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On the distributional Jacobian of maps from $\mathbb{S}^N$ into $\mathbb{S}^N$ in fractional Sobolev and Hölder spaces

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Abstract

H. Brezis and L. Nirenberg proved that if $(g_k) \subset C^0(\mathbb{S}^N, \mathbb{S}^N)$ and $g \in C^0(\mathbb{S}^N, \mathbb{S}^N)$ $(N \geq 1)$ are such that $g_k \to g$ in $\mathrm{BMO}(\mathbb{S}^N)$, then $\deg g_k \to \deg g$. On the other hand, if $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$, then Kronecker's formula asserts that $\deg g = \frac{1}{|\mathbb{S}^N|} \int_{\mathbb{S}^N} \det(\nabla g)\, d\sigma$. Consequently, $\int_{\mathbb{S}^N} \det(\nabla g_k)\, d\sigma$ converges to $\int_{\mathbb{S}^N} \det(\nabla g)\, d\sigma$ provided $g_k \to g$ in $\mathrm{BMO}(\mathbb{S}^N)$. In the same spirit, we consider the quantity $\mathbf{J}(g, \psi) := \int_{\mathbb{S}^N} \psi \det(\nabla g)\, d\sigma$, for all $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$ and study the convergence of $\mathbf{J}(g_k, \psi)$. In particular, we prove that $\mathbf{J}(g_k, \psi)$ converges to $\mathbf{J}(g, \psi)$ for any $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$ if g_k converges to g in $C^{0,\alpha}(\mathbb{S}^N)$ for some $\alpha > \frac{N-1}{N}$. Surprisingly, this result is "optimal" when $N > 1$. In the case $N = 1$ we prove that if $g_k \to g$ almost everywhere and $\limsup_{k \to \infty} |g_k - g|_{\mathrm{BMO}}$ is sufficiently small, then $\mathbf{J}(g_k, \psi) \to \mathbf{J}(g, \psi)$ for any $\psi \in C^1(\mathbb{S}^1, \mathbb{R})$. We also establish bounds for $\mathbf{J}(g, \psi)$ which are motivated by the works of J. Bourgain, H. Brezis, and H.-M. Nguyen and H.-M. Nguyen. We pay special attention to the case $N = 1$.

Contents

1. Introduction 1142
2. The main bounds: Proofs of Theorem 2 and Corollary 5 1151
3. Optimality of the bounds: Proofs of the statements in Remarks 4, 5, and 6 1154
3.1. Proof of the statement in Remark 4 1154
3.2. Proof of the statement in Remark 5 1155
3.3. Proof of the statement in Remark 6 1156
4. The main convergence results: Proofs of Theorem 1 and Corollary 3 1157
4.1. Proofs of Theorem 1 and Corollary 3 1157
4.2. A remark on the degree 1160
5. Proofs of Proposition 1 and Remark 2 1161
5.1. Proof of Proposition 1 1161
5.2. Proof of Remark 2: Optimality of Theorem 1 1162

6. Definition and properties of $g^*\omega$ for $g \in (W \cap \mathrm{VMO})(\mathbb{S}^N, \mathbb{S}^N)$ 1163
7. The case N=1 1165
7.1. Proofs of Theorems 4 and 3 1165
Proofs of Theorems 4 and 3 1166
7.2. Definition and properties of $g^*\omega$ for $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$ 1166
7.3. An improvement of Theorem 3: A partial answer to Open Question 2 1167
Appendix A. A basic estimate for the lifting: Proof of Theorem 6 1170
References 1179

1. Introduction

H. Brezis and L. Nirenberg [20] proved that if $(g_k) \subset C^0(\mathbb{S}^N, \mathbb{S}^N)$ and $g \in C^0(\mathbb{S}^N, \mathbb{S}^N)$ $(N \geq 1)$ are such that $\lim_{k\to 0} |g_k - g|_{\mathrm{BMO}} = 0$, then

$$\lim_{k\to\infty} \deg g_k = \deg g. \tag{1.1}$$

Hereafter in this paper, we use the following BMO-semi-norm:

$$|f|_{\mathrm{BMO}(\Omega)} := \sup_{B(x,r)\subset\subset\Omega} \fint_{B(x,r)} \left| f(\xi) - \fint_{B(x,r)} f(\eta)\, d\eta \right| d\xi, \quad \forall\, f \in \mathrm{BMO}(\Omega),$$

where $B(x,r)$ denotes the ball in Ω of radius r centered at x and $|\ |$ denotes the Euclidean norm. In fact we will establish a slightly better result (see §4.2, Prop. 4), namely if $\limsup_{k\to\infty} |g_k - g|_{\mathrm{BMO}} < 1$ and g_k converges to g a.e., then (1.1) holds. On the other hand, the well-known Kronecker formula asserts that

$$\deg g = \frac{1}{|\mathbb{S}^N|} \int_{\mathbb{S}^N} \det(\nabla g)\, d\sigma, \tag{1.2}$$

for any $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$. In this integral " det " denotes the determinant of an $N \times N$ matrix, once an orientation has been chosen on $\mathbb{S}^N$. Note that one has

$$\det(\nabla g) = \det(\nabla g, g) \quad \text{ on } \mathbb{S}^N, \tag{1.3}$$

where " det " in the right-hand side denotes the determinant of an $(N+1) \times (N+1)$ matrix and g is considered as a map with values into $\mathbb{R}^{N+1}$. Equality (1.3) holds provided we choose an orientation on $\mathbb{S}^N$ such that at every point $\xi \in \mathbb{S}^N$, (B_ξ, n_ξ) is a direct basis of $\mathbb{R}^{N+1}$, where B_ξ is a direct basis in the tangent hyperplane to $\mathbb{S}^N$ at ξ with the orientation inherited from the one of $\mathbb{S}^N$ and n_ξ is the outward normal at ξ. Consequently, $\int_{\mathbb{S}^N} \det(\nabla g_k)\, d\sigma$ converges to $\int_{\mathbb{S}^N} \det(\nabla g)\, d\sigma$ provided $g_k \to g$ in $\mathrm{BMO}(\mathbb{S}^N)$.

In the same spirit we consider the quantity

$$\mathbf{J}(g, \psi) := \int_{\mathbb{S}^N} \psi \det(\nabla g)\, d\sigma, \quad \forall\, \psi \in C^1(\mathbb{S}^N, \mathbb{R})$$

and study the convergence of $\mathbf{J}(g_k, \psi)$ for fixed $\psi \in C^\infty(\mathbb{S}^N, \mathbb{R})$ under various assumptions on the convergence of (g_k). As a consequence, we will be able to give a "robust" meaning to the quantity $\mathbf{J}(g, \psi)$ even in the case where $g : \mathbb{S}^N \mapsto \mathbb{S}^N$ is *not differentiable* but $\psi \in C^\infty(\mathbb{S}^N, \mathbb{R})$. Roughly speaking, the main assumptions will be that g belongs to $VMO \cap W^{s,p}(\mathbb{S}^N, \mathbb{S}^N)$ with $s = \frac{N-1}{N}$ and $p = N$. It is convenient to present our results first in the case $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$. We will explain in Sections 6 and 7 how to weaken this assumption. In view of the results mentioned above one may ask whether $\mathbf{J}(g_k, \psi) \to \mathbf{J}(g, \psi)$ if $g_k \to g$ for example in C^0. This is not true even if $g_k \to g$ in $C^{0,\alpha}$ for any $\alpha < \frac{N-1}{N}$ (see Proposition 1 below). To present our result, we first introduce the following notation.

Notation 1. Let $N \geq 1$ and Ω be an N-dimensional smooth manifold of $\mathbb{R}^{N+1}$ or an open subset of $\mathbb{R}^N$, and $g : \Omega \to \mathbb{R}^k$ ($k \geq 1$) be a measurable map. Define

$$|g|_W^N := \int_\Omega \int_\Omega \frac{|g(x) - g(y)|^N}{|x - y|^{2N-1}} \, dx \, dy. \tag{1.4}$$

It is clear that

$$W(\Omega) := \{g \in L^1(\Omega); |g|_W < \infty\} \tag{1.5}$$

is a normed space with the following norm:

$$\|g\|_W := |g|_W + \|g\|_{L^1}, \quad \forall\, g \in W(\Omega).$$

When $N \geq 2$, the semi-norm $|\,|_W$ corresponds to the semi-norm in the fractional Sobolev space $W^{s,p}$ with $s = \frac{N-1}{N}$ and $p = N$ (also called the Slobodeckij semi-norm; see e.g. [58]).

We recall that for $0 < s < 1$ and $p > 1$,

$$|g|_{W^{s,p}(\Omega)}^p := \int_\Omega \int_\Omega \frac{|g(x) - g(y)|^p}{|x - y|^{N+sp}} \, dx \, dy, \quad \forall\, g \in W^{s,p}(\Omega).$$

We have

Theorem 1. *Let $N \geq 1$, $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$, and $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$ be such that*

i) $\limsup_{k\to\infty} \|g_k - g\|_{\mathrm{BMO}(\mathbb{S}^N)} < 1$

and

ii) $\lim_{k\to\infty} \|g_k - g\|_W = 0$.

Then

$$\lim_{k\to\infty} \mathbf{J}(g_k, \psi) = \mathbf{J}(g, \psi), \quad \forall\, \psi \in C^1(\mathbb{S}^N, \mathbb{R}).$$

Remark 1. The proof of Theorem 1 is inspired from the work of J. Bourgain, H. Brezis, and P. Mironescu [9], [10], J. Bourgain, H. Brezis, and H.-M. Nguyen [7], and H.-M. Nguyen [54].

Remark 2. If one of the assumptions of Theorem 1 fails, the conclusion need not be true. More precisely, one can construct the following examples (see §§5.2 and 5.1):

a) There exists a sequence $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ $(N \geq 1)$ such that $g_k \to g := (0, \dots, 0, 1)$ in $W(\mathbb{S}^N)$, $g_k \to g$ almost everywhere, $\sup_k \|\nabla g_k\|_{L^N} < +\infty$, $\lim_{k\to\infty} \|g_k - g\|_{\mathrm{BMO}} = 1$, and for all k, $\deg g_k = 1 > 0 = \deg g$.
b) There exists a sequence $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ $(N \geq 2)$ such that $g_k \to g := (0, \dots, 0, 1)$ weakly in $W(\mathbb{S}^N)$, $(g_k) \to g$ uniformly on $\mathbb{S}^N$, and $\liminf_{k\to\infty} \mathbf{J}(g_k, x_{N+1}) > 0 = \mathbf{J}(g, x_{N+1})$.

As a consequence of Theorem 1 one has

Corollary 1. *Let $N \geq 3$, $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$, and $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$ be such that*

i) $\lim_{k\to\infty} \|g_k - g\|_{\mathrm{BMO}} = 0$

and

ii) $\sup_k \|\nabla g_k\|_{L^{N-1}} < \infty$.

Then

$$\lim_{k\to\infty} \mathbf{J}(g_k, \psi) = \mathbf{J}(g, \psi), \quad \forall\, \psi \in C^1(\mathbb{S}^N, \mathbb{R}).$$

Proof. We have (see [18]), for $N \geq 3$,

$$\|g_k - g\|_W \leq C \|g_k - g\|_{W^{1,N-1}}^{\frac{N-1}{N}} \|g_k - g\|_{\mathrm{BMO}}^{\frac{1}{N}} \to 0 \text{ as } k \to \infty.$$

The conclusion now follows from Theorem 1. □

Open question 1. We do not know whether Corollary 1 holds when $N = 2$ even if condition i) is replaced by the stronger assumption $\lim\limits_{k\to\infty} \|g_k - g\|_{L^\infty} = 0$.

Another consequence of Theorem 1 is

Corollary 2. *Let $N \geq 2$, $\frac{N-1}{N} < \alpha < 1$, $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$, and $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$ be such that g_k converges to g in $C^{0,\alpha}(\mathbb{S}^N)$. Then*

$$\lim_{k\to\infty} \mathbf{J}(g_k, \psi) = \mathbf{J}(g, \psi), \quad \forall\, \psi \in C^1(\mathbb{S}^N, \mathbb{R}).$$

Proof of Corollary 2. Since (g_k) converges to g in $C^{0,\alpha}(\mathbb{S}^N)$ and $\alpha > \frac{N-1}{N}$, it follows that (g_k) and g satisfy conditions i) and ii) of Theorem 1. □

Corollary 2 is optimal in the following sense.

Proposition 1. *Let $N \geq 2$. There exist a sequence $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ and $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$ such that g_k converges to $g := (0, \ldots, 0, 1)$ in $C^{0, \frac{N-1}{N}}(\mathbb{S}^N)$,* $\sup_k \|g_k\|_W < +\infty$, *and*

$$\liminf_{k\to\infty} \mathbf{J}(g_k, \psi) > 0 = \mathbf{J}(g, \psi).$$

Hereafter $|\;|_{0,\alpha}$ denotes the usual semi-norm in the Hölder space $C^{0,\alpha}$.

There is a natural quantity which appears in the study of N-forms. Consider a smooth N-form on $\mathbb{S}^N$

$$\omega := F(y)\, dy.$$

The pullback $g^*\omega$ of ω under a smooth map $g : \mathbb{S}^N \to \mathbb{S}^N$ is given by

$$g^*\omega = F(g(x)) \det \nabla g(x)\, dx.$$

Recall (see e.g. [55]) that

$$\deg g = \int_{\mathbb{S}^N} g^*\omega \quad \text{ if } \int_{\mathbb{S}^N} \omega = 1.$$

Using Theorem 1, in Section 4.1 we will establish the following convergence result for the quantity

$$g \mapsto \int_{\mathbb{S}^N} F(g(x)) \det \nabla g(x) \psi(x)\, dx,$$

where $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$.

Corollary 3. *Let $N \geq 1$, $F \in C^{0,\alpha}(\mathbb{S}^N, \mathbb{R})$ $(0 < \alpha < 1)$, $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$, and $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$ be such that*

i) $\lim_{k\to\infty} |g_k - g|_{\mathrm{BMO}(\mathbb{S}^N)} = 0$

and

ii) $\lim_{k\to\infty} |g_k - g|_{W(\mathbb{S}^N)} = 0$.

Then

$$\lim_{k\to\infty} \int_{\mathbb{S}^N} F(g_k(x)) \det(\nabla g_k)(x) \psi(x)\, dx = \int_{\mathbb{S}^N} F(g(x)) \det(\nabla g)(x) \psi(x)\, dx,$$

for all $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$.

We next establish bounds for $\mathbf{J}(g, \psi)$ which are motivated by the works of J. Bourgain, H. Brezis, and H.-M. Nguyen [7] and H.-M. Nguyen [54]. We first recall a new estimate for the topological degree of maps from $\mathbb{S}^N$ into $\mathbb{S}^N$ established by J. Bourgain, H. Brezis, and H.-M. Nguyen in [7].

Proposition 2. *Let $g : \mathbb{S}^N \to \mathbb{S}^N$ be a continuous map. Then, for every $0 < \delta < \sqrt{2}$, there exists a constant $C = C(\delta, N)$, independent of g, such that*

$$|\deg g| \leq C \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} \frac{1}{|x-y|^{2N}}\, dx\, dy.$$
$$\scriptstyle |g(x)-g(y)|>\delta$$

Subsequently, H.-M. Nguyen improved this result and showed in [54] that

PROPOSITION 3. *There exists a positive constant $C = C(N)$, depending only on N, such that*

$$|\deg g| \le C \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g(x)-g(y)|\ge \ell_N} \frac{1}{|x-y|^{2N}}\,dx\,dy, \quad \forall\, g \in C(\mathbb{S}^N, \mathbb{S}^N), \tag{1.6}$$

where

$$\ell_N = \sqrt{2 + \frac{2}{N+1}}. \tag{1.7}$$

Moreover, this estimate is optimal in the sense that there exists a sequence of maps $(g_k) \subset C(\mathbb{S}^N, \mathbb{S}^N)$ such that

$$\deg g_k = 1$$

and

$$\lim_{k\to\infty} \int_{\mathbb{S}^N}\int_{\mathbb{S}^N}\limits_{|g_k(x)-g_k(y)|> \ell_N} \frac{1}{|x-y|^{2N}}\,dx\,dy = 0.$$

Remark 3. ℓ_N is the edge of an $(N+1)$-dimensional regular simplex inscribed in $\mathbb{S}^N$, i.e., an equilateral triangle when $N = 1$, a regular tetrahedron when $N = 2$, etc.

The following notation will be useful.

Notation 2. Let $N \ge 1$ and Ω be an N-dimensional smooth manifold of $\mathbb{R}^{N+1}$ or an open subset of $\mathbb{R}^N$, and $g : \Omega \to \mathbb{R}^k$ $(k \ge 1)$ be a measurable map. Define

$$T_\delta(g) := \int_{\Omega}\int_{\Omega}\limits_{|g(x)-g(y)|\ge \delta} \frac{1}{|x-y|^{2N}}\,dx\,dy, \quad \forall\, \delta > 0. \tag{1.8}$$

The following result provides an estimate for $\mathbf{J}(g, \psi)$.

THEOREM 2. *Let $N \ge 1$, $g \in C^1(\mathbb{S}^N, \mathbb{S}^N)$, and $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$. Then*

$$|\mathbf{J}(g,\psi)| \le C\left(\|\psi\|_{L^\infty} T_{\ell_N}(g) + \|\nabla\psi\|_{L^\infty} |g|_W^N\right), \tag{1.9}$$

for some positive constant $C = C(N)$.

Here ℓ_N is defined by (1.7), $T_{\ell_N}(g)$ is defined by (1.8) with $\delta = \ell_N$, and $|\,|_W$ is defined in (1.4). Clearly, Theorem 2 implies Proposition 3. One cannot deduce Theorem 2 from Proposition 3 (see Remark 5 below). However, the proof of Theorem 2 borrows many ideas from the proof of Proposition 3 in [54] and also from the earlier papers of J. Bourgain, H. Brezis, and P. Mironescu [9], [10], and J. Bourgain, H. Brezis, and H.-M. Nguyen [7].

