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Distribution of periodic torus orbits and Duke's theorem for cubic fields

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Abstract

We study periodic torus orbits on spaces of lattices. Using the action of the group of adelic points of the underlying tori, we define a natural equivalence relation on these orbits, and show that the equivalence classes become uniformly distributed. This is a cubic analogue of Duke's theorem about the distribution of closed geodesics on the modular surface: suitably interpreted, the ideal classes of a cubic totally real field are equidistributed in the modular 5-fold $\mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3$. In particular, this proves (a stronger form of) the folklore conjecture that the collection of maximal compact flats in $\mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3$ of volume $\leq V$ becomes equidistributed as $V \to \infty$.

The proof combines subconvexity estimates, measure classification, and local harmonic analysis.

Contents

1. **Introduction**

1.1. *Historical perspective.* In the preface to his book "Ergodic properties of algebraic fields," Linnik [32] writes

> *...In the present book other applications of the ergodic concepts are presented. Constructing "flows" of integral points on certain algebraic manifolds given by systems of integral polynomials, we are able to prove individual ergodic theorems and mixing theorems in certain cases. These theorems permit asymptotic calculations of the distribution of integral points on such manifolds and we arrive at results inaccessible up to now by the usual methods of analytic number theory. Typical in this respect is this theorem concerning the asymptotic distribution and ergodic behavior of the set of integral points on the sphere*

$$x^2 + y^2 + z^2 = m \tag{$*$}$$

> *for increasing m.*

This presents what Linnik called "the ergodic method"; it enabled Linnik to show that solutions to $(*)$ become equidistributed upon projection to the unit sphere — at least, for the m satisfying an explicit congruence condition. Subsequently, using that method, Skubenko solved the related problem for the solutions of the equation

$$y^2 - xz = m \tag{$**$}$$

also under similar congruence conditions on m.

Both of these problems are related to the distribution of ideal classes of orders in quadratic fields: in the case of points on the sphere $(*)$, one deals with imaginary quadratic fields, while $(**)$ corresponds either to real or imaginary quadratic fields depending on the sign of m. The latter problem for $m > 0$ is also equivalent to the problem of the distribution of closed geodesics on the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$.

Since the time of Linnik's work, the tools of analytic number theory have developed tremendously. In particular W. Duke [12], using a breakthrough of H. Iwaniec, proved that the integer solutions of $(*)$ as well as $(**)$ become equidistributed as $m \to \infty$.

In [32, Chaps. VI–VII], Linnik considers in detail the corresponding questions for number fields of higher degree, particularly cubic fields. However, he was able to prove comparatively little compared to the quadratic case. In modern terms, he established, by a remarkable elementary calculation, a special case of the equidistribution of Hecke points.

In this paper, we revisit Linnik's problems for cubic (and higher degree) fields, settling them for totally real cubic fields (among other cases).

We give precise statements later in the introduction; for now, we note that just as $(**)$ relates to the ideal class group of quadratic fields and to closed geodesics on the modular surface, the higher rank analogues will pertain to ideal class groups of higher degree fields and to periodic orbits of maximal tori in the space of lattices $X_n = \mathrm{PGL}_n(\mathbb{Z})\backslash\mathrm{PGL}_n(\mathbb{R})$. We will, in fact, explain our main result in this language in the introduction; there is also an interpretation analogous to $(**)$, which we postpone to Corollary 3.3 in Section 3.

Interestingly, we do not know how to prove our main equidistribution result using purely analytic techniques, nor using purely ergodic theoretic techniques, though each of these methods does give some partial information in this direction. Our proof works by combining these two very different techniques; to handle the case of nonmaximal orders we also have to prove new estimates involving local Fourier analysis.[1]

This paper is part of a series of papers we have been writing on the distribution properties of compact torus orbits on homogeneous spaces. In [18], we present a general setup for the study of the periodic orbits and prove results regarding the distribution of individual orbits as well as fairly arbitrary collections of periodic orbits. In [17], we give a modern reincarnation of Linnik's original argument, giving in particular a purely dynamical proof of equidistribution in the problem $(**)_{m>0}$; without an auxiliary congruence condition. We still do not know how to give a purely dynamic proof of Duke's theorems regarding the equidistribution of the solutions to $(*)$ or to $(**)_{m<0}$. Each of these papers is self-contained and can be read independently; related discussions can also be found in [16] and [36].

1.2. *Geometric perspective.* For clarity, we shall continue to focus on the "$\mathbb{R}$-split case" of our main questions (i.e. problem $(**)_{m>0}$, totally real fields, orbits of $\mathbb{R}$-split tori, etc.). We introduce our main result in geometric terms. Later, in Section 3, we discuss interpreting it in "arithmetic" terms (akin to the interpretations $(*)$ and $(**)$).

Let $M = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic Riemann surface, and

$$X := S^1M = \Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$$

the unit tangent bundle of M. Bowen and Margulis, independently [4], [34], proved that the set of geodesics of length $\leq L$, considered as closed orbits of the geodesic flow on X, are equidistributed with respect to Liouville measure as $L \to +\infty$.

[1] It is also conceivable that an ergodic approach to these estimates may exist.

On the other hand, for a Riemann surface, if the latter is "arithmetic" in a suitable sense (see §4.5 below), then the results of Bowen/Margulis are valid in a much stronger form. The most basic instance of an arithmetic surface is the modular surface,

$$M = M_2 = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}.$$

The equidistribution theorem of Duke already mentioned implies[2] that the collection of geodesics *of fixed length* ℓ becomes equidistributed in $X = X_2$ as $\ell \to \infty$. Note that the lengths of closed geodesics can have high multiplicity; indeed, the lengths are of the form $\log(d + \sqrt{d^2 - 1})$, for $d \in \mathbb{N}_{>0}$, and the set of geodesics of this length is parametrized by the class group of the quadratic order of discriminant d.

Our main result establishes the analogues of both of these equidistribution theorems to the setting of the rank two Riemannian manifold

$$M_3 = \mathrm{PGL}_3(\mathbb{Z})\backslash\mathrm{PGL}_3(\mathbb{R})/\mathrm{PO}_3.$$

The role of "closed geodesics" is replaced by "maximal compact flats", and the role of "quadratic order" is replaced by "cubic order".

At least with our present understanding, the rank two case seems to be much more difficult than the rank one case. This difficulty manifests itself from all the perspectives; crudely, the smaller the *acting* group, relative to the *ambient* group, the more difficult the question.

1.3. *Statement of results.* Now let us give a precise statement of our main result, at least in the "$\mathbb{R}$-split, cubic field" case. (Our most general theorem is stated in Theorem 4.9; what follows is a specialization of this.)

Let H be the diagonal subgroup of $\mathrm{PGL}_3(\mathbb{R})$. In [18], we attached to each closed orbit xH on

$$X_3 = \mathrm{PGL}_3(\mathbb{Z})\backslash\mathrm{PGL}_3(\mathbb{R})$$

a discriminant $\mathrm{disc}(xH)$ as a way to measure the "arithmetic complexity" of that orbit; let us briefly recall how it is obtained. Writing xH into the form $\Gamma\backslash\Gamma gH$, we set $\mathbf{T} = \overline{\Gamma \cap gHg^{-1}}^{\mathrm{Zar}}$; $\mathbf{T}$ is a maximal (anisotropic) $\mathbb{Q}$-torus. The discriminant is then closely related to the "denominator" of the $\mathbb{Q}$-point $\mathbf{T}$ inside the variety of maximal tori of PGL_3. In Section 4, we will review this construction in the adelic setting. We prove:

1.4. Theorem. *The periodic orbits of H on X_3 are grouped into equivalence classes, equivalent orbits having the same volume and discriminant. For*

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[2]Strictly speaking, Duke's theorem establishes equidistribution on the modular surface; equidistribution at the level of the unit tangent bundle was established in the unpublished Ph.D. thesis of R. Chelluri, [7].

each periodic orbit xH, *let* Y_{xH} *be the union of all compact orbits equivalent to* xH.

If $\{x_iH\}_i$ *is a sequence of compact orbits, with* $\mathrm{disc}(x_iH) \to +\infty$, *then*:

(1) $\mathrm{vol}(Y_{x_iH}) = \mathrm{disc}(x_iH)^{1/2+o(1)}$,

(2) *the* Y_{x_iH} *become uniformly distributed in* X_3.

In particular, for $V > 0$, *let* $Y(V)$ *denotes the collection of all* H*-orbits with volume* V; *as* $V_i \to \infty$ *through any sequence for which* $Y(V_i) \neq \emptyset$, *then the* $Y(V_i)$ *become uniformly distributed in* X_3.

Noting that the projection of $Y(V)$ to M_3 is the collection of all maximal compact flats on M_3 of volume V, Theorem 1.4 implies (a stronger form of) the rank two analogue of the Bowen/Margulis theorem, indicated above.

1.5. *About the proof; adelization.* Viewed from the classical point of view, the grouping of periodic torus orbits into packets is rather mysterious and can be quite tricky to define in the nonmaximal case. It turns out that the adeles give a powerful and concise language to describe these equidistribution results, and we have written the bulk of this paper consistently in the adelic language.

In particular, as we shall see in Section 5 the full equivalence class Y_{x_iH} of a periodic H-orbit (H being a maximal split torus) is essentially the projection to the infinite place of a single periodic orbit of an *adelic* torus. The precise connection between packets and adelic tori is contained in Theorem 5.2; Theorem 1.4 is then immediate from the adelic results Theorems 4.8 and 4.9.

We will, indeed, go to some length to set equidistribution questions in a genuinely adelic framework, which has the pleasing side effect that we are able to address simultaneously many different equidistribution questions (cf. §4).

To aid the reader, we give an outline of the main ideas that enter into its proof in purely classical terms in Section 2. For the moment, we only observe an important contrast between X_3 and X_2: while for X_2, the analogue of our main Theorem is "purely" a result about L-functions, for X_3 this is not so. To fill this gap, we will need to combine results from measure rigidity, L-functions, and harmonic analysis on Lie groups.

1.6. *Scope of the method.* We shall discuss certain natural generalizations of Theorem 1.4 and interesting questions associated to them.

1.6.1. *S-arithmetic variants.* Theorem 1.4 is derived from the underlying adelic result — Theorem 4.9 — and there is therefore no difficulty in replacing $\mathrm{PGL}_3(\mathbb{Z})$ by a congruence subgroup, or $\mathbb{Q}$ by a number field, or passing to an S-arithmetic context.

However, although this is not apparent from the statement of Theorem 1.4, the general statement Theorem 4.9 is not as satisfactory as the corresponding statement for PGL_2 (given in Theorem 4.6). Indeed, our general PGL_3-theorem imposes local conditions, akin to a Linnik-type condition, which happen to be automatically satisfied in the setting of Theorem 1.4.

To dispense with these local conditions seems to be a very interesting and fundamental question.

1.6.2. *Sparse equidistribution.* Assuming a suitable subconvex estimate on L-functions[3] one can obtain the following "sparse equidistribution" result by our methods:

Notation as in Theorem 1.4, there is $\alpha < 1/2$ so that, if $\mathrm{vol}(x_iH) > \mathrm{disc}(x_iH)^\alpha$ then the x_iH become uniformly distributed on X_3.

Presumably assuming the full force of Generalized Riemann Hypothesis (GRH) would yield $\alpha = 1/4$, although to prove this requires more careful local analysis than we have done.

Conjecturally, however, a much stronger statement should hold: we conjecture ([18]) that this equidistribution statement for single H-orbits remains true for *any* $\alpha > 0$. We refer to [18] for discussion of this conjecture, some partial results towards it, and counterexamples to more optimistic conjectures.

1.6.3. *Spaces of higher dimensional lattices.* Much of our analysis carries through from X_3 to

$$X_n = \mathrm{PGL}_n(\mathbb{Z})\backslash\mathrm{PGL}_n(\mathbb{R}).$$

There are two obstacles, however, to obtaining a complete generalization of Theorem 1.4:

(1) the lack of available subconvex bounds,
(2) the lack of suitable technology to rule out "intermediate limit measures."

At the moment we have little to offer concerning the second point. In the case when n is prime, the issue of intermediate measures does not occur; if we suppose Hypothesis A.1 (i.e., a subconvex bound for Dedekind ζ-functions of degree n number fields), then *the analog of Theorem 1.4 holds*; i.e., packets of periodic torus orbits become equidistributed on X_n.

1.6.4. *An almost-subconvexity bound for class group L-functions of cubic fields.* Let K be a real cubic field, and let ψ be a nontrivial character of the class group of K.

We have an associated L-function

$$L(K,\psi,s) = \sum_{\mathfrak{a}} \psi(\mathfrak{a})\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})^{-s},$$

[3]The specific case of subconvexity needed is subconvexity in the level aspect, on GL(3).

the sum being extended over all integral ideals $\mathfrak{a}$, and $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})$ denoting the norm of $\mathfrak{a}$. This L-function has conductor D_K, the discriminant of K.

One corollary to our main result is that, for any fixed $\delta > 0$,

$$\frac{\left|\sum_{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})\leq\delta\sqrt{D_K}} \psi(\mathfrak{a})\right|}{\left|\sum_{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})\leq\delta\sqrt{D_K}} 1\right|} = o(1). \tag{1}$$

To put this in perspective, a subconvex bound for the degree 3 L-function $L(K, \psi, s)$ would guarantee that the same is true if we replace $\delta\sqrt{D_K}$ by $D_K^{0.499}$. The result (1) could therefore be considered as a "nonquantitative" form of subconvexity for this degree 3 L-function.

Since this paper was submitted, K. Soundararajan [41] proved a very general weak subconvex bound, valid for a wide class of L-functions. His result does not imply (1) with currently known bounds: what is needed is any improvement of Stark's result [42]:

$$\mathrm{res}_{s=1}\zeta_K(s) \gg \log(\mathrm{disc}(K))^{-1};$$

e.g. any larger exponent would suffice.

1.6.5. *The cocompact case.* If one considers the quotient of $\mathrm{PGL}_n(\mathbb{R})$ by a lattice associated to a $\mathbb{R}$-split division algebra, then one obtains a *compact* quotient. One certainly believes the analogue of Theorem 1.4 to be valid, but in this case the methods of the present paper which use Eisenstein series in an essential way do not apply.

This is an instance in which the cocompact case seems harder that the noncompact case. We refer to [18] where we obtain (weaker) results in the cocompact case by different methods.

1.7. *Connection to existing work.* In the rank one case of PGL_2, the analogs of the questions we consider have been intensively studied from many perspectives, both from the perspective of the work of Linnik and that of Iwaniec and Duke.

Concerning PGL_n for $n \geq 3$, the direct ancestor of our work is that of Linnik, who devotes several chapters of his book [32] to the question of distribution of the packets Y_{x_iH}. The paper of Oh and Benoist [2] considers problems similar to those we consider. Both [32] and [2] give results about the problem in the special case when the $\mathbb{Q}$-torus attached to x_iH remains constant.

1.8. *Organization of the paper.* In Section 2, we present an outline of the proof of Theorem 1.4 in entirely classical language.

In Section 3, we discuss some of the arithmetic manifestations of our result.

In Section 4, we present a systematic framework for thinking about adelic equidistribution problems. We then explain, in this context, our main results:

Theorems 4.8 and 4.9. These imply the first and second assertions of Theorem 1.4.

In Section 5, we explain the grouping of periodic orbits into packets. This uses the setup of Section 4.

In Section 6, we specialize to the case of the group $\mathbf{G} = \mathrm{PGL}_n$ and explain the parametrization of packets of periodic orbits.

In Section 7, we give a brief recollection of properties of the building of PGL_n, over a local field.

In Section 9, we explain the local harmonic analysis that will be needed.

In Section 8, we set up notational conventions about number fields, ideles and adeles (especially: normalizations of measures).

In Section 10, we set up general notation about Eisenstein series (these are the generalization of the functions "E_f" discussed in §2).

In Section 11, we prove, in adelic language, the estimates for the integral of an Eisenstein series over a torus orbit (this is the adelic version of (5) from §2).

In Section 12, we translate the results of Section 11 from the adelic to the S-arithmetic context, obtaining Proposition 12.5 (this is the S-arithmetic form of (5) from §2).

In Section 13, we complete the proofs of Proposition 4.8 and Theorem 4.9, and therefore also of Theorem 1.4.

In Section A, we briefly recall basic facts about the subconvexity problem for L-functions.

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2. An overview of the proof for $\mathrm{PGL}_3(\mathbb{Z})\backslash\mathrm{PGL}_3(\mathbb{R})$

The majority of this paper is presented in an "adelized" framework, and the results presented are substantially more general than Theorem 1.4. Nonetheless, in this section, we would like to explain the ideas that go into Theorem 1.4 in as classical a setting as possible.

2.1. *Parametrizing compact orbits of maximal tori.* Let us briefly recall how, to a number field K with $[K:\mathbb{Q}] = n$, we may associate a compact orbit of a maximal torus inside $\mathrm{PGL}_n(\mathbb{R})$, on the space $\mathrm{PGL}_n(\mathbb{Z})\backslash\mathrm{PGL}_n(\mathbb{R})$. If the field K is totally real, the torus will be $\mathbb{R}$-split, then we will be in the situation described in Theorem 1.4, and the construction will specialize to that discussed in our prior paper [18, Cor. 4.4].

Fix a subalgebra $A_{r,s} \subset M_n(\mathbb{R})$ isomorphic to $\mathbb{R}^r \oplus \mathbb{C}^s$. Then $H_{r,s} := A_{r,s}^\times/\mathbb{R}^\times$ is a maximal torus in $\mathrm{PGL}_n(\mathbb{R})$; as (r,s) vary through pairs satisfying $r + 2s = n$, they exhaust maximal tori, up to conjugacy. In the case $(r,s) = (n,0)$, $H_{n,0}$ is conjugate to the diagonal subgroup H.

Let $[K:\mathbb{Q}] = n$. Let r and s be the number of real and complex embeddings of K, respectively. We shall say K has *signature* (r,s).

Suppose given data (K, L, θ), where K has signature (r,s), $\theta : K \otimes \mathbb{R} \to A_{r,s}$ is an algebra isomorphism, and L is a "K-equivalence class of lattices" in K, i.e. a free $\mathbb{Z}$-submodule of rank $[K:\mathbb{Q}]$, up to multiplication by $K^\times$.

We associate to this data the $H_{r,s}$-orbit $\overline{\iota(L)}H_{r,s}$; here ι is any map $K \otimes \mathbb{R} \to \mathbb{R}^n$ satisfying $(ab)^\iota = a^\iota.\theta(b)$. The resulting orbit is independent of choice of ι. The stabilizer in $H_{r,s}$ of any point in the orbit is $\theta(\mathscr{O}_L^\times)$ where $\mathscr{O}_L = \{\lambda \in K : \lambda L \subset L\}$, and the volume of the orbit is $\mathrm{reg}(\mathscr{O}_L)$, the regulator of $\mathscr{O}_L$.

As explained in [18, Cor. 4.4], in the totally real case $(r,s) = (n,0)$, *all* compact H-orbits in X_n are obtained in that way. This does not hold for the other signature (think of an imaginary quadratic field); to recover a form of this property, one has to consider more general S-arithmetic quotients of PGL_n.

For clarity, we specialize in the remainder of this section to the case $(r,s) = (n,0)$ *and* $H = H_{n,0}$ *a maximal* $\mathbb{R}$*-split torus. Interpreted in an appropriate* S*-arithmetic context* (*cf.* §12), *most of the discussion carries over to general signature so long as the field* K *admits a fixed split place, and this is how our main result is proven in general.*

2.2. *Packets for* PGL_n. Consider an order $\mathscr{O}$ inside a totally real field K; assume we have fixed an identification θ as above.

Let us denote by $\widetilde{Y}_{\mathscr{O}}$ the set of H-orbits associated to data (K, L, θ) such that $\mathscr{O}_L = \mathscr{O}$. Varying K, $\mathscr{O}$ and θ, the collections $\widetilde{Y}_{\mathscr{O}}$ define a partition of the set of compact H-orbits. As it turns out, these form a slightly *coarser* partition of the compact H-orbits than the one alluded to in Theorem 1.4; i.e., there is a *further* natural equivalence relation on the set of lattices with $\mathscr{O}_L = \mathscr{O}$, which is not trivial in general[4]. This equivalence is discussed and explicated in more detail in Section 5.5, and we use the term *packets* to refer to its equivalence classes, or to the associated collections of H-orbits.

Assuming this for now, let $Y_{\mathscr{O}}$ be any packet of compact H-orbits contained in $\widetilde{Y}_{\mathscr{O}}$, and let $\mu_{\mathscr{O}}$ the corresponding measure.

It is not difficult to verify that $\mathrm{reg}(\mathscr{O})$ goes to infinity with $\mathrm{disc}(\mathscr{O})$. In the totally real case, the equidistribution statement of Theorem 1.4 is thereby equivalent to the statement, in the $n = 3$ case:

$$\text{As } \mathrm{disc}(\mathscr{O}) \to +\infty,\ \mu_{\mathscr{O}} \text{ approaches Haar measure on } X_3. \tag{2}$$

2.3. *Overview of proof for X_2 via analytic number theory.* To put things in perspective, it will be useful to recall the principle of Duke's proof for X_2.

Duke verifies *Weyl's equidistribution criterion*, i.e. for a suitable basis $\{\varphi\}$ for the functions in $L^2(\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R}))$with integral zero, he shows:

$$\mu_{\mathscr{O}}(\varphi) := \int_{X_2} \varphi(g)d\mu_{\mathscr{O}}(g) \to 0,\ \mathrm{disc}(\mathscr{O}) \to +\infty. \tag{3}$$

The basis is chosen to consist of automorphic forms – either cusp forms or Eisenstein series. Duke proved (3) by interpreting the period integral $\mu_{\mathscr{O}}(\varphi)$ in terms of the $\mathrm{disc}(\mathscr{O})$-th Fourier coefficient of an half-integral weight form $\tilde{\varphi}$, proving nontrivial bounds for such coefficients by generalizing a method of Iwaniec [26]. In its most general form, the formula relating the period integral to Fourier coefficients is due to Waldspurger [47].

Soon thereafter, another proof emerged that turned out to work in greater generality. Namely, by a result of Waldspurger [48], one has the relation

$$|\mu_{\mathscr{O}}(\varphi)|^2 = I_{\mathscr{O}}(\varphi)\frac{L(\pi, 1/2)L(\pi \otimes \chi_K, 1/2)}{\mathrm{disc}(\mathscr{O})^{1/2}},$$

where

(1) π is the automorphic representation to which φ belongs;
(2) $L(\pi, s)$ and $L(\pi \otimes \chi_K, s)$ are, respectively, the Hecke L-function of π and the Hecke L-function of the twist of π by the quadratic character associated with K;

[4]However, when $\mathscr{O}$ is a *Gorenstein* ring — e.g. when $n = 2$, or $\mathscr{O}$ is the maximal order, or $\mathscr{O}$ is monogenic — this further equivalence is trivial; i.e. $\widetilde{Y}_{\mathscr{O}}$ is a single packet in the sense of Theorem 1.4.

(3) $I_{\mathscr{O}}(\varphi)$ is a product of local integrals supported at the place at ∞ and at the primes dividing $\operatorname{disc}(\mathscr{O})$.

Then (3) is a consequence of the estimates

$$L(\pi \otimes \chi_K, 1/2) \ll \operatorname{disc}(\mathscr{O}_K)^{1/2-\eta}, \; I_{\mathscr{O}}(\varphi) \ll \Big(\frac{\operatorname{disc}(\mathscr{O})}{\operatorname{disc}(\mathscr{O}_K)}\Big)^{1/2-\eta} \tag{4}$$

for some absolute $\eta > 0$; here $\mathscr{O}_K$ denote the maximal order of K.

The first bound in (4) is called a *subconvex* bound and is due to Duke, Friedlander and Iwaniec [13]; it is a special instance of the so-called *subconvexity problem* for automorphic L-functions (see [27] for a discussion of that problem).

The second bound in greater generality is due to Clozel and Ullmo [8] and we will call it a *local* subconvex bound. It is somewhat easier than the first, but it addresses the issue of $\mathscr{O}$ nonmaximal.

It is tempting to try to generalize this approach to the space X_n of lattices of higher rank. However this does not seem within reach of the current technology. In particular, it is not expected that the corresponding "Weyl sums" are related to (GL_n) L-functions. Even were this the case, we do not know how to solve the corresponding subconvexity problems.[5] There is however an exception to this which will turn out to be crucial for our coming argument.

2.4. *Overview of the proof for* X_3. In summary, our strategy is to check Weyl's equidistribution criterion against test functions taken from a *tiny* portion of $L^2(X_3)$, by using/proving global and local subconvex bounds, and to *bootstrap* this information to a full equidistribution statement using results on measures invariant under higher rank torus actions.

The input we use from ergodic theory is the following measure classification result regarding invariant measures in rank ≥ 2 — as well as a p-adic variant of it — by the first two authors and A. Katok [15]:

2.5. THEOREM. *Let $n \geq 3$ and let μ be an ergodic H-invariant probability measure on X_n, where H denotes a maximal $\mathbb{R}$-split torus in* $\mathrm{PGL}_n(\mathbb{R})$. *Assume that for at least one $a \in H$ the ergodic theoretic entropy $h_\mu(a)$ is positive. Then μ is homogeneous: there exists a reductive group $H \subsetneq L \subset$* $\mathrm{PGL}_n(\mathbb{R})$ *such that μ is the L invariant probability measure on a single periodic L-orbit. In particular, if n is prime, μ is Haar measure on X_n.*

[5] Another possibility, more in line with Duke's original proof would be to use results of Gan, Gross and Savin [23] which relate the Weyl's sums to Fourier coefficient to automorphic forms on G_2; unfortunately our state of knowledge concerning bounds for these Fourier coefficients is rather limited.

The main content of this paper will be to show that assumptions of this theorem are satisfied when $n = 3$.

It is conjectured that the following substantially stronger statement holds:

2.6. CONJECTURE (Furstenberg, Katok-Spatzier [28], Margulis [33]). *Let $n \geq 3$ and let μ be an ergodic H-invariant probability measure on X_n, H as above. Then μ is homogeneous.*

Note that in Conjecture 2.6, the measure μ can certainly be the natural measure on a periodic H-orbit, a possibility that is ruled out in Theorem 2.5. We refer the interested reader to [16] or to the original paper [15] for a historical background and for an exposition of some of the ideas that enter into the proof.

