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Livšic Theorem for matrix cocycles

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Abstract

We prove the Livšic Theorem for arbitrary $\mathrm{GL}(m,\mathbb{R})$ cocycles. We consider a hyperbolic dynamical system $f : X \to X$ and a Hölder continuous function $A : X \to \mathrm{GL}(m,\mathbb{R})$. We show that if A has trivial periodic data, i.e. $A(f^{n-1}p)\cdots A(fp)A(p) = \mathrm{Id}$ for each periodic point $p = f^n p$, then there exists a Hölder continuous function $C : X \to \mathrm{GL}(m,\mathbb{R})$ satisfying $A(x) = C(fx)C(x)^{-1}$ for all $x \in X$. The main new ingredients in the proof are results of independent interest on relations between the periodic data, Lyapunov exponents, and uniform estimates on growth of products along orbits for an arbitrary Hölder function A.

1. Introduction

For a hyperbolic dynamical system $f : X \to X$ and a group G we consider the question of when a Hölder continuous function $A : X \to G$ is a *coboundary*, i.e. there exists a (continuous or Hölder continuous) function $C : X \to G$ satisfying

$$A(x) = C(fx)C(x)^{-1} \qquad \text{for all } \ x \in X.$$

This is equivalent to the fact that the G-valued cocycle $\mathcal{A}$ generated by A (see (2.2) and (2.3)) over the $\mathbb{Z}$ action generated by f is cohomologous to the identity cocycle. Since any coboundary A must have trivial periodic data, i.e.

(1.1)
$$\mathcal{A}(p,n) \stackrel{\text{def}}{=} A(f^{n-1}p)\,\cdots\, A(fp)\,A(p) = \mathrm{Id} \qquad \forall\, p \in X,\, n \in \mathbb{N} \ \text{ with } f^n p = p,$$

the question is whether this necessary condition is also sufficient. Cocycles appear naturally in many important problems in dynamics. A. Livšic was first to study cohomology of dynamical systems in his seminal papers [10], [11]. In the case of Abelian G he obtained positive answers for this and related questions. Similar questions for non-abelian groups are substantially more difficult and, despite some progress, were not successfully resolved. Non-abelian cohomology of hyperbolic systems has since been extensively studied; some of

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the highlights are [5], [2], [3], [12], [13], [14], [15], [16], [17], [19]. We refer the reader to [4] and to the upcoming book [9] for some of the most recent results and overview of historical development in this area. The natural difficulty in non-abelian Livšic-type arguments is related to the growth of the cocycle along orbits. In particular, the sufficiency of condition (1.1) was established when G is compact or when A is either sufficiently close to identity or satisfies some growth assumptions. For example, specific *localization assumptions* are given in [4] for various cases of groups and metrics on them.

In this paper we prove the sufficiency of (1.1) for an arbitrary $\mathrm{GL}(m, \mathbb{R})$ cocycle, which has been a long standing open problem. We also obtain an important result for cocycles with uniformly bounded periodic data. Our theorems cover most classes of groups with interesting applications, except for groups of diffeomorphisms. To prove these theorems we establish new relations between the periodic data, Lyapunov exponents and uniform estimates of the growth for an arbitrary Hölder cocycle. These results are of independent interest and have wide applicability.

To include various classes of hyperbolic systems $f : X \to X$ and streamline the notation we formulate explicitly the property to be used.

Definition. We call orbit segments $x, fx, \dots, f^n x$ and $p, fp, \dots, f^n p$ *exponentially δ close with exponent* $\lambda > 0$ if for every $i = 0, \dots, n$ we have

$$\mathrm{dist}(f^i x, f^i p) \le \delta \cdot \exp(-\lambda \min\{i, n-i\}). \tag{1.2}$$

Definition. We say that a homeomorphism f of a metric space X satisfies the *closing property* if there exist $c, \lambda, \delta_0 > 0$ such that for any $x \in X$ and $n > 0$ with $\mathrm{dist}(x, f^n x) < \delta_0$ there exists a point $p \in X$ with $f^n p = p$ such that the orbit segments $x, fx, \dots, f^n x$ and $p, fp, \dots, f^n p$ are exponentially $\delta = c\, \mathrm{dist}(x, f^n x)$ close with exponent λ and there exists a point $y \in X$ such that for every $i = 0, \dots, n$

$$\mathrm{dist}(f^i p, f^i y) \le \delta\, e^{-\lambda i} \quad \text{and} \quad \mathrm{dist}(f^i y, f^i x) \le \delta\, e^{-\lambda (n-i)}. \tag{1.3}$$

The Anosov Closing Lemma and the local product structure yield the closing property for smooth hyperbolic systems such as hyperbolic automorphisms of tori and nilmanifolds, Anosov diffeomorphisms, and locally maximal hyperbolic sets (basic sets of axiom-A systems) [8]. Another class satisfying the closing property includes symbolic dynamical systems such as subshifts of finite type.

We now state our main result, the Livšic Theorem for matrix cocycles. Recall that a homeomorphism is called *topologically transitive* if it has a dense orbit.

THEOREM 1.1. *Let f be a topologically transitive homeomorphism of a compact metric space X satisfying the closing property. Let $A : X \to \mathrm{GL}(m, \mathbb{R})$ be an α-Hölder function such that*

$$A(f^{n-1}p) \dots A(fp)\, A(p) = \mathrm{Id} \qquad \forall\, p \in X,\, n \in \mathbb{N} \;\; \textit{with } f^n p = p.$$

Then there exists an α-Hölder function $C : X \to \mathrm{GL}(m, \mathbb{R})$ such that

$$A(x) = C(fx)C(x)^{-1} \qquad \textit{for all } \; x \in X. \tag{1.4}$$

Remark. Note that a value of C at a point x uniquely determines by (1.4) the values of C on the orbit of x. Hence, by the topological transitivity of f, C is unique up to a translation; i.e., any other C' satisfying (1.4) is of the form $C'(x) = C(x)B$ for some $B \in \mathrm{GL}(m, \mathbb{R})$. Also, [14, Th. 2.4] implies that such C is smooth if so are A and (X, f).

Remark. As we note in the end of the proof, if A takes values in a closed subgroup G of $\mathrm{GL}(m, \mathbb{R})$, then C can be naturally chosen to take values in G. Thus Theorem 1.1 holds if $\mathrm{GL}(m, \mathbb{R})$ is replaced by such a group G. In fact, the theorem holds for any connected Lie group G as follows from the remark after the next theorem.

Next we consider a more general case when the periodic data is not trivial but is uniformly bounded, for example is contained in a compact subgroup. In this case we prove that the cocycle itself is also bounded.

THEOREM 1.2. *Let f be a transitive homeomorphism of a compact metric space X satisfying the closing property and let $A : X \to \mathrm{GL}(m, \mathbb{R})$ be an α-Hölder function. Suppose that there exists a compact set $K \subset \mathrm{GL}(m, \mathbb{R})$ such that $\mathcal{A}(p, n) \in K$ for all $p \in X$ and $n \in \mathbb{N}$ with $f^n p = p$. Then there exists a compact set K' such that $\mathcal{A}(x, n) \in K'$ for all $x \in X$ and $n \in \mathbb{Z}$.*

In particular, this theorem allows one to obtain further cohomology information for $\mathrm{GL}(m, \mathbb{R})$ cocycles with uniformly bounded periodic data by using results obtained in [19] for cocycles that distort a distance on the group in a bounded fashion. For further results on cocycles with bounded or conformal periodic data, see the subsequent paper [6].