Remark 4. Obviously, from the definition of $\mathbf{J}(g,\psi)$, we have

$$|\mathbf{J}(g,\psi)| \le \|\nabla g\|_{L^N}^N \|\psi\|_{L^\infty}. \tag{1.10}$$

Therefore, for fixed $\psi \in C^1(\mathbb{S}^N)$, $\mathbf{J}(g,\psi)$ is controlled when $\|g\|_{W^{1,N}} \le C$. Similarly, $\mathbf{J}(g_k,\psi) \to \mathbf{J}(g,\psi)$ for $\psi \in C^1(\mathbb{S}^N,\mathbb{R})$ if $g_k \to g$ in $W^{1,N}(\mathbb{S}^N)$. In some sense these facts are optimal in the scale of the Sobolev spaces $W^{1,p}$: there exists a sequence $(g_k) \subset C^1(\mathbb{S}^N,\mathbb{S}^N)$ $(N \ge 2)$ such that $\lim_{k\to\infty} \|\nabla g_k\|_{L^p} = 0$, for all $p < N$ and $|\deg g_k| \to +\infty$ as $k \to +\infty$. This is proved in Section 3.1.

Remark 5. In view of Proposition 3 one may wonder whether it is possible to replace $T_{\ell_N}(g)$ by $\deg g$ in (1.9). The answer is negative. More precisely: Let $N \ge 1$. Then there exists a sequence $(g_k) \subset C^1(\mathbb{S}^N,\mathbb{S}^N)$ and $\psi \in C^1(\mathbb{S}^N,\mathbb{R})$ such that $\lim_{k\to\infty}(|\deg g_k| + |g_k|_W + |g_k|_{W^{1,p}}) = 0$ for all $p < N$, $g_k \to g := (0,\dots,0,1)$ a.e., while

$$\lim_{k\to\infty} \mathbf{J}(g_k,\psi) = +\infty.$$

This will be proved in Section 3.2.

An immediate consequence of Theorem 2 is

Corollary 4. *Let* $N \ge 1$, $\frac{N-1}{N} < \alpha < 1$, $g \in C^1(\mathbb{S}^N,\mathbb{S}^N)$, *and* $\psi \in C^1(\mathbb{S}^N,\mathbb{R})$. *Then*

$$|\mathbf{J}(g,\psi)| \le C\left(\|\psi\|_{L^\infty}|g|_{0,\alpha}^{\frac{N}{\alpha}} + \|\nabla\psi\|_{L^\infty}|g|_{0,\alpha}^N\right),$$

for some positive constant $C = C(\alpha,N)$, *depending only on* α *and* N.

Proof of Corollary 4. Since $\alpha > \frac{N-1}{N}$, it follows that

$$|g|_W \le C_{\alpha,N}|g|_{0,\alpha}.$$

On the other hand, by a direct computation, one has

$$T_{\ell_N}(g) \le C_N|g|_{0,\alpha}^{\frac{N}{\alpha}}. \qquad \square$$

Remark 6. Corollary 4 is optimal in the following sense: Let $N \ge 2$ and $g = (0,\dots,0,1) \in \mathbb{S}^N$. There exist a sequence $(g_k) \subset C^1(\mathbb{S}^N,\mathbb{S}^N)$ and $\psi \in C^1(\mathbb{S}^N,\mathbb{R})$ such that $\lim_{k\to\infty} \|g_k - g\|_{0,\frac{N-1}{N}} = 0$ and

$$\lim_{k\to\infty} \mathbf{J}(g_k,\psi) = +\infty.$$

This will be proved in Section 3.3.

Using Theorem 2, we will establish in Section 2

Corollary 5. *Let* $N \ge 1$, $F \in C^{0,\alpha}(\mathbb{S}^N,\mathbb{R})$ $(0 < \alpha < 1)$, $g \in C^1(\mathbb{S}^N,\mathbb{S}^N)$, *and* $\psi \in C^1(\mathbb{S}^N,\mathbb{R})$. *Then there exists* $\delta > 0$, *depending only on* $\|F\|_{C^{0,\alpha}}$, *such* *that*

$$\left|\int_{\mathbb{S}^N} F(g(x))\det(\nabla g)(x)\psi(x)\,dx\right| \le C\left(\|\psi\|_{L^\infty}T_\delta(g) + \|\nabla\psi\|_{L^\infty}|g|_W^N\right),$$

for some positive constant $C = C(N, \|F\|_{C^{0,\alpha}})$.

Remark 7. Assume $N \ge 3$; then $W^{1,N-1}(\mathbb{S}^N,\mathbb{S}^N) \subset W(\mathbb{S}^N,\mathbb{S}^N)$. Indeed

$$W^{1,N-1}(\mathbb{S}^N,\mathbb{S}^N) \subset W^{1,N-1}(\mathbb{S}^N) \cap L^\infty(\mathbb{S}^N) \subset W(\mathbb{S}^N),$$

with the corresponding inequality

$$\|g\|_W^N \lesssim \|g\|_{W^{1,N-1}}^{N-1}\|g\|_{L^\infty} \tag{1.11}$$

(here we use the fact that $N-1>1$). This is a special case of the following more general case

$$\|g\|_{W^{s,q}} \lesssim \|g\|_{W^{1,p}}^s\|g\|_{L^\infty}^{1-s}, \tag{1.12}$$

with $p = sq$, $p > 1$, $0 < s < 1$; see [16, Cor. 2] (see also [47]). Therefore, in Theorem 2 and Corollary 5 we may replace $|g|_W^N$ by $\|\nabla g\|_{L^{N-1}}^{N-1}$. Inequality (1.11) fails when $N = 2$. However, we do not know whether the following inequality

$$|\mathbf{J}(g,\psi)| \lesssim \left(\|\psi\|_{L^\infty}T_{\ell_N}(g) + \|\nabla\psi\|_{L^\infty}|g|_{W^{1,N-1}}^{N-1}\right) \quad \forall\,\psi \in C^1(\mathbb{S}^N,\mathbb{R})$$

holds when $N = 2$.

On the other hand, for every $N \ge 1$, we have trivially

$$T_\delta(g) \le \delta^{-p}|g|_{W^{s,p}}^p \quad \forall\, 0 < s < 1 \text{ and } sp = N,$$

and thus, in Theorem 2 and Corollary 5, we may replace $T_\delta(g)$ by $\delta^{-p}|g|_{W^{s,p}}^p$. When $\psi \equiv 1$ we obtain an estimate for $|\deg g|$ originally due to J. Bourgain, H. Brezis, and P. Mironescu [10].

We can give a meaning to $\det(\nabla g)$ as a distribution assuming only that $g \in (W\cap\mathrm{VMO})(\mathbb{S}^N,\mathbb{S}^N)$ (when $N = 1$, it suffices to assume that $g \in \mathrm{VMO}(\mathbb{S}^1,\mathbb{S}^1)$). More generally, $F(g)\det(\nabla g)$ is also well-defined if in addition $F \in C^{0,\alpha}(\mathbb{S}^N,\mathbb{R})$ for some $\alpha > 0$ (resp. $F \in C^0(\mathbb{S}^1,\mathbb{R})$) when $N \ge 2$ (resp. $N = 1$). In particular the pullback $g^*\omega$ is well-defined as a distribution when ω is a (smooth) N-form on $\mathbb{S}^N$ and $g \in (W\cap\mathrm{VMO})(\mathbb{S}^N,\mathbb{S}^N)$. All the preceding results remain valid in this framework (see §§6 and 7).

Remark 8. In a subsequent paper [18], we will define $\det(\nabla h)$ for any $h \in W(\mathbb{R}^N,\mathbb{R}^N)$; or more generally, for maps $h \in W(\Omega,\mathbb{R}^N)$, where Ω is an open subset of $\mathbb{R}^N$. A major difference is that VMO is irrelevant there but the space W will play a crucial role. When $g \in W(\mathbb{S}^N,\mathbb{S}^N)\cap C^0(\mathbb{S}^N,\mathbb{S}^N)$ we could use the result of [18] to define directly the distribution $\det(\nabla g)$ as follows. Given a point $x_0 \in \mathbb{S}^N$, fix small spherical caps $\Sigma_r(x_0)$ and $\Sigma_R(g(x_0))$ centered at x_0 and $g(x_0)$ such that $g(\Sigma_r(x_0)) \subset \Sigma_R(g(x_0))$. Then choose $r' > 0$, $R' > 0$, and smooth maps $\pi_1 : B_{r'}(0) \to \Sigma_r(x_0)$ and $\pi_2 : \Sigma_R(g(x_0)) \to B_{R'}(0)$, where $B_\rho(0)$ denotes the ball in $\mathbb{R}^N$ of radius ρ, centered at 0, such that

$$\det(\nabla\pi_1) \equiv 1 \quad \text{and} \quad \det(\nabla\pi_2) \equiv 1.$$

Set $h = \pi_2 \circ g \circ \pi_1 : B_{r'}(0) \to B_{R'}(0)$, so that

$$\det(\nabla g) = (\det \nabla h) \circ \pi_1^{-1} \text{ on } \Sigma_r(x_0).$$

We conclude that $\det(\nabla g)$ is a well-defined distribution on $\mathbb{S}^N$ using a partition of unity. We do not know how to adapt this argument if $g \in W \cap \mathrm{VMO}(\mathbb{S}^N, \mathbb{S}^N)$.

Remark 9. F. Hang and F. Lin [36] considered a notion of distributional Jacobian for maps $g \in W^{\frac{N}{N+1}, N+1}(\mathbb{R}^m, \mathbb{S}^N)$ for $m \geq N+1 \geq 2$ (see also an earlier work of R. Jerrard and M. Soner [40]). In their work the condition $g \in \mathrm{VMO}$ is not necessary. They also proved that if this distribution has finite total mass, then it is an integer multiplicity rectifiable current. In the case $m = N+1$, this result was improved by J. Bourgain, H. Brezis, and P. Mironescu [10]. They proved that, for every $g \in W^{s,p}(\mathbb{S}^{N+1}, \mathbb{S}^N)$ with $sp = N$ for any $0 < s < 1$, $\det(\nabla g)$ is a distribution of the form $\omega_{N+1} \sum_i (\delta_{P_i} - \delta_{N_i})$ in $\mathbb{S}^{N+1}$ such that

$$\sum_i |P_i - N_i| \leq C|g|^p_{W^{s,p}}$$

($\omega_{N+1} = |\mathbb{S}^{N+1}|$). In the special case $N = 1$, this result had been previously established by J. Bourgain, H. Brezis, and P. Mironescu in [9] for maps in $H^{\frac{1}{2}}$. In the same esprit, H. Brezis, P. Mironescu, and A. Ponce [17] studied the distributional Jacobian for maps $g \in W^{1,1}(\Omega, \mathbb{S}^1)$, where Ω is the boundary of a simply connected domain of $\mathbb{R}^3$ and they obtained similar results. Our situation in this paper is completely different: we handle the case $m = N \geq 1$. In our framework, we need the *two conditions*: g must belong to $\mathrm{VMO}(\mathbb{S}^N, \mathbb{S}^N)$ and to $W = W^{\frac{N-1}{N}, N}(\mathbb{S}^N, \mathbb{S}^N)$. The reader may also wonder whether our condition $g \in W = W^{\frac{N-1}{N}, N}$ could be replaced by $W^{s,p}$ with $sp = N-1$ and $0 < s < \frac{N-1}{N}$. However this is not true (see Proposition 1). Finally, let us mention that the case $g : \mathbb{R}^{N+1} \to \mathbb{S}^N$ could be considered as a special case of the situation where $g : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$. In this general setting, we are able to define the distributional Jacobian provided $g \in W^{\frac{N}{N+1}, N+1}(\mathbb{R}^{N+1})$ (which is the same space as in [36]) and this condition is optimal (see [18]).

Finally, we present further properties in the case $N = 1$. Here we have

$$\det(\nabla g) = \det(g, g') = g \wedge g' = \varphi', \tag{1.13}$$

provided we choose the standard positive orientation on $\mathbb{S}^1$ and a locally smooth lifting φ of g ($g = e^{i\varphi}$). We have variants of the above results, which do not involve the space W.

Theorem 3. *Let* $F \in C^0(\mathbb{S}^1, \mathbb{R})$, $(g_k) \subset C^1(\mathbb{S}^1, \mathbb{S}^1)$, $g \in C^1(\mathbb{S}^1, \mathbb{S}^1)$ *be such that*

$$\lim_{k\to\infty} |g_k - g|_{\mathrm{BMO}} = 0.$$

Then

$$\lim_{k\to\infty} \int_{\mathbb{S}^1} F(g_k(x)) \det(\nabla g_k)(x)\psi(x)\,dx = \int_{\mathbb{S}^1} F(g(x)) \det(\nabla g)(x)\psi(x)\,dx, \quad \forall\, \psi \in C^1(\mathbb{S}^1, \mathbb{R}).$$

When $F \equiv 1$, we still have an open problem motivated by Theorems 1 and 3:

Open question 2. Let $(g_k) \subset C^1(\mathbb{S}^1, \mathbb{S}^1)$ and $g \in C^1(\mathbb{S}^1, \mathbb{S}^1)$ be such that

i) $\limsup_{k\to\infty} |g_k - g|_{\mathrm{BMO}(\mathbb{S}^1)} < 1$

and

ii) g_k converges to g a.e. on $\mathbb{S}^1$.

Is it true that

$$\lim_{k\to\infty} \int_{\mathbb{S}^1} \det(\nabla g_k)\psi\,dx = \int_{\mathbb{S}^1} \det(\nabla g)\psi\,dx, \quad \forall\, \psi \in C^1(\mathbb{S}^1, \mathbb{R})? \tag{1.14}$$

Remark 10. We can prove that (1.14) holds in two cases:

a) if $\psi \equiv 1$ (see Proposition 4 in §4.2),
b) if the constant 1 in i) is replaced by some small (universal) constant (see Proposition 10 in §7.3).

Concerning the bound, we have

Theorem 4. *Let* $F \in C^0(\mathbb{S}^1, \mathbb{R})$. *Then there exist constants* $\delta > 0$ *and* C *depending only on* $\|F\|_{C^0}$ *such that for all* $g \in C^1(\mathbb{S}^1, \mathbb{S}^1)$ *and for all* $\psi \in C^1(\mathbb{S}^1, \mathbb{R})$,

$$\left| \int_{\mathbb{S}^1} F(g(x)) \det(\nabla g)(x)\psi(x)\,dx \right| \le C\|\psi\|_{W^{1,\infty}}\Big(T_\delta(g) + 1\Big).$$

Other types of results concerning $\mathbb{S}^N$-valued maps can be found in e.g. [13], [26], [27], [5], [31], [20], [32], [33], [15], [36], [40], [42], [41], [2], [12], [9], [44], [1], [29], [6], [11], [45], [46], and [48].

The Jacobian determinant of maps from $\mathbb{R}^N$ into $\mathbb{R}^N$ has been extensively studied in the literature; see e.g. [49], [56], [3], [4], [51], [52], [24], [39], [14], [21], [53], [35], [37], [32], [33], [38], [28], and [30].

We first present the proof of Theorem 2 which is inspired by the works of J. Bourgain, H. Brezis, and P. Mironescu [9], [10], J. Bourgain, H. Brezis, and H.-M. Nguyen [7], and H.-M. Nguyen [54] related to an estimate for the topological degree. We then turn to the proof of Theorem 1 which uses a similar device.

2. The main bounds: Proofs of Theorem 2 and Corollary 5

We first give another representation of $\mathbf{J}(g,.)$. This representation is inspired by the work of J. Bourgain, H. Brezis, and P. Mironescu in [9, Lemma 3]. Their idea is also used in [36].

Hereafter in this paper, B denotes the unit ball in $\mathbb{R}^{N+1}$.

LEMMA 1. *Let* $N \geq 1$. *Assume that* $g \in W^{1,N}(\mathbb{S}^N, \mathbb{S}^N)$ (*and in addition* $g \in H^{\frac{1}{2}}(\mathbb{S}^1)$ *if* $N = 1$), $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$, $u \in W^{1,N+1}(B, \mathbb{R}^{N+1}) \cap L^\infty(B)$, *and* $\varphi \in C^1(\bar{B}, \mathbb{R})$. *Suppose that*

$$u_{|\mathbb{S}^N} = g \quad \textit{and} \quad \varphi_{|\mathbb{S}^N} = \psi.$$

Then

$$\mathbf{J}(g,\psi) = (N+1)\int_B \varphi \det(\nabla u)\,dx + \sum_{i=1}^{N+1}\int_B \partial_i\varphi \mathbf{D}_i(u)\,dx, \tag{2.1}$$

where $\mathbf{D}_i(v)$ *is given by*

$$\mathbf{D}_i(v) = \det\left(\partial_1 v, \ldots, \partial_{i-1}v, v, \partial_{i+1}v, \ldots, \partial_{N+1}v\right), \quad \forall\, v \in W^{1,N+1}(B, \mathbb{R}^{N+1}) \cap L^\infty(B),$$

for $1 \leq i \leq N+1$.

Proof. Case 1: $g \in C^2(\mathbb{S}^N, \mathbb{S}^N)$ *and* $u \in C^2(\bar{B})$. We first note that

$$(N+1)\det(\nabla u) = \operatorname{div}\mathbf{D}.$$

Hence by Green's formula, one has

$$\int_B (N+1)\varphi\det(\nabla u)\,dx = -\int_B \sum_{i=1}^{N+1}\partial_i\varphi\mathbf{D}_i(u)\,dx + \int_{\mathbb{S}^N}\sum_{i=1}^{N+1}\varphi\mathbf{D}_i(u)n_i\,dx.$$

However, $\sum_{i=1}^{N+1}\mathbf{D}_i n_i = \det(\nabla g, g)$ on $\mathbb{S}^N$, and the conclusion follows.

