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Heisenberg uniqueness pairs and the Klein-Gordon equation

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Abstract

A Heisenberg uniqueness pair (HUP) is a pair (Γ, Λ), where Γ is a curve in the plane and Λ is a set in the plane, with the following property: any finite Borel measure μ in the plane supported on Γ, which is absolutely continuous with respect to arc length, and whose Fourier transform $\widehat{\mu}$ vanishes on Λ, must automatically be the zero measure. We prove that when Γ is the hyperbola $x_1x_2 = 1$ and Λ is the lattice-cross

$$\Lambda = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

where α, β are positive reals, then (Γ, Λ) is an HUP if and only if $\alpha\beta \le 1$; in this situation, the Fourier transform $\widehat{\mu}$ of the measure solves the one-dimensional Klein-Gordon equation. Phrased differently, we show that

$$\mathrm{e}^{\pi \mathrm{i}\alpha n t},\ \mathrm{e}^{\pi \mathrm{i}\beta n/t}, \qquad n \in \mathbb{Z},$$

span a weak-star dense subspace in $L^\infty(\mathbb{R})$ if and only if $\alpha\beta \le 1$. In order to prove this theorem, some elements of linear fractional theory and ergodic theory are needed, such as the Birkhoff Ergodic Theorem. An idea parallel to the one exploited by Makarov and Poltoratski (in the context of model subspaces) is also needed. As a consequence, we solve a problem on the density of algebras generated by two inner functions raised by Matheson and Stessin.

1. Introduction

Heisenberg uniqueness pairs. Let μ be a finite complex-valued Borel measure in the plane $\mathbb{R}^2$. Associate to it the Fourier transform

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} \mathrm{e}^{\pi \mathrm{i}\langle x, \xi\rangle} \mathrm{d}\mu(x),$$

where $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2)$, with inner product

$$\langle x, \xi\rangle = x_1\xi_1 + x_2\xi_2.$$

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The Heisenberg uncertainty principle states that μ and $\widehat{\mu}$ cannot both be too concentrated to a point (see [6] for the original paper of Heisenberg, and [5] for a more general treatment); in particular, they cannot both have compact support (unless $\mu = 0$). Here, we shall study a variation on that theme. Let Γ be a smooth curve in $\mathbb{R}^2$, or, more generally, a finite disjoint union of smooth curves. Suppose that $\operatorname{supp}\mu \subset \Gamma$ and that μ is absolutely continuous with respect to arc length measure on Γ. Which sets $\Lambda \subset \mathbb{R}^2$ have the property that

$$\widehat{\mu}|_\Lambda = 0 \qquad \Longrightarrow \qquad \mu = 0?$$

If this is the case, we say that (Γ, Λ) is a *Heisenberg uniqueness pair*. A dual formulation is that (Γ, Λ) is a Heisenberg uniqueness pair if and only if the functions

$$e_\xi(x) = \mathrm{e}^{\pi \mathrm{i}\langle x, \xi\rangle}, \qquad \xi \in \Lambda$$

span a weak-star dense subspace in $L^\infty(\Gamma)$. This concept of Heisenberg uniqueness pairs has many features in common with the notion of (weakly) mutually annihilating pairs of Borel measurable sets having positive area measure, which appears, for instance, in the book by Havin and Jöricke [5].

The properties of the Fourier transform with respect to translation and multiplication by complex exponentials show that for all points $x^*, \xi^* \in \mathbb{R}^2$, we have

$$\text{(inv-1)} \qquad (\Gamma + \{x^*\}, \Lambda + \{\xi^*\}) \quad \text{is an HUP} \quad \Longleftrightarrow \quad (\Gamma, \Lambda) \quad \text{is an HUP},$$

where HUP is short for "Heisenberg uniqueness pair". Likewise, it is also straightforward to see that if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear transformation with adjoint T^*, then

$$\text{(inv-2)} \qquad (T^{-1}(\Gamma), T^*(\Lambda)) \quad \text{is an HUP} \quad \Longleftrightarrow \quad (\Gamma, \Lambda) \quad \text{is an HUP}.$$

Algebraic curves and partial differential equations. Algebraic curves Γ are of particular interest because of their connection to partial differential equations. That connection follows from the observation that for polynomials p of two variables,

$$p\left(\frac{\partial_1}{\pi \mathrm{i}}, \frac{\partial_2}{\pi \mathrm{i}}\right)\widehat{\mu}(\xi) = \int_{\mathbb{R}^2} \mathrm{e}^{\pi \mathrm{i}\langle x, \xi\rangle}\, p(x_1, x_2)\, \mathrm{d}\mu(x),$$

so that if p is real-valued and Γ is the locus of the equation

$$p(x_1, x_2) = 0,$$

then

$$p(x_1, x_2)\, \mathrm{d}\mu(x_1, x_2) = 0$$

identically. Therefore $\widehat{\mu}$ solves the partial differential equation (PDE)

$$p\left(\frac{\partial_1}{\pi \mathrm{i}}, \frac{\partial_2}{\pi \mathrm{i}}\right)\widehat{\mu}(\xi) = 0 \tag{1.1}$$

in the plane. In fact, equation (1.1) encodes the requirement that $\operatorname{supp}\mu \subset \Gamma$.

Conic sections. We shall consider the case when Γ is a *conic section*, that is, the locus of a quadratic equation

$$ax_1^2 + bx_2^2 + cx_1x_2 + dx_1 + ex_2 + f = 0,$$

where a, b, c, d, e, f are real constants. As we only consider the case when Γ is a curve, this leaves us with the following cases: a straight line, two parallel straight lines, a cross, an ellipse, a parabola, or a hyperbola.

The line. Let us look at the line first, as a model example. By the invariance properties (inv-1) and (inv-2), we may assume that $\Gamma = \mathbb{R} \times \{0\}$, the x_1-axis. In this case, $\widehat{\mu}(\xi)$ depends only on ξ_1, and it is easy to see that (Γ, Λ) is a Heisenberg uniqueness pair if and only if $\pi_1(\Lambda)$, the orthogonal projection of Λ to the ξ_1-axis, is dense.

Two parallel lines. If Γ is the union of two parallel lines, we may without loss of generality assume that

$$\Gamma = \mathbb{R} \times \{0, 1\}.$$

In this case, we see from the example of the line that in order for (Γ, Λ) to be a Heisenberg uniqueness pair, it is necessary that $\pi_1(\Lambda)$ be dense. But something more is needed. An absolutely continuous measure μ (with respect to arc length) on Γ may be written in the form

$$\mathrm{d}\mu(x) = f(x_1)\mathrm{d}x_1\mathrm{d}\delta_0(x_2) + g(x_1)\mathrm{d}x_1\mathrm{d}\delta_1(x_2),$$

where $f, g \in L^1(\mathbb{R})$ (δ_y denotes the unit point mass at the point y), so that

$$\widehat{\mu}(\xi) = \widehat{f}(\xi_1) + \mathrm{e}^{\pi \mathrm{i} \xi_2}\widehat{g}(\xi_1).$$

Next, we split

$$\pi_1(\Lambda) = \pi_1^a(\Lambda) \cup \pi_1^b(\Lambda),$$

where the two sets are disjoint: $t \in \pi_1^a(\Lambda)$ if there are two lifted points $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$ in Λ, with $\xi_1 = \eta_1 = t$ and $\xi_2 - \eta_2 \notin 2\mathbb{Z}$, whereas $t \in \pi_1^b(\Lambda)$ if the latter does not happen. We quickly find that

$$\widehat{f}(t) = \widehat{g}(t) = 0, \qquad t \in \pi_1^a(\Lambda). \tag{1.2}$$

On the other hand, for $t \in \pi_1^b(\Lambda)$, the expression $\mathrm{e}^{\pi \mathrm{i} \xi_2}$ is a well-defined function of $\xi_1 = t$, where ξ_2 stands for any of the points with $(\xi_1, \xi_2) \in \Lambda$; we write $\chi(t)$ for this unimodular function. If E is a closed subset of $\mathbb{R}$ and $t_0 \in E$, we say that a function $\varphi : E \to \mathbb{C}$ is *locally the Fourier transform of an* $L^1(\mathbb{R})$ *function around* t_0 provided that there exists a small open interval I around t_0 and a function ψ which is the Fourier transform of an $L^1(\mathbb{R})$ function, such that $\psi = \varphi$ on $E \cap I$. Let $\pi_1^c(\Lambda)$ consist of those points $t_0 \in \pi_1^b(\Lambda)$ where $\chi : \pi_1^b(\Lambda) \to \mathbb{C}$ is locally the Fourier transform of an $L^1(\mathbb{R})$ function around t_0.