Remark. For a cocycle with values in a connected Lie group G, Theorem 1.2 can be applied to the adjoint representation. For example, if the periodic data is trivial (1.1), then the theorem implies that all $\mathrm{Ad}(\mathcal{A}(x, n))$ are uniformly bounded and hence the cocycle distorts a right invariant metric on G in a bounded fashion. It follows from [19] or classical arguments [11], [9, Th. 5.3.1] that Theorem 1.1 holds for such G.

To prove Theorems 1.1 and 1.2 we first establish the following growth estimates for a cocycle in terms of its periodic data. This result gives new tools for further study of cohomology for non-abelian cocycles, in particular for the case when the periodic data has exponents close to zero. We think that Theorem 1.3 will also be useful for various problems in smooth dynamics of hyperbolic systems and actions, such as existence of invariant geometric structures and rigidity.

THEOREM 1.3. *Let f be a homeomorphism of a compact metric space X satisfying the closing property and let $\mathcal{A}$ be a Hölder* $\mathrm{GL}(m,\mathbb{R})$ *cocycle over f. Let $\chi_{\min}$ and $\chi_{\max}$ be real numbers such that for every periodic point p, every eigenvalue ρ of $\mathcal{A}(p,n)$ satisfies $\chi_{\min} \le \frac{1}{n}\log|\rho| \le \chi_{\max}$, where n is the period of p. Then for any $\varepsilon > 0$ there exists a constant c_ε such that for all $x \in X$ and $n \in \mathbb{N}$*

$$\|\mathcal{A}(x,n)\| \le c_\varepsilon \exp(n\chi_{\max} + \varepsilon n) \quad \textit{and} \quad \|\mathcal{A}(x,n)^{-1}\| \le c_\varepsilon \exp(-n\chi_{\min} + \varepsilon n). \tag{1.5}$$

The proof of this theorem relies on our next result which resembles Theorem 3.1 in [21] on approximation of Lyapunov exponents of a hyperbolic invariant measure for a diffeomorphism that follows earlier results in [7]. Note that in our case there is no assumption on hyperbolicity of the cocycle and, in fact, our main application is to cocycles with all Lyapunov exponents equal to zero.

THEOREM 1.4. *Let f be a homeomorphism of a compact metric space X satisfying the closing property, let $\mathcal{A}$ be a Hölder* $\mathrm{GL}(m,\mathbb{R})$ *cocycle over f, and let μ be an ergodic invariant measure for f. Then the Lyapunov exponents $\chi_1 \le \cdots \le \chi_m$ (listed with multiplicities) of $\mathcal{A}$ with respect to μ can be approximated by the Lyapunov exponents of $\mathcal{A}$ at periodic points. More precisely, for any $\varepsilon > 0$ there exists a periodic point $p \in X$ for which the Lyapunov exponents $\chi_1^{(p)} \le \cdots \le \chi_m^{(p)}$ of $\mathcal{A}$ satisfy $|\chi_i - \chi_i^{(p)}| < \varepsilon$ for $i = 1, \ldots, m$.*

Remark. Theorems 1.3 and 1.4 use only a weaker version of the closing property without the existence of a point y. Also, $\delta = c\,\mathrm{dist}(x, f^n x)$ in the closing property could be replaced by $\delta = c\,\mathrm{dist}(x, f^n x)^\beta$ with $\beta > 0$. The proofs of Theorems 1.2, 1.3, and 1.4 work in the same way with proper modifications of exponents. Similarly, Theorem 1.1 holds in this case with C being $(\alpha\beta)$-Hölder.

Remark. More generally, Theorems 1.1, 1.2, 1.3, and 1.4 hold for an extension $\mathcal{A}$ of f by linear transformations of a vector bundle $\mathcal{B}$ over X. The arguments are essentially identical since we compare the values of A and related structures only at nearby points. This can be done if one can identify fibers at nearby points Hölder-continuously via local trivialization or connection. In particular, the theorems apply to the derivative cocycle of a smooth hyperbolic system, as well as to its restriction to a Hölder continuous invariant distribution, without any global trivialization assumptions.

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2. Cocycles over $\mathbb{Z}$ actions

In this section we review some basic definitions and facts of the Oseledec theory of cocycles over $\mathbb{Z}$ actions. We use [1] as a general reference.

2.1. *Cocycles.* Let f be an invertible transformation of a space X. A function $\mathcal{A} : X \times \mathbb{Z} \to \mathrm{GL}(m, \mathbb{R})$ is called a *linear cocycle* or a *matrix–valued cocycle* over f if for all $x \in X$ and $n, k \in \mathbb{Z}$ we have $\mathcal{A}(x, 0) = \mathrm{Id}$ and

$$\mathcal{A}(x, n+k) = \mathcal{A}(f^k x, n) \cdot \mathcal{A}(x, k). \tag{2.1}$$

We consider only matrix-valued cocycles and simply call them *cocycles.* Any cocycle $\mathcal{A}(x, n)$ is uniquely determined by its generator $A : X \to \mathrm{GL}(m, \mathbb{R})$, which we sometimes also call a cocycle. The generator is defined by $A(x) = \mathcal{A}(x, 1)$, and the cocycle can be reconstructed from its generator as follows: for any $n > 0$

$$\mathcal{A}(x, n) = A(f^{n-1}x) \cdots A(fx) \cdot A(x), \tag{2.2}$$

$$\mathcal{A}(x, -n) = A(f^{-n}x)^{-1} \cdots A(f^{-2}x)^{-1} \cdot A(f^{-1}x)^{-1} = \mathcal{A}(f^{-n}x, n)^{-1}. \tag{2.3}$$

A cocycle $\mathcal{A}$ over a homeomorphism f of a metric space X is called α-Hölder if its generator $A : X \to \mathrm{GL}(m, \mathbb{R})$ is Hölder continuous with exponent α. To consider this notion we need to introduce a metric on $\mathrm{GL}(m, \mathbb{R})$, for example as follows:

$$\mathrm{dist}_{\mathrm{GL}(m,\mathbb{R})}(A, B) = \|A - B\| + \|A^{-1} - B^{-1}\|, \quad \text{where} \tag{2.4}$$

$$\|A\| = \sup\{\|Au\| \cdot \|u\|^{-1} : \ 0 \neq u \in \mathbb{R}^m\}.$$

We note that on any compact set in $\mathrm{GL}(m, \mathbb{R})$ the norms $\|A^{-1}\|$ and $\|B^{-1}\|$ are uniformly bounded and hence this distance is Lipschitz equivalent to $\|A - B\|$. Therefore, for a compact X, a cocycle $\mathcal{A}$ is α-Hölder if and only if

$$\|A(x) - A(y)\| \leq c \, \mathrm{dist}(x, y)^\alpha$$

for all $x, y \in X$. For a noncompact X certain caution is needed as in the proof of Theorems 1.1 and 1.2.

2.2. *Lyapunov exponents and Lyapunov metric.* Cocycles can be considered in various categories. Even though in this paper we mostly study Hölder cocycles, a general theory is developed for measurable cocycles over measure-preserving transformations.