Let μ_∞ denote a weak* limit[6] of the $\{\mu_{\mathscr{O}}\}_{\mathscr{O}}$. There are two main issues to verify:

(A) The measure μ_∞ is a probability measure (i.e. the sequence of measures $\{\mu_{\mathscr{O}}\}_{\mathscr{O}}$ is *tight*).

(B) Almost every ergodic component of μ_∞ has positive entropy with respect to some $a \in H$.

Even assuming the stronger conjectured measure classification given by Conjecture 2.6, one needs to overcome pretty much the same obstructions; in that case the following weaker form of (B) would suffice:

(B′) $\mu_\infty(xH) = 0$ for any periodic H-orbit xH.

In the context of this paper (B′) does not seem to be much easier to verify than the weaker statement in (B).

2.7. *Weyl's equidistribution criterion.* Our method for verifying both (A) and (B) is by checking *Weyl's equidistribution criterion* for a special class of functions from which follows *a priori bounds* for the μ_∞-volumes of certain sets. We shall be able to obtain such bounds on the mass of neighborhoods of the cusp in X_3 (which addresses (A)) or of ε-balls around *any* $x \in X_3$ (which addresses (B)).

2.7.1. *The Siegel-Eisenstein series.* Let us recall that we can identify X_3 with the space of lattices in $\mathbb{R}^3$ of co-volume 1. We shall make use of this identificaition throughout what follows. Let f be any continuous, compactly supported function on $\mathbb{R}^3$ and let E_f be the Siegel-Eisenstein series

$$E_f(L) = \sum_{\lambda \in L - \{0\}} f(\lambda);$$

[6] recall that a sequence of probability measures $\{\mu_i\}_i$ weak* converge to some measure μ_∞ if, for any compactly supported function f, $\mu_i(f) \to \mu_\infty(f)$ as $i \to +\infty$.

we shall prove

$$\mu_\infty(E_f) := \lim_{\mathrm{disc}(\mathscr{O})\to+\infty} \mu_{\mathscr{O}}(E_f) = \int_{\mathbb{R}^3} f(x)dx. \tag{5}$$

Observe that, by *Siegel's formula*, (see eg. [49]), $\int_{\mathbb{R}^3} f(x)dx = \mu_{\mathrm{Haar}}(E_f)$; in particular (5) is consistent with (2).

By taking suitable choices of f, (5) yields the necessary *a priori* bounds. Indeed, take $v \in \mathbb{R}^3$. Take f to be a smooth nonnegative function supported in the 2ε-ball $B(v, 2\varepsilon)$, which takes value 1 on $B(v,\varepsilon)$. When $v = 0$, $E(f)$ dominates a neighborhood of the cusp (in fact approaches infinity near the cusp); when $v \neq 0$, $E(f)$ dominates the characteristic function of an ε-neighborhood of any lattice class x that contains v. In the latter case, we deduce that for, ε small enough

$$\mu_\infty(B(x,\varepsilon)) \ll \varepsilon^3.$$

This improves over the trivial bound $\mu_\infty(B(x,\varepsilon)) \ll \varepsilon^2$, arising from the fact that μ_∞ is invariant by the two-parameter group H.

This improvement[7] from 2 to 3 already shows that μ_∞ cannot be supported along a compact H-orbit. More importantly, this implies that for a generic $a \in H$, almost every ergodic component of μ_∞ has positive entropy with respect to the action of a from which we deduce the full equidistribution by Theorem 2.5. To finish this section, we remark that the principle of testing Weyl's criterion against Einsenstein series appears in other contexts, for example, in [20] and [45].

2.7.2. *Connection to L-functions.* The key point, for establishing (5), is that the $\mu_{\mathscr{O}}(E_f)$ is indeed related to L-functions. One has the following formula, which goes back to Hecke [25][8]:

$$\mu_{\mathscr{O}}(E_f) = \int_{\Re s\gg 1} \widehat{f}(s)\frac{I_{\mathscr{O}}(f,s)\zeta_K(s)}{\mathrm{disc}(\mathscr{O})^{1-s}}ds, \tag{6}$$

where $\widehat{f}(s)$ is a certain Mellin-like transform of f, $\zeta_K(s)$ is the Dedekind zeta function of K and $I_{\mathscr{O}}(f,s)$ is a product of local integrals supported at the place ∞ and at the primes dividing $\mathrm{disc}(\mathscr{O})$.

Shifting the contour to $\Re s = 1/2$, we pick up a residue at $s = 1$ which equals $\mu_{\mathrm{Haar}}(E_f)$; the fact that the remaining integral goes to 0 as $\mathrm{disc}(\mathscr{O}) \to$

[7] In passing, we note that the test functions "E_f" considered are *not* invariant under the maximal compact $K = \mathrm{PO}_3(\mathbb{R})$, in general (unlike many problems using the classical theory of modular forms). This feature is essential to improve the trivial bound $\mu_\infty(B(x,\varepsilon)) = O(\varepsilon^2)$ to $O(\varepsilon^3)$.

[8] Hecke proved that way the analytic continuation and the functional equation of Grössencharacter L-functions.

$+\infty$ follows from the global and local subconvex bounds

$$\zeta_K(s) \ll_s \operatorname{disc}(\mathscr{O}_K)^{1/2-\eta}, \tag{7}$$

$$I_{\mathscr{O}}(f,s) \ll_s \Big(\frac{\operatorname{disc}(\mathscr{O})}{\operatorname{disc}(\mathscr{O}_K)}\Big)^{1/2-\eta} \tag{8}$$

for some absolute $\eta > 0$ and $\Re s = 1/2$.

For $n = 3$ the bound (7) follows from the work of Burgess [6] if K is abelian and (essentially) from the deep work of Duke, Friedlander and Iwaniec if K is cubic not abelian ([3], [14], [37]).

The local bound (8) is new and occupies a good part of the present paper; moreover, it is valid for any n. Let us describe how it is proved.

2.7.3. *Local estimates: harmonic analysis on p-adic homogeneous spaces.* Our approach to bounding the local integrals $I_{\mathscr{O}}$ is based on[9], first of all, relating $I_{\mathscr{O}}$ to integrals of matrix coefficients (inspired by ideas of Waldspurger and Ichino-Ikeda) and then bounding the integrals of matrix coefficients using the local building (inspired by ideas of Clozel and Ullmo).

Let us make a remark in representation theory to explain this. Let V be an irreducible, unitary representation of a group G, and let $H \subset G$ be a subgroup. Suppose that there exists a unique scaling class of invariant functionals $L : V \to \mathbb{C}$ invariant by H.

It is sometimes possible to understand something about functionals L simply by studying matrix coefficients. Indeed, if convergent, $\int_{h\in H}\langle hv_1, v_2\rangle$ defines a functional (on v_1) invariant by H and a conjugate-linear functional (on v_2) invariant by H. We conclude by the uniqueness assumption that:

$$\int_{h\in H}\langle hv_1, v_2\rangle dh = c_V L(v_1)\overline{L(v_2)} \tag{9}$$

for some constant c_V. Thereby, L can be studied through matrix coefficients.

It turns out that computing $I_{\mathscr{O}}$ amounts to computing with such functionals L, when $G = \mathrm{GL}(n,k)$ and H is a maximal torus inside G; the representations V we are concerned with are those that occur inside $L^2(k^n)$. Thereby, (9) allows us to reduce understanding $I_{\mathscr{O}}$ to computations with matrix coefficients.

3. Number theoretic interpretation

We discuss how our main result can be interpreted in terms of integral points on varieties, generalizing the equations $(*)$ and $(**)$ from the introduction.

[9] In fact, the estimates needed can be proved in a direct and elementary way; this was the original approach of the paper. However, the approach carried out, although requiring more input, has the advantage of being very general.

3.1. *Integral points on homogeneous varieties.* In this section, we interpret our results in terms of distribution of algebraically defined sets of integral matrices, which was one of Linnik's original motivations. This is part of a more general problem of studying the structure of the set of integral points $\mathbf{V}(\mathbb{Z})$ on an algebraic variety $\mathbf{V}$.

A particularly structured situation occurs when V is homogeneous; i.e., $\mathbf{V}(\mathbb{C})$ possesses a transitive action of a linear algebraic group $\mathbf{G}$. In that case, it is expected that there are many points which are rather well distributed. Similar results are found in [19], [21], [22], [32], and [40].

3.2. *Solving polynomial equations in matrices.* To motivate what follows, note that for $(x, y_1, y_2, z) \in \mathbb{Z}^4$ and m not a perfect square,

$$\begin{pmatrix} y_1 & z \\ x & y_2 \end{pmatrix}^2 = m\,\mathrm{Id} \Longleftrightarrow y_1 = -y_2, y_1^2 + xz = m.$$

Thereby, $(**)$ is a statement concerning 2×2 integral matrices satisfying a prescribed quadratic equation.

Given P a monic integral irreducible polynomial of degree n with integral coefficients, we let

$$Z_P = \{M \in M_n,\ P(\lambda) = \det(M - \lambda I)\}.$$

Thus integral points $Z_P(\mathbb{Z})$ can be identified simply with integral solutions to $P = 0$ in $n \times n$ matrices.

The *signature* (r, s) of such a polynomial will be the number of real roots, resp. conjugate pairs of complex roots; thus $r + 2s = n$. Let $Z_{r,s}$ be the space of all splittings of $\mathbb{R}^n$ into r real lines and s complex planes.

If P has signature (r, s), the space $Z_P(\mathbb{R})$ is identified with $Z_{r,s}$ by, first, fixing an ordering of the real and complex roots of P; and then associating to a matrix $M \in Z_P(\mathbb{R})$ its eigenspaces.[10]

The spaces $Z_{r,s}$ carry a $\mathrm{PGL}_n(\mathbb{R})$-invariant measure, unique up to scaling (indeed, $Z_{r,s} \cong \mathrm{PGL}_n(\mathbb{R})/H_{r,s}$), which we denote by $\mathrm{vol}(\cdot)$.

3.3. Corollary. *Let $\{P_i\}_i$ be a sequence of cubic, monic, integral, irreducible polynomials of signature (r, s) and of discriminant satisfying* $\mathrm{disc}(P_i) \to +\infty$.

(1) *If $(r, s) = (3, 0)$, then $Z_{P_i}(\mathbb{Z})$ becomes uniformly distributed on $Z_{3,0}$.*
(2) *If $(r, s) = (1, 1)$, and there exists a fixed prime number p so that P_i has three p-adic roots, then $Z_{P_i}(\mathbb{Z})$ becomes uniformly distributed on $Z_{1,1}$.*

[10] For a complex eigenvalue, we take the eigenspaces corresponding to that eigenvalue and its complex conjugate, and intersect their sum with $\mathbb{R}^n$.

Here, we say, a sequence of discrete sets $\mathcal{Z}_i \subset Z_{r,s} \cong G/H_{r,s}$ becomes *equidistributed on* $Z_{r,s}$ if, for any compact sets $\Omega_1, \Omega_2 \subset Z_{r,s}$ with boundary measure zero and $\mathrm{vol}(\Omega_2) > 0$, one has

$$\frac{|\mathcal{Z}_i \cap \Omega_1|}{|\mathcal{Z}_i \cap \Omega_2|} \to \frac{\mathrm{vol}(\Omega_1)}{\mathrm{vol}(\Omega_2)}, \text{ as } i \to \infty.$$

Implicit in this statement is the fact that $|\mathcal{Z}_i \cap \Omega_2|$ is nonzero if i is large enough: for instance, in Corollary 3.3, one can show that for any $\varepsilon > 0$ and i sufficiently large (depending on Ω_2)

$$|Z_{P_i}(\mathbb{Z}) \cap \Omega_2| \gg_\varepsilon \mathrm{disc}(P_i)^{1/2-\varepsilon}\mathrm{vol}(\Omega_2).$$

3.4. *Cube roots of integers in* 3×3 *matrices.* Let us specialize further, to make this even more concrete. For $d > 0$ not a perfect cube, consider the polynomials $P_d(X) = X^3 - d$ and set $Z_d(\mathbb{Z}) = Z_{P_d}(\mathbb{Z})$.

Here, it is convenient to explicitly interpret $Z_{1,1} \cong G/H_{1,1}$ as the space of "matrix cube roots of unity":

$$Z_{1,1} = \{M \in M_3(\mathbb{R}),\ M^3 = \mathrm{Id}, M \neq \mathrm{Id}\}.$$

This being so, our previous corollary can be stated in terms of the "radial projections" to the latter space:

3.5. Corollary. *Let $p > 3$ be a fixed prime. As $d \to +\infty$ amongst the integers which are not perfect cubes and such that*

$$p \textit{ is } \text{totally split} \textit{ in the field } \mathbb{Q}(\sqrt[3]{d}),$$

then the sequence of sets $\frac{1}{d^{1/3}}.\{M \in M_3(\mathbb{Z}), M^3 = d\}$ *becomes equidistributed in the space* $\{M \in M_3(\mathbb{R}),\ M^3 = \mathrm{Id}\}$.

3.6. *Translations.* Let us explain how the above corollaries follow, indeed, from our main theorems. Set $G = \mathrm{PGL}_n(\mathbb{R})$ and $\Gamma = \mathrm{PGL}_n(\mathbb{Z})$.

Let P have signature (r, s); let $\mathscr{O}_P$ be the ring $\mathbb{Z}[t]/P$ and $K_P = \mathscr{O}_P \otimes \mathbb{Q}$. By a *coarse ideal class for* $\mathscr{O}_P$, we understand a lattice $L \subset K_P$ so that $\mathscr{O}_P.L \subset L$, considered up to multiplication by $K_P^\times$. With this convention, there are maps:

$$\begin{aligned} (10) \qquad \Gamma\text{-orbits on } Z_P(\mathbb{Z}) &\leftrightarrow \text{ coarse ideal classes for } \mathscr{O}_P \\ &\to \text{ compact } H_{r,s}\text{-orbits on } X_n. \end{aligned}$$

The first map is a bijection. The composite of the two arrows amounts to the identification between Γ-orbits on $G/H_{r,s}$, and $H_{r,s}$-orbits on $\Gamma\backslash G$.

In arithmetic terms, we can understand the maps as follows:

(1) If we fix $M \in Z_P(\mathbb{Z})$, then the map $t \mapsto M$ makes $\mathbb{Z}^n$ into a $\mathscr{O}_P$-module; there is a unique coarse ideal class L so that L and $\mathbb{Z}^n$ are isomorphic as $\mathscr{O}_P$-modules. (This is very classical; see, e.g. [31].)

(2) The second injection associates to the class of $L \subset K_P$ the compact orbit associated to (K_P, L, θ), in the notation of Section 2.2. Here $\theta : K_P \to A_{r,s}$ is the identification arising from the chosen ordering of the roots of P.

The composite map associates to the set of Γ-orbits on $Z_P(\mathbb{Z})$ a set of $H_{r,s}$-orbits on X_n; in the notation of Section 2.2, this set is:

$$Y_P = \bigcup_{\mathscr{O}_P \subset \mathscr{O} \subset \mathscr{O}_{K_P}} \tilde{Y}_{\mathscr{O}}, \tag{11}$$

corresponding to all packets whose associated order in K_P contains $\mathscr{O}_P$. The possibility of intermediate orders corresponds to the fact that the ideals that arise need not be *proper* $\mathscr{O}_P$-ideals.

Under these bijections, the equidistribution assertion about Z_P translates to an equidistribution assertion about Y_P, as we now recall.

3.7. *Integral-points interpretations.* As is well known (cf. [2, §8]) the (tautological) equivalences

$$\Gamma\backslash\Gamma g H_{r,s} \longleftrightarrow \Gamma g H_{r,s} \longleftrightarrow \Gamma g H_{r,s}/H_{r,s}$$

can be used to transfer equidistribution results about periodic $H_{r,s}$-orbits on $\Gamma\backslash G$, to equidistribution of the corresponding Γ-orbits on $G/H_{r,s}$. Taking into account (10), the first assertion of Corollary 3.3 reduces to the following:

3.8. Corollary. *Let $\{P_i\}_i$ be a sequence of cubic, monic, integral, irreducible, $\mathbb{R}$-split polynomials of discriminant* $\mathrm{disc}(P_i) \to +\infty$. *Then the set of compact H-orbits defined by* (11) *becomes equidistributed on X_3 (here $H = H_{3,0}$).*

This is indeed a corollary to Theorem 1.4, taking into account the fact that the total number of compact H-orbits on X_3 with bounded volume is finite.

Similarly, the other assertion of Corollary 3.3 follows from the more general adelic Theorem 4.9.

4. **Homogeneous subsets in the adelic context**

This paper has been written consistently in the *adelic* framework. It is therefore appropriate for us to discuss adelic equidistribution problems. We confine ourselves to equidistribution problems associated to tori, although much of the discussion applies in greater generality.

Let us emphasize that the adeles are simply a linguistic tool: all statements and results could be readily stated in the S-arithmetic context. The advantage of the adeles, rather, is that they provide a unified approach to broad classes of questions. For instance, consider the following equidistribution questions on the modular surface:

(1) Equidistribution of CM points,
(2) Equidistribution of large hyperbolic circles, centered at the point $i \in \mathbb{H}$,
(3) Equidistribution of closed geodesics (see §1.2),
(4) On $\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R})$, equidistribution of the translate of a *fixed* closed H-orbit by a "large" group element in $\mathrm{PGL}_2(\mathbb{R})$.[11]

These situations are all closely related, although they are often treated separately, and our aim is to discuss them as specializiations of a single adelic context.

Similarly, in [8], two classes of equidistribution problems are considered: "equidistribution on the group" and "equidistribution on the symmetric space"; these two problems again become unified in our presentation.

Explicitly, the goals of this section are as follows. We define "homogeneous toral sets:; roughly, as will be clarified in Theorem 5.2, these generalize the groupings Y_{x_iH} of compact orbits discussed in Theorem 1.4. We then define two important invariants ("volume" and "discriminant") for homogeneous toral sets, formulate the main question about their distribution (§4.4) and state our main theorems — Theorem 4.8 and Theorem 4.9 — in these terms. These theorems imply immediately Theorem 1.4, and their proofs comprise most of this paper.

4.1. *Homogeneous sets: definitions.* Let F be a number field with adele ring $\mathbb{A}$. Let $\mathbf{G}$ be a F-group with Lie algebra $\mathfrak{g}$. Set $X = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$.

A *homogeneous toral subset* of $X_{\mathbb{A}}$ will be, by definition, one of the form

$$Y = \mathbf{T}(F)\backslash\mathbf{T}(\mathbb{A}).g_{\mathbb{A}},$$

when $g_{\mathbb{A}} \in \mathbf{G}(\mathbb{A})$ and $\mathbf{T} \subset \mathbf{G}$ is a maximal torus. We shall consider only the case where the torus $\mathbf{T}$ is anisotropic over F.

Then Y supports a natural probability measure μ_Y: the pushforward of the Haar probability measure on $\mathbf{T}(F)\backslash\mathbf{T}(\mathbb{A})$ by $h \mapsto hg_{\mathbb{A}}$.

We shall associate to Y two additional invariants: a *discriminant* $\mathrm{disc}(Y)$, measuring its arithmetic complexity, and a *volume* $\mathrm{vol}(Y)$, measuring how "large" it is.

4.2. *Discriminant.* Let $r = \dim \mathbf{T}$. Let V be the affine space $(\wedge^r\mathfrak{g})^{\otimes 2}$. We fix a compatible system of norms $\|\cdot\|_v$ on $V \otimes_F F_v$, for each place v (for a discussion of norms, see Section 7; "compatible" means that, for almost all v, the unit balls of the norms coincide with the closure inside $V \otimes F_v$ of a fixed $\mathscr{O}_F$-lattice within V).

[11] See [21] for very general theorems concerning this setting, and also [2] for related results.

To the Lie algebra $\mathfrak{t}$ of $\mathbf{T}$, we associate a point in the affine space $(\wedge^r \mathfrak{g})^{\otimes 2}$:

$$\iota(\mathfrak{t}) = (e_1 \wedge \cdots \wedge e_r)^{\otimes 2} (\det B(e_i, e_j))^{-1}, \tag{12}$$

where $e_1, \ldots, e_r$ is a basis for $\mathfrak{t}$, and B the Killing form. We set

$$\mathrm{disc}_v(Y) = \|\mathrm{Ad}(g_v^{-1})\iota(\mathfrak{t})\|_v.$$

The discriminant $\mathrm{disc}(Y)$ is defined to be the product

$$\prod_v \mathrm{disc}_v(Y).$$

4.3. *Volume.* The definition of "volume", for a homogeneous toral subset, will depend on the choice of a compact neighborhood $\Omega_0 \subset \mathbf{G}(\mathbb{A})$ of the identity. This notion depends on Ω_0, but the notions arising from two different choices of Ω_0 are comparable to each other, in the sense that their ratio is bounded above and below. We define

$$\mathrm{vol}(Y) := \mathrm{vol}\Big(\{t \in \mathbf{T}(\mathbb{A}) : g_{\mathbb{A}}^{-1} t g_{\mathbb{A}} \in \Omega_0\}\Big)^{-1}, \tag{13}$$

where we endow $\mathbf{T}(\mathbb{A})$ with the measure that assigns the quotient $\mathbf{T}(F)\backslash\mathbf{T}(\mathbb{A})$ total mass 1.

4.4. *Desideratum.* We shall say that a measure on X is *homogeneous* if it is supported on a single orbit of its stabilizer.

The kind of problem we are interested in is the following:

When $\mathrm{disc}(Y_i) \to \infty$ (*equivalently* $\mathrm{vol}(Y_i) \to \infty$), *show that* μ_{Y_i} *converges to an homogeneous measure.*

There are certain cases of this problem which are easier. For instance (the "depth" aspect) we might consider a sequence of homogeneous toral sets for which there exists a fixed place v with $\mathrm{disc}_v(Y_i) \to \infty$. In this case, a limit of the μ_{Y_i}s will be invariant under a unipotent subgroup. This special case is interesting from the point of view of many applications, as for instance in the work of Vatsal [44]. Eskin, Mozes and Shah [21], and also Benoist and Oh [2] study this aspect.

4.5. *The case of a quaternion algebra.* The current state of knowledge concerning quaternion algebras implies the following theorem:

4.6. THEOREM. *Let* $\mathbf{G}$ *be the projectivized group of units in a quaternion algebra over a number field* F. *Let* $\{Y_i\}_i$ *be a sequence of homogeneous toral sets whose discriminant approaches* ∞ *with* $i \to +\infty$. *Then*

$$\mathrm{vol}(Y_i) = \mathrm{disc}(Y_i)^{1/2+o(1)}, \ i \to +\infty.$$

Moreover, any weak* *limit of the measures* μ_{Y_i} *is a homogeneous probability measure on* $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$, *invariant under the image of* $\widetilde{\mathbf{G}}(\mathbb{A}) \mapsto \mathbf{G}(\mathbb{A})$.

Here, $\widetilde{\mathbf{G}}$ denotes the simply-connected covering-group of $\mathbf{G}$. Indeed, one even knows this in a quantitative form: if $f \in C^{\infty}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))$ generates an irreducible, infinite-dimensional $\mathbf{G}(\mathbb{A})$-representation, then $|\mu_{Y_i}(f) - \mu(f)|$ is bounded by $O_f(\mathrm{disc}(Y_i)^{-\delta})$ for a positive δ.

The reason that we cannot simply assert that μ_{Y_i} converge to the Haar measure has to do with "connected component issues."

Theorem 4.6 is a consequence of works by several authors:

(1) Siegel's lower bounds (for the statement concerning volumes),
(2) Iwaniec [26], Duke [12], Duke-Friedlander-Iwaniec [13], Cogdell-Piatetski-Shapiro-Sarnak [9], and the fourth-named author [46] (for the pertinent subconvexity bounds),
(3) Waldspurger [48] (see also [29]), Clozel-Ullmo [8], Popa [38], S.-W. Zhang [51] and P. Cohen [10].

It conceals a unified statement of a large number of "instances" of that theorem, corresponding to varying the parameter disc in different ways: e.g. [7], [8], and [37]. For instance, if the quaternion algebra is defined over a totally real field and the quaternion algebra is ramified at one place, then one obtains in that way, equidistribution results for closed geodesics on an arithmetic Riemannian surface. Another example: it implies the solution (outlined by Cogdell-Piatetsky-Shapiro-Sarnak in [9]) to the representability question for ternary quadratic forms over number fields.

4.7. *Results for* $\mathbf{G} = \mathrm{PGL}_n$. Let $\{Y_i\}_i$ be any sequence of homogeneous (maximal) toral sets on $X = \mathrm{PGL}_n(F)\backslash\mathrm{PGL}_n(\mathbb{A})$ whose discriminant approaches ∞ with $i \to +\infty$. Let $\mathbf{T}_i$ be the associated tori; then

$$\mathbf{T}_i = \mathrm{Res}_{K_i/F}\mathbb{G}_{m,K_i}/\mathbb{G}_{m,F},$$

for a field extension K_i/F of degree n, unique up to isomorphism. We show:

4.8. THEOREM. *For* $\{Y_i\}_i$ *as above, one has*

$$\mathrm{vol}(Y_i) = \mathrm{disc}(Y_i)^{1/2+o(1)}, \text{ as } i \to +\infty.$$

This result is easy given well-known (but difficult) bounds on class numbers. It shows that the definitions of adelic volume and discriminant proposed are compatible.

Now let us describe our result on the distribution of homogeneous toral sets. First suppose there exists a fixed place v with one of the following properties:

(1) The local discriminant $\mathrm{disc}_v(Y_i) \to \infty$,

(2) Every $\mathbf{T}_i$ is split at v and a sub-convexity result *in the discriminant aspect* is known for values, along the critical line, of the Dedekind-ζ-functions associated to the fields K_i. (See (71) for the precise requirement.)

Then our results establish that any weak* limit of the measures μ_{Y_i} is a convex combination of homogeneous probability measures. However, the precise shape of such a homogeneous probability measure appears to be somewhat complicated in the adelic setting.

Rather than attempt a precise statement of the above, let us simply give the result in the simple case $n = 3$. It is simple for two reasons: first of all, the necessary subconvexity is known; secondly, the fact that n is prime forces there to be very few intermediate measures. We prove:[12]

4.9. Theorem. *Suppose* $n = 3$ *and let* $\{Y_i\}_i$ *be a sequence of homogeneous toral sets such that* $\operatorname{disc}(Y_i) \to +\infty$ *with* i; *suppose there exists a place* v *so that* $\operatorname{disc}_v(Y_i) \to \infty$ *or so that each* $\mathbf{T}_i$ *is split at* v. *Then any* weak* *limit of the* μ_{Y_i}, *as* $i \to +\infty$, *is a homogeneous probability measure on* X, *invariant by the image of* $\mathrm{SL}_3(\mathbb{A})$.