THEOREM 1.1. (Γ, Λ) *is a Heisenberg uniqueness pair if and only if* $\pi_1^a(\Lambda) \cup (\pi_1^b(\Lambda) \setminus \pi_1^c(\Lambda))$ *is dense in* $\mathbb{R}$.

Proof. We observe that

$$\widehat{f}(t) = -\chi(t)\widehat{g}(t), \qquad t \in \pi_1^b(\Lambda). \tag{1.3}$$

If $t \in \pi_1^b(\Lambda) \setminus \pi_1^c(\Lambda)$, this is only possible if $\widehat{g}(t) = 0$, so that

$$\widehat{f}(t) = \widehat{g}(t) = 0, \qquad t \in \pi_1^b(\Lambda) \setminus \pi_1^c(\Lambda). \tag{1.4}$$

A combination of (1.2) and (1.4) shows that $f = g = 0$ (so that $\mu = 0$) if the set $\pi_1^a(\Lambda) \cup (\pi_1^b(\Lambda) \setminus \pi_1^c(\Lambda))$ is dense in $\mathbb{R}$.

As for the other direction, suppose that $\pi_1(\Lambda)$ is dense in $\mathbb{R}$, while $\pi_1^a(\Lambda) \cup (\pi_1^b(\Lambda) \setminus \pi_1^c(\Lambda))$ fails to be dense in $\mathbb{R}$. We then pick a point $t_0 \in \mathbb{R}$ such that an open interval J around it has empty intersection with

$$\pi_1^a(\Lambda) \cup (\pi_1^b(\Lambda) \setminus \pi_1^c(\Lambda)).$$

But then $\pi_1^c(\Lambda) \cap J$ is dense in J, and χ is locally the Fourier transform of an $L^1(\mathbb{R})$ function around t_0. We thus find a function χ_1 which coincides with χ on some open interval $I \subset J$ with $t_0 \in I$, while χ_1 is the Fourier transform of an $L^1(\mathbb{R})$ function. Next, we pick $g \in L^1(\mathbb{R})$ with $\widehat{g}(t_0) \neq 0$, such that $\operatorname{supp} \widehat{g} \Subset I$, and define $f \in L^1(\mathbb{R})$ via $\widehat{f} = -\chi_1 \widehat{g}$, so that (1.3) holds. This gives us a nontrivial measure μ with the required properties, and so (Γ, Λ) cannot be a Heisenberg uniqueness pair. □

The cross. If Γ is a cross, the PDE (1.1) expresses the wave equation. By the invariance properties (inv-1) and (inv-2), we may restrict our attention to the case when

$$\Gamma = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$$

is the union of the two axes. Here, it appears that the characterization of uniqueness pairs (Γ, Λ) may get quite complicated. Obviously, it is a necessary condition that $\pi_1(\Lambda)$ and $\pi_2(\Lambda)$ be dense ($\pi_2(\Lambda)$ is the orthogonal projection to the ξ_2-axis). This is far from sufficient, because if Λ is contained in a smooth graph, we may run into trouble. For instance, if Λ is contained in the diagonal $\xi_1 = \xi_2$, then we may choose

$$\mathrm{d}\mu(x_1, x_2) = f(x_1)\, \mathrm{d}x_1 \mathrm{d}\delta_0(x_2) - f(x_2)\, \mathrm{d}x_2 \mathrm{d}\delta_0(x_1),$$

where $f \in L^1(\mathbb{R})$, which is supported on Γ and nontrivial generically, while $\widehat{\mu}(\xi_1, \xi_2) = 0$ for $\xi_1 = \xi_2$.

The ellipse. If Γ is an ellipse, the invariances (inv-1) and (inv-2) allow us to focus on the circle

$$\Gamma = \{x = (x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 = 1\}.$$

The corresponding PDE (1.1) is the Helmholtz equation (the eigenvalue equation for the Laplacian). Here, the fact that Γ is compact entails that $\widehat{\mu}(\xi)$ extends to an entire function of exponential type in $\mathbb{C}^2$. It would seem that reasonable criteria on Λ may be found that are at least close to being necessary and sufficient for (Γ, Λ) to be a Heisenberg uniqueness pair.

The parabola. If Γ is a parabola, the invariances (inv-1) and (inv-2) allow us to focus on the parabola

$$\Gamma = \{x = (x_1, x_2) \in \mathbb{R}^2 : \; x_2 = x_1^2\}.$$

The corresponding PDE (1.1) is the one-dimensional Schrödinger equation without potential. Here, the problem of characterizing the Heisenberg uniqueness pairs (Γ, Λ) appears quite challenging.

The hyperbola. We shall focus most of our attention to the case when Γ is a hyperbola. The corresponding PDE (1.1) is the one-dimensional Klein-Gordon equation. We will see that the situation with Heisenberg uniqueness pairs is dramatically different from that of the cross. By the invariances (inv-1) and (inv-2), we may assume that the hyperbola is given by

$$x_1 x_2 = 1.$$

THEOREM 1.2. *Suppose Γ is the hyperbola $x_1 x_2 = 1$ and that Λ is the lattice-cross*

$$\Lambda = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

where α, β are positive reals. Then (Γ, Λ) is a Heisenberg uniqueness pair if and only if $\alpha\beta \leq 1$.

The remainder of this work is devoted to proving this assertion. But before we turn to the proof, let us consider a generalization which is more or less immediate.

COROLLARY 1.3. *Suppose Γ_ε is the hyperbola $x_1 x_2 = \varepsilon$, where $\varepsilon \neq 0$ is real, and that Λ is the lattice-cross*

$$\Lambda = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

where α, β are positive reals. Then $(\Gamma_\varepsilon, \Lambda)$ is a Heisenberg uniqueness pair if and only if $\alpha\beta \leq 1/|\varepsilon|$.

The eccentricity of the hyperbola Γ_ε is $\sqrt{2}$ independently of ε. The condition of the corollary ($\alpha\beta \leq 1/|\varepsilon|$) gets weaker as $|\varepsilon|$ decreases. However, in the limit situation $\varepsilon = 0$ — the cross — the situation changes dramatically: if Λ is contained in the dual cross $(\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$, then Λ must actually be dense in the cross for (Γ_0, Λ) to be a Heisenberg uniqueness pair.

Remark 1.4. Consider for a moment the sets

$$\Lambda' = ([\theta, +\infty[\times] - \infty, 0]) \cup (] - \infty, 0] \times [0, +\infty[)$$

and

$$\Lambda'' = \Big\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \ a_1\xi_1 + a_2\xi_2 = 0\Big\},$$

where θ, a_1, a_2 are all real parameters, subject to $\theta > 0$ and $a_1 a_2 > 0$. The set Λ' is arguably more massive than the lattice-cross Λ of Corollary 1.3. Nevertheless, if Γ_ε is as in Corollary 1.3, with ε positive, it can be shown that $(\Gamma_\varepsilon, \Lambda')$ fails to be a Heisenberg uniqueness pair, no matter what positive values ε and θ assume. Analogously, $(\Gamma_\varepsilon, \Lambda'')$ also fails to be a Heisenberg uniqueness pair, for all $\varepsilon > 0$ and $a_1 a_2 > 0$ (however, it can be shown that $(\Gamma_\varepsilon, \Lambda' \cup \Lambda'')$ is a Heisenberg uniqueness pair). This suggests that it is crucial that the points of the lattice-cross Λ of Corollary 1.3 are located along the characteristic directions for the Klein-Gordon equation (the two axes).