THEOREM 2.1 (Oseledec Multiplicative Ergodic Theorem; [1, Th. 3.4.3]). *Let f be an invertible ergodic measure-preserving transformation of a Lebesgue probability measure space (X, μ). Let $\mathcal{A}$ be a measurable cocycle whose generator satisfies $\log \|A(x)\| \in L^1(X,\mu)$ and $\log \|A(x)^{-1}\| \in L^1(X,\mu)$. Then there exist numbers $\chi_1 < \cdots < \chi_l$, an f-invariant set $\mathcal{R}^\mu$ with $\mu(\mathcal{R}^\mu) = 1$, and an A-invariant Lyapunov decomposition of $\mathbb{R}^m$ for $x \in \mathcal{R}^\mu$,*

$$\mathbb{R}^m_x = E_{\chi_1}(x) \oplus \cdots \oplus E_{\chi_l}(x)$$

with $\dim E_{\chi_i}(x) = m_i$, such that for any $i = 1, \dots, l$ and any $0 \neq v \in E_{\chi_i}(x)$ one has

$$\lim_{n\to\pm\infty} n^{-1} \log \|\mathcal{A}(x,n)v\| = \chi_i \quad \textit{and} \quad \lim_{n\to\pm\infty} n^{-1} \log \det \mathcal{A}(x,n) = \sum_{i=1}^{l} m_i \chi_i.$$

Definitions. The numbers $\chi_1, \dots, \chi_l$ are called the *Lyapunov exponents* of A and the dimension m_i of the space $E_{\chi_i}(x)$ is called the *multiplicity* of the exponent χ_i. The points of the set $\mathcal{R}^\mu$ are called *regular*.

We denote the standard scalar product in $\mathbb{R}^m$ by $\langle \cdot, \cdot \rangle$. For a fixed $\varepsilon > 0$ and a regular point x we introduce *the ε-Lyapunov scalar product* (*or metric*) $\langle \cdot, \cdot \rangle_{x,\varepsilon}$ in $\mathbb{R}^m$ as follows. For $u \in E_{\chi_i}(x)$, $v \in E_{\chi_j}(x)$, $i \neq j$ we set $\langle u, v \rangle_{x,\varepsilon}, = 0$. For $i = 1, \dots, l$ and $u, v \in E_{\chi_i}(x)$ we define

$$\langle u, v \rangle_{x,\varepsilon} = m \sum_{n\in\mathbb{Z}} \langle \mathcal{A}(x,n)u, \mathcal{A}(x,n)v \rangle \exp(-2\chi_i n - \varepsilon |n|).$$

Note that the series converges exponentially for any regular x. The constant m in front of the conventional formula is introduced for more convenient comparison with the standard scalar product. Usually, ε will be fixed and we will denote $\langle \cdot, \cdot \rangle_{x,\varepsilon}$ simply by $\langle \cdot, \cdot \rangle_x$ and call it the *Lyapunov scalar product.* The norm generated by this scalar product is called the *Lyapunov norm* and is denoted by $\| \cdot \|_{x,\varepsilon}$ or $\| \cdot \|_x$.

We summarize below some important properties of the Lyapunov scalar product and norm; for more details see [1, §§3.5.1–3.5.3]. A direct calculation shows [1, Th. 3.5.5] that for any regular x and any $u \in E_{\chi_i}(x)$

(2.5)
$$\exp(n\chi_i - \varepsilon|n|)\|u\|_{x,\varepsilon} \le \|\mathcal{A}(x,n)u\|_{f^n x,\varepsilon} \le \exp(n\chi_i + \varepsilon|n|)\|u\|_{x,\varepsilon} \quad \forall n \in \mathbb{Z},$$

$$\exp(n\chi - \varepsilon n) \le \|\mathcal{A}(x,n)\|_{f^n x \leftarrow x} \le \exp(n\chi + \varepsilon n) \quad \forall n \in \mathbb{N}, \tag{2.6}$$

where $\chi = \chi_l$ is the maximal Lyapunov exponent and $\|\cdot\|_{f^n x \leftarrow x}$ is the operator norm with respect to the Lyapunov norms. It is defined for any matrix A and any regular points x, y as follows:

$$\|A\|_{y \leftarrow x} = \sup\{\|Au\|_{y,\varepsilon} \cdot \|u\|_{x,\varepsilon}^{-1} : \ 0 \neq u \in \mathbb{R}^m\}.$$

We emphasize that, for any given $\varepsilon > 0$, Lyapunov scalar product and Lyapunov norm are defined only for regular points with respect to the given measure. They depend only measurably on the point even if the cocycle is Hölder. Therefore, comparison with the standard norm becomes important. The uniform lower bound follows easily from the definition: $\|u\|_{x,\varepsilon} \geq \|u\|$. The upper bound is not uniform, but it changes slowly along the regular orbits [1, Prop. 3.5.8]: there exists a measurable function $K_\varepsilon(x)$ defined on the set of regular points $\mathcal{R}^\mu$ such that

$$\|u\| \leq \|u\|_{x,\varepsilon} \leq K_\varepsilon(x)\|u\| \qquad \forall x \in \mathcal{R}^\mu, \ \forall u \in \mathbb{R}^m \tag{2.7}$$

and

$$K_\varepsilon(x)e^{-\varepsilon|n|} \leq K_\varepsilon(f^n x) \leq K_\varepsilon(x)e^{\varepsilon|n|} \qquad \forall x \in \mathcal{R}^\mu, \ \forall n \in \mathbb{Z}. \tag{2.8}$$

These estimates are obtained in [1] using the fact that $\|u\|_{x,\varepsilon}$ is *tempered*, but they can also be checked directly using the definition of $\|u\|_{x,\varepsilon}$ on each Lyapunov space and noting that angles between the spaces change slowly.

For any matrix A and any regular points x, y, inequalities (2.7) and (2.8) yield

$$K_\varepsilon(x)^{-1}\|A\| \leq \|A\|_{y \leftarrow x} \leq K_\varepsilon(y)\|A\|. \tag{2.9}$$

When ε is fixed we will usually omit it and write $K(x) = K_\varepsilon(x)$. For any $l > 1$ we also define the following sets of regular points

$$\mathcal{R}^\mu_{\varepsilon,l} = \{x \in \mathcal{R}^\mu : \ \ K_\varepsilon(x) \leq l\}. \tag{2.10}$$

Note that $\mu(\mathcal{R}^\mu_{\varepsilon,l}) \to 1$ as $l \to \infty$. Without loss of generality, we can assume that the set $\mathcal{R}^\mu_{\varepsilon,l}$ is compact and that Lyapunov splitting and Lyapunov scalar product are continuous on $\mathcal{R}^\mu_{\varepsilon,l}$. Indeed, by Luzin's theorem we can always find a subset of $\mathcal{R}^\mu_{\varepsilon,l}$ satisfying these properties with arbitrarily small loss of measure. (In fact, for standard Pesin sets these properties are automatically satisfied.)

3. **Proof of Theorem 1.4**

We begin with Lemma 3.1 below which gives a general estimate of the norm of $\mathcal{A}$ along *any* orbit segment close to a regular one. In fact, its proof does not use the measure μ and relies only on the estimates for $\mathcal{A}$ and the Lyapunov norm along the orbit segment $x, fx, \dots, f^n x$ that follow from the fact that $x, f^n x \in \mathcal{R}^\mu_{\varepsilon,l}$.