The equidistribution assertion of Theorem 1.4 is a consequence of Theorem 4.9, applied with $F = \mathbb{Q}, v = \infty$; translation from Theorem 4.9 is provided in the next section.

We conclude this section by observing that it remains a very interesting problem to remove the usage of the place v in Theorem 4.9, i.e., obtaining for PGL_3 a result as strong as Theorem 4.6 (even without a rate).

5. Packets

In this section we will clarify the relationship between the adelic perspective of Section 4 and the classical perspective of [18].

We will therefore exhibit a natural equivalence relation on the set of compact H-orbits on $\Gamma\backslash G$ for which the equivalence classes are (almost) finite abelian groups.

The equivalence classes will be called packets, and the union of compact torus orbits in a packet corresponds, roughly speaking, to an adelic torus orbit.

5.1. *Notation.* We recall the data prescribed in [18]. Let $\mathbf{G}$ be a semisimple group over $\mathbb{Q}$ that is $\mathbb{R}$-split; $G = \mathbf{G}(\mathbb{R}), \Gamma \subset G$ a congruence lattice, and

[12] It should be observed that this relies on an extension of [14] that has been announced by the latter two authors [36], and a theorem announced in [16], but neither of the proofs have yet appeared. With $F=\mathbb{Q}$, the proofs exist in print and are contained in [3], [14], and [15].

H a Cartan subgroup of G. To simplify notation we will write $\Gamma g H$ for the right H-orbit $\Gamma\backslash\Gamma g H \subset \Gamma\backslash G$.

We fix a lattice $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$ that is stable by the (adjoint) action of Γ, as well as a G-invariant bilinear form $B(\cdot,\cdot)$ on $\mathfrak{g}$ with $B(\mathfrak{g}_{\mathbb{Z}},\mathfrak{g}_{\mathbb{Z}}) \subset \mathbb{Z}$. Finally, we fix a Euclidean norm on $\mathfrak{g}_{\mathbb{R}}$.

Let r be the rank of G. Take $V = \wedge^r \mathfrak{g}$. For all finite p, we endow $V \otimes \mathbb{Q}_p$ with the norm which has as unit ball the closure of $\left(\wedge^r \mathfrak{g}_{\mathbb{Z}}\right)^{\otimes 2} \otimes \mathbb{Z}_p$. For $p = \infty$, we give $V \otimes \mathbb{R}$ the Euclidean norm derived from that on $\mathfrak{g}$. These choices allow one to define the notion of discriminant of a homogeneous toral set, as in Section 4.

In addition to this, we shall take *as given* one further piece of data. Let $\mathbb{A}_f$ be the ring of finite adeles. Choose a compact open subgroup $K_f \subset \mathbf{G}(\mathbb{A}_f)$ so that $K_f \cap \mathbf{G}(\mathbb{Q}) = \Gamma$ and so that $\mathfrak{g}_{\mathbb{Z}}$ is stable under the (adjoint) action of K_f.

Let $X_{\mathbb{A}}$ denote the double quotient $X_{\mathbb{A}} := \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K_f$. Clearly G acts on $X_{\mathbb{A}}$ and we shall refer to its G-orbits as the *components* of $X_{\mathbb{A}}$ and to the orbit of the identity double coset as the *identity component* (these need not be topologically connected); the identity component is identified with $\Gamma\backslash G$. The set of components is finite and is parametrized by the double quotient $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)/K_f$.

5.2. THEOREM. (1) *Each compact H-orbit $\Gamma g H \subset \Gamma\backslash G \subset X_{\mathbb{A}}$ is contained in the projection to $X_{\mathbb{A}}$ of a homogeneous toral set $Y \subset \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$. The set Y is unique up to translation by K_f; in particular, all such Y's have the same projection to $X_{\mathbb{A}}$. Moreover, the discriminant of $\Gamma g H$, in the sense of* [18], *and the discriminant of Y, in the sense of* Section 4, *coincide up to a positive multiplicative factor, the latter factor depending only on H, on the choice of $B(\cdot,\cdot)$ and on the Euclidean norm on $\mathfrak{g}_{\mathbb{R}}$.*

(2) *Declare two compact H-orbits to be* equivalent *if they both are contained in the projection to $X_{\mathbb{A}}$ of a homogeneous toral set Y.*

An equivalence class of compact H-orbits we refer to as a packet. *Packets are finite; indeed, the packet of $\Gamma g H$ is parametrized by the fiber of the map*

$$\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A}_f)/(K_f \cap \mathbf{T}(\mathbb{A}_f)) \to \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)/K_f$$

above the identity double coset; here $\mathbf{T}$ is the unique $\mathbb{Q}$-torus so that $\mathbf{T}(\mathbb{R}) = gHg^{-1}$. In particular, if $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K_f$ has a single component, every packet naturally has the structure of a principal homogeneous space for a finite abelian group.

(3) *Compact orbits in the same packet have the same stabilizer and the same discriminant.*

The proof of this theorem is straightforward. However, its content is quite beautiful: the collection of compact Cartan orbits on $\Gamma\backslash G$ group themselves into equivalence classes, each (almost – see assertion (2) of the theorem) parametrized by finite abelian groups, and the latter are themselves closely related to ideal class groups in number fields.

As such, this is a natural generalization of the situation described in the introduction to our paper: the set of geodesics of fixed length on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is parametrized by the class group of a real quadratic field.

5.3. *Proofs.* In order to keep notation minimal, for $g \in \mathbf{G}(\mathbb{A})$ we shall write $[g] = \mathbf{G}(\mathbb{Q}).g.K_f \in X_{\mathbb{A}}$ for the associated double coset. If $\mathfrak{S} \subset \mathbf{G}(\mathbb{A})$ is a subset, we often write simply $[\mathfrak{S}]$ for $[g] : g \in \mathfrak{S}$.

Let us recall from [18]:

5.4. Proposition (Basic correspondence). *There is a canonical bijection between*

1. *periodic H-orbits ΓgH on $\Gamma\backslash G$,*
2. *Γ-orbits on pairs $(\mathbf{T}, g)$ where $\mathbf{T}$ is an anisotropic torus defined over $\mathbb{Q}$ and $g \in G/H$ is so that $gHg^{-1} = \mathbf{T}(\mathbb{R})$.*

The bijection associates to ΓgH the pair $(\mathbf{T}, g)$, where $\mathbf{T}$ is the unique $\mathbb{Q}$-torus whose real points are gHg^{-1}.

Proof of the first statement of the theorem. Given $(\mathbf{T}, g)$, clearly ΓgH is contained in the projection to $X_{\mathbb{A}}$ of the homogeneous toral set $Y_0 := (\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A})).(g, 1)$. (Here $(g, 1) \in \mathbf{G}(\mathbb{A})$ is the element that equals g at the real place, and is the identity elsewhere; in what follows we abbreviate it simply to g.)

Now let us show that Y_0 is the only such homogeneous toral set, up to K_f. Take any homogeneous toral set $Y = (\mathbf{T}'(\mathbb{Q})\backslash\mathbf{T}'(\mathbb{A}))g'_{\mathbb{A}}$ whose projection to $X_{\mathbb{A}}$ contains $\Gamma gH = \Gamma\mathbf{T}(\mathbb{R})g$. Therefore,

$$\mathbf{T}(\mathbb{R}) \subset \bigcup_{\delta \in \mathbf{G}(\mathbb{Q})} \delta\mathbf{T}'(\mathbb{A})g'_{\mathbb{A}}K_f g^{-1}.$$

It follows that there exists $\delta \in \mathbf{G}(\mathbb{Q})$ so that

$$\mathbf{T}(\mathbb{R})^{(0)} \subset \delta\mathbf{T}'(\mathbb{A})g'_{\mathbb{A}}K_f g^{-1}.$$

Therefore there exists $t' \in \mathbf{T}'(\mathbb{A})$ and $k \in K_f$ so that $\delta t' g'_{\mathbb{A}} k g^{-1} = 1$ and $\mathbf{T} = \delta\mathbf{T}'\delta^{-1}$ and, moreover, we conclude that

$$(\mathbf{T}'(\mathbb{Q})\backslash\mathbf{T}'(\mathbb{A}))g'_{\mathbb{A}} = \delta^{-1}(\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A}))\delta t' g'_{\mathbb{A}} = (\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A}))gk^{-1}$$

where we treated these as subsets of $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$.

It follows that any homogeneous toral set Y, whose projection to $X_{\mathbb{A}}$ contains ΓgH, is necessarily of the form $(\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A})).(g,1)$, up to modification by K_f.

The equality of discriminants asserted in the theorem is a direct consequence of the definitions of the discriminant (see [18, (2.2)] for the definition for compact Cartan orbits). For this note that the p-adic discriminant measures the power of p in the denominator of $\iota(\mathfrak{t})$, while the discriminant at ∞ equals the norm of $\iota(\mathfrak{h})$ and so is constant.

Proof of the second assertion of the theorem. Now let us observe that the packet of the compact orbit ΓgH consists of all compact orbits $\Gamma\delta gH$ with $\delta \in \mathbf{G}(\mathbb{Q}) \cap K_f\mathbf{T}(\mathbb{A}_f)$ (where the intersection is taken in $\mathbf{G}(\mathbb{A}_f)$). Here $\mathbf{T}$ is the torus corresponding to ΓgH. We shall verify that the packet of ΓgH is parametrized by the fiber of the map

$$\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A}_f)/K_f \cap \mathbf{T}(\mathbb{A}_f) \to \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_f)/K_f$$

above the identity double coset. The finiteness assertion is then an immediate consequence of the finiteness of class numbers for algebraic groups over number fields.

With notation being as above, two $\delta, \delta' \in \mathbf{G}(\mathbb{Q}) \cap K_f\mathbf{T}(\mathbb{A}_f)$ define the same compact orbit $\Gamma\delta gH$ if and only if $\Gamma\delta\mathbf{T}(\mathbb{Q}) = \Gamma\delta'\mathbf{T}(\mathbb{Q})$; therefore compact orbits are parametrized by the double quotient

$$\mathbf{G}(\mathbb{Q}) \cap K_f\backslash(\mathbf{G}(\mathbb{Q}) \cap K_f\mathbf{T}(\mathbb{A}_f))/\mathbf{T}(\mathbb{Q}) = K_f\backslash(K_f\mathbf{G}(\mathbb{Q}) \cap K_f\mathbf{T}(\mathbb{A}_f))/\mathbf{T}(\mathbb{Q}),$$

but this is (after inverting) precisely what is described by the theorem.

Proof of the final assertion of the theorem. Let $\Gamma gH, \Gamma\delta gH$ be in the same packet. We have already verified the equality of discriminants. To verify the equality of stabilizers, we check that $\mathbf{T}(\mathbb{R}) \cap \Gamma = \mathbf{T}(\mathbb{R}) \cap \delta^{-1}\Gamma\delta$. By assumption, $\Gamma = \mathbf{G}(\mathbb{Q}) \cap K_f$. Thus $\mathbf{T}(\mathbb{R}) \cap \Gamma = \mathbf{G}(\mathbb{Q}) \cap \mathbf{T}(\mathbb{R})K_f$ and $\mathbf{T}(\mathbb{R}) \cap \delta^{-1}\Gamma\delta = \mathbf{G}(\mathbb{Q}) \cap \mathbf{T}(\mathbb{R})(\delta_f^{-1}K_f\delta_f)$, where δ_f is the image of δ under $\mathbf{G}(\mathbb{Q}) \hookrightarrow \mathbf{G}(\mathbb{A}_f)$. There exists $t_f \in \mathbf{T}(\mathbb{A}_f), k_f \in K_f$ so that $\delta_f = k_f t_f$; therefore, we need to prove

$$\mathbf{G}(\mathbb{Q}) \cap \mathbf{T}(\mathbb{R})K_f = \mathbf{G}(\mathbb{Q}) \cap \mathbf{T}(\mathbb{R})(t_f^{-1}K_f t_f).$$

An element on the left-hand side belongs to $\mathbf{T}(\mathbb{Q})$, so it automatically commutes with t_f, and also belongs to the right-hand side. Reversing this reasoning shows the equality. □

5.5. *Example*: *packets for* GL_n. Let us explain, by way of illustration, the equivalence relation explicitly in the case of $\mathrm{PGL}_n(\mathbb{Z})\backslash\mathrm{PGL}_n(\mathbb{R})$.

More precisely, we take $\mathbf{G} = \mathrm{PGL}_n$, $\Gamma = \mathrm{PGL}_n(\mathbb{Z})$, H the diagonal subgroup, and K_f the closure of Γ in $\mathrm{PGL}_n(\mathbb{A}_f)$. We may identify the Lie algebra $\mathfrak{pgl}_n$ with the quotient of $n \times n$ matrices by diagonal matrices; for $\mathfrak{g}_{\mathbb{Z}}$; we take then the projection of $M_n(\mathbb{Z})$ to $\mathfrak{pgl}_n$. For B we take simply the Killing form, and we take the Euclidean norm on $\mathfrak{pgl}_{n,\mathbb{R}}$ to be that derived from the Hilbert-Schmidt norm on $M_n(\mathbb{R})$. (This differs by a factor of 2 from that chosen in [18, §4].)

As discussed in [18], and recalled previously, compact H-orbits are parametrized by data (K, L, θ).

The equivalence corresponding to *packets* can be described in elementary terms as follows: Declare $(K, L, \theta) \sim (K, L', \theta)$ whenever L, L' are *locally homothetic*; here, we say that two lattices $L, L' \subset K$ are *locally homothetic* for every prime number p, and there exists $\lambda_p \in (K \otimes \mathbb{Q}_p)^\times$ so that $L_p = \lambda_p L'_p$. Here L_p, L'_p denote their respective closures in $K \otimes \mathbb{Q}_p$.

Observe that $L \sim_{loc} L'$ implies that $\mathscr{O}_L = \mathscr{O}_{L'}$. However, the converse is *a priori* not true (unless $\mathscr{O}_L$ is Gorenstein, see e.g. [1, (6.2), (7.3)]: for instance if $\mathscr{O}_L$ is the maximal order). Thus, the grouping into packet refines several plausible cruder groupings, e.g., grouping compact orbits with the same volume, or grouping compact orbits for which the order $\mathscr{O}_L$ is fixed.

Moreover, one can see at a heuristic level why the "packet" grouping has better formal properties than the "fixed order" grouping. Namely the set of proper $\mathscr{O}_L$ ideals up to homothety (i.e. up to multiplication by $K^\times$) is not *a priori* a group, nor a principal homogeneous space under a group. By contrast, the set of lattices up to homothety, within the local-homothety class of L, *does* form a principal homogeneous space for a certain abelian group, the *Picard group* $\mathrm{Pic}(\mathscr{O}_L)$, i.e. the group of homothety classes of ideals locally homothetic to $\mathscr{O}_L$. Of course, if $\mathscr{O}_L$ happens to be Gorenstein, the packet is parametrized simply by $\mathrm{Pic}(\mathscr{O}_L)$.

6. Homogeneous toral sets for GL_n

We shall now consider explicitly the case of $\mathbf{G} = \mathrm{PGL}_n$ over a global field F, introducing data $\mathscr{D}$ which parametrizes homogeneous toral sets, which we term global torus data.

By localization, such data will give rise to certain *local* data over each place of v, which we term local torus data. We shall explain how to compute the discriminant of the homogeneous toral set — in the sense of Section 4 — very explicitly in terms of the local torus data.

6.1. *The local data. Local torus data $\mathscr{D}$*, over a local field k, consists simply of an étale k-algebra $A \subset M_n(k)$ of dimension n: i.e., a direct sum $A = \oplus_i K_i$ where the K_i are field extensions of k.

Set $\mathfrak{g} = M_n(k)/k$ identified as the Lie algebra of PGL_n. Let V be the affine space

$$V = (\wedge^{n-1}\mathfrak{g})^{\otimes 2}.$$

To the data $\mathscr{D}/k$, we associate a point in $x \in V$ via:

$$x_{\mathscr{D}} = [(f_1 \wedge \cdots \wedge f_{n-1})^{\otimes 2} (\det B(f_i, f_j))^{-1}], \tag{14}$$

where $f_1, \ldots, f_{n-1}$ is a basis for $A/k \subset M_n(k)/k$, and B is the Killing form on $\mathfrak{g}$.

We set $\operatorname{disc}(\mathscr{D}/k) := \|x_{\mathscr{D}}\|_V$. Here $\|\cdot\|_V$ is (see §7)

- the norm on V whose unit ball is the closure of $\wedge^r M_n(\mathscr{O}_k)/\mathscr{O}_k$, for k non-archimedean;
- the norm on V which descends from the Hilbert-Schmidt norm on $M_n(k)$ for k archimedean.

Explicitely: let $f_0, f_1, \ldots, f_{n-1}$ be a k-basis of for A with $f_0 \in k$ and which span $\Lambda = A \cap M_n(\mathscr{O}_k)$ as an $\mathscr{O}_k$-module (when k is nonarchimedean) or which is orthonormal with respect to the Hilbert-Schmidt norm on $M_n(k)$ (when k is archimedean). If this is so, then one may compute $\operatorname{disc}(\mathscr{D}/k)$ by the rule:

$$\operatorname{disc}(\mathscr{D}/k) := |(2n)^{1-n} \det(\operatorname{tr}(f_i f_j))^{-1}|. \tag{15}$$

In particular, for k nonarchimedean, the discriminant of $\mathscr{D}/k$ differs, by a constant, from the discriminant of the ring Λ.

6.2. *Proof of equivalence between* (15) *and* (14). This is, in essence, proved in [18]. Let us reprise it here. Let f_i be a basis for A, as chosen above. Let $\bar{f}_i$, for $1 \leq i \leq n-1$, be the projection of f_i to $\mathfrak{g}$. Norms as above,

$$\|(\bar{f}_1 \wedge \cdots \wedge \bar{f}_{n-1})^{\otimes 2}\|_V = 1.$$

In fact, in the nonarchimedean case, we may extend $\bar{f}_i$ $(1 \leq i \leq n-1)$ to an $\mathscr{O}_k$-basis for $M_n(\mathscr{O}_k)/\mathscr{O}_k$, which makes the result obvious. In the archimedean case, the claim is an immediate consequence of the fact that $f_0, \ldots, f_n$ are orthonormal with respect to the Hilbert-Schmidt norm.

The Killing form on $\mathfrak{g}$ evaluates to: $B(\bar{f}_i, \bar{f}_j) = 2(n\operatorname{tr}(f_i f_j) - \operatorname{tr}(f_i)\operatorname{tr}(f_j))$. Therefore, the determinant $\det B(\bar{f}_i, \bar{f}_j)_{1 \leq i,j \leq n-1}$ equals

$$(2n)^{n-1} \det(\operatorname{tr}(f_i f_j)_{0 \leq i,j \leq n-1}.$$

The discussion of Section 4.2 associates to $\mathfrak{t} = A/k \subset \mathfrak{g}$ the point

$$\iota(\mathfrak{t}) = (2n)^{1-n} \det(\operatorname{tr}(f_i f_j)_{0 \leq i \leq n-1})^{-1} (\bar{f}_1 \wedge \cdots \bar{f}_{n-1})^{\otimes 2}$$

The required compatibility follows.

6.3. *Global data.* Let F be a global field. We define *global torus data* $\mathscr{D}$ to consist of a subfield $K \subset M_n(F)$ and an element $g_{\mathbb{A}} = (g_\infty, g_f) \in \mathbb{A}_K^\times \backslash \mathrm{GL}_n(\mathbb{A})$.

To the global data we may associate (cf. §4):

(1) A homogeneous toral set $Y_{\mathscr{D}} = (\mathbf{T}_K(F) \backslash \mathbf{T}_K(\mathbb{A}))g$, and the probability measure $\mu_{\mathscr{D}}$ on $Y_{\mathscr{D}}$. Here $\mathbf{T}_K$ is the unique subtorus of PGL_n with Lie algebra K/F.
(2) A (global) discriminant $\operatorname{disc}(\mathscr{D}) := \operatorname{disc}(Y_{\mathscr{D}})$, depending on $K, g_{\mathbb{A}}$.

The global discriminant $\mathrm{disc}(\mathscr{D})$ can be computed in terms of our discussion above. Indeed, the global torus data $\mathscr{D} = (K, g)$ also gives rise to a collection of local torus data $(\mathscr{D}_v)_v$: for every place v, $\mathscr{D}_v$ consists of the subalgebra

$$A_v = g_v^{-1}(K \otimes F_v)g_v \subset M_n(F_v)$$

and it follows from the definitions that

$$\mathrm{disc}(\mathscr{D}) = \prod_v \mathrm{disc}(\mathscr{D}_v).$$

7. **The local building**

In this section, we are going to recall the basic theory of the building attached to the general linear group, over a local field k. We will follow the beautiful old ideas of Goldman and Iwahori, [24], interpreting this by norms on a k-vector space.

7.1. *Notation concerning local fields.* We denote by k a local field. Let us normalize once and for all an absolute value on it. If $k = \mathbb{R}$ or $\mathbb{C}$, then let $|\cdot|$ denote the usual absolute value and if k is non-archimedean, we normalize $|\cdot|$ to be the module of k: $|\cdot| = q^{-v_\pi(\cdot)}$, where q is the cardinality of the residue field and $\pi \in k$ any uniformizer, and $v_\pi(\cdot)$, the corresponding valuation.

7.2. *Definition of the building by norms.* Let k be a local field and V vector space over k of finite dimension. A norm on V is a function $N : V \to \mathbb{R}^+$ into the nonnegative reals that satisfies

$$N(v) = 0 \Leftrightarrow v = 0_V, \quad N(\lambda x) = |\lambda| N(x),\ \lambda \in k$$

$$N(x+y) \leq \begin{cases} N(x) + N(y) & \text{(if } k \text{ is archimedean)} \\ \max(N(x), N(y)) & \text{(if } k \text{ is nonarchimedean).} \end{cases}$$

For N a norm, we denote its homothety class by $[N]$:

$$[N] = \{\mu N,\ \mu \in \mathbb{R}_{>0}\}.$$

If k is real (respectively complex), then we call a norm on V *good* if it is quadratic (respectively Hermitian). If k is nonarchimedean, we shall refer to any norm as *good*.

We let $\mathcal{B}(V)$ and $\bar{\mathcal{B}}(V)$ be the building of $\mathrm{GL}(V)$ and $\mathrm{PGL}(V)$ respectively; specifically $\mathcal{B}(V)$ is the set of good norms on V, and $\bar{\mathcal{B}}(V)$ the set of such norms up to homothety.

7.3. *Action of the group.* The group $\mathrm{GL}(V)$ acts transitively on $\mathcal{B}(V)$ via the rule

$$g.N(x) = N(xg).$$

This induces a transtive action of $\mathrm{PGL}(V)$ on $\bar{\mathcal{B}}(V)$.

If k is archimedean, then the stabilizer of a good norm is a maximal compact subgroup, and any (good) norm is determined by its unit ball.

If k is nonarchimedean, then neither of these are true: consider, for instance, the norm on $\mathbb{Q}_p^2$ given by $(x, y) \mapsto \max(|x|, p^{-1/2}|y|)$. We say that a norm is *standard* if is satisfies

$$N(x) = \inf\{|\lambda| : \lambda \in k, N(x) \leq |\lambda|\}.$$

A standard norm is determined by its unit ball, and its stabilizer is a maximal compact subgroup of $\mathrm{GL}(V)$. Standard norms, are, equivalently (q the cardinality of the residue field of k):

(1) Those which take values in $q^{\mathbb{Z}}$,
(2) Those which look like $x = (x_1, \ldots, x_n) \mapsto \max_i |x_i|$ in suitable coordinates,
(3) Those that correspond to (special) vertices of the building.

The action of $\mathrm{GL}(V)$ preserves standard norms.

7.4. *Direct sums. Apartments.* Given norms N_V on V and N_W on W, they determine a norm $N_V \oplus N_W$ on $V \oplus W$, defined by $\sqrt{N_V^2 + N_W^2}$ if k is archimedean, and $\max(N_V(v), N_W(w))$ if k is nonarchimedean.

Any splitting of V into one-dimensional subspaces determines an *apartment*. This consists of all norms that are direct sums of norms on the one-dimensional subspaces. Any two norms belong to an apartment. Apartments are in bijection with split tori within $\mathrm{GL}(V)$: a splitting of V into one-dimensional spaces determines a split torus, namely, those automorphisms of V preserving each one-dimensional space.

If H is a split torus with co-character lattice $X_\star(H)$, then the vector space $\mathfrak{h} := X_\star(H) \otimes \mathbb{R}$ acts simply transitively on the apartment. It suffices to explicate this when V is one-dimensional; in that case, the action of the tautological character $\mathbf{G}_m \to \mathrm{GL}(V) = H$ on the set of norms is multiplication by cardinality of the residue field if k is nonarchimedean, and multiplication by (e.g.) $e = 2.718$, when k is archimedean.

In explicit terms, we can phrase this as follows: the apartment in the building of k^n, corresponding to the diagonal torus in $\mathrm{GL}(n, k)$, consists of all norms of the following form:

$$N(x_1, \ldots, x_n) = \begin{cases} \max_i q^{t_i}|x_i|, & \text{nonarchimedean}, \ q := \text{size of residue field.} \\ \left(\sum_i e^{2t_i}|x_i|^2\right)^{1/2}, & \text{archimedean}, \end{cases}$$

for some $(t_1, \dots, t_n) \in \mathbb{R}^n$. Therefore, this apartment is parametrized by the affine space $\mathbb{R}^n$.

7.5. *The canonical norm on an algebra.* Let A be a (finite dimensional) étale algebra over k, i.e. $A = \oplus_i K_i$ is a direct sum of field extensions of k, we equip it with a norm which we shall call the *canonical* norm.

– If k is nonarchimedean: let $\mathscr{O}_A = \oplus_i \mathscr{O}_{K_i}$ denote the maximal compact subring of A:

$$N_A(t) = \inf\{|\lambda|,\ \lambda \in k : t \in \lambda \mathscr{O}_A\}. \tag{16}$$

That norm is standard and has unit ball $\mathscr{O}_A$. Moreover, for t decomposing as $t = (t_1, \dots, t_i, \dots)$, $t_i \in K_i$, one has

$$N_A(t) = \max_i N_{K_i}(t_i). \tag{17}$$

Here N_{K_i} denote again the standard norm of the k-algebra K_i.

