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All automorphisms of the Calkin algebra are inner

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Dedicated to my wife Tatiana Velasevic and Dr. Carl J. Vaughan and Dr. Leonard N. Girardi of New York-Presbyterian Hospital. Without them I would not be around to prove Theorem 1.

Abstract

We prove that it is relatively consistent with the usual axioms of mathematics that all automorphisms of the Calkin algebra are inner. Together with a 2006 Phillips-Weaver construction of an outer automorphism using the Continuum Hypothesis, this gives a complete solution to a 1977 problem of Brown-Douglas-Fillmore. We also give a simpler and self-contained proof of the Phillips-Weaver result.

Fix a separable infinite-dimensional complex Hilbert space H. Let $\mathcal{B}(H)$ be its algebra of bounded linear operators, $\mathcal{K}(H)$ its ideal of compact operators and $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ the Calkin algebra. Let $\pi\colon \mathcal{B}(H) \to \mathcal{C}(H)$ be the quotient map. In [7, 1.6(ii)] (also [35], [43]) it was asked whether all automorphisms of the Calkin algebra are inner. Phillips and Weaver ([33]) gave a partial answer by constructing an outer automorphism using the Continuum Hypothesis. We complement their answer by showing that a well-known set-theoretic axiom implies all automorphisms are inner. Neither the statement of this axiom nor the proof of Theorem 1 involve set-theoretic considerations beyond the standard functional analyst's toolbox.

Theorem 1. *Todorcevic's Axiom*, TA, *implies that all automorphisms of the Calkin algebra of a separable Hilbert space are inner.*

Todorcevic's Axiom (also known as the Open Coloring Axiom, OCA) is stated in Section 2.3. Every model of ZFC has a forcing extension in which TA holds ([41]). TA also holds in Woodin's canonical model for negation of the Continuum Hypothesis ([44], [28]) and it follows from the Proper Forcing Axiom, PFA ([40]). The latter is a strengthening of the Baire Category Theorem and besides its applications to the theory of liftings it can be used to find other combinatorial reductions ([40, §8], [31]).

The Calkin algebra provides both a natural context and a powerful tool for studying compact perturbations of operators on a Hilbert space. The original motivation for the problem solved in Theorem 1 comes from a classification problem for normal operators. By results of Weyl, von Neumann, Berg and Sikonia, if a and b are normal operators in $\mathcal{B}(H)$ then one is untarily equivalent to a compact perturbation of the other if and only if their essential spectra coincide (see the introduction to [6] or [8, §IX]). The *essential spectrum*, $\sigma_e(a)$, of a is the set of all accumulation points of its spectrum $\sigma(a)$, together with all of its isolated points of infinite multiplicity. It is known to be equal to the spectrum of $\pi(a)$ in the Calkin algebra. Therefore the map $a \mapsto \sigma_e(a)$ provides a complete invariant for the unitary equivalence of those operators in the Calkin algebra that lift to normal operators in $\mathcal{B}(H)$.

An operator a is said to be *essentially normal* if $aa^* - a^*a$ is compact, or equivalently, if its image in the Calkin algebra is normal. Not every essentially normal operator is a compact perturbation of a normal operator. For example, an argument using the Fredholm index shows that the unilateral shift S is not a compact perturbation of a normal operator ([6]) while its image in $\mathcal{C}(H)$ is clearly a unitary. Since the essential spectra of S and its adjoint are both equal to the unit circle, the above mentioned classification does not extend to all normal operators in $\mathcal{C}(H)$. For an essentially normal operator a and $\lambda \in \mathbb{C} \setminus \sigma_e(a)$ the operator $a - \lambda I$ is Fredholm. In [6] (see also [7] or [8, §IX]) it was proved that the function $\lambda \mapsto \operatorname{index}(a - \lambda I)$ together with $\sigma_e(a)$ provides a complete invariant for the relation of unitary equivalence modulo a compact perturbation on essentially normal operators.

It is interesting to note that the unitary equivalence of normal (even self-adjoint) operators is of much higher complexity than the unitary equivalence of normal (or even essentially normal) operators modulo the compact perturbation. By the above, the latter relation is *smooth*: a complete invariant is given by a Borel-measurable map into a Polish space. On the other hand, the complete invariant for the former given by the spectral theorem is of much higher complexity. As a matter of fact, in [26] it was proved that the unitary equivalence of self-adjoint operators does not admit any effectively assigned complete invariants coded by countable structures.

Instead of the unitary equivalence modulo compact perturbation, one may consider a coarser relation which we temporarily denote by $\sim$. Let $a \sim b$ if there is an automorphism Φ of the Calkin algebra sending $\pi(a)$ to $\pi(b)$. It is clear that $a \sim b$ implies $\sigma_e(a) = \sigma_e(b)$, and therefore two relations coincide on normal operators. By [6] these two relations coincide on normal operators, and the conclusion of Theorem 1 implies that they coincide on all of $\mathcal{B}(H)$. The outer automorphism Φ constructed in [33], as well as the one in Section 1 below, is *pointwise inner*: $\Phi(\pi(a)) = \Phi(\pi(b))$ implies an inner automorphism sends

$\pi(a)$ to $\pi(b)$. It is not known whether $\sim$ can differ from the unitary equivalence modulo a compact perturbation in some model of set theory. In particular, it is still open whether the Continuum Hypothesis implies the existence of an automorphism of the Calkin algebra sending the image of the unilateral shift to its adjoint. See [33] for a discussion and related open problems.

Theorem 1 belongs to a line of results starting with Shelah's groundbreaking construction of a model of set theory in which all automorphisms of the quotient Boolean algebra $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial ([36]). An equivalent reformulation states that it is impossible to construct a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ without using some additional set-theoretic axiom. Through the work of Shelah-Steprāns, Velickovic, Just, and the author this conclusion was extended to many other quotient algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$. The progress was made possible by replacing Shelah's intricate forcing construction by the PFA ([37]) and then in [42] by Todorcevic's Axiom ([40, §8]) in conjunction with Martin's Axiom. A survey of these results can be found in [16]. See also [24] for closely related rigidity results in the Borel context (cf. §6 below).

0.1. *Terminology and notation.* All the necessary background on operator algebras can be found e.g., in [32] or [43]. Throughout we fix an infinite dimensional separable complex Hilbert space H and an orthonormal basis (e_n). Let $\pi\colon \mathcal{B}(H) \to \mathcal{C}(H)$ be the quotient map. If F is a closed subspace of H then proj_F denotes the orthogonal projection to F. Fix an increasing family of finite-dimensional projections $(\mathbf{R}_n)$ such that $\bigvee_n \mathbf{R}_n = I$, and consider a nonincreasing family of seminorms $\|a\|_n = \|(I-\mathbf{R}_n)a\|$. Let $\|a\|_{\mathcal{K}} = \lim_{n\to\infty} \|a\|_n$. Note that $\|a\|_{\mathcal{K}} = \|\pi(a)\|$, with the norm of $\pi(a)$ computed in the Calkin algebra. Projections P and Q are *almost orthogonal* if PQ is compact. This is equivalent to $QP = (PQ)^*$ being compact.

Let $\mathcal{A}$, $\mathcal{B}$ be C*-algebras, J_1, J_2 their ideals and let $\Phi\colon \mathcal{A}/J_1 \to \mathcal{B}/J_2$ be a *-homomorphism. A map $\Psi\colon \mathcal{A} \to \mathcal{B}$ such that (π_{J_i} is the quotient map)

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Psi"] \arrow[d, "\pi_{J_1}"] & \mathcal{B} \arrow[d, "\pi_{J_2}"] \\ \mathcal{A}/J_1 \arrow[r, "\Phi"'] & \mathcal{B}/J_2 \end{tikzcd}$$

commutes, is a *representation* of Φ. Since we do not require Ψ to be a *-homomorphism, the Axiom of Choice implies every Φ has a representation.

For a partition $\vec{E}$ of $\mathbb{N}$ into finite intervals (E_n) let $\mathcal{D}[\vec{E}]$ be the von Neumann algebra of all operators in $\mathcal{B}(H)$ for which each $\overline{\operatorname{span}}\{e_i \mid i \in E_n\}$ is invariant. We always assume E_n are consecutive, so that $\max(E_n) + 1 = \min(E_{n+1})$ for each n. If $E_n = \{n\}$ then $\mathcal{D}[\vec{E}]$ is the *standard atomic masa* von Neumann algebra of all operators diagonalized by the standard basis.

These FDD (short for 'finite dimensional decomposition') von Neumann algebras play an important role in the proof of Theorem 1. For $M \subseteq \mathbb{N}$ let $\mathbf{P}_M^{\vec{E}}$ (or $\mathbf{P}_M$ if $\vec{E}$ is clear from the context) be the projection to the closed linear span of $\bigcup_{i\in M}\{e_n \mid n \in E_i\}$ and let $\mathcal{D}_M[\vec{E}]$ be the ideal $\mathbf{P}_M\mathcal{D}[\vec{E}]\mathbf{P}_M = \mathbf{P}_M\mathcal{D}[\vec{E}]$ of $\mathcal{D}[\vec{E}]$. It is not difficult to see that an operator a in $\mathcal{D}[\vec{E}]$ is compact if and only if $\lim_i \|\mathbf{P}_{\{i\}}^{(\vec{E})}a\| = 0$. The strong operator topology coincides with the product of the norm topology on the unitary group of $\mathcal{D}[\vec{E}]$, $\mathcal{U}[\vec{E}] = \prod_i \mathcal{U}(E_i)$ and makes it into a compact metric group.

If $\mathcal{A}$ is a unital C*-algebra then $\mathcal{U}(\mathcal{A})$ denotes its unitary group. We shall write $\mathcal{U}[\vec{E}]$ for $\mathcal{U}(\mathcal{D}[\vec{E}])$ and $\mathcal{U}_A[\vec{E}]$ for $\mathcal{U}(\mathcal{D}_A[\vec{E}])$. Similarly, we shall write $\mathcal{C}[\vec{E}]$ for $\mathcal{D}[\vec{E}]/(\mathcal{D}[\vec{E}] \cap \mathcal{K}(H))$. For a C^*-algebra $\mathcal{D}$ and $r < \infty$ write

$$\mathcal{D}_{\leq r} = \{a \in \mathcal{D} \mid \|a\| \leq r\}.$$

The set of self-adjoint operators in $\mathcal{D}$ is denoted by $\mathcal{D}_{\mathrm{sa}}$.

The *spectrum* of a normal operator b in a unital C*-algebra is

$$\sigma(b) = \{\lambda \in \mathbb{C} \mid b - \lambda I \text{ is not invertible}\}.$$

A rough outline of the proof of Theorem 1. If $\mathcal{D}$ is a subset of $\mathcal{B}(H)$, we say that Φ is *inner on* $\mathcal{D}$ if there is an inner automorphism Φ' of $\mathcal{C}(H)$ such that the restrictions of Φ and Φ' to $\pi[\mathcal{D}]$ coincide. In Theorem 1.1 we use CH to construct an outer automorphism of the Calkin algebra whose restriction to each $\mathcal{D}[\vec{E}]$ is inner. In Theorem 3.2 we use TA to show that for any outer automorphism Φ there is $\vec{E}$ such that Φ is not inner on $\mathcal{D}[\vec{E}]$. Both of these proofs involve the analysis of 'coherent families of unitaries' (§1).

Fix an automorphism of the Calkin algebra Φ. Fix $\vec{E}$ such that the sequence $\#E_n$ is nondecreasing. A simple fact that Φ is inner on $\mathcal{D}[\vec{E}]$ if and only if it is inner on $\mathcal{D}_M[\vec{E}]$ for some infinite M is given in Lemma 6.2. Hence we only need to find an infinite M such that the restriction of Φ to $\mathcal{D}_M[\vec{E}]$ is inner. This is done in Proposition 7.1. Its proof proceeds in several stages and it involves the notion of an ε-approximation (with respect to $\|\cdot\|_{\mathcal{K}}$) to a representation (see §4) and the family $\mathcal{J}^n(\vec{E}) = \{A \subseteq \mathbb{N} \mid \Phi$ has a C-measurable 2^{-n}-approximation on $\mathcal{D}[\vec{E}]\}$. In Lemma 7.2, TA is used again to prove that $\mathcal{J}^n(\vec{E})$ is so large for every n that $\bigcap_n \mathcal{J}^n(\vec{E})$ contains an infinite set M. The Jankov, von Neumann uniformization theorem (Theorem 2.1) is used to produce a C-measurable representation of Φ on $\mathcal{D}_M[\vec{E}]$ as a 'limit' of given 2^{-n}-approximations. This C-measurable representation is turned into a conjugation by a unitary in Theorem 6.3. This result depends on the Ulam-stability of approximate *-homomorphisms (Theorem 5.1).

Part of the present proof that deals with FDD von Neumann algebras owes much to the proof of the 'main lifting theorem' from [13] and a number of elegant improvements from Fremlin's account [20]. In particular, the proof of Claim 6.5 is based on the proof of [20, Lemma 1P] and Section 7.1 closely follows [20, Lemma 3C].

1. An outer automorphism from the Continuum Hypothesis

We first prove a slight strengthening of the Phillips-Weaver result. Lemmas 1.2, 1.3 and 1.6, as well as definitions of ρ and Δ_I, will be needed in the proof of Theorem 1.

THEOREM 1.1. *The Continuum Hypothesis implies there is an outer automorphism of the Calkin algebra. Moreover, the restriction of this automorphism to the standard atomic masa and to any separable subalgebra is inner.*

If $\vec{E}$ and $\vec{F}$ are partitions of $\mathbb{N}$ into finite intervals, we write $\vec{E} \leq^* \vec{F}$ if for all but finitely many i there is j such that $E_i \cup E_{i+1} \subseteq F_j \cup F_{j+1}$. A family $\mathcal{E}$ of partitions is *cofinal* if for every $\vec{F}$ there is $\vec{E} \in \mathcal{E}$ such that $\vec{F} \leq^* \vec{E}$.

Let $\mathbb{T}$ denote the circle group, $\{z \in \mathbb{C} \mid |z| = 1\}$, and let $\mathbb{T}^{\mathbb{N}}$ be its countable power. It is isomorphic to the unitary group of the standard atomic masa. For $\alpha \in \mathbb{T}^{\mathbb{N}}$ let u_α be the unitary operator on H that sends e_n to $\alpha(n)e_n$. For a unitary u let Ψ_u be the conjugation by u, $\Psi_u(a) = uau^*$ (usually denoted by $\operatorname{Ad} u$ in the operator algebras literature.) If $u = u_\alpha$ we write Ψ_α for Ψ_{u_α}. We say that Ψ_α and Ψ_β *agree modulo compacts* on $\mathcal{D}$ if $\Psi_\alpha(a) - \Psi_\beta(a)$ is compact for every $a \in \mathcal{D}$.

Given $\vec{E}$ define two coarser partitions: $\vec{E}^{\text{even}}$, whose entries are $E_{2n} \cup E_{2n+1}$ and $\vec{E}^{\text{odd}}$, whose entries are $E_{2n-1} \cup E_{2n}$ (with $E_{-1} = \emptyset$). Let

$$\mathcal{F}[\vec{E}] = \mathcal{D}[\vec{E}^{\text{even}}] \cup \mathcal{D}[\vec{E}^{\text{odd}}].$$

I have proved Lemma 1.2 below using the methods of [3]. George Elliott pointed out that the proof of this lemma (in a more general setting) is contained in the proof of [11, Theorem 3.1], as remarked in [12].

LEMMA 1.2. *For a sequence (a_n) in $\mathcal{B}(H)$ there are a partition $\vec{E}$, $a_n^0 \in \mathcal{D}[\vec{E}^{\text{even}}]$ and $a_n^1 \in \mathcal{D}[\vec{E}^{\text{odd}}]$ such that $a_n - a_n^0 - a_n^1$ is compact for each n.*

Proof. For $A \subseteq \mathbb{N}$ write $\mathbf{P}_A^{(e_n)}$ for the projection to the closed linear span of $\{e_i \mid i \in A\}$. Fix $m \in \mathbb{N}$ and $\varepsilon > 0$. Since $a\mathbf{P}_{[0,m)}$ is compact, we can find $n > m$ large enough to have $\|\mathbf{P}_{[n,\infty)} a \mathbf{P}_{[0,m)}\| < \varepsilon$ and similarly $\|\mathbf{P}_{[n,\infty)} a^* \mathbf{P}_{[0,m)}\| < \varepsilon$. Therefore $\|\mathbf{P}_{[0,m)} a \mathbf{P}_{[n,\infty)}\| < \varepsilon$ as well. Recursively find a strictly increasing $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $m \leq n$ and $i \leq n$ we have $\|\mathbf{P}_{[f(n+1),\infty)} a_i \mathbf{P}_{[0,f(m))}\| < 2^{-n}$ and $\|\mathbf{P}_{[0,f(m))} a_i \mathbf{P}_{[f(n+1),\infty)}\| < 2^{-n}$. We shall check that $\vec{E}$ defined by $E_n = [f(n), f(n+1))$ is as required. Write $Q_n = \mathbf{P}_{[f(n),f(n+1))}$ (with $f(0) = 0$). Fix $a = a_i$ and define

$$a^0 = \textstyle\sum_{n=0}^\infty (Q_{2n} a Q_{2n} + Q_{2n} a Q_{2n+1} + Q_{2n+1} a Q_{2n}),$$
$$a^1 = \textstyle\sum_{n=0}^\infty (Q_{2n+1} a Q_{2n+1} + Q_{2n+1} a Q_{2n+2} + Q_{2n+2} a Q_{2n+1}).$$

Then $a^0 \in \mathcal{D}[\vec{E}^{\text{even}}]$, $a^1 \in \mathcal{D}[\vec{E}^{\text{odd}}]$. Let $c = a - a^0 - a^1$. For every n we have

$$\begin{aligned}\|\mathbf{P}_{[f(n),\infty)}c\| &\leq \left\|\textstyle\sum_{i=n}^{\infty} \mathbf{P}_{[f(i),\infty)} a \mathbf{P}_{[0,f(i-1))}\right\| + \left\|\textstyle\sum_{i=n+1}^{\infty} \mathbf{P}_{[f(n),f(i))} a \mathbf{P}_{[f(i+1),\infty)}\right\| \\ &\leq 2^{-n+2} + 2^{-n+1},\end{aligned}$$

and therefore c is compact. □

Whenever possible we collapse the subscripts/superscripts and write e.g., Ψ_ξ for Ψ_{α^ξ} (which is of course $\Psi_{u_{\alpha^\xi}}$).

LEMMA 1.3. *Assume $(\vec{E}^\xi)_{\xi\in\Lambda}$ is a directed cofinal family of partitions and α^ξ, $\xi \in \Lambda$, are such that Ψ_η and Ψ_ξ agree modulo compacts on $\mathcal{F}[\vec{E}^\xi]$ for $\xi \leq \eta$. Then there is an automorphism Φ of $\mathcal{C}(H)$ such that Ψ_ξ is a representation of Φ on $\mathcal{F}[\vec{E}^\xi]$ for every $\xi \in \Lambda$. Moreover, Φ is unique.*

Proof. By Lemma 1.2, for each $a \in \mathcal{B}(H)$ there is a partition $\vec{E}$ with $a_0 \in \mathcal{D}[\vec{E}^{\text{even}}]$ and $a_1 \in \mathcal{D}[\vec{E}^{\text{odd}}]$ such that $a - a_0 - a_1$ is compact. Fix $\vec{F} = \vec{E}^\xi$ such that $\vec{E} \leq^* \vec{F}$ and let $\Phi(\pi(a)) = \pi(\Psi_{\vec{F}}(a))$.