Lemma 3.1. *Let $\mathcal{A}$ be an α-Hölder cocycle over a homeomorphism f of a compact metric space X and let μ be an ergodic measure for f with the largest Lyapunov exponent χ. Then for any positive λ and ε satisfying $\lambda > \varepsilon/\alpha$ there exists $c > 0$ such that for any $n \in \mathbb{N}$, any regular point x with both x and $f^n x$ in $\mathcal{R}^{\mu}_{\varepsilon,l}$, and any point $y \in X$ such that the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n y$ are exponentially δ close with exponent λ we have*

$$\|\mathcal{A}(y,n)\|_{f^n x \leftarrow x} \le e^{c\,l\delta^{\alpha}} e^{n(\chi+\varepsilon)} \le e^{2n\varepsilon + c\,l\delta^{\alpha}}\, \|\mathcal{A}(x,n)\|_{f^n x \leftarrow x} \tag{3.1}$$

and

$$\|\mathcal{A}(y,n)\| \le l\, e^{c\,l\delta^{\alpha}} e^{n(\chi+\varepsilon)} \le l^2 e^{2n\varepsilon + c\,l\delta^{\alpha}}\, \|\mathcal{A}(x,n)\|. \tag{3.2}$$

The constant c depends only on the cocycle $\mathcal{A}$ and on the number $(\alpha\lambda - \varepsilon)$.

Proof. We denote $x_i = f^i x$ and $y_i = f^i y$, $i = 0, \dots, n$, and estimate the Lyapunov norm

$$\begin{aligned}
\|\mathcal{A}(y,n)\|_{x_n \leftarrow x_0} &= \|A(y_{n-1}) \;\cdots\; A(y_1)\, A(y_0)\|_{x_n \leftarrow x_0} \\
&= \|A(x_{n-1})\, [A(x_{n-1})^{-1} A(y_{n-1})] \;\cdots\; A(x_0)\, [A(x_0)^{-1} A(y_0)]\|_{x_n \leftarrow x_0} \\
&\le \|A(x_{n-1})\|_{x_n \leftarrow x_{n-1}} \|A(x_{n-1})^{-1} A(y_{n-1})\|_{x_{n-1} \leftarrow x_{n-1}} \cdots \\
&\quad \cdots \|A(x_0)\|_{x_1 \leftarrow x_0} \|A(x_0)^{-1} A(y_0)\|_{x_0 \leftarrow x_0}.
\end{aligned}$$

Since $\|A(x_i)\|_{x_{i+1} \leftarrow x_i} \le e^{\chi+\varepsilon}$ by (2.6), where χ is the maximal exponent of $\mathcal{A}$ at x, we conclude that

$$\|\mathcal{A}(y,n)\|_{x_n \leftarrow x_0} \le e^{n(\chi+\varepsilon)} \prod_{i=0}^{n-1} \|A(x_i)^{-1} A(y_i)\|_{x_i \leftarrow x_i}. \tag{3.3}$$

To estimate the product term we consider $D_i = A(x_i)^{-1} A(y_i) - \mathrm{Id}$. Since $A(x)$ is α-Hölder on the compact space X, and hence $\|A(x)^{-1}\|$ is uniformly bounded, we obtain, using the closeness of the orbit segments, that

$$\|D_i\| \le \|A(x_i)^{-1}\| \cdot \|A(y_i) - A(x_i)\| \le c' \mathrm{dist}(x_i, y_i)^{\alpha} \le c' \left(\delta e^{-\lambda \min\{i, n-i\}}\right)^{\alpha}, \tag{3.4}$$

where the constant c' depends only on the cocycle $\mathcal{A}$. Since both x and $f^n x$ are in $\mathcal{R}^{\mu}_{\varepsilon,l}$ we have $K(x_i) \le l e^{\varepsilon \min\{i,n-i\}}$ by (2.8) and (2.10). Hence for the Lyapunov norms we can conclude that

$$\begin{aligned}
\|D_i\|_{x_i \leftarrow x_i} \le K(x_i) \|D_i\| &\le l e^{\varepsilon \min\{i,n-i\}}\, \|D_i\| \\
&\le l e^{\varepsilon \min\{i,n-i\}}\, c' \delta^{\alpha} e^{-\lambda\alpha \min\{i,n-i\}}
\end{aligned} \tag{3.5}$$

and

$$\|A(x_i)^{-1} A(y_i)\|_{x_i \leftarrow x_i} \le 1 + \|D_i\|_{x_i \leftarrow x_i} \le 1 + c' l\, \delta^{\alpha}\, e^{(\varepsilon - \alpha\lambda)\, \min\{i,n-i\}}. \tag{3.6}$$

Now using (3.3) and (3.6) we obtain

$$\log(\|\mathcal{A}(y,n)\|_{x_n\leftarrow x_0}) - n(\chi+\varepsilon) \le \sum_{i=0}^{n-1} \log \|A(x_i)^{-1}A(y_i)]\|_{x_i\leftarrow x_i}$$
$$\le c' l\delta^\alpha \sum_{i=0}^{n-1} \exp\left[(\varepsilon-\alpha\lambda)\min\{i,n-i\}\right] \le c\, l\delta^\alpha$$

since the sum is uniformly bounded due to the assumption $\varepsilon < \alpha\lambda$. The constant c depends only on the cocycle $\mathcal{A}$ and on $(\alpha\lambda-\varepsilon)$. We conclude using (2.6) that

$$\|\mathcal{A}(y,n)\|_{x_n\leftarrow x_0} \le e^{c\, l\delta^\alpha} e^{n(\chi+\varepsilon)} \le e^{2n\varepsilon + c\, l\delta^\alpha} \|\mathcal{A}(x,n)\|_{x_n\leftarrow x_0}. \tag{3.7}$$

Since $K(x_0)\le l$ and $K(x_n)\le l$ we can also estimate the standard norm

$$\begin{aligned}\|\mathcal{A}(y,n)\| &\le K(x_0)\|\mathcal{A}(y,n)\|_{x_n\leftarrow x_0}\\ &\le l e^{c\, l\delta^\alpha} e^{n(\chi+\varepsilon)} \le l e^{2n\varepsilon + c\, l\delta^\alpha} \|\mathcal{A}(x,n)\|_{x_n\leftarrow x_0}\\ &\le l e^{2n\varepsilon + c\, l\delta^\alpha} K(x_n)\, \|\mathcal{A}(x,n)\| \le l^2 e^{2n\varepsilon + c\, l\delta^\alpha} \|\mathcal{A}(x,n)\|.\end{aligned} \tag{3.8}$$

Estimates (3.7) and (3.8) complete the proof of Lemma 3.1. □

The main part of the proof of Theorem 1.4 is the following proposition which gives approximation for the largest Lyapunov exponent of $\mathcal{A}$. We use it to complete the proof of Theorem 1.4 at the end of Section 3.