– If k is non-archimedean, $A = \oplus K_i$, for certain subfield $K_i \subset \mathbb{C}$. We define

$$N_A(\sum_i x_i) = \left(\sum_i |x_i|^2\right)^{1/2}.$$

To keep our notation consistent with the non-archimedean case, we *define* $\mathscr{O}_A$ to be the unit ball of N_A:

$$\mathscr{O}_A = \{x \in A,\ N_A(x) \leq 1\}. \tag{18}$$

When $A = k^n$, we denote the canonical norm by N_0.

7.6. *The metric and operator norms.* We may equip $\mathcal{B}(V)$ with a $\mathrm{GL}(V)$-invariant metric by using the notion of operator norm:

If N_1, N_2 are any two norms, then we let $\exp(\mathrm{dist}(N_1, N_2))$ be the smallest constant $\alpha \geq 1$ such that N_2 satisfies $\alpha^{-1} N_1 \leq N_2 \leq \alpha N_1$.

Given any two norms N_1, N_2, there exists an apartment that contains them both; thus, to understand dist, it suffices to understand it on each apartment. In Section 7.4, we explicitly parametrized each apartment by an affine space $(t_1, \dots, t_n) \in \mathbb{R}^n$. In terms of that parametrization,

$$\mathrm{dist}((t_1, \dots, t_n), (t_1', \dots, t_n')) := \log(q) \max_i |t_i - t_i'|.$$

Here we understand $q = e$ for k archimedean. Therefore, this amounts to an L^∞-metric on each apartment.

We equip $\bar{\mathcal{B}}(V)$ with the quotient metric[13]. In particular, if N_1, N_2 are two norms and $[N_1], [N_2]$ the corresponding elements in $\bar{\mathcal{B}}(V)$, then, for any

[13] Recall that if X is a metric space and G acts by isometries on X, we may define the metric on X/G via $d(x_1, x_2) = \inf_{g \in G} d(x_1 g, x_2)$.

$v_1, v_2 \in V$:

$$\mathrm{dist}([N_1],[N_2]) \geq \frac{1}{2}\log\frac{N_1(v_1)N_2(v_2)}{N_2(v_1)N_1(v_2)}. \tag{19}$$

Indeed, one may define dist as the supremum of the quantities appearing on the right-hand side.

7.7. *Harish-Chandra spherical function.* We shall make use of the Harish-Chandra spherical function on $\mathrm{GL}(V)$. It is defined with reference to a maximal compact subgroup $K \subset \mathrm{GL}(V)$, which we take to be the stabilizer of a standard norm N.

Fix an apartment containing N, and let $H \subset \mathrm{GL}(V)$ be the split torus corresponding to this apartment. Let $B \supset H$ be Borel containing H, with unipotent radical U corresponding to all positive roots. We have a decomposition $\mathrm{GL}(V) = UHK$. Let $\mathrm{H} : \mathrm{GL}(V) \to H$ be the projection according to this decomposition and let $\rho : H \to \mathbb{R}_+$ be defined by

$$\rho : a \mapsto \prod_{\alpha\in\Phi^+} |\alpha(a)|^{1/2}$$

be the "half-sum of positive roots" character with respect to B.

The Harish-Chandra spherical function is defined as:

$$\Xi(g) := \int_{k\in K} \mathrm{H}(kg)^\rho dk,$$

where the measure on K is the Haar measure with total volume 1. We will be needing the following bound: for any $\alpha < 1$,

$$\Xi(g) \ll_\alpha \exp(-\alpha.\mathrm{dist}([gN],[N])). \tag{20}$$

Indeed, it suffices to prove (20) when gN belongs to a fixed apartment containing N. Identifying this with an affine space, with N as origin, the point gN has coordinates $(t_1, \dots, t_n)$; without loss of generality, we may assume that $t_1 \leq t_2 \leq \dots t_n$, and (cf. [30, Prop. 7.15] for real semisimple Lie groups)

$$\Xi(g) \ll_\alpha \left(q^{-\frac{1}{2}\sum_{i<j} t_i - t_j}\right)^{-\alpha}$$

for any $\alpha < 1$. On the other hand, by (19)

$$\mathrm{dist}([gN],[N]) = \frac{1}{2}\log q.(t_n - t_1),$$

whence our conclusion.

7.8. *Action of invertible linear maps.* If $\iota : V \to W$ is an invertible map between vector spaces, which we understand as acting on the right (i.e. $v.i$ or $v^\iota \in W$), and N is a norm on W, then we denote by ιN the norm $v \mapsto N(v.\iota)$, a norm on V.

8. Notation

In this section, we set up some fairly standard notation concerning number fields. We set up local notation first, and then global notation. In Section 9 we shall use only the local notation; in the rest of the paper, we make use of the global notation.

8.1. *Local notation and normalizations.* Let k be a local field and $A \subset M_n(k)$ local torus data, with $[A : k] = n$. The absolute value on k is normalized as in Section 7.1.

We denote by $|\cdot|_A$ the "module" of A, i.e., the factor by which the map $y \mapsto yx$ multiplies Haar measure on A if $x \in A^\times$ and $|x|_A = 0$ if $x \in A - A^\times$. Consequently, writing $A = \oplus_i K_i$,

$$|x|_A = \prod_i |x_i|_{K_i},\ x = (\dots, x_i, \dots),\ x_i \in K_i;$$

in particular, for[14] $x \in k \subset A$, $|x|_A = |x|_k^n$. Observe also that the module $|x|_A$ coincide with $|\det(x)|$ when we view x as an element of $M_n(k)$.

We fix an additive character $e : k \to \mathbb{C}$; it induces the additive character on A

$$a \to e_A(a) := e(\mathrm{tr}_{A/k}(a)) = e(\mathrm{tr}(a)),\ a \in A.$$

Observe that for $a \in A$, the A/k-trace coincide with the restriction to $A \subset M_n(k)$ to the matrix trace; thus there is no ambiguity in refereing to the trace.

We fix Haar measure dx, da on k, A; for definiteness, we normalize them to be self-dual with respect to the characters $e(\cdot)$ and $e(\mathrm{tr}(\cdot))$ respectively. Sometimes we will write $d_k x$ and $d_A x$ to emphasize the measures on k and A respectively. We will often write vol_k or vol_A for volume of a set with respect to these measures.

Even though we have normalized vol_k and vol_A to be self-dual, it is occasionally more conceptually clear and helpful — for instance, when working with Fourier transforms — to introduce a separate notation for the dual measures. Thus, we shall denote by $\widehat{\mathrm{vol}_A}$ the Haar measure dual to vol_A, with respect to the character e_A; $\widehat{\mathrm{vol}_k}$ is defined similarly. Our normalizations are so that $\widehat{\mathrm{vol}} = \mathrm{vol}$, but we try to keep the notions conceptually separate.

We normalize multiplicative Haar measure $d_k^\times x, d_A^\times x$ on $k^\times$ and $A^\times$, respectively, by the rules

$$d_k^\times x = \zeta_k(1)|x|_k^{-1} d_k x, d_A^\times a = \zeta_A(1)|x|_A^{-1} d_A a.$$

(See §8.2 below, for a recollection of the definition of ζ.)

[14] At this point we must observe a small ugliness of notation: for $x \in k$, the "module" $|x|_k$ coincides with our normalization of $|x|$ from Section 7.1 if $k \neq \mathbb{C}$. If $k = \mathbb{C}$, however, $|x|_{\mathbb{C}} = |x|^2$. This unfortunate notational clash seems somewhat unavoidable, for the module of $\mathbb{C}$ does not coincide with what is usually termed the absolute value.

These normalizations have the following effect: for k nonarchimedean,

$$\mathrm{vol}_{A^\times}(\mathscr{O}_A^\times) = \mathrm{vol}_A(\mathscr{O}_A), \mathrm{vol}_{k^\times}(\mathscr{O}_k^\times) = \mathrm{vol}(\mathscr{O}_k). \tag{21}$$

For k archimedean, we shall not define $\mathscr{O}_A^\times$ or $\mathscr{O}_k^\times$; however, it will be convenient (to uniformize notation) to *define* their volumes $\mathrm{vol}_A(\mathscr{O}_A^\times) := \mathrm{vol}_A(\mathscr{O}_A)$ and $\mathrm{vol}_k(\mathscr{O}_k^\times) := \mathrm{vol}_k(\mathscr{O}_k)$ so that the equality (21) remains valid. Recall that $\mathscr{O}_A, \mathscr{O}_k$ in the archimedean case are defined by the convention (18).

8.2. *Local ζ-functions.* The local ζ-function of the field k and of A are denoted $\zeta_k(s), \zeta_A(s)$ ($s \in \mathbb{C}$): for $A = \oplus_i K_i$,

$$\zeta_A(s) = \prod_i \zeta_{K_i}(s).$$

We recall that the local ζ-function of a local field k is defined by $\zeta_k(s) = \pi^{-s/2}\Gamma(s/2)$ if $k = \mathbb{R}$, $2(2\pi)^{-s}\Gamma(s)$ if $k = \mathbb{C}$, and finally $\zeta_k(s) = (1-q^{-s})^{-1}$, where q is the size of the residue field, if k is nonarchimedean.

More generally for ψ a character of $A^\times$ ($\psi = (\dots, \psi_i, \dots)$, ψ_i a character of $K_i^\times$), we denote by $L(A, \psi)$ the local L-function of ψ:

$$L(A, \psi) = \prod_i L(K_i, \psi_i).$$

See [5, Chap. 3] for definition and discussion.

For $s \in \mathbb{C}$, we will also write

$$L(A, \psi, s) := L(A, \psi |\cdot|_A^s).$$

In particular $\zeta_A(s) = L(A, |\cdot|_A^s)$.

If k is nonarchimedean, we attach to ψ a discriminant $\mathrm{disc}(\psi)$. This may be defined directly as follows: we write $A = \oplus K_i$ and $\psi = (\dots, \psi_i, \dots)$; for each i, let t_i be the largest integer so that ψ_i is trivial on $1 + \mathfrak{q}_{K_i}^{t_i}$; here $\mathfrak{q}_v$ is the prime ideal of the ring of integers in K_i. Then $\mathrm{disc}(\psi) := \prod_i q_i^{t_i}$, where q_i is the residue field size of K_i.

For k archimedean, we set by definition $\mathrm{disc}(\psi) \equiv 1$.

8.3. *Number fields.* We now pass to a global setting.

Let F be a number field. We denote by $\mathbb{A}$ and $\mathbb{A}_{F,f}$ the ring of adeles and of finite adeles, respectively.

We will work with global data $\mathscr{D}$, as in Section 6.3, which will consist of: $K \subset M_n(F)$ and $g_{\mathscr{D}} \in \mathbb{A}_K^\times \backslash \mathrm{GL}_n(\mathbb{A})$. We shall fix an identification $\iota : K \to F^n$ of right K-modules; this means that $(ab)^\iota = a^\iota.b$, where, on the right-hand side, we understand $b \in M_n(F)$.

We obtain from this data an embedding of the F-torus

$$\mathbf{T}_K := \mathrm{Res}_{K/F}\mathbb{G}_m/\mathbb{G}_m \hookrightarrow \mathrm{PGL}_{n,F}.$$

Here $\mathrm{PGL}_{n,F}$ denotes the algebraic group PGL_n over the field F.

We will use the letter v for a place of F and w for a place of K. If v is a place of F, then we denote by F_v the completion of F at v, and $K_v := K \otimes_F F_v$.

By localization, the global data $\mathscr{D}$ gives rise to local data $\mathscr{D}_v$ (in the sense of §6.1) for each place v of F; i.e., we take $A_v = g_{\mathscr{D},v}^{-1} K_v g_{\mathscr{D},v} \subset M_n(F_v)$. We will write A instead of A_v when the dependence on v is clear.

Let us note that the map

$$\iota_v : a \in A_v \mapsto (g_{\mathscr{D},v} a g_{\mathscr{D},v}^{-1})^{\iota} . g_{\mathscr{D},v} \tag{22}$$

from A_v to F_v^n is then an identification for the A_v-module structures.

8.4. *Adeles, ideles and their characters.* There is a natural norm map, the module, $\mathbb{A}^\times \xrightarrow{|\cdot|} \mathbb{R}_{>0}$. We write $|x|_{\mathbb{A}}$ or sometimes simply $|x|$; this will cause no confusion so long as it is clear that the variable x belongs to $\mathbb{A}^\times$. We denote by $\mathbb{A}^{(1)}$ the kernel of the norm map, and similarly for $\mathbb{A}_K$.

Let $\Omega(C_F)$ resp. $\Omega(C_K)$ denote the group of homomorphisms from $\mathbb{A}^\times/F^\times$ resp. $\mathbb{A}_K^\times/K^\times$ to $\mathbb{C}^\times$. For $\psi \in \Omega(C_K)$ we shall denote by $\psi|_F$ the restriction of ψ to $\mathbb{A}^\times/F^\times$.

The group $\mathbb{C}$ is identified with the connected component of $\Omega(C_F)$, via identifying $s \in \mathbb{C}$ with the character $x \mapsto |x|_{\mathbb{A}}^s$. Given $\chi \in \Omega(C_F)$, we set $\chi_s(x) := \chi(x)|x|_{\mathbb{A}}^s$. This $\mathbb{C}$-action on $\Omega(C_F)$ gives $\Omega(C_F)$ the structure of a complex manifold.

For any $\chi \in \Omega(C_F)$, there exists unique $s \in \mathbb{R}$ so that $|\chi(x)| = |x|_{\mathbb{A}}^s$. We shall denote this s by $\Re\chi$, the "real part" of χ. Thus $\Re\chi = 0$ if and only if χ is unitary.

Finally, there is a natural map $\mathbb{R}_{>0} \to C_{\mathbb{Q}}$ (inclusion at the infinite place). Thus there is also a map $\mathbb{R}_{>0} \to C_K, C_F$. We say that a character of C_K or C_F is *normalized* if its pullback to $\mathbb{R}_{>0}$ is trivial.

If $\omega : F_v^\times \to \mathbb{C}^\times$ is a multiplicative character, then we denote by $L(F_v, \omega, s)$ the corresponding L-factor, and by $L(F_v, \omega)$ its value when $s = 0$. In particular, when $\omega = |x|_v^s$, we get the local ζ-factor: $L(F_v, \omega) = \zeta_{F_v}(s)$. Corresponding definitions also hold for K. If v is a place of F, we will write $\zeta_{K,v} := \prod_{w|v} \zeta_{K,w}$.

8.5. *Discriminants.* Suppose $\psi \in \Omega(C_K)$. For v any place of F, we have discriminants:

$$\mathrm{disc}_v(F), \mathrm{disc}_v(K/F), \mathrm{disc}_v(K), \mathrm{disc}_v(\mathscr{D}_v), \mathrm{disc}_v(\psi|_{K_v}).$$

Namely, $\mathrm{disc}_v(F)$ is the discriminant of $F_v/\mathbb{Q}_p$ (where p is the prime of $\mathbb{Q}$ below v) and $\mathrm{disc}_v(K)$ the discriminant of $K_v/\mathbb{Q}_p$. We set

$$\mathrm{disc}_v(K/F) = \mathrm{disc}_v(K)\mathrm{disc}_v(F)^{-[K:F]}.$$

By convention, we shall understand $\mathrm{disc}_v(F) = \mathrm{disc}_w(K) = 1$ if v or w are archimedean. $\mathrm{disc}_v(\mathscr{D}_v)$ is as in Section 6 and $\mathrm{disc}_v(\psi|_{K_v})$ is defined in Section 8.2.

For any of the objects above, we set $\mathrm{disc}(\dots) = \prod_v \mathrm{disc}_v(\dots)$. We note that $\mathrm{disc}(\mathscr{D}) \gg_{F,n} \mathrm{disc}(K)$; this will follow from Lemma 9.5 and the fact that $\mathrm{disc}_v(\mathscr{D})$ is bounded from below at each archimedean place.

8.6. *Measure normalizations.* Let $e_{\mathbb{Q}} : \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to \mathbb{C}$ be the unique character whose restriction to $\mathbb{R}$ is $x \mapsto e^{2\pi i x}$. Set $e_F = e_{\mathbb{Q}} \circ \mathrm{tr}_{F/\mathbb{Q}}$. We then normalize local measures according to the prescription of Section 8.1 with $k = F_v$, $A = K_v$.

Let us explicate this to be precise.

We choose for each v the measure on F_v that is self-dual with respect to e_F. The product of these measures, then, assigns volume 1 to $\mathbb{A}/F$. We define a measure on $F_v^\times$ by $d^\times x := \zeta_{F,v}(1)\frac{dx}{|x|_v}$. We make the corresponding definitions for K, replacing the character e_F by the character $e_K := e_F \circ \mathrm{tr}_{K/F}$. Taking the product of these measures yields measures on $\mathbb{A}, \mathbb{A}_K, \mathbb{A}^\times, \mathbb{A}_K^\times$. This fixes, in particular, a quotient measure on $\mathbf{T}_K(\mathbb{A}) = \mathbb{A}_K^\times/\mathbb{A}^\times$.

We obtain a measure on A_v through the identification $x \mapsto g_{\mathscr{D},v}^{-1} x g_{\mathscr{D},v}$ from K_v to A_v.

We will often denote by $d_K x$ the measure on K_v and by $d_F x$ the measure on F_v or F_v^n.

With these definitions, it is not difficult to verify that for finite places v, w:

$$\int_{\mathscr{O}_{K,w}^\times} d_K^\times x = \mathrm{disc}_w(K)^{-1/2}, \int_{\mathscr{O}_{F,v}^\times} d_F^\times x = \mathrm{disc}_v(F)^{-1/2} \tag{23}$$

$$\int_{\mathscr{O}_{K,w}} d_K x = \mathrm{disc}_w(K)^{-1/2}, \int_{\mathscr{O}_{F,v}} d_F x = \mathrm{disc}_v(F)^{-1/2}. \tag{24}$$

Moreover, (24) remains valid for v archimedean, if we replace equality by $\asymp$, and we interpret $\mathscr{O}_{K,w}$ resp. $\mathscr{O}_{F,v}$ as the unit balls for the canonical norms associated to K_w resp. F_v.

9. **Local theory of torus orbits**

In this section, we are going to explicate certain estimates over a local field, which are what are needed for "local subconvexity" discussed in Section 2. Indeed, the main result of this section, Proposition 9.9, is designed precisely to bound the (general version of the) quantities "$I_{\mathscr{O}}$" that occur in (6) and (8).

Our methodology is quite general (i.e., would apply to estimates for more general period integrals of automorphic forms) and is inspired, in part, by the paper of Clozel and Ullmo [8].

9.1. *Explanation in classical terms.* A simple case of our estimates is the following result:

Let Q be a positive definite quadratic form on $\mathbb{R}^3$. Consider:

$$I(Q) = \frac{\int_{Q(x,y,z)\le 1} |xyz|^{-1/2} dx\, dy\, dz.}{\mathrm{vol}(\mathbf{x} \in \mathbb{R}^3 :\ Q(\mathbf{x}) \le 1)^{1/2}}. \tag{25}$$

It will transpire — this will follow from Lemma 10.4 that we prove later — that the quantity $I_{\mathscr{O}}$, defined in (6), will be be bounded by products of integrals like (25) and nonarchimedean analogues. (More precisely, in the notation of (6): if the function f is invariant by the rotation group of Q, then we will be able to bound $I_{\mathscr{O}}(f, s)$ in terms of (25); the general case will reduce to this). Our goal will be to get good bounds for $I(Q)$. Evidently, the function I is invariant under coordinate dilations, and thus our bounds should also be so.

9.2. *The generalization to local fields.* The general setting we consider will be: we replace $\mathbb{R}, \mathbb{R}^3$ by a local field k and an etale k-algebra A of dimension n (say).

For an arbitrary norm N on A, we consider the integral

$$I(N) := \frac{\int_{x \in A : N(x) \leq 1} |x|_A^{-1/2} d_A x}{\mathrm{vol}(x \in A : N(x) \leq 1)^{1/2}}, \tag{26}$$

where $d_A x$ denote a Haar measure on A and $|x|_A$ denote the "module" of A (the factor by which $d_A x$ is transformed under $y \mapsto xy$). We consider the variation of $I(N)$ with N. Again, $I(N)$ is invariant by scaling and so defines a function on the building of $\mathrm{PGL}(A)$.[15] We shall show that:

(1) $I(N)$ decays exponentially fast with the distance of N to a certain subspace in the building, viz. the $A^\times$-orbit of the norm N_A;
(2) The distance to this subspace measures the discriminant of local torus data:

Choose an identification $\iota : A \to k^n$ which carries the unit ball for $N(x)$, to the unit ball of the standard norm on k^n. The identification ι, together with the action of A on itself by multiplication, can be used to embed $A \subset M_n(k)$. Thus, we have specified local torus data. Roughly speaking (Lemma 9.12), the discriminant of this local torus orbit, is a measure of the distance of N to our distinguished subspace of the building.

Our final result is presented in Proposition 9.9; the reader may find it helpful to interpret it in terms of the language above, in order to better absorb its content.

9.3. *Local torus data.* In the rest of this section, we fix local torus data $\mathscr{D}$ consisting of $A \subset M_n(k)$; we shall follow the notation of Section 8. Throughout this section, we allow the notation $\ll$ and $O(\cdot)$ to indicate an implicit constant that remains bounded, if k is restricted to be of bounded degree over $\mathbb{R}$ or $\mathbb{Q}_p$.

[15] Throughout this paper, we use $\mathrm{GL}(A)$ to denote k-linear automorphisms of A, thought of as a k-vector space; similarly, $\mathrm{PGL}(A)$.

By the inclusion, $A \subset M_n(k)$, k^n is a right A-module. For the rest of this section, we fix an identification of right A-modules $\iota : A \to k^n$; in other words,

$$(ab)^\iota = a^\iota . b, a \in A, b \in A \tag{27}$$

where, on the right-hand side, we regard $b \in M_n(k)$. The identification ι is unique up to multiplication by elements of $A^\times$. We write ι on the right to be as consistent as possible with our other notation.

In this setting, we have described (cf. §7.5) two norms: the canonical norm on the algebra A, to be denoted N_A, and the canonical norm on the algebra k^n, to be denoted N_0. We have denoted their unit balls by $\mathscr{O}_A$ and $\mathscr{O}_{k^n}$ respectively. These unit balls coincide with the maximal compact subgroups of these algebras in the non-archimedean case.

We are going to introduce an element $h \in \mathrm{GL}_n(k)$ with quantifies the relation between these norms. Choose $h \in \mathrm{GL}_n(k)$ so that:

$$hN_0(x)(:= N_0(xh)) = N_A(x^{\iota^{-1}}), \quad x \in k^n. \tag{28}$$

This is possible because, by choice, the norm N_A corresponds to a vertex of the building of $\mathrm{GL}(A)$, i.e. takes values in $q^{\mathbb{Z}}$ in the nonarchimedean case. Observe that choice of h depends on the choice of ι; given ι, the quantity $|\det h|$ is uniquely determined.

This definition implies that $\mathscr{O}_A{}^\iota = \mathscr{O}_{k^n} h^{-1}$; thus

$$\mathrm{vol}_A((\mathscr{O}_{k^n})^{\iota^{-1}}) = \mathrm{vol}_A(\mathscr{O}_A)|\det h|,$$

or

$$\frac{\iota_* \mathrm{vol}_A}{\mathrm{vol}_{k^n}} = \frac{\mathrm{vol}_A(\mathscr{O}_A)}{\mathrm{vol}_{k^n}(\mathscr{O}_{k^n})}|\det h|. \tag{29}$$

Similarly, we define $h_A \in \mathrm{GL}(A)$ by the following rule:

$$(yh_A)^\iota = y^\iota h, y \in A.$$

This means that

$$h_A \iota N_0 = N_A = \iota h N_0, \tag{30}$$

and so $|\det h_A|_k = |\det h|_k$.

With these conventions, and those of Section 7.8,

$$t\iota^{-1} N_A = \iota^{-1} t N_A, (t \in A^\times). \tag{31}$$

Let us be completely explicit, because t is acting in two different ways on the two sides of this equation. According to the conventions set out in Section 7.8, the left-hand side is the norm on k^n defined by $\mathbf{x} \mapsto N_A(\mathbf{x}t\iota^{-1})$; in particular, t is acting as an endomorphism of k^n. The right-hand side is the norm on k^n defined by $\mathbf{x} \mapsto N_A(\mathbf{x}\iota^{-1}t)$; here t is acting by right multiplication on A. The coincidence of the two sides follows from (27).

9.4. *Discriminant vs. discriminant.* We have two notions of "discriminant" attached to the local data $A \subset M_n(k)$:

On the one hand, we have the *absolute* discriminant, $\mathrm{disc}(A)$, of the k-algebra A:

– If k is non-archimedean, then it is given by

$$\mathrm{disc}(A) = [\mathscr{O}_A^* : \mathscr{O}_A] = \frac{\mathrm{vol}_A(\mathscr{O}_A^*)}{\mathrm{vol}_A(\mathscr{O}_A)},$$

Here $\mathscr{O}_A^*$ denote the dual lattice of $\mathscr{O}_A$ in A

$$\mathscr{O}_A^* := \{a \in A : |\mathrm{tr}_{A/k}(a\mathscr{O}_A)| \leq 1\} \supset \mathscr{O}_A.$$

– If k is archimedean, then we set $\mathrm{disc}(A) = 1$.

In particular in either case, we have

$$\mathrm{disc}(A) \asymp_n \frac{\mathrm{vol}_A(\mathscr{O}_A^*)}{\mathrm{vol}_A(\mathscr{O}_A)}. \tag{32}$$

On the other hand, in our previous discussion Section 6, we have defined a notion of discriminant $\mathrm{disc}(\mathscr{D})$ which is *relative* to the embedding $A \subset M_n(k)$. We shall presently compare the two notions and for this we shall interpret $\mathrm{disc}(\mathscr{D})$ in more geometric terms.

Let Λ be the set of $x \in A$ with operator norm ≤ 1 with respect to the norm N_0; here we regard k^n as an A-module via the *right* multiplication of $A \subset M_n(k)$. If k is nonarchimedean, then Λ is an *order* (and thus is contained in $\mathscr{O}_A$); indeed, $\Lambda = A \cap M_n(\mathscr{O}_k)$. Let Λ^* be the dual to Λ,

$$\Lambda^* = \{y \in A : |\mathrm{tr}_{A/k}(y\Lambda)| \leq 1\}. \tag{33}$$

9.5. Lemma. *We have* (*compare with* (32))

$$\mathrm{disc}(\mathscr{D}) \asymp_n \frac{\mathrm{vol}_A(\Lambda^*)}{\mathrm{vol}_A(\Lambda)}.$$

Moreover, $\asymp_n$ may be replaced by equality *if k is nonarchimedean, of residue characteristic exceeding n.*

Proof. The definition of $\mathrm{disc}(\mathscr{D})$ is explicated in (14). Choose, first of all, a k-basis $f_0 = 1, f_1, f_2, \ldots, f_{n-1}$ for A so that the unit cube C on basis f_i; i.e.,

$$\sum_i \lambda_i f_i, \ |\lambda_i| \leq 1, \tag{34}$$

is equal to Λ (nonarchimedean case) and comparable to Λ as a convex body (archimedean case).