Then Φ is well-defined by the agreement of Ψ_ξ's. For every pair of operators a, b there is a single partition $\vec{E}$ with a_0 and b_0 in $\mathcal{D}[\vec{E}^{\text{even}}]$ and a_1 and b_1 in $\mathcal{D}[\vec{E}^{\text{odd}}]$ such that both $a - a_0 - a_1$ and $b - b_0 - b_1$ are compact. This readily implies Φ is a *-homomorphism.

The inverse maps $\Psi_\xi^* = \Psi_{(u_{\alpha^\xi})^*}$ also satisfy the assumptions of the lemma and there is a *-homomorphism Φ^* such that Ψ_ξ^* is a representation of Φ^* on $\mathcal{F}[\vec{E}^\xi]$ for every ξ. Then $\Phi\Phi^* = \Phi^*\Phi$ is the identity on $\mathcal{C}(H)$; hence Φ is an automorphism. The uniqueness follows from Lemma 1.2. □

Let $\mathbb{T}$ be the unitary group of the 1-dimensional complex Hilbert space. Recall that every inner automorphism of $\mathcal{C}(H)$ has a representation of the form Ψ_u for u which is an isometry between subspaces of H of finite codimension. The proof of the following lemma was suggested by Nik Weaver.

LEMMA 1.4. *Assume u and v are isometries between subspaces of H of finite codimension. If $\Psi_u(a) - \Psi_v(a)$ is compact for every a diagonalized by (e_n), then there is $\alpha \in (\mathbb{T})^{\mathbb{N}}$ for which the linear map w defined by $w(e_n) = \alpha(n)v(e_n)$ for all n is such that $\Psi_w(a) - \Psi_u(a)$ is compact for all a in $\mathcal{B}(H)$.*

Proof. Let $\mathcal{A} = \{a \in \mathcal{B}(H) \mid a \text{ is diagonalized by } (e_n)\}$. By our assumption, $\pi(v^*u)$ commutes with $\pi(a)$ for all $a \in \mathcal{A}$. Since $\pi[\mathcal{A}]$ is, by [22], a maximal abelian self-adjoint subalgebra of the Calkin algebra we have $\pi(w_0) = \pi(v^*u)$ for some $w_0 \in \mathcal{A}$. Let $w_0 = bw_1$ be the polar decomposition of w_0 in $\mathcal{A}$. Since $\pi(b) = I$ we have $\pi(w_1) = \pi(v^*u)$. Since the Fredholm index of w_1 is 0 and $\pi(w_1)$ is a unitary, we may assume w_1 is a unitary. Fix $\alpha \in \mathbb{T}^{\mathbb{N}}$ such that $w_1 = u_\alpha$, i.e., $w_1(e_n) = \alpha(n)e_n$ for all n.

Let $w = vw_1$. Then $\pi(w) = \pi(vv^*u) = \pi(u)$. Hence $\Psi_w(a) - \Psi_u(a)$ is compact for all $a \in \mathcal{B}(H)$. Also, for each n we have $w(e_n) = vw_1(e_n) = v(\alpha(n)e_n) = \alpha(n)v(e_n)$. □

For i, j in $\mathbb{N}$ and α, β in $\mathbb{T}^{\mathbb{N}}$, let

$$\rho(i, j, \alpha, \beta) = |\alpha(i)\overline{\alpha(j)} - \beta(i)\overline{\beta(j)}|.$$

For fixed i, j the function $f \equiv \rho(i, j, \cdot, \cdot)$ satisfies the triangle inequality:

$$f(\alpha, \beta) + f(\beta, \gamma) \geq f(\alpha, \gamma).$$

We also have

$$\rho(i, j, \alpha, \beta) = |\rho(i, j, \alpha, \beta)\alpha(j)\overline{\beta(i)}| = |\alpha(i)\overline{\beta(i)} - \alpha(j)\overline{\beta(j)}|.$$

Hence for fixed α, β the function $f_1 \equiv \rho(\cdot, \cdot, \alpha, \beta)$ also satisfies the triangle inequality:

$$f_1(i, j) + f_1(j, k) \geq f_1(i, k).$$

For $I \subseteq \mathbb{N}$ and α and β in $\mathbb{T}^{\mathbb{N}}$ write

$$\Delta_I(\alpha, \beta) = \sup_{i \in I,\ j \in I} \rho(i, j, \alpha, \beta).$$

The best picture Δ_I is furnished by (5) of the following lemma.

Lemma 1.5. *For all I, α, β,*

(1) $\Delta_I(\alpha, \beta) \leq 2 \sup_{i \in I} |\alpha(i) - \beta(i)|$.
(2) $\Delta_I(\alpha, \beta) \geq \sup_{j \in I} |\alpha(j) - \beta(j)| - \inf_{i \in I} |\alpha(i) - \beta(i)|$. *In particular if $\alpha(i_0) = \beta(i_0)$ for some $i_0 \in I$, then $\Delta_I(\alpha, \beta) \geq \sup_{j \in I} |\alpha(j) - \beta(j)|$.*
(3) *If $z \in \mathbb{T}$, then $\Delta_I(\alpha, \beta) = \Delta_I(\alpha, z\beta)$.*
(4) *If $I \cap J$ is nonempty, then $\Delta_{I \cup J}(\alpha, \beta) \leq \Delta_I(\alpha, \beta) + \Delta_J(\alpha, \beta)$.*
(5) $\inf_{z \in \mathbb{T}} \sup_{i \in I} |\alpha(i) - z\beta(i)| \leq \Delta_I(\alpha, \beta) \leq 2 \inf_{z \in \mathbb{T}} \sup_{i \in I} |\alpha(i) - z\beta(i)|$.

Proof. Since

$$\rho(i, j, \alpha, \beta) = |\alpha(i)\overline{\beta(i)} - \alpha(j)\overline{\beta(j)}| = |\overline{\beta(i)}(\alpha(i) - \beta(i)) + \overline{\beta(j)}(\beta(j) - \alpha(j))|$$

and $|\beta(i)| = |\beta(j)| = 1$, we have

$$||\alpha(i) - \beta(i)| - |\alpha(j) - \beta(j)|| \leq \rho(i, j, \alpha, \beta) \leq |\alpha(i) - \beta(i)| + |\alpha(j) - \beta(j)|.$$

This implies

$$\begin{aligned}\Delta_I(\alpha, \beta) &= \sup_{i \in I,\ j \in I} \rho(i, j, \alpha, \beta) \\ &\leq \sup_{i \in I,\ j \in I} (|\alpha(i) - \beta(i)| + |\alpha(j) - \beta(j)|) \leq 2 \sup_{i \in I} |\alpha(i) - \beta(i)|\end{aligned}$$

and (1) follows. For (2) we have

$$\begin{aligned}\Delta_I(\alpha,\beta) &= \sup_{i\in I,\ j\in I} \rho(i,j,\alpha,\beta)\\ &\geq \sup_{i\in I,\ j\in I} ||\alpha(i)-\beta(i)|-|\alpha(j)-\beta(j)||\\ &= \sup_{i\in I}|\alpha(i)-\beta(i)| - \inf_{i\in I}|\alpha(i)-\beta(i)|.\end{aligned}$$

Clause (3) is an immediate consequence of the equality $\rho(i,j,\alpha,z\beta)=\rho(i,j,\alpha,\beta)$. It is not difficult to see that in order to prove (4), we only need to check $\rho(i,j,\alpha,\beta) \leq \Delta_I(\alpha,\beta)+\Delta_J(\alpha,\beta)$ for all $i\in I$ and $j \in J$. Pick $k \in I\cap J$. Then we have

$$\rho(i,j,\alpha,\beta) \leq \rho(i,k,\alpha,\beta)+\rho(k,j,\alpha,\beta) \leq \Delta_I(\alpha,\beta)+\Delta_J(\alpha,\beta),$$

completing the proof.

Now we prove (5). By the definition, for every $j \in I$ we have $\Delta_I(\alpha,\beta) \geq \sup_{i\in I}|\alpha(i)-(\alpha(j)\overline{\beta(j)})\beta(i)|$. Therefore $\Delta_I(\alpha,\beta) \geq \inf_{z\in\mathbb{T}}\sup_{i\in I}|\alpha(i)-\beta(i)|$. On the other hand, for all i and j,

$$|\alpha(i)-(\alpha(j)\overline{\beta(j)})\beta(i)| \leq |\alpha(i)-z\beta(i)|+|\alpha(j)-z\beta(j)|$$

for every $z\in\mathbb{T}$, which immediately implies the other inequality. $\square$

Recall that for $\alpha\in\mathbb{T}^{\mathbb{N}}$ by u_α we denote the unitary such that $u_\alpha(e_n) = \alpha(n)e_n$ and that $\Psi_\alpha = \Psi_{u_\alpha}$ is the conjugation by u_α.

Lemma 1.6. (a) *If* $\lim_n|\alpha(n)-\beta(n)|=0$ *then* $\Psi_\alpha(a)-\Psi_\beta(a)$ *is compact for all* $a\in\mathcal{B}(H)$.

(b) *The difference* $\Psi_\alpha(a)-\Psi_\beta(a)$ *is compact for all* $a \in \mathcal{D}[\vec{E}]$ *if and only if* $\limsup_n \Delta_{E_n}(\alpha,\beta)=0$.

Proof. (a) Since $\lim_n|\alpha(n)-\beta(n)|=0$ implies $\pi(u_\alpha)=\pi(u_\beta)$, we have that $\Psi_\alpha(a)-\Psi_\beta(a)=(u_\alpha-u_\beta)a(u_\alpha^*-u_\beta^*)$ is compact.

(b) Assume $\limsup_n\Delta_{E_n}(\alpha,\beta)=0$. For each n let $m_n=\min(E_n)$ and define $\gamma\in\mathbb{T}^{\mathbb{N}}$ by

$$\gamma(i)=\beta(i)\overline{\beta(m_n)}\alpha(m_n), \quad \text{if } i\in E_n.$$

The operator $\sum_{n\in\mathbb{N}}\overline{\beta(m_n)}\alpha(m_n)\operatorname{proj}_{E_n}$ (with the obvious interpretation of the infinite sum) is central in $\mathcal{D}[\vec{E}]$ and therefore for every $a\in\mathcal{D}[\vec{E}]$ we have $\Psi_\gamma(a)=\Psi_\beta(a)$. By clauses (2) and (3) of Lemma 1.5 we have $|\gamma(i)-\alpha(i)| \leq \Delta_{E_n}(\alpha,\gamma)=\Delta_{E_n}(\alpha,\beta)$ for $i\in E_n$. Therefore $\lim_i|\gamma(i)-\alpha(i)|=0$ and the conclusion follows by (a).

Now assume $\limsup_n\Delta_{E_n}(\alpha,\beta)>0$. Fix $\varepsilon>0$, an increasing sequence $n(k)$ and $i(k)<j(k)$ in $E_{n(k)}$ such that $\rho(i(k),j(k),\alpha,\beta)\geq\varepsilon$ for all k. The partial

isometry a defined by $a(e_{i(k)}) = e_{j(k)}$, $a(e_{j(k)}) = e_{i(k)}$, and $a(e_j) = 0$ for other values of $j \in E_{n(k)}$ belongs to $\mathcal{D}[\vec{E}]$. Then

$$\begin{aligned}\Psi_\alpha(a)(e_{i(k)}) = (u_\alpha a u_\alpha^*)(e_{i(k)}) &= u_\alpha(a(\overline{\alpha(i(k))}e_{i(k)}) \\ &= u_\alpha(\overline{\alpha(i(k))}e_{j(k)}) = \alpha(j(k))\overline{\alpha(i(k))}e_{j(k)}.\end{aligned}$$

Similarly $\Psi_\beta(a)(e_{i(k)}) = \beta(j(k))\overline{\beta(i(k))}e_{j(k)}$ for all k. Therefore

$$\|(\Psi_\alpha(a) - \Psi_\beta(a))(e_{i(k)})\| \geq \rho(i(k), j(k), \alpha, \beta) \geq \varepsilon$$

for all k, and the difference $\Psi_\alpha(a) - \Psi_\beta(a)$ is not compact. □

Proof of Theorem 1.1. Enumerate $\mathbb{T}^{\mathbb{N}}$ as β^ξ for $\xi < \omega_1$ and all partitions of $\mathbb{N}$ into finite intervals as $\vec{F}^\xi$, with $\xi < \omega_1$. Construct a $\leq^*$-increasing cofinal chain $\vec{E}^\xi$ of partitions and $\alpha^\xi \in \mathbb{T}^{\mathbb{N}}$ such that for all $\xi < \eta$ we have

(1) $\limsup_n \Delta_{E_n^\xi \cup E_{n+1}^\xi}(\alpha^\xi, \alpha^\eta) = 0$.
(2) $\limsup_n \Delta_{E_n^{\xi+1}}(\alpha^{\xi+1}, \beta^\xi) \geq \sqrt{2}$.
(3) E^η is eventually coarser than E^ξ in the sense that E_m^η is equal to a union of intervals from E^ξ for all but finitely many m.

In order to describe the recursive construction, we consider two cases. First, assume $\zeta < \omega_1$ and $\vec{E}^\xi$ and α^ξ were chosen for all $\xi \leq \zeta$. Let $\vec{E}^{\zeta+1}$ be such that $F_n = E_n^{\zeta+1}$ is the union of $2n+1$ consecutive intervals of $\vec{E}^\zeta$, denoted by $F_0^n, \dots, F_{2n}^n$. Fix n. If $\Delta_{\vec{E}_n^{\zeta+1}}(\alpha^\zeta, \beta^\zeta) \geq \sqrt{2}$, let $\alpha^{\zeta+1}$ coincide with α^ζ on $E_n^{\zeta+1}$. Now assume $\Delta_{E_n^{\zeta+1}}(\alpha^\zeta, \beta^\zeta) < \sqrt{2}$. Let $\gamma_n = \exp(i\pi/n)$. Let $\alpha^{\zeta+1}(j) = \gamma_n^k \alpha^\zeta(j)$ for $j \in F_k^n$. If $i \in F_0^n$ and $j \in F_n^n$, then $\alpha^{\zeta+1}(i) = \alpha^\zeta(i)$ and $\alpha^{\zeta+1}(j) = -\alpha^\zeta(j)$. Since $|\alpha^\zeta(j)\overline{\alpha^\zeta(i)} - \beta^\zeta(j)\overline{\beta^\zeta(i)}| < \sqrt{2}$, we have $\Delta_{E_n^{\zeta+1}}(\alpha^{\zeta+1}, \beta^\zeta) \geq |\alpha^\zeta(j)\overline{\alpha^\zeta(i)} + \beta^\zeta(j)\overline{\beta^\zeta(i)}| > \sqrt{2}$.

Hence (2) holds. We need to check $\limsup_m \Delta_{E_m^\zeta \cup E_{m+1}^\zeta}(\alpha^\zeta, \alpha^{\zeta+1}) = 0$. We have $\Delta_{E_m^\zeta}(\alpha^\zeta, \alpha^{\zeta+1}) = 0$ for all m. Since $\alpha^{\zeta+1}$ and α^ζ coincide on F_0^n and on F_{2n}^n for each n, $\Delta_{F_{2n}^n \cup F_0^{n+1}}(\alpha^\zeta, \alpha^{\zeta+1}) = 0$ for all n. If $0 \leq k < 2n$ then $\Delta_{F_k^n \cup F_{k+1}^n}(\alpha^\zeta, \alpha^{\zeta+1}) \leq |\gamma_n| \leq |\sin(\pi/n)| \leq \pi/n$. Hence clause (1) is satisfied with $\xi = \zeta$ and $\eta = \zeta + 1$, and therefore it holds for all ξ and $\eta = \zeta + 1$ by transitivity.

Now assume $\zeta < \omega_1$ is a limit ordinal such that $\vec{E}^\xi$ and α^ξ have been defined for $\xi < \zeta$. Let ξ_n, for $n \in \mathbb{N}$, be an increasing sequence with supremum ζ and write $\vec{E}^n, \alpha^n$ for $\vec{E}^{\xi_n}, \alpha^{\xi_n}$. Find a strictly increasing function $f\colon \mathbb{N} \to \mathbb{N}$ such that

(4) $f(0) = 0$.
(5) $f(n+1)$ is large enough so that each E_i^{n+1} disjoint from $[0, f(n+1))$ is the union of finitely many intervals of E^n.

(6) For all $k \leq n$ the interval $F_n = [f(n), f(n+1))$ is the union of finitely many intervals from E^k.

(7) If $l < k \leq n$, $j \in \mathbb{N}$, and $\Delta_{E^l_j \cup E^l_{j+1}}(\alpha^l, \alpha^k) \geq 1/n$, then $\max E^l_j \leq f(n)$.

(8) $F^\zeta_i \not\supseteq F_n$ for all i and all n.

The values $f(n)$ for $n \in \mathbb{N}$ are chosen recursively. If $f(n)$ has been chosen then each of the clauses (5), (6), (7), and (8) can be satisfied by choosing $f(n+1)$ to be larger than the maximum of a finite subset of $\mathbb{N}$.

Assume f has been chosen to satisfy (4)–(8). By (6) for m and $i \geq m$ we have $E^m_i \cup E^m_{i+1} \subseteq F_n \cup F_{n+1}$ if n is the maximal such that $f(n) < \min E^m_i$. Therefore with $\vec{E}^\zeta = \vec{F}$ we have $\vec{E}^n \leq^* \vec{E}^\zeta$ for all n, and therefore $\vec{E}^\xi \leq^* \vec{E}^\zeta$ for all $\xi < \zeta$. By (8) we have that for each $i \in \mathbb{N}$ the interval F^ζ_i intersects at most two of the intervals F_n nontrivially and therefore $\vec{F}^\xi \leq^* \vec{E}^\xi$.

Recursively define $\gamma_n \in \mathbb{T}$ for $n \in \mathbb{N}$ and $\alpha^\zeta(j)$ for $j \in \mathbb{N}$ so that for all n we have

(9) $\alpha^\zeta(j) = \gamma_n \alpha^n(j)$ for $j \in F_n \cup \{f(n+1)\}$.

To this end, let

$$\begin{aligned} \alpha^\zeta(j) &= \alpha^0(j) && \text{for } j \in F_0 \cup \{f(1)\}, \\ \gamma_1 &= \alpha^0(f(1))\overline{\alpha^1(f(1))} && \\ \alpha^\zeta(j) &= \gamma_1 \alpha^1(j) && \text{for } j \in F_1 \cup \{f(2)\}, \end{aligned}$$

and in general for $n \geq 0$ let

$$\gamma_{n+1} = \alpha_n(f(n+1))\gamma_n \overline{\alpha^{n+1}(f(n+1))}$$

and let

$$\alpha^\zeta(j) = \gamma_{n+1}\alpha^{n+1}(j) \qquad \text{for } j \in F_{n+1} \cup \{f(n+2)\}.$$

This sequence satisfies (9).