Let f be a homeomorphism of a compact metric space X satisfying the closing property with exponent λ, let $\mathcal{A}$ be an α-Hölder $\mathrm{GL}(m,\mathbb{R})$ cocycle over f, and let μ be an ergodic invariant measure for f. We denote by χ the largest Lyapunov exponent of $\mathcal{A}$ with respect to μ. Similarly, for any periodic point p we denote by $\chi^{(p)}$ the largest Lyapunov exponent of $\mathcal{A}$ at p. We set $\varepsilon_0 = \min\{\lambda\alpha, (\chi-\nu)/2)\}$, where $\nu<\chi$ is the second largest Lyapunov exponent with respect to μ. In the case when χ is the only Lyapunov exponent of $\mathcal{A}$ with respect to μ, we take $\varepsilon_0 = \lambda\alpha$.

Proposition 3.2. *Let f, $\mathcal{A}$, μ, and ε_0 be as above. Then for any positive l and $\varepsilon<\varepsilon_0$ there exist $N,\delta>0$ such that if a periodic orbit $p, fp, \dots, f^n p = p$ is exponentially δ close to an orbit segment $x, fx, \dots, f^n x$, with $x, f^n x$ in $\mathcal{R}^\mu_{\varepsilon,l}$ and $n>N$, then $|\chi-\chi^{(p)}|\le 3\varepsilon$.*

Proof. To estimate $\chi^{(p)}$ from above we apply Lemma 3.1 with $p=y$. Note that the largest exponent at p satisfies

$$\chi^{(p)} \le n^{-1}\log\|\mathcal{A}(p,n)\|.$$

From the first inequality in (3.2) we obtain that

$$n^{-1}\log\|\mathcal{A}(p,n)\| \le \chi+\varepsilon+n^{-1}\log(l\, e^{c\, l\delta^\alpha}).$$

We conclude that $\chi^{(p)} \leq \chi + 2\varepsilon$ provided that δ is small enough and n is large enough compared to l.

To estimate $\chi^{(p)}$ from below we will estimate the growth of vectors in a certain cone $K \subset \mathbb{R}^m$ invariant under $\mathcal{A}(p,n)$. As in Lemma 3.1 we first consider an arbitrary orbit segment close to a regular one. Let x be a point in $\mathcal{R}^{\mu}_{\varepsilon,l}$ and $y \in X$ be a point such that the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n y$ are exponentially δ close with exponent λ. We denote $x_i = f^i x$ and $y_i = f^i y$, $i = 0, \dots, n$. For each i we have orthogonal splitting $\mathbb{R}^m = E_i \oplus F_i$, where E_i is the Lyapunov space at x_i corresponding to the largest Lyapunov exponent χ and F_i is the direct sum of all other Lyapunov spaces at x_i corresponding to the Lyapunov exponents less than χ. For any vector $u \in \mathbb{R}^m$ we denote by $u = u' + u^{\perp}$ the corresponding splitting with $u' \in E_i$ and $u^{\perp} \in F_i$; the choice of i will be clear from the context. To simplify notation, we write $\|.\|_i$ for the Lyapunov norm at x_i. For each $i = 0, \dots, n$ we consider cones

$$K_i = \{u \in \mathbb{R}^m : \|u^{\perp}\|_i \leq \|u'\|_i\} \quad \text{and} \quad K_i^{\eta} = \{u \in \mathbb{R}^m : \|u^{\perp}\|_i \leq (1-\eta)\|u'\|_i\}$$

with $\eta > 0$. We will consider the case when χ is *not* the only Lyapunov exponent of $\mathcal{A}$ with respect to μ. Otherwise $F_i = \{0\}$, $K_i^{\eta} = K_i = \mathbb{R}^m$, and the argument becomes simpler. Recall that $\varepsilon < \varepsilon_0 = \min\{\lambda\alpha, (\chi - \nu)/2)\}$, where $\nu < \chi$ is the second largest Lyapunov exponent of $\mathcal{A}$ with respect to μ.

Lemma 3.3. *In the notation above, for any regular set $\mathcal{R}^{\mu}_{\varepsilon,l}$ there exist $\eta, \delta > 0$ such that if $x, f^n x \in \mathcal{R}^{\mu}_{\varepsilon,l}$ and the orbit segments $x, fx, \dots, f^n x$ and $y, fy, \dots, f^n y$ are exponentially δ close with exponent λ, then for every $i = 0, \dots, n-1$ we have $A(y_i)(K_i) \subset K_{i+1}^{\eta}$ and $\|\,(A(y_i)u)'\,\|_{i+1} \geq e^{\chi - 2\varepsilon}\|u'\|_i$ for any $u \in K_i$.*

Proof. We fix $0 \leq i < n$ and write

$$A(y_i) = A(y_i)A(x_i)^{-1}A(x_i) = (\mathrm{Id} + D_i)\,A(x_i),$$

where similarly to (3.4) we have

(3.9)
$$\|D_i\| = \|A(y_i)A(x_i)^{-1} - \mathrm{Id}\| \leq \|A(y_i) - A(x_i)\|\,\|A(x_i)^{-1}\| \leq c_1 \mathrm{dist}(x_i, y_i)^{\alpha}.$$

For any $u = u' + u^{\perp} \in K_i$ we consider $v = A(x_i)u$ and its splitting $v = v' + v^{\perp}$ with $v' \in E_{i+1}$ and $v^{\perp} \in F_{i+1}$. Then by (2.5) we have $\|v\|_{i+1} \leq e^{\chi+\varepsilon}\|u\|_i$ as well as

$$\|v'\|_{i+1} = \|A(x_i)u'\|_{i+1} \geq e^{\chi-\varepsilon}\|u'\|_i$$

and

$$\|v^{\perp}\|_{i+1} = \|A(x_i)u^{\perp}\|_{i+1} \leq e^{\nu+\varepsilon}\|u^{\perp}\|_i.$$

Now we consider $w = A(y_i)u = (\mathrm{Id} + D_i)v = v + D_i v$ and its splitting $w = w' + w^\perp$ with $w' \in E_{i+1}$ and $w^\perp \in F_{i+1}$. Then we have

$$w' = v' + (D_i v)' \quad \text{and} \quad w^\perp = v^\perp + (D_i v)^\perp. \tag{3.10}$$

Now using (3.9) we obtain

$$\begin{aligned}\|D_i v\|_{i+1} &\le \|D_i\|_{x_{i+1}\leftarrow x_{i+1}} \|v\|_{i+1} \le K(x_{i+1})\|D_i\|\, e^{\chi+\varepsilon}\|u\|_i \\ &\le l e^{\varepsilon \min\{i+1, n-i-1\}}\, c_1 \mathrm{dist}(x_i, y_i)^\alpha\, e^{\chi+\varepsilon}\sqrt{2}\,\|u'\|_i\,,\end{aligned}$$

as both x_0 and x_n are in $\mathcal{R}^\mu_{\varepsilon,l}$. Since $\mathrm{dist}(x_i, y_i) \le \delta e^{-\lambda \min\{i,n-i\}}$ we conclude that

$$\|D_i v\|_{i+1} \le \sqrt{2}\, l c_1 e^\varepsilon \delta^\alpha e^{(-\lambda\alpha+\varepsilon)\min\{i,n-i\}}\|u'\|_i \le c_2 l\, \delta^\alpha \|u'\|_i\,, \tag{3.11}$$

since $-\lambda\alpha + \varepsilon < 0$. Now using (3.10) and (3.11) we obtain that for small enough δ,

$$\|w'\|_{i+1} \ge e^{\chi-\varepsilon}\|u'\|_i - c_2 l\, \delta^\alpha \|u'\|_i \ge e^{\chi-2\varepsilon}\|u'\|_i\,,$$

which gives the inequality in the lemma. Similarly we obtain an upper estimate

$$\|w'\|_{i+1} \le e^{\chi+\varepsilon}\|u'\|_i + c_2 l\, \delta^\alpha \|u'\|_i \le c_3 \|u'\|_i. \tag{3.12}$$

