of X. Then the set of basic pre-special subvarieties Y of X having a translate that is maximal among quasi-pre-special subvarieties of $\mathcal{Z} = \pi^{-1}(V)$ for some $V \in \mathbf{V}$ is a finite set.*

Proof. Since $\mathbf{V}$ is a definable family, the set of such Y is a definable subset of the appropriate Grassmannian parametrizing basic linear subvarieties of X. As in Proposition 10.2, it consists entirely of algebraic points, and so must be a finite set. □

Let us call a variety X as in Theorem 1.1 a variety of AOMML *type*. If Y is a basic special subvariety of a variety X of AOMML type, then the translates of Y in X are prameterized by another variety X_Y of AOMML type (possibly empty), and, as in Section 11.3, we may take X_Y to be a product over some subset of the constituent varieties of X. (The parametrization is not unique: there may be several (but finitely many) $y \in X_Y$ giving the same translate of Y in X.) Such an X_Y will be called an AOMML *subvariety* of X, and the translate of a basic Y by a point $a \in X_Y$ will be denoted $\mathrm{tr}(Y, a) \subset X$.

We may think of a special point in X as a translate of the trivial basic special subvariety, which we denote $\mathbf{0}$, consisting of the trivial subgroup of any elliptic and multiplicative factors of X and the empty subset of the modular variables.

13.2. THEOREM. *Let X be variety as in Theorem* 1.1 *and $\mathbf{V}$ a definable family of subvarieties of X. Let δ be a positive integer. There is a finite family $\mathcal{Y}$ of basic special subvarieties Y of X, and for each $Y \in \mathcal{Y}$ there is an* AOMML *subvariety X_Y of X and a constant $C(X, \mathbf{V}, \delta, Y)$ with the following property. Let V be a variety in the family $\mathbf{V}$ with $\delta(V) \leq \delta$, $Y \in \mathcal{Y}$, and $a \in X_Y$ a special point. Suppose that* $\mathrm{tr}(\mathrm{Y}, \mathrm{a})$ *is a maximal special subvariety of V; then*

$$\Delta(a) \leq C(X, \mathbf{V}, \delta, Y), \quad \text{and} \quad \delta(\mathrm{tr}(\mathrm{Y}, \mathrm{a})) \leq C(X, \mathbf{V}, \delta, Y).$$

In particular, the number of maximal special subvarieties is uniformly bounded for $V \in \mathbf{V}$ with $\delta(V) \leq \delta$.

Proof. We prove the theorem by induction on $\dim X$. It is evidently true if $\dim X = 1$, in which case a subvariety of X is X itself or a finite number of points whose number is uniformly bounded as a consequence of the definability of the family. Suppose then that $\dim X \geq 2$ and the theorem holds for all X of smaller dimension. By Proposition 13.1 there is a finite collection $\mathcal{Y}$ of basic special subvarieties of X containing all those that have a translation that is a maximal quasi-special subvariety of any $V \in \mathbf{V}$. Each $Y \in \mathcal{Y}$ is defined over $\overline{\mathbb{Q}}$. By increasing δ by some bounded factor depending on $\mathbf{V}$ we may assume that X and all the $Y \in \mathcal{Y}$ are defined with V over a number field K of degree $\leq \delta$. Suppose $Y \in \mathcal{Y}$ has positive dimension. Then maxmimal translates of Y in subvarieties $V \subset X$ correspond to maximal special subvarieties of dimension 0 (i.e. special points outside the special set) of a suitable subvariety V' of $X' = X_Y$ as in the proof of Section 11.3. For $V \in \mathbf{V}$ the subvarieties V' form a definable family, and $\dim X' < \dim X$, and since X' and V' may be defined over K, we get by induction a bound $C(X', \mathbf{V}', \delta, \mathbf{0})$ for the complexity of maximal translates of the trivial basic special subvariety, i.e. the translates of Y that occur maximally in V, and for the degree of a field of definition for them over $\mathbb{Q}$. This gives a uniform bound on the complexity and degree (over $\mathbb{Q}$) of V^{sp} for $V \in \mathbf{V}$. Now the proof of Theorem 11.2 gives a uniform upper bound on the complexity of a translate of the trivial basic subvariety in V, and thence on the complexity and degree over $\mathbb{Q}$ of a field of definition for all special subvarieties of $V \in \mathbf{V}$. □

For MM for semi-abelian varieties (even for commutative algebraic groups) defined over a number field, explicit uniform bounds are given by Hrushovski [43]. Explicit uniform bounds for the number of special subvarieties in ML for an abelian variety (over $\overline{\mathbb{Q}}$) are given by Rémond [78]. For explicit bounds for ML in $\mathbb{G}^{\ell}$, see Evertse [37].