Fix m and write I_n for $E^m_n \cup E^m_{n+1}$. We want to show

$$\lim_{n\to\infty} \Delta_{I_n}(\alpha^\zeta, \alpha^m) = 0.$$

By (6), for all $n \geq m$ we have $E^m_n \subseteq F_k$, for some $k = k(n)$ and therefore $I_n \subseteq F_k \cup F_{k+1}$. This implies, by Lemma 1.5(4) and Lemma 1.5(3), that

$$\Delta_{I_n}(\alpha^\zeta, \alpha^m) \leq \Delta_{E^m_n \cup \{f(k+1)\}}(\alpha^k, \alpha^m) + \Delta_{E^m_{n+1}}(\alpha^{k+1}, \alpha^m)$$

and by (7) the right-hand side is $\leq \frac{1}{k} + \frac{1}{k+1}$, since $n \geq m$. Moreover $\lim_{n\to\infty} k(n) = \infty$, and we can conclude that $\lim_{n\to\infty} \Delta_{I_n}(\alpha^\zeta, \alpha^m) = 0$. Therefore the conditions of Lemma 1.6(b) are satisfied and α^ζ satisfies (1).

This finishes the description of the construction of $\vec{E}^\xi$ and α^ξ satisfying (1) and (2). By Lemma 1.3 there is an automorphism Φ of $\mathcal{C}(H)$ that has Ψ_ξ as its representation on $\mathcal{F}[\vec{E}^\xi]$ for each ξ. Assume this automorphism is inner. Then it has a representation of the form Ψ_u for some partial isometry u.

By Lemma 1.4 applied to Ψ_0 and Ψ_u there is $\beta \in \mathbb{T}^{\mathbb{N}}$ such that Ψ_β is a representation of Φ. But β is equal to β^ξ for some $\xi < \omega_1$, and by (2) and Lemma 1.6(b) the mapping Ψ_β is not a representation of Φ on $\mathcal{F}[\vec{E}^\zeta]$.

By construction, the constructed automorphism is inner on the standard atomic masa, and actually on each $\mathcal{D}[\vec{E}]$. In addition, Lemma 1.2 shows that for every countable subset of $\mathcal{C}(H)$ there is an inner automorphism of $\mathcal{C}(H)$ that sends a to $\Phi(a)$. □

In the proof of Theorem 1.1, CH was used only in the first line to find enumerations $(\vec{F}^\xi)$ and (β_ξ), $\xi < \omega_1$. The first enumeration was used to find a $\leq^*$-cofinal ω_1-sequence of partitions $\vec{E}^\xi$ and the second to assure that $\Phi \neq \Psi_{\beta_\xi}$ for all ξ. A weakening of CH known as $\mathfrak{d} = \aleph_1$ (see e.g., [4]) suffices for the first task. Stefan Geschke pointed out that the proof of Theorem 1.1 easily gives $2^{\aleph_1}$ automorphisms and therefore that the existence of the second enumeration may be replaced with another cardinal inequality, $2^{\aleph_0} < 2^{\aleph_1}$ (so-called *weak Continuum Hypothesis*). Therefore the assumptions $\mathfrak{d} = \aleph_1$ and $2^{\aleph_0} < 2^{\aleph_1}$ together imply the existence of an outer automorphism of the Calkin algebra. It not known whether these assumptions imply the existence of a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$.

2. **The toolbox**

2.1. *Descriptive set theory.* The standard reference is [27]. A topological space is *Polish* if it is separable and completely metrizable. We consider $\mathcal{B}(H)$ with the strong operator topology. For every $M < \infty$ the strong operator topology on $(\mathcal{B}(H))_{\leq M} = \{a \in \mathcal{B}(H) \mid \|a\| \leq M\}$ is Polish. Throughout 'Borel' refers to the Borel structure on $\mathcal{B}(H)$ induced by the strong operator topology.

Fix a Polish space X. A subset of X is *meager* (or, it is of *first category*) if it can be covered by countably many closed nowhere-dense sets. A subset of X has the *Property of Baire* (or, is *Baire-measurable*) if its symmetric difference with some open set is meager. A subset of X is *analytic* if it is a continuous image of a Borel subset of a Polish space. Analytic sets (as well as their complements, *coanalytic sets*), share the classical regularity properties of Borel sets such as the Property of Baire and measurability with respect to Borel measures. A function f between Polish spaces is C-*measurable* if it is measurable with respect to the smallest σ-algebra generated by analytic sets. C-measurable functions are Baire-measurable (and therefore continuous on a dense G_δ subset of the domain) and, if the domain is also a locally compact topological group, Haar-measurable. The following uniformization theorem will be used a large number of times in the proof of Theorem 1; for its proof see e.g. [27, Theorem 18.1].

Theorem 2.1 (Jankov, von Neumann). *If X and Y are Polish spaces and $A \subseteq X \times Y$ is analytic, then there is a* C*-measurable function $f\colon X \to Y$ such that for all $x \in X$, if $(x, y) \in A$ for some y, then $(x, f(x)) \in A$.* □

A function f as above *uniformizes* A. In general it is impossible to uniformize a Borel set by a Borel-measurable function, but the following two special cases of [27, Theorem 8.6] (applied with $\mathcal{I}_x$ being the meager ideal or the null ideal, respectively, for each $x \in X$) will suffice for our purposes.

Theorem 2.2. *Assume X and Y are Polish spaces, $A \subseteq X \times Y$ is Borel and for each $x \in X$ the vertical section $A_x = \{y \mid (x, y) \in A\}$ is either empty or nonmeager. Then A can be uniformized by a Borel-measurable function.* □

Theorem 2.3. *Assume X and Y are Polish spaces, Y carries a Borel probability measure, $A \subseteq X \times Y$ is Borel and for each $x \in X$ the vertical section $A_x = \{y \mid (x, y) \in A\}$ is either empty or has a positive measure. Then A can be uniformized by a Borel-measurable function.* □

A topological group is *Polish* if has a compatible complete separable metric. The unitary group of $\mathcal{B}(H)$, for a separable H, is a Polish group with respect to the strong operator topology (see [27, 9.B(6)]). Also, the unitary group of every separably acting von Neumann algebra $\mathcal{D}$ is Polish with respect to strong operator topology. A complete separable metric on $\mathcal{U}(\mathcal{D})$ is given by $d'(a, b) = d(a, b) + d(a^*, b^*)$, where d is the usual complete metric on $\mathcal{D}_{\leq 1}$ compatible with the strong operator topology. The following is Pettis's theorem (see [27, Theorem 9.10]).

Theorem 2.4. *Every Baire-measurable homomorphism from a Polish group into a second-countable group is continuous.* □

We end this subsection with a simple computation.

Lemma 2.5. *Consider $\mathcal{B}(H)$ with the strong operator topology. Fix $M < \infty$.*

(a) *The set of compact operators of norm $\leq M$ is a Borel subset of $\mathcal{B}(H)_{\leq M}$.*
(b) *For $\varepsilon \geq 0$ the set of operators a of norm $\leq M$ such that $\|a\|_{\mathcal{K}} \leq \varepsilon$ is a Borel subset of $\mathcal{B}(H)_{\leq M}$.*

Proof. (a) Recall that $\mathbf{R}_n$ is a fixed increasing sequence of finite-rank projections such that $\bigvee_n \mathbf{R}_n = I$. For a projection P the set $\{a \mid \|Pa\| \leq x\}$ is strongly closed for every $x \geq 0$, and

$$\mathcal{K}(H) = \{a \mid (\forall m)(\exists n)\|(I - \mathbf{R}_n)a\| < 1/m\}.$$

Hence $\mathcal{K}(H) \cap \mathcal{B}(H)_{\leq M}$ is a relatively $F_{\sigma\delta}$ subset of $\mathcal{B}(H)_{\leq M}$ for each M.

The proof of (b) is almost identical. □

2.2. *Set theory of the power-set of the natural numbers.* A metric d on $\mathcal{P}(\mathbb{N})$ is defined by $d(A,B) = 2^{-\min(A\Delta B)}$, where $A\Delta B$ is the symmetric difference of A and B. This turns $(\mathcal{P}(\mathbb{N}), \Delta)$ into a compact metric topological group, and the natural identification of subsets of $\mathbb{N}$ with infinite sequences of zeros and ones is a homeomorphism into the triadic Cantor set.

2.3. *Todorcevic's axiom.* Let X be a separable metric space and let

$$[X]^2 = \{\{x,y\} | x \neq y \text{ and } x, y \in X\}.$$

Subsets of $[X]^2$ are naturally identified with the symmetric subsets of $X \times X$ minus the diagonal. A *coloring* $[X]^2 = K_0 \cup K_1$ is *open* if K_0, when identified with a symmetric subset of $X \times X$, is open in the product topology. If $K \subseteq [X]^2$, then a subset Y of X is K*-homogeneous* if $[Y]^2 \subseteq K$. Since $K_1 = [X]^2 \setminus K_0$ is closed, a closure of a K_1-homogeneous set is always K_1-homogeneous. The following axiom was introduced by Todorcevic in [40] under the name Open Coloring Axiom, OCA.

TA. If X is a separable metric space and $[X]^2 = K_0 \cup K_1$ is an open coloring, then X either has an uncountable K_0-homogeneous subset or it can be covered by a countable family of K_1-homogeneous sets.

The instance of TA when X is analytic follows from the usual axioms of mathematics (see e.g., [19]). In this case the uncountable K_0-homogeneous set can be chosen to be perfect; hence this variant of TA is a generalization of the classical perfect-set property for analytic sets ([27]).

Note that K_1 is not required to be open. We should say a word to clarify our use of the phrase 'open coloring.' In order to be able to apply TA to some coloring $[X]^2 = K_0 \cup K_1$ it suffices to know that there is a separable metric topology τ on X which makes K_0 open. For example, for $X \subseteq \mathcal{P}(\mathbb{N})$ and for each $x \in X$ we fix an $f_x \in \mathbb{N}^{\mathbb{N}}$ and consider the coloring $[X]^2 = K_0 \cup K_1$ defined by

$$\{x,y\} \in K_0 \text{ if and only if } f_x(n) \neq f_y(n) \text{ for some } n \in x \cap y.$$

This K_0 is not necessarily open in the topology inherited from $\mathcal{P}(\mathbb{N})$ (§2.2). However, it is open in the topology obtained by identifying X with a subspace of $\mathcal{P}(\mathbb{N}) \times \mathbb{N}^{\mathbb{N}}$ via the embedding $x \mapsto \langle x, f_x \rangle$. We shall use such refinements tacitly quite often and say only that the coloring $[X]^2 = K_0 \cup K_1$ is open, meaning that it is open in some separable metric topology.

Assume $K_0 \subseteq [X]^2$ is equal to a union of countably many rectangles $K_0 = \bigcup_i U_i \times V_i$. If sets U_i and V_i separate points of C, then this is equivalent to K_0 being open in some separable metric topology on X. Even without this assumption, by [13, Prop. 2.2.11], TA is equivalent to its apparent strengthening to colorings K_0 that can be expressed as a union of countably many rectangles. Reformulations of TA are discussed at length in [13, §2].

2.4. *Absoluteness.* A *vertical section* of $B \subseteq X \times Y$ is a set of the form $B_x = \{y \mid (x, y) \in B\}$ for some $x \in X$.

Theorem 2.6. *Assume X and Y are Polish spaces and $B \subseteq X \times Y$ is Borel. Truth (or falsity) of the assertion that some vertical section of B is empty cannot be changed by going to a forcing extension.*

In particular, if there is a proof using TA *that B has an empty vertical section, then B has an empty vertical section.*

Proof. The first part is a special case of Shoenfield's Absoluteness Theorem (see e.g., [23, Theorem 13.15]). The second part follows from the fact that every model of ZFC has a forcing extension in which TA holds ([41]). $\square$

3. Coherent families of unitaries

If u is a partial isometry between cofinite-dimensional subspaces of H we write $\Psi_u(a) = uau^*$. An operator in $\mathcal{C}(H)$ is invertible if and only if it is of the form $\pi(a)$ for some Fredholm operator a (this is Atkinson's theorem, [32, Theorem 3.11.11]; see also [6, §3]). Therefore, inner automorphisms of $\mathcal{C}(H)$ are exactly the ones of the form Ψ_u for a partial isomorphism u between cofinite-dimensional subspaces of H. A family $\mathcal{F}$ of pairs $(\vec{E}, u)$ satisfying conditions (1)–(3) below is called a *coherent family of unitaries*:

(1) If $(\vec{E}, u) \in \mathcal{F}$ then $\vec{E}$ is a partition of $\mathbb{N}$ into finite intervals and u is a partial isometry between cofinite-dimensional subspaces of H,
(2) for all $(\vec{E}, u)$ and $(\vec{F}, v)$ in $\mathcal{F}$ and all $a \in \mathcal{D}[\vec{E}] \cap \mathcal{D}[\vec{F}]$ the operator $\Psi_u(a) - \Psi_v(a)$ is compact,
(3) for every partition $\vec{E}$ of $\mathbb{N}$ into finite intervals there is u such that $(\vec{E}, u) \in \mathcal{F}$.

(By (1) above, $\pi(u)$ is a unitary in the Calkin algebra for each $(\vec{E}, u) \in \mathcal{F}$.) The following is an immediate consequence of Lemmas 1.4 and 1.3.

Lemma 3.1. *If $\mathcal{F}$ is a coherent family of unitaries, then there is a unique automorphism Φ of $\mathcal{C}(H)$ such that Ψ_u is a representation of Φ on $\mathcal{D}[\vec{E}]$ for all $(\vec{E}, u) \in \mathcal{F}$.* $\square$

Such an automorphism Φ is *determined by a coherent family of unitaries.* Since $\mathcal{D}[\vec{E}] \subseteq \mathcal{D}[\vec{F}]$ whenever $\vec{F}$ is coarser than $\vec{E}$, Φ is uniquely determined by those $(\vec{E}, u) \in \mathcal{F}$ such that $\#E_n$ is strictly increasing. In Theorem 1.1 we have seen that the Continuum Hypothesis implies the existence of an outer automorphism determined by a coherent family of unitaries. The following result, which is the main result of this section, complements Theorem 1.1.

Theorem 3.2. *Assume* TA. *Then every automorphism of $\mathcal{C}(H)$ determined by a coherent family of unitaries is inner.*

We shall first show that it suffices to prove Theorem 3.2 in the case when each u is of the form u_α for $\alpha \in \mathbb{T}^{\mathbb{N}}$, as constructed in Section 1. If Φ is determined by $\mathcal{F}$, fix $(\vec{E}_0, u_0) \in \mathcal{F}$. Then $\mathcal{F}' = \{(\vec{F}, v(u_0)^*) \mid (\vec{F}, v) \in \mathcal{F}\}$ is a coherent family of unitaries. The automorphism Φ' determined by $\mathcal{F}'$ is inner if and only if Φ is inner. Also, Φ' is equal to the identity on the standard atomic masa. In the proof of Theorem 3.2 we may therefore assume Φ is equal to the identity on the standard atomic masa. Recall that $\mathbb{T}$ is the circle group. For $\alpha \in \mathbb{T}^{\mathbb{N}}$ by u_α denote the unitary that sends e_n to $\alpha_n e_n$. By our convention and Lemma 1.4, for every $(\vec{E}, u) \in \mathcal{F}$ there is α such that Ψ_{u_α} and Ψ_u agree modulo compacts on $\mathcal{B}(H)$. We may therefore identify $\mathcal{F}$ with the family $\{(\vec{E}, \alpha) \mid (\vec{E}, u) \in \mathcal{F}, \Psi_u \text{ and } \Psi_{u_\alpha} \text{ agree modulo compacts}\}$. It will also be convenient to code partitions $\vec{E}$ by functions $f\colon \mathbb{N} \to \mathbb{N}$.

3.1. *The directed set* $(\mathbb{N}^{\uparrow\mathbb{N}}, \leq^*)$. Let $\mathbb{N}^{\uparrow\mathbb{N}}$ denote the set of all strictly increasing functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(0) > 0$. (The reader should be warned that the requirement $f(0) > 0$ is *nonstandard* and *important*.) Such a function can code a partition of $\mathbb{N}$ into finite intervals in more than one way. It will be convenient to use the following quantifiers: $(\forall^\infty n)$ stands for $(\exists n_0)(\forall n \geq n_0)$ and $(\exists^\infty n)$ stands for the dual quantifier, $(\forall n_0)(\exists n \geq n_0)$. For f and g in $\mathbb{N}^{\uparrow\mathbb{N}}$ write $f \geq^* g$ if $(\forall^\infty n) f(n) \geq g(n)$. A diagonal argument shows that $\mathbb{N}^{\uparrow\mathbb{N}}$ is *σ-directed* in the sense that for each sequence (f_n) in $\mathbb{N}^{\uparrow\mathbb{N}}$ there is $g \in \mathbb{N}^{\uparrow\mathbb{N}}$ such that $f_n \leq^* g$ for all n.

For $f \in \mathbb{N}^{\uparrow\mathbb{N}}$ recursively define f^+ by $f^+(0) = f(0)$ and $f^+(i+1) = f(f^+(i))$. Some $\mathcal{X} \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ is *$\leq^*$-cofinal* if $(\forall f \in \mathbb{N}^{\uparrow\mathbb{N}})(\exists g \in \mathcal{X}) f \leq^* g$.

LEMMA 3.3. *Assume $\mathcal{X} \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ is $\leq^*$-cofinal.*

(1) *If $\mathcal{X}$ is partitioned into countably many pieces, then at least one of the pieces is $\leq^*$-cofinal.*
(2) $(\exists^\infty n)(\exists i)(\forall k \geq n)(\exists f \in \mathcal{X})(f(i) \leq n$ *and* $f(i+1) \geq k)$.
(3) $\{f^+ \mid f \in \mathcal{X}\}$ *is $\leq^*$-cofinal.*

Proof. (1) Assume the contrary, let $\mathcal{X} = \bigcup_n \mathcal{Y}_n$ be such that no $\mathcal{Y}_n$ is cofinal. Pick f_n such that $f_n \not\leq^* g$ for all $g \in \mathcal{Y}_n$. If $f \geq^* f_n$ for all n, then there is no $g \in \mathcal{X}$ such that $f \leq^* g$ — a contradiction.

(2) We first prove

(*) $(\exists^\infty n)(\forall k \geq n)(\exists i)(\exists f \in \mathcal{X})(f(i) \leq n$ and $f(i+1) \geq k)$.

Assume not and fix n_0 such that for all $n \geq n_0$ there is $k = g(n)$, such that for all $f \in \mathcal{X}$ and all i, if $f(i) \leq n$ then $f(i+1) \leq g(n)$. Define functions h_m for $m \in \mathbb{N}$ recursively by $h_m(0) = \max(m, n_0)$ and $h_m(i+1) = g(h_m(i))$. For $f \in \mathcal{X}$ we have $f \leq^* h_{f(0)}$. By recursion we prove $f(i) \leq h_{f(0)}(i)$ for all i. For $i = 0$ this is immediate. Assume $f(i) \leq h_{f(0)}(i)$. Then $f(i+1) \leq g(f(i)) \leq$

$g(h_{f(0)}(i)) = h_{f(0)}(i+1)$. Now fix $h \in \mathbb{N}^{\uparrow\mathbb{N}}$ such that $h_m \leq^* h$. By the above, there is no $f \in \mathcal{X}$ such that $h \leq^* f$, a contradiction.