We need a result of algebraic nature.

LEMMA 1.5. *Let $z_1, z_2 \in \mathbb{C}$ be two points such that*

$$z_1 - z_2 = am \in a\mathbb{Z}, \quad \frac{1}{z_1} - \frac{1}{z_2} = bn \in b\mathbb{Z},$$

for some positive reals a, b. Then, unless $z_1 = z_2$,

$$z_1 = \frac{am}{2}\left(1 \pm \sqrt{1 - \frac{4}{abmn}}\right), \quad z_2 = z_1 - am.$$

The proof is a simple exercise, and therefore omitted.

Remark 1.6. Let us consider the singular measure $\mu = \delta_u - \delta_v$, where

$$u = (u_1, 1/u_1) \in \Gamma, \qquad v = (v_1, 1/v_1) \in \Gamma.$$

Then

$$\widehat{\mu}(\xi) = \mathrm{e}^{\pi\mathrm{i}(\xi_1 u_1 + \xi_2/u_1)} - \mathrm{e}^{\pi\mathrm{i}(\xi_1 v_1 + \xi_2/v_1)},$$

so that

$$\widehat{\mu}(\xi_1, 0) = \mathrm{e}^{\pi\mathrm{i}\xi_1 u_1} - \mathrm{e}^{\pi\mathrm{i}\xi_1 v_1}, \qquad \widehat{\mu}(0, \xi_2) = \mathrm{e}^{\pi\mathrm{i}\xi_2/u_1} - \mathrm{e}^{\pi\mathrm{i}\xi_2/v_1}.$$

Suppose we try to achieve that

$$\widehat{\mu}(\alpha j, 0) = \widehat{\mu}(0, \beta k) = 0, \qquad j, k \in \mathbb{Z}, \tag{1.5}$$

for some positive reals α, β. We see that this amounts to

$$\mathrm{e}^{\pi\mathrm{i}\alpha u_1} = \mathrm{e}^{\pi\mathrm{i}\alpha v_1}, \qquad \mathrm{e}^{\pi\mathrm{i}\beta/u_1} = \mathrm{e}^{\pi\mathrm{i}\beta/v_1},$$

which we rewrite in the form

$$u_1 - v_1 \in \frac{2}{\alpha}\mathbb{Z}, \qquad \frac{1}{u_1} - \frac{1}{v_1} \in \frac{2}{\beta}\mathbb{Z}.$$

In view of Lemma 1.5, there are plenty of such points $u_1, v_1 \in \mathbb{R}$ with $u_1 \neq v_1$, for any given α, β. This shows that the requirement that the measure μ be absolutely continuous with respect to arc length measure on Γ is essential; without it, Theorem 1.2 would simply not be true.

2. **Dynamics of a Gauss-type map**

A Gauss-type map. In order to prove our main theorem (Theorem 1.2), we will need to study the invariant measures of a particular map. We shall consider a map on the interval $]-1,1]$, which we think of as $\mathbb{R}/2\mathbb{Z}$ (topologically as well). The map in question is defined by $U(0)=0$ and

$$U(x) = \left\{ -\frac{1}{x} \right\}_2, \qquad x \neq 0,$$

where for real t, the expression $\{t\}_2 \in]-1,1]$ is the unique number such that $t - \{t\}_2 \in 2\mathbb{Z}$. The function U is locally strictly increasing and continuous, except for being interrupted by jumps. The map $U :]-1,1] \to]-1,1]$ is associated with *continued fractions with even partial quotients* (see [10], [11], [7], [3]). We see that, for $j = \pm1, \pm2, \pm3, \dots$,

$$U(x) = -\frac{1}{x} + 2j, \qquad \frac{1}{2j+1} < x \leq \frac{1}{2j-1},$$

and hence U maps the interval $]\frac{1}{2j+1}, \frac{1}{2j-1}]$ onto $]-1,1]$ in a one-to-one fashion. The derivative of U is locally

$$U'(x) = \frac{1}{x^2}, \qquad x \in]-1,1] \setminus \frac{1}{2\mathbb{Z}+1}.$$

The point 1 is a fixed point for U, and $U'(1^-) = U'(-1^+) = 1$, which makes 1 a *weakly repelling fixed point.* This means that when we iterate U, once we are close to 1 (which is the same point as -1 in $\mathbb{R}/2\mathbb{Z}$), the successive iterates will remain near 1 for a long time. If $x \in]-1,1]$ is rational, then after a finite number of steps, the U-iterate of x is either 0 or 1 (see, for instance [7]). This illuminates why irrational numbers tend to spend a large portion of their U-orbits near 1.

Invariant measures. If φ is a continuous function on $\mathbb{R}/2\mathbb{Z}$ and ν is a finite complex Borel measure on $]-1,1]$, then the integral

$$\int_{]-1,1]} \varphi(x)\, \mathrm{d}\nu(x) \tag{2.1}$$

is well-defined. However, the integral (2.1) makes sense under weaker assumptions on φ. Suppose E is an open subset of $]-1,1]$ such that the complement $]-1,1] \setminus E$ is countable and that φ is bounded on $]-1,1]$ and continuous on E. Then (2.1) makes sense for φ and we call the function φ *pseudo-continuous.*

We recall the familiar notion that a finite complex Borel measure ν on $]-1,1]$ is U-*invariant* provided that

$$\int_{]-1,1]} \varphi(U(x))\,\mathrm{d}\nu(x) = \int_{]-1,1]} \varphi(x)\,\mathrm{d}\nu(x) \tag{2.2}$$

holds for all pseudo-continuous test functions φ; it is easy to see that $\varphi\circ U$ is pseudo-continuous if φ is pseudo-continuous, so that (2.2) makes sense. We shall reformulate this criterion in more concrete terms. First, we note that

$$\begin{aligned}\int_{]-1,1]\setminus\{0\}} \varphi(U(x))\,\mathrm{d}\nu(x) &= \sum_{j\in\mathbb{Z}^*}\int_{]\frac{1}{2j+1},\frac{1}{2j-1}]} \varphi(U(x))\,\mathrm{d}\nu(x)\\ &= \sum_{j\in\mathbb{Z}^*}\int_{]\frac{1}{2j+1},\frac{1}{2j-1}]} \varphi\biggl(-\frac{1}{x}+2j\biggr)\,\mathrm{d}\nu(x),\end{aligned}$$

where $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$, and that

$$\int_{]\frac{1}{2j+1},\frac{1}{2j-1}]} \varphi\biggl(-\frac{1}{x}+2j\biggr)\,\mathrm{d}\nu(x) = \int_{]-1,1]} \varphi(t)\,\mathrm{d}\nu_j(t),$$

where

$$\mathrm{d}\nu_j(t) = \mathrm{d}\nu\biggl(\frac{1}{2j-t}\biggr), \qquad -1<t\le 1, \tag{2.3}$$

so that we have

$$\int_{]-1,1]\setminus\{0\}} \varphi(U(x))\,\mathrm{d}\nu(x) = \sum_{j\in\mathbb{Z}^*}\int_{]-1,1]} \varphi(t)\,\mathrm{d}\nu_j(t).$$

It follows that ν is U-invariant if and only if

$$\nu = \nu(\{0\})\delta_0 + \sum_{j\in\mathbb{Z}^*} \nu_j. \tag{2.4}$$

More generally, given $\lambda\in\mathbb{C}$, we want to talk about (U,λ)-invariant measures, defined by the requirement that

$$\int_{]-1,1]} \varphi(U(x))\,\mathrm{d}\nu(x) = \lambda\int_{]-1,1]} \varphi(x)\,\mathrm{d}\nu(x) \tag{2.5}$$

holds for all test functions φ; specifically, this means that

$$\lambda\nu = \nu(\{0\})\delta_0 + \sum_{j\in\mathbb{Z}^*} \nu_j. \tag{2.6}$$

It is easy to see that for $|\lambda|>1$, there are no (U,λ)-invariant measures except for the zero measure.