13.3. *Aside*. Let $V \subset \mathbb{P}^N$ defined over $\mathbb{C}$. Then V has a maximal subvariety $\overline{V}$ defined over $\overline{\mathbb{Q}}$ whose total degree (sum of degrees of components) is bounded in terms of $d(V)$ and N. This can be phrased as an analogue of AO/MM: if we call irreducible subvarieties defined over $\overline{\mathbb{Q}}$ *special*, then V contains finitely many maximal special subvarieties, whose number and complexity (=degree) are uniformly bounded in terms of $d(V)$ and the ambient space $\mathbb{P}^N$. If V is in a definable family, then $\overline{V}$ also lies in a definable family, and the conclusion of Theorem 13.2 holds for all $V \in \mathbf{V}$ with $\delta = \delta(\overline{V})$.

Let us finally make some comments on effectivity. The question arises whether the ineffective lower bound for class numbers is the only ineffective element in the proof. The upper bound for rational points, which comes via the reparametrization in [70], would seem to be effective if one has *effective o-minimality* of the structure involved, as defined in Berarducci-Servi [7]. It seems an interesting — by no means trivial — problem to establish effective o-minimality for the structure $\mathbb{R}_j$ generated by the graph of the modular invariant j on its fundamental domain considered as a subset of $\mathbb{R}^4$. Note that $\mathbb{R}_{\exp} \subset \mathbb{R}_j$ by a result of Miller [57]. The result of Peterzil-Starchenko establishes the o-minimality of this structure by showing it is contained in $\mathbb{R}_{\text{an,exp}}$, but the latter is too big to expect any reasonable form of effectivity. In $\mathbb{R}_j$, the definable sets lie in countably many families that are definable without parameters, and one would try to bound the number of connected components of a set X in such a family (or the somewhat finer invariant $\gamma(X)$ of [7]) by an effective function of the defining formula.

Siegel's lower bound for class numbers can be made effective if one admits one possible exceptional quadratic field (Tatuzawa [87] see e.g. [39]): if $\epsilon > 0$, there is an effective constant $c(\epsilon) > 0$ such that

$$h(D) \geq c(\epsilon)|D|^{1/2-\epsilon}$$

for all negative discriminants D except possibly those corresponding to orders in one imaginary quadratic field. (I thank a referee for the suggestion to explore the consequences of this result.)

13.4. *Definition*. For $\epsilon > 0$, an *ϵ-restricted special point* in a product of modular curves $Y_1 \times \cdots \times Y_n$ will mean a special point such that each pre-special coordinate is not in the quadratic field that is exceptional for ϵ in a bound as above. An ϵ-restricted special subvariety is a special subvariety such that all the special points in the defining data are ϵ-special.

Equivalently, an ϵ-restricted special subvariety is a subvariety that contains at least one (equivalently a Zariski-dense set of) ϵ-restricted special points.

If we now assume that Theorem 3.6 (for $\mathbb{R}_j$) and Proposition 13.1 (for $X = Y_1 \times \cdots \times Y_n$) can be made effective, then we get an (unconditional and) effective version of Theorem 13.2 for ϵ-restricted special points.

13.5. Proposition. *Suppose an effective version of Theorem* 3.6 *for sets definable in* $\mathbb{R}_j$, *and an effective version of Proposition* 13.1 *for algebraic families of subvarieties of products of modular curves. Let* X *be a product of modular curves*, $V \subset X$ *defined over* $\overline{\mathbb{Q}}$, *and* $\epsilon > 0$. *Then there is an effective upper bound on the number* (*and complexity*) *of maximal* ϵ*-restricted special subvarieties of* V. *Moreover*, *this bound depends only on* $X, \epsilon, d(V), \delta(V)$.

For special points and subvarieties corresponding to any fixed given quadratic field, one has effective lower bounds for the class number of orders, and the result would also be effective under the assumptions of Proposition 13.5. Note that in this case the results of Edixhoven [34] are unconditional and surely effective as well. Under GRH, the uniformity in the conclusion for curves of fixed degree was observed in [32], and this was shown to be effective and extended to curves in $\mathbb{C}^n$ by Breuer [19].

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(Received: July 21, 2009)
(Revised: April 8, 2010)

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