For each n the set $\{i \mid (\exists f \in \mathcal{X}) f(i) \leq n\}$ is finite. Therefore in (*) the same i works for infinitely many k. An easy induction shows that for $f \in \mathbb{N}^{\uparrow\mathbb{N}}$ we have $f(i) \leq f^+(i)$ for all i, and (3) follows. $\square$

LEMMA 3.4. *If $f, g \in \mathbb{N}^{\uparrow\mathbb{N}}$ and $f \geq^* g$ then for all but finitely many n there is i such that $f^i(0) \leq g(n) < g(n+1) \leq f^{i+2}(0)$. If moreover $f(m) \geq g(m)$ for all m, then for every n there is such an i.*

Proof. If n is such that $f(m) \geq g(m)$ for all $m \geq n$, let i be the minimal such that $f^{i+1}(0) \geq g(n)$. Then $f^{i+2}(0) \geq f(g(n))$, and since $g \in \mathbb{N}^{\uparrow\mathbb{N}}$ implies $g(n) \geq n+1$ this is $\geq f(n+1) \geq g(n+1)$. $\square$

To $f \in \mathbb{N}^{\uparrow\mathbb{N}}$ associate sequences of finite intervals of $\mathbb{N}$:

$$\begin{aligned} E_n^f &= [f(n), f(n+1)), \\ F_n^f &= [f^n(0), f^{n+2}(0)), \\ F_n^{f,\mathrm{even}} &= [f^{2n}(0), f^{2n+2}(0)), \\ F_n^{f,\mathrm{odd}} &= [f^{2n+1}(0), f^{2n+3}(0)) \end{aligned}$$

('F' is for 'fast'). By Lemma 3.4, if $\mathcal{X} \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ is $\leq^*$-cofinal, then each one of $\{\vec{F}^{f,\mathrm{even}} \mid f \in \mathcal{X}\}$ and $\{\vec{F}^{f,\mathrm{odd}} \mid f \in \mathcal{X}\}$ is a cofinal family of partitions as defined in Section 1. Notation $\Delta_I(\alpha, \beta)$ used in the following proof was defined before Lemma 1.5.

LEMMA 3.5. *Assume Φ is an automorphism of $\mathcal{C}(H)$ whose restriction to the standard atomic masa is equal to the identity and which is determined by a coherent family of unitaries. For each $f \in \mathbb{N}^{\uparrow\mathbb{N}}$ there is $\alpha \in \mathbb{T}^{\mathbb{N}}$ such that Ψ_α is a representation of Φ on both $\mathcal{D}[\vec{F}^{f,\mathrm{even}}]$ and $\mathcal{D}[\vec{F}^{f,\mathrm{odd}}]$.*

Proof. By Lemma 1.4, for each f there are β and γ in $\mathbb{T}^{\mathbb{N}}$ such that Ψ_β is a representation of Φ on $\mathcal{D}[\vec{F}^{f,\mathrm{even}}]$ and Ψ_γ is a representation of Φ on $\mathcal{D}[\vec{F}^{f,\mathrm{odd}}]$. Define β' and γ' recursively as follows. For $i \in [f^0(0), f^2(0))$, let $\beta'(i) = \beta(i)$. If $\beta'(i)$ has been defined for $i < f^{2n}(0)$, then for $i \in [f^{2n-1}(0), f^{2n+1}(0))$ let

$$\gamma'(i) = \gamma(i)\overline{\gamma(f^{2n-1}(0))}\beta'(f^{2n-1}(0)).$$

If $\gamma'(i)$ has been defined for $i < f^{2n+1}(0)$ then for $i \in [f^{2n}(0), f^{2n+2}(0))$ let

$$\beta'(i) = \beta(i)\overline{\beta(f^{2n}(0))}\gamma'(f^{2n}(0)).$$

Then $\gamma'(f^j(0)) = \beta'(f^j(0))$ for all j, $\Psi_{\beta'}(a) = \Psi_\beta(a)$ for all $a \in \mathcal{D}[\vec{F}^{f,\mathrm{even}}]$, and $\Psi_{\gamma'}(a) = \Psi_\gamma(a)$ for all $a \in \mathcal{D}[\vec{F}^{f,\mathrm{odd}}]$. Let $J_n = [f^n(0), f^{n+1}(0))$. Then $\sup_{i \in J_n} |\beta'(i) - \gamma'(i)| \leq \Delta_{J_n}(\beta, \gamma')$ by Lemma 1.5(2). Since $\Psi_{\beta'}$, Ψ_β, $\Psi_{\gamma'}$ and Ψ_γ are all representations of Φ on $\mathcal{D}[\vec{J}]$, by Lemma 1.6(b) we conclude that

$\Delta_{J_{2n+1}}(\beta',\gamma) \to 0$ and $\Delta_{J_{2n}}(\beta,\gamma') \to 0$ as $n \to \infty$. Thus $\lim_i |\beta'(i) - \gamma'(i)| = 0$, and therefore Lemma 1.6(a) implies that $\Psi_{\gamma'}$ and $\Psi_{\beta'}$ agree on $\mathcal{B}(H)$ modulo the compact operators. Therefore $\alpha = \beta'$ is as required. □

Proof of Theorem 3.2. As pointed out after the statement of Theorem 3.2, we may assume Φ is equal to the identity on the standard atomic masa and that the unitary u is of the form u_α for each (f, u) in the coherent family defining Φ. Let $\mathcal{X} \subseteq \mathbb{N}^{\uparrow\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}$ be the set of all pairs (f, α) such that Ψ_α is a representation of Φ on both $\mathcal{D}[\vec{F}^{f,\mathrm{even}}]$ and $\mathcal{D}[\vec{F}^{f,\mathrm{odd}}]$. By Lemma 3.5, for every $f \in \mathbb{N}^{\uparrow\mathbb{N}}$ there is α such that $(f,\alpha) \in \mathcal{X}$.

For $\varepsilon > 0$ define $[\mathcal{X}]^2 = K_0^\varepsilon \cup K_1^\varepsilon$ by assigning a pair (f,α), (g,β) to K_0^ε if

(K_0^ε): There are m, n such that with $J = F_m^f \cap F_n^g$ we have $\Delta_J(\alpha,\beta) > \varepsilon$.

We consider $\mathbb{N}^{\uparrow\mathbb{N}}$ with the *Baire space topology*, induced by the metric

$$d(f,g) = 2^{-\min\{n \mid f(n) \neq g(n)\}}.$$

This is a complete separable metric. Consider $\mathbb{T}^{\mathbb{N}}$ in the product of strong operator topology and $\mathcal{X}$ in the product topology. If K_0^ε is identified with a symmetric subset of $\mathcal{X}^2$ off the diagonal, then it is open in this topology.

Claim 3.6. TA *implies that for* $\varepsilon > 0$ *there are no uncountable* K_0^ε*-homogeneous subsets of* $\mathcal{X}$.

Proof. Assume the contrary and fix $\varepsilon > 0$ and an uncountable K_0^ε-homogeneous set $\mathcal{H}$. We shall refine $\mathcal{H}$ to an uncountable subset several times, until we reach a contradiction. In order to keep the notation under control, each successive refinement will be called $\mathcal{H}$. Let

$$\mathcal{F} = \{g^+ \mid (\exists\alpha)(g,\alpha) \in \mathcal{H}\}.$$

We may assume $\mathcal{H}$ has size $\aleph_1$, hence TA and [40, Ths. 3.4 and 8.5] imply that $\mathcal{F}$ is $\leq^*$-bounded by some $\bar{f} \in \mathbb{N}^{\uparrow\mathbb{N}}$. For each $g \in \mathcal{F}$ fix l_g such that $\bar{f}(n) \geq g(n)$ for all $n \geq l_g$ and let $s_g = g \restriction l_g$. Fix $(\bar{l}, \bar{s})$ such that $\{g \in \mathcal{F} \mid (l_g, s_g) = (\bar{l}, \bar{s})\}$ is uncountable. By refining $\mathcal{H}$ and increasing $\bar{f} \restriction \bar{l}$ to $\bar{f}(\bar{l})$ we may assume $\bar{f}(n) \geq g^+(n)$ for all $g^+ \in \mathcal{F}$ and all $n \in \mathbb{N}$. Lemma 3.4 implies that for every $(g,\alpha) \in \mathcal{H}$ and every n there is i such that $F_n^g \cup F_{n+1}^g \subseteq F_i^{\bar{f}} \cup F_{i+1}^{\bar{f}}$. By Lemma 3.5 we may fix $\alpha \in \mathbb{T}^{\mathbb{N}}$ such that Ψ_α is a representation of Φ on both $\mathcal{D}[\vec{F}^{\bar{f},\mathrm{even}}]$ and $\mathcal{D}[\vec{F}^{\bar{f},\mathrm{odd}}]$. Lemma 1.6(b) implies that for every $(g,\beta) \in \mathcal{H}$ we have $\limsup_n \Delta_{E_n^g}(\alpha,\beta) = 0$. Fix $\bar{k}, \bar{m} \in \mathbb{N}$ for which the set $\mathcal{H}'$ of all $(g,\beta) \in \mathcal{H}$ such that $g^{\bar{m}+1}(0) = \bar{k}$ and $\Delta_{E_n^g}(\alpha,\beta) < \varepsilon/2$ whenever $n \geq \bar{m}$ is uncountable. By the separability of $\mathbb{T}^{\mathbb{N}}$ we can find distinct (g,β) and (h,γ) in $\mathcal{H}'$ such that $g^i(0) = h^i(0)$ for all $i \leq \bar{m}+1$ and $|\beta(i) - \gamma(i)| < \varepsilon/2$ for all $i \leq \bar{k}$.

Fix i and j such that $J = F_i^g \cap F_j^h \neq \emptyset$. We shall prove $\Delta_J(\beta,\gamma) < \varepsilon$. Since $\max_{i<\bar{k}} |\beta(i) - \gamma(i)| < \varepsilon/2$, we may assume that $J \setminus [0,\bar{k}) \neq \emptyset$ and therefore $J \cap [0, g^{\bar{m}}(0)) = J \cap [0, h^{\bar{m}}(0)) = \emptyset$. Find l such that $F_i^g \subseteq F_l^{\bar{f}}$, and therefore $J \subseteq F_l^{\bar{f}}$. Then

$$\Delta_J(\beta,\gamma) \leq \Delta_{F_i^g \cap F_l^{\bar{f}}}(\beta,\alpha) + \Delta_{F_j^h \cap F_l^{\bar{f}}}(\gamma,\alpha) < \varepsilon.$$

Since i and j were arbitrary we conclude that $\{(g,\beta),(h,\gamma)\} \in K_1^\varepsilon$, contradicting our assumption on $\mathcal{H}$. □

Since K_0^ε is an open partition, by TA and Claim 3.6, for each $\varepsilon > 0$ there is a partition of $\mathcal{X}$ into countably many K_1^ε-homogeneous sets.

We fix n and let $\varepsilon_n = 2^{-n}$. Repeatedly using Lemma 3.3, find sets $\mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \dots$ and $0 = m(0) < m(1) < \dots$ so that (1) each $\mathcal{X}_n$ is $K_1^{\varepsilon_n}$-homogeneous, (2) each set $\{f \mid (\exists\alpha)(f,\alpha) \in \mathcal{X}_n\}$ is $\leq^*$-cofinal, and (3) for all n and all $k > m(n)$ there are $j \in \mathbb{N}$ and $(f,\alpha) \in \mathcal{X}_n$ such that $f^j(0) \leq m(n)$ and $f^{j+1}(0) \geq k$.

In $\mathcal{X}_n$ pick a sequence $(f_{n,i},\alpha_{n,i})$ and $j(i)$, for $i \in \mathbb{N}$, such that

(4) $(f_{n,i})^{j(i)}(0) \leq m(n) < m(n+i) \leq (f_{n,i})^{j(i)+1}(0)$ for all i.

By compactness we may assume $\alpha_{n,i}$ converge to $\alpha_n \in \mathbb{T}^{\mathbb{N}}$. We claim that

(5) $\Delta_{[m(l),\infty)}(\alpha_k,\alpha_l) \leq \varepsilon_k$ whenever $k \leq l$.

Assume not and fix $m(l) \leq n_1 < n_2$ such that $\rho(n_1,n_2,\alpha_k,\alpha_l) > \varepsilon_k$. By (4), for all large enough i we have $(f_{k,i})^{j(i)+1}(0) > n_2$ and $(f_{l,i})^{j(i)+1}(0) > n_2$. Since $\lim_i \alpha_{k,i} = \alpha_k$ and $\lim_i \alpha_{l,i} = \alpha_l$ we have $\rho(n_1,n_2,\alpha_{k,i},\alpha_{l,i}) > \varepsilon_k$ for large enough i. These facts imply $\Delta_{F_{j(i)}^{f_{k,i}} \cap F_{j(i)}^{f_{l,i}}}(\alpha_{k,i},\alpha_{l,i}) > \varepsilon_k$ for a large enough i. However, $(f_{k,i},\alpha_{k,i}) \in \mathcal{X}_k$ and $(f_{l,i},\alpha_{l,i}) \in \mathcal{X}_l \subseteq \mathcal{X}_k$, contradicting the homogeneity of $\mathcal{X}_k$.

By (5) and Lemma 1.5(5), for $k < l$ we can fix $z_{k,l} \in \mathbb{T}$ such that

(6) $\sup_{i \geq m(l)} |z_{k,l}\alpha_k(i) - \alpha_l(i)| \leq \varepsilon_k$,

with $z_{k,k} = 1$. We claim that

(7) $|z_{k,l} - z_{k,m}\overline{z_{l,m}}| \leq 3\varepsilon_{\min\{k,l,m\}}$ for all k,l and m.

For β and γ in $\mathbb{T}^{\mathbb{N}}$ and $\varepsilon > 0$, in the proof of (7) we write $\beta \sim_\varepsilon \gamma$ if $\sup_{i \geq m(\max\{k,l,m\})} |\beta(i) - \gamma(i)| \leq \varepsilon$. Letting $\varepsilon = \varepsilon_{\min\{k,l,m\}}$, by (6) we have

$$z_{k,l}\alpha_k \sim_\varepsilon \alpha_l \sim_\varepsilon \overline{z_{l,m}}\alpha_m \sim_\varepsilon \overline{z_{l,m}}z_{k,m}\alpha_k$$

and therefore $|z_{k,l} - z_{k,m}\overline{z_{l,m}}| \leq 3\varepsilon$.

We want to find an infinite $Y \subseteq \mathbb{N}$ such that for all $i < j$ in Y we have $|1 - z_{k(i),k(j)}| \leq 4\varepsilon_{k(i)}$. To this end, define a coloring $M_0 \cup M_1$ of triples in $\mathbb{N}$ by putting a triple $k < l < m$ into M_0 if

$$|z_{l,m} - 1| \leq 4\varepsilon_k.$$

We claim there are no infinite sets Y such that every triple of elements from Y belongs to M_1. Assume the contrary. For such Y let $k = \min(Y)$ and pick $n \in Y$ such that Y has at least $2\pi/\varepsilon_k$ strictly between k and n. Then there are distinct l and m in Y between k and n such that $|z_{l,n} - z_{m,n}| \leq \varepsilon_k$. Using (7) in the second inequality we have

$$|z_{l,m} - 1| \leq |z_{l,m} - z_{l,n}\overline{z_{m,n}}| + |1 - z_{l,n}\overline{z_{m,n}}| \leq 4\varepsilon_k.$$

Therefore there is no infinite M_1-homogeneous set of triples. By using Ramsey's theorem we can find an infinite $Y \subseteq \mathbb{N}$ such that all triples of elements of Y belong to M_0. Enumerate Y increasingly as $k(i)$, for $i \in \mathbb{N}$. We may assume $k(0) \geq 2$ and therefore $4\varepsilon_{k(i)} \leq \varepsilon_i$. Since $|1 - z_{k(i),k(j)}| \leq \varepsilon_i$, we have

(8) $\sup_{l \geq m(k(i))} |\alpha_{k(i)} - \alpha_{k(j)}| \leq \varepsilon'_i$ for all $i < j$.

Define $\gamma \in \mathbb{T}^{\mathbb{N}}$ by $\gamma(l) = \alpha_{k(i)}(l)$ if $l \in [m(k(i)), m(k(i+1)))$ and $\gamma(l) = \alpha_{k(0)}(l)$ if $l < m(k(0))$. By (8) we have

(9) $|\gamma(l) - \alpha_{k(i)}(l)| \leq \varepsilon_i$ for all i and all $l \geq m(k(i))$.

We claim that for all j (recall that $F_i^f = [f^i(0), f^{i+1}(0)]$)

(10) If $(f, \beta) \in \mathcal{X}_{k(j)}$, then for all i we have $\Delta_{F_i^f \setminus m_{k(j)}}(\beta, \gamma) \leq 5\varepsilon_j$.

Write $n = k(j)$. Since $(f_{n,l}, \alpha_{n,l}) \in \mathcal{X}_n$, for l large enough to have

$$[(f_{n,l})^{j(l)}(0), (f_{n,l})^{j(l)+1}(0)) \supseteq F_i^f \setminus m(n),$$

we have $\Delta_{F_i^f \setminus m(n)}(\beta, \alpha_{n,l}) \leq \varepsilon_n$. The continuity implies $\Delta_{F_i^f \setminus m(n)}(\beta, \alpha_n) \leq \varepsilon_n$ and (10) implies $\Delta_{F_i^f \setminus m_{k(j)}}(\beta, \gamma) \leq 5\varepsilon_j$.

By Lemma 3.1, it suffices to prove Ψ_γ is a representation of Φ on every $\mathcal{D}[\vec{E}]$. Fix g such that $\vec{E} = \vec{E}^g$ (with $E_n^g = [g(n), g(n+1))$). Fix β such that Ψ_β is a representation of Φ on $\mathcal{D}[\vec{E}^g]$. By Lemma 1.6(b) it suffices to prove $\Delta_{E_n^g}(\beta, \gamma) \to 0$ as $n \to \infty$. Fix $m \in \mathbb{N}$ and $(f, \alpha) \in \mathcal{X}_m$ such that $f \geq^* g$. By Lemma 1.6(b) we have $\lim_n \Delta_{E_n^g}(\alpha, \beta) \to 0$. By (10) we have $\limsup_n \Delta_{E_n^g}(\beta, \gamma) \leq \limsup_n \Delta_{F_n^f}(\alpha, \gamma) \leq 5\varepsilon_n$. Since n was arbitrary, the conclusion follows. $\square$

The construction in Theorem 1.1 hinges on the existence of a nontrivial coherent family of unitaries under CH. Theorem 3.2 shows that under TA, every coherent family of unitaries is 'uniformized' by a single unitary. There is an analogy to the effect of CH/TA on the additivity of the strong homology as exhibited in [30]/[9] and [40, Theorem 8.7]. In the latter, uniformizing certain families of functions from subsets of $\mathbb{N}$ into $\{0,1\}$ that are coherent modulo finite plays the key role. For more on such uniformizations see [13, §§2.2–2.4].

4. **Representations and ε-approximations**

The present section is a loose collection of results showing that a sufficiently measurable representation, or an approximation to a representation, can be further improved in one way or another.

Lemma 4.1 below illustrates how drastically different automorphisms of the Calkin algebra are from the automorphisms of Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and is directly responsible for the fact that Martin's Axiom is not needed in the proof of Theorem 1. Recall that for $\mathcal{D} \subseteq \mathcal{B}(H)$ we say Φ is *inner on* $\mathcal{D}$ if there is an inner automorphism Φ' of $\mathcal{C}(H)$ such that the restrictions of Φ and Φ' to $\pi[\mathcal{D}]$ coincide.