PROPOSITION 2.1. *Suppose that ν is a finite (U,λ)-invariant measure on $]-1,1]$ and write $\nu=\nu_a+\nu_s$, where ν_a is absolutely continuous, while ν_s is singular. Then ν_a and ν_s are also (U,λ)-invariant. Moreover, if $|\lambda|=1$, then $|\nu|$, $|\nu_a|$, and $|\nu_s|$ are all U-invariant measures.*

Proof. The relation (2.6) splits:

$$\nu_a = \lambda \sum_{j\in\mathbb{Z}^*} (\nu_j)_a, \qquad \nu_s = \lambda\bigg(\nu(\{0\})\delta_0 + \sum_{j\in\mathbb{Z}^*} (\nu_j)_s\bigg), \tag{2.7}$$

where the subscripts a and s indicate the absolutely continuous and singular parts, respectively, of the measure in question. We easily realize that $(\nu_j)_a = (\nu_a)_j$ and $(\nu_j)_s = (\nu_s)_j$, so that (2.7) expresses that ν_a and ν_s are both U-invariant.

Next, we suppose $|\lambda| = 1$, and turn to the assertion that $|\nu|$ is U-invariant. Taking absolute values, we have

$$|\mathrm{d}\nu(t)| \le |\nu(\{0\})|\mathrm{d}\delta_0(t) + \sum_{j\in\mathbb{Z}^*} |\mathrm{d}\nu_j(t)|, \tag{2.8}$$

and so

$$\begin{aligned}\int_{]-1,1]} |\mathrm{d}\nu(t)| &\le |\nu(\{0\})| + \sum_{j\in\mathbb{Z}^*}\int_{]-1,1]} |\mathrm{d}\nu_j(t)| \\ &= |\nu(\{0\})| + \sum_{j\in\mathbb{Z}^*}\int_{]\frac{1}{2j+1},\frac{1}{2j-1}]} |\mathrm{d}\nu(t)| \\ &= |\nu(\{0\})| + \int_{]-1,1]\setminus\{0\}} |\mathrm{d}\nu(t)| = \int_{]-1,1]} |\mathrm{d}\nu(t)|.\end{aligned}$$

This is only possible if we have in fact equality in (2.8):

$$|\mathrm{d}\nu(t)| = |\nu(\{0\})|\mathrm{d}\delta_0(t) + \sum_{j\in\mathbb{Z}^*} |\mathrm{d}\nu_j(t)|.$$

This relation expresses that $|\nu|$ is U-invariant; that $|\nu_a|$ and $|\nu_s|$ are U-invariant is a simple consequence of this fact. $\square$

A smooth invariant measure of infinite mass. We now consider the positive σ-finite smooth positive measure

$$\mathrm{d}\omega(x) = \frac{\mathrm{d}x}{1-x^2},$$

which has infinite mass. The criterion (2.4) makes sense although ω is not finite. The following assertion was essentially found by Schweiger [10].

PROPOSITION 2.2. *The measure ω is U-invariant.*

Proof. We supply the simple proof, checking that

$$\mathrm{d}\omega_j(t) = \mathrm{d}\omega\bigg(\frac{1}{2j-t}\bigg) = \frac{\mathrm{d}t}{(2j-t)^2-1}.$$

Since

$$\sum_{j\in\mathbb{Z}^*}\frac{1}{(2j-t)^2-1}=\frac{1}{2}\sum_{j\in\mathbb{Z}^*}\left(\frac{1}{2j-t-1}-\frac{1}{2j-t+1}\right)$$
$$=\frac{1}{2}\left(\frac{1}{1+t}+\frac{1}{1-t}\right)=\frac{1}{1-t^2},$$

we find from (2.4) that ω is U-invariant. □

Schweiger [10] actually focused on the related map $|U|:[0,1]\to[0,1]$ given by $|U|(x)=|U(x)|$. He obtained the following basic result.

Proposition 2.3. *The measure ω is invariant also with respect to $|U|$. Moreover, $|U|$ is ergodic; that is, if $E\subset[0,1]$ is a $|U|$-invariant set, then either $\omega(E)=0$ or $\omega([0,1]\setminus E)=0$.*

Consequences of ergodic theory. The Birkhoff Ergodic Theorem — in this setting of an infinite invariant ergodic measure [1] — states that if φ is Borel measurable and even with

$$\int_{-1}^{1}\frac{|\varphi(t)|}{1-t^2}\,\mathrm{d}t<+\infty,$$

then

$$\frac{1}{N}\sum_{k=0}^{N-1}\varphi(U^{\langle k\rangle}(t))\to 0\quad\text{as } N\to+\infty$$

almost everywhere on $]-1,1]$. Here, $U^{\langle k\rangle}$ stands for the k-th iterate of U. We observe that we do not need to know whether U is ergodic, just that $|U|$ is, if we use that $|U(-x)|=|U(x)|$. We pick $\varphi(t)=1-t^2$ and get:

$$\frac{1}{N}\sum_{k=0}^{N-1}\left(1-|U^{\langle k\rangle}(t)|^2\right)\to 0\quad\text{as } N\to+\infty \tag{2.9}$$

almost everywhere on $]-1,1]$. Suppose ν is a *positive*, *finite*, *and absolutely continuous* U-invariant measure on $]-1,1]$. By the U-invariance,

$$\int_{]-1,1]}\left(1-|U^{\langle k\rangle}(t)|^2\right)\mathrm{d}\nu(t)=\int_{]-1,1]}(1-t^2)\,\mathrm{d}\nu(t) \tag{2.10}$$

and so

$$\int_{]-1,1]}\frac{1}{N}\sum_{k=0}^{N-1}\left(1-|U^{\langle k\rangle}(t)|^2\right)\mathrm{d}\nu(t)=\int_{]-1,1]}(1-t^2)\,\mathrm{d}\nu(t).$$

By the Lebesgue dominated convergence theorem, it follows from (2.9) that

$$\int_{]-1,1]}\frac{1}{N}\sum_{k=0}^{N-1}\left(1-|U^{\langle k\rangle}(t)|^2\right)\mathrm{d}\nu(t)\to 0,\quad\text{as } N\to+\infty,$$

which combined with (2.10) leads to

$$\int_{]-1,1]} (1-t^2)\,\mathrm{d}\nu(t) = 0.$$

This is only possible if $\nu = 0$.

We formalize this in a proposition.

PROPOSITION 2.4. *Suppose* $\lambda \in \mathbb{C}$ *has* $|\lambda| = 1$ *and that* ν *is an absolutely continuous finite complex* (U, λ)*-invariant Borel measure on* $]-1,1]$. *Then* $\nu = 0$.

Proof. By Proposition 2.1, $|\nu|$ is a U-invariant measure. In view of the above argument, $|\nu| = 0$ and so $\nu = 0$. □

3. Extension of the trigonometric system

The trigonometric system. The trigonometric system $\{e_n(x)\}_{n\in\mathbb{Z}}$, with $e_n(x) = \mathrm{e}^{\pi \mathrm{i} n x}$, is very successful in describing 2-periodic functions on the line. Harald Bohr — the brother of Niels Bohr, the physicist — developed, over a number of years in the 1920's and 1930's, the theory of almost periodic functions based on more general real frequencies rather than the integer frequencies of the trigonometric system.

An extension of the trigonometric system. Here, we consider another extension of the trigonometric system, connected with the theory of composition operators. Let β be a positive real parameter. We introduce, for integers n,

$$e_n^{\langle\beta\rangle}(x) = e_n\left(\frac{\beta}{x}\right) = \mathrm{e}^{\pi \mathrm{i}\beta n/x}$$

and note that these functions are bounded on the real line.

After a dilation of the line, Theorem 1.2 is equivalent to the following statement.

THEOREM 3.1. *As* n *ranges over the integers, the functions* $e_n(x)$ *and* $e_n^{\langle\beta\rangle}(x)$ *form a weak-star-spanning system in* $L^\infty(\mathbb{R})$ *if and only if* $0 < \beta \le 1$.