Case 2: *The general case.* Let $(g_k) \subset C^\infty(\mathbb{S}^N, \mathbb{S}^N)$ be a sequence converging to g in $W^{1,N}(\mathbb{S}^N)$. Let $\tilde{u}$ and $\tilde{u}_k$ be the harmonic extensions of g and g_k on B. From (1.12), we have $g \in W^{\frac{N}{N+1}, N+1}(\mathbb{S}^N)$ when $N \geq 2$ since $g \in W^{1,N} \cap L^\infty$. This implies $\tilde{u} \in W^{1,N+1}(B)$ and $\tilde{u}_k \to \tilde{u}$ in $W^{1,N+1}(B)$. Let (v_k) be a sequence of $C^\infty_c(B, \mathbb{R}^{N+1})$, such that (v_k) converges to $u - \tilde{u}$ in $W^{1,N+1}(B)$ and $\sup_k \|v_k\|_{L^\infty(B)} < \infty$. Since $u - \tilde{u} \in W^{1,N+1}_0(B)$ and $u - \tilde{u} \in L^\infty(B)$ such a sequence exists. Set $u_k = v_k + \tilde{u}_k$ for $k \geq 1$. Then $u_{k|\mathbb{S}^N} = g_k$, u_k converges to u in $W^{1,N+1}(B)$, and $\sup_k \|u_k\|_{L^\infty(B)} < \infty$. From Case 1, one has

$$\mathbf{J}(g_k,\psi) = (N+1)\int_B \varphi\det(\nabla u_k)\,dx + \sum_{i=1}^{N+1}\int_B \partial_i\varphi\mathbf{D}_i(u_k)\,dx.$$

Letting k go to infinity yields

$$\mathbf{J}(g,\psi) = (N+1)\int_B \varphi \det(\nabla u)\,dx + \sum_{i=1}^{N+1}\int_B \partial_i\varphi \mathbf{D}_i(u)\,dx. \qquad \square$$

We are ready to present the

Proof of Theorem 2. Let $\tilde{u} : B \mapsto \mathbb{R}^{N+1}$ be the extension by average of g, i.e.,

$$\tilde{u}(rx) = \fint_{B(x,2(1-r))\cap\mathbb{S}^N} g(y)\,dy, \quad \forall\, x \in \mathbb{S}^N, r \in (0,1),$$

and $\varphi \in C^1(\bar{B})$ be an extension of ψ such that

$$\|\varphi\|_{L^\infty(B)} \lesssim \|\psi\|_{L^\infty(\mathbb{S}^N)} \quad \text{and} \quad \|\nabla\varphi\|_{L^\infty(B)} \lesssim \|\nabla\psi\|_{L^\infty(\mathbb{S}^N)}.$$

Hereafter in this proof $B(x,r) := \{y \in \mathbb{S}^N; |y-x| \le r\}$. The notation $a \lesssim b$ means that there exists a constant C depending only on N such that $a \le Cb$. The notation $a \gtrsim b$ means that $b \lesssim a$.

It is well-known that the mapping $g \mapsto \tilde{u}$ is a bounded linear operator from $W(\mathbb{S}^N)$ into $W^{1,N}(B)$ and

$$|\nabla\tilde{u}(ry)| \lesssim \frac{1}{1-r} \quad \forall\, y \in \mathbb{S}^N, r \in (0,1). \tag{2.2}$$

Set $\alpha = \frac{1}{250(N+1)}$. Define $u : B \to \mathbb{R}^{N+1}$ as follows:

$$u(X) = \begin{cases} \dfrac{\tilde{u}(X)}{|\tilde{u}(X)|} & \text{if } |\tilde{u}(X)| \ge \alpha, \\ \dfrac{1}{\alpha}\tilde{u}(X) & \text{otherwise.} \end{cases} \tag{2.3}$$

From Lemma 1, we have

$$|\mathbf{J}(g,\psi)| \lesssim \|\psi\|_{L^\infty(\mathbb{S}^N)}\int_B |\det(\nabla u)|\,dx + \|\nabla\psi\|_{L^\infty(\mathbb{S}^N)}\int_B |\nabla u(x)|^N\,dx. \tag{2.4}$$

Since $\det(\nabla u) = 0$ if $|\tilde{u}| \ge \alpha$, it follows from (2.2) that

$$\int_B |\det(\nabla u)|\,dx \lesssim \int_{\mathbb{S}^N}\int_0^{1-\rho(y)} \frac{1}{(1-r)^{N+1}}\,dr\,dy,$$

where $\rho : \mathbb{S}^N \mapsto \mathbb{R}$ is defined by

$$\rho(y) = \sup\{r;\, |\tilde{u}((1-s)y)| \ge \alpha \ \text{ for all}\, 0 < s < r\}.$$

This implies, as in [10] (see also [7] and [54]),

$$\int_B |\det(\nabla u)|\,dx \lesssim \int_{\substack{\mathbb{S}^N\\ \rho(y)<1}} \frac{1}{\rho(y)^N}\,dy. \tag{2.5}$$

Using the idea in the proof of [54, Lemma 6] (see also [7] when T_{ℓ_N} is replaced by T_δ, for the case $0 < \delta < \sqrt{2}$), one has

$$\int_{\substack{\mathbb{S}^N \\ \rho(y)<1}} \frac{1}{\rho(y)^N}\, dx \lesssim T_{\ell_N}(g). \tag{2.6}$$

Thus it follows from (2.5) and (2.6) that

$$\int_B |\det(\nabla u)|\, dx \lesssim T_{\ell_N}(g). \tag{2.7}$$

On the other hand, from (2.3), one has

$$\int_B |\nabla u|^N\, dx \lesssim \int_B |\nabla \tilde{u}|^N\, dx \tag{2.8}$$

and, since $\tilde{u}$ is the extension by average of g,

$$\int_B |\nabla \tilde{u}(x)|^N\, dx \lesssim |g|_W^N.$$

Thus

$$\int_B |\nabla u(x)|^N\, dx \lesssim |g|_W^N. \tag{2.9}$$

The conclusion follows from (2.4), (2.7), and (2.9). □

Remark 11. By the same proof, we can also obtain that

$$|\mathbf{J}(g,\psi)| \lesssim \Big(\|\psi\|_{L^\infty} T_{\ell_N}(g) + |\psi|_{W^{1-\frac{1}{q},q}} |g|^N_{W^{1-\frac{1}{Np},Np}}\Big),$$

where $\frac{1}{q} + \frac{1}{p} = 1$, $1 \le p < \infty$ (the case $p = 1$ corresponds to Theorem 2). Thus

$$|\mathbf{J}(g,\psi)| \lesssim \Big(\|\psi\|_{L^\infty} T_{\ell_N}(g) + |\psi|_{0,\beta} |g|^N_{0,\alpha}\Big),$$

for any $\alpha > \frac{N-1}{N}$ and $1 > \beta > N(1-\alpha)$.

We next turn to

Proof of Corollary 5. We first recall the fact, due to B. Dacorogna and J. Moser [23] (see also [50] and [34]), that if $G \in C^{0,\alpha}(\mathbb{S}^N, \mathbb{R})$ $(0 < \alpha < 1)$ is such that $G > 0$ on $\mathbb{S}^N$ and $⨍_{\mathbb{S}^N} G = 1$, then there exists $\mathcal{G} \in C^{1,\alpha}(\mathbb{S}^N, \mathbb{S}^N)$ such that $\det(\nabla \mathcal{G}) = G$ with a bound for $\|\mathcal{G}\|_{C^{1,\alpha}}$ depending only on $\|G\|_{C^{0,\alpha}}$. Define

$$G = c_1(F + c_2),$$

where $c_1 > 0$ and $c_2 > 0$, depending only on $\|F\|_{C^0}$, are chosen such that $G > 0$ and $⨍_{\mathbb{S}^N} G = 1$. Hence there exists $\mathcal{G} : \mathbb{S}^N \mapsto \mathbb{S}^N$ such that $\det(\nabla \mathcal{G}) = G$ (with a bound for $\|\mathcal{G}\|_{C^{1,\alpha}}$ depending only on $\|F\|_{C^{0,\alpha}}$). This implies

$$G(g) \det(\nabla g) = \det \nabla \mathcal{G}(g).$$

Next, applying Theorem 2 to $\mathcal{G}(g)$, one has

$$|\mathbf{J}(\mathcal{G}(g),\psi)| \lesssim T_{\ell_N}(\mathcal{G}(g))\|\psi\|_{L^\infty} + |\mathcal{G}(g)|_W^N \|\nabla\psi\|_{L^\infty}.$$

However,

$$T_{\ell_N}(\mathcal{G}(g)) \le T_\delta(g),$$

for some $\delta > 0$, e.g., $\delta = \ell_N/(\|\nabla\mathcal{G}\|_{L^\infty}+1)$, and

$$|\mathcal{G}(g)|_W \le C|g|_W,$$

for some $C > 0$. Hence

$$|\mathbf{J}(\mathcal{G}(g),\psi)| \lesssim T_\delta(g)\|\psi\|_{L^\infty} + |g|_W^N \|\nabla\psi\|_{L^\infty}. \tag{2.10}$$

On the other hand, from the definition of G,

$$\begin{aligned}\Big|\int_{\mathbb{S}^N} F(g)\det(\nabla g)\psi\Big| &\le \frac{1}{c_1}\Big|\int_{\mathbb{S}^N} G(g)\det(\nabla g)\psi\Big| + c_2\Big|\int_{\mathbb{S}^N}\det(\nabla g)\psi\Big| \\ &= \frac{1}{c_1}\Big|\mathbf{J}(\mathcal{G}(g),\psi)\Big| + c_2\Big|\int_{\mathbb{S}^N}\det(\nabla g)\psi\Big|.\end{aligned}$$

Applying Theorem 2 to g and using (2.10), one has

$$\Big|\int_{\mathbb{S}^N} F(g)\det(\nabla g)\psi\Big| \lesssim T_\delta(g)\|\psi\|_{L^\infty} + |g|_W^N\|\nabla\psi\|_{L^\infty}. \qquad \square$$

3. **Optimality of the bounds: Proofs of the statements in Remarks 4, 5, and 6**

3.1. *Proof of the statement in Remark* 4. Let $\mathcal{N}$ and $\mathcal{S}$ be the north pole and the south pole of $\mathbb{S}^N$; i.e., $\mathcal{N} = (0,\dots,0,1)$, $\mathcal{S} = (0,\dots,0,-1)$. Let exp be the exponential map on $\mathbb{S}^N$ (see e.g. [25]). For $k \gg 1$, take $m \in \mathbb{N}$ such that $m \approx \ln k$. Consider $g_k \in W^{1,\infty}(\mathbb{S}^N)$ such that $g_k(x) = \mathcal{N}$ for $x \ne \exp_{\mathcal{N}}(tv)$ for $v \in \mathbb{R}^N$, $|v| = 1$ and $t \in [0, 2m/k]$, and satisfying (a) and (b) below:

(a) $g_k(\exp_{\mathcal{N}}(tv)) = \exp_{\mathcal{N}}(k\pi(t-i/k)w_i)$ if $v \in \mathbb{R}^N, |v|=1, t \in [i/k,(i+1)/k]$, for $0 \le i \le 2m-1$ and i even, where w_i is chosen in the set $\{(v', v_N), (v', -v_N)\}$ $(v = (v', v_N) \in \mathbb{R}^{N-1}\times\mathbb{R})$ such that

$$\text{sign } \det(\nabla g_k(\exp_{\mathcal{N}}(t_i v))) = 1$$

with $t_i = i/k + 1/(4k)$.

(b) $g_k(\exp_{\mathcal{N}}(tv)) = \exp_{\mathcal{S}}(k\pi(t-i/k)w_i)$ if $|v| = 1, t \in [i/k,(i+1)/k]$, for $0 \le i \le 2m-1$ and i odd, where w_i is chosen in the set $\{(v', v_N), (v', -v_N)\}$ $(v = (v', v_N) \in \mathbb{R}^{N-1}\times\mathbb{R})$ such that

$$\text{sign } \det\nabla g_k((\exp_{\mathcal{N}}(t_i v))) = 1$$

with $t_i = i/k + 1/(4k)$.

Then

$$\deg g_k = 2m,$$

and

$$|\nabla g_k| \lesssim \begin{cases} k & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$A = \Big\{ \exp_{\mathcal{N}}(tv) \text{ with } t \in [0, 2m/k] \text{ and } |v| = 1 \Big\} \subset \mathbb{S}^N.$$

Since $|A| \lesssim (m/k)^N$ and $m \approx \ln k$, we obtain

$$\lim_{k\to\infty} \int_{\mathbb{S}^N} |\nabla g_k|^p \, dx \lesssim \lim_{k\to\infty} (\ln k/k)^N k^p = 0.$$

The conclusion follows by a standard regularization argument. □

3.2. *Proof of the statement in Remark* 5. We use the same notations as above for $\mathcal{S}$, $\mathcal{N}$, and exp. For $k \gg 1$, take $m \in \mathbb{N}$ such that $m \approx \ln k$. Define $g_k \in W^{1,\infty}(\mathbb{S}^N, \mathbb{S}^N)$ as follows:

(i) For $x \neq \exp_{\mathcal{N}}(tv)$ and $x \neq \exp_{\mathcal{S}}(tv)$, where $v \in \mathbb{R}^N$, $|v| = 1$, and $t \in [0, 2m/k]$, $g_k(x) := \mathcal{N}$.

(ii) For $x = \exp_{\mathcal{N}}(tv)$ with $t \in [0, 2m/k]$ and $|v| = 1$, we define g_k as follows:

$$g_k(\exp_{\mathcal{N}}(tv)) = \exp_{\mathcal{N}}(k\pi(t - i/k)w_i) \text{ if } v \in \mathbb{R}^N, |v| = 1, t \in [i/k, (i+1)/k],$$

for $0 \le i \le 2m-1$ and i even, where w_i is chosen in the set $\{(v', v_N), (v', -v_N)\}$ $(v = (v', v_N) \in \mathbb{R}^{N-1} \times \mathbb{R})$ such that

$$\text{sign } \det(\nabla g_k(\exp_{\mathcal{N}}(t_i v))) = 1$$

with $t_i = i/k + 1/(4k)$, and

$$g_k(\exp_{\mathcal{N}}(tv)) = \exp_{\mathcal{S}}(k\pi(t - i/k)w_i) \quad \text{if } |v| = 1, t \in [i/k, (i+1)/k],$$

for $0 \le i \le 2m-1$ and i odd, where w_i is chosen in the set $\{(v', v_N), (v', -v_N)\}$ $(v = (v', v_N) \in \mathbb{R}^{N-1} \times \mathbb{R})$ such that

$$\text{sign } \det \nabla g_k((\exp_{\mathcal{N}}(t_i v))) = 1$$

with $t_i = i/k + 1/(4k)$.

(iii) For $x = \exp_{\mathcal{S}}(tv)$ with $t \in [0, 2m/k]$ and $|v| = 1$, we define g_k as follows:

$$g_k(\exp_{\mathcal{S}}(tv)) = \exp_{\mathcal{N}}(k\pi(t - i/k)w_i) \text{ if } v \in \mathbb{R}^N, |v| = 1, t \in [i/k, (i+1)/k],$$

for $0 \le i \le 2m-1$ and i even, where w_i is chosen in the set $\{(v', v_N), (v', -v_N)\}$ $(v = (v', v_N) \in \mathbb{R}^{N-1} \times \mathbb{R})$ such that

$$\text{sign } \det \nabla g_k((\exp_{\mathcal{S}}(t_i v))) = -1 \tag{3.1}$$

with $t_i = i/k + 1/(4k)$, and

$$g_k(\exp_{\mathcal{S}}(tv)) = \exp_{\mathcal{S}}(k\pi(t - i/k)w_i) \quad \text{if } |v| = 1, t \in [i/k, (i+1)/k],$$

for $0 \le i \le 2m-1$ and i odd, where w_i is chosen in the set $\{(v', v_N), (v', -v_N)\}$ ($v = (v', v_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$) such that

$$\text{sign } \det \nabla g_k((\exp_{\mathcal{S}}(t_i v))) = -1 \tag{3.2}$$

with $t_i = i/k + 1/(4k)$.

From the definition of g_k, we have

$$\deg g_k = 0$$

and $g_k \to g := (0, \dots, 0, 1)$ a.e.

Define $h_k \in W^{1,\infty}(\mathbb{S}^N, \mathbb{S}^N)$ similarly as g_k, however in the right-hand side of (3.1) and (3.2), we take $+1$ instead of -1. Fix $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$ such that $\psi = -1$ if $x_{N+1} \le -1/2$ and $\psi = 1$ if $x_{N+1} > 1/2$. Then

$$\frac{1}{|\mathbb{S}^N|}\mathbf{J}(g_k, \psi) = \frac{1}{|\mathbb{S}^N|}\mathbf{J}(h_k, 1) = \deg(h_k) = 2m.$$

On the other hand, since g_k is Lipschitz with $\text{Lip}\,(g_k) \lesssim k$ and $m \approx \ln k$, it follows from the construction of g_k that, for $1 \le p < N$,

$$\int_{\mathbb{S}^N} |\nabla g_k|^p \lesssim k^p \Big(\frac{m}{k}\Big)^N \lesssim \ln^p k / k^{N-p} \to 0 \text{ as } k \to \infty.$$

Therefore, since $\|g_k\|_{L^\infty} = 1$ it follows from interpolation inequalities that $\lim_{k\to\infty} |g_k|_W = 0$.

By a standard regularization argument, we may construct a sequence in $C^1(\mathbb{S}^N, \mathbb{S}^N)$ with similar properties. □

3.3. *Proof of the statement in Remark* 6. We only prove here that there exist $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ and $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$ such that $\sup_k \|g_k\|_{0,\alpha} < +\infty$ for all $0 < \alpha < \frac{N-1}{N}$, $g_k \to g$ uniformly on $\mathbb{S}^N$, and

$$\lim_{k\to\infty} \mathbf{J}(g_k, \psi) = +\infty.$$

The proof in the general case, which is more involved, uses the same technique as in [18, Prop. 4].