LEMMA 4.1. *Assume $\mathcal{D}_1$ and $\mathcal{D}_2$ are subsets of $\mathcal{B}(H)$ such that for some partial isometry u we have $u\mathcal{D}_2u^* \subseteq \mathcal{D}_1$ and $P = u^*u$ satisfies $P\mathcal{D}_2P = \mathcal{D}_2$. If Φ is inner on $\mathcal{D}_1$, then it is inner on $\mathcal{D}_2$.*

Proof. Fix v such that $a \mapsto vav^*$ is a representation of Φ on $\mathcal{D}_1$ and w such that $\pi(w) = \Phi(\pi(u))$. If $b \in \mathcal{D}_2$, then $ubu^* \in \mathcal{D}_1$ and $u^*ubu^*u = b$. If Ψ is any representation of Φ, then we have (writing $c \sim_{\mathcal{K}} d$ for '$c - d$ is compact')

$$\Psi(b) \sim_{\mathcal{K}} \Psi(u^*)\Psi(ubu^*)\Psi(u) \sim_{\mathcal{K}} w^*vubu^*v^*w.$$

Therefore w^*vu shows that Φ is inner on $\mathcal{D}_2$. □

An analogue of Lemma 4.1 fails for automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. For example, in [38] (see also [39]) it was proved that a weakening of the Continuum Hypothesis implies the existence of a nontrivial automorphism whose ideal of trivialities is a maximal ideal.

4.1. *ε-approximations.* Assume $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras, J_1 and J_2 are their ideals, $\Phi\colon \mathcal{A}/J_1 \to \mathcal{B}/J_2$ is a *-homomorphism, and $\mathcal{X} \subseteq \mathcal{A}$. A map Θ whose domain contains $\mathcal{X}$ and is contained in $\mathcal{A}$ and whose range is contained in $\mathcal{B}$ is an *ε-approximation to Φ on $\mathcal{X}$* if for all $a \in \mathcal{X}$ we have

$$\|\Phi(\pi_{J_1}(a)) - \pi_{J_2}(\Theta(a))\| \leq \varepsilon$$

for all $a \in \mathcal{X}$. If $\mathcal{X} = \mathcal{A}$, we say Θ is an *ε-approximation to* Φ.

LEMMA 4.2. *Assume $\mathcal{A}$ and $\mathcal{B}$ are* C*-*subalgebras of $\mathcal{B}(H)$ containing $\mathcal{K}(H)$ and Φ is a *-homomorphism from $\mathcal{A}/\mathcal{K}(H)$ into $\mathcal{B}/\mathcal{K}(H)$. Then*

(1) *Φ has a Borel-measurable representation if and only if it has a Borel-measurable representation on $\mathcal{U}(\mathcal{A})$.*
(2) *If Φ has a Borel-measurable ε-approximation on $\mathcal{U}(\mathcal{A})$, then it has a Borel-measurable 4ε-approximation on $\mathcal{A}_{\leq 1}$.*

Proof. (1) We only need to prove the reverse implication. There are norm-continuous maps $\gamma_i\colon \mathcal{A}_{\leq 1} \to \mathcal{U}(\mathcal{A})$ for $i < 4$ such that $a = \sum_{i<4} \gamma_i(a)$ for all

$a \in \mathcal{A}$. This is because if $a \in \mathcal{A}$, then $b = (a + a^*)/2$ and $c = i(a - a^*)/2$ are self-adjoint of norm $\leq \|a\|$ such that $b + ic = a$. If b is self-adjoint of norm ≤ 1, then the operators $b_1 = b + i\sqrt{I - b^2}$ and $b_2 = b - i\sqrt{I - b^2}$ have norm ≤ 1 and their product is equal to I. Therefore they are both unitaries. Also, their mean is equal to b. It is now clear how to define γ_i. Assume Ψ is a representation of Φ on $\mathcal{U}(\mathcal{A})$. Then let $\Psi_1(0) = 0$ and $\Psi_1(a) = \|a\| \sum_{i<4} \Psi(\gamma_i(a/\|a\|))$ for $a \neq 0$. This is the required Borel representation. The proof of (2) uses the same formula and the obvious estimates. □

Lemma 4.3. *Assume $\mathcal{D}$ is a von Neumann subalgebra of $\mathcal{B}(H)$ and assume $\Phi\colon \mathcal{D}/(\mathcal{K}(H) \cap \mathcal{D}) \to \mathcal{B}(H)/\mathcal{K}(H)$ is a *-homomorphism.*

(1) *If Φ has a* C*-measurable ε-approximation Ψ on $\mathcal{U}(\mathcal{D})$, then it has a Borel-measurable 8ε-approximation on $\mathcal{U}(\mathcal{D})$.*
(2) *If Φ has a* C*-measurable representation on $\mathcal{U}(\mathcal{D})$, then it has a Borel-measurable representation on $\mathcal{D}$.*
(3) *If there are* C*-measurable maps Ψ_i for $i \in \mathbb{N}$ whose graphs cover an ε-approximation to Φ on $\mathcal{D}_{\leq 1}$, then there are Borel-measurable maps Ψ'_i for $i \in \mathbb{N}$ whose graphs cover an 8ε-approximation to Φ on $\mathcal{D}_{\leq 1}$.*

Proof. (1) Consider $\mathcal{U}(\mathcal{D})$ with respect to the strong operator topology. It is a Polish group. Since Ψ is Baire-measurable we may fix a dense G_δ subset $\mathcal{X}$ of $\mathcal{U}(\mathcal{D})$ on which Ψ is continuous. The set

$$\mathcal{Y} = \{(a, b) \in \mathcal{U}(\mathcal{D})^2 \mid b \in \mathcal{X} \cap a^*\mathcal{X}\}$$

is Borel and it has co-meager sections. By Theorem 2.2 there is a Borel uniformization h for $\mathcal{Y}$. Then for each a both $h(a)$ and $ah(a)$ belong to $\mathcal{X}$ and therefore $\Psi'(a) = \Psi(ah(a))\Psi(h(a))^*$ is a Borel-measurable 2ε-approximation to Φ on $\mathcal{U}(\mathcal{D})$. By Lemma 4.2, Φ has an 8ε-approximation. (2) follows from the case $\varepsilon = 0$ of (1) plus Lemma 4.2(1). To prove (3), find a dense G_δ subset $\mathcal{X}$ of $\mathcal{U}(\mathcal{D})$ on which each Ψ_i is continuous. Define $\mathcal{Y}$ and h as above and consider the maps $\Psi'_{ij}(a) = \Psi_i(ah(a))\Psi_j(h(a))^*$. □

Lemma 4.4. *Let $\mathcal{D} \subseteq \mathcal{B}(H)$ be a von Neumann algebra, $\Phi\colon \mathcal{D}/(\mathcal{K}(H) \cap \mathcal{D}) \to \mathcal{B}(H)/\mathcal{K}(H)$ a *-homomorphism, and $\mathcal{Y} \subseteq \mathcal{D}_{\leq 1}$. Assume Φ has a Borel-measurable 2^{-n}-approximation Ξ_n on $\mathcal{Y}$ for every $n \in \mathbb{N}$. Then Φ has a* C*-measurable representation on $\mathcal{Y}$.*

Proof. Let $\mathcal{X} = \{(a, b) \in \mathcal{D}_{\leq 1} \times \mathcal{B}(H)_{\leq 1} \mid (\forall n)\|\Xi_n(a) - b\|_{\mathcal{K}} \leq 2^{-n+1}\}$. By Lemma 2.5, this is a Borel set. If Ψ is a C-measurable uniformization of $\mathcal{X}$ provided by Theorem 2.1, then Ψ is a representation of Φ on $\mathcal{Y}$. □

5. **Approximate *-homomorphisms**

Assume $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras. A map $\Lambda\colon \mathcal{A} \to \mathcal{B}$ in an *ε-approximate *-homomorphism* if for all a, b in $\mathcal{A}_{\leq 1}$ we have the following:

(1) $\|\Lambda(ab) - \Lambda(a)\Lambda(b)\| < \varepsilon$,
(2) $\|\Lambda(a+b) - \Lambda(a) - \Lambda(b)\| < \varepsilon$,
(3) $\|\Lambda(a^*) - \Lambda(a)^*\| < \varepsilon$,
(4) $|\|a\| - \|\Lambda(a)\|| < \varepsilon$.

We say Λ is *unital* if both $\mathcal{A}$ and $\mathcal{B}$ are unital and $\Lambda(I) = I$. We say Λ is *δ-approximated* by Θ if $\|\Lambda(a) - \Theta(a)\| < \delta$ for all $a \in \mathcal{A}_{\leq 1}$. Theorem 5.1 is the main result of this section and may be of independent interest. The numerical value of the constant K is irrelevant for our purposes and we shall make no attempt to provide a sharp bound.

A shorter proof of Theorem 5.1 can be given by using a special case of a result of Sakai ([34]). After applying the Grove-Karcher-Roh/Kazhdan result on ε-representations to obtain a representation Θ of $\Lambda \restriction \mathcal{U}(\mathcal{A})$ that is a norm-continuous group homomorphism, use [34] to extend Θ to a *-homomorphism or a conjugate *-homomorphism of $\mathcal{A}$ into $\mathcal{B}$. An argument included in the proof below shows that this extension has to be a *-homomorphism. Parts of the proof of Theorem 5.1 resemble parts of Sakai's proof, of which I was not aware at the time of preparing this manuscript.

THEOREM 5.1. *There is a universal constant $K < \infty$ such that the following holds. If $\varepsilon < 1/1000$, $\mathcal{A}$ is a finite-dimensional C*-algebra, $m \in \mathbb{N}$, and $\Lambda\colon \mathcal{A} \to M_m(\mathbb{C})$ is a Borel-measurable unital ε-approximate *-homomorphism, then Λ can be $K\varepsilon$-approximated by a unital *-homomorphism.*

In the terminology of S. Ulam, the approximate *-homomorphisms are *stable* (see e.g., [24]). Connection between lifting theorems and Ulam-stability of approximate homomorphisms between Boolean algebras was first exploited in [13]. Analogous results for groups appear in [14] and [24]. The following is a special case of a well-known result (see [29]) but we include a proof for the reader's convenience.

LEMMA 5.2. *Assume $\varepsilon < 1$ and a is an element of a finite-dimensional C*-algebra A such that $\|a^*a - I\| < \varepsilon$. If $a = bu$ is the polar decomposition of a, then u is a unitary and $\|a - u\| < \varepsilon$.*

Proof. We have $\|u^*b^2u - I\| < 1$. Then $P = u^*u$ is a projection and $\|I - P\| = \|(I - P)(u^*bu - I)\| < 1$. Therefore $u^*u = I$ and since A is finite-dimensional u is a unitary. Hence we have $\|b^2 - I\| < \varepsilon$ and $\|b - I\| < \varepsilon$. Clearly, $\|a - u\| = \|bu - u\| = \|b - I\|$. $\square$

Proof of Theorem 5.1. We shall write $\mathcal{B}$ for $M_m(\mathbb{C})$ and consider its representation on the m-dimensional Hilbert space, denoted K. We also write $a \approx_\delta b$ for $\|a-b\| < \delta$. Fix a unitary u in $\mathcal{A}$ and let $a = \Lambda(u)$. Then $aa^* \approx_{\varepsilon(1+\varepsilon)} \Lambda(u)\Lambda(u^*) \approx_\varepsilon \Lambda(uu^*)$; thus $\|aa^* - I\| < 2\varepsilon + \varepsilon^2$. Similarly $\|a^*a - I\| < 2\varepsilon + \varepsilon^2$. Therefore, by Lemma 5.2 there is a unitary $v \in \mathcal{B}$ such that $\|\Lambda(u) - v\| < 2\varepsilon + \varepsilon^2 = \varepsilon_1$.

Let $\mathcal{X}$ be the set of all pairs $(u,v) \in \mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$ such that $\|\Lambda(u)-v\| < \varepsilon_1$. Since Λ is Borel-measurable, $\mathcal{X}$ is a Borel set. By Theorem 2.1 there is a C-measurable $\Lambda' \colon \mathcal{U} \to \mathcal{V}$ uniformizing $\mathcal{X}$. Note that $\|\Lambda'(u) - \Lambda(u)\| < \varepsilon_1$ for all unitaries u.

We have $\Lambda'(u)\Lambda'(v) \approx_{(2+\varepsilon)\varepsilon_1} \Lambda(u)\Lambda(v) \approx_\varepsilon \Lambda(uv) \approx_{\varepsilon_1} \Lambda'(uv)$. Thus $\|\Lambda'(uv) - \Lambda'(u)\Lambda'(v)\| < (3+\varepsilon)\varepsilon_1 + \varepsilon = \varepsilon_2$. In the terminology of [25], Λ' is a $2\varepsilon_2$-*representation* of $\mathcal{U}(\mathcal{A})$ on K. In [25] and [21] it was proved (among other things) that if $\delta < 1/100$, then every strongly continuous 2δ-representation ρ of a compact group can be 2δ-approximated by a (strongly continuous) representation ρ'. See [1, Theorem 5.13] for a more streamlined presentation of this proof. The approximating representation is obtained as a limit of a succession of integrals with respect to the Haar measure and the assumption that ρ is continuous can be weakened to the assumption that ρ is Haar measurable without altering the proof (or the conclusion). In particular, the proof given in [1] taken verbatim covers the case of a Haar-measurable approximation.

Let Θ be a continuous homomorphism between the unitary groups of $\mathcal{A}$ and $\mathcal{B}$ that is a $2\varepsilon_2$-aproximation to Λ' on $\mathcal{U}(\mathcal{A})$. For all u we then have

$$\|\Theta(u) - \Lambda(u)\| < 2\varepsilon_2 + \varepsilon_1 = \varepsilon_3.$$

For a self-adjoint $a \in \mathcal{A}$ the map $\mathbb{R} \ni r \mapsto \exp(ira) \in \mathcal{U}(K)$ defines a norm-continuous one-parameter group. By Stone's theorem (e.g., [32, Theorem 5.3.15]), there is the unique $\rho(a) \in \mathcal{U}(K)$ such that

$$\Theta(\exp(ira)) = \exp(ir\rho(a)).$$

Since $\mathcal{B}$ is a von Neumann algebra, we can conclude that $\rho(a) \in \mathcal{B}$.

Claim 5.3. $\rho(I) = I$, *hence for all* $r \in \mathbb{R}$ *we have* $\Theta(e^{ir}I) = e^{ir}I$.

Proof. Let $b = \rho(I)$ and assume $b \neq I$. Since b is self-adjoint, there is $s \neq 1$ in the spectrum of b. Pick $r \in \mathbb{R}$ such that $r(1-s) = \pi + 2k\pi$ for some $k \in \mathbb{N}$. Let ξ be the unital eigenvector of $\exp(irb)$ corresponding to the eigenvalue e^{irs}. Then the vector

$$\exp(irb)(\xi) - e^{ir}(\xi) = e^{irs}\xi - e^{ir}\xi = e^{irs}(\xi - e^{ir(1-s)}\xi) = 2e^{irs}\xi$$

has norm 2; hence $\|\exp(irb) - e^{ir}I\| = 2$. However,

$$\|\Theta(e^{ir}I) - e^{ir}I\| \leq \|\Theta(e^{ir}I) - \Lambda(e^{ir}I)\| + \|\Lambda(e^{ir}I) - e^{ir}I\| \leq \varepsilon_3 + \varepsilon < 1,$$

a contradiction. □

Let u be a self-adjoint unitary. Then $u = I - 2P$ for some projection P and therefore $\exp(iru) = e^{ir}u$ and by Claim 5.3 we have $\Theta(u) = \rho(u)$. Also $P = \frac{1}{2}(I - u)$ and one straightforwardly checks that $\rho(P) = \frac{1}{2}(I - \rho(u))$ and that $\|\rho(P) - \Lambda(P)\| \leq \varepsilon$.

Claim 5.4. *If projections P and Q commute, then $\rho(P)$ and $\rho(Q)$ commute. If $PQ = 0$, then $\rho(P)\rho(Q) = 0$.*

Proof. Since $I - 2P$ and $I - 2Q$ commute if and only if P and Q commute, the first sentence follows. If $PQ = 0$, then

$$I - 2\rho(P + Q) = \Theta((I - 2P)(I - 2Q)) = (I - 2\rho(P))(I - 2\rho(Q))$$

and we have $\rho(P + Q) - \rho(P) - \rho(Q) = 2\rho(P)\rho(Q)$. The left-hand side has norm $< 4\varepsilon < 1$. As a product of two commuting projections, $\rho(P)\rho(Q)$ is a projection, and so it has to be 0. $\square$

We say that projections P and Q in a C*-algebra $\mathcal{A}$ are *Murray-von Neumann equivalent* and write $P \sim Q$ if there is a partial isometry u such that $uPu^* = Q$. In the case when $\mathcal{A}$ is finite-dimensional this is equivalent to asserting that there is a unitary u such that $uPu^* = Q$.

Claim 5.5. *If P and Q are Murray-von Neumann equivalent projections, then $\rho(P)$ and $\rho(Q)$ are Murray-von Neumann equivalent projections.*

Proof. Fix a unitary u such that $uPu^* = Q$. Write $v_P = I - 2P$ ad $v_Q = I - 2Q$. Then (with $w = \Theta(u)$) we have $w\Theta(v_P)w^* = \Theta(v_Q)$, and therefore $w\Theta(P)w^* = \Theta(Q)$. $\square$

Let u be a unitary in $\mathcal{A}$ and let e^{ir_j}, for $0 \leq j \leq n-1$, be the spectrum of u. Fix projections P_j, for $0 \leq j \leq n-1$, such that $u = \sum_{j=0}^{n-1} e^{ir_j} P_j = \prod_{j=0}^{n-1} \exp(ir_j P_j)$. Then (by Claim 5.4 in the last equality)

$$\Theta(u) = \prod_{j=0}^{n-1} \Theta(\exp(ir_j P_j)) = \prod_{j=0}^{n-1} \exp(ir\rho(P_j)) = \sum_{j=0}^{n-1} e^{ir_j}\rho(P_j).$$

Claim 5.6. *If A is isomorphic to $M_k(\mathbb{C})$ for some natural number k, then Θ preserves the normalized trace*, Tr, *of the unitaries.*

Proof. By the above computation, we only need to show there is $d \in \mathbb{N}$ such that for every m, every projection of rank m in $\mathcal{A}$ is mapped to a projection of rank dm in $\mathcal{B}$. But this is an immediate consequence of Claim 5.5 and the obvious equality $\rho(P + Q) = \rho(P) + \rho(Q)$[1] for commuting projections P and Q, since in $M_k(\mathbb{C})$ two projections are Murray-von Neumann equivalent if and only if they have the same rank. $\square$

[1]This equality holds for any two self-adjoint operators, but we shall not need it.

CLAIM 5.7. *The map* $\Upsilon\colon \mathcal{A} \to \mathcal{B}$ *given by* $\Upsilon(\sum_j \alpha_j u_j) = \sum_j \alpha_j \Theta(u_j)$ *whenever* α_j *are scalars and* u_j *are unitaries is a well-defined* **-homomorphism from* $\mathcal{A}$ *into* $\mathcal{B}$.