If μ is a positive finite absolutely continuous Borel measure on $\mathbb{R}$, then a bounded function in $L^\infty(\mathbb{R})$ is automatically in $L^p(\mathbb{R},\mu)$ for $1 < p < +\infty$, and the weak-star closure of a subspace in $L^\infty(\mathbb{R})$ is contained in the norm closure in $L^p(\mathbb{R},\mu)$. We then have the following consequence of Theorem 3.1. The necessity part just requires mimicking the corresponding argument involving harmonic extensions in Section 4 below.

Corollary 3.2. *Suppose* $1 < p < +\infty$ *and that* $\mathrm{d}\mu(x) = M(x)\mathrm{d}x$, *where* $M(x) \geq 0$ *is Borel measurable, with*

$$0 < \int_{-\infty}^{+\infty} M(x)\,\mathrm{d}x < +\infty.$$

Then, as n *ranges over the integers, the functions* $e_n(x)$ *and* $e_n^{\langle\beta\rangle}(x)$ *form a spanning system in* $L^p(\mathbb{R},\mu)$ *provided that* $0 < \beta \leq 1$. *If, in addition,*

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^{p/(p-1)} M(x)^{1/(p-1)}} < +\infty,$$

the condition $0 < \beta \leq 1$ *is also necessary in order to have a spanning system.*

4. Necessity of the condition $0 < \beta \leq 1$

Harmonic extension. We extend the functions e_n harmonically and boundedly to the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \ \mathrm{Im}\, z > 0\}$:

$$e_n(z) = \mathrm{e}^{\pi \mathrm{i} n z}, \qquad \mathrm{Im}\, z \geq 0, \ \ n \geq 0,$$

while

$$e_n(z) = \mathrm{e}^{\pi \mathrm{i} n \bar{z}}, \qquad \mathrm{Im}\, z \geq 0, \ \ n < 0.$$

Likewise, the harmonic extension of $e_n^{\langle\beta\rangle}$ is

$$e_n^{\langle\beta\rangle}(z) = \mathrm{e}^{\pi \mathrm{i} \beta n/\bar{z}}, \qquad \mathrm{Im}\, z \geq 0, \ \ n \geq 0,$$

and

$$e_n^{\langle\beta\rangle}(z) = \mathrm{e}^{\pi \mathrm{i} \beta n/z}, \qquad \mathrm{Im}\, z \geq 0, \ \ n < 0.$$

Point separation. A general $L^\infty(\mathbb{R})$ function is extended harmonically and boundedly to $\mathbb{C}_+$ via the Poisson kernel; for each $z_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C}_+$, the point evaluation functional $f \mapsto f(z_0)$ is given by

$$f(z_0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} P(t, z_0)\, f(t)\,\mathrm{d}t, \qquad P(t, z_0) = \frac{y_0}{(x_0 - t)^2 + y_0^2},$$

where $t \mapsto P(t, z_0)$ is in $L^1(\mathbb{R})$. The functional is therefore weak-star continuous on $L^\infty(\mathbb{R})$. As we harmonically extend all the functions in $L^\infty(\mathbb{R})$, we get the space of all bounded harmonic functions in $\mathbb{C}_+$. The bounded harmonic functions in $\mathbb{C}_+$ separate the points of $\mathbb{C}_+$, so if we can find two points $z_1, z_2 \in \mathbb{C}_+$ with $z_1 \neq z_2$, such that

$$e_n(z_1) = e_n(z_2), \qquad e_n^{\langle\beta\rangle}(z_1) = e_n^{\langle\beta\rangle}(z_2), \tag{4.1}$$

for all $n \in \mathbb{Z}$, then the linear span of $e_n, e_n^{\langle\beta\rangle}$ cannot be weak-star dense in $L^\infty(\mathbb{R})$. The condition (4.1) boils down to

$$z_1 - z_2 \in 2\mathbb{Z}, \qquad \frac{1}{z_1} - \frac{1}{z_2} \in \frac{2}{\beta}\mathbb{Z},$$

where we may apply Lemma 1.5, with $m = n = 1$, $a = 2$, and $b = 2/\beta$. Assuming that $1 < \beta < +\infty$, we get that

$$z_1 = 1 + \mathrm{i}\sqrt{\beta - 1}, \qquad z_2 = -1 + \mathrm{i}\sqrt{\beta - 1},$$

are points in $\mathbb{C}_+$ satisfying (4.1). It follows that the requirement $0 < \beta \le 1$ is necessary in Theorem 3.1.

5. **Periodic and inverted-periodic functions**

Periodic and inverted-periodic functions. The weak-star closure in $L^\infty(\mathbb{R})$ of the linear span of the functions $e_n(x) = \mathrm{e}^{\pi i n x}$, $n \in \mathbb{Z}$, equals $L^\infty_2(\mathbb{R})$, the subspace of 2-periodic functions. Similarly, the weak-star closure of the linear span of the functions $e^{\langle\beta\rangle}(x) = e_n(\beta/x)$, $n \in \mathbb{Z}$, equals the subspace $L^\infty_{\langle\beta\rangle}(\mathbb{R})$ of all functions $f \in L^\infty(\mathbb{R})$ with $x \mapsto f(\beta/x)$ being 2-periodic. Let us tacitly extend all functions in $L^\infty(\mathbb{R})$ harmonically to $\mathbb{C}_+$ using the Poisson kernel.

The intersection space. Let us, for a moment, consider the intersection

$$L^\infty_2(\mathbb{R}) \cap L^\infty_{\langle\beta\rangle}(\mathbb{R}).$$

We introduce $\mathfrak{G}(\beta)$ as the group of Möbius transformations preserving $\mathbb{C}_+$ generated by the translation $z \mapsto z + 2$ and the mapping $z \mapsto \beta z/(\beta - 2z)$; then the elements of $L^\infty_2(\mathbb{R}) \cap L^\infty_{\langle\beta\rangle}(\mathbb{R})$ are precisely the functions in $L^\infty(\mathbb{R})$ that are invariant under $f \mapsto f \circ \gamma$, for $\gamma \in \mathfrak{G}(\beta)$. This situation is investigated in Section 11.4 of Beardon's book [2]. For $0 < \beta \le 1$, the group $\mathfrak{G}(\beta)$ is discrete and free (see, e.g., Gilman and Maskit [4]), and the fundamental domain (hyperbolic polygon) associated with $\mathbb{C}_+/\mathfrak{G}(\beta)$ is given by

$$\mathfrak{D}(\beta) = \left\{ z \in \mathbb{C}_+ : \ |\operatorname{Re} z| < 1, \ \left| z - \frac{\beta}{2} \right| > \frac{\beta}{2}, \ \left| z + \frac{\beta}{2} \right| > \frac{\beta}{2} \right\}. \tag{5.1}$$

The domain $\mathfrak{D}(\beta)$ has a cusp at infinity and at the origin. In addition, it has cusp(s) at ± 1 for $\beta = 1$. For $0 < \beta < 1$, the fundamental domain has two boundary line segments $]-1, -\beta[$ and $]\beta, 1[$, which is enough for $\mathbb{C}_+/\mathfrak{G}(\beta)$ to carry plenty of bounded harmonic (holomorphic as well) functions. A cusp is a removable singularity for a bounded harmonic function on $\mathbb{C}_+/\mathfrak{G}(\beta)$ (it is just an isolated removed point on the Riemann surface), which means that only constants are bounded and harmonic on $\mathbb{C}_+/\mathfrak{G}(\beta)$ for $\beta = 1$. For $1 < \beta < +\infty$, the group $\mathfrak{G}(\beta)$ is discrete if and only if

$$\beta = \frac{1}{\cos^2(p\pi/(2q))} \tag{5.2}$$

for some coprime positive integers p, q with $p < q$ and $p \in \{1, 2\}$. In case $p = 1$, the fundamental domain is still given by (5.1), while for $p = 2$ it is smaller, but retains two of the cusps. Anyway, under (5.2), only cusps (two or three) occur in $\mathbb{C}_+/\mathfrak{G}(\beta)$ and all bounded harmonic functions are constant. In the remaining case, when (5.2) fails, the group $\mathfrak{G}(\beta)$ is nondiscrete, and then every harmonic function which is invariant under $\mathfrak{G}(\beta)$ is necessarily constant.