Let $v_k = (v_{1,k}, \dots, v_{N,k}) : \mathbb{R}^N \to \mathbb{R}^N$ ($k \ge 1$) be defined as follows:

$$v_{i,k}(x) = k^{-\alpha} \sin(kx_i), \quad \forall\, 1 \le i \le N-1$$

and

$$v_{N,k}(x) = k^{-\alpha} x_N \prod_{i=1}^{N-1} \cos(kx_i).$$

We have

$$\det \nabla v_k = k^{(N-1)(1-\alpha)-\alpha} \prod_{i=1}^{N-1} \cos^2(kx_i) \ge 0. \tag{3.3}$$

Set

$$G_k = \varphi v_k,$$

where $\varphi \in C^1(\mathbb{R}^N)$ is such that $\operatorname{supp}\varphi \subset B^N_{1/2}$ and $\varphi = 1$ in $B^N_{1/4}$. Hereafter in this proof B^N_r denotes the open ball of $\mathbb{R}^N$ of radius r centered at the origin. Since $\|G_k\|_{L^\infty} \lesssim k^{-\alpha}$ and $\|\nabla G_k\|_{L^\infty} \lesssim k^{1-\alpha}$, it is clear that

$$\|G_k\|_{0,\alpha} \lesssim 1. \tag{3.4}$$

Set

$$\Sigma = \{x \in \mathbb{S}^N; |x'| < 1/2 \text{ and } x_{N+1} > 0\},$$

where $x = (x', x_{N+1}) \in \mathbb{R}^N \times \mathbb{R}$. Let $\phi : B^N_{1/2} \to \Sigma$ be defined by $\phi(x') = (x', \sqrt{1-|x'|^2})$. Clearly ϕ is bijective, ϕ, ϕ^{-1} are smooth, and $\det(\nabla\phi) \gtrsim 1$, $\det \nabla\phi^{-1} \gtrsim 1$.

Define $g_k : \mathbb{S}^N \to \mathbb{S}^N$ (for k large) as follows:

$$g_k(x) = \begin{cases} \phi \circ G_k \circ \phi^{-1}(x) & \text{if } x \in \Sigma, \\ (0, \dots, 0, 1) & \text{otherwise.} \end{cases}$$

Let $\psi \in C^1(\mathbb{S}^N)$ be such that $\operatorname{supp}\psi \subset \phi(B^N_{1/4})$, $0 \le \psi \le 1$, and $\psi = 1$ in $\phi(B^N_{1/8})$. Then

$$\begin{aligned}
&\int_{\mathbb{S}^N} \det(\nabla g_k)\psi \\
&\quad = \int_{\mathbb{S}^N} \det(\nabla\phi)(G_k \circ \phi^{-1}(x)) \det(\nabla G_k)(\phi^{-1}(x)) \det(\nabla\phi^{-1})(x)\psi(x)\, dx \\
&\quad = \int_{B^N_{1/4}} \det(\nabla\phi)(G_k(y)) \det(\nabla G_k)(y)\psi(\phi(y))\, dy.
\end{aligned}$$

Thus from the definition of g_k, v_k, ψ, φ, and (3.3) we have

$$\int_{\mathbb{S}^N} \det(\nabla g_k)\psi \gtrsim \int_{B^N_{1/8}} \det(\nabla v_k) \gtrsim k^{(N-1)(1-\alpha)-\alpha}. \tag{3.5}$$

Since $0 < \alpha < \frac{N-1}{N}$, the conclusion follows from (3.4) and the definition of g_k. □

4. **The main convergence results: Proofs of Theorem 1 and Corollary 3**

4.1. *Proofs of Theorem* 1 *and Corollary* 3.

Proof of Theorem 1. Set

$$a = \limsup_{k\to\infty} |g_k - g|_{\mathrm{BMO}(\mathbb{S}^N)} < 1$$

and define

$$\varepsilon_0 = \frac{1-a}{4}.$$

Let $\tilde{u}$ and $\tilde{u}_k$ be the extensions by average of g, as in the proof of Theorem 2, and g_k ($k \in \mathbb{N}$) respectively. Fix $\alpha \in (\varepsilon_0, 2\varepsilon_0)$ and let u and u_k be the functions defined on B as follows:

$$u(X) = \begin{cases} \dfrac{\tilde{u}(X)}{|\tilde{u}(X)|} & \text{if } |\tilde{u}(X)| \geq \alpha, \\ \dfrac{1}{\alpha}\tilde{u}(X) & \text{otherwise} \end{cases}$$

and

$$u_k(X) = \begin{cases} \dfrac{\tilde{u}_k(X)}{|\tilde{u}_k(X)|} & \text{if } |\tilde{u}_k(X)| \geq \alpha, \\ \dfrac{1}{\alpha}\tilde{u}_k(X) & \text{otherwise,} \end{cases}$$

for all $k \geq 1$. Let $\varphi \in C^1(\bar{B})$ be an extension of ψ in $\bar{B}$. Then by Lemma 1, one has

$$\mathbf{J}(g, \psi) = (N+1) \int_B \varphi \det \nabla u \, dx + \sum_{i=1}^{N+1} \int_B \partial_i \varphi \mathbf{D}_i(u) \, dx \tag{4.1}$$

and

$$\mathbf{J}(g_k, \psi) = (N+1) \int_B \varphi \det \nabla u_k \, dx + \sum_{i=1}^{N+1} \int_B \partial_i \varphi \mathbf{D}_i(u_k) \, dx. \tag{4.2}$$

We claim that

$$\lim_{k \to \infty} \int_B \varphi \det \nabla u_k \, dx = \int_B \varphi \det \nabla u \, dx. \tag{4.3}$$

Indeed, since $g \in \mathrm{VMO}(\mathbb{S}^N, \mathbb{S}^N)$, there exists $d > 0$ such that

$$\fint_{B(y,r)} \left| g(\xi) - \fint_{B(y,r)} g(\eta) \, d\eta \right| d\xi \leq \varepsilon_0, \quad \forall\, y \in \mathbb{S}^N, \forall\, r \in (0, 2d).$$

Thus

$$|\tilde{u}(ry)| \geq 1 - \varepsilon_0 > \alpha, \quad \forall\, y \in \mathbb{S}^N, \forall\, r \in (1-d, 1),$$

which shows that

$$\det \nabla u(ry) = 0, \quad \forall\, y \in \mathbb{S}^N, \forall\, r \in (1-d, 1). \tag{4.4}$$

Moreover, since $\limsup_{k \to \infty} |g_k - g|_{\mathrm{BMO}(\mathbb{S}^N)} = a$, there exists $m \in \mathbb{N}$ such that

$$\fint_{B(y,r)} \left| g_k(\xi) - \fint_{B(y,r)} g_k(\eta) \, d\eta \right| d\xi \\ \leq |g_k - g|_{\mathrm{BMO}} + \fint_{B(y,r)} \left| g(\xi) - \fint_{B(y,r)} g(\eta) \, d\eta \right| d\xi \leq a + 2\varepsilon_0$$

for all $y \in \mathbb{S}^N$, $r \in (0, 2d)$, and for all $k \geq m$. Thus

$$|\tilde{u}_k(ry)| \geq 1 - a - 2\varepsilon_0 = 2\varepsilon_0 > \alpha, \quad \forall\, y \in \mathbb{S}^N, \forall\, r \in (1-d, 1), \forall\, k \geq m,$$

which shows that

$$\det \nabla u_k(ry) = 0, \quad \forall\, y \in \mathbb{S}^N, \forall\, r \in (1-d, 1), \forall\, k \geq m. \tag{4.5}$$

Combining (4.4) and (4.5) yields

$$\lim_{k\to\infty} \int_{|x|\geq 1-d} \varphi \det \nabla u_k \; dx = \int_{|x|\geq 1-d} \varphi \det \nabla u \, dx = 0. \tag{4.6}$$

On the other hand, since g_k converges to g in $L^1(\mathbb{S}^N)$, we may assume (passing to a subsequence still denoted (g_k)) that $g_k \to g$ a.e. on $\mathbb{S}^N$. This implies

$$\lim_{k\to\infty} \nabla u_k(x) = \nabla u(x), \quad \text{for a.e. } x \in B.$$

Moreover, since $\tilde{u}_k$ is the extension by average of g_k,

$$|\nabla u_k(ry)| \lesssim 1/(1-r), \quad \forall\, y \in \mathbb{S}^N, \forall\, r \in (0,1).$$

Thus applying the Lebesgue dominated convergence theorem, one gets

$$\lim_{k\to\infty} \int_{|x|\leq 1-d} \varphi \det \nabla u_k \, dx = \int_{|x|\leq 1-d} \varphi \det \nabla u \, dx, \quad \forall\, d \in (0,1). \tag{4.7}$$

Combining (4.6) and (4.7) yields

$$\lim_{k\to\infty} \int_B \varphi \det \nabla u_k \, dx = \int_B \varphi \det \nabla u \, dx.$$

Thus (4.3) is established.

Next, we claim that

$$\lim_{k\to\infty} \int_B \partial_i \varphi \mathbf{D}_i(u_k) \, dx = \int_B \partial_i \varphi \mathbf{D}_i(u) \, dx, \quad \forall\, 1 \leq i \leq N+1. \tag{4.8}$$

This is obvious since $u_k \to u$ in $W^{1,N}(B)$, $|u_k| \leq 1$ and $u_k \to u$ a.e. in B.

Combining (4.1), (4.2), (4.3), and (4.8), we obtain

$$\lim_{k\to\infty} \int_{\mathbb{S}^N} \psi \det(g_k, \nabla g_k) \, dy = \int_{\mathbb{S}^N} \psi \det(g, \nabla g) \, dy. \qquad \square$$

Proof of Corollary 3. Corollary 3 is a consequence of Theorem 1. Indeed, it suffices to prove that any subsequence of g_k (still denoted by g_k) there exists a subsequence g_{n_k} such that

$$\lim_{k\to\infty} \int_{\mathbb{S}^N} F(g_k(x)) \det(\nabla g_k)(x) \psi(x) \, dx = \int_{\mathbb{S}^N} F(g(x)) \det(\nabla g)(x) \psi(x) \, dx,$$

for all $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$. Since $\lim_{k\to\infty} |g_k - g|_{\mathrm{BMO}} = 0$, there exists a subsequence (g_{n_k}) of g_k and $c \in \mathbb{R}^{N+1}$ such that g_{n_k} converges to $g + c$ in $L^1(\mathbb{S}^N)$. It is clear that $g + c \in \mathbb{S}^N$ for almost every $x \in \mathbb{S}^N$. Hence either $c = 0$, or $c \neq 0$ and $g \cdot c = \text{constant}$.

Case 1: $c = 0$. Then g_k converges to g in W. Let $\mathcal{G}$ be the function defined as in the proof of Corollary 5. Since

$$\lim_{k\to\infty} |g_k - g|_{\mathrm{BMO}} = 0 \text{ and } \lim_{k\to\infty} \|g_k - g\|_W = 0,$$

it follows that $\lim_{k\to\infty} |\mathcal{G}(g_k) - \mathcal{G}(g)|_{\mathrm{BMO}} = 0$ and $\lim_{k\to\infty} \|\mathcal{G}(g_k) - \mathcal{G}(g)\|_W = 0$. Thus one can apply Theorem 1 and the conclusion follows.

Case 2: $c \neq 0$ and $g \cdot c = \text{constant}$. Then $\det(\nabla g) = 0$, which implies

$$\int_{\mathbb{S}^N} F(g(x)) \det(\nabla g)(x) \psi(x)\, dx = 0$$

and, as in the case $c = 0$,

$$\lim_{k\to\infty} \int_{\mathbb{S}^N} F(g_k(x)) \det(\nabla g_k)(x) \psi(x)\, dx = \int_{\mathbb{S}^N} F(g(x) + c) \det(\nabla (g + c))(x) \psi(x)\, dx = 0.$$

The conclusion follows. □

4.2. *A remark on the degree.* Motivated by Theorem 1 we prove

Proposition 4. *Let $(g_k) \subset C(\mathbb{S}^N, \mathbb{S}^N)$ and $g \in C(\mathbb{S}^N, \mathbb{S}^N)$. Suppose that g_k converges to g for almost every $x \in \mathbb{S}^N$ and*

$$\limsup_{k\to\infty} |g_k - g|_{\mathrm{BMO}(\mathbb{S}^N)} < 1.$$

Then

$$\lim_{k\to\infty} \deg g_k = \deg g.$$

Proof. We could prove Proposition 4 by using the same method as in the proof of Theorem 1. Nevertheless, we present here a direct argument.

Since $\limsup_{k\to\infty} |g_k - g|_{\mathrm{BMO}(\mathbb{S}^N)} < 1$ and $g \in C(\mathbb{S}^N, \mathbb{S}^N)$, there exist $r_0 > 0$, $k_0 > 0$, and $\varepsilon_0 > 0$ such that $\left| \fint_{B(x,r)} g_k \, d\sigma \right| > \varepsilon_0$ and $\left| \fint_{B(x,r)} g \, d\sigma \right| > \varepsilon_0$, for all $r \le r_0$ and $k > k_0$. Set

$$g_{k,r_0}(x) = \frac{\displaystyle\fint_{B(x,r_0)} g_k \, d\sigma}{\left| \displaystyle\fint_{B(x,r_0)} g_k \, d\sigma \right|} \quad \text{and} \quad g_{r_0}(x) = \frac{\displaystyle\fint_{B(x,r_0)} g \, d\sigma}{\left| \displaystyle\fint_{B(x,r_0)} g \, d\sigma \right|},$$

where $B(x,r) = \{y \in \mathbb{S}^N; |y - x| < r\}$. Then $\deg g_{k,r_0} = \deg g_k$, $\deg g_{r_0} = \deg g$, and g_{k,r_0} converges uniformly to g_{r_0} in $\mathbb{S}^N$. This implies

$$\lim_{k\to\infty} \deg g_k = \deg g. \qquad \square$$

Remark 12. Proposition 4 is a slight improvement of the result of H. Brezis and L. Nirenberg [20] which asserts that if $\lim_{k\to\infty} |g_k - g|_{\mathrm{BMO}} = 0$, then $\lim_{k\to\infty} \deg g_k = \deg g$.

5. **Proofs of Proposition 1 and Remark 2**

5.1. *Proof of Proposition* 1. We only prove here that there exist $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ and $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$ such that $\lim_{k\to\infty} \|g_k - g\|_{0,\alpha} < +\infty$ for all $0 < \alpha < \frac{N-1}{N}$, $\sup_k \|g_k\|_{0,\frac{N-1}{N}} < +\infty$, $\sup_k \|g_k\|_{W^{\frac{N-1}{N},N}} < +\infty$, and

$$\lim_{k\to\infty} \mathbf{J}(g_k, \psi) = +\infty.$$

The proof in the general case, which is more involved, uses the same technique as in [18, Prop. 4].

Define (g_k) and ψ as in the proof of the statement in Remark 6 (see §3.3) with $\alpha = \frac{N-1}{N}$. Then

$$\sup_k \|g_k\|_{0,\frac{N-1}{N}} < \infty,$$

g_k converges uniformly to $g := (0, \dots, 0, 1)$, and

$$\liminf_{k\to\infty} \mathbf{J}(g_k, \psi) > 0 = \mathbf{J}(g, \psi),$$

by (3.5).

It remains to check that $\sup_{k\in\mathbb{N}} |g_k|_W < \infty$. Indeed, from the definition of g_k, it suffices to prove that, with $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$,

$$\sup_k \left(|k^{-\frac{N-1}{N}} \sin(kx_N)|_{W([-1,1]^N)} + |k^{-\frac{N-1}{N}} \cos(kx_N)|_{W([-1,1]^N)} \right) < +\infty.$$

A standard computation yields

$$\int_{[-1,1]^{N-1}} \int_{[-1,1]^{N-1}} \frac{1}{|x-y|^{2N-1}}\, dx'\, dy' \lesssim \frac{1}{|x_N - y_N|^N},$$

where $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and $y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. This implies (5.1)

$$|k^{-\frac{N-1}{N}} \sin(kx_N)|^N_{W([-1,1]^N)} \lesssim \int_{-1}^1 \int_{-1}^1 \frac{k^{-(N-1)} |\sin(kt) - \sin(ks)|^N}{|t-s|^N}\, ds\, dt.$$

On the other hand,

$$\int_{-1}^1 \int_{-1}^1 \frac{k^{-(N-1)} |\sin(kt) - \sin(ks)|^N}{|t-s|^N}\, ds\, dt = \frac{1}{k} \int_{-k}^k \int_{-k}^k \frac{|\sin(t) - \sin(s)|^N}{|t-s|^N}\, ds\, dt.$$

Since

$$\frac{|\sin(t) - \sin(s)|^N}{|t-s|^N} \lesssim \frac{1}{(1+|t-s|)^N}, \quad \forall\, t, s \in (-k, k)$$

and

$$\int_{-k}^k \int_{-k}^k \frac{ds\, dt}{(1+|t-s|)^N} \lesssim k,$$

it follows that

$$\int_{-1}^1 \int_{-1}^1 \frac{k^{-(N-1)} |\sin(kt) - \sin(ks)|^N}{|t-s|^N}\, ds\, dt \lesssim 1.$$

Thus from (5.1),

$$|k^{-\frac{N-1}{N}} \sin(kx_N)|_{W([-1,1]^N)} \lesssim 1.$$

Similarly,

$$|k^{-\frac{N-1}{N}} \cos(kx_N)|_{W([-1,1]^N)} \lesssim 1. \qquad \square$$

5.2. *Proof of Remark* 2: *Optimality of Theorem* 1. It suffices to prove that condition i) is necessary since the importance of condition ii) was already discussed in Proposition 1.

PROPOSITION 5. *Let* $N \ge 1$. *There exists a sequence* $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ $(N \ge 1)$ *such that* $g_k \to g := (0, \dots, 0, 1)$ *in* $W(\mathbb{S}^N, \mathbb{S}^N)$, $g_k \to g$ *a.e.*, $\sup_k \|\nabla g_k\|_{L^N} < +\infty$, $\lim_{k\to\infty} |g_k - g|_{\mathrm{BMO}} = 1$, *and* $\deg g_k = 1 > 0 = \deg g$.