Proof. Since every operator in $\mathcal{A}$ is a linear combination of four unitaries (cf. the proof of Lemma 4.2), in order to see that Υ is well defined we only need to check that $\sum_j \alpha_j u_j = 0$ implies $\sum_j \alpha_j \Theta(u_j) = 0$.

Let us first consider the case when $\mathcal{A}$ is a full matrix algebra. The following argument is taken from Dye ([10, Lemma 3.1]).

Assume $a = \sum_i \alpha_i u_i = 0$. Then $0 = \operatorname{Tr}(aa^*) = \sum_{i,j} \alpha_i \bar{\alpha}_j \operatorname{Tr}(u_i u_j^*)$. Also with $b = \sum_i \alpha_i \Theta(u_i)$ we have

$$\operatorname{Tr}(bb^*) = \textstyle\sum_{i,j} \alpha_i \bar{\alpha}_j \operatorname{Tr}(\Theta(u_i u_j^*)) = \sum_{i,j} \alpha_i \bar{\alpha}_j \operatorname{Tr}(u_i u_j^*),$$

which is 0 by Claim 5.6. Therefore $b = 0$, proving that Υ is well-defined when $\mathcal{A}$ is a full matrix algebra.

In order to prove the general case, let $S_0, \dots, S_{m-1}$ list all minimal central projections of $\mathcal{A}$. Then $S_i \mathcal{A} S_i$ is isomorphic to some $M_{k(i)}(\mathbb{C})$ and therefore Υ is well-defined on this subalgebra. However, $\Theta(u) = \sum_{j=0}^{m-1} \rho(S_j)\Theta(u)$ for all unitaries u in $\mathcal{A}$, and therefore $\Upsilon(a) = \sum_{j=0}^{m-1} \rho(S_j)\Upsilon(a)$ is well-defined for all $a \in \mathcal{A}$.

Clearly Υ is a complex vector space homomorphism and $\Upsilon(u) = \Theta(u)$ for a unitary u in $\mathcal{A}$. It is straightforward to check that Υ is multiplicative and a *-homomorphism. □

Any $a \in \mathcal{A}_{\leq 1}$ can be written as $b + ic$, where b and c are self-adjoints of norm ≤ 1, and $\Lambda(a) \approx_{3\varepsilon} \Lambda(b) + i\Lambda(c)$. If b is self-adjoint of norm ≤ 1, then there are unitaries u and v such that $b = u + v$ (cf. the proof of Lemma 4.2). Therefore $\Lambda(b) \approx_{\varepsilon} \Lambda(u) + \Lambda(v) \approx_{2\varepsilon_3} \Xi(u) + \Xi(v) = \Upsilon(b)$. All in all, we have $\|\Lambda(a) - \Upsilon(a)\| \leq \varepsilon + \varepsilon_3$ for $a \in \mathcal{A}_{\leq 1}$. Since for small ε each ε_i for $1 \leq i \leq 3$ is bounded by a linear function of ε, this concludes the proof. □

The assumption that Λ is unital was not necessary in Theorem 5.1. This is an easy consequence of Theorem 5.1 and the following well-known lemma, whose proof can also be found in [29].

LEMMA 5.8. *If* $0 < \varepsilon < 1/8$, *then in every* C**-algebra* $\mathcal{A}$ *the following holds. For every* a *satisfying* $\|a^2 - a\| < \varepsilon$ *and* $\|a^* - a\| < \varepsilon$ *there is a projection* Q *such that* $\|a - Q\| < 4\varepsilon$.

Proof. We claim that $M = \|a\| < 2$. Let $b = (a + a^*)/2$. Then $\|a - b\| < \varepsilon/2$, b is self-adjoint, and $\|b\| > M - \varepsilon/2$. Consider $\mathcal{A}$ as a concrete C*-algebra acting on some Hilbert space H. In the weak closure of $\mathcal{A}$ find a spectral projection R of b corresponding to $(M - \varepsilon/2 + \|a - b\|, \|b\|]$. If ξ is a unit vector in the range of R, then $\|b\xi - M\xi\| < \varepsilon/2 - \|a - b\|$. If $\eta = a\xi - M\xi$, then $\|\eta\| < \varepsilon/2$, and $M - \varepsilon/2 < \|a\xi\| \leq M$. Also, $a^2\xi = a\eta + Ma\xi = a\eta + M\eta + M^2\xi$, and therefore $\|a^2\xi\| \geq M^2 - M\varepsilon$. Since $\|a^2\xi - a\xi\| < \varepsilon$, we have $\|a^2\xi\| < \|a\xi\| + \varepsilon < M + 3\varepsilon/2$. Therefore $M + 3\varepsilon/2 > M^2 - M\varepsilon$, and with $\varepsilon < 1/4$ this implies $M < 2$ as claimed.

Therefore $\|aa^* - a\| < 2\|a^* - a\| + \|a^2 - a\| < 3\varepsilon$. So we have

$$4\|b^2 - b\| = \|a^2 + aa^* + a^*a + (a^*)^2 - 2a - 2a^*\| < 8\varepsilon. \tag{1}$$

We may assume $\mathcal{A}$ is unital. Since b is self-adjoint, via the function calculus in $C^*(b, I)$, the subalgebra of $\mathcal{A}$ generated by b and I, b corresponds to the identity function on its spectrum $\sigma(b)$. By (1), for every $x \in \sigma(b)$ we have $|x^2 - x| < 2\varepsilon$. Therefore $1/2 \notin \sigma(b)$, $U = \{x \in \sigma(b) \mid |x - 1| < 1/2\}$ is a relatively closed and open subset of $\sigma(b)$, and the projection Q corresponding to the characteristic function of this set in $C^*(b, I)$ belongs to $\mathcal{A}$. Then $\|Q - b\| = \sup_{x\in\sigma(b)} \min(|x|, |1 - x|)$. If $\delta(\varepsilon) = \sup\{\min(|x|, |1 - x|) \mid |x^2 - x| < 2\varepsilon\} + \varepsilon/2$, then $\|a - Q\| < \delta(\varepsilon)$. Clearly $\delta(\varepsilon) < 4\varepsilon$ for $\varepsilon < 1$. □

6. Automorphisms with C-measurable representations are inner

Each known proof that all automorphisms of a quotient structure related to $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ or $\mathcal{B}(H)/\mathcal{K}(H)$ are 'trivial' proceeds in two stages. In the first, some additional set-theoretic axioms are used to prove that all automorphisms are 'topologically simple.' In the second, all 'topologically simple' automorphisms are shown to be trivial, without use of any additional set-theoretic axioms (see [16]). The present proof is not an exception and the present section deals with the second step. Even though no additional set-theoretic axioms are needed for its conclusion, the proof of Theorem 6.1 given at the end of this section will take a metamathematical detour via TA and Shoenfield's theorem (Theorem 2.6). Note that the latter is not needed for the proof of Theorem 1, since TA follows from its assumptions.

THEOREM 6.1. *Every automorphism of $\mathcal{C}(H)$ with a* C*-measurable representation on $\mathcal{U}(H)$ is inner.*

6.1. *Inner on* FDD *von Neumann algebras.* If v is a linear isometry between cofinite-dimensional subspaces of H then $\Psi_v(a) = vav^*$ is a representation of an automorphism of $\mathcal{C}(H)$. We use notation $\vec{E}$, $\mathcal{D}[\vec{E}]$, and $\mathcal{D}_M[\vec{E}]$ from Section 0.1.

LEMMA 6.2. *Assume $\#E_n$ is a nondecreasing sequence. If an automorphism Φ of the Calkin algebra is inner on $\mathcal{D}_M[\vec{E}]$ for some infinite M, then it is inner on $\mathcal{D}[\vec{E}]$.*

Proof. By Lemma 4.1, it will suffice to find a partial isometry u such that $u\mathcal{D}[\vec{E}]u^* \subseteq \mathcal{D}_M[\vec{E}]$ and $u^*u = I$. If (m_j) is an increasing enumeration of M, then $\#E_j \leq \#E_{m_j}$ by our assumption. Let

$$u_j \colon \overline{\operatorname{span}}\{e_i \mid i \in E_j\} \to \overline{\operatorname{span}}\{e_i \mid i \in E_{m_j}\}$$

be a partial isometry. Then $u = \sum_j u_j$ is as required. □

Theorem 6.3. *Assume Φ is an automorphism of $\mathcal{C}(H)$, $\vec{E}$ is a partition of $\mathbb{N}$ into finite intervals, and Φ has a* C*-measurable representation on $\mathcal{D}[\vec{E}]$. Then Φ has a representation which is a *-homomorphism from $\mathcal{D}[\vec{E}]$ into $\mathcal{B}(H)$. Moreover, there is a partial isomorphism v of cofinite-dimensional subspaces of H such that Ψ_v is a representation of Φ on $\mathcal{D}[\vec{E}]$.*

Proof. By coarsening $\vec{E}$ we may assume the sequence $\#E_n$ is nondecreasing. Since $\vec{E}$ is fixed, we write $\mathbf{P}_A$ for $\mathbf{P}_A^{\vec{E}}$. The proof proceeds by successively constructing a sequence of representations, each one with more adequate properties than the previous ones, until we reach one that is a *-homomorphism between the underlying algebras. This is similar to the proofs in [13, §1]. Some of the arguments may also resemble those from [3].

Let $\varepsilon_i = 2^{-i}$. Fix a finite ε_i-dense in norm subset

$$\mathbf{a}_i \subseteq \mathcal{B}(\overline{\operatorname{span}}\{e_i \mid i \in E_n\})_{\leq 1}$$

containing the identity and zero. Note that $\prod_{j=l}^{l+m} \mathbf{a}_j$ is $2\varepsilon_l$-dense in

$$\prod_{j=l}^{l+m} \mathcal{B}(\overline{\operatorname{span}}\{e_i \mid i \in E_j\})_{\leq 1}.$$

Let $\mathbf{A} = \prod_i \mathbf{a}_i$. We shall identify $a \in \mathbf{a}_i$ with $\bar{a} \in \mathcal{D}[\vec{E}]$ such that $\mathbf{P}_{\{i\}}\bar{a} = a$ and $(I - \mathbf{P}_{\{i\}})\bar{a} = 0$. For $J \subseteq \mathbb{N}$ and $x \in \mathbf{A}$ it will be convenient to write $x \restriction J$ for the projection of x to $\prod_{i \in J} \mathbf{a}_i$, identified with $\mathbf{P}_J x$.

Claim 6.4. *There is a strongly continuous representation Ψ_1 of Φ on $\mathbf{A}$.*

Proof. Since each $\mathbf{a}_i$ is finite, the strong operator topology on $\mathbf{A}$ coincides with its Cantor-set topology which is compact metric. Let $\mathcal{X} \subseteq \mathbf{A}$ be a dense G_δ set on which Ψ is continuous. Write $\mathcal{X}$ as an intersection of dense open sets U_n, $n \in \mathbb{N}$. Since each $\mathbf{a}_i$ is finite, a straightforward diagonalization argument produces an increasing sequence (n_i) in $\mathbb{N}$, with $J_i = [n_i, n_{i+1})$, $\mathbf{b}_i = \mathbf{a}_{J_i} = \prod_{k \in J_i} \mathbf{a}_i$, and $s_i \in \mathbf{b}_i$ such that for all $x \in \mathbf{A}$ and all i we have $x \restriction J_i = s_i$ implies $x \in \bigcap_{j=0}^{i} U_j$. Therefore $\{x \mid (\exists^\infty i) x \restriction J_i = s_i\} \subseteq \mathcal{X}$.

Let $C_0 = \bigcup_{j \text{ even}} J_j$, $C_1 = \bigcup_{j \text{ odd}} J_j$, $R_0 = \mathbf{P}_{C_0}$, and $R_1 = \mathbf{P}_{C_1}$. Let $S_0 = \sum_{j \text{ odd}} s_j$ and let $S_1 = \sum_{j \text{ even}} s_j$. Note that $R_i u = u R_i = R_i u R_i$ for all $u \in \mathcal{D}[\vec{E}]$ and $i \in \{0, 1\}$. For $u \in \mathbf{A}$ let

$$(*) \quad \Psi_1(u) = \Psi(uR_0 + S_0) - \Psi(S_0) + \Psi(uR_1 + S_1) - \Psi(S_1).$$

Then Ψ_1 is a continuous representation of Φ on $\mathbf{A}$. □

Our next task is to find a representation Ψ_2 of Φ on $\mathbf{A}$ which is stabilized (in a sense to be made very precise below) and then extend it to a representation of Φ on $\mathcal{D}[\vec{E}]$. Start with Ψ_1 as provided by Claim 6.4. By possibly replacing Ψ_1 with $b \mapsto \Psi_1(b)\Psi_1(I)^*$, we may assume $\Psi_1(I) = I$.

The sequence of projections $(\mathbf{R}_k)$ was fixed in Section 0.1.

Claim 6.5. *For all n and $\varepsilon > 0$ there are $k > n$ and $u \in \prod_{i=n}^{k-1} \mathbf{a}_i$ such that for all a and b in $\mathbf{A}$ satisfying $a \restriction [n, \infty) = b \restriction [n, \infty)$ and $a \restriction [n, k) = u$, we have*

(1) $\|(\Psi_1(a) - \Psi_1(b))(I - \mathbf{R}_k)\| \leq \varepsilon$ *and*
(2) $\|(I - \mathbf{R}_k)(\Psi_1(a) - \Psi_1(b))\| \leq \varepsilon$.

Proof. Write $\mathbf{c} = \prod_{i=0}^{n-1} \mathbf{a}_i$. For $a \in \mathbf{A}$ and $s \in \mathbf{c}$ write $a[s] = s + \mathbf{P}_{[n,\infty)}a$. For $k > n$ let

$$V_k = \{a \in \mathbf{A} \mid (\exists s \in \mathbf{c})(\exists t \in \mathbf{c})\|(\Psi_1(a[s]) - \Psi_1(a[t]))(I - \mathbf{R}_k)\| > \varepsilon \text{ or } \|(I - \mathbf{R}_k)(\Psi_1(a[s]) - \Psi_1(a[t]))\| > \varepsilon\}.$$

Since Ψ_1 is continuous, each V_k is an open subset of $\mathbf{A}$. If $a \in \mathcal{D}[\vec{E}]$ and s and t are in $\mathbf{c}$, then $\Psi_1(a[s]) - \Psi_1(a[t])$ is compact. Therefore

$$\|(\Psi_1(a[s]) - \Psi_1(a[t]))(I - \mathbf{R}_k)\| \leq \varepsilon$$

and

$$\|(I - \mathbf{R}_k)(\Psi_1(a[s]) - \Psi_1(a[t]))\| \leq \varepsilon$$

for a large enough $k = k(a, s, t)$. Since $\mathbf{c}$ is finite, for some large enough $k = k(a)$ we have $a \notin V_k$. Therefore the G_δ set $\bigcap_k V_k$ is empty. By the Baire Category Theorem, we may fix l such that V_l is not dense. There is a basic open set disjoint from V_l. Since $a \in V_l$ if and only if $a[s] \in V_l$ for all a and $s \in \mathbf{c}$, for some $k \geq l$ there is a $u \in \prod_{i=n}^{k-1} \mathbf{a}_i$ such that $\{a \in \mathbf{A} \mid a \restriction [n, k) = u\}$ is disjoint from V_k (note that $V_k \subseteq V_l$). Then k and u are as required. $\square$

We shall find two increasing sequences of natural numbers, (n_i) (unrelated to the one appearing in the proof of Claim 6.4) and (k_i) so that $n_i < k_i < n_{i+1}$ for all i. These sequences will be chosen according to the requirements described below. With $J_i = [n_i, n_{i+1})$ write $\mathbf{b}_i = \mathbf{a}_{J_i} = \prod_{j \in J_i} \mathbf{a}_j$.

Let $\varepsilon_i = 2^{-i}$. A $u_i \in \mathbf{b}_i$ is an *ε_i-stabilizer for Ψ_1* (or a *stabilizer*) if for all a, b in $\mathbf{A}$ such that $a \restriction [n_i, n_{i+1}) = b \restriction [n_i, n_{i+1}) = u_i$ the following hold.

(a) If $a \restriction [n_i, \infty) = b \restriction [n_i, \infty)$, then
  (a1) $\|(\Psi_1(a) - \Psi_1(b))(I - \mathbf{R}_{k_i})\| < \varepsilon_i$ and
  (a2) $\|(I - \mathbf{R}_{k_i})(\Psi_1(a) - \Psi_1(b))\| < \varepsilon_i$.
(b) If $a \restriction [0, n_{i+1}) = b \restriction [0, n_{i+1})$, then
  (b1) $\|(\Psi_1(a) - \Psi_1(b))\mathbf{R}_{k_i}\| < \varepsilon_i$ and
  (b2) $\|\mathbf{R}_{k_i}(\Psi_1(a) - \Psi_1(b))\| < \varepsilon_i$.

We shall find (n_i), (k_i), J_i, $\mathbf{b}_i$ as above and a stabilizer $u_i \in \mathbf{b}_i$ for all i. Assume all of these objects up to and including n_i, k_{i-1} and u_{i-1} have been chosen to satisfy the requirements. Applying Claim 6.5, find $k_i \geq n_i$ and $u_i^0 \in \prod_{j=n_i}^{k_i-1} \mathbf{a}_j$ such that (a1) and (a2) hold. Then apply the continuity of Ψ_1 to find $n_{i+1} \geq k_i$ and $u_i \in \prod_{j=n_i}^{n_{i+1}-1} \mathbf{a}_j$ such that $u_i \restriction [n_i, k_i) = u_i^0$, and (b1) and (b2) hold as well.

Once the sequences n_{i+1}, k_i and $u_i \in \mathbf{b}_i = \prod_{j \in J_i} \mathbf{a}_j$ are chosen, let

$$\mathcal{V}_i = \bigoplus_{j \in J_i} \mathcal{B}(E_j).$$

Then $\mathcal{D}[\vec{E}] = \prod_i \mathcal{V}_i$. We identify $\mathcal{V}_j$ with $\mathbf{P}_{J_i}\mathcal{D}[\vec{E}]$ and $b \in \mathcal{D}[\vec{E}]_{\leq 1}$ with the sequence $\langle b_j \rangle_j$ such that $b_j \in \mathcal{V}_j$ and $b = \sum_j b_j$. Let I_j denote the identity of $\mathcal{V}_j$. Note that $I_j \in \mathbf{b}_j$. Recall that $\mathbf{b}_i$ is $2\varepsilon_i$-dense in $(\mathcal{V}_i)_{\leq 1}$ and fix a linear ordering of each $\mathbf{b}_i$. Define

$$\sigma_i \colon \mathcal{V}_i \to \mathbf{b}_i$$

by letting $\sigma_i(c)$ be the first element of $\mathbf{b}_i$ that is within $2\varepsilon_i$ of c. For $c \in \mathcal{D}[\vec{E}]_{\leq 1}$ let

$$c_{\text{even}} = \sum_i \sigma_{2i}(c_{2i}) \qquad \text{and} \qquad c_{\text{odd}} = \sum_i \sigma_{2i+1}(c_{2i+1}).$$

Both of these elements belong to $\mathbf{A}$ and $c - c_{\text{even}} - c_{\text{odd}}$ is compact.