If $\bar{f}_i$ denotes the projection of f_i to A/k, then $\det B(\bar{f}_i, \bar{f}_j)$ and $\det(\mathrm{tr}(f_i f_j))$ differ by $(2n)^{n-1}$ (see [18, 4.1.3] or §6.2). Let $f_0^*, f_1^*, \ldots, f_{n-1}^* \in A$ be the dual basis to the f_i, that is to say: $\mathrm{tr}(f_i f_j^*) = \delta_{ij}$.

Then the unit cube C^* on basis f_i^* equals Λ^* (nonarchimedean case) and is comparable with Λ^* as a convex body (archimedean case).

On the other hand, $\mathrm{vol}(C)/\mathrm{vol}(C^*) = \det \mathrm{tr}(f_i f_j)$. Our claimed conclusion follows. ☐

9.6. Lemma. *Suppose k is non-archimedean, of the residue characteristic greater than n and that $\mathrm{disc}(\mathscr{D}) = 1$. Then A is unramified (i.e. a sum of unramified field extensions of k) and $\mathscr{O}_{k^n}$ is stable under $\mathscr{O}_A$. In particular, $(\mathscr{O}_{k^n})^{\iota^{-1}} = \lambda \mathscr{O}_A$, for some $\lambda \in A^\times$.*

Proof. Since k is non-archimedean, we have the chain of inclusions

$$\Lambda \subset \mathscr{O}_A \subset \mathscr{O}_A^* \subset \Lambda^*; \tag{35}$$

our assumption and the prior lemma shows that equality holds.

That is to say, $\mathscr{O}_A$ is self-dual with respect to the trace form; this implies that A is unramified over k.

As for the latter statement, $\mathscr{O}_{k^n}$ is stable by Λ by definition, and therefore by $\mathscr{O}_A$ since $\Lambda = \mathscr{O}_A$. It is equivalent to say that $(\mathscr{O}_{k^n})^{\iota^{-1}}$ is stable under $\mathscr{O}_A$, whence the final statement. ☐

Therefore, the quantity $\mathrm{disc}(\mathscr{D})$ measures the *distance of the data $\mathscr{D}$ from the most pleasant situation.*

9.7. Lemma. *If k nonarchimedean, let $\Lambda^\times$ be the units of the order Λ. Then*

$$\frac{\mathrm{vol}_A(\Lambda^\times)}{\mathrm{vol}_A(\mathscr{O}_A^\times)} \geq c(n) \max(1 - n/q, q^{-n}) \left(\frac{\mathrm{disc}(\mathscr{D})}{\mathrm{disc}(A)}\right)^{-1/2}.$$

Here $c(n) = 1$ if the residue characteristic of k exceeds n.

Proof. We have by (35)

$$[\Lambda^* : \mathscr{O}_A^*] = \frac{\mathrm{vol}_A(\Lambda^*)}{\mathrm{vol}_A(\mathscr{O}_A^*)} = [\mathscr{O}_A : \Lambda] = \frac{\mathrm{vol}_A(\mathscr{O}_A)}{\mathrm{vol}_A(\Lambda)};$$

hence

$$\frac{\mathrm{vol}_A(\Lambda)}{\mathrm{vol}_A(\mathscr{O}_A)} \asymp \left(\frac{\mathrm{disc}(\mathscr{D})}{\mathrm{disc}(A)}\right)^{-1/2},$$

where $\asymp$ may be replaced by equality when the residue characteristic is greater than n. On the other hand, our normalizations of measure are so that $\mathrm{vol}_A(\mathscr{O}_A^\times) = \mathrm{vol}_A(\mathscr{O}_A)$.

It remains (cf. (32)) to show

$$\mathrm{vol}_A(\Lambda^\times) \geq \max(q^{-n}, 1 - n/q)\mathrm{vol}_A(\Lambda).$$

That is effected by the following comments:

(1) Let π be any uniformizer of k. Then $1 + \pi\Lambda \subset \Lambda^\times$.

(2) The fraction of elements in Λ which are invertible is $\geq 1 - nq^{-1}$.

Both of these statements would remain valid when Λ is replaced by any suborder of $\mathscr{O}_A$, containing $\mathscr{O}_k$. (1) follows, because one may use the Taylor series expansion of $(1+\pi\lambda)^{-1}$, for $\lambda \in \Lambda$, to invert it.

Note that $\mathscr{O}_A$ has at most n maximal ideals. If $\lambda \in \Lambda$ does not belong to any of these ideal, then it is annihilated by a monic polynomial of degree $\leq n$ with coefficients in $\mathscr{O}_k$ and constant term in $\mathscr{O}_k^\times$. Therefore $\lambda^{-1} \in \Lambda$ also. The volume of each maximal ideal is, as a fraction of the volume of $\mathscr{O}_A$, at most q^{-1}. This shows (2). ☐

9.8. *Local bounds.* For any Schwartz function Ψ on k^n, we shall write Ψ_A for the function $\iota\Psi$ on A, that is to say,

$$\Psi_A \ : \ x \mapsto \Psi(x^\iota).$$

Our final goal is to discuss bounds for integrals of the form

$$\int_{x \in A^\times} \Psi_A(x)\psi(x) d_A^\times x,$$

where ψ is a character of $A^\times$ (we will eventually normalize it to make it independent of ι). We wish to find useful bounds for this, when Ψ is fixed and the data $\mathscr{D}$ is allowed to vary. We will do so in terms of the following norms: let $\check{\Psi}$ to be the Fourier transform of Ψ, with respect to the character e; i.e., $\check{\Psi}(y) = \int_{y \in k^n} \Psi(y)e(xy)dy$. Put

$$\|\Psi\| = \max\left(\int_{k^n} |\Psi(x)| d\mathrm{vol}_k(x), \int_{k^n} |\check{\Psi}(x)| d\widehat{\mathrm{vol}}_k(x)\right).$$

The following proposition presents bounds. The reader should ignore the many constants and focus on the fact that these bounds decay as $\mathrm{disc}(A)$ or $\mathrm{disc}(\mathscr{D})$ increase. In our present language, this is the analogue of the discussion of (25).

9.9. Proposition (Local bounds). *Let ψ be a unitary character of $A^\times$, Ψ a Schwartz function on k^n. Set:*

$$I(\Psi) = |\det h|^{-1/2} \int_{A^\times} \Psi(x^\iota)|x|_A^{1/2} \psi(x) d_A^\times x$$

(note that $|I(\Psi)|$ is independent of the choice of ι). Then:

(1) *We have*

$$|I(\Psi)| \ll_n C_\Psi \cdot C_V \cdot \left(\frac{\mathrm{disc}(\mathscr{D})}{\mathrm{disc}(A)}\right)^{-\frac{1}{16n^2}} \tag{36}$$

where $C_V = \mathrm{vol}(\mathscr{O}_A^\times)$, and moreover, we may take $C_\Psi = 1$ when Ψ is the characteristic function of $\mathscr{O}_{k^n}$.

(2) *We have*

$$|I(\Psi)| \ll_{e,n} C_V \|\Psi\| (\mathrm{disc}(\psi)\mathrm{disc}(A))^{-1/4}, \tag{37}$$

where $C_V = \left(\frac{\mathrm{vol}(\mathscr{O}_A)}{\mathrm{vol}(\mathscr{O}_k)^n}\right)^{1/2}$ (*the notion of* $\mathrm{disc}(\psi)$ *is defined in* §8.2.)

In inequality (36) the implicit constant depends at most on n and the degree[16] *of k over $\mathbb{R}$ or $\mathbb{Q}_p$; in (37) it depends at most on these and on, in addition, the additive character e of k.*

9.10. *Local harmonic analysis and the local discriminant.* We now work on the building of $\mathrm{PGL}_n(k)$ and on $\mathrm{PGL}(A)$. In particular, to simplify notation, if N_1 and N_2 are norms either on A or k^n, $\mathrm{dist}(N_1, N_2)$ refers to the distance between their respective homothety classes, $\mathrm{dist}([N_1], [N_2])$, as defined in Section 7.

Let us recall that the norm N_0, defined by $N_0(x_1, \dots, x_n) = \max |x_i|$, defines a point in the building of $\mathrm{PGL}_n(k)$, and the norm N_A defines a point in the building of $\mathrm{PGL}(A)$.

9.11. Lemma. *Write $A = \oplus_i K_i$ as a direct sum of fields. Let $\|\cdot\|_i = \|\cdot\|_{K_i}$ be the absolute value on K_i extending $|\cdot|$ on k.*[17] *For $t \in A^\times$, set*

$$\|t\| := \max_i \|t_i\|_i / \min_i \|t_i\|_i,$$

where $t = (t_1, \dots, t_i, \dots)$ with $t_i \in K_i^\times$. Then, for any $t \in A^\times$, we have

$$\mathrm{dist}(tN_A, N_A) \geq \frac{1}{2} \log \|t\|.$$

Proof. It is a consequence of the definition of dist and (19).

Let K be any of the fields K_i let N_K be the canonical norm attached to K. It is easy to see that for any $t \in K^\times$,

$$N_K(t^{-1})^{-1} \leq \|t\|_K \leq N_K(t). \tag{38}$$

Given $t = (t_1, \dots, t_i, \dots) \in A^\times$, $t_i \in K_i^\times$. Let $i_{\max}$ and $i_{\min}$ be, respectively, those values of i for which $\|t\|_i$ is maximized and minimized. Let $x_{\max} \in A$ be the element whose $i_{\max}$-th component is 1 and whose other components are 0 and let $x_{\min}$ be the element whose $i_{\min}$-th component is $t_{i_{\min}}^{-1}$ and whose other components are 0. Then by (19) applied to $v_1 = x_{\max}$, $v_2 = x_{\min}$

[16] We have already remarked that the implicit constants in this section may depend on this degree without explicit mention; thus this is not denoted explicitly in (36) or (37).

[17] Thus $\|\cdot\|_i = |\cdot|_{K_i}^{1/[K_i:k]}$, in the case when $k \neq \mathbb{C}$; when $k = \mathbb{C}$ we have simply $K_i = \mathbb{C}$ and the absolute value on both k and K_i is the usual absolute value on $\mathbb{C}$, according to Section 7.1.

and by (17) and (38), one has

$$\begin{aligned}\operatorname{dist}(tN_A, N_A) &\geq \frac{1}{2}\log\left(\frac{N_A(tx_{\max})}{N_A(x_{\max})}\frac{N_A(x_{\min})}{N_A(tx_{\min})}\right)\\ &= \frac{1}{2}\log\bigl(N_{K_{i_{\max}}}(t_{i_{\max}})N_{K_{i_{\min}}}(t_{i_{\min}}^{-1})\bigr)\\ &\geq \frac{1}{2}\log\left(\frac{\|t_{i_{\max}}\|_{i_{\max}}}{\|t_{i_{\min}}\|_{i_{\min}}}\right) = \frac{1}{2}\log\|t\|. \qquad \square\end{aligned}$$

The following lemma shows that the discriminant of local torus data is related to distance-measurements on the building.

9.12. Lemma. *One has the lower bound*

$$\inf_{t\in A^\times}\operatorname{dist}(N_0, t\iota^{-1}N_A) \geq \frac{1}{4n}\log\left(\frac{\operatorname{disc}(\mathscr{D})}{\operatorname{disc}(A)}\right) + O_n(1);$$

here one may ignore the $O_n(1)$ term when k is nonarchimedean and of residue characteristic larger than n.

Proof. The action of $\mathrm{PGL}_n(k)$ on the building is proper and so the infimum is attained. Let t_0 attain the infimum and put $\Delta = \operatorname{dist}(t_0\iota^{-1}N_A, N_0)$.

We are going to use the characterization of $\operatorname{disc}(\mathscr{D})$ from Lemma 9.5.

Adjusting t_0 as necessary by an element of $k^\times$, we may assume:

$$e^{-\Delta}N_0 \leq t_0\iota^{-1}N_A \leq e^{\Delta}N_0. \tag{39}$$

Suppose that $x \in k$ satisfies $|x| = \exp(-2\Delta)$. Such an x exists (cf. (19) and the subsequent comment; Δ is a half-integral multiple of $\log q$ in the non-archimedean case). If $y \in A$ has operator norm ≤ 1 with respect to N_0, then (39) shows that xy has operator norm ≤ 1 with respect to $t_0\iota^{-1}N_A$, and vice versa.

The set of $a \in A$ which have operator norm ≤ 1 with respect to N_0 is exactly Λ.

The set of $a \in A$ which have operator norm ≤ 1 with respect to $t_0\iota^{-1}N_A$ is by definition the set of a such that for all $x \in k^n$

$$t_0\iota^{-1}N_A(xa) \leq t_0\iota^{-1}N_A(x);$$

that is by (31)

$$N_A(x^{\iota^{-1}}at_0) \leq N_A(x^{\iota^{-1}}t_0).$$

Using that $x^{\iota^{-1}}at_0 = x^{\iota^{-1}}t_0a$ and changing $x^{\iota^{-1}}t_0$ to t, we see that this set is the set of $a \in A$ satisfying for any $t \in A$

$$N_A(ta) \leq N_A(t)$$

which is precisely $\mathscr{O}_A$.

We conclude:

$$x\mathscr{O}_A \subset \Lambda \subset \mathscr{O}_A, \quad \mathscr{O}_A^* \subset \Lambda^* \subset x^{-1}\mathscr{O}_A^*.$$

The second equation is obtained by duality from the first; here Λ^* and $\mathscr{O}_A^*$ are dual in the sense of (33). Thereby,

$$\frac{\mathrm{vol}_A(\Lambda^*)}{\mathrm{vol}_A(\Lambda)} \leq \exp(4n\Delta)\frac{\mathrm{vol}_A(\mathscr{O}_A^*)}{\mathrm{vol}_A(\mathscr{O}_A)},$$

whence the result (cf. (32) and Lemma 9.5). □

9.13. Lemma. *Let $R \geq 0$. Then:*

$$\frac{\mathrm{vol}\{t \in A^\times/k^\times : \log\|t\| \in [R, R+1]\}}{\mathrm{vol}_A(\mathscr{O}_A^\times)/\mathrm{vol}_k(\mathscr{O}_k^\times)} \ll_n (1+R)^{n-1}. \tag{40}$$

Here, $\|t\|$ is as in the statement of Lemma 9.11.

Proof. The archimedean case may be verified by direct computation.

Consider k nonarchimedean. We may write $A = \oplus_{i=1}^r K_i$ as a sum of r fields. The map

$$a = \oplus a_i \mapsto \frac{\log\|a_i\|_i}{\log(q)}$$

gives an isomorphism of $A^\times/\mathscr{O}_A^\times$ with a finite index sublattice Q of $\frac{1}{n!}\mathbb{Z}^r$. Moreover, $k^\times/\mathscr{O}_k^\times$ is identified with the sublattice $Q' \subset Q$ generated by $(1, 1, 1, \ldots, 1) \in \mathbb{Z}^r$. Finally, the norm $\log\|t\|$ on $A^\times/k^\times$ descends to a function on Q/Q' and is described explicitly as:

$$(\mu_1, \ldots, \mu_r) \in Q \mapsto \log(q)\left(\max_i \mu_i - \min_i \mu_i\right).$$

The left-hand side of (40) is thereby bounded by:

$$\#\left\{\mu \in \frac{1}{n!}\mathbb{Z}^r/\mathbb{Z} : \left(\max_i \mu_i - \min_i \mu_j\right) \leq \frac{R+1}{\log 2}\right\},$$

which is bounded as indicated, since $r \leq n$. □

9.14. Lemma. *For any $\alpha \in (0, 1)$,*

$$\frac{\int_{t\in A^\times/k^\times} \exp(-\alpha \mathrm{dist}(N_0, t\iota^{-1}N_A))}{\mathrm{vol}_A(\mathscr{O}_A^\times)/\mathrm{vol}_k(\mathscr{O}_k^\times)} \ll_{\alpha,n} \left(\frac{\mathrm{disc}(\mathscr{D})}{\mathrm{disc}(A)}\right)^{-\frac{\alpha}{8n}}. \tag{41}$$

Proof. Let Δ, t_0 be as in Lemma 9.12. Using it, (31), Lemma 9.11, and the triangle inequality:

$$\begin{aligned}\mathrm{dist}(N_0, t\iota^{-1}N_A) &\geq \mathrm{dist}(t\iota^{-1}N_A, t_0\iota^{-1}N_A) - \mathrm{dist}(t_0\iota^{-1}N_A, N_0)\\ &\geq \frac{1}{2}\log(\|t/t_0\|) - \Delta.\end{aligned} \tag{42}$$

To estimate (41), we split the t-integral into regions when $\log \|t/t_0\| \leq [4\Delta]$ and $\log \|t/t_0\| \in [R, R+1]$, where R ranges through integers $\geq [4\Delta]$. Here $[4\Delta]$ is the greatest integer $\leq 4\Delta$. Thereby, the left-hand side of (41) is bounded by

$$C(n)\Bigg((1+\Delta)^n \exp(-\alpha\Delta) + \sum_{R \geq [4\Delta]} \exp(\alpha(\Delta - R/2))(1+R)^{n-1}\Bigg).$$

To conclude, we bound Δ from below using Lemma 9.12. □

9.15. *The action of* $\mathrm{GL}(A)$ *on* $L^2(A)$. Let V comprise $-n/2$-homogeneous functions on A; i.e.,

$$V = \{f : A \to \mathbb{C} : f(\lambda x) = |\lambda|_k^{-n/2} f(x), \lambda \in k, x \in A\}.$$

The group $\mathrm{PGL}(A)$ acts on V, via

$$gf(x) = f(xg)|\det g|_k^{1/2}.$$

The space V possesses a (unique up to scaling) natural $\mathrm{GL}(A)$-invariant inner product. We shall normalize it as follows: for any Schwartz function Φ on A, let $\widetilde{\Phi}$ be its projection to V, defined as

$$\widetilde{\Phi}(x) = \int_{\lambda \in k^\times} \Phi(\lambda x)|\lambda|_k^{n/2} d^\times \lambda.$$

We normalize the inner product $\langle \cdot, \cdot \rangle$ on V so that for any $v \in V$:

$$\langle v, \widetilde{\Phi} \rangle = \int_{x \in A} v(x)\overline{\Phi(x)} d_A x.$$

In particular, let Φ_1, Φ_2 be Schwartz functions on A. We have:

(43)

$$\begin{aligned}
\int_{t \in A^\times/k^\times} \langle t \cdot \widetilde{\Phi_1}, \widetilde{\Phi_2} \rangle &= \int_{t \in A^\times} \int_{x \in A} \Phi_1(xt)|t|_A^{1/2} \overline{\Phi_2(x)} d_A x d_A^\times t \\
&= \Big(\int_{y \in A^\times} \Phi_1(y)|y|_A^{1/2} d_A^\times y\Big) \overline{\Big(\int_{x \in A^\times} \Phi_2(x)|x|_A^{-1/2} d_A x\Big)} \\
&= \zeta_A(1)\Big(\int_{y \in A} \Phi_1(y)|y|_A^{-1/2} d_A y\Big) \overline{\Big(\int_{x \in A} \Phi_2(x)|x|_A^{-1/2} d_A x\Big)}.
\end{aligned}$$

Let us note that we use, in the above reasoning and at various other points in the text, the evident fact that the measure of $A - A^\times$ is zero.

Let $K \subset \mathrm{GL}(A)$ be a maximal compact subgroup, corresponding to the stabilizer of ιN_0, i.e. in the nonarchimedean case, the stabilizer of the lattice $(\mathscr{O}_{k^n})^{\iota^{-1}}$. Let $\Xi_0 : \mathrm{GL}(A) \to \mathbb{C}$ be the Harish-Chandra spherical function with respect to K. For two vectors $v_1, v_2 \in V$, and $\sigma \in \mathrm{GL}(A)$, we have the bound (see [11] and also equation (20)):

$$\begin{aligned}(44)\quad \langle \sigma v_1, v_2\rangle &\leq (\dim K v_1)^{1/2}(\dim K v_2)^{1/2}\Xi_0(\sigma)^{1/n}\|v_1\|_2\|v_2\|_2\\ &\ll_\alpha (\dim K v_1)^{1/2}(\dim K v_2)^{1/2}\exp(-\alpha \mathrm{dist}(\sigma\iota N_0, \iota N_0))\|v_1\|_2\|v_2\|_2\end{aligned}$$

for any $\alpha < 1/n$. Here distances are measured between homothety classes of norms.

9.16. *Proof of the first estimate in Proposition* 9.9. Since ψ is a unitary character, it is sufficient (by taking absolute values) to assume that ψ is trivial and Ψ, nonnegative.

Let Ψ be a nonnegative Schwartz function on k^n; put $\Psi_A := \Psi\circ\iota$, Φ the characteristic function of the unit ball of ιN_0, and $\Phi_2 = 1_{\mathscr{O}_A}$.

These are all Schwartz functions on A. In the nonarchimedean case Φ is the characteristic function of $(\mathscr{O}_{k^n})^{\iota^{-1}}$. Also, the definition of h_A (see (29)) shows that $\Phi_2 = h_A\Phi$. Indeed, $h_A\Phi$ is the characteristic function of

$$\{x\in A : N_0((xh_A)^\iota)\leq 1\} = \{x \in A : N_A(x)\leq 1\} = \mathscr{O}_A.$$

Consequently, $h_A\widetilde{\Phi} = |\det h|_k^{1/2}\widetilde{h_A\Phi} = |\det h|_k^{1/2}\widetilde{\Phi}_2$.

We shall proceed in the case when k nonarchimedean, the archimedean case being similar (the only difference: one needs to decompose Ψ as a sum of K-finite functions in the archimedean case, and the implicit constant will be bounded by a Sobolev norm of Ψ).

Let us observe that there is a constant $C_\Psi \geq 0$, equal to 1 when $\Psi = \Phi$, so that

$$\langle \widetilde{\Psi}_A, \widetilde{\Psi}_A\rangle \leq C_\Psi^2\langle\widetilde{\Phi},\widetilde{\Phi}\rangle.$$

Indeed for some $\lambda_\Psi \in k$, one has $\Psi(x)\leq \|\Psi\|_\infty 1_{\mathscr{O}_{k^n}}(\lambda_\Psi x)$, $x\in k^n$; this bounds Ψ_A in terms of Φ and leads to the above-claimed bound.

For $t\in A^\times/k^\times$,

$$\begin{aligned}(45)\quad |\det h|_k^{1/2}\langle t\widetilde{\Psi_A}, \widetilde{\Phi_2}\rangle &= \langle t\widetilde{\Psi_A}, h_A\widetilde{\Phi}\rangle = \langle h_A^{-1}t\widetilde{\Psi_A},\widetilde{\Phi}\rangle\\ &\ll_\alpha \|\widetilde{\Psi_A}\|_2\|\widetilde{\Phi}\|_2\dim(K\widetilde{\Psi_A})\exp(-\alpha\mathrm{dist}(h_A^{-1}t\iota N_0,\iota N_0)),\quad \alpha<\frac{1}{n}.\end{aligned}$$

We have applied (44) with $v_1 = \widetilde{\Psi_A}, v_2 = \widetilde{\Phi}, \sigma = h_A^{-1}t$; observe that our choice of Φ means $\dim(K\widetilde{\Phi}) = 1$.

We observe, using the definitions (28), (29) and the compatibility (31), that we have $\mathrm{dist}(h_A^{-1}t\iota N_0, \iota N_0) = \mathrm{dist}(N_0, t^{-1}\iota^{-1}N_A)$. To write out every step, this follows from the chain of equalities

$$\begin{aligned}\mathrm{dist}(h_A^{-1}t\iota N_0,\iota N_0) &= \mathrm{dist}(t\iota N_0, h_A\iota N_0) = \mathrm{dist}(t\iota N_0, N_A) = \mathrm{dist}(N_0, \iota^{-1}t^{-1}N_A)\\ &= \mathrm{dist}(N_0, t^{-1}\iota^{-1}N_A) = \mathrm{dist}(tN_0,\iota^{-1}N_A) = \mathrm{dist}(N_0, t^{-1}\iota^{-1}N_A).\end{aligned}$$

We now integrate over $t \in A^\times/k^\times$. By (43) and (45) we have, for any $\alpha < \frac{1}{n}$,

$$(46)\quad \left|\int \Psi(x^\iota)|x|_A^{-1/2} d_A x\right| \left|\int \Phi_2(x)|x|_A^{-1/2} d_A x\right| = \zeta_A(1)^{-1} \int_{A^\times/k^\times} \langle t\widetilde{\Psi}, \widetilde{\Phi}\rangle$$
$$\ll_{n,\alpha} C_\Psi |\det h|_k^{-1/2} (\dim \mathrm{GL}_n(\mathscr{O}_k).\Psi) \|\widetilde{\Phi}\|_2^2 \int_{t\in A^\times/k^\times} e^{-\alpha \mathrm{dist}(N_0, t^{-1}\iota^{-1}N_A)}.$$

We show below that

$$(47)\qquad \frac{\|\widetilde{\Phi}\|_2^2}{\int_{x\in A} \Phi_2(x)|x|_A^{-1/2} d_A x} \ll_n |\det h|_k \mathrm{vol}_k(\mathscr{O}_k^\times).$$

Combining (46) and (47) with Lemma 9.14, we establish the first claim of Proposition 9.9.