Proof. For $k \ge 1$, define

$$g_k(x) = \begin{cases} (0, \dots, 0, 1) & \text{if } x_{N+1} > -1 + \frac{1}{k}, \\ (z', z_{N+1}) & \text{otherwise,} \end{cases}$$

where $x = (x', x_{N+1}) \in \mathbb{R}^N \times \mathbb{R}$, $z_{N+1} = 2kx_{N+1} + 2k - 1$ and $z' = \sqrt{1 - z_{N+1}^2}\,\dfrac{x'}{|x'|}$. It is clear that g_k is Lipschitz with $\|g_k\|_{\mathrm{Lip}} \lesssim \sqrt{k}$. Hence, since $g_k(x) = (0, \dots, 0, 1)$ if $x_{N+1} > -1 + \frac{1}{k}$, it follows that

$$|g_k|^p_{W^{1,p}} \lesssim k^{\frac{p}{2}} \left| \{x; x_{N+1} \le -1 + \frac{1}{k}\} \right| \lesssim k^{\frac{p-N}{2}}.$$

Therefore $\sup_k \|\nabla g_k\|_{L^N} < +\infty$ and $\lim_{k\to\infty} \|\nabla g_k\|_{L^p} = 0$ for all $1 \le p < N$. By interpolation, we obtain

$$\lim_{k\to\infty} |g_k|_W = 0.$$

On the other hand, from the construction of g and g_k, one has

$$\deg g = 0 \quad \text{and} \quad \deg g_k = 1, \quad \forall\, k \ge 1.$$

It remains to prove that $|g_k|_{\mathrm{BMO}} = 1$. We first note that

$$\fint_{B(x,r)} \left| g_k(y) - \fint_{B(x,r)} g_k \right|^2 dy = \fint_{B(x,r)} |g_k|^2\, dy - \left| \fint_{B(x,r)} g_k\, dy \right|^2 \le 1,$$

for any ball $B(x,r) \subset \mathbb{S}^N$. Thus $|g_k|_{\mathrm{BMO}} \le 1$. Next, we recall that for any $h \in C^1(\mathbb{S}^N, \mathbb{S}^N)$ if $\deg h \neq 0$, then for any $v \in C^1(\bar{B})$ extension of h, there exists a point $X \in B$ such that $v(X) = 0$. Thus, since $\deg g_k = 1$, there exist $B(x_0, r_0)$ for some $x_0 \in \mathbb{S}^N$ and $r_0 > 0$ such that $\fint_{B(x_0,r_0)} g_k\, dy = 0$. This implies that $\fint_{B(x_0,r_0)} |g_k(y) - \fint_{B(x_0,r_0)} g_k|^2\, dy = 1$ for such a ball. Therefore $|g_k|_{\mathrm{BMO}} \ge 1$. Consequently, $|g_k|_{\mathrm{BMO}} = 1$. $\square$

6. Definition and properties of $g^*\omega$ for $g \in (W \cap \mathrm{VMO})(\mathbb{S}^N, \mathbb{S}^N)$

In this section we extend the previous results to the case $g \in \mathrm{VMO}(\mathbb{S}^N, \mathbb{S}^N) \cap W(\mathbb{S}^N, \mathbb{S}^N)$; $\mathrm{VMO}(\mathbb{S}^N, \mathbb{S}^N) \cap W(\mathbb{S}^N, \mathbb{S}^N)$ is denoted by $(\mathrm{VMO} \cap W)(\mathbb{S}^N, \mathbb{S}^N)$ or $(W \cap \mathrm{VMO})(\mathbb{S}^N, \mathbb{S}^N)$, and $\mathrm{VMO}(\mathbb{S}^N) \cap W(\mathbb{S}^N)$ is denoted by $(\mathrm{VMO} \cap W)(\mathbb{S}^N)$ or $(W \cap \mathrm{VMO})(\mathbb{S}^N)$.

We begin with

LEMMA 2. *Let $N \geq 1$ and $g \in (\mathrm{VMO} \cap W)(\mathbb{S}^N, \mathbb{S}^N)$. Then there exists a sequence $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ such that $g_k \to g$ in $W(\mathbb{S}^N)$ and $\mathrm{BMO}(\mathbb{S}^N)$.*

Proof. For $k \gg 1$, define

$$\bar{g}_k(x) = \fint_{B(x,1/k)} g_k(s)\, ds$$

and

$$g_k(x) = \bar{g}_k(x)/|\bar{g}_k(x)|.$$

Since $g \in \mathrm{VMO}(\mathbb{S}^N, \mathbb{S}^N)$, g_k is well-defined when k is large enough. In addition $g_k \to g$ in $\mathrm{BMO}(\mathbb{S}^N)$ (see e.g. [20, Cor. 4]). Moreover, $g_k = F(\bar{g}_k)$, where $F(\xi) = \xi/|\xi|$ is a Lipschitz map on $\{\xi \in \mathbb{R}^{N+1}; |\xi| \geq 1/2\}$. We conclude (see e.g. [9, Claim (5.43)]) that $g_k = F(\bar{g}_k) \to F(\bar{g}) = g$ in $W(\mathbb{S}^N)$ since $\bar{g}_k \to \bar{g}$ in $W(\mathbb{S}^N)$. □

LEMMA 3. *Let $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ be a Cauchy sequence in $(W \cap \mathrm{VMO})(\mathbb{S}^N)$. Then $\mathbf{J}(g_k, \psi)$ is a Cauchy sequence for any $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$.*

Proof. Let u_k be the extension of g_k as in the proof of Theorem 1. Then

$$\mathbf{J}(g_k, \psi) = (N+1) \int_B \varphi \det \nabla u_k \, dx + \sum_{i=1}^{N+1} \int_B \partial_i \varphi \mathbf{D}_i(u_k)\, dx,$$

where $\varphi \in C^1(B)$ is an extension of ψ. Applying the method used in the proof of Theorem 4, one can show that $\det \nabla u_k$ is a Cauchy sequence in $L^1(B)$. Here we only use the fact that g_k is a Cauchy sequence in VMO. Next we see that $\mathbf{D}_i(u_k)$ is a Cauchy sequence in $L^1(B)$ for $1 \leq i \leq N+1$. Here we only use the fact that g_k is a Cauchy sequence in $W(\mathbb{S}^N)$ so that u_k is Cauchy in $W^{1,N}(B)$. The conclusion follows. □

Definition 1. Let $N \geq 1$ and $g \in (\mathrm{VMO} \cap W)(\mathbb{S}^N, \mathbb{S}^N)$. Then for any $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$, we can define $\mathbf{J}(g, \psi)$ as the limit of $\mathbf{J}(g_k, \psi)$ for any sequence $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ such that $g_k \to g$ in $(W \cap \mathrm{VMO})(\mathbb{S}^N)$. This object is well-defined according to Lemmas 2 and 3.

Remark 13. Let $g \in (W \cap \mathrm{VMO})(\mathbb{S}^N, \mathbb{S}^N)$ and let u be the extension of g as in the proof of Theorem 1. Then $\det \nabla u \in L^\infty(B)$ since $g \in \mathrm{VMO}(\mathbb{S}^N, \mathbb{S}^N)$

and $u \in W^{1,N}(B)$ since $g \in W(\mathbb{S}^N)$. Moreover,
(6.1)

$$\mathbf{J}(g,\psi) = (N+1)\int_B \varphi \det \nabla u \, dx + \sum_{i=1}^{N+1} \int_B \partial_i \varphi \mathbf{D}_i(u)\, dx, \quad \forall\, \psi \in C^1(\mathbb{S}^N, \mathbb{R}),$$

where $\varphi \in C^1(\bar{B}, \mathbb{R})$ is any extension of ψ.

Similarly, the quantity

$$\int_{\mathbb{S}^N} F(g) \det(\nabla g) \psi \, dx$$

is well-defined in the distributional sense when $F \in C^{0,\alpha}(\mathbb{S}^N, \mathbb{R})$, $g \in (\mathrm{VMO} \cap W)(\mathbb{S}^N, \mathbb{S}^N)$, and $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$ (see the proof of Corollary 5). Moreover, if $(g_k) \subset (\mathrm{VMO} \cap W)(\mathbb{S}^N, \mathbb{S}^N)$ and $g \in (\mathrm{VMO} \cap W)(\mathbb{S}^N, \mathbb{S}^N)$ are such that $g_k \to g$ in $(\mathrm{VMO} \cap W)(\mathbb{S}^N)$, then

$$\lim_{k \to \infty} \int_{\mathbb{S}^N} F(g_k) \det(\nabla g_k) \psi \, dx = \int_{\mathbb{S}^N} F(g) \det(\nabla g) \psi \, dx.$$

We next state some properties of $\mathbf{J}(g,\psi)$ (resp. $\int_{\mathbb{S}^N} F(g) \det(\nabla g) \psi \, dx$) in the case $g \in (W \cap \mathrm{VMO})(\mathbb{S}^N, \mathbb{S}^N)$ (resp. $g \in (W \cap \mathrm{VMO})(\mathbb{S}^N, \mathbb{S}^N)$ and $F \in C^{0,\alpha}(\mathbb{S}^N, \mathbb{R})$, for some $\alpha > 0$). The proofs are left to the reader.

Proposition 6. *Let $N \geq 1$, $g \in W(\mathbb{S}^N, \mathbb{S}^N) \cap C^0(\mathbb{S}^N, \mathbb{S}^N)$, and $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$. Then*

$$|\mathbf{J}(g,\psi)| \leq C \left(\|\psi\|_{L^\infty} T_{\ell_N}(g) + \|\nabla \psi\|_{L^\infty} |g|_W^N \right),$$

for some positive constant $C = C(N)$.

As a consequence of Proposition 6, we have

Proposition 7. *Let $N \geq 1$, $g \in W(\mathbb{S}^N, \mathbb{S}^N) \cap C^0(\mathbb{S}^N, \mathbb{S}^N)$, $F \in C^{0,\alpha}(\mathbb{S}^N, \mathbb{R})$ for some $\alpha > 0$, and $\psi \in C^1(\mathbb{S}^N, \mathbb{R})$. Then*

$$\left| \int_{\mathbb{S}^N} F(g) \det(\nabla g) \psi \, dx \right| \leq C \left(\|\psi\|_{L^\infty} T_\delta(g) + \|\nabla \psi\|_{L^\infty} |g|_W^N \right), \tag{6.2}$$

for some positive constants $C = C(N, \|F\|_{0,\alpha})$ and $\delta = \delta(N, \|F\|_{0,\alpha})$.

Propositions 6 and 7 are still valid for $g \in (W \cap \mathrm{VMO})(\mathbb{S}^N, \mathbb{S}^N)$. For the proofs we go back to the formula (6.1) and use the same method as in the one of Theorem 2. It would be natural to construct a sequence $(g_k) \subset C^1(\mathbb{S}^N, \mathbb{S}^N)$ such that $g_k \to g$ in $(W \cap \mathrm{VMO})(\mathbb{S}^N)$ and then use (6.2). The left-hand side in (6.2) converges to the desired quantity. However, we do not know whether $\liminf_{k \to \infty} T_\delta(g_k) \lesssim T_\delta(g)$, even for a particular sequence (see Remark 16 at the end of the appendix). We warn the reader that the corresponding estimates are sometimes useless. More precisely, there exists $g \in (W \cap \mathrm{VMO})(\mathbb{S}^N, \mathbb{S}^N)$ such that $T_\delta(g) = \infty$ for every $0 < \delta < 1$ (see [19]).

7. The case N=1

7.1. *Proofs of Theorems* 4 *and* 3. We continue our study of $\mathbf{J}(g,\psi)$ and establish further properties valid when $N=1$. Our main estimate is

Theorem 5. *Let* $g\in C^1(\mathbb{S}^1,\mathbb{S}^1)$ *and* $\psi\in C^1(\mathbb{S}^1,\mathbb{R})$. *Then for all* $0<\delta<\ell_1=\sqrt{3}$, *there exists a constant* $C_\delta>0$, *depending only on* δ, *such that*

$$|\mathbf{J}(g,\psi)|\le C_\delta\|\psi\|_{W^{1,\infty}}\left(T_\delta(g)+1\right).$$

The limiting case $\delta=\sqrt{3}$ in Theorem 5 is open (this is in contrast with Theorem 2).

Open question 3. Is it true that

$$|\mathbf{J}(g,\psi)|\le C\|\psi\|_{W^{1,\infty}}\left(T_{\sqrt{3}}(g)+1\right)\quad\forall\, g\in C^1(\mathbb{S}^1,\mathbb{S}^1),$$

for some positive constant C?

Our main ingredient in the proof of Theorem 5 is the following:

Theorem 6. *For each* $\delta\in(0,\sqrt{3})$, *there exists a positive constant* C_δ *such that*

$$\int_0^1\int_0^1|\varphi(x)-\varphi(y)|\,dx\,dy\le C_\delta\Big[T_\delta(e^{i\varphi})+1\Big],\quad\forall\,\varphi\in\mathrm{VMO}((0,1),\mathbb{R}).$$

Theorem 6 was first established when δ is very small and φ is continuous by J. Bourgain, H. Brezis, and P. Mironescu [8]. Their (unpublished) proof is quite involved. Our proof is also very technical and totally different from theirs. We will present it in the appendix. We will also prove there that $\sqrt{3}$ is optimal in the sense that for any $\delta>\sqrt{3}$, the conclusion fails.

Proof of Theorem 5. Fix P a point of $\mathbb{S}^1$. Let $\varphi\in C^1(\mathbb{S}^1\setminus\{P\},\mathbb{R})$ be a lifting of g; i.e., $g=e^{i\varphi}$ on $\mathbb{S}^1\setminus\{P\}$. Then, by (1.13),

$$\mathbf{J}(g,\psi)=-\int_{\mathbb{S}^1}\varphi(s)\psi'(s)\,ds+2\pi\psi(P)\deg g.$$

It follows that

$$|\mathbf{J}(g,\psi)|\le\left|\int_{\mathbb{S}^1}\varphi(s)\psi'(s)\,ds\right|+2\pi|\deg g||\psi(P)|. \tag{7.1}$$

We have, since $\int_{\mathbb{S}^1}\psi'(s)\,ds=0$,

$$\int_{\mathbb{S}^1}\varphi(s)\psi'(s)\,ds=\int_{\mathbb{S}^1}\Big(\varphi(s)-\int_{\mathbb{S}^1}\varphi(t)\,dt\Big)\psi'(s)\,ds=\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}[\varphi(s)-\varphi(t)]\psi'(s)\,ds.$$

This implies

$$\left|\int_{\mathbb{S}^1}\varphi(s)\psi'(s)\,ds\right|\le\|\psi'\|_{L^\infty}\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}|\varphi(s)-\varphi(t)|\,ds\,dt.$$

Applying Theorem 6, one has, for any $0 < \delta < \sqrt{3}$,

$$\left| \int_{\mathbb{S}^1} \varphi(s)\psi'(s)\,ds \right| \le C_\delta \|\psi'\|_{L^\infty}(T_\delta(g) + 1),$$

for some positive constant C_δ. On the other hand, by Proposition 3,

$$|\deg g||\psi(P)| \le \|\psi\|_{L^\infty} T_\delta(g).$$

The conclusion follows from (7.1). □

THEOREM 7. *Let* $(g_k) \subset C^1(\mathbb{S}^1, \mathbb{S}^1)$ *and* $g \in C^1(\mathbb{S}^1, \mathbb{S}^1)$. *Suppose that* $\lim_{k\to\infty} \|g_k - g\|_{\mathrm{BMO}} = 0$. *Then*

$$\lim_{k\to\infty} \mathbf{J}(g_k, \psi) = \mathbf{J}(g, \psi), \quad \forall\, \psi \in C^1(\mathbb{S}^1, \mathbb{R}).$$

Proof. Fix $P \in \mathbb{S}^1$. Since $g_k, g \in C^1(\mathbb{S}^1, \mathbb{S}^1)$ and $g_k \to g$ in $\mathrm{BMO}(\mathbb{S}^1, \mathbb{S}^1)$, there exist φ_k, $\varphi \in C^1(\mathbb{S}^1 \setminus \{P\})$ such that $e^{i\varphi_k} = g_k$, $e^{i\varphi} = g$ on $\mathbb{S}^1 \setminus \{P\}$, and $\varphi_k \to \varphi$ in $\mathrm{BMO}(\mathbb{S}^1 \setminus \{P\})$ by [20, Th. 3]. Thus since

$$\mathbf{J}(g_k, \psi) = -\int_{\mathbb{S}^1} \varphi_k(s)\psi'(s)\,ds + 2\pi\psi(P)\deg g_k$$

and

$$\mathbf{J}(g, \psi) = -\int_{\mathbb{S}^1} \varphi(s)\psi'(s)\,ds + 2\pi\psi(P)\deg g_k,$$

the conclusion follows. Here we use the fact that if $(g_k) \subset C^1(\mathbb{S}^1, \mathbb{S}^1)$ converges to $g \in C^1(\mathbb{S}^1, \mathbb{S}^1)$ in $\mathrm{BMO}(\mathbb{S}^1)$, then $\lim_{k\to\infty} \deg g_k = \deg g$ according to Proposition 4 (see also [20]). □

Proofs of Theorems 4 *and* 3. Theorems 4 and 3 are consequences of Theorems 5 and 7 respectively (see the proofs of Corollaries 5 and 3).

7.2. *Definition and properties of* $g^*\omega$ *for* $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$. In this section we will extend the result in Section 7.1 to $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$. We begin with (see e.g. [20, Cor. 4])

LEMMA 4. *Let* $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$. *There exists* $(g_k) \subset C^1(\mathbb{S}^1, \mathbb{S}^1)$ *such that* $g_k \to g$ *in* $\mathrm{BMO}(\mathbb{S}^1)$.