Finally, from (3.10) we have

$$\|w'\|_{i+1} \ge \|v'\|_{i+1} - \|D_i v\|_{i+1} \quad \text{and} \quad \|w^\perp\|_{i+1} \le \|v^\perp\|_{i+1} + \|D_i v\|_{i+1}\,,$$

so that using (3.11) again we can estimate

$$\begin{aligned}\|w'\|_{i+1} - \|w^\perp\|_{i+1} &\ge \|v'\|_{i+1} - \|v^\perp\|_{i+1} - 2\|D_i v\|_{i+1} \\ &\ge e^{\chi-\varepsilon}\|u'\|_i - e^{\nu+\varepsilon}\|u^\perp\|_i - 2c_2 l\, \delta^\alpha \|u'\|_i \\ &\ge (e^{\chi-\varepsilon} - e^{\nu+\varepsilon} - 2c_2 l\, \delta^\alpha)\|u'\|_i \ge \eta'\, \|u'\|_i\end{aligned}$$

for any fixed $\eta' < (e^{\chi-\varepsilon} - e^{\nu+\varepsilon})$ provided that δ is small enough. Now using (3.12) we conclude that $\|w'\|_{i+1} - \|w^\perp\|_{i+1} \ge \eta\, \|w'\|_{i+1}$ with $\eta = \eta'/c_3$. This shows that $w \in K^\eta_{i+1}$ and hence $A(y_i)(K_i) \subset K^\eta_{i+1}$. This completes the proof of Lemma 3.3. □

We now apply this lemma to the periodic orbit $p, fp, \dots, f^n p = p$ and conclude that $A(p,n)(K_0) \subset K^\eta_n$. Since the Lyapunov splitting and Lyapunov metric are continuous on the compact set $\mathcal{R}^\mu_{\varepsilon,l}$, the cones K^η_0 and K^η_n are close if x and $f^n x$ are close enough. Therefore we can ensure that $K^\eta_n \subset K_0$ if δ small enough and thus $A(p,n)(K) \subset K$ for $K = K_0$. Finally, using the norm estimate in the lemma we obtain for any $u \in K$,

$$\begin{aligned}\|A(p,n)\, u\|_n \ge \|(A(p,n)\, u)'\|_n &\ge e^{n(\chi-2\varepsilon)}\|u'\|_0 \\ &\ge \frac{1}{\sqrt{2}} e^{n(\chi-2\varepsilon)}\|u\|_0 \ge \frac{1}{2} e^{n(\chi-2\varepsilon)}\|u\|_n\end{aligned}$$

since Lyapunov norms at x and $f^n x$ are close if δ is small enough. Since $A(p,n)\,u \in K$ for any $u \in K$, we can iteratively apply $A(p,n)$ and use the inequality above to estimate the largest Lyapunov exponent at p

$$\chi^{(p)} \geq \chi(u) = \lim_{k\to\infty} \frac{1}{kn} \log \|\mathcal{A}(p,kn)u\|_n \geq \frac{1}{n} \lim_{k\to\infty} \frac{1}{k} \log\left(\left(\frac{1}{2}e^{n(\chi-2\varepsilon)}\right)^k \|u\|_n\right)$$
$$\geq \frac{1}{n}\left[n(\chi-2\varepsilon) - \log 2\right] + \frac{1}{n}\lim_{k\to\infty}\frac{\|u\|_n}{k} \geq (\chi-2\varepsilon) - \frac{\log 2}{n} \geq \chi - 3\varepsilon$$

provided that n is large enough. This gives the desired lower estimate and completes the proof of Proposition 3.2. $\square$

We will now complete the proof of Theorem 1.4. We apply Proposition 3.2 to cocycles $\wedge^i \mathcal{A}$ induced by $\mathcal{A}$ on the i-fold exterior powers $\wedge^i \mathbb{R}^m$, for $i = 1, \dots, m$. This trick is related to Ragunatan's proof of the Multiplicative Ergodic Theorem [1, §3.4.4] and was also used in [21]. We note that the largest Lyapunov exponent of $\wedge^i \mathcal{A}$ is equal to $(\chi_m + \cdots + \chi_{m-i+1})$, where $\chi_1 \leq \cdots \leq \chi_m$ are the Lyapunov exponents of $\mathcal{A}$ listed with multiplicities.

For any positive $\varepsilon < \varepsilon_0$ we choose l so that $\mu(R) > 0$, where R is the intersection of the sets $\mathcal{R}^{\mu}_{\varepsilon,l}$ for all cocycles $\wedge^i \mathcal{A}$, $i = 1, \dots, m$. We may assume that μ is not atomic since the theorem is trivial otherwise. We take $x \in R$ to be a nonperiodic point with $\mu(B_r(x) \cap R) > 0$ for any $r > 0$, where $B_r(x)$ is the ball of radius r centered at x. Then by Poincaré recurrence there exist iterates $f^n x$, with n growing to infinity, returning to R arbitrarily close to x. Therefore, by the closing property, for any $\delta > 0$ there exists a periodic point p with $f^n p = p$ such that orbit segments $x, fx, \dots, f^n x$ and $p, fp, \dots, f^n p$ are exponentially δ close with exponent λ. Then Proposition 3.2 implies that for small enough δ such a periodic point p gives the approximation

$$\left|(\chi_m + \cdots + \chi_{m-i+1}) - \left(\chi_m^{(p)} + \cdots + \chi_{m-i+1}^{(p)}\right)\right| \leq 3\varepsilon$$

for all $i = 1, \dots, m$. This yields the simultaneous approximation for all χ_i, $i = 1, \dots, m$, and completes the proof of Theorem 1.4. $\square$

4. **Proof of Theorem 1.3**

The assumption on the eigenvalues of $\mathcal{A}(p,n)$ implies that all Lyapunov exponents of $\mathcal{A}$ at all periodic orbits are in the interval $[\chi_{\min}, \chi_{\max}]$. It follows from Theorem 1.4 that the Lyapunov exponents of $\mathcal{A}$ are in $[\chi_{\min}, \chi_{\max}]$ for *any* ergodic f-invariant measure. Such control on exponents gives the desired uniform estimates on the growth of the norm of the cocycle. This uses a result on subadditive sequences obtained in [20]. We formulate here a weaker version sufficient for our purposes, which appeared with a short proof in [18].