To prove (47), proceed as follows: First of all,

$$\int_y \Phi_2(y)|y|_A^{-1/2} d_A y = \int_{\mathscr{O}_A} |y|_A^{-1/2} d_A y = \mathrm{vol}_A(\mathscr{O}_A^\times)\zeta_A(1/2).$$

Noting that $\widetilde{\Phi}(x) = \iota N_0(x)^{-n/2} \int_{|\lambda|_k \le 1} |\lambda|_k^{n/2} d^\times \lambda$, we see:

$$\|\widetilde{\Phi}\|_2^2 = \int_{|\lambda|_k\le 1} |\lambda|_k^{n/2} d^\times\lambda \cdot \int_{\iota N_0(x)\le 1} \iota N_0(x)^{-n/2} d_A x$$
$$= \mathrm{vol}_k(\mathscr{O}_k^\times)\zeta_k(n/2) \int_{\iota N_0(x)\le 1} \iota N_0(x)^{-n/2} d_A x.$$

Noting that $\iota N_0 = h_A^{-1} N_A$, we have

$$\int_{\iota N_0(x)\le 1} \iota N_0(x)^{-n/2} d_A x = \int_{N_A(x h_A^{-1})\le 1} N_A(x h_A^{-1})^{-n/2} d_A x,$$

so that, making the change of variable $x' = x h_A^{-1}$, the previous integral equals

$$|\det h_A|_k \int_{N_A(x)\le 1} N_A(x)^{-n/2} d_A x = |\det h|_k \int_{\mathscr{O}_A} N_A(x)^{-n/2} d_A x.$$

For k nonarchimedean, the last integral equals (π denote an uniformizer of k)

$$\sum_{j\ge 0} |\pi^j|_k^{-n/2} \int_{N_A(x)=|\pi^j|_k} d_A x = \sum_{j\ge 0} |\pi^j|_k^{-n/2} \int_{N_A(x\pi^{-j})=1} d_A x$$
$$= \mathrm{vol}(\{x,\ N_A(x)=1\}) \sum_{j\ge 0} |\pi_k^j|^{n/2}$$
$$= \mathrm{vol}(\{x,\ N_A(x)=1\})\zeta_k(n/2).$$

Combining these, the left-hand side of (47) is bounded by:

$$\ll_n |\det h|_k \mathrm{vol}_k(\mathscr{O}_k^\times) \frac{\mathrm{vol}_A(\{x\in A : N_A(x)=1\})}{\mathrm{vol}_A(\mathscr{O}_A^\times)}.$$

The last ratio is easily seen to be bounded above by $(1 + n/(q-1))$; in particular, it is bounded above in terms of n and the claim (47) follows. $\square$

9.17. *Proof of the second estimate in Proposition 9.9.* For any character ψ of $A^\times$ we set $\psi_s(x) = \psi(x)|x|_A^s$. Let us comment that the notation ψ_s^{-1} always denotes the character $x \mapsto \psi^{-1}(x)|x|_A^s$. In other words, we apply the operation of twisting by $|x|_A^s$ *after* the operation of inverting ψ.

The following result is proved in Tate's thesis. See [43, (3.2.1), (3.2.6.3), (3.4.7)].

9.18. Lemma (Local functional equation). *Let Φ be a Schwartz function on A, and set*

$$\widehat{\Phi}(x) = \int_{y \in A} e_A(xy)\Phi(y) d_A y.$$

Then, for a unitary character ψ of $A^\times$,

$$\epsilon(A, \psi, s, e_A) \frac{\int_{A^\times} \Phi(x)\psi_s(x) d_A^\times x}{L(A, \psi, s)} = \frac{\int_{A^\times} \widehat{\Phi}(x)\psi_{1-s}^{-1}(x) d_A^\times x}{L(A, \psi^{-1}, 1-s)}, \tag{48}$$

where $s \mapsto \epsilon(A, \psi, s, e_A)$ is a holomorphic function of exponential type. More precisely, both sides of (48) are holomorphic, and

(1) *If $\widehat{\mathrm{vol}}_A$ denotes the Haar measure dual to vol_A under the Fourier transform*[18]*, then $|\epsilon(A, \psi, s, e_A)|^2 = \frac{\mathrm{vol}_A}{\widehat{\mathrm{vol}}_A}$ when $\Re(s) = 1/2$.*

(2) *$|\epsilon(A, \psi, s, e_A)| = |\epsilon(A, \psi, 0, e_A)|(\delta(e)^n \mathrm{disc}(\psi)\mathrm{disc}(A))^{-\Re(s)}$, where $\delta(e)$ is a positive constant depending on the additive character e of k.*

We remark that one may take $\delta(e) = 1$ in the unramified case, i.e. when k is nonarchimedean and e is an unramified character of k.

We now proceed to the proof of the second estimate in Proposition 9.9. Recall that ψ is unitary. To ease our notation, we suppose (as we may do, without loss of generality) that $\|\Psi\| = 1$.

For $\Re(s) = 1$, we have:

$$\left| \int_{A^\times} \Psi_A(x)\psi_s(x) d_A^\times x \right| \le \zeta_A(1) \frac{\iota_* \mathrm{vol}_A}{\mathrm{vol}_{k^n}} \ll_n \frac{\iota_* \mathrm{vol}_A}{\mathrm{vol}_{k^n}}. \tag{49}$$

For $\Re(s) = 0$ we apply (48) to reduce to (49), obtaining:

$$\left| \frac{\int_{A^\times} \Psi_A(x)\psi_s(x) d_A^\times x}{L(A, \psi, s)/L(A, \psi^{-1}, 1-s)} \right| \ll_{e,n} (\mathrm{disc}(\psi)\mathrm{disc}(A))^{-1/2}, \tag{50}$$

where the constant implied depend at most on the additive character e of k and on n

[18] With our choice of normalizations, $\widehat{\mathrm{vol}}_A = \mathrm{vol}_A$, but we prefer to phrase the Lemma in a fashion that is independent of choice of measures.

We may now apply the maximum modulus principle to interpolate between (49) and (50). The simplest thing to do would be to apply the maximum modulus principle to the holomorphic quotients that occur in (48). This would be fine for nonarchimedean places; however, for archimedean places, this would run into some annoyances owing to the decay of Γ-factors. We proceed in a slightly different way.

Let $F(s)$ be an analytic function in a neighbourhood of the strip $0 \leq \Re(s) \leq 1$ so that $F(s)L(A,\psi,s)$ is holomorphic. We shall choose $F(s)$ momentarily. Let U be the right-hand side of (49), and V the right-hand side of (50).

Take

$$h(s) := U^{-s}V^{-(1-s)}F(s)\int_{A^\times}\Psi_A(x)\psi_s(x)d_A^\times x.$$

Note that $h(s)$ is holomorphic in $0 \leq \Re(s) \leq 1$. We have

$$|h(s)| \ll_{e,n} \begin{cases} |F(s)|, \Re(s) = 1. \\ \frac{|F(s)L(A,\psi,s)|}{|L(A,\psi^{-1},1-s)|}, \Re(s) = 0. \end{cases}$$

(1) k nonarchimedean. We choose $F(s) = L(A,\psi,s)^{-1}$. Then

$$\sup_{\Re(s)=1}|F(s)| \ll_n 1,$$

whereas $\sup_{\Re(s)=0}\frac{|F(s)L(A,\psi,s)|}{|L(A,\psi^{-1},1-s)|}$ is also bounded by $\zeta_A(1)$.

Therefore, by the maximum modulus principle, one has for $\Re(s) = 1/2$, $|h(s)| \ll_{e,n} 1$. On the other hand, for $\Re(s) = 1/2$, $|F(s)| \geq \zeta_A(1/2)^{-1}$. Therefore, for $\Re(s) = 1/2$,

$$\left|\int_{A^\times}\Psi_A(x)\psi_s(x)d_A^\times x\right| \ll_{n,e} \sqrt{UV}.$$

(2) k archimedean. In explicit terms, $L(A,\psi_s)$ is a product of a finite number of Γ-factors

$$\prod_i \Gamma_{K_i}(s+\nu_i),$$

where $\Re(\nu_i)$ are nonnegative integers, and

$$\Gamma_K(s) = \begin{cases} \pi^{-s/2}\Gamma(s/2) = \Gamma_{\mathbb{R}}(s) & \text{if } K = \mathbb{R} \\ 2(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) & \text{if } K = \mathbb{C}. \end{cases} \tag{51}$$

We take $F(s) = \prod_{\Re(\nu_i)=0}(s+\nu_i)(s+\nu_i-100)^{-1}$. Then $\sup_{\Re(s)=1}|F(s)|$ and $\sup_{\Re(s)=0}\frac{|F(s)L(A,\psi,s)|}{|L(A,\check{\psi},1-s)|}$ are both bounded above by functions of $[A:\mathbb{R}] \leq 2n$. The first is clear; for the second:

$$\frac{|F(s)L(A,\psi,s)|}{|L(A,\check{\psi},1-s)|} = \prod_{\Re(\nu_i)=0}\frac{s+\nu_i}{(s+\nu_i-100)}\frac{\Gamma_{K_i}(s+\nu_i)}{\Gamma_{K_i}(1-s+\overline{\nu}_i)}\cdot\prod_{\Re(\nu_i)\neq 0}\frac{\Gamma_{K_i}(s+\nu_i)}{\Gamma_{K_i}(1-s+\overline{\nu}_i)}.$$

It is not hard to see that the right-hand side is, indeed, bounded above when $\Re(s) = 0$, by a function of $[A : \mathbb{R}]$.

We conclude that, for $\Re(s) = 1/2$, $|h(s)| \ll_{e,n} 1$. But, for $\Re(s) = 1/2$, we also see $|F(s)| \gg_n 1$. We conclude that for $\Re(s) = 1/2$:

$$\left| \int \Psi_A(x) \psi_s(x) d_A^\times x \right| \ll_{n,e} \sqrt{UV}.$$

We have therefore shown that, for $\|\Psi\| = 1$,

$$| \det h|^{-1/2} \left| \int \Psi_A(x) \psi(x) |x|_A^{1/2} d_A^\times(x) \right| \\ \ll_{e,n} | \det h|^{-1/2} \left(\frac{\iota_* \mathrm{vol}_A}{\mathrm{vol}_{k^n}} \right)^{1/2} (\mathrm{disc}(\psi) \mathrm{disc}(A))^{-1/4}.$$

Taking into account (29), we see that the proof of the second assertion of Proposition 9.9 is complete.

10. **Eisenstein series: definitions and torus integrals**

In this section, we define the *Eisenstein series* on GL_n and give a formula (Lemma 10.4) for their integrals over tori. This formula will later be used to derive (6). This section is included merely to make the paper self-contained. Indeed these computations go back to Hecke (see also [50]).

10.1. *Eisenstein series — definition and meromorphic continuation.* We follow the notation of Section 8 throughout this section.

For each place v of F, let Ψ_v be a Schwartz function[19] on F_v^n. We suppose that, for almost all v, the function Ψ_v coincides with the characteristic function of $\mathscr{O}_{F,v}^n$. Let $\Psi := \prod_v \Psi_v$ be the corresponding Schwartz function on $\mathbb{A}^n$.

Put, for $g \in \mathrm{GL}_n(\mathbb{A})$ and $\chi \in \Omega(C_F)$,

$$E_\Psi(\chi, g) = \int_{t \in \mathbb{A}^\times / F^\times} \sum_{\mathbf{v} \in F^n - \{0\}} \Psi(\mathbf{v} t g) \chi(t) d^\times t. \tag{52}$$

The integral is convergent when $\Re\chi$ is sufficiently large. Note that $E_\Psi(\chi, g)$ has central character χ^{-1}.

To avoid conflicting notation, we shall set occasionally, for $s \in \mathbb{C}$

$$E_\Psi(\chi, s, g) := E_\Psi(\chi_s, g) = E_\Psi(\chi | \cdot |_{\mathbb{A}}^s, g).$$

Let $[\cdot, \cdot]$ be a nondegenerate bilinear pairing on F^n. The pairing $[\cdot, \cdot]$ gives also a nondegenerate bilinear pairing $\mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}$.

[19] Recall that this has the usual meaning if v is archimedean, and means: locally constant of compact support, otherwise.

Let g^* be defined so that $[\mathbf{v}_1 g, \mathbf{v}_2 g^*] = [\mathbf{v}_1, \mathbf{v}_2]$. Therefore $|\det g|_{\mathbb{A}}|\det g^*|_{\mathbb{A}} = 1$ for $g \in \mathrm{GL}_n(\mathbb{A})$. The pairing defines a Fourier transform:

$$\widehat{\Psi}(\mathbf{v}^*) = \int_{\mathbb{A}^n} \Psi(\mathbf{v}) e[\mathbf{v}, \mathbf{v}^*] \, d\mathbf{v};$$

in particular, the Fourier transform of $\mathbf{v} \mapsto \Psi(\mathbf{v}tg)$ is

$$\mathbf{v} \mapsto |t|_{\mathbb{A}}^{-n} |\det g|_{\mathbb{A}}^{-1} \widehat{\Psi}(\mathbf{v}t^{-1}g^*).$$

Recall that Poisson summation formula shows that

$$\sum_{\mathbf{v} \in F^n} \Psi(\mathbf{v}) = \sum_{\mathbf{v} \in F^n} \widehat{\Psi}(\mathbf{v}).$$

10.2. Proposition. *$E_\Psi(\chi, g)$ continues to a meromorphic function in the variable χ, with simple poles when $\chi = 1$ and $\chi = |\cdot|_{\mathbb{A}}^n$. One has*

$$\mathrm{res}_{\chi=1} E_\Psi(\chi, g) = -\mathrm{vol}(\mathbb{A}^{(1)}/F^\times)\Psi(0)$$
$$\mathrm{res}_{\chi=|\cdot|_{\mathbb{A}}^n} E_\Psi(\chi, g) = |\det g|_{\mathbb{A}}^{-1} \mathrm{vol}(\mathbb{A}^{(1)}/F^\times) \int_{\mathbb{A}^n} \Psi(x) dx.$$

Moreover,

$$|\det g|_{\mathbb{A}} E_\Psi(\chi, g) = E_{\widehat{\Psi}}(\chi^{-1}, n, g^*).$$

Proof. Split the defining integral $E_\Psi(\chi, g)$ into $|t|_{\mathbb{A}} \le 1$ and $|t|_{\mathbb{A}} \ge 1$. Apply Poisson summation to the former, and then substitute $t \leftarrow t^{-1}$. The result, valid for $\Re\chi \gg 1$, is:

$$\begin{aligned}
(53) \quad & \int_{|t|_{\mathbb{A}} \ge 1} d^\times t \left(\sum_{v \in F^n} \Psi(\mathbf{v}tg) - \Psi(0) \right) \chi(t) \\
& + |\det g^*|_{\mathbb{A}} \int_{|t|_{\mathbb{A}} \ge 1} d^\times t \left(\sum_{\mathbf{v} \in F^n} \widehat{\Psi}(\mathbf{v}tg^*) - \widehat{\Psi}(0) \right) \chi_n^{-1}(t) \\
& - \Psi(0) \int_{|t|_{\mathbb{A}} \ge 1} \chi(t) d^\times t + \widehat{\Psi}(0) \int_{|t|_{\mathbb{A}} \le 1} \chi_{-n}(t) d^\times t.
\end{aligned}$$

The last two terms can be explicitly evaluated. If χ is of the form $|x|_{\mathbb{A}}^s$, then they are equal to $-\frac{\Psi(0)}{s}\mathrm{vol}(\mathbb{A}^{(1)}/F^\times)$ and $|\det g^*|_{\mathbb{A}} \frac{\int \Psi}{s-n} \mathrm{vol}(\mathbb{A}^{(1)}/F^\times)$, respectively; otherwise, they are identically zero. The former two terms define holomorphic functions of χ. □

10.3. *Torus integrals of Eisenstein series.* Put $\Psi_K = \Psi(x^\iota g_{\mathscr{D}})$, a function on $\mathbb{A}_K$. Let us recall we have fixed global torus data $\mathscr{D} = (K \subset M_n(F), g_{\mathscr{D}} \in \mathrm{GL}_n(\mathbb{A}))$.

10.4. Lemma (Integration of Eisenstein series over a torus). *Let $\psi \in \Omega(C_K)$ be so that $\chi = \psi|_F$. Then:*

$$\int_{\mathbf{T}_K(F)\backslash\mathbf{T}_K(\mathbb{A})} E_\Psi(\chi, tg_{\mathscr{D}})\psi(t)dt = \int_{\mathbb{A}_K^\times} \Psi_K(y)\psi(y)d^\times y. \tag{54}$$

Let us observe that, owing to the restriction $\chi = \psi|_F$, the map $t \mapsto E_\Psi(\chi, tg)\psi(t)$ indeed defines a function on $\mathbf{T}_K(F)\backslash\mathbf{T}(\mathbb{A})$.

The integral on the right-hand side can be expressed as a product over places of K; for almost all places, the resulting (local) integral equals an L-function. This is explicitly carried out in (62). Thus, the lemma indeed gives the reduction of torus integrals of Eisenstein series to L-functions, as discussed in Section 2.

Proof. We unfold.

$$\begin{aligned}
&\int_{\mathbf{T}_K(F)\backslash\mathbf{T}(\mathbb{A})} \psi(t)E_\Psi(\chi, tg_{\mathscr{D}})dt \\
&\quad= \int_{u\in\mathbb{A}^\times/F^\times} du \int_{t\in\mathbf{T}_K(F)\backslash\mathbf{T}_K(\mathbb{A})} dt\psi(t) \sum_{\mathbf{x}\in F^n-\{0\}} \Psi^{g_{\mathscr{D}}}(\mathbf{x}.u.t)\chi(u) \\
&\quad= \int_{t\in(\mathbb{A}_K^\times/K^\times)/(\mathbb{A}^\times/F^\times)} \int_{u\in\mathbb{A}^\times/F^\times} \sum_{x\in K^\times} \Psi((x.u.t)^\iota g_{\mathscr{D}})\chi(u)\psi(t)dudt \\
&\quad= \int_{t\in\mathbb{A}_K^\times/K^\times} \sum_{x\in K^\times} \Psi_K(x.t)\psi(t)d^\times t = \int_{t\in\mathbb{A}_K^\times} \Psi_K(t)\psi(t)d^\times t.
\end{aligned} \tag{55}$$

$\square$

10.5. Lemma (Class number formula). *The measure of $\mathbf{T}_K(F)\backslash\mathbf{T}_K(\mathbb{A})$ equals*

$$n\frac{\mathrm{Res}_{s=1}\zeta_K(s)}{\mathrm{vol}(\mathbb{A}^{(1)}/F^\times)}.$$

Proof. We set $g = 1$, $\psi = |\cdot|_{\mathbb{A}_K}^s, \chi = |\cdot|_{\mathbb{A}}^{ns}$ and take residues in (54) as $s \to 1$.

We first remark that, for almost all v, the local integral $\int_{K_v^\times} \Psi_{K,v}(t)|t|_v^s d^\times t$ equals the local zeta function $\zeta_{K,v}(s) := \prod_{w|v} \zeta_{K_w}(s)$. Taking residues yields:

$$\frac{\mathrm{vol}(\mathbf{T}_K(F)\backslash\mathbf{T}_K(\mathbb{A}))\cdot\mathrm{vol}(\mathbb{A}^{(1)}/F^\times)}{n}\cdot\int_{\mathbb{A}^n}\Psi = \mathrm{Res}_{s=1}\zeta_K(s).\prod_v \frac{\int_{K_v^\times}\Psi_{K,v}(t)|t|_v d^\times t}{\zeta_{K,v}(1)},$$

where almost all factors in the infinite product are identically 1. The result follows from the choice of the measure. $\square$

11. Eisenstein series: estimates

Let us explain by reference to Section 2.7.2 the contents of this section:

In Section 10 we have established the general form of (6). We are going to assume known a subconvexity bound (57), which one can see as a generalization of the bound (7)) from the introduction. The results of Section 9 in effect establish the analog of (8).

Putting these together, we shall obtain in the present section — Proposition 11.3 — a slightly disguised form of (5). This disguised form is translated to a more familiar S-arithmetic context in the next section.

11.1. *Assumed subconvexity.* Our result makes the assumption of a certain *sub-convexity estimate.* In order to state what this means, we need to recall the notion of archimedean conductor. For a character ω of a archimedean local field k, we define the archimedean conductor

$$C(\omega) = \prod_i (1 + |\nu_i|), \tag{56}$$

where the $\nu_i \in \mathbb{C}$ are so that $L(\omega, s) = \prod_i \Gamma_{\mathbb{R}}(s + \nu_i)$, say. (See (51) for definitions of $\Gamma_{\mathbb{R}}$.) If ω is unitary, then $\Re(\nu_i) \in \frac{1}{2}\mathbb{N}$. For $\chi \in \Omega(C_F)$, and v archimedean, we put $C_v(\chi)$ to be the archimedean conductor of $\chi|_{F_v}$, and let $C_\infty(\chi) = \prod_{v|\infty} C_v(\chi)$. Similarly, one defines $C_\infty(\psi)$ for $\psi \in \Omega(C_K)$.

Given a unitary character $\psi \in \Omega(C_K)$, we shall assume known the following bound

$$|L(K, s, \psi)| \ll C_\infty(\psi_s)^N \mathrm{disc}(K)^{1/4-\theta} \mathrm{disc}(\psi)^{1/4-\theta}, \Re(s) = 1/2 \tag{57}$$

for some constants N, $\theta > 0$ which depend only on F and $n = [K : F]$.

The validity of (57) is a consequence of the generalized Riemann hypothesis. The generality in which (57) is known unconditionally is fairly slim, but it is enough for some applications. For a recollection of what is known unconditionally, see Appendix A.

11.2. *The main estimate.* In this section, we use notation as in Section 8. We regard F, n as fixed throughout; thus we allow implicit constants $\ll$ to depend both on n and F. In particular, any discriminants depending only on F, e.g. $\mathrm{disc}_v(F)$, will often be incorporated into $\ll$ notation.

For typographical simplicity, we write $D_\psi, D_{\mathscr{D}}, D_K, D_F$ in place of $\mathrm{disc}(\psi)$, $\mathrm{disc}(\mathscr{D})$, etc. in the following Proposition.

11.3. Proposition. *Let $\mathscr{D}$ be global torus data, given by $K \subset M_n(F)$ and $g_{\mathscr{D}} \in \mathbb{A}_K^\times \backslash \mathrm{GL}_n(\mathbb{A})$. Let Ψ be a Schwartz function on $\mathbb{A}^n$ and $\psi \in \Omega(C_K)$ a normalized unitary character, $\chi = \psi|_F$.*

Suppose known (57). *There exists* $\beta > 0$, *depending on* n *and the exponent* θ *of* (57), *so that*: *for* $\Re(s) = n/2$ *and any* $M \geq 1$, *one has*

$$\left|\frac{|\det g_{\mathscr{D}}|_{\mathbb{A}}^{s/n}}{\mathrm{vol}(\mathbf{T}_K(F)\backslash\mathbf{T}_K(\mathbb{A}))}\int_{\mathbf{T}_K(F)\backslash\mathbf{T}_K(\mathbb{A})}\psi_{s/n}(t)E_\Psi(\chi,s,tg)dt\right| \ll_{\Psi,M} \frac{C_\infty(\psi_{s/n})^N}{C_\infty(\chi_s)^M}D_\psi^{-\beta}D_{\mathscr{D}}^{-\beta}. \tag{58}$$

The main point of this is the decay in $D_\psi, D_{\mathscr{D}}$; the reader should ignore the various factors of C_∞, which are a technical matter. In words, (58) asserts that varying sequence of homogeneous toral sets, on GL_n, become equidistributed "as far as Eisenstein series are concerned," if we suppose the pertinent subconvexity hypothesis.

In terms of the discussion of the introduction, it is (58) that proves (5).[20]

Proof. Let us begin by clarifying volume normalizations. As in Section 8, we fix an identification $\iota : K \to F^n$ which is an isomorphism for the right K-module structures (see §8.3).

Define, for each v, local torus data A_v and identifications ι_v according to the discussion around (22). It is important to keep in mind that A_v is *not* $(K \otimes F_v)$ but rather its conjugate by g_v; similarly ι_v is *not* simply "$\iota \otimes_F F_v$" but is rather twisted by g_v.

The discussion of (28) yields elements $h_v \in \mathrm{GL}_n(F_v)$; e.g., for v nonarchimedean, we have: $\mathscr{O}_{F_v}^n h_v^{-1} = (\mathscr{O}_{A_v})^{\iota_v}$.

Observe that

$$\prod_v |\det h_v|_v |\det g_{\mathscr{D},v}|_v \asymp (\mathrm{disc}(K/F))^{1/2}. \tag{59}$$

To verify (59), note that $\iota : K \to F^n$ induces

$$\iota_{\mathbb{A}} : \mathbb{A}_K = K \otimes_F \mathbb{A} \to \mathbb{A}^n.$$

This identification is measure-preserving, because, with our choice of measures, both $\mathbb{A}/F$ and $\mathbb{A}_K/K$ have measure 1. In view of the definition (22) of ι_v, this remark implies $\prod_v |\det g_{\mathscr{D},v}|_v \frac{\iota_{v*}\mathrm{vol}_{A_v}}{\mathrm{vol}_{F_v^n}} = 1$. Therefore, taking product of (29) over all places v,

$$\prod_v \frac{\mathrm{vol}(\mathscr{O}_{K_v})}{\mathrm{vol}(\mathscr{O}_{F_v}^n)} = \prod_v |\det g_{\mathscr{D},v}|_v^{-1} |\det h_v|_v^{-1}$$

which, in combination with (23), yields our claim. (Recall that $\mathscr{O}_{F,v}$ and $\mathscr{O}_{K,v}$ are defined, at archimedean places, by the unit balls for suitable norms; cf. §9.3).

[20](58) also delivers uniformity in the ψ-variable; e.g., Section 1.6.2 would use this aspect.

Without loss of generality $\Psi = \Psi_\infty \times \prod_v \Psi_v$ where, for each finite place v, Ψ_v is the characteristic function of an $\mathscr{O}_v$-lattice in F_v^n. (In the general case, one may express Ψ as a sum of such, the implicit cost being absorbed into the $\ll_\Psi$.)

Let B be the union of the following sets of places:

(1) B_∞: v is archimedean.
(2) B_{ram}: $\mathrm{disc}_v(\psi)\mathrm{disc}_v(\mathscr{D}) > 1$, or F_v is ramified over $\mathbb{Q}$, or the residue field at v has size $\leq n$.
(3) B_Ψ: Ψ_v does not coincide with the characteristic function of $\mathscr{O}_{F,v}^n$.

Let us note that

$$\exp(|B|) \ll_{F,\Psi} (D_\psi D_{\mathscr{D}})^\varepsilon, \tag{60}$$

this being a consequence of the fact that the number of prime factors of an integer N is $o(\log N)$.