We also have

LEMMA 5. *Let* $(g_k) \subset C^1(\mathbb{S}^1, \mathbb{S}^1)$ *be such that* (g_k) *is a Cauchy sequence in* $\mathrm{BMO}(\mathbb{S}^1)$. *Then for any* $\psi \in C^1(\mathbb{S}^1, \mathbb{R})$, $\mathbf{J}(g_k, \psi)$ *is a Cauchy sequence.*

Proof. Fix $P \in \mathbb{S}^1$. Since $(g_k) \subset C^1(\mathbb{S}^1, \mathbb{S}^1)$ and (g_k) is a Cauchy sequence in $\mathrm{BMO}(\mathbb{S}^1)$, there exists $\varphi_k \in C^1(\mathbb{S}^1 \setminus \{P\}, \mathbb{R})$ such that φ_k is a Cauchy sequence in $\mathrm{BMO}(\mathbb{S}^1 \setminus \{P\}, \mathbb{R})$ (see [20, Th. 3]). Thus since

$$\mathbf{J}(g_k, \psi) = -\int_{\mathbb{S}^1} \varphi_k(s)\psi'(s)\,ds + 2\pi\psi(P)\deg g_k, \quad \forall\, \psi \in C^1(\mathbb{S}^1, \mathbb{R}),$$

the conclusion follows. □

Definition 2. Let $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$. Then for any $\psi \in C^1(\mathbb{S}^1, \mathbb{R})$, we can define $\mathbf{J}(g, \psi)$ as the limit of $\mathbf{J}(g_k, \psi)$ for any sequence $(g_k) \subset C^1(\mathbb{S}^1, \mathbb{S}^1)$ such that $g_k \to g$ in $\mathrm{VMO}(\mathbb{S}^1)$. This object is well-defined according to Lemmas 4 and 5.

Remark 14. Let $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$. Then

$$\mathbf{J}(g, \psi) = -\int_{\mathbb{S}^1} \varphi(s)\psi'(s)\, ds + 2\pi\psi(P) \deg g, \tag{7.2}$$

for any $P \in \mathbb{S}^1$ and for any $\varphi \in \mathrm{VMO}(\mathbb{S}^1 \setminus \{P\}, \mathbb{R})$ such that $e^{i\varphi} = g$ on $\mathbb{S}^1 \setminus \{P\}$.

Similarly, the quantity,

$$\int_{\mathbb{S}^1} F(g) \det(\nabla g)\psi\, ds$$

is well-defined in the distributional sense when $F \in C(\mathbb{S}^1, \mathbb{R})$, $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$, and $\psi \in C^1(\mathbb{S}^1, \mathbb{R})$ (see the proof of Corollary 5).

Moreover, if $(g_k) \subset \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$ and $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$ are such that $g_k \to g$ in $\mathrm{BMO}(\mathbb{S}^1)$, then

$$\lim_{k\to\infty} \int_{\mathbb{S}^1} F(g_k) \det(\nabla g_k)\psi\, ds = \int_{\mathbb{S}^1} F(g) \det(\nabla g)\psi\, ds.$$

We next state some properties of $\mathbf{J}(g, \psi)$ (resp. $\int_{\mathbb{S}^N} F(g) \det(\nabla g)\psi\, dx$) in the case $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$ (resp. $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$ and $F \in C(\mathbb{S}^1, \mathbb{R})$).

Proposition 8. *Let $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$ and $\psi \in C^1(\mathbb{S}^1, \mathbb{R})$. Then for all $0 < \delta < \ell_1 = \sqrt{3}$, there exists a constant $C_\delta > 0$ depending only on δ such that*

$$|\mathbf{J}(g, \psi)| \le C_\delta \|\psi\|_{W^{1,\infty}} \left(T_\delta(g) + 1\right).$$

Proof. The proof is the same as that of Theorem 5 by using Theorem 6. □

As a consequence of Proposition 8, we have

Proposition 9. *Let $g \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$, $F \in C(\mathbb{S}^1, \mathbb{R})$, and $\psi \in C^1(\mathbb{S}^1, \mathbb{R})$. Then*

$$\left| \int_{\mathbb{S}^1} F(g) \det(\nabla g)\psi\, dx \right| \le C \|\psi\|_{W^{1,\infty}} \left(T_\delta(g) + 1\right),$$

for some positive positive constants C and δ depending only on $\|F\|_{L^\infty}$.

7.3. *An improvement of Theorem* 3: *A partial answer to Open Question* 2. In this section, we prove

Proposition 10. *There exists a constant $c > 0$ such that if $(g_k) \subset C^1(\mathbb{S}^1, \mathbb{S}^1)$, $g \in C^1(\mathbb{S}^1, \mathbb{S}^1)$, g_k converges to g a.e. in $\mathbb{S}^1$, and*

$$\limsup_{k\to\infty} |g_k - g|_{\mathrm{BMO}(\mathbb{S}^1)} < c,$$

then

$$\lim_{k\to\infty} \int_{\mathbb{S}^1} \det(\nabla g_k)\psi\, dx = \int_{\mathbb{S}^1} \det(\nabla g)\psi\, dx.$$

This proposition is a consequence of

PROPOSITION 11. *There exists a constant $c > 0$ such that if $(g_k) \subset C^1((0,1), \mathbb{S}^1)$, $g \in C^1((0,1), \mathbb{S}^1)$, g_k converges to g a.e. in $(0,1)$, and*

$$\limsup_{k \to \infty} |g_k - g|_{\mathrm{BMO}} < c,$$

then

$$\lim_{k \to \infty} \int_0^1 \int_0^1 |[\varphi_k(x) - \varphi(x)] - [\varphi_k(y) - \varphi(y)]| \, dx \, dy = 0.$$

Here φ_k and $\varphi \in C^1((0,1), \mathbb{R})$ are respectively liftings of g_k and g.

We first accept Proposition 11 and turn to the

Proof of Proposition 10. Fix $P \in \mathbb{S}^1$. Let ψ, $\psi_k \in C^1(\mathbb{S}^1 \setminus \{P\}, \mathbb{R})$ be liftings of g and g_k. Then

$$\mathbf{J}(g_k, \psi) = -\int_{\mathbb{S}^1} \varphi_k \psi' \, ds + 2\pi \deg g_k \, \psi(P).$$

From the assumption of Proposition 10, by Proposition 4,

$$\lim_{k \to \infty} \deg g_k = \deg g.$$

It suffices to prove that

$$\lim_{k \to \infty} \int_{\mathbb{S}^1} \varphi_k \psi' \, ds = \int_{\mathbb{S}^1} \varphi \psi' \, dx.$$

We have

$$\int_{\mathbb{S}^1} \varphi_k \psi' \, dx = \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} [\varphi_k(x) - \varphi_k(y)] \psi'(x) \, dx \, dy$$

and

$$\int_{\mathbb{S}^1} \varphi \psi' \, dx = \int_{\mathbb{S}^1} \int_{\mathbb{S}^1} [\varphi(x) - \varphi(y)] \psi'(x) \, dx \, dy.$$

It follows from Proposition 11 that

$$\lim_{k \to \infty} \int_{\mathbb{S}^1} \varphi_k \psi' \, ds = \int_{\mathbb{S}^1} \varphi \psi' \, dx. \qquad \square$$

We now return to

Proof of Proposition 11. In this proof I denotes the interval $(0,1)$. For all x, $y \in I$, one has

$$|[\varphi_k(x) - \varphi(x)] - [\varphi_k(y) - \varphi(y)]| \lesssim |\exp(i[\varphi_k(x) - \varphi(x)] - i[\varphi_k(y) - \varphi(y)]) - 1| + |[\varphi_k(x) - \varphi(x)] - [\varphi_k(y) - \varphi(y)]|^2.$$

However,

$$|\exp(i[\varphi_k(x)-\varphi(x)]-i[\varphi_k(y)-\varphi(y)])-1| = |g_k(x)/g(x)-g_k(y)/g(y)| \leq |g_k(x)-g(x)|+|g_k(y)-g(y)|.$$

Hence

$$|[\varphi_k(x)-\varphi(x)]-[\varphi_k(y)-\varphi(y)]| \lesssim |g_k(x)-g(x)|+|g_k(y)-g(y)|+|[\varphi_k(x)-\varphi(x)]-[\varphi_k(y)-\varphi(y)]|^2.$$

This implies

$$\int_I\int_I |[\varphi_k(x)-\varphi(x)]-[\varphi_k(y)-\varphi(y)]|\,dx\,dy \lesssim \int_I |g_k(x)-g(x)|\,dx + \int_I\int_I |[\varphi_k(x)-\varphi(x)]-[\varphi_k(y)-\varphi(y)]|^2\,dx\,dy.$$

On the other hand, as a consequence of inequality $(2)'$ in F. John and L. Nirenberg [43] we have

$$\int_I\int_I |[\varphi_k(x)-\varphi(x)]-[\varphi_k(y)-\varphi(y)]|^2\,dx\,dy \lesssim |\varphi_k-\varphi|_{\mathrm{BMO}}\int_I\int_I |[\varphi_k(x)-\varphi(x)]-[\varphi_k(y)-\varphi(y)]|\,dx\,dy.$$

Thus

$$\int_I\int_I |[\varphi_k(x)-\varphi(x)]-[\varphi_k(y)-\varphi(y)]|\,dx\,dy \lesssim \int_I |g_k(x)-g(x)|\,dx + |\varphi_k-\varphi|_{\mathrm{BMO}}\int_I\int_I |[\varphi_k(x)-\varphi(x)]-[\varphi_k(y)-\varphi(y)]|\,dx\,dy.$$

Finally, we use an inequality of R. Coifman and Y. Meyer [22] (see also [20, Th. 4]):

$$|\varphi_k-\varphi|_{\mathrm{BMO}} \leq 4|g_k-g|_{\mathrm{BMO}}$$

when $|g_k-g|_{\mathrm{BMO}}$ is sufficiently small. Hence, there exists a positive constant c such that if $|g_k-g|_{\mathrm{BMO}} < c$, then

$$\int_I\int_I |[\varphi_k(x)-\varphi(x)]-[\varphi_k(y)-\varphi(y)]|\,dx\,dy \lesssim \int_I |g_k(x)-g(x)|\,dx.$$

Since g_k converges to g for almost every $x \in I$,

$$\lim_{k\to\infty}\int_I\int_I |[\varphi_k(x)-\varphi(x)]-[\varphi_k(y)-\varphi(y)]|\,dx\,dy = 0. \qquad \square$$

Appendix A. A basic estimate for the lifting: Proof of Theorem 6

This section is devoted to the proof of the following fundamental estimate in Theorem 6:

(A.1)
$$\int_0^1\int_0^1 |\varphi(x)-\varphi(y)|\,dx\,dy \le C_\delta\big[T_\delta(e^{i\varphi})+1\big], \quad \forall\,\varphi\in \mathrm{VMO}((0,1),\mathbb{R}), \forall\,\delta<\sqrt{3}.$$

We recall that

$$T_\delta(e^{i\varphi}) = \mathop{\int_0^1\int_0^1}_{|e^{i\varphi(x)}-e^{i\varphi(y)}|\ge\delta} \frac{1}{|x-y|^2}\,dx\,dy, \quad \forall\,\delta>0.$$

The constant $\sqrt{3}$ in estimate (A.1) is optimal in the sense that for any $\delta>\sqrt{3}$, the conclusion fails. Indeed, one can construct as in [54] a sequence $(\varphi_k)\subset C^1([0,1],\mathbb{R})$ such that φ_k is increasing, $\varphi_k(0)=0$, $\varphi_k(1/k)=2\pi$

$$\varphi_k(x+1/k)=\varphi_k(x)+2\pi, \forall\,x\in[0,1-1/k],$$

and

$$\lim_{k\to\infty} \mathop{\int_0^1\int_0^1}_{|e^{i\varphi_k(x)}-e^{i\varphi_k(y)}|>\sqrt{3}} \frac{1}{|x-y|^2}\,dx\,dy = 0.$$

It is easy to see that $|\varphi_k|_{\mathrm{BMO}}\approx k$.

The limiting case $\delta=\sqrt{3}$ is open:

Open question 4. Is it true that

$$\int_0^1\int_0^1 |\varphi(x)-\varphi(y)|\,dx\,dy \le C\big[T_{\sqrt{3}}(e^{i\varphi})+1\big], \quad \forall\,\varphi\in\mathrm{VMO}((0,1),\mathbb{R}),$$

for some positive constant C?

Proof of (A.1). We follow the strategy of J. Bourgain, H. Brezis, and P. Mironescu in the proof of [10, Th. 0.1] and use ideas inspired from [7] and [54].

In this proof the notation $a\lesssim b$ means that there exists a positive constant C_δ such that $a\le C_\delta b$. The notation $a\gtrsim b$ means that $b\lesssim a$ and I denotes the unit open interval $(0,1)$.

Set $g=e^{i\varphi}$. Extending g by symmetry to the interval $(-1,0)$, and then by periodicity to all of $\mathbb{R}$, one may assume, without loss of generality, that $g\in\mathrm{VMO}(\mathbb{R},\mathbb{S}^1)$, and it suffices to prove that

(A.2)
$$\int_0^1\int_0^1 |\varphi(x)-\varphi(y)|\,dx\,dy \lesssim T_\delta(g)+1, \quad \forall\,\delta\in(0,\sqrt{3}),$$

where

$$T_\delta(g) := \mathop{\int_{-2}^3\int_{-2}^3}_{|g(x)-g(y)|>\delta} \frac{1}{|x-y|^2}\,dx\,dy.$$

We only need to prove (A.2) for $\delta < \sqrt{3}$ and δ is close to $\sqrt{3}$. Hereafter, we assume this.

Step 1: *Proof of* (A.1) *when* g *is continuous.* Let $u : I^2 \to \mathbb{R}^2$ be the extension by average of g, i.e.

$$u(x,r) = \fint_{x-r}^{x+r} g(z)\, dz,$$

and set $\alpha = \frac{\delta^2-2}{2} > 0$.

For each $x \in I$, define $\rho(x)$ by

$$\rho(x) = \sup\{r;\ |u(x,s)| \geq \alpha \,\text{for all}\, 0 < s < r\}.$$

We adapt here the ideas used in the proof of [10, Theorem 0.1]. Set

$$G = \{X = (x,r);\ x \in I \text{ and } r \in (\rho(x),1)\}.$$

Since $|\nabla u(x,r)| \leq 1/r$ for $(x,r) \in I^2$, one has

$$\int_G |\nabla u(x,r)|^2\, dr\, dx \lesssim \int\limits_{\rho(x)<1} \frac{1}{\rho(x)}\, dx. \tag{A.3}$$

To obtain an estimate for the right-hand side of (A.3), we follow the same argument as in the proof of [54, Th. 1]. Recall that if J is a nonempty set and $(A_j)_{j\in J}$ is a collection of points in $\mathbb{S}^1$ such that $\text{dist}(\text{conv}\,(\{A_j;\ j\in J\}), O) \leq 1/2$, then there exist $j_1, j_2 \in J$ such that $|A_{j_1} - A_{j_2}| \geq \sqrt{3}$ (see [54, Cor. 4]). Here $O = (0,0) \in \mathbb{R}^2$ and conv (.) denotes the convex hull of a subset of $\mathbb{R}^2$. Thus, since $\alpha < 1/2$, if $\rho(x) < 1$,

$$\left| \fint_{x-\rho(x)}^{x+\rho(x)} g(s)\, ds \right| < \alpha,$$

which implies, as in the proof of [54, Lemma 6],

$$\Big|\{(\xi,\eta) \in (x-\rho(x), x+\rho(x))^2;\ |g(\xi)-g(\eta)| \geq \delta\}\Big| \gtrsim \rho(x)^2. \tag{A.4}$$

Hence, for some positive constant τ, independent of g and x, one has

$$\int\limits_{\substack{x-\rho(x) \\ |g(\xi)-g(\eta)|\geq\delta \\ |\xi-\eta|\geq\tau\rho(x)}}^{x+\rho(x)} \int_{x-\rho(x)}^{x+\rho(x)} \frac{1}{|\xi-\eta|^2}\, d\xi\, d\eta \gtrsim 1.$$

It follows that

$$\int_{\rho(x)<1} \frac{1}{\rho(x)}\, dx \lesssim \int_{\rho(x)<1} \frac{1}{\rho(x)} \int\limits_{\substack{x-\rho(x) \\ |g(\xi)-g(\eta)|\geq\delta \\ |\xi-\eta|\geq\tau\rho(x)}}^{x+\rho(x)} \int_{x-\rho(x)}^{x+\rho(x)} \frac{1}{|\xi-\eta|^2}\, d\xi\, d\eta\, dx.$$

A simple computation gives

$$\int_{\rho(x)<1} \frac{1}{\rho(x)}\, dx \lesssim T_\delta(g). \tag{A.5}$$

Combining (A.3) and (A.5) yields

$$\int_G |\nabla u|^2\, dX \lesssim T_\delta(g). \tag{A.6}$$

Using the co-area formula, one has

$$\int_{1/4}^{\alpha} \int_{\{\sigma\in I^2;\, |u(\sigma)|=\beta\}} |\nabla u|\, d\sigma\, d\beta = \int_{\{X\in I^2;\, 1/4<|u(X)|<\alpha\}} |\nabla u| |\nabla |u||\, dX. \tag{A.7}$$

However, from the definition of G and ρ it is clear that

$$\{X \in I^2;\, 1/4 < |u(X)| < \alpha\} \subset G. \tag{A.8}$$

Combining (A.6), (A.7), and (A.8) yields

$$\int_{1/4}^{\alpha} \int_{\{\sigma\in I^2;\, |u(\sigma)|=\beta\}} |\nabla u|\, d\sigma\, d\beta \lesssim T_\delta(g).$$

Thus, by Sard's theorem, there exists a regular value β of $|u|$ ($1/4 < \beta < \alpha$) such that

$$\int_\Gamma |\nabla u|\, d\sigma \lesssim T_\delta(g), \tag{A.9}$$

where $\Gamma = \{X \in B;\, |u(X)| = \beta\}$.