Let us concentrate on $\mathcal{V}_{2i+1}$. Define $\Lambda_{2i+1} \colon \mathcal{V}_{2i+1} \to \mathcal{B}(H)$:

$$\Lambda_{2i+1}(b) = \Psi_1(u_{\text{even}} + \sigma_{2i+1}(b)) - \Psi_1(u_{\text{even}}).$$

Since both σ_i and Ψ are Borel-measurable, Λ_{2i+1} is Borel-measurable as well. Let $\mathbf{Q}_i = \mathbf{R}_{k_{i+1}} - \mathbf{R}_{k_{i-1}}$, with $k_{-1} = 0$.

Claim 6.6. *For $b \in \mathcal{D}[\vec{E}]_{\leq 1}$ such that $b_{2i} = 0$ for all i, the operator $\Psi_1(b) - \sum_{i=0}^{\infty} \mathbf{Q}_{2i+1}\Lambda_{2i+1}(b_{2i+1})\mathbf{Q}_{2i+1}$ is compact. In particular the latter operator is bounded.*

Proof. Since $b - b_{\text{odd}}$ is compact, so is $\Psi_1(b) + \Psi_1(u_{\text{even}}) - \Psi_1(u_{\text{even}} + b_{\text{odd}})$. By applying (a1) and (b1) to b_{odd}, $b^+ = \sum_{j=i}^{\infty} \sigma_{2j+1}(b)$ and $\sigma_{2i+1}(b_{2i+1})$ we see that

$$\begin{aligned}
&\|(\Lambda_{2i+1}(b_{2i+1}) + \Psi_1(u_{\text{even}}) - \Psi_1(u_{\text{even}} + b_{\text{odd}}))\mathbf{Q}_{2i+1}\| \\
&\qquad = \|\Psi_1((u_{\text{even}} + \sigma_{2i+1}(b)) - \Psi_1(u_{\text{even}} + b_{\text{odd}}))\mathbf{Q}_{2i+1})\| \\
&\qquad \leq \|(\Psi_1(u_{\text{even}} + \sigma_{2i+1}(b)) - \Psi_1(u_{\text{even}} + b^+))\mathbf{Q}_{2i+1}\| \\
&\qquad\quad + \|(\Psi_1(u_{\text{even}} + b^+) - \Psi_1(u_{\text{even}} + b_{\text{odd}}))\mathbf{Q}_{2i+1}\| \\
&\qquad < 2\varepsilon_{2i+1}.
\end{aligned}$$

Since $\sum_i (\varepsilon_{2i+1})^2 < \infty$ and $I - \sum_i \mathbf{Q}_{2i+1}$ is a compact operator, the operator $\Psi_1(u_{\text{even}} + b_{\text{odd}}) - \Psi_1(u_{\text{even}}) - \sum_{i=0}^{\infty} \Lambda_{2i+1}(b_{2i+1})\mathbf{Q}_{2i+1}$ is compact. An analogous proof using (a2) and (b2) instead of (a1) and (b1) gives that $\Psi_1(u_{\text{even}} + b_{\text{odd}}) - \Psi_1(u_{\text{even}}) - \sum_{i=0}^{\infty} \mathbf{Q}_{2i+1}\Lambda_{2i+1}(b_{2i+1})\mathbf{Q}_{2i+1}$ is compact. $\square$

Define $\Lambda'_{2i+1}\colon \mathcal{V}_{2i+1} \to \mathcal{B}(H)$ by

$$\Lambda'_{2i+1}(b) = \mathbf{Q}_{2i+1}\Lambda_{2i+1}(b)\mathbf{Q}_{2i+1}.$$

With $a_{2i+1} = \Lambda'_{2i+1}(I_{2i+1})$ let $\varepsilon_i = \max(\|a^2_{2i+1} - a_{2i+1}\|, \|a^*_{2i+1} - a_{2i+1}\|)$. We claim that $\limsup_i \varepsilon_i = 0$. Assume not and find $\varepsilon > 0$ and an infinite $M \subseteq 2\mathbb{N}+1$ such that for all $i \in M$ we have $\max(\|a_i^2 - a_i\|, \|a_i^* - a_i\|) > \varepsilon$. With $a = \sum_{i\in M} a_i$ the operator $\Psi_1(\mathbf{P}_M) - a$ is compact, thus $a^* - a$ and $a^2 - a$ are both compact. Since $a_i = \mathbf{Q}_i a \mathbf{Q}_i$ and $\mathbf{Q}_i\mathbf{Q}_j = 0$ for distinct i and j in M, we have $a^* = \sum_{i\in M} a_i^*$ and $a^2 = \sum_{i \in M} a_i^2$. By the choice of M and ε at least one of $a - a^*$ and $a^2 - a$ is not compact, a contradiction.

Applying Lemma 5.8 to a_{2i+1} such that ε_i is small enough, obtain projections $\mathbf{S}_{2i+1} \leq \mathbf{Q}_{2i+1}$ such that $\limsup_{i\to\infty} \|\mathbf{S}_{2i+1} - \Lambda'_{2i+1}(I_{2i+1})\| = 0$. With Lemma 4.1 in mind, we shall ignore all the even-numbered $\mathcal{V}_i$ and Λ_i. Let

$$\Lambda''_i(a) = \mathbf{S}_{2i+1}\Lambda'_{2i+1}(a)\mathbf{S}_{2i+1}$$

for $a \in \mathcal{V}_{2i+1}$ and let $\mathbf{S}''_i = \mathbf{S}_{2i+1}$ and $\mathcal{V}''_i = \mathcal{V}_{2i+1}$ for all i.

Then $\Lambda''(a) = \sum_i \Lambda''_i(a_i)$ is a representation of Φ on $\bigoplus_i \mathcal{V}''_i$. For $j \in \mathbb{N}$ let

$$\delta_j^0 = \sup_{a,b\in(\mathcal{V}''_j)_{\leq 1}} \|\Lambda''_j(ab) - \Lambda''_j(a)\Lambda''_j(b)\|,$$

$$\delta_j^1 = \sup_{a,b\in(\mathcal{V}''_j)_{\leq 1}} \|\Lambda''_j(a+b) - \Lambda''_j(a) - \Lambda''_j(b)\|,$$

$$\delta_j^2 = \sup_{a\in(\mathcal{V}''_j)_{\leq 1}} \|\Lambda''_j(a^*) - \Lambda''_j(a)^*\|,$$

$$\delta_j^4 = \sup_{a\in(\mathcal{V}''_j)_{\leq 1}} |\|a\| - \|\Lambda''_j(a)\||.$$

We claim that $\lim_j \max_{0\leq k\leq 4} \delta_j^k = 0$. We shall prove only $\lim_j \delta_j^0 = 0$ since the other proofs are similar. Assume the limit is nonzero, and for each j fix b_j and c_j in $(\mathcal{V}''_j)_{\leq 1}$ such that $\|\Lambda''_j(b_jc_j) - \Lambda''_j(b_j)\Lambda''_j(c_j)\| \geq \delta_j^0/2$ for all j. Let b and c in $\mathcal{B}[\vec{E}]_{\leq 1}$ be such that $\mathbf{P}_{J_j} b = b_j$ and $\mathbf{P}_{J_j} c = c_j$ for all j. Then $\Psi_1(bc) - \Psi_1(b)\Psi_1(c)$ is compact. By Claim 6.6, so is $\sum_j \Lambda''_j(b_jc_j) - \Lambda''_j(b_j)\Lambda''_j(c_j)$. This implies $\lim_j \delta_j^0 = 0$, a contradiction.

Each Λ''_j is a $2\delta_j$-approximate *-homomorphism as defined in Section 5. Since $\lim_j 2\delta_j = 0$ and each Λ''_j is Borel-measurable, by applying Theorem 5.1 to Λ''_j for j larger than some n_0, we find a $2K\delta_j$-approximation to Λ''_j which is a unital *-homomorphism, $\Xi_i\colon \mathcal{D}_{2i+1} \to \mathcal{B}(\mathbf{S}''_i[H])$. For $i \leq n_0$ let Ξ_i be identically equal to 0. Since $\lim_j 2K\delta_j = 0$ and $\mathbf{S}''_i$ are pairwise orthogonal, the diagonal Ξ of Ξ_i is a *-homomorphism and a representation of Φ on $\mathcal{D}_{\bigcup_{i \text{ odd}} J_i}[\vec{E}]$.

Still ignoring the even-numbered $\mathcal{V}_j$'s, we address the second part of Theorem 6.3 by showing Φ is inner on $\mathcal{D}_{\bigcup_{i \text{ odd}} J_i}[\vec{E}]$. Let $F_i = \mathbf{P}_{J_i}[H]$ and $G_i = \mathbf{S}''_i[H]$.

Claim 6.7. *For all but finitely many i there is a linear isometry $v_i\colon F_i \to G_i$ such that $\Xi_i(a) = v_i a v_i^*$ for all $a \in \mathcal{D}[(E_j)_{j\in J_i}]$.*

Proof. Let ξ_n, for $n \in \mathbb{N}$, be an orthonormal sequence such that each ξ_n belongs to some F_i and no two ξ_n belong to the same F_i. Let $P = \operatorname{proj}_{\overline{\operatorname{span}}\{\xi_n | n\in\mathbb{N}\}}$ and consider the masa $\mathcal{A}$ of $\mathcal{B}(P[H])$ consisting of all operators diagonalized by ξ_n, for $n \in \mathbb{N}$. The image under the quotient map of $\mathcal{A}$ in the Calkin algebra $\mathcal{C}(P[H])$ is a masa ([22]). It is contained in the domain of Ξ. The image of the Ξ-image of $\mathcal{A}$ is a masa in $\mathcal{C}(\Xi(P)[H])$. Because of this, for all but finitely many n the projection $\Xi(\operatorname{proj}_{\mathbb{C}\xi_n})$ has rank 1. Since (ξ_n) was arbitrary, for all but finitely many n and all one-dimensional projections $R \le \operatorname{proj}_{F_n}$ the rank of $\Xi(R)$ is equal to 1. Fix such n and a basis $(\eta_j \mid j < \dim(E_n))$ of F_n. Let $P_j = \Xi(\operatorname{proj}_{\mathbb{C}\eta_j})$. For all but finitely many n we have $\sum_{j<\dim(F_n)} P_j = \operatorname{proj}_{G_n}$. Consider n large enough for this to hold. Fix a unit vector ξ_0 in the range of P_0. Let $a \in \mathcal{U}(F_n)$ be generated by a cyclic permutation of $\{\eta_j\}$, so that $a(\eta_j) = \eta_{j+1}$ (with $\eta_{\dim(F_n)} = \eta_0$). With $b = \Xi(a)$ let $\xi_j = b^j(\xi_0)$ (here b^j is the j-th power of b). Then (ξ_j) form a basis of G_n. It is clear that $\eta_j \mapsto \xi_j$ defines an isometry v_n as required. □

For a large enough m the sum $v = \bigoplus_{n=m}^{\infty} v_n$ is a partial isometry from $\bigoplus_{n=m}^{\infty} F_n$ to $\bigoplus_{n=m}^{\infty} G_n$ such that $\Xi(a) - vav^*$ has finite rank for all $a \in \mathcal{D}[\vec{E}]$. Lemma 6.2 implies Φ is inner on $\mathcal{D}[\vec{E}]$. □

Proof of Theorem 6.1. Fix an automorphism Φ of $\mathcal{C}(H)$ with a C-measurable representation. By Lemma 4.3(2) we may assume that Φ has a Borel-measurable representation Ψ. Let $B \subseteq \mathcal{B}(H)_{\le 1} \times \mathcal{B}(H)_{\le 1}$ be the set of all pairs (a, b) such that $\Psi(b) - aba^*$ is not compact. Then the assertion of Theorem 6.1 is equivalent to $(\exists a)(\forall b)(a, b) \notin B$. Lemma 2.5 implies B is Borel and therefore by Theorem 2.6 we may use TA in the proof of Theorem 6.1.

By Theorem 6.3, Φ is inner on $\mathcal{D}[\vec{E}]$ for each finite-dimensional decomposition $\vec{E}$ of H. By Theorem 3.2, Φ is inner. □

7. **Locally inner automorphisms**

Fix an automorphism Φ of $\mathcal{C}(H)$. Proposition 7.1 below is roughly modeled on the proof of [13, Prop. 3.12.1]. Its main components are Lemma 7.2, Proposition 7.7, and Theorem 6.3. The key device in the proof of Lemma 7.2 is the partition defined in (K1)–(K3). It is a descendant of Velickovic's partition ([42]) and the partitions used in [13, p. 100].

If u is a partial isomorphism we write Ψ_u for the conjugation, $\Psi_u(a) = uau^*$. Fix a partition $\vec{E}$ of $\mathbb{N}$ into finite intervals such that the sequence $\#E_n$ is nondecreasing.

Proposition 7.1. TA *implies Φ is inner on $\mathcal{D}[\vec{E}]$.*

Using the Axiom of Choice, find a representation $\Psi\colon \mathcal{B}(H) \to \mathcal{B}(H)$ of Φ. It is not assumed that Ψ is C-measurable or that it is a homomorphism, but we may assume $\Psi(P)$ is a projection whenever P is a projection. This is because every projection in the Calkin algebra is the image of some projection in $\mathcal{B}(H)$ via the quotient map ([43, Lemma 3.1]). We may also assume $\|\Psi(a)\| \leq \|a\|$ for all a: find the polar decomposition of $\Psi(a)$, apply the spectral theorem to its positive part, and truncate the function to $\|a\|$.

For $M \subseteq \mathbb{N}$ let $\mathcal{U}_M[\vec{E}]$ denote the unitary group of $\mathcal{D}_M[\vec{E}]$ and let

$$\begin{aligned}\mathcal{J}^n(\vec{E}) = \{M \subseteq \mathbb{N} \mid & \text{ there is a Borel-measurable } \Xi\colon \mathcal{U}_M[\vec{E}] \to \mathcal{B}(H)\\ &(\forall a \in \mathcal{U}_M[\vec{E}])\|\Phi(\pi(a)) - \pi(\Xi(a))\| \leq 2^{-n}\},\\ \mathcal{J}^n_\sigma(\vec{E}) = \{M \subseteq \mathbb{N} \mid & \text{ there are Borel-measurable } \Psi_i\colon \mathcal{U}_M[\vec{E}] \to \mathcal{B}(H),\ i \in \mathbb{N}\\ &(\forall a \in \mathcal{U}_M[\vec{E}])(\exists i)\|\Phi(\pi(a)) - \pi(\Psi_i(a))\| \leq 2^{-n}\}.\end{aligned}$$

In the terminology of Section 4.1, Ξ is a 2^{-n}-approximation to Φ on $\mathcal{U}_M[\vec{E}]$. Each $\mathcal{J}^n(\vec{E})$ and each $\mathcal{J}^n_\sigma(\vec{E})$ is hereditary and closed under finite changes of its elements, but these sets are not necessarily closed under finite unions.

Given $\vec{E} = (E_n)_{n=0}^\infty$, write $F_n = \overline{\operatorname{span}}\{e_i \mid i \in E_n\}$ and $\mathbf{P}^{\vec{E}}_A$ for the projection to $\bigoplus_{n\in A} F_n$. While $\vec{E}$ is fixed we drop the superscript and write $\mathbf{P}_A$. A family of subsets of $\mathbb{N}$ is *almost disjoint* if $A \cap B$ is finite for all distinct A and B in the family. An almost disjoint family $\mathcal{A}$ is *tree-like* if there is a partial ordering $\preceq$ of $\mathbb{N}$ such that $(\mathbb{N}, \preceq)$ is isomorphic to $(2^{<\mathbb{N}}, \subseteq)$ and each element of $\mathcal{A}$ is a maximal branch of this tree. If J_s $(s \in 2^{<\mathbb{N}})$ are pairwise disjoint finite subsets of $\mathbb{N}$ and $X \subseteq 2^{\mathbb{N}}$, then the family of all $M_x = \bigcup_n J_{x\restriction n}$, $x \in X$, is tree-like, and every tree-like family is of this form.

Lemma 7.2. TA *implies that for every k every tree-like family of $\mathcal{J}^k_\sigma(\vec{E})$-positive sets is at most countable.*

Proof. Fix an uncountable tree-like family $\mathcal{A}$ and a partial ordering $\preceq$ on $\mathbb{N}$ such that $(\mathbb{N}, \preceq)$ is isomorphic to $(2^{<\mathbb{N}}, \subseteq)$ and all elements of $\mathcal{A}$ are maximal branches in $(\mathbb{N}, \preceq)$. Let

$$\mathcal{X} = \{(S, a) \mid S \text{ is infinite and } (\exists B(S) \in \mathcal{A})(S \subseteq B(S) \text{ and } a \in \mathcal{U}_S[\vec{E}])\}.$$

Note that $(S, a) \in \mathcal{X}$ implies $\mathbf{P}_S a = a\mathbf{P}_S = \mathbf{P}_S a \mathbf{P}_S = a$. Also, for $i \in S$ we have that $\mathbf{P}_{\{i\}} a \in \mathcal{B}(F_i)$. If moreover $(T, b) \in \mathcal{X}$, then $\mathbf{P}_S\mathbf{P}_T = \mathbf{P}_{S\cap T}$ and for each i we have $(a-b)\mathbf{P}_{\{i\}} = \mathbf{P}_{\{i\}}(a-b) = \mathbf{P}_{\{i\}}(a-b)\mathbf{P}_{\{i\}}$.

Modify Ψ as follows. If $a \in \mathcal{D}_B[\vec{E}] \setminus \mathcal{K}(H)$ for some $B \in \mathcal{A}$, then replace $\Psi(a)$ with $\Psi(\mathbf{P}_B)\Psi(a)\Psi(\mathbf{P}_B)$. Since a is not compact, such B is unique and since $\mathbf{P}_B a \mathbf{P}_B = a$, the modified Ψ is a representation of Φ which satisfies $\|\Psi(a)\| \leq \|a\|$ for all a and $\Psi(a)\Psi(\mathbf{P}_B) = \Psi(\mathbf{P}_B)\Psi(a)$ for a and B as above.

Fix $n \in \mathbb{N}$. Define a partition $[\mathcal{X}]^2 = K_0^n \cup K_1^n$ by letting $\{(S,a),(T,b)\}$ in K_0^n if and only if the following three conditions hold:

(K1) $B(S) \neq B(T)$.
(K2) For each $i \in S \cap T$ we have $\|(a-b)\mathbf{P}_{\{i\}}\| < 2^{-i}$.
(K3) $\|\Psi(a)\Psi(\mathbf{P}_T) - \Psi(\mathbf{P}_S)\Psi(b)\| > 2^{-n}$ or
$\|\Psi(\mathbf{P}_T)\Psi(a) - \Psi(b)\Psi(\mathbf{P}_S)\| > 2^{-n}$.

The definition is clearly symmetric. Consider $\mathcal{P}(\mathbb{N})$ with the Cantor-set topology (§2.2) and $\mathcal{B}(H)_{\leq 1}$ with the strong operator topology.

Claim 7.3. *The coloring K_0^n is open in the topology on $\mathcal{X}$ obtained by identifying (S,a) with $(B(S), S, a, \Psi(\mathbf{P}_S), \Psi(a)) \in \mathcal{P}(\mathbb{N})^2 \times (\mathcal{B}(H)_{\leq 1})^3$.*

Proof. Assume the pair $(S,a),(T,b)$ satisfies (K1). Since S and T are infinite subsets of disjoint branches of $(\mathbb{N}, \preceq)$, their intersection is finite and we may fix $s \in S \cap (B(S) \setminus B(T))$ and $t \in T \cap (B(T) \setminus B(S))$. Then it follows that $U = \{(S',a') \mid s \in S'\}$ and $V = \{(T',b') \mid s \in T'\}$ are open neighborhoods of (S,a) and (T,b), and any pair in $U \times V$ satisfies (K1).