We denote by $L^{(B)}$ an L-function with the omission of those places inside B. Suppose $\Re(s) = n/2$. In view of (54),

$$|\det g|_{\mathbb{A}}^{1/2} \left| \int_{\mathbf{T}_K(F)\backslash \mathbf{T}_K(\mathbb{A})} E_\Psi(\chi, s, tg)\psi_{s/n}(t) \right|$$

factors as:

$$\begin{aligned} &\prod_v |\det g_{\mathscr{D},v}|_v^{1/2} \left| \int_{K_v} \Psi_{K,v}(t)\psi_{s/n}(t)d^\times t \right| \\ &\quad = |L^{(B)}(K, \psi_{s/n})| \\ &\qquad \cdot \prod_{v\notin B} |\det g_{\mathscr{D},v} \det h_v|_v^{1/2} \left| \prod_{v\in B} |\det g_{\mathscr{D},v}|_v^{1/2} \int_{K_v} \Psi_{K,v}(t)\psi_{s/n}(t)d^\times t \right| \\ &\quad = |L^{(B)}(K, \psi_{s/n})| \\ &\qquad \cdot \prod_v |\det g_{\mathscr{D},v} \det h_v|_v^{1/2} \left| \prod_{v\in B} |\det h_v|_v^{-1/2} \int_{K_v} \Psi_{K,v}(t)\psi_{s/n}(t)d^\times t \right|. \end{aligned} \tag{61}$$

Here $\Psi_{K,v}(x) := \Psi_v(x^\iota g_{\mathscr{D},v})$, so that $\prod_v \Psi_{K,v} = \Psi_K$. Moreover, we have used the following evaluation for $v \notin B$: For such v, Ψ_v is the characteristic function of $\mathscr{O}_{F,v}^n$, and Lemma 9.6 implies that $(\mathscr{O}_{F,v}^n)^{\iota_v^{-1}} = \lambda_1 \mathscr{O}_{A_v,v}$ (some $\lambda_1 \in A_v^\times$); because of the definition (22) of ι_v, this means that there is $\lambda \in K_v^\times$ with $(\lambda\mathscr{O}_{K,v})^\iota g_{\mathscr{D},v} = \mathscr{O}_{F,v}^n$. Thus $x \mapsto \Psi_v(x^\iota g_{\mathscr{D},v})$ coincides with the characteristic function of $\lambda\mathscr{O}_{K,v}$. Because, by assumption, both ψ and K/F are unramified at such v,

$$|\det g_{\mathscr{D},v}|_v^{s/n} \left| \int \Psi_{K,v}(t)\psi_{s/n}(t)d^\times t \right| = |L_v(K, \psi, s/n)||\lambda \det(g_{\mathscr{D},v})|_v^{s/n}. \tag{62}$$

By (22) and the discussion preceding (29), $(\mathscr{O}_{K,v})^\iota g_{\mathscr{D},v} = (\mathscr{O}_{A_v})^{\iota_v} = \mathscr{O}_{F_v}^n h_v^{-1}$. Comparing this with $(\lambda\mathscr{O}_{K,v})^\iota g_{\mathscr{D},v} = \mathscr{O}_{F,v}^n$, we deduce that $|\lambda_v|_v = |\det(h_v)|_v$.

The left-hand side of (62) thereby has the same absolute value as

$$|\det g_{\mathscr{D},v} \det h_v|_v^{s/n} L_v(K, \psi_{s/n}),$$

concluding our justification of (61).

From the assumed subconvexity bound (57), together with (60),

$$|L^{(B)}(K, \psi_{s/n})| \ll_\Psi C_\infty(\psi_{s/n})^N (D_K D_\psi)^{1/4-\theta}, \quad \Re(s) = n/2. \tag{63}$$

Proposition 9.9, with our measure normalizations, and after identifying a K_v-integral to an A_v-integral in the obvious way, shows for arbitrary $M \geq 1$:

$$\left| \prod_{v \in B} |\det h_v|_v^{-1/2} \int_{K_v} \Psi_{K,v}(t) \psi_{s/n}(t) d^\times t \right| \tag{64}$$
$$\ll_{M,\Psi,\varepsilon} (D_\psi D_{\mathscr{D}})^\varepsilon \begin{cases} C_\infty(\chi_s)^{-M} D_K^{-1/4} (D_\psi D_K)^{-1/4}, \\ D_K^{-1/2} (D_{\mathscr{D}}/D_K)^{-\frac{1}{16n^2}}. \end{cases}$$

The factor $C_\infty(\chi_s)^{-M}$ we have interposed on the right-hand side amounts to "integrating by parts" at archimedean places, before applying Proposition 9.9; it will be useful for convergence purposes later. Indeed, suppose v is archimedean. The integral of (64) is unchanged if we replace $\Psi_v(x)$ by $\psi_{s/n}(\lambda)\Psi_v(\lambda x)$, for $\lambda \in F_v^\times$. On the other hand, $\psi|_F = \chi$, so $\psi_{s/n}(\lambda) = \chi_v(\lambda)|\lambda|_v^s$ ($\lambda \in F_v^\times$). Thus if ν is any probability measure on $F_v^\times$, the integral of (64) is unchanged by the substitution:

$$\Psi_v \mapsto \Psi_v', \Psi_v'(x) = \int_\lambda \Psi_v(\lambda x) \chi_s(\lambda) d\nu(\lambda).$$

Now compare $\|\Psi_v'\|$ and $\|\Psi_v\|$, the norm $\|\cdot\|$ being defined as in the the second statement of Proposition 9.9. An elementary computation shows that, for a smooth measure ν, we must have:

$$\|\Psi_v'\| \ll_{\Psi_v} \mathrm{Cond}_v(\chi_s)^{-M}. \tag{65}$$

Let us note that the implicit constants here depend on higher derivatives of Ψ_v; this is permissible.[21]

[21] In explicit terms, (65) for v real amounts to a bound of the type:

$$\int_{x \in \mathbb{R}} \left| \int_{1/2 \lesssim \lambda \lesssim 2} f(\lambda x) |\lambda|^{it} d\lambda \right| \ll_f (1 + |t|)^{-M},$$

for a Schwartz function f, which is easily verified by integration by parts.

Combining (63), (64) and (59), we see that for $\Re(s) = n/2$:

$$\left| |\det g_{\mathscr{D}}|_{\mathbb{A}}^{1/2} \int_{\mathbf{T}_K(F)\backslash\mathbf{T}_K(\mathbb{A})} E_\Psi(\chi, s, tg)\psi_{s/n}(t) d^\times t \right| \ll_{M,\Psi,\varepsilon} C_\infty(\psi_{s/n})^N (D_\psi D_{\mathscr{D}})^\varepsilon \begin{cases} C_\infty(\chi_s)^{-M} D_K^{-\theta} D_\psi^{-\theta}, \\ D_\psi^{1/4} D_K^{-\theta} (D_{\mathscr{D}}/D_K)^{-\frac{1}{16n^2}}. \end{cases} \tag{66}$$

Pick $0 < p < 1$. Using the obvious fact $\min(U, V) \leq U^p V^{1-p}$, we may replace the right-hand side (ignoring ε-exponents) by:

$$\begin{aligned} &C_\infty(\psi_{s/n})^N C_\infty(\chi_s)^{-pM} D_K^{-\theta} D_\psi^{-\theta p + \frac{1-p}{4}} (D_{\mathscr{D}}/D_K)^{-\frac{1-p}{16n^2}} \\ &\qquad \leq C_\infty(\psi_{s/n})^N C_\infty(\chi_s)^{-pM} D_{\mathscr{D}}^{-a} D_\psi^{-b}, \end{aligned} \tag{67}$$

where $a = \min(\frac{1-p}{16n^2}, \theta), b = \theta p - \frac{1-p}{4}$. For p sufficiently close to 1, these are all positive. Making M arbitrarily large, and using Lemma 10.5 together with bounds for Dedekind ζ-functions, yields the desired conclusion.

12. The reaping: *a priori* bounds

In this section, we translate Proposition 11.3 into a form that very explicitly generalizes (5).

The result is Proposition 12.5. The work has already been done; this section simply translates between adelic and S-arithmetic.

We begin by explicating the connection of the Eisenstein series E_Ψ with the classical "Siegel-Eisenstein" series, in the case when the base field is $\mathbb{Q}$. We then carry out the analogue in an S-arithmetic setting over an arbitrary base field F.

12.1. *Explications over* $\mathbb{Q}$. The (Siegel)-Eisenstein series on the quotient $\mathrm{PGL}_n(\mathbb{Z})\backslash\mathrm{PGL}_n(\mathbb{R})$ often appears in the following guise. Let f be a Schwartz function on $\mathbb{R}^n$. To each $L \in \mathrm{PGL}_n(\mathbb{Z})\backslash\mathrm{PGL}_n(\mathbb{R})$ thought of as a lattice $L \subset \mathbb{R}^n$ of covolume 1, we associate the number

$$E_f(L) := \sum_{v \in L - \{0\}} f(v). \tag{68}$$

We shall explicate the connection of this construction with the Eisenstein series that we defined previously.

Specialize to the case $F = \mathbb{Q}, \chi = |\cdot|_{\mathbb{A}}^s$. We take Ψ_v to coincide with the characteristic function of $\mathbb{Z}_v^n$ for all finite v, and $\Psi_\infty = f$. Define f_s on $\mathbb{R}^n$ via the rule

$$f_s(\mathbf{v}) = \int_{t \in \mathbb{R}^\times} |t|^s f(\mathbf{v}t) d^\times t.$$

Then f_s satisfies the transformation property $f_s(\lambda \mathbf{v}) = |\lambda|^{-s} f_s(\mathbf{v})$. Taking into account the fact that the natural map $\mathbb{R}_{>0} \times \prod_p \mathbb{Z}_p^\times \to \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$ is a homeomorphism, we see that for $g_\infty \in \mathrm{GL}_n(\mathbb{R})$,

$$|\det g_\infty|^{s/n} E_\Psi(|\cdot|^s, g_\infty) = |\det g_\infty|^{s/n} \sum_{\mathbf{v} \in \mathbb{Z}^n . g_\infty} f_s(\mathbf{v}).$$

Note that, by Mellin inversion, $f = \int_s f_s ds$, where the s-integration is taken over a line of the form $\Re(s) = \sigma \gg 1$. Consequently, the Siegel-Eisenstein series (68) corresponds to the function on $\mathrm{PGL}_n(\mathbb{Q})\backslash \mathrm{PGL}_n(\mathbb{A})$ defined by

$$g \mapsto \int_{\Re(s)=\sigma\gg 1} |\det g|_\mathbb{A}^{s/n} E_\Psi(s, g) ds.$$

12.2. *S-arithmetic setup.* We revert to the general setting of a number field F. We shall henceforth pass from an adelic setup, to an S-arithmetic setup.

Fix, therefore, a finite set of places S, containing all archimedean ones. Set $F_S = \prod_{v\in S} F_v$. We assume that S is chosen so large that

$$\mathbb{A}^\times = F^\times F_S^\times \prod_{v\notin S} \mathscr{O}_{F,v}^\times.$$

Under these assumptions, we may identify $\mathbb{A}^\times/F^\times \prod_{v\notin S} \mathscr{O}_{F,v}^\times$ to $F_S^\times/\mathscr{O}^\times$, where $\mathscr{O} := F \cap \prod_{v\notin S} \mathscr{O}_{F,v}$. By Dirichlet's theorem, the quotient of $\mathscr{O}$ by roots of unity comprises a free abelian group of rank $|S| - 1$.

Similarly, making use of the strong approximation theorem for the group SL_n, we can identify

$$\mathrm{PGL}_n(F)\backslash \mathrm{PGL}_n(\mathbb{A})/ \prod_{v\notin S} \mathrm{PGL}_n(\mathscr{O}_{F,v})$$

to the quotient

$$\mathrm{PGL}_n(\mathscr{O})\backslash \mathrm{PGL}_n(F_S).$$

If μ is a measure on $\mathrm{PGL}_n(F)\backslash \mathrm{PGL}_n(\mathbb{A})$, then we shall often abuse notation and identify μ with the projected measure on $\mathrm{PGL}_n(\mathscr{O})\backslash \mathrm{PGL}_n(F_S)$.

12.3. *The S-arithmetic Eisenstein series.* It will take us a little work to unravel the Eisenstein series into a form which we can easily use in the S-arithmetic case. There will be some complications arising from the failure of strong approximation for PGL_n.

Let $F_S^{(1)}$ consist of those elements of $F_S^\times$ with $|x| = 1$. Then $F_S^\times$ can be identified with $F_S^{(1)} \times \mathbb{R}_{>0}$. In this way, we can identify a character of the compact group $F_S^{(1)}/\mathscr{O}^\times$, to a character of $F_S^\times/\mathscr{O}^\times$: by extending trivially on $\mathbb{R}_{>0}$. Thus the group of all normalized characters of C_F, unramified away from S, is identified to the character group of $F_S^{(1)}/\mathscr{O}^\times$.

We have already normalized measures on $F_S^\times$. We normalize the measure on $F_S^{(1)}$ by differentiating along fibers of the map $F_S^\times \to \mathbb{R}_{>0}$ given by $x \mapsto |x|$; here we equip $\mathbb{R}_{>0}$ with the measure $\frac{d\lambda}{\lambda}$. The measure of $F_S^{(1)}/\mathscr{O}^\times$ then equals the measure of $\mathbb{A}^{(1)}/F^\times$.

We now construct the S-arithmetic version of the Eisenstein series. Let Ψ_S is a Schwartz function on F_S; put $\Psi = \Psi_S \times \prod_{v \notin S} 1_{\mathscr{O}_v^n}$. Let χ be a character of $F_S^{(1)}/\mathscr{O}^\times$, identified, via the remarks above, with a character of $\mathbb{A}^\times/F^\times$.

Set

$$\overline{E}_\Psi(\chi, g) = \int_{\Re(s)=2n} E_\Psi(\chi, s, g)|\det g|_{\mathbb{A}}^{s/n} \frac{ds}{2\pi i}.$$

Then, for $g \in \mathrm{GL}_n(F_S)$, $E_\Psi(\chi, g)$ equals:

$$\begin{aligned}(69) \quad &\int_{t\in\mathbb{A}^\times/F^\times:|t|_{\mathbb{A}}^n=|\det g|_{\mathbb{A}}} \chi^{-1}(t) \sum_{\mathbf{v}\in F^n-\{0\}} \Psi(\mathbf{v}t^{-1}g) \\ &= \int_{t\in F_S^\times/\mathscr{O}^\times:|t|_{F_S}^n=|\det g|_{F_S}} \chi^{-1}(t) \sum_{\mathbf{v}\in\mathscr{O}^n-\{0\}} \Psi_S(\mathbf{v}t^{-1}g).\end{aligned}$$

In the first expression, the t-integral is taken with respect to the measure that is transported from the measure on $\mathbb{A}^{(1)}/F^\times$; in the second expression, the t-integral here is taken over a compact abelian Lie group of dimension $|S|-1$, with respect to the measure previously discussed.

Fix $a \in F_S^{(1)}$ and put, for $g \in \mathrm{GL}_n(F_S)$,

$$\begin{aligned}(70) \quad \overline{E}_{\Psi,a}(g) &= \frac{1}{\mathrm{vol}(F_S^{(1)}/\mathscr{O}^\times)} \sum_{\chi:F_S^{(1)}/\mathscr{O}^\times\to S^1} \chi(a\det g)\overline{E}_\Psi(\chi^n, g) \\ &= \sum_{\substack{t^n=a\det g\\ t\in F_S^\times/\mathscr{O}^\times}} \sum_{\mathbf{v}\in\mathscr{O}^n-\{0\}} \Psi_S(\mathbf{v}t^{-1}g).\end{aligned}$$

Indeed, the χ-sum restricts to those $t \in F_S^\times/\mathscr{O}^\times$ so that t^n and $a\det g$ differ by an element of $\mathbb{R}_{>0}$; however, since $|t^n|_{F_S} = |a\det g|_{F_S}$, this forces $t^n = a\det g$. The t-sum is finite, for the quotient $F_S^{(1)}/\mathscr{O}^\times$ is a compact abelian Lie group.

The function $\overline{E}_{\Psi,a}(g)$ defines a function on $\mathrm{PGL}_n(\mathscr{O})\backslash\mathrm{PGL}_n(F_S)$; it is the S-arithmetic version of our degenerate Eisenstein series.

12.4. *Bounding the mass of $\overline{E}_{\Psi,a}$.* In order to bound the $\mu_{\mathscr{D}}$-measures of functions of the type $\overline{E}_{\Psi,a}$, via Proposition 11.3, we require a subconvex bound for the L-functions $L(K,\psi)$, when ψ is pulled back from a fixed character of F via the norm map, i.e. we require the following estimate for $\Re(s) = 1/2$:

$$(71) \quad |L(K, \chi\circ\mathrm{N}_{K/F}, s)| \ll (C_\infty(\chi_s)D_\chi)^N D_K^{1/4-\eta}, \ \chi \in \Omega(C_F) \text{ unitary}$$

for some constants $N, \eta > 0$ which depend at most on F and $n = [K : F]$. The quantity $C_\infty(\chi)$ is defined as in (56). Notice that unlike (57), we do not require in (71) a subconvex bound in the "D_χ-aspect".

The bound (71) is known in more cases than (57). For instance, it is known[22] when $F = \mathbb{Q}$ and $[K : \mathbb{Q}] \leq 3$ (cf. Appendix A).

12.5. Proposition. *Suppose (71) is known. Then, for homogeneous toral data $\mathscr{D}$,*

$$\left| \mu_{\mathscr{D}}(\overline{E}_{\Psi,a}) - \delta \int_{\mathbb{A}_F^n} \Psi \right| \ll_\Psi \mathrm{disc}(\mathscr{D})^{-\beta} \tag{72}$$

for some $\beta > 0$ — depending on n and the exponent η of (71), and where $\delta = \delta_a \in \mathbb{Q}$ belongs to a finite set of rational numbers, depending on S, F and n.

The δ arise from the "connected components." We could be a little more precise about their value, but there is no point. This result implies the generalization, to an arbitrary base field F and an S-arithmetic setting, of (5), discussed in the introduction.

Proof. The data $\mathscr{D}$ is defined by a field $K \subset M_n(F)$ and an element $g_{\mathscr{D}} \in \mathrm{GL}_n(\mathbb{A})$.

Recall

$$\begin{aligned} \overline{E}_\Psi(\chi, g) &= \int_{\Re(s) \gg 1} E_\Psi(\chi, s, g) |\det g|_{\mathbb{A}}^{s/n} \frac{ds}{2\pi i}, \\ \overline{E}_{\Psi,a}(g) &= \frac{1}{\mathrm{vol}(F_S^{(1)}/\mathscr{O}^\times)} \sum_{\chi : F_S^{(1)}/\mathscr{O}^\times \to S^1} \chi(a \det g) \overline{E}_\Psi(\chi^n, g). \end{aligned} \tag{73}$$

As we have already commented, we are going to identify $\mu_{\mathscr{D}}$ with its projection to $\mathrm{PGL}_n(\mathscr{O})\backslash\mathrm{PGL}_n(F_S)$. This being so, let us consider $\mu_{\mathscr{D}}(\overline{E}_{\Psi,a})$. Shift contours to $\Re(s) = n/2$ in the defining integrals and apply the bounds of Proposition 11.3. (There are no concerns with convergence; the support of $\mu_{\mathscr{D}}$ is compact.) The function $E_\Psi(\chi^n, s, g)$ has a pole (by Proposition 10.2) precisely when χ^n is the trivial character and $s \in \{0, n\}$; moreover, Proposition 10.2 computes the residue in those cases.

The result is

$$\mu_{\mathscr{D}}(\overline{E}_{\Psi,a}) = \sum_{\substack{\chi : F_S^{(1)}/\mathscr{O}^\times \to S^1 \\ \chi^n = 1}} \mu_{\mathscr{D}}(\chi(a \det g)) \int_{\mathbb{A}^n} \Psi(x) dx + \mathrm{Error} \tag{74}$$

[22] even in the D_χ-aspect

(note that $g \mapsto \chi(a \det g)$ indeed defines a function on $\mathrm{PGL}_n(\mathscr{O})\backslash\mathrm{PGL}_n(F_S)$), and

$$
\text{(75)}\quad |\mathrm{Error}| \ll \frac{1}{\mathrm{vol}(F_S^{(1)}/\mathscr{O}^\times)} \times \sum_{\chi: F_S^{(1)}/\mathscr{O}^\times \to S^1} \max_{\Re(s)=1/2} (1+|s|)^2 \left| \mu_{\mathscr{D}} \left(E_\Psi(\chi^n, s, \cdot)\chi_{s/n} \circ \det \right) \right|.
$$

For χ as above, put $\psi := \chi \circ N_{K/F}$, a character of C_K. We have

$$
\begin{aligned}
&|\mu_{\mathscr{D}} \left(E_\Psi(\chi^n, s, \cdot)\chi_{s/n} \circ \det \right)| \\
&\qquad = |\det g_{\mathscr{D}}|_{\mathbb{A}}^{s/n} \left| \frac{\int_{\mathbf{T}_K(F)\backslash\mathbf{T}_K(\mathbb{A})} \psi_{s/n}(t) E_\Psi(\chi^n, s, tg_{\mathscr{D}})dt}{\mathrm{vol}(\mathbf{T}_K(F)\backslash\mathbf{T}_K(\mathbb{A}))} \right|.
\end{aligned}
$$

By (the proof of) Proposition 11.3, we have under (71) for any $M > 1$ and some $\beta, N > 0$

$$
|\mu_{\mathscr{D}} \left(E_\Psi(\chi^n, s, \cdot)\chi_{s/n} \circ \det \right)| \ll_{M,\Psi} (C_\infty(\chi_s^n))^{-M} D_\chi^N D_{\mathscr{D}}^{-\beta}.
$$

We have utilized the notation $\chi_s^n := (\chi^n)_s$, the character $x \mapsto \chi(x)^n |x|^s$. We have used the fact that, in the present context, $C_\infty(\chi_s^n)$ and $C_\infty(\psi_{s/n})$ are bounded within powers of each other.

Taking M large enough, we obtain the following bound:[23]

$$
\mathrm{Error} \ll_\Psi D_{\mathscr{D}}^{-\beta}.
$$

Let us now analyze the right-hand side of (74). The set of elements of C_F of the form $\mathrm{N}_{K/F}(x).\det g_{\mathscr{D}}$ (for some $x \in C_K$) is a coset of a subgroup of C_F of finite index. It projects to a coset of a subgroup of $F_S^\times/\mathscr{O}^\times$ which contains the nth powers, which we may identify with a coset of a subgroup of the finite group $F_S^\times/\mathscr{O}^\times(F_S^\times)^n$. Call this subgroup Q and the pertinent coset bQ. If $\chi : F_S^\times/\mathscr{O}^\times \to S^1$ is so that $\chi^n = 1$, then

$$
\mu_{\mathscr{D}}(\chi(a \det g)) = \frac{1}{|Q|} \sum_{x \in Q} \chi(abx).
$$

Therefore, the coefficient of $\int \Psi$, on the right-hand side of (74), is given by:

$$
\sum_{\chi^n = 1} \frac{1}{|Q|} \sum_{x \in Q} \chi(abx).
$$

In particular, this lies in a finite set of rational numbers. □

[23]The number of possibilities for $\chi|_{F_\infty}$ is, in general, infinite. However, if χ contributes nontrivially to Error, then D_χ is bounded above depending on Ψ. Moreover, the number of such χs with $\inf_{\Re(s)=1/2} C_\infty(\chi_s^n) \leq T$ is bounded polynomially in T.

13. Proof of Theorem 4.9 and Theorem 4.8

Let F be a number field, and let $\mathscr{D}_i$ be a sequence of homogeneous toral data on PGL_n over F. Let Y_i be the associated homogeneous toral sets, and $\mu_i = \mu_{\mathscr{D}_i}$ the corresponding probability measures. Recall that $\mathscr{D}_i$ is defined by a torus $\mathbf{T}_i \subset \mathrm{PGL}_n$ together with $g_i \in \mathrm{PGL}_n(\mathbb{A})$. Let K_i be the corresponding (degree n) field extensions. See Section 6.

When convenient we may drop the subscript i, referring simply to $\mathscr{D}$, $\mu_{\mathscr{D}}$, K, $\mathbf{T}$, etc.

13.1. *Bounds on the mass of small balls and the cusp.* Fix a set of representatives $1 = a_1, \dots, a_r \in F_S^{(1)}$ for $F_S^\times/(F_S^\times)^n \mathscr{O}^\times$. Such representatives may indeed be chosen in $F_S^{(1)}$.

Take $x \in \mathrm{PGL}_n(\mathscr{O})\backslash\mathrm{PGL}_n(F_S)$; we say that a lattice in F_S^n (i.e. an $\mathscr{O}$-submodule of rank n) *corresponds to* x if it is of the form $\mathscr{O}^n.g_0 t_0^{-1}$ where g_0 is a representative for x in $\mathrm{GL}_n(F_S)$; and $t_0 \in F_S^\times/\mathscr{O}^\times$ is so that $t_0^n = \det(g_0)a_i$, some $1 \le i \le r$. There are only finitely many lattices corresponding to a given x.

We say a set $\mathcal{Q}$ in F_S^n is nice if there exists a Schwartz function Ψ_S on F_S^n so that $\Psi_S \ge 1$ on $\mathcal{Q}$ with $\int \Psi_S \le 2\mathrm{vol}(\mathcal{Q})$.

13.2. Lemma. *Let $\mathcal{Q} \subset F_S^n$ be nice; set $\mathcal{L}_{\mathcal{Q}} \subset \mathrm{PGL}_n(\mathscr{O})\backslash\mathrm{PGL}_n(F_S)$ to comprise those $x \in \mathrm{PGL}_n(\mathscr{O})\backslash\mathrm{PGL}_n(F_S)$ so that a lattice corresponding to x contains an element of $\mathcal{Q}$.*

If (71) is known, then

$$\mu_{\mathscr{D}}(\mathcal{L}_{\mathcal{Q}}) \ll_{F,S,n} \mathrm{vol}(\mathcal{Q}) + O_{\mathcal{Q}}(\mathrm{disc}(\mathscr{D})^{-\beta}). \tag{76}$$

Proof. Indeed, choose Ψ_S as remarked. As is clear from (70), the function $\sum_{i=1}^r \overline{E}_{\Psi,a_i}$ dominates the characteristic function of $\mathcal{L}_{\mathcal{Q}}$. The result is a consequence of Proposition 12.5. □

Let $\underline{\epsilon} = (\epsilon_v)_{v\in S}$ be a choice of $\epsilon_v \in (0,1)$ for each $v \in S$. Set $\|\underline{\epsilon}\| = \prod_{v\in S} |\epsilon_v|_v$. For each $v \in S$, let $B_v(\epsilon_v)$ be an ϵ_v-neighbourhood of the identity in $\mathrm{PGL}_n(F_v)$. Here, we equip $\mathrm{PGL}_n(F_v)$ with the metric that arises from the adjoint embedding $\mathrm{PGL}_n \hookrightarrow M_{n^2}$; and we equip M_{n^2} with the metric that arises from norm: supremum of all matrix entries.