Fix x and y in I ($x < y$). Set

$$U = \{(z, r) \in [x, y] \times I;\, |u(z, r)| > \beta\}.$$

Let W be the connected component of U such that $[x, y] \times \{0\} \subset \partial W$ and γ be the connected component of ∂W such that $[x, y] \times \{0\} \subset \gamma$ (see Figure 1). Set

$$h = \frac{u}{|u|} \quad \text{on}\, W.$$

Let $\psi \in C(\gamma - \{y\}, \mathbb{R})$ be such that $h = e^{i\psi}$ on γ and $\psi = \varphi$ on $(x, y) \times \{0\}$. Then

$$\Big|\psi(y, 0_+) - \psi(y-, 0)\Big| \le \int_{\Gamma\setminus\gamma} |\nabla u|\, d\sigma, \tag{A.10}$$

where $\psi(y, 0_+) = \lim_{r\to 0_+} \psi(y, r)$ and $\psi(y-, 0) = \lim_{z\to y_-} \psi(z, 0)$. Hence from (A.10) one has

$$|\varphi(x)-\varphi(y)| = |\psi(x, 0)-\psi(y-, 0)| \le \Big|\psi(y, 0_+)-\psi(x, 0)\Big| + \int_\Gamma |\nabla u|\, d\sigma. \tag{A.11}$$

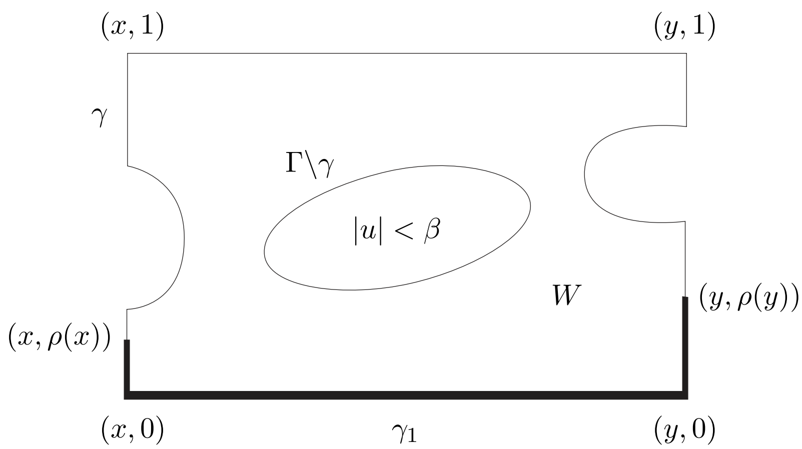

Figure 1.

However,

$$\Big|\psi(y,0_+)-\psi(x,0)\Big| \le \Big|\psi(y,0_+)-\psi(y,\rho(y))\Big| + \Big|\psi(y,\rho(y))-\psi(x,\rho(x))\Big| + \Big|\psi(x,\rho(x))-\psi(x,0)\Big|$$

and, with $\gamma_1 := ([x,y]\times\{0\})\cup(\{x\}\times[0,\rho(x)])\cup(\{y\}\times[0,\rho(y)])$,

$$\Big|\psi(y,\rho(y))-\psi(x,\rho(x))\Big| \le \int_{\gamma\setminus\gamma_1}|\nabla h|\,d\sigma \lesssim \int_\Gamma |\nabla u|\,d\sigma + \frac{1}{\rho(x)} + \frac{1}{\rho(y)}.$$

It follows from (A.11) that

$$|\varphi(x)-\varphi(y)| \lesssim \int_\Gamma |\nabla u|\,dy + \frac{1}{\rho(x)} + \frac{1}{\rho(y)} + |\psi(x,\rho(x))-\psi(x,0)| + |\psi(y,\rho(y))-\psi(y,0_+)|. \tag{A.12}$$

We claim that

$$|\psi(x,\rho(x))-\psi(x,0_+)| \lesssim \int_{|g(z)-g(x)|>\delta} \frac{1}{|z-x|^2}\,dz + 1 \tag{A.13}$$

and

$$|\psi(y,\rho(y))-\psi(y,0_+)| \lesssim \int_{|g(z)-g(y)|\ge\delta} \frac{1}{|z-y|^2}\,dz + 1. \tag{A.14}$$

To prove the claim, we proceed as follows (this is inspired from [7] and [54]): Let $k\in\mathbb{Z}$ be such that

$$2k\pi \le \psi(x,\rho(x))-\psi(x,0) < 2k\pi + 2\pi. \tag{A.15}$$

Without loss of generality, one may assume that $k \ge 0$ and $\psi(x, 0) = 0$. It follows from (A.15) that there exist $0 < t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} \le \rho(x)$ such that

$$\text{(A.16)} \qquad \begin{cases} \psi(x, t_{2m-1}) = 2m\pi - \pi, \\ \psi(x, t_{2m}) = 2m\pi, \end{cases} \qquad \forall\, 1 \le m \le k.$$

Set

$$A_{x,m} = \{z \in \mathbb{R};\, t_{2m} < |z - x| < t_{2m+1}\}, \quad \forall\, m \ge 1$$

and

$$B_{x,m} = \{z \in \mathbb{R};\, |z - x| < t_m\}, \quad \forall\, m \ge 1.$$

Since $|u| > \alpha$ on $\{x\} \times [0, \rho(x)]$, with the notation $g = (g_1, g_2)$, it follows from (A.16) that

$$\text{(A.17)} \qquad \fint_{B_{x,2m}} g_1\, dz \ge \alpha \quad \text{and} \quad \fint_{B_{x,2m+1}} g_1\, dz \le -\alpha.$$

However,

$$\text{(A.18)} \qquad \fint_{B_{x,2m+1}} g_1\, dz = \frac{|B_{x,2m}|}{|B_{x,2m+1}|} \fint_{B_{x,2m}} g_1\, dz + \frac{|A_{x,m}|}{|B_{x,2m+1}|} \fint_{A_{x,m}} g_1\, dz.$$

Combining (A.17) and (A.18) yields

$$\text{(A.19)} \qquad |A_{x,m}| \gtrsim |B_{x,2m+1}| \ge t_{2m+1}$$

and

$$\text{(A.20)} \qquad \fint_{A_{x,m}} g_1\, dz \le -\alpha.$$

From (A.20), one has

$$|\{z \in A_{x,m};\, g_1(z) \le -\alpha\}| \gtrsim |A_{x,m}|,$$

which implies, since $2 + 2\alpha = \delta^2$ and $g(x) = (1, 0)$,

$$\text{(A.21)} \qquad |\{z \in A_{x,m};\, |g(z) - g(x)| \ge \delta\}| \gtrsim |A_{x,m}|.$$

Using (A.19), (A.20), and (A.21) we obtain

$$|\{z \in A_{x,m};\, |g(z) - g(x)| \ge \delta\}| \gtrsim t_{2m+1}.$$

This implies, since $|z - x| \le t_{2m+1}$ for $z \in A_{x,m}$,

$$\int\limits_{A_{x,m} \cap \{z;\, |g(z) - g(x)| \ge \delta\}} \frac{1}{|z - x|^2}\, dz \gtrsim 1.$$

Consequently,

$$\int\limits_{|g(z) - g(x)| \ge \delta} \frac{1}{|z - x|^2}\, dz \gtrsim \sum_{m=1}^{k-1} \int\limits_{A_{x,m} \cap \{z;\, |g(z) - g(x)| \ge \delta\}} \frac{1}{|z - x|^2}\, dz \gtrsim k - 1.$$

This shows that

$$|\psi(x,\rho(x)) - \psi(x,0_+)| \lesssim \int\limits_{|g(z)-g(x)|\ge\delta} \frac{1}{|z-x|^2}\,dz + 1.$$

Similarly,

$$|\psi(y,\rho(y)) - \psi(y,0_+)| \lesssim \int\limits_{|g(z)-g(y)|\ge\delta} \frac{1}{|z-y|^2}\,dz + 1.$$

Thus (A.13) and (A.14) are proved.

Combining (A.9), (A.12), (A.13), and (A.14) yields

$$|\varphi(x)-\varphi(y)| \lesssim T_\delta(g) + \frac{1}{\rho(x)} + \frac{1}{\rho(y)} + \int\limits_{|g(z)-g(x)|\ge\delta} \frac{1}{|z-x|^2}\,d\eta$$
$$+ \int\limits_{|g(z)-g(y)|\ge\delta} \frac{1}{|z-y|^2}\,dz + 1.$$

Integrating the above inequality with respect to (x,y) over $I\times I$, one has

$$\int_I\int_I |\varphi(x)-\varphi(y)|\,dx\,dy \lesssim T_\delta(g) + \int\limits_{\rho(x)<1} \frac{1}{\rho(x)}\,dx + \int\limits_{\rho(y)<1} \frac{1}{\rho(y)}\,dy + 1.$$

It follows from (A.5) that

$$\int_I\int_I |\varphi(x)-\varphi(y)|\,dx\,dy \lesssim T_\delta(g) + 1.$$

Remark 15. Inequality (A.4) is equivalent to the existence of a constant $C_\delta > 0$ such that

$$\Big|\{(\xi,\eta)\in(x-\rho(x),x+\rho(x));\,|g(\xi)-g(\eta)|\ge\delta\}\Big| \ge C_\delta\rho(x)^2. \tag{A.22}$$

One cannot deduce from the assumptions $g\in L^1((a,b),\mathbb{S}^1)$ and $|\fint_a^b g(z)\,dz| = \alpha < 1/2$, that there exists a universal positive constant C such that

$$\Big|\{(\xi,\eta)\in(x-r,x+r);\,|g(\xi)-g(\eta)|\ge\sqrt{3}\}\Big| \ge Cr^2.$$

Here is an example. Assume $a=0$ and $b=2$. Let $A=(1,0)$, $B=e^{i(2\pi/3+\tau)}$, and $C=e^{i(4\pi/3-\tau)}$ where $\tau>0$ (small) is chosen such that $|O-\frac{B+C}{2}| = 1-\alpha > 1/2$. Let g be a function defined on $(0,2)$ such that $|\{y;\,g(y)=B\}| = |\{y;\,g(y)=C\}| = \frac{1+\alpha}{2-\alpha}$, and $|\{y;\,g(y)=A\}| = \frac{2-4\alpha}{2-\alpha}$. Then

$$\Big|\{(\xi,\eta)\in(0,2);\,|g(\xi)-g(\eta)|\ge\sqrt{3}\}\Big| = 2\frac{2-4\alpha}{2-\alpha}\frac{1+\alpha}{2-\alpha} \to 0 \text{ as } \alpha\to 1/2.$$

This is the reason why we cannot establish estimate (A.1) for $\delta=\sqrt{3}$ using this method.

Step 2: *The general case.* Define $g_\varepsilon : [-1,2] \mapsto \mathbb{S}^1$ (ε small) as follows:

$$g_\varepsilon(x) = \fint_{x-\varepsilon}^{x+\varepsilon} g(s)\,ds \Big/ \left| \fint_{x-\varepsilon}^{x+\varepsilon} g(s)\,ds \right| .$$

Let $\varphi_\varepsilon \in \mathrm{VMO}(I, \mathbb{R})$ be the lifting of g_ε such that φ_ε converges to φ in L^1, let u_ε be the extension by average of g_ε as in Step 1, and let $0 < \lambda < \frac{1}{2}$ be such that $2 + 2\lambda = (3 + \delta^2)/2$. Then, since $2 + 2\alpha = \delta^2 < 3$, one has $\alpha < \lambda < 1/2$.

For each $x \in I$, define $\rho_\varepsilon(x)$ by

$$\rho_\varepsilon(x) = \sup\{r;\ |u_\varepsilon(x,s)| \ge \lambda \text{ for all } 0 < s < r\}.$$

If $\rho_\varepsilon(x) < 1$, then

$$\left| \fint_{x-\rho_\varepsilon(x)}^{x+\rho_\varepsilon(x)} g_\varepsilon(z)\,dz \right| = \lambda. \tag{A.23}$$

We claim that $\rho_\varepsilon(x) \ge r_0$ for some $r_0 > 0$ independent of ε and x as ε is small. In fact, from (A.23), $\varepsilon \lesssim \rho_\varepsilon(x)$. Since

$$\begin{aligned}\fint_{x-\rho_\varepsilon(x)}^{x+\rho_\varepsilon(x)} |g_\varepsilon(r) - g(r)|\,dr \le & \fint_{x-\rho_\varepsilon(x)}^{x+\rho_\varepsilon(x)} \fint_{r-\varepsilon}^{r+\varepsilon} |g(\xi) - g(r)|\,d\xi\,dr \\ & + \fint_{x-\rho_\varepsilon(x)}^{x+\rho_\varepsilon(x)} \left| \fint_{r-\varepsilon}^{r+\varepsilon} g(\xi) \right| \Big(\frac{1}{\left| \fint_{r-\varepsilon}^{r+\varepsilon} g \right|} - 1 \Big)\,d\xi\,dr,\end{aligned}$$

it follows that

$$\fint_{x-\rho_\varepsilon(x)}^{x+\rho_\varepsilon(x)} |g_\varepsilon(r) - g(r)|\,dr \le 1/8, \tag{A.24}$$

when ε small (by $g \in \mathrm{VMO}$). On the other hand,

$$\fint_{x-\rho_\varepsilon(x)}^{x+\rho_\varepsilon(x)} g_\varepsilon(r)\,dr = \fint_{x-\rho_\varepsilon(x)}^{x+\rho_\varepsilon(x)} (g_\varepsilon(r) - g(r))\,dr + \fint_{x-\rho_\varepsilon(x)}^{x+\rho_\varepsilon(x)} g(r)\,dr$$

and $\left| \fint_{y-s}^{y+s} g(r)\,dr \right|$ converges to 1 uniformly in $[0,1]$ as s goes to 0. It follows from (A.23) and (A.24) that $\rho_\varepsilon(x) \ge r_0$ for some $r_0 > 0$ independent of ε and x.

Since $\rho_\varepsilon \ge r_0$, there exists ε_1 such that for all $\varepsilon \le \varepsilon_1$,

$$\left| \fint_{x-\rho_\varepsilon(x)}^{x+\rho_\varepsilon(x)} g(y)\,dy \right| \le \frac{1}{2}\left(\lambda + \frac{1}{2}\right) < 1/2.$$

Hence, as in the proof of Step 1, one has

$$\int\limits_{\rho_\varepsilon(x) < 1} \frac{1}{\rho_\varepsilon(x)}\,dx \lesssim T_\delta(g), \tag{A.25}$$

for $\varepsilon \le \varepsilon_1$.

Let $r_1 > 0$ be such that

$$\iint_D |\varphi(x) - \varphi(y)|\,dx\,dy \le \frac{1}{10}\int_I\int_I |\varphi(x) - \varphi(y)|\,dx\,dy, \tag{A.26}$$

for every measurable subset D of I^2 such that $|D| \le r_1^2$.

Set $\tau_0 = (\lambda - \alpha)/2 > 0$. We have

(A.27)

$$\text{if} \fint_{x-r}^{x+r} |g_\varepsilon(z) - g(z)|\,dz \le \tau_0 \text{ and } \left|\fint_{x-r}^{x+r} g_\varepsilon(z)\,dz\right| \ge \lambda, \text{ then } \left|\fint_{x-r}^{x+r} g(z)\,dz\right| \ge \alpha.$$

Define

$$\mathcal{A}_\varepsilon = \left\{x \in I;\, \fint_{x-r}^{x+r} |g_\varepsilon(z) - g(z)|\,dz \ge \tau_0 \text{ for some } r \in (0,1)\right\}. \tag{A.28}$$

From the theory of maximal functions (see e.g. [57, Th. 1, p. 5]), we infer that

$$|\mathcal{A}_\varepsilon| \lesssim \frac{1}{\tau_0}\int_{-1}^{2} |g_\varepsilon(s) - g(s)|\,ds.$$

Thus, since g_ε converges to g in L^1, there exists $\varepsilon_2 > 0$ such that

$$|\mathcal{A}_\varepsilon| \le r_1^2/2, \quad \forall\, \varepsilon \le \varepsilon_2. \tag{A.29}$$

Since φ_ε converges to φ in $L^1(I)$, there exists $\varepsilon_3 > 0$ such that

$$|\mathcal{B}_\varepsilon| \le r_1^2/2, \quad \forall\, \varepsilon \le \varepsilon_3, \tag{A.30}$$

where

$$\mathcal{B}_\varepsilon = \left\{x \in I; |\varphi_\varepsilon(x) - \varphi(x)| \ge \frac{1}{4}\right\}. \tag{A.31}$$

Set $\mathcal{C}_\varepsilon = I \setminus \mathcal{A}_\varepsilon$. We claim that, for $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$,

$$\int_{\mathcal{C}_\varepsilon}\int_{\mathcal{C}_\varepsilon} |\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|\,dx\,dy \lesssim T_\delta(g) + 1. \tag{A.32}$$

Indeed, fix $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. As in Step 1, it follows from (A.25) that there exists $\beta_\varepsilon \in (1/4, \lambda)$ such that

$$\int_{\Gamma_\varepsilon} |\nabla u_\varepsilon| d\sigma \lesssim T_\delta(g), \tag{A.33}$$

where $\Gamma_\varepsilon = \{X \in I^2;\, |u_\varepsilon(X)| = \beta\}$.