We gather some of the above observations in a proposition.

PROPOSITION 5.1. $L^\infty_2(\mathbb{R}) \cap L^\infty_{\langle\beta\rangle}(\mathbb{R}) = \{\text{constants}\}$ *if and only if* $1 \le \beta < +\infty$. *Moreover, for* $0 < \beta < 1$, $L^\infty_2(\mathbb{R}) \cap L^\infty_{\langle\beta\rangle}(\mathbb{R})$ *is infinite-dimensional.*

The sum space. Next, we turn to the study of the sum space. In order to obtain Theorem 3.1, we show that

$$L^\infty_2(\mathbb{R}) + L^\infty_{\langle\beta\rangle}(\mathbb{R})$$

is weak-star dense in $L^\infty(\mathbb{R})$ if and only if $0 < \beta \le 1$. In Section 4, we saw that the sum fails to be weak-star dense for $1 < \beta < +\infty$. In the sequel, we therefore *assume that* $0 < \beta \le 1$. We now make a basic observation. Functions in $L^\infty_2(\mathbb{R})$ may be prescribed freely on $]-1,1]$, but then they are uniquely determined everywhere else, due to periodicity. Likewise, functions in $L^\infty_{\langle\beta\rangle}(\mathbb{R})$ are free on $\mathbb{R}\backslash]-\beta,\beta]$ and extend by "periodicity" everywhere else. This allows us to define operators $\mathbf{S}$, $\mathbf{T}_\beta$ as follows. The first operator

$$\mathbf{S} : L^\infty(]-1,1]) \to L^\infty(\mathbb{R}\backslash]-1,1])$$

is obtained by extending the function to be 2-periodic on $\mathbb{R}$ and then restricting the extended function to $\mathbb{R}\backslash]-1,1]$. The second operator

$$\mathbf{T}_\beta : L^\infty(\mathbb{R}\backslash]-\beta,\beta]) \to L^\infty(]-\beta,\beta])$$

is the analogous extension associated with the "periodicity" in $L^\infty_{\langle\beta\rangle}(\mathbb{R})$; in symbols, we may express it as

$$\mathbf{T}_\beta[f] = (\mathbf{S}[f \circ I_\beta]) \circ I_\beta,$$

where $I_\beta(x) = -\beta/x$.

Next, we agree on a useful convention. For a Lebesgue measurable subset X of the real line of positive linear measure, we identify $L^\infty(X)$ with a weak-star closed subspace of $L^\infty(\mathbb{R})$ by extending the functions to vanish on the complement $\mathbb{R} \setminus X$.

LEMMA 5.2. *If* $\mathbf{I}$ *is the identity operator and if the operator*

$$\mathbf{I} - \mathbf{T}_\beta\mathbf{S} : L^\infty(]-1,1]) \to L^\infty(]-1,1])$$

has weak-star dense range, then the sum space $L^\infty_2(\mathbb{R}) + L^\infty_{\langle\beta\rangle}(\mathbb{R})$ *is weak-star dense in* $L^\infty(\mathbb{R})$.

Proof. We write $\mathcal{R}$ for the range of the operator $\mathbf{I} - \mathbf{T}_\beta\mathbf{S}$. Pick an arbitrary $F_2 \in L^\infty(\mathbb{R}\backslash]-1,1])$, and ask of $F_1 \in L^\infty(]-1,1])$ that $F_1 = \mathbf{T}_\beta[F_2] + R$, where $R \in \mathcal{R}$. The set of all sums $F = F_1 + F_2 \in L^\infty(\mathbb{R})$ obtained in this fashion is denoted by $\mathcal{F}$. The following straightforward argument shows that $\mathcal{F}$ is weak-star dense in $L^\infty(\mathbb{R})$, provided that $\mathcal{R}$ is. Suppose $K \in L^1(\mathbb{R})$ has

$$\langle F, K\rangle_{\mathbb{R}} = 0$$

for all $F \in \mathcal{F}$. We decompose $K = K_1 + K_2 \in L^1(\mathbb{R})$, where

$$K_1 \in L^1(]-1,1]) \quad \text{and} \quad K_2 \in L^1(\mathbb{R}\backslash]-1,1]).$$

Then

$$0 = \langle F, K\rangle_{\mathbb{R}} = \langle F_1, K_1\rangle_{\mathbb{R}} + \langle F_2, K_2\rangle_{\mathbb{R}} = \langle \mathbf{T}_\beta[F_2], K_1\rangle_{\mathbb{R}} + \langle R, K_1\rangle_{\mathbb{R}} + \langle F_2, K_2\rangle_{\mathbb{R}}.$$

We rewrite this in the form

$$\langle R, K_1\rangle_{\mathbb{R}} = -\langle F_2, K_2\rangle_{\mathbb{R}} - \langle \mathbf{T}_\beta[F_2], K_1\rangle_{\mathbb{R}}.$$

Only the left-hand side depends on $R \in \mathcal{R}$; by linearity, the only way this is possible is if $\langle R, K_1\rangle_{\mathbb{R}} = 0$ for all $R \in \mathcal{R}$. But as $\mathcal{R}$ is dense, we get that $K_1 = 0$. The remaining relationship now reads

$$\langle F_2, K_2\rangle_{\mathbb{R}} = 0.$$

As F_2 was arbitrary, we conclude that $K_2 = 0$ as well.

To finish the proof, we show that

$$\mathcal{F} \subset L_2^\infty(\mathbb{R}) + L_{\langle\beta\rangle}^\infty(\mathbb{R}).$$

For $F = F_1 + F_2 \in \mathcal{F}$ as above, the fact that $F_1 - \mathbf{T}_\beta[F_2] = R \in \mathcal{R}$ means that there exists a $g \in L^\infty(]-1,1])$ such that

$$g - \mathbf{T}_\beta\mathbf{S}[g] = (\mathbf{I} - \mathbf{T}_\beta\mathbf{S})[g] = F_1 - \mathbf{T}_\beta[F_2]. \tag{5.3}$$

Also, let $h \in L^\infty(\mathbb{R}\backslash]-1,1])$ be given by

$$h = F_2 - \mathbf{S}[g]. \tag{5.4}$$

Then, by (5.4),

$$F_2 = h + \mathbf{S}[g], \tag{5.5}$$

so that

$$\mathbf{T}_\beta[F_2] = \mathbf{T}_\beta[h] + \mathbf{T}_\beta\mathbf{S}[g].$$

If we combine this with (5.3), we get

$$F_1 = \mathbf{T}_\beta[F_2] + g - \mathbf{T}_\beta\mathbf{S}[g] = \mathbf{T}_\beta[h] + \mathbf{T}_\beta\mathbf{S}[g] + g - \mathbf{T}_\beta\mathbf{S}[g] = g + \mathbf{T}_\beta[h]. \tag{5.6}$$

It follows from (5.5) and (5.6) that

$$\begin{aligned} F = F_1 + F_2 &= (g + \mathbf{T}_\beta[h]) + (h + \mathbf{S}[g]) \\ &= (g + \mathbf{S}[g]) + (h + \mathbf{T}_\beta[h]) \in L_2^\infty(\mathbb{R}) + L_{\langle\beta\rangle}^\infty(\mathbb{R}). \end{aligned}$$