Let

$$B_S(\underline{\epsilon}) = \prod_{v\in S} B_v(\epsilon_v) \subset \mathrm{PGL}_n(F_S).$$

The following is a consequence of Lemma 13.2, for a suitable choice of $\mathcal{Q}$; we leave the details to the reader.

13.3. Lemma (Bounds for the mass of small balls). *Suppose* (71) *is known. Let* $x_0 \in \mathrm{PGL}_n(\mathscr{O})\backslash\mathrm{PGL}_n(F_S)$. *Then*

$$\mu_{\mathscr{D}}(x_0 B_S(\underline{\epsilon})) \ll_{F,S,n} \|\underline{\epsilon}\|^n + O_{\underline{\epsilon}}(\mathrm{disc}(\mathscr{D})^{-\beta}).$$

Moreover, the implicit constant in $O_{\underline{\epsilon}}(\dots)$ *is bounded uniformly when* x_0 *belongs to any fixed compact.*

Now let $N_{0,v}$ be the standard norm on F_v^n (cf. §7). For $g \in \mathrm{GL}_n(\mathbb{A})$, we set

$$\mathrm{ht}(g)^{-1} = |\det(g)|_{\mathbb{A}}^{-1/n} \inf_{\lambda \in F^n - \{0\}} \prod_v N_{0,v}(\lambda g_v).$$

Let $K_{\max} = \prod_v \mathrm{Stabilizer}(N_{0,v})$ be the standard maximal compact subgroup of $\mathrm{GL}_n(\mathbb{A})$, and let $\bar{K}_{\max}$ be its image in PGL_n. Then ht descends to a proper map from $\mathrm{PGL}_n(F)\backslash\mathrm{PGL}_n(\mathbb{A})/\bar{K}_{\max}$ to $\mathbb{R}_{>0}$. In particular, sets of the type $\mathrm{ht}^{-1}\big([R,\infty)\big)$, for large $R > 0$, define "the cusp."

The next result is again a consequence of Lemma 13.2.

13.4. Lemma (Bounds for the cusp). *Suppose* (71) *is known.*

$$\mu_{\mathscr{D}}(\mathrm{ht}^{-1}[R,\infty)) \ll R^{-n} + O_R(\mathrm{disc}(\mathscr{D})^{-\beta}).$$

13.5. *Proof of Theorem* 4.8. The volume of the homogeneous toral set associated to $\mathscr{D}$ is defined in (13). We take the set Ω_0 to be the product $\prod_{v|\infty} \Omega_v \times \prod_{v \text{ finite}} \mathrm{PGL}_n(\mathscr{O}_{F,v})$. Here, we set $\Omega_v \subset \mathrm{PGL}_n(F_v)$, for v archimedean, to equal the image, in PGL_n, of $\exp(E_v)$; here

$$E_v := \{Y \in M_n(F_v) : Y \text{ has operator norm} \le \tfrac{1}{10}\},$$

and the operator norm is taken with respect to the canonical norm on F_v^n. Let us note that $\exp : E_v \to \exp(E_v)$ is a diffeomorphism onto its image, being inverted by log.

We claim that

$$\log \mathrm{vol}(Y) = \sum_v \log \mathrm{vol}\{t \in \mathbf{T}(F_v) : g_v^{-1} t g_v \in \Omega_0\}^{-1} + o(\log \mathrm{disc}(\mathscr{D})), \tag{77}$$

where, on the right-hand side, the measure on $\mathbf{T}(F_v)$ is normalized as indicated in Section 8; and we understand $\mathbf{T}(F_v)$ as being embedded in $\mathbf{T}(\mathbb{A})$ in the natural way.

To verify (77) we need to consider our measure normalizations. In the definition of "vol(Y)", in Section 4.3, we endowed $\mathbf{T}(F)\backslash\mathbf{T}(\mathbb{A})$ with a probability measure. If we normalize the measures on $\mathbf{T}(F_v)$ according to Section 8, the product measure is not a probability measure on $\mathbf{T}(F)\backslash\mathbf{T}(\mathbb{A})$; its mass is given by Lemma 10.5 to be a certain ζ-value. By a result of Siegel, $\log|\zeta_K(1)| = o(\log \mathrm{disc} K)$, and, by Lemma 9.6, $\mathrm{disc}(K) \ll \mathrm{disc}(\mathscr{D})$. This establishes (77).

Let v be a finite place. Let us recall that we defined an order $\Lambda_v \subset g_v^{-1} K_v g_v$ via $\Lambda_v = g_v^{-1} K_v g_v \cap M_n(\mathscr{O}_{F,v})$ (see Section 9.4). Therefore, $\Lambda_v^\times = g_v^{-1} K_v g_v \cap$

$\mathrm{GL}_n(\mathscr{O}_{F,v})$. Thus $\{t \in \mathbf{T}(F_v) : g_v^{-1} t g_v \in \Omega_0\}$ is identified, via $t \mapsto g_v^{-1} t g_v$, to $\Lambda_v^\times F_v^\times / F_v^\times$. Since $\Lambda_v \cap F_v = \mathscr{O}_{F,v}$, we see, given the measure normalizations of Section 8, that $\mathrm{vol}\{t \in \mathbf{T}(F_v) : g_v^{-1} t g_v \in \Omega_0\} = \mathrm{vol}(\Lambda_v^\times)/\mathrm{vol}(\mathscr{O}_{F,v}^\times)$. Taking into account Lemmas 9.5, 9.7 and (23), this becomes:

$$\log \mathrm{vol}\{t \in \mathbf{T}(F_v) : g_v^{-1} t g_v \in \Omega_0\} = \frac{1}{2} \log \mathrm{disc}(\mathscr{D}_v) + O_F(1), \quad v \text{ finite.} \tag{78}$$

Here the error term $o_F(1)$ is identically zero if the residue characteristic of v is larger than n, F is unramified at v, and $\mathrm{disc}_v(\mathscr{D}_v) = 1$.

Now we establish an (approximate) analog of (78) at archimedean places. We claim that, for archimedean v,

$$\begin{aligned} \log \mathrm{vol}\{t \in \mathbf{T}(F_v) : g_v^{-1} t g_v \in \Omega_0\} &= \log \mathrm{vol}(\Lambda_v) + O_F(1) \\ &= \frac{1}{2} \log \mathrm{disc}_v(\mathscr{D}_v) + O_F(1). \end{aligned} \tag{79}$$

The second equality follows readily from Lemma 9.5, so we need to verify the first equality. Put $A_v = g_v^{-1}(K \otimes F_v) g_v \subset M_n(F_v)$. Because A_v is a *subalgebra*, we have $\exp(A_v) \subset A_v$, and $\log(A_v) \subset A_v$ when log is defined. Therefore,

$$A_v \cap \exp(E_v) = \exp(A_v \cap E_v).$$

The set $\{t \in \mathbf{T}(F_v) : g_v^{-1} t g_v \in \Omega_0\}$ is identified, via $t \mapsto g_v^{-1} t g_v$, to $F_v^\times \exp(A_v \cap E_v)/F_v^\times$. Its measure is therefore easily seen to be bounded above and below by constant multiples by the A_v-measure of $\exp(A_v \cap E_v).\{F_v^\times \cap \exp(E_v)\}$. This set contains $\exp(A_v \cap E_v)$ and is contained in $\exp(A_v \cap 2E_v)$. The measure of $\exp(A_v \cap E_v)$ and $\exp(A_v \cap 2E_v)$ are bounded above and below by constant multiples of the volume of Λ_v by constants, for the map $\exp : A_v \to A_v^\times$ preserves (up to a constant) measure. This establishes (79).

Combining (77), (78) and (79), we see that

$$\log \mathrm{vol}(Y) = \frac{1}{2} \sum_v \log \mathrm{disc}(\mathscr{D}_v) + o_F(\log \mathrm{disc} \mathscr{D}).$$

The conclusion of Theorem 4.8 follows. □

13.6. *Proof of Theorem* 4.9. Let v be a place as indicated in the proof of Theorem 4.9. Let S be a finite set of places of F as in Section 12.2; enlarging S, we may suppose $v \in S$ without loss of generality.

Let $H_i = g_{i,v}^{-1} \mathbf{T}_i(\mathbb{Q}_v) g_{i,v}$. The measure $\mu_i := \mu_{\mathscr{D}_i}$, upon projection to $\mathrm{PGL}_3(\mathscr{O}) \backslash \mathrm{PGL}_3(F_S)$, is invariant under H_i. Denote by $\bar{\mu}_i$ this projection.

Let $\bar{\mu}_\infty$ be any weak* limit of the measures $\bar{\mu}_{\mathscr{D}_i}$, which we may assume is the projection of a limit μ_∞ of the original sequence. Because the bounds of Lemma 13.4 are uniform in $\mathscr{D}_i$, the measure $\bar{\mu}_\infty$ is a probability measure.

It will suffice to show that $\bar{\mu}_\infty$ is $\mathrm{SL}_3(F_v)$-invariant. In fact, it then follows that $\bar{\mu}_\infty$ is $\mathrm{SL}_3(F_S)$-invariant by irreducibility of the lattice $\mathrm{PGL}_3(\mathscr{O}) \subset \mathrm{PGL}_3(F_S)$ and, S being arbitrary, that μ_∞ is $\mathrm{SL}_3(\mathbb{A})$-invariant. Once μ_∞

is $\mathrm{SL}_3(\mathbb{A})$-invariant, it is determined by its projection to the compact group $\mathbb{A}_F^\times/F^\times(\mathbb{A}_F^\times)^3$. We are reduced to showing that limits of homogeneous measures on a compact abelian group are of the same type, which is easy.

We use the following observation, which is a consequence of Lemma 13.2: Let $\mathbf{P} \subset \mathrm{PGL}_3$ be the stabilizer of a line in F^3; let $\mathbf{P}'$ be the stabilizer of a plane in F^3. Thus $\mathbf{P}, \mathbf{P}'$ are F-parabolic subgroups.

Let $Z \subset \mathbf{P}(F_w)\backslash\mathrm{PGL}_3(F_w), Z' \subset \mathbf{P}'(F_w)\backslash\mathrm{PGL}_3(F_w)$ be the F_w-points of varieties of dimension ≤ 1. Then:

$$\bar{\mu}_\infty((\mathbf{P}(\mathscr{O})\backslash\mathbf{P}(F_S)).Z) = 0,\ \bar{\mu}_\infty(\mathbf{P}'(\mathscr{O})\backslash\mathbf{P}'(F_S).Z') = 0. \tag{80}$$

Indeed, the first assertion of (80) follows directly from Lemma 13.2, taking for $\mathcal{Q}$ a suitable sequence of sets. The second assertion may be deduced from the first by applying the outer automorphism (transpose-inverse) of PGL_3 to the entire situation.

Case 1. Suppose $\mathrm{disc}_v(\mathscr{D}_i) \to \infty$. Let $\mathfrak{h}_i = \mathrm{Lie}(H_i)$; let $\mathfrak{h}_\infty$ be any limit of $\mathfrak{h}_i$ inside the Grassmannian of $\mathfrak{pgl}_3$. It is a two-dimensional commutative Lie subalgebra. The measure $\bar{\mu}_\infty$ is invariant by $\exp(\mathfrak{h}_\infty)$. Necessarily $\mathfrak{h}_\infty$ contains a nilpotent element.

Identify $\mathfrak{pgl}_3$ with trace-free 3×3 matrices. There are two conjugacy classes of nontrivial nilpotents in $\mathfrak{pgl}_3$ according to the two possible Jordan blocks. Suppose that $\mathfrak{h}_\infty$ contains a conjugate of $\left(\begin{smallmatrix} 0&1&0\\0&0&1\\0&0&0\end{smallmatrix}\right)$ (i.e. a generic nilpotent element). Then since the centralizer of this generic element is two-dimensional, it follows in this case by commutativity of $\mathfrak{h}_\infty$ that $\mathfrak{h}_\infty$ is this centralizer. That is, $\mathfrak{h}_\infty$ contains in any case a conjugate $\mathfrak{n}$ of the Lie algebra spanned by $\left(\begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right)$.

We make the following observation: Suppose $\mathfrak{j}$ is a proper Lie subalgebra of $\mathfrak{sl}_3$ containing $\mathfrak{n}$. Then $\mathfrak{j}$ is reducible over $\bar{F}_v^3$, i.e. fixes a line or a plane over the algebraic closure. Indeed, the only proper Lie subalgebra of $\mathfrak{sl}_3$ that acts irreducibly over the algebraic closure is $\mathfrak{so}_q$, for q a nondegenerate quadratic form. But $\mathfrak{so}_q$ does not contain any conjugate of $\mathfrak{n}$.

By [35] and [39], $\bar{\mu}_\infty$ may be expressed as a convex linear combination of Haar probability measures μ_ι on closed orbits $x_\iota H_\iota$ with ι belonging to some probability space I; here H_ι is a closed subgroup of $\mathrm{PGL}_3(F_S)$. Moreover, all the measures μ_ι are ergodic under the action of $N = \exp(\mathfrak{n})$.

Suppose $\bar{\mu}_\infty$ is not $\mathrm{SL}_3(F_v)$-invariant. Then for a positive proportion of the ι, say for $\iota \in I'$, the subgroup H_ι does not contain $\mathrm{SL}_3(F_v)$. Therefore, $\bar{\mu}_\infty$ dominates the convex combination:

$$\bar{\mu}_\infty \geq \int_{I'} \mu_\iota.$$

Fix some $\iota \in I'$ and let $x_\iota = \mathrm{PGL}_3(\mathscr{O})g$, where we may assume that x_ι has dense orbit in $x_\iota H_\iota$ under the action of N. We claim:

(81) There exists a proper F-subgroup $\mathbf{J} \subset \mathrm{PGL}_3$

so that $x_\iota H_\iota \subset (\mathbf{J}(\mathscr{O})\backslash \mathbf{J}(F_S)).g.$

For the proof of the claim, consider first H_ι as a subgroup of a product of real and p-adic Lie groups: if $\bar{S}$ is the set of places of $\mathbb{Q}$ below S, then we consider $\mathrm{PGL}_3(F_S) = \prod_{p\in\bar{S}} \prod_{w|p, w\in S} \mathrm{PGL}_3(F_w)$. It may be seen that, in a neighbourhood of the identity H_ι is itself a product of real and p-adic subgroups. We *define* the Lie algebra $\mathfrak{h}$ of H_ι to be the product of the real Lie algebra and the various p-adic ones; this is, by definition, a $\mathbb{Q}_{\bar{S}}$-submodule of $\oplus_{w\in S}\mathfrak{pgl}_3(F_w)$. (Here, and in what follows, we use $\mathfrak{pgl}_3(k)$ to denote the k-points of the vector space $\mathfrak{pgl}_3$.)

In general, if the map $S \to \bar{S}$ is not bijective, $\mathfrak{h}$ may not be a direct sum of its projections to the $\mathfrak{pgl}_3(F_w)$. We claim, however, that the projection of $\mathfrak{h}$ to $\mathfrak{pgl}_3(F_v)$ is a proper subalgebra. Indeed, $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{h}\cap\mathfrak{pgl}_3(F_v)$; it follows that all conjugates of $\mathfrak{n}$ by elements of H_ι again belong to $\mathfrak{h}\cap\mathfrak{pgl}_3(F_v)$. Were the projection of $\mathfrak{h}$ to $\mathfrak{pgl}_3(F_v)$ surjective, it would follow – by the simplicity of $\mathfrak{pgl}_3(F_v)$ as a module over itself — that $\mathfrak{pgl}_3(F_v)$ is contained in $\mathfrak{h}$; contradiction.

Next let $\mathbf{J}'$ be the Zariski closure of $gH_\iota g^{-1}\cap\mathrm{PGL}_3(\mathscr{O})$; by definition, this is an F-algebraic subgroup of PGL_3. $\mathbf{J}'$ preserves the projection of $\mathrm{Ad}(g)\mathfrak{h}$ to $\mathfrak{pgl}_3(F_v)$, and is therefore a proper subgroup of PGL_3.

Just as in the Borel density theorem it follows that gNg^{-1} is contained in $\mathbf{J}'(F_v)$. In fact, by Chevalley's theorem there is an algebraic representation ϕ of PGL_3 on V and a vector $v_\phi \in V_\phi$ such that $\mathbf{J}'$ is the stabilizer of the line generated by v_ϕ. Fix some parametrization n_t of N as a one-parameter unipotent group. Note that the line spanned by $\phi(gn_tg^{-1})(v_\phi)$ approaches the line spanned by an eigenvector v_N of $\phi(gNg^{-1})$ if $|t|\to\infty$. By our assumption on x_ι we have a sequence $t_k \in F_v$ with $|t_k|_v \to \infty$ for which $x_\iota n_{t_k} \to x_\iota$ as $k\to\infty$. Therefore, there exists some sequence $\gamma_k \in \mathrm{PGL}_3(\mathscr{O})\cap gH_\iota g^{-1}$ and $g_k \in \mathrm{PGL}_3(F_v)$ with $gn_{t_k} = \gamma_k g g_k$ and g_k approaching the identity. This implies that $\phi(gn_{t_k}^{-1}g^{-1})(v_\phi) = \phi(gg_k^{-1}g^{-1}\gamma_k^{-1})(v_\phi)$ both approaches v_N and v_ϕ, i.e. that $v_N = V_\phi$ and so $gNg^{-1} \subset \mathbf{J}'(F_v)$.

To prove (81), we proceed as follows. Let $\mathbf{J}'' \subset \mathbf{J}'$ be the preimage, in $\mathbf{J}'$, of the commutator subgroup of $\mathbf{J}'/R_u(\mathbf{J}')$. Since $\mathbf{J}''$ is F-algebraic without F-characters, it follows that $\mathrm{PGL}_3(\mathscr{O})\mathbf{J}''(F_S)$ is closed. Therefore, the same holds for $\mathrm{PGL}_3(\mathscr{O})\mathbf{J}''(F_S)g$ which is invariant under N. We see that $\mathrm{PGL}_3(\mathscr{O})\mathbf{J}''(F_S)g$ contains $x_\iota H_\iota$ by our choice of x_ι. We can take $\mathbf{J} := \mathbf{J}''$.

Next we claim that $\mathbf{J}$ is contained in an F-parabolic subgroup $\mathbf{P}$. For this we need to show that $\mathbf{J}$ preserves a line in F^3 or a line in the dual $(F^3)^*$. Indeed, as noted before $\operatorname{Lie}\mathbf{J} \otimes_F \bar{F}_v$ preserves a line or a dual line in $\bar{F}_v^3$, and so $\operatorname{Lie}\mathbf{J}$ preserves a line or a dual line over the algebraic closure. If the Galois conjugates of this line are not contained in a plane, then $\mathbf{J}$ would be a torus, contradicting the fact it contains unipotents. Otherwise, the Galois conjugates of the line span either a line or a plane; this yields a preserved line[24] in F^3 or in $(F^3)^*$.

Therefore,

$$\mathrm{PGL}_3(\mathscr{O})\mathbf{J}(F_S)g \subset \mathrm{PGL}_3(\mathscr{O})\mathbf{P}(F_S)g,$$

for some F-parabolic subgroup $\mathbf{P}$, the stabilizer of a line or a dual line.

Moreover, we know that $\mathfrak{n} \subset \operatorname{Ad}(g^{-1})\mathfrak{p}$ where $\mathfrak{p}$ is the Lie algebra of $\mathbf{P}$. Thus the fixed line (or dual line) for the parabolic $\operatorname{Ad}(g_v^{-1})\mathbf{P}(F_v)$ is also preserved by $\mathfrak{n}$ acting on F^3 or $(F^3)^*$, i.e. belongs to the kernel of $\mathfrak{n}$. The kernel has dimension 2, and so we see that the coset $\mathbf{P}.g_v$ belongs to a one-dimensional subvariety of $\mathbf{P}(F_v)\backslash\mathrm{PGL}_3(F_v)$.

Applying (80) and noting that there are only countably many F-parabolic subgroups, we derive a contradiction.

Case 2. There exists a place v so that the associated tori $\mathbf{T}_i$ are all F_v-split. We may assume that $\operatorname{disc}_v(\mathscr{D}_i)$ remain bounded. The subgroups H_i then remain in a compact set within the space of tori in $\mathrm{PGL}_3(\mathbb{Q}_v)$. Let H be any limit of the subgroups H_i. Then $\bar{\mu}_\infty$ is H-invariant and H is an F_v-split torus inside $\mathrm{PGL}_3(F_v)$.

By Lemma 13.3, every H-ergodic component of $\bar{\mu}_\infty$ has positive entropy with respect to the action of a regular element in H. It follows by [16, Th. 2.6] (which generalizes [15] to the S-algebraic setting) that $\bar{\mu}_\infty$ is $\mathrm{SL}_3(F_v)$-invariant. □

Appendix A. Recollections on subconvexity

In this section, we are going to briefly recall the *subconvexity problem* for L-functions and some of the progress towards it. We refer to [27] for a more complete description of the subconvexity problem.

Let $L(\pi, s)$ denote an L-function attached to some arithmetic object π (of degree $n \geq 1$),

$$L(\pi, s) = \prod_p L(\pi_p, s) = \prod_p \prod_{i=1}^n (1 - \alpha_{\pi,i}(p)p^{-s})^{-1}.$$

$L(\pi, s)$ is expected to have meromorphic continuation to $\mathbb{C}$ with (under an appropriate normalization) finitely many poles located on the lines $\Re s = 0, 1$. It satisfies a functional equation of the form

$$q_\pi^{s/2} L(\pi_\infty, s)L(\pi, s) = \omega(\pi)q_\pi^{(1-s)/2}\overline{L(\pi_\infty, 1-\bar{s})L(\pi, 1-\bar{s})}.$$

[24]Implicitly, we use Hilbert's Theorem 90.

Here

$$L(\pi_\infty, s) = \prod_{i=1}^{n} \Gamma_{\mathbb{R}}(s + \mu_{\pi,i}),\ \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2),$$

$q_\pi \geq 1$ is an integer (the conductor of π) and $|\omega(\pi)| = 1$.

A subconvex bound (in the conductor aspect), is a bound of the form

$$L(\pi, s) \ll (C_\infty(\pi_s))^N q_\pi^{1/4-\theta} \tag{82}$$

for some absolute constants $N > 0$ and $\theta > 0$. Here we denote by $C_\infty(\pi_s)$ the quantity

$$C_\infty(\pi_s) = \prod_{i=1}^{d}(1 + |\mu_{\pi,i} + s|). \tag{83}$$

The bound (82) is named subconvex by comparison with the (easier) *convexity* bound — which may be deduced from the Phragmén-Lindelöf convexity principle — in which the exponent $1/4 - \delta$ is replaced by any exponent $> 1/4$.

In this paper the main class of L-functions for which we consider the subconvexity problem are the *Dedekind* ζ-function of a number field K: the *Dedekind ζ-function* of K is a function of a single complex variable. For $\Re(s) > 1$ it is defined by the rule

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathscr{O}_K} \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})^{-s},$$

the sum being taken over the nonzero ideals of $\mathscr{O}_K$. It extends to a meromorphic function of s with a simple pole at $s = 1$. In that case the conductor of ζ_K is the (absolute value of the) discriminant of K:

A.1. Hypothesis. *Let K be a number field of fixed degree n. There exists $\theta, N > 0$ (depending at most on n) such that for $\Re s = 1/2$,*

$$\zeta_K(s) \ll_n |s|^N \mathrm{disc}(K)^{1/4-\theta}.$$

By now Hypothesis A.1 is known for a restriced class of number fields K:

- when K is an abelian extensions of $\mathbb{Q}$ of fixed degree (n say); this follows from the Kronecker-Weber theorem and from Burgess's subconvex bound for Dirichlet L-functions [6]. More generally, Hypothesis 5.1 holds if K varies through the abelian extensions of a *fixed* number field F by the work of fourth named author [46].
- when K is a cubic extension of $\mathbb{Q}$: when K is not abelian, $\zeta_K(s)$ factors as $\zeta(s)L(\rho, s)$ where ρ is a dihedral (two-dimensional) irreducible complex Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; more precisely $L(\rho, s)$ is the L-function of a cubic *ring class* character of the unique quadratic extension contained in the closure of K. By quadratic base change,

$L(\rho, s)$ is the L-function of a $\mathrm{GL}_{2,\mathbb{Q}}$-automorphic form and the subconvex bound for the latter class of L-functions follows from the works of Duke, Friedlander, Iwaniec [14] and Blomer, Harcos and the third author [3]. By the work of the third and fourth named authors [37], this now holds when K is a cubic extension of a *fixed* number field F.
– More generally, by the above quoted works, Hypothesis A.1 is known if $K/\mathbb{Q}$ is contained in a ring class field of an arbitrary quadratic extension of an arbitrary ground field F.

We also need to consider the L-function, $L(K, \psi, s)$, associated to a Hecke Grössencharacter of K ψ (in other words a character of the idèles of K, $\mathbb{A}_K^\times/K^\times$). The conductor of $L(K, \psi, s)$ is the product of $\mathrm{disc}(K)$ and the "discriminant of ψ." (Usually, the conductor of ψ is defined as a certain integral ideal of K; the norm of this ideal is the discriminant of ψ.)

A.2. Hypothesis. *Let K be a number field of degree n and ψ a unitary character of the idèles of K. Then there exists $\theta, N > 0$ (depending at most on n) so that for $\Re s = 1/2$,*

$$L(K, \psi, s) \ll_n C_\infty(\psi, s)^N (\mathrm{disc}(\psi)\mathrm{disc}(K))^{1/4-\theta},$$

where $\mathrm{disc}(\psi)$ *denote the conductor of ψ.*

Hypothesis A.2 is known in even fewer cases:

– when K is an *fixed* number field and ψ is varying : this is a consequence of Burgess work if $K = \mathbb{Q}$ and of [46] in general.
– when K is a (possibly varying) quadratic extension of the base field F: again this follows (by quadratic base change) from the works [14], [3] and [37].
– when K/F is an extension of given degree which is either, abelian, cubic or contained in a ring class field of an arbitrary quadratic extension of F and ψ *factors through the norm map*: that is $\psi = \chi \circ N_{K/F}$ for some Hecke character (over F). In that case $L(K, \psi, s)$ (viewed as an L-function "over" F) equals the twist of $\zeta_K(s)$ by the character χ and the subconvex bound follows from a combination of the above quoted works.

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