Set

$$U_\varepsilon = \{(z,r) \in [x,y] \times I;\, |u_\varepsilon(z,r)| > \beta\}.$$

Let W_ε be the connected component of U_ε such that $[x,y] \times \{0\} \subset \partial W_\varepsilon$ and γ_ε be the connected component of ∂W_ε such that $[x,y] \times \{0\} \subset \gamma_\varepsilon$. Set

$$h_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|} \quad \text{on}\, W_\varepsilon.$$

Let $\psi_\varepsilon \in C(\gamma_\varepsilon - \{y\}, \mathbb{R})$ be such that $h_\varepsilon = e^{i\psi_\varepsilon}$ on γ_ε and $\psi_\varepsilon = \varphi_\varepsilon$ on $(x, y) \times \{0\}$ (we recall that φ_ε is a lifting of g_ε). As in the proof of (A.12), one has

$$|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| \lesssim \int_{\Gamma_\varepsilon} |\nabla u_\varepsilon|\, dy + \frac{1}{\rho_\varepsilon(x)} + \frac{1}{\rho_\varepsilon(y)}$$
$$+ |\psi_\varepsilon(x, \rho_\varepsilon(x)) - \psi_\varepsilon(x, 0_+)| + |\psi_\varepsilon(y, \rho_\varepsilon(y)) - \psi_\varepsilon(y, 0_+)|. \tag{A.34}$$

We claim that

$$|\psi_\varepsilon(x, \rho(x)) - \psi_\varepsilon(x, 0)| \lesssim \int\limits_{|g(z)-g(x)|\geq\delta} \frac{1}{|z-x|^2}\, dz + 1 \tag{A.35}$$

and

$$|\psi_\varepsilon(y, \rho(x)) - \psi_\varepsilon(y, 0_+)| \lesssim \int\limits_{|g(z)-g(y)|\geq\delta} \frac{1}{|z-y|^2}\, dz + 1. \tag{A.36}$$

We follow the strategy presented in Step 1. Take x, $y \in \mathcal{C}_\varepsilon$ and let $k \in \mathbb{Z}$ such that

$$2k\pi \leq \psi_\varepsilon(x, \rho(x)) - \psi_\varepsilon(x, 0) < 2k\pi + 2\pi. \tag{A.37}$$

Let $\psi \in C(\{x\} \times [0, \rho_\varepsilon(x)])$ be such that $e^{i\psi} = u/|u|$ on $\{x\} \times [0, \rho_\varepsilon(x)]$ (u is the extension by average of g as in Step 1). From (A.28), ψ is well-defined since $x \notin \mathcal{A}_\varepsilon$. Without loss of generality, one may assume that $k \geq 0$ and $\psi(x, 0) = 0$ and $\psi_\varepsilon(x, 0) \in [-\pi, \pi]$. It follows from (A.37) that there exist $0 < t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} \leq \rho_\varepsilon(x)$ such that

$$\begin{cases} \psi_\varepsilon(x, t_{2m-1}) = 2m\pi - \pi, \\ \psi_\varepsilon(x, t_{2m}) = 2m\pi, \end{cases} \quad \forall\, 1 \leq m \leq k-1.$$

Since $x \notin \mathcal{A}_\varepsilon$, it follows from (A.28) that there exist $s_1, \ldots, s_{2k-2}$ such that $0 < t_1 < s_1 < s_2 \cdots < s_{2k-2} < \rho_\varepsilon(x)$ and

$$\begin{cases} \psi(x, s_{2m-1}) = 2m\pi - \pi, \\ \psi(x, s_{2m}) = 2m\pi, \end{cases} \quad \forall\, 1 \leq m \leq k-1.$$

Thus since $x \notin \mathcal{A}_\varepsilon$, according to (A.27) and (A.28), one has

$$\fint_{x-s_{2m}}^{x+s_{2m}} g_1(z)\, dz \geq \alpha \quad \text{and} \quad \fint_{x-s_{2m}}^{x+s_{2m}} g_1(z)\, dz \leq -\alpha.$$

Applying the same method used to obtain (A.13) in Step 1, one has

$$|\psi_\varepsilon(x, \rho(x)) - \psi_\varepsilon(x, 0)| \lesssim \int\limits_{|g(z)-g(x)|\geq\delta} \frac{1}{|z-x|^2}\, dz + 1.$$

Similarly,

$$|\psi_\varepsilon(y,\rho(x)) - \psi_\varepsilon(y,0_+)| \lesssim \int\limits_{|g(z)-g(y)|\geq\delta} \frac{1}{|z-y|^2}\,dz + 1.$$

Thus (A.35) and (A.36) are proved.

Integrating (A.34) with respect to x and y on I^2 and using (A.25) and (A.33), one obtains (A.32).

Combining (A.29), (A.30), (A.31), and (A.32) yields

$$\int_{\mathcal{C}_\varepsilon\setminus\mathcal{B}_\varepsilon}\int_{\mathcal{C}_\varepsilon\setminus\mathcal{B}_\varepsilon} |\varphi(x)-\varphi(y)|\,dx\,dy \lesssim T_\delta(g)+1.$$

Therefore, the conclusion follows from (A.26). □

Remark 16. A natural strategy for Step 2 would be to construct, for any given $g \in \mathrm{VMO}(\mathbb{S}^1,\mathbb{S}^1)$, a sequence $(g_k) \subset C(\mathbb{S}^1,\mathbb{S}^1)$ such that $g_k \to g$ in $\mathrm{BMO}(\mathbb{S}^1)$ and

$$\lim_{k\to\infty} T_\delta(g_k) = T_\delta(g). \tag{A.38}$$

We warn the reader that there exist $g \in C([0,1],\mathbb{R})$ and a sequence $(g_k) \subset C([0,1],\mathbb{R})$ such that $g_k \to g$ in BMO and

$$\lim_{k\to\infty} T_\delta(g_k) = +\infty, \quad \forall\,\delta > 0.$$

(see [19]). However, it might be true that (A.38) holds for a *special* sequence (g_k); this is an open problem (see [19]).

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References

[1] G. Alberti, Distributional Jacobian and singularities of Sobolev maps, *Ricerche Mat.* **54** (2005), 375–394 (2006). MR 2289488. Zbl pre05230855.

[2] G. Alberti, S. Baldo, and G. Orlandi, Functions with prescribed singularities, *J. Eur. Math. Soc.* (*JEMS*) **5** (2003), 275–311. MR 2002215. Zbl 1033.46028. doi: 10.1007/s10097-003-0053-5.

[3] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1976/77), 337–403. MR 0475169. Zbl 0368.73040. doi: 10.1007/BF00279992.

[4] J. M. Ball and F. Murat, $W^{1,p}$-quasiconvexity and variational problems for multiple integrals, *J. Funct. Anal.* **58** (1984), 225–253. MR 0759098. Zbl 0549.46019. doi: 10.1016/0022-1236(84)90041-7.

[5] F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg-Landau Vortices*, *Progr. Nonlinear Differential Equations Appl.* **13**, Birkhäuser, Boston, MA, 1994. MR 1269538. Zbl 0802.35142.

[6] F. Bethuel and D. Chiron, Some questions related to the lifting problem in Sobolev spaces, in *Perspectives in Nonlinear Partial Differential Equations*, *Contemp. Math.* **446**, Amer. Math. Soc., Providence, RI, 2007, pp. 125–152. MR 2373727. Zbl 05831662.

[7] J. Bourgain, H. Brezis, , and H.-M. Nguyen, A new estimate for the topological degree, *C. R. Math. Acad. Sci. Paris* **340** (2005), 787–791. MR 2139888. Zbl 1071.55002. doi: 10.1016/j.crma.2005.04.007.

[8] J. Bourgain, H. Brezis, and P. Mironescu, Complements to the paper: "Lifting, degree, and distributional Jacobian revisited", unpublished), posted on the website http://math.univ-lyon1.fr/ mironescu/2.pdf.

[9] ———, $H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation, *Publ. Math. Inst. Hautes Études Sci.* (2004), 1–115. MR 2075883. Zbl 1051.49030. doi: 10.1007/s10240-004-0019-5.

[10] ———, Lifting, degree, and distributional Jacobian revisited, *Comm. Pure Appl. Math.* **58** (2005), 529–551. MR 2119868. Zbl 1077.46023. doi: 10.1002/cpa.20063.

[11] P. Bousquet, Topological singularities in $W^{s,p}(S^N, S^1)$, *J. Anal. Math.* **102** (2007), 311–346. MR 2346560. Zbl 1158.58018. doi: 10.1007/s11854-007-0023-z.

[12] H. Brezis, The interplay between analysis and topology in some nonlinear PDE problems, *Bull. Amer. Math. Soc.* **40** (2003), 179–201. MR 1962295. Zbl 1161.35354. doi: 10.1090/S0273-0979-03-00976-5.

[13] H. Brezis, J.-M. Coron, and E. H. Lieb, Harmonic maps with defects, *Comm. Math. Phys.* **107** (1986), 649–705. MR 0868739. Zbl 0608.58016. Available at http://projecteuclid.org/getRecord?id=euclid.cmp/1104116234.

[14] H. Brezis, N. Fusco, and C. Sbordone, Integrability for the Jacobian of orientation preserving mappings, *J. Funct. Anal.* **115** (1993), 425–431. MR 1234399. Zbl 0847.26012. doi: 10.1006/jfan.1993.1098.

[15] H. Brezis, Y. Li, P. Mironescu, and L. Nirenberg, Degree and Sobolev spaces, *Topol. Methods Nonlinear Anal.* **13** (1999), 181–190. MR 1742219. Zbl 0956.46024.

[16] H. Brezis and P. Mironescu, Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces, *J. Evol. Equ.* **1** (2001), 387–404. MR 1877265. Zbl 1023.46031. doi: 10.1007/PL00001378.

[17] H. Brezis, P. Mironescu, and A. C. Ponce, $W^{1,1}$-maps with values into S^1, in *Geometric Analysis of* PDE *and Several Complex Variables*, *Contemp. Math.* **368**, Amer. Math. Soc., Providence, RI, 2005, pp. 69–100. MR 2126464. Zbl 1078.46020.

[18] H. Brezis and H.-M. Nguyen, The Jacobian determinant revisited, Invent. Math. (to appear). doi: 10.1007/s00222-010-0300-9.

[19] ———, On a new class of functions related to VMO, *C. R. Acad. Sci. Paris Sér.* I *Math.* **349** (2011), 157–160. doi: 10.1016/j.crma.2010.11.026.

[20] H. Brezis and L. Nirenberg, Degree theory and BMO; Part I: Compact manifolds without boundaries, *Selecta Math.* **1** (1995), 197–263. MR 1354598. Zbl 0852.58010. doi: 10.1007/BF01671566.

[21] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, Compensated compactness and Hardy spaces, *J. Math. Pures Appl.* **72** (1993), 247–286. MR 1225511. Zbl 0864.42009.

[22] R. R. Coifman and Y. Meyer, Une généralisation du théorème de Calderón sur l'intégrale de Cauchy, in *Fourier Analysis*, *Asoc. Mat. Espa nola* **1**, Asoc. Mat. Espa nola, Madrid, 1980, pp. 87–116. MR 0582251. Zbl 0541.42008.

[23] B. Dacorogna and J. Moser, On a partial differential equation involving the Jacobian determinant, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **7** (1990), 1–26. MR 1046081. Zbl 0707.35041.

[24] B. Dacorogna and F. Murat, On the optimality of certain Sobolev exponents for the weak continuity of determinants, *J. Funct. Anal.* **105** (1992), 42–62. MR 1156669. Zbl 0769.46025. doi: 10.1016/0022-1236(92)90071-P.

[25] M. P. do Carmo, *Riemannian Geometry*, *Mathematics*: *Theory Appl.*, Birkhäuser, Boston, MA, 1992. MR 1138207. Zbl 0752.53001.

[26] M. J. Esteban, A direct variational approach to Skyrme's model for meson fields, *Comm. Math. Phys.* **105** (1986), 571–591. MR 0852091. Zbl 0621.58035. Available at http://projecteuclid.org/getRecord?id=euclid.cmp/1104115501.

[27] M. J. Esteban and S. Müller, Sobolev maps with integer degree and applications to Skyrme's problem, *Proc. Roy. Soc. London Ser. A* **436** (1992), 197–201. MR 1177129. Zbl 0757.49010. doi: 10.1098/rspa.1992.0014.

[28] I. Fonseca, N. Fusco, and P. Marcellini, On the total variation of the Jacobian, *J. Funct. Anal.* **207** (2004), 1–32. MR 2027634. Zbl 1041.49016. doi: 10.1016/S0022-1236(03)00111-3.

[29] ———, Topological degree, Jacobian determinants and relaxation, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* **8** (2005), 187–250. MR 2122983. Zbl 1177.49066.

[30] I. Fonseca, G. Leoni, and J. Malý, Weak continuity and lower semicontinuity results for determinants, *Arch. Ration. Mech. Anal.* **178** (2005), 411–448. MR 2196498. Zbl 1081.49013. doi: 10.1007/s00205-005-0377-2.

[31] M. Giaquinta, G. Modica, and J. Souček, Remarks on the degree theory, *J. Funct. Anal.* **125** (1994), 172–200. MR 1297018. Zbl 0822.55003. doi: 10.1006/jfan.1994.1121.

[32] M. Giaquinta, *Cartesian Currents in the Calculus of Variations*. I, *Ergeb. Math. Grenzgeb.* **37**, Springer-Verlag, New York, 1998. MR 1645086. Zbl 0914.49001.

[33] ———, *Cartesian Currents in the Calculus of Variations*. II, *Ergeb. Math. Grenzgeb.* **38**, Springer-Verlag, New York, 1998. MR 1645082. Zbl 0914.49002.

[34] M. L. Gromov and Y. Eliashberg, Construction of a smooth mapping with a prescribed Jacobian. I, *Funct. Anal. Appl.* **7** (1973), 27–33. MR 0353357. Zbl 0285.57015.

[35] P. Hajłasz, A note on weak approximation of minors, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **12** (1995), 415–424. MR 1341410. Zbl 0910.49025.

[36] F. Hang and F. Lin, A remark on the Jacobians, *Commun. Contemp. Math.* **2** (2000), 35–46. MR 1753137. Zbl 1033.49047. doi: 10.1142/S0219199700000037.

[37] T. Iwaniec, Integrability theory of the Jacobians, Lipschitz lectures in Bonn, preprint 36, Sonderforschungsbereich 256, Bonn, 1995.

[38] T. Iwaniec and J. Onninen, $\mathcal{H}_1$-estimates of Jacobians by subdeterminants, *Math. Ann.* **324** (2002), 341–358. MR 1933861. Zbl 1055.42011. doi: 10.1007/s00208-002-0341-5.

[39] T. Iwaniec and C. Sbordone, On the integrability of the Jacobian under minimal hypotheses, *Arch. Rational Mech. Anal.* **119** (1992), 129–143. MR 1176362. Zbl 0766.46016. doi: 10.1007/BF00375119.

[40] R. L. Jerrard and H. M. Soner, Functions of bounded higher variation, *Indiana Univ. Math. J.* **51** (2002), 645–677. MR 1911049. Zbl 1057.49036. doi: 10.1512/iumj.2002.51.2229.

[41] ———, The Jacobian and the Ginzburg-Landau energy, *Calc. Var. Partial Differential Equations* **14** (2002), 151–191. MR 1890398. Zbl 1034.35025. doi: 10.1007/s005260100093.

[42] ———, Limiting behavior of the Ginzburg-Landau functional, *J. Funct. Anal.* **192** (2002), 524–561. MR 1923413. Zbl 1028.49015. doi: 10.1006/jfan.2001.3906.

[43] F. John and L. Nirenberg, On functions of bounded mean oscillation, *Comm. Pure Appl. Math.* **14** (1961), 415–426. MR 0131498. Zbl 0102.04302. doi: 10.1002/cpa.3160140317.

[44] F. Lin and Y. Yang, Existence of two-dimensional skyrmions via the concentration-compactness method, *Comm. Pure Appl. Math.* **57** (2004), 1332–1351. MR 2070206. Zbl 1059.81184. doi: 10.1002/cpa.20038.

[45] ———, Analysis on Faddeev knots and Skyrme solitons: recent progress and open problems, in *Perspectives in Nonlinear Partial Differential Equations*, *Contemp. Math.* **446**, Amer. Math. Soc., Providence, RI, 2007, pp. 319–344. MR 2376667. Zbl 05831673.

[46] ———, Energy splitting, substantial inequality, and minimization for the Faddeev and Skyrme models, *Comm. Math. Phys.* **269** (2007), 137–152. MR 2274465. Zbl 1113.81096. doi: 10.1007/s00220-006-0123-0.

[47] V. Maz′ya and T. Shaposhnikova, On the Brezis and Mironescu conjecture concerning a Gagliardo-Nirenberg inequality for fractional Sobolev norms,

J. Math. Pures Appl. **81** (2002), 877–884. MR 1940371. Zbl 1036.46026. doi: 10.1016/S0021-7824(02)01262-X.

[48] P. Mironescu, Sobolev maps on manifolds: degree, approximation, lifting, in *Perspectives in Nonlinear Partial Differential Equations*, *Contemp. Math.* **446**, Amer. Math. Soc., Providence, RI, 2007, pp. 413–436. MR 2376670. Zbl 1201.46032.

[49] C. B. Morrey, Jr., *Multiple Integrals in the Calculus of Variations*, *Grundlehren Math. Wis.* **130**, Springer-Verlag, New York, 1966. MR 0202511. Zbl 0142.38701.

[50] J. Moser, On the volume elements on a manifold, *Trans. Amer. Math. Soc.* **120** (1965), 286–294. MR 0182927. Zbl 0141.19407. doi: 10.2307/1994022.

[51] S. Müller, Higher integrability of determinants and weak convergence in L^1, *J. Reine Angew. Math.* **412** (1990), 20–34. MR 1078998. Zbl 2376667. doi: 10.1515/crll.1990.412.20.

[52] ———, Det = det. A remark on the distributional determinant, *C. R. Acad. Sci. Paris Sér. I Math.* **311** (1990), 13–17. MR 1062920. Zbl 0717.46033.

[53] ———, On the singular support of the distributional determinant, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **10** (1993), 657–696. MR 1253606. Zbl 0792.46027.

[54] H.-M. Nguyen, Optimal constant in a new estimate for the degree, *J. Anal. Math.* **101** (2007), 367–395. MR 2346551. Zbl 1147.47046. doi: 10.1007/s11854-007-0014-0.

[55] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, *Courant Lect. Notes Math.* **6**, New York University Courant Institute of Mathematical Sciences, New York, 2001, Chapter 6 by E. Zehnder, Notes by R. A. Artino, Revised reprint of the 1974 original. MR 1850453. Zbl 0992.47023.

[56] J. G. Rešetnjak, The weak convergence of completely additive vector-valued set functions, *Sibirsk. Mat. Ž.* **9** (1968), 1386–1394. MR 0240274.

[57] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, *Princeton Math. Series* **30**, Princeton Univ. Press, Princeton, N.J., 1970. MR 0290095. Zbl 0207.13501.

[58] H. Triebel, *Theory of Function Spaces*, *Monogr. Math.* **78**, Birkhäuser, Basel, 1983. MR 0781540. Zbl 0546.46027. doi: 10.1007/978-3-0346-0416-1.

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