The proof is complete. $\square$

The operator $\mathbf{T}_\beta\mathbf{S}$. For $x \in \mathbb{R}$, let $\{x\}_2$ denote the number with $-1 < \{x\}_2 \le 1$ and $x - \{x\} \in 2\mathbb{Z}$. Then, since for $\varphi \in L^\infty(]-1,1])$,

$$\mathbf{S}[\varphi](x) = \varphi(\{x\}_2)\; 1_{\mathbb{R}\setminus]-1,1]}(x), \qquad x \in \mathbb{R},$$

where 1_E denotes the characteristic function of the set $E \subset \mathbb{R}$, we find that for $\psi \in L^\infty(\mathbb{R}\setminus]-\beta,\beta])$,

$$\mathbf{T}_\beta[\psi](x) = \psi\!\left(\frac{\beta}{\{\beta/x\}_2}\right) 1_{]-\beta,\beta]}(x), \qquad x \in \mathbb{R}.$$

It follows that

$$\mathbf{T}_\beta\mathbf{S}[\varphi](x) = \varphi\!\left(\left\{\frac{\beta}{\{\beta/x\}_2}\right\}_2\right) 1_{E_\beta}(x), \qquad x \in \mathbb{R}, \tag{5.7}$$

where

$$E_\beta = \left\{x \in]-\beta,\beta] \setminus \{0\} :\; \frac{\beta}{\{\beta/x\}_2} \in \mathbb{R}\setminus]-1,1]\right\}.$$

6. Analysis of a related composition operator

The Gauss-type map. Let U_β be the mapping

$$U_\beta(x) = \{-\beta/x\}_2, \qquad x \ne 0,$$

with $U_\beta(0) = 0$. We consider the associated compressed composition operator

$$\mathbf{C}_\beta :\; L^\infty(]-1,1]) \to L^\infty(]-1,1])$$

given by

$$\mathbf{C}_\beta[f](x) = f(U_\beta(x))\, 1_{]-\beta,\beta]}(x).$$

We quickly realize from (5.7) that

$$\mathbf{T}_\beta\mathbf{S} = \mathbf{C}_\beta^2$$

and turn to analyzing $\mathbf{C}_\beta$. The identity

$$\mathbf{I} - \mathbf{T}_\beta\mathbf{S} = \mathbf{I} - \mathbf{C}_\beta^2 = (\mathbf{I} + \mathbf{C}_\beta)(\mathbf{I} - \mathbf{C}_\beta)$$

shows that if $\mathbf{I} + \mathbf{C}_\beta$ and $\mathbf{I} - \mathbf{C}_\beta$ both have weak-star dense range, then $\mathbf{I} - \mathbf{C}_\beta^2$ has weak-star dense range as well. By elementary Functional Analysis, the operators $\mathbf{I} + \mathbf{C}_\beta$ and $\mathbf{I} - \mathbf{C}_\beta$ both have weak-star dense range if and only if for $\lambda = \pm 1$, the (predual) adjoint

$$\lambda\mathbf{I} - \mathbf{C}_\beta^* :\; L^1(]-1,1]) \to L^1(]-1,1])$$

has null kernel, that is, if the points ± 1 both fail to be eigenvalues of $\mathbf{C}_\beta^*$. The following result shows that this is the case for $0 < \beta \le 1$, making $\mathbf{I} - \mathbf{T}_\beta\mathbf{S}$ have weak-star dense range, and thus, in view of Lemma 5.2,

$$L_2^\infty(\mathbb{R}) + L_{\langle\beta\rangle}^\infty(\mathbb{R})$$

is weak-star dense in $L^\infty(\mathbb{R})$. Given that we have verified the necessity of the condition $0<\beta\le 1$ in the context of Theorem 3.1, the rest of the assertion of Theorem 3.1 follows.

PROPOSITION 6.1. *For $0<\beta\le 1$, the point spectrum $\sigma_p(\mathbf{C}_\beta^*)$ of $\mathbf{C}_\beta^*: L^1(]-1,1])\to L^1(]-1,1])$ is contained in the open unit disk $\mathbb{D}$. In particular, ±1 are not eigenvalues of $\mathbf{C}_\beta^*$.*

Proof. It is clear that

$$\sigma_p(\mathbf{C}_\beta^*)\subset\sigma(\mathbf{C}_\beta^*)\subset\bar{\mathbb{D}},$$

where $\bar{\mathbb{D}}$ is the closed unit disk, so we just need to show that a $\lambda\in\mathbb{C}$ with $|\lambda|=1$ cannot be an eigenvalue.

The treatment of the cases $\beta=1$ and $0<\beta<1$ will be different.

The case $\beta=1$. For $\beta=1$, $U_\beta=U_1=U$ (the map we studied in §2) and $\mathbf{C}_1$ is

$$\mathbf{C}_1[f](x)=f(U(x))\,1_{]-1,1]}(x).$$

The defining relation for the (predual) adjoint $\mathbf{C}_1^*: L^1(]-1,1])\to L^1(]-1,1])$ is

$$\int_{]-1,1]} f(x)\,\mathbf{C}_1^*[g](x)\,\mathrm{d}x=\langle \mathbf{C}_1^*[g],f\rangle_{\mathbb{R}}=\langle g,\mathbf{C}_1[f]\rangle_{\mathbb{R}}=\int_{]-1,1]} f(U(x))\,g(x)\,\mathrm{d}x.$$

If g is a nontrivial eigenfunction for $\mathbf{C}_1^*$ with eigenvalue λ, then $\mathbf{C}_1^*[g]=\lambda g$ and so

$$\lambda\int_{]-1,1]} f(x)\,g(x)\,\mathrm{d}x=\int_{]-1,1]} f(U(x))\,g(x)\,\mathrm{d}x;$$

this expresses that the absolutely continuous finite measure $\mathrm{d}\nu(x)=g(x)\mathrm{d}x$ is a (U,λ)-invariant measure. By Proposition 2.4, there are no finite (U,λ)-invariant measures except the null measure, for $|\lambda|=1$. Consequently, $\lambda\in\mathbb{C}$ is not an eigenvalue of $\mathbf{C}_1^*$ for $|\lambda|=1$.

The case $0<\beta<1$. The same analysis reduces the problem to studying the absolutely continuous finite measures ν on $]-1,1]$ with

$$\lambda\int_{]-1,1]} f(t)\,\mathrm{d}\nu(t)=\int_{]-1,1]}\mathbf{C}_\beta[f](t)\,\mathrm{d}\nu(t)=\int_{]-\beta,\beta]} f(U_\beta(t))\,\mathrm{d}\nu(t) \tag{6.1}$$

for all $f\in L^\infty(]-1,1])$, where $\lambda\in\mathbb{C}$ is fixed with $|\lambda|=1$. In more concrete terms, this amounts to

$$\lambda\,\mathrm{d}\nu(t)=\sum_{j\in\mathbb{Z}^*}\mathrm{d}\nu_j(t),\qquad t\in]-1,1],$$

where

$$\mathrm{d}\nu_j(t)=\mathrm{d}\nu\biggl(\frac{\beta}{2j-t}\biggr),\qquad t\in]-1,1].$$

Taking absolute values, we have, for $|\lambda|=1$,

$$|\mathrm{d}\nu(t)| \le \sum_{j\in\mathbb{Z}^*} |\mathrm{d}\nu_j(t)|, \qquad t \in]-1,1].$$

Integrating over $]-1,1]$, we find that

$$\int_{]-1,1]} |\mathrm{d}\nu(t)| \le \sum_{j\in\mathbb{Z}^*} \int_{]-1,1]} |\mathrm{d}\nu_j(t)| = \int_{[-\beta,\beta]} |\mathrm{d}\nu(t)|, \qquad t \in]-1,1],$$

which is only possible if we have the equality

$$|\mathrm{d}\nu(t)| = \sum_{j\in\mathbb{Z}^*} |\mathrm{d}\nu_j(t)|, \qquad t \in]-1,1],$$

as well as

$$\mathrm{d}\nu(t) = 0, \qquad t \in]-1,1] \setminus [-\beta,\beta].$$

If we iterate the relation (6.1), we get

$$\lambda^n \int_{]-1,1]} f(t)\,\mathrm{d}\nu(t) = \int_{]-1,1]} \mathbf{C}_\beta^n[f](t)\,\mathrm{d}\nu(t) = \int_{E_\beta(n)} f(U_\beta^{\langle n\rangle}(t))\,\mathrm{d}\nu(t), \tag{6.2}$$

where set $E_\beta(n)$ is given by

$$E_\beta(n) = \Big\{ t \in]-1,1] : U_\beta^{\langle k\rangle}(t) \in [-\beta,\beta] \text{ for } k=0,\ldots,n-1 \Big\}.$$