[18, Prop. 3.4] *Let $f : X \to X$ be a continuous map of a compact metric space. Let $a_n : X \to \mathbb{R}$, $n \geq 0$, be a sequence of continuous functions such that*

$$a_{n+k}(x) \leq a_n(f^k(x)) + a_k(x) \text{ for every } x \in X, \;\; n, k \geq 0 \tag{4.1}$$

and such that there is a sequence of continuous functions $b_n : X \to \mathbb{R}$, $n \geq 0$, satisfying

$$a_n(x) \leq a_n(f^k(x)) + a_k(x) + b_k(f^n(x)) \text{ for every } x \in X, \;\; n, k \geq 0. \tag{4.2}$$

If $\inf_n \left(\frac{1}{n}\int_X a_n d\mu\right) < 0$ for every ergodic f-invariant measure, then there is $N \geq 0$ such that $a_N(x) < 0$ for every $x \in X$.

We take $\varepsilon > 0$ and apply this result to $a_n(x) = \log \|\mathcal{A}(x, n)\| - (\chi_{\max} + \varepsilon)n$. It is easy to see that a_n satisfy (4.1). Then the Subadditive Ergodic Theorem (or [1, Th. 3.5.5], or equations (2.6), (2.8), and (2.9)) implies that for every f-invariant ergodic measure μ, its maximal exponent χ, and μ-a.e. $x \in X$

$$\inf_n \frac{1}{n}\int_X a_n d\mu = \lim_{n\to\infty} \frac{1}{n} a_n(x) = \chi - (\chi_{\max} + \varepsilon) < 0,$$

and thus the assumptions on a_n are satisfied. Taking into account (4.1) we see that (4.2) holds once $a_n(x) \leq a_{n+k}(x) + b_k(f^n x)$ is satisfied. This is easily verified for $b_k(x) = \log \|\mathcal{A}(x, k)^{-1}\|$ since by the cocycle identity (2.1) we have

$$\|\mathcal{A}(x, n)\| \leq \|\mathcal{A}(f^n x, k)^{-1}\| \cdot \|\mathcal{A}(x, n + k)\|.$$

We conclude from the proposition above that for any $\varepsilon > 0$ there exists N_ε such that $a_{N_\varepsilon}(x) < 0$, i.e. $\|\mathcal{A}(x, N_\varepsilon)\| \leq e^{(\chi_{\max}+\varepsilon)N_\varepsilon}$ for all $x \in X$. Hence (1.5) is satisfied for all x in X and n in $\mathbb{N}$, where $c_\varepsilon = \max \|\mathcal{A}(x, k)\|$ with the maximum taken over all $x \in X$ and $1 \leq k < N_\varepsilon$. The other estimate in (1.5) is obtained similarly, for example by applying the same argument to the cocycle generated by A^{-1} over f^{-1}. This completes the proof of Theorem 1.3. □

5. Proofs of Theorem 1.1 and Theorem 1.2

We follow the usual approach of extension along a dense orbit. Our proof is similar to the one in [4] with some modifications for the case of bounded periodic data. The main difference is that Theorem 1.3 enables us to apply the following proposition. This allows us to complete the proof without extra assumptions on the cocycle $\mathcal{A}$.

Proposition 5.1. *Let f be a homeomorphism of a compact metric space X and let A be an α-Hölder $\mathrm{GL}(m, \mathbb{R})$ cocycle over f such that for some $\varepsilon > 0$ and c_ε,*

$$\|\mathcal{A}(x, n)\| \leq c_\varepsilon e^{\varepsilon n} \quad \text{and} \quad \|\mathcal{A}(x, n)^{-1}\| \leq c_\varepsilon e^{\varepsilon n} \qquad \forall\, x \in X, n \in \mathbb{N}. \tag{5.1}$$

Then for any $\lambda > 2\varepsilon/\alpha$ *there exists a constant* c, *which depends only on* A, c_ε, *and* $(\alpha\lambda - 2\varepsilon)$, *such that for any* δ *and any orbit segments* $x, fx, \dots, f^n x$ *and* $y, fy, \dots, f^n y$,

(5.2)

$$\text{if } \operatorname{dist}(f^i x, f^i y) \le \delta e^{-\lambda i},\ i = 0, \dots, n,\ \text{ then } \|\mathcal{A}(x,n)^{-1}\mathcal{A}(y,n) - \mathrm{Id}\| \le c\,\delta^\alpha$$

and

$$\text{if } \operatorname{dist}(f^i x, f^i y) \le \delta e^{-\lambda(n-i)},\ i = 0, \dots, n,\ \text{ then } \|\mathcal{A}(x,n)\mathcal{A}(y,n)^{-1} - \mathrm{Id}\| \le c\,\delta^\alpha.$$

Proof. We will consider the case when $\operatorname{dist}(f^i x, f^i y) \le \delta e^{-\lambda i}$ for $i = 0, \dots, n$. The other case can be proved similarly. Denoting

$$D_i = A(f^i x)^{-1} A(f^i y) - \mathrm{Id}, \qquad i = 0, \dots, n-1,$$

we can write

$$\begin{aligned}
\mathcal{A}(x,n)^{-1}\mathcal{A}(y,n) &= \mathcal{A}(x,n-1)^{-1} A(f^{n-1}x)^{-1} A(f^{n-1}y)\, \mathcal{A}(y,n-1) \\
&= \mathcal{A}(x,n-1)^{-1}(\mathrm{Id} + D_{n-1})\, \mathcal{A}(y,n-1) \\
&= \mathcal{A}(x,n-1)^{-1}\mathcal{A}(y,n-1) + \mathcal{A}(x,n-1)^{-1} D_{n-1}\, \mathcal{A}(y,n-1) \\
&= \cdots = \mathrm{Id} + \sum_{i=0}^{n-1} \mathcal{A}(x,i)^{-1} D_i\, \mathcal{A}(y,i).
\end{aligned}$$

Therefore, using assumption (5.1) we obtain

$$\|\mathcal{A}(x,n)^{-1}\mathcal{A}(y,n) - \mathrm{Id}\| \le \sum_{i=0}^{n-1} \|\mathcal{A}(x,i)^{-1}\| \cdot \|D_i\| \cdot \|\mathcal{A}(y,i)\| \le \sum_{i=0}^{n-1} (c_\varepsilon e^{\varepsilon i})^2\, \|D_i\|.$$

Similarly to (3.4), we can estimate

$$\|D_i\| = \|A(f^i x)^{-1} A(f^i y) - \mathrm{Id}\| \le c_1 \operatorname{dist}(f^i x, f^i y)^\alpha \le c_1 \delta^\alpha e^{-\alpha\lambda i}.$$

Using the two estimates above and the assumption $\lambda > 2\varepsilon/\alpha$ we conclude that

$$\|\mathcal{A}(x,n)^{-1}\mathcal{A}(y,n) - \mathrm{Id}\| \ \le \sum_{i=0}^{n-1} c_1\, c_\varepsilon^2\, \delta^\alpha\, e^{(2\varepsilon - \alpha\lambda) i} \le c\,\delta^\alpha,$$

where the constant c depends only on A, c_ε, and $(\alpha\lambda - 2\varepsilon) > 0$. □

We will now prove Theorems 1.2 and 1.1. Note that the condition on the periodic data of $\mathcal{A}$ in either theorem implies that the assumptions of Theorem 1.3 are satisfied with $\chi_{\min} = \chi_{\max} = 0$ and hence (1.5) gives (5.1) with any $\varepsilon > 0$. Therefore, we can take $\varepsilon < \alpha\lambda/2$, where λ is the exponent in the closing property for f.