A repetition of the above argument involving $U_\beta^{\langle n\rangle}$ in place of U_β shows that if $|\lambda|=1$, then

$$\mathrm{d}\nu(t) = 0, \qquad t \in]-1,1] \setminus E_\beta(n).$$

As $n \to +\infty$, the set $E_\beta(n)$ shrinks down to

$$E_\beta(\infty) = \Big\{ t \in]-1,1] : U_\beta^{\langle k\rangle}(t) \in [-\beta,\beta] \text{ for } k=0,1,2,3,\ldots \Big\}.$$

This final set $E_\beta(\infty)$ is U_β-invariant, and it is not hard to show that it must have zero length. But the measure ν vanishes everywhere else, and being absolutely continuous, it must be the zero measure. In particular, $\lambda \in \mathbb{C}$ with $|\lambda|=1$ cannot be eigenvalues of $\mathbf{C}_\beta^*$. The proof is complete. □

A remark on model subspaces. Given an inner function Θ in the upper half-plane $\mathbb{C}_+$, one considers the model subspaces $K_\Theta(\mathbb{C}_+) = H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+)$. Uniqueness sets for model subspaces have been studied recently by Makarov and Poltoratski [8], and the injectivity of the Toeplitz operator with symbol $\bar\Theta B_\Lambda$ is equivalent to $\Lambda \subset \mathbb{C}_+$ being a uniqueness set. In our setting, we use mainly that the operators $\lambda\mathbf{I} - \mathbf{C}_\beta^*$ are injective for $\lambda = \pm 1$, so that apparently, these operators are analogous to the Toeplitz operators from the model subspace case.

7. Applications and open problems

An application to BMO. Let $\mathrm{BMOA}(\mathbb{C}_+)$ be the (weak-star closed) subspace of the space $\mathrm{BMO}(\mathbb{R})$ consisting of functions whose Poisson extensions to $\mathbb{C}_+$ are holomorphic in $\mathbb{C}_+$. We recall that $\mathrm{BMO}(\mathbb{R})$ denotes the space of functions with bounded mean oscillation. The Cauchy-Szegö (analytic) projection

$$\mathbf{P} : L^\infty(\mathbb{R}) \to \mathrm{BMOA}(\mathbb{C}_+)$$

is bounded and surjective. Here, we think of $\mathrm{BMOA}(\mathbb{C}_+)$ as the dual of $H^1(\mathbb{C}_+)$ with respect to the sesquilinear form of $L^2(\mathbb{R})$; in particular, $\mathrm{BMOA}(\mathbb{C}_+)$ is a space of functions modulo the constants. We observe that if $\mathcal{X}$ is a linear subspace of $L^\infty(\mathbb{R})$ which is weak-star dense, then $\mathbf{P}(\mathcal{X})$ is is weak-star dense in $\mathrm{BMOA}(\mathbb{C}_+)$. Moreover, let $\mathrm{BMOA}_2(\mathbb{C}_+)$ be the subspace of $\mathrm{BMOA}(\mathbb{C}_+)$ of functions invariant under $z \mapsto z+2$, and let $\mathrm{BMOA}_{\langle\beta\rangle}(\mathbb{C}_+)$ be the subspace of $\mathrm{BMOA}(\mathbb{C}_+)$ of functions invariant under $z \mapsto \beta z/(\beta - 2z)$. We quickly check that $\mathbf{P}$ maps $L^\infty_2(\mathbb{R}) \to \mathrm{BMOA}_2(\mathbb{C}_+)$ and $L^\infty_{\langle\beta\rangle}(\mathbb{R}) \to \mathrm{BMOA}_{\langle\beta\rangle}(\mathbb{C}_+)$.

In view of our main theorem (Theorem 3.1), we have the following.

Corollary 7.1. *The sum* $\mathrm{BMOA}_2(\mathbb{C}_+) + \mathrm{BMOA}_{\langle\beta\rangle}(\mathbb{C}_+)$ *is weak-star dense in* $\mathrm{BMOA}(\mathbb{C}_+)$ *if and only if* $0 < \beta \le 1$. *In other words, the functions*

$$e_n(x) = \mathrm{e}^{\pi i n x}, \quad e^{\langle\beta\rangle}_{-n}(x) = \mathrm{e}^{-\pi\beta i n/x}, \qquad n = 0, 1, 2, 3, \dots$$

span a weak-star dense subspace of $\mathrm{BMOA}(\mathbb{C}_+)$ *if and only if* $0 < \beta \le 1$.

By the Möbius invariance of BMO, we may transfer this result to the setting of the unit disk and answer Problem 2 of Matheson and Stessin [9] in the affirmative.

Four open problems. (a) Suppose that in the context of Theorem 1.2 we consider a lattice-cross

$$\Lambda = ((\alpha\mathbb{Z} + \{\theta\}) \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

where $\theta \in \mathbb{R}$ is fixed. It seems that Theorem 1.2 should remain true with this new Λ, with only moderate modifications in the proof. But what happens if the lattice-cross is less regular, that is, if the two spacings α and β are allowed to fluctuate a bit along the cross?

(b) In the context of Corollary 3.2, as n ranges over the integers, do the functions $e_n(x)$ and $e^{\langle\beta\rangle}_n(x)$ form a spanning system in $L^p(\mathbb{R}, \mu)$ for all $0 < \beta < +\infty$ provided that

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(1+x^2)^{p/(p-1)} M(x)^{1/(p-1)}} = +\infty?$$

(c) Let $H^\infty_2(\mathbb{C}_+)$ denote the (weak-star closed) subspace of $L^\infty_2(\mathbb{R})$ consisting of those functions whose Poisson extensions to the upper half-plane $\mathbb{C}_+$ are analytic. Analogously, let $H^\infty_{\langle\beta\rangle}(\mathbb{C}_+)$ be the (weak-star closed) subspace of $L^\infty_{\langle\beta\rangle}(\mathbb{R})$ consisting of those functions whose Poisson extensions to the upper half-plane $\mathbb{C}_+$ are analytic. Is the sum

$$H^\infty_2(\mathbb{C}_+) + H^\infty_{\langle\beta\rangle}(\mathbb{C}_+)$$

weak-star dense in $H^\infty(\mathbb{C}_+)$ for $0 < \beta \le 1$? This does not seem to follow from our Theorem 3.1, and, if answered in the affirmative (for $0 < \beta < 1$), it would solve Problem 1 of [9].

(d) In the context of Corollary 1.3, let the finite Borel measure μ be supported on the hyperbola $x_1x_2 = \varepsilon$ and let Λ be the lattice-cross given by the positive parameters α, β. If μ is absolutely continuous with respect to arc length measure on the hyperbola, an argument involving curvature considerations shows that

$$\widehat{\mu}(\xi) \to 0 \quad \text{as } |\xi| \to +\infty.$$

In a sense, this expresses the absence of point masses in μ. Moreover, $\widehat{\mu}$ solves the Klein-Gordon equation $\partial_1\partial_2\widehat{\mu} + \varepsilon\pi^2\widehat{\mu} = 0$. It would be desirable to remove, to the extent possible, the Fourier analysis ingredient in Corollary 1.3. Let u be a bounded continuous complex-valued function on $\mathbb{R}^2$ with $u(\xi) \to 0$ as $|\xi| \to +\infty$. Suppose, in addition, that $\partial_1\partial_2 u + \varepsilon\pi^2 u = 0$ holds in the sense of distribution theory. The problem: is the lattice-cross Λ, with $\alpha\beta \le 1/|\varepsilon|$, a uniqueness set for u (that is, $u|_\Lambda = 0 \implies u = 0$)?

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