In the proof we will abbreviate $d_G = \operatorname{dist}_{\mathrm{GL}(m,\mathbb{R})}$. Since f is transitive, there exists a point $z \in X$ with dense orbit $\mathcal{O} = \{f^k z\}_{k \in \mathbb{Z}}$. We will show that $d_G(\mathcal{A}(z,k), \mathrm{Id})$ is uniformly bounded in $k \in \mathbb{Z}$. Since $\mathcal{O}$ is dense and $\mathcal{A}$

is continuous this implies that $d_G(\mathcal{A}(x,n),\mathrm{Id})$ is uniformly bounded in $x \in X$ and $n \in \mathbb{Z}$. This yields Theorem 1.2.

Consider any two points of $\mathcal{O}$ for which $\mathrm{dist}(f^{k_1}z, f^{k_2}z) < \delta_0$, where δ_0 is as in the closing property. Assume $k_1 < k_2$ and denote $x = f^{k_1}z$ and $n = k_2 - k_1$, so that $\delta = \mathrm{dist}(x, f^n x) < \delta_0$. By the closing property there exist points $p, y \in X$ with $f^n p = p$ such that for $i = 0, \dots, n$

$$\mathrm{dist}(f^i y, f^i p) \le c\,\delta\, e^{-\lambda i} \qquad \text{and} \qquad \mathrm{dist}(f^i y, f^i x) \le c\,\delta\, e^{-\lambda(n-i)}.$$

Now using Proposition 5.1 we obtain

$$\|\mathcal{A}(p,n)^{-1}\mathcal{A}(y,n) - \mathrm{Id}\| \le c_1\delta^\alpha \quad \text{and} \quad \|\mathcal{A}(x,n)\mathcal{A}(y,n)^{-1} - \mathrm{Id}\| \le c_1\delta^\alpha. \tag{5.3}$$

We want to show that these inequalities imply that there exists c_2 such that

$$d_G(\mathcal{A}(p,n),\mathcal{A}(y,n)) \le c_2\delta^\alpha \quad \text{and} \quad d_G(\mathcal{A}(y,n),\mathcal{A}(x,n)) \le c_2\delta^\alpha \tag{5.4}$$

uniformly in x, p, y, n. We use the following simple estimate.

LEMMA 5.2. *If $d_G(A,\mathrm{Id}) \le M$ and either $\|A^{-1}B - \mathrm{Id}\| \le \xi$ or $\|AB^{-1} - \mathrm{Id}\| \le \xi$, with $\xi < 1/2$, then $d_G(A,B) \le 3(M+1)\xi$.*

Proof. We prove the first case; the second case follows similarly. From the assumption we have $\|A\| \le M+1$ and $\|A^{-1}\| \le M+1$. Then

$$\|A - B\| \le \|A\| \cdot \|\mathrm{Id} - A^{-1}B\| \le (M+1)\xi.$$

Denoting $Y = \mathrm{Id} - A^{-1}B$ we obtain $B^{-1}A = (\mathrm{Id} - Y)^{-1} = \mathrm{Id} + Y + Y^2 + \cdots$. Then

$$\|B^{-1}A - \mathrm{Id}\| \le \sum_{k=1}^{\infty} \|Y^k\| \le \sum_{k=1}^{\infty} \xi^k = \frac{\xi}{1-\xi} \le 2\xi$$

and

$$\|A^{-1} - B^{-1}\| \le \|A^{-1}\| \cdot \|\mathrm{Id} - B^{-1}A\| \le (M+1)2\xi$$

so that $d_G(A,B) = \|A - B\| + \|A^{-1} - B^{-1}\| \le 3(M+1)\xi$. □

Since the periodic data is in a compact subset of $\mathrm{GL}(m,\mathbb{R})$ there exists c_0 so that

$$d_G(\mathcal{A}(p,n),\mathrm{Id}) \le c_0 \tag{5.5}$$

for all p and n. Now Lemma 5.2 and the first equation in (5.3) give the first equation in (5.4) which implies, in particular, that $d_G(\mathcal{A}(y,n),\mathrm{Id})$ is also uniformly bounded. Then the lemma and the second equation in (5.3) give the second equation in (5.4). This establishes (5.4), which implies that

$$d_G(\mathcal{A}(p,n),\mathcal{A}(x,n)) \le 2c_2\delta^\alpha \tag{5.6}$$

and hence

$$d_G(\mathcal{A}(x,n),\mathrm{Id}) \le c_0 + 2c_2\delta^\alpha \le c_3 \tag{5.7}$$

for all $x \in \mathcal{O}$ and $n \in \mathbb{Z}$ with $\delta = \mathrm{dist}(x, f^n x) < \delta_0$. The case of negative n follows from the corresponding estimate for positive n.

By density of $\mathcal{O}$ we can take its finite piece $\mathcal{O}_L = \{f^k z\}_{k\in[-L,L]}$ which forms a δ_0 net in X and choose $c_4 = \max_{k\in[-L,L]} d_G(\mathcal{A}(z,k),\mathrm{Id})$. Then for any $N \in \mathbb{Z}$ there exists $k \in [-L, L]$ such that $\mathrm{dist}(f^k z, f^N z) < \delta_0$. Denoting $x = f^k z$ and $n = N - k$ we have $\mathrm{dist}(x, f^n x) < \delta_0$, so that (5.7) applies. The cocycle property (2.1) gives

$$\mathcal{A}(z,N) = \mathcal{A}(x,n)\,\mathcal{A}(z,k).$$

Since the distance from Id to the terms on the right is bounded by c_3 and c_4, we conclude that $d_G(\mathcal{A}(z,N),\mathrm{Id})$ is also uniformly bounded. This completes the proof of Theorem 1.2.

To prove Theorem 1.1 we define a function $C : \mathcal{O} \to \mathrm{GL}(m,\mathbb{R})$ by $C(f^n z) = \mathcal{A}(z,n)$. Note that C satisfies (1.4) for $x \in \mathcal{O}$ and that $d_G(C,\mathrm{Id})$ is uniformly bounded by the previous argument. It remains to show that C is α-Hölder on $\mathcal{O}$ with uniform constant and hence extends uniquely to an α-Hölder function on X, which also satisfies (1.4). Indeed, consider any $x \in \mathcal{O}$ and $n \in \mathbb{Z}$ with $\mathrm{dist}(x, f^n x) = \delta < \delta_0$. Since $\mathcal{A}(p,n) = \mathrm{Id}$ by the assumption, using (5.6) we obtain

$$\|C(f^n x)C(x)^{-1} - \mathrm{Id}\| < d_G(C(f^n x)C(x)^{-1},\mathrm{Id}) = d_G(\mathcal{A}(x,n),\mathrm{Id}) \le 2c_2\delta^\alpha.$$

Now, since $d_G(C,\mathrm{Id})$ is uniformly bounded, Lemma 5.2 gives the desired Hölder continuity of $C : \mathcal{O} \to \mathrm{GL}(m,\mathbb{R})$. This completes the proof of Theorem 1.1.

Note that if the function $A : X \to \mathrm{GL}(m,\mathbb{R})$ takes values in a subgroup $G \subset \mathrm{GL}(m,\mathbb{R})$, then so does the function C on $\mathcal{O}$ and, if G is closed, then so does the extension $C : X \to \mathrm{GL}(m,\mathbb{R})$. □

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