We shall show (K2) is open relative to (K1). Fix (S,a) and (T,b) satisfying (K1) and (K2) and U, V as above. Let $U' = \{(S',a') \mid (\forall r \preceq s) r \in S'$ if and only if $r \in S\}$ and $V' = \{(T',b') \mid (\forall r \preceq t) r \in T'$ if and only if $r \in T\}$. These two sets are open and for $(S',a') \in U'$ and $(T',b') \in V'$ we have $S' \cap T' = S \cap T$. For each i in this intersection, $\mathbf{P}_{\{i\}}$ has finite rank. In a finite-dimensional space the norm topology coincides with the strong operator topology; therefore (K2) is open on $\mathcal{X}$ modulo (K1).

It remains to prove (K3) is open. Assuming the pair $\{(S,a),(T,b)\}$ satisfies one of the alternatives of (K3) (without a loss of generality, the first one) one only needs to fix a unit vector ξ such that $\|(\Psi(a)\Psi(\mathbf{P}_T) - \Psi(\mathbf{P}_S)\Psi(b))\xi\| > 2^{-n}$; this defines an open neighborhood consisting of pairs satisfying (K3). $\square$

Claim 7.4. *There are no uncountable K_0^n-homogeneous sets for any n.*

Proof. Assume the contrary. Fix $n \in \mathbb{N}$ and an uncountable K_0^n-homogeneous $\mathbf{H}$. For $i \in M = \bigcup_{(S,a)\in\mathbf{H}} S$ fix $(S_i, a_i) \in \mathbf{H}$ such that $i \in S_i$ and let $c = \sum_{i \in M} a_i \mathbf{P}_{\{i\}}$. Then $c \in \mathcal{D}_M[\vec{E}]_{\leq 1}$ and $\|(c-a)\mathbf{P}_{\{i\}}\| = \|(a_i - a)\mathbf{P}_{\{i\}}\| < 2^{-i}$ for all $(S,a) \in \mathbf{H}$. For $(S,a) \in \mathbf{H}$ we have $M \supseteq S$ and the operator $\mathbf{P}_S c - a = c\mathbf{P}_S - a$ is compact. Therefore, the operators $\Psi(c)\Psi(\mathbf{P}_S) - \Psi(a)$ and $\Psi(\mathbf{P}_S)\Psi(c) - \Psi(a)$ are in $\mathcal{K}(H)$. There is a finite-dimensional projection $\mathbf{R} = \mathbf{R}(S,a)$ such that $\|(I-\mathbf{R})(\Psi(c)\Psi(\mathbf{P}_S) - \Psi(a))\| < 2^{-n-2}$ and $\|(I-\mathbf{R})(\Psi(\mathbf{P}_S)\Psi(c) - \Psi(a))\| < 2^{-n-2}$. Since $\Psi(\mathbf{P}_S)$ is a projection, we may choose $\mathbf{R}$ so that $\mathbf{R}\Psi(\mathbf{P}_S) = \Psi(\mathbf{P}_S)\mathbf{R}$.

Let $\delta = 2^{-n-4}$. By the separability of $\mathcal{K}(H)$ there are a projection $\bar{\mathbf{R}}$ and an uncountable $\mathbf{H}' \subseteq \mathbf{H}$ such that $\|\bar{\mathbf{R}} - \mathbf{R}(S,a)\| < \delta$ for all (S,a) in $\mathbf{H}'$. By the norm-separability of the range of $\bar{\mathbf{R}}$ we may find an uncountable $\mathbf{H}'' \subseteq \mathbf{H}'$

such that for all (S,a) and (T,b) in $\mathbf{H}''$ we have $\|\bar{\mathbf{R}}(\Psi(\mathbf{P}_S)-\Psi(\mathbf{P}_T))\|<\delta$ and $\|\bar{\mathbf{R}}(\Psi(a)-\Psi(b))\|<\delta$.

Write $a\approx_\varepsilon b$ for $\|a-b\|<\varepsilon$. Fix distinct (S,a) and (T,b) in $\mathbf{H}''$. Recalling that $\|\Psi(d)\|=\|d\|$ for all d, we have

$$\begin{aligned}
(I-\bar{\mathbf{R}})\Psi(a)\Psi(\mathbf{P}_T) &\approx_\delta && (I-\mathbf{R}(S,a))\Psi(a)\Psi(\mathbf{P}_T)\\
&\approx_{2^{-n-2}} && (I-\mathbf{R}(S,a))\Psi(\mathbf{P}_S)\Psi(c)\Psi(\mathbf{P}_T)\\
&= && \Psi(\mathbf{P}_S)(I-\mathbf{R}(S,a))\Psi(c)\Psi(\mathbf{P}_T)\\
&\approx_{2\delta} && \Psi(\mathbf{P}_S)(I-\mathbf{R}(T,b))\Psi(c)\Psi(\mathbf{P}_T)\\
&\approx_{2^{-n-2}} && \Psi(\mathbf{P}_S)(I-\mathbf{R}(T,b))\Psi(b)\\
&\approx_{2\delta} && \Psi(\mathbf{P}_S)(I-\mathbf{R}(S,a))\Psi(b)\\
&= && (I-\mathbf{R}(S,a))\Psi(\mathbf{P}_S)\Psi(b)\\
&\approx_\delta && (I-\bar{\mathbf{R}})\Psi(\mathbf{P}_S)\Psi(b).
\end{aligned}$$

Hence $\|(I-\bar{\mathbf{R}})(\Psi(a)\Psi(\mathbf{P}_T)-\Psi(\mathbf{P}_S)\Psi(b))\|<6\delta+2^{-n-1}$. Also

$$\bar{\mathbf{R}}\Psi(a)\Psi(\mathbf{P}_T)\approx_\delta \bar{\mathbf{R}}\Psi(b)\Psi(\mathbf{P}_T)=\bar{\mathbf{R}}\Psi(\mathbf{P}_T)\Psi(b)\approx_\delta \bar{\mathbf{R}}\Psi(\mathbf{P}_S)\Psi(b)$$

and $\|\Psi(a)\Psi(\mathbf{P}_T)-\Psi(\mathbf{P}_S)\Psi(b)\|<8\delta+2^{-n-1}<2^{-n}$. Since an analogous argument shows $\|\Psi(\mathbf{P}_T)\Psi(a)-\Psi(b)\Psi(\mathbf{P}_S)\|<2^{-n}$, the pair $\{(S,a),(T,b)\}$ satisfies (K3). Since (K1) and (K2) are automatic, we have $\{(S,a),(T,b)\}\in K_1^n$, a contradiction. □

With k as in the statement of Lemma 7.2 let $\bar{n}=k+3$. By Claim 7.4 and TA, $\mathcal{X}$ can be covered by the union of $K_1^{\bar{n}}$-homogeneous sets $\mathcal{X}_i$ for $i\in\mathbb{N}$. For each i fix a countable $\mathbf{D}_i\subseteq\mathcal{X}_i$ dense in the separable metric topology from Claim 7.3. It suffices to prove that every $B\in\mathcal{A}\setminus\{B(S):(\exists a)(S,a)\in\bigcup_i\mathbf{D}_i\}$ belongs to $\mathcal{J}_\sigma^{\bar{n}-3}(\vec{E})$.

Fix a dense set of projections Q_i, for $i\in\mathbb{N}$, in the projections of $\mathcal{K}(H)$. We also assume that $Q_0=0$ and that for every i the set $\{Q_m : Q_m\geq Q_i\}$ is dense in $\{P : P\geq Q_i$ and P is a projection in $\mathcal{K}(H)\}$. For example, we may let Q_m enumerate all finite rank projections belonging to some countable elementary submodel of $H_{\mathfrak{c}^+}$.

For $m\in\mathbb{N}$ define a relation $\sim_m$ on $\mathcal{X}$ by letting

$$(S,a)\sim_m(T,d)$$

if and only if all of the following conditions are satisfied.

($\sim_m$1) $S\cap m=T\cap m$,
($\sim_m$2) $\|(a-b)\mathbf{P}_{\{i\}}\|<2^{-i-1}$ for all $i<m$,
($\sim_m$3) $\|Q_j(\Psi(\mathbf{P}_S)-\Psi(\mathbf{P}_T))Q_j\|\leq 1/m$ for all $j\leq m$, and
($\sim_m$4) $\|Q_j(\Psi(a)-\Psi(b))Q_j\|\leq 1/m$ for all $j\leq m$.

We should emphasize that this is not an equivalence relation.

For p and m in $\mathbb{N}$ and $(S,a) \in \mathcal{X}_p$, let

$$m^+(S,a,p) = \min\{j > m : (\exists (T,d) \in \mathbf{D}_p)((T,d) \sim_m (S,a) \text{ and } T \cap B \subseteq j)\}.$$

If $(S,a) \in \mathcal{X}_p$, then (T,d) as in the definition of $m^+(S,a,p)$ exists and $T \cap B$ is finite. Therefore $m^+(S,a,p)$ is well-defined whenever $(S,a) \in \mathcal{X}_p$.

We check that for every m and every p there is a finite set $F_m \subseteq \mathbf{D}_p$ such that for every $(S,a) \in \mathcal{X}_p$, there is $(T,d) \in F_m$ satisfying $(S,a) \sim_m (T,d)$. Clearly there are only finitely many possibilities for $S \cap m$. The projections $\mathbf{P}_{\{i\}}$ and Q_j are finite-dimensional and therefore the unit ball of the range of any of these projections is totally bounded. Finally, note that in $(\sim_m 2)$ we have $(a-b)\mathbf{P}_{\{i\}} = \mathbf{P}_{\{i\}}(a-b)\mathbf{P}_{\{i\}}$. Therefore for $m \in \mathbb{N}$ we have that

$$m^+ = \max\{m^+(S,a,p) : (S,a) \in \mathcal{X}_p \text{ for some } p < m\}$$

is well-defined. Let $m(0) = 0$ and $m(j+1) > m(j)^+$ for all j. Let

$$B_0 = B \cap \textstyle\bigcup_{j=0}^{\infty} [m(2j), m(2j+1))$$

and find a nondecreasing sequence $k(j)$, for $j \in \mathbb{N}$, such that the following conditions are satisfied.

(1) $\delta(j) = \|Q_{k(j)}\Psi(\mathbf{P}_{B_0}) - \Psi(\mathbf{P}_{B_0})Q_{k(j)}\|$ satisfies $\lim_{j\to\infty} \delta(j) = 0$,
(2) $k(j) \leq m(2j+1)$, and
(3) $Q_{k(j)}$ strongly converge to the identity.

Let us describe the construction of the sequence $k(j)$, for $j \in \mathbb{N}$. Since we can write R as a strong limit of an increasing sequence of finite rank projections, there is an increasing sequence of finite rank projections R_i, for $i \in \mathbb{N}$, that strongly converge to the identity and such that

$$\lim_{j\to\infty} \|R_j \Psi(\mathbf{P}_{B_0}) - \Psi(\mathbf{P}_{B_0}) R_j\| = 0.$$

Let $k(0) = 0$ and using the density of Q_i, for $i \in \mathbb{N}$, pick a nondecreasing sequence $l(j)$ such that $\|Q_{l(j)} - R_j\| \to 0$ as $j \to \infty$ and $Q_{l(j)}$ converge to the identity in the strong operator topology as $j \to \infty$. Letting $k(j) = \max\{l(i) : l(i) \leq m(2j+1)\}$ we have that (1)–(3) hold.

For $a \in \mathcal{U}_{B_0}[\vec{E}]$ and $p \in \mathbb{N}$ let

$$\begin{aligned} \mathcal{Y}_{a,p} = \{c : (\forall j > p)(\exists (S,d) \in & \mathbf{D}_p) \text{ so that} \\ & \text{(i) } S \cap B_0 \subseteq m(2j+1), \\ & \text{(ii) } S \cap m(2j+1) = B_0 \cap m(2j+1), \\ & \text{(iii) } \|(a-d)\mathbf{P}_{\{i\}}\| < 2^{-i} \text{ for } i \in S \cap B_0, \\ \text{and for all } l \leq m(2j+1) \text{ we have } & \text{(iv) } \|Q_l(\Psi(\mathbf{P}_{B_0}) - \Psi(\mathbf{P}_S))Q_l\| < 2/j \\ & \text{and (v) } \|Q_l(c - \Psi(d))Q_l\| < 2/j\}. \end{aligned}$$

Since $\mathbf{D}_p$ is countable, the set

$$\mathcal{Y}(\bar{n},p)=\bigcup\{\{a\}\times\mathcal{Y}_{a,p}:a\in\mathcal{U}_{B_0}[\vec{E}]\}$$

is Borel for all p.

Claim 7.5. *Assume $a\in\mathcal{U}_{B_0}[\vec{E}]$ is such that $(B_0,a)\in\mathcal{X}_p$. Then*

(4) $\Psi(a)\in\mathcal{Y}_{a,p}$ *and*
(5) $\|\Psi(\mathbf{P}_{B_0})c-\Psi(a)\Psi(\mathbf{P}_{B_0})\|<2^{-\bar{n}+1}$ *for all $c\in\mathcal{Y}_{a,p}$.*

Proof. (4) Fix j. By the definition of $\sim_{m(2j+1)}$ and the choice of $m(2j+2)$ we can choose $(S,d)\in\mathbf{D}_p$ such that (i)–(v) are satisfied with $c=\Psi(a)$.

(5) Assume the contrary, that $\|\Psi(\mathbf{P}_{B_0})c-\Psi(a)\Psi(\mathbf{P}_{B_0})\|>2^{-\bar{n}+1}$. Fix j large enough to have $2\delta(j)<2^{-\bar{n}}$ and

(6) $\|Q_{k(j)}(\Psi(\mathbf{P}_{B_0})c-\Psi(a)\Psi(\mathbf{P}_{B_0}))Q_{k(j)}\|>2^{-\bar{n}+1}$.

Fix $i\geq j$. By the definition of $\mathcal{Y}_{a,p}$ we can pick $(S,d)=(S(i),d(i))\in\mathbf{D}_p$ such that

(7) $S\cap B_0\subseteq m(2i+1)$,
(8) $S\cap m(2i+1)=B_0\cap m(2i+1)$,
(9) $\|(a-d)\mathbf{P}_{\{r\}}\|<2^{-r}$ for all $r\in S\cap B_0$,
(10) $\|Q_l(\Psi(\mathbf{P}_{B_0})-\Psi(\mathbf{P}_S))Q_l\|<2/i$ for all $l\leq m(2i+1)$, and
(11) $\|Q_l(c-\Psi(d))Q_l\|<2/i$ for all $l\leq m(2i+1)$.

Since the pair $\{(B_0,a),(S,d)\}$ belongs to $K_1^{\bar{n}}$ and the corresponding instances of (K1) and (K2) hold, we must have

(12) $\|\Psi(\mathbf{P}_{B_0})\Psi(d)-\Psi(a)\Psi(\mathbf{P}_S)\|<2^{-\bar{n}}$.

The proof is concluded by a computation. Writing $x\approx^j_\varepsilon y$ for

$$\|Q_{k(j)}(x-y)Q_{k(j)}\|\leq\varepsilon,$$

by applying (1), (11), (1), and then (12) in this order we obtain the following estimates:

$$\begin{aligned}\Psi(\mathbf{P}_{B_0})c\approx^j_{\delta(j)}\Psi(\mathbf{P}_{B_0})Q_{k(j)}c\approx^j_{2/i}\Psi(\mathbf{P}_{B_0})Q_{k(j)}\Psi(d)&\\ \approx^j_{\delta(j)}\Psi(\mathbf{P}_{B_0})\Psi(d)\approx^j_{2^{-\bar{n}}}\Psi(a)\Psi(\mathbf{P}_S)&\end{aligned}$$

and therefore

(13) $\|Q_{k(j)}(\Psi(\mathbf{P}_{B_0})c-\Psi(a)\Psi(\mathbf{P}_S))Q_{k(j)}\|\leq 2^{-\bar{n}}+\frac{2}{i}+2\delta(j)$.

Recall that $(S,d)=(S(i),d(i))$ depends on i and note that (10) implies that $\Psi(\mathbf{P}_{S(i)})$ converge to $\Psi(\mathbf{P}_{B_0})$ in the strong operator topology as $i\to\infty$. Since the range of $Q_{k(j)}$ is finite-dimensional,

$$\lim_{i\to\infty}\|Q_{k(j)}(\Psi(a)\Psi(\mathbf{P}_{S(i)})-\Psi(a)\Psi(\mathbf{P}_{B_0}))Q_{k(j)}\|=0.$$

Together with (13) this implies

$$\|Q_{k(j)}(\Psi(\mathbf{P}_{B_0})c - \Psi(a)\Psi(\mathbf{P}_{B_0}))Q_{k(j)}\| < 2^{-\bar{n}+1},$$

a contradiction. □

By Theorem 2.1 there is a C-measurable uniformization $\Theta^0_p\colon \mathcal{U}_{B_0}[\vec{E}] \to \mathcal{B}(H)$ of $\mathcal{Y}(\bar{n}, p)$. By Claim 7.5 the graphs of functions

$$\Upsilon^0_p(a) = \Psi(\mathbf{P}_{B_0})\Theta^0_p(a)$$

for $p \in \mathbb{N}$ cover a graph of a $2^{-\bar{n}+1}$-approximation to Φ. By Lemma 4.3(1) there are Borel-measurable functions witnessing $B_0 \in \mathcal{J}^{\bar{n}-2}_\sigma(\vec{E})$. An analogous argument gives $(\Upsilon^1_i)_i$ witnessing $B_1 = B \setminus B_0 \in \mathcal{J}^{\bar{n}-2}_\sigma(\vec{E})$. Since $a \in \mathcal{U}_B[\vec{E}]$ implies that both $a\mathbf{P}_{B_0} = \mathbf{P}_{B_0} a \mathbf{P}_{B_0} \in \operatorname{dom}(\Upsilon^0_i)$ and $a\mathbf{P}_{B_1} = \mathbf{P}_{B_1} a \mathbf{P}_{B_1} \in \operatorname{dom}(\Upsilon^1_j)$, functions $\Upsilon_{ij}(a) = \Upsilon^0_i(a\mathbf{P}_{B_0}) + \Upsilon^1_j(a\mathbf{P}_{B_1})$ witness $B \in \mathcal{J}^{\bar{n}-3}_\sigma(\vec{E})$. □

7.1. *Uniformizations.* An automorphism Φ of $\mathcal{C}(H)$ and its representation Ψ are fixed. The unitary group $\mathcal{U}_A[\vec{E}]$ of $\mathcal{D}_A[\vec{E}]$ is compact metric with respect to its strong operator topology. Let $\nu_{\vec{E}}$ denote the normalized Haar measure on this group.

LEMMA 7.6. *Assume* $\mathbf{K}$ *is a positive Haar-measurable subset of* $\mathcal{U}[\vec{E}]$ *such that* Φ *has a measurable* ε*-approximation* Ξ *on* $\mathbf{K}$. *Then* Φ *has a Borel-measurable* 2ε*-approximation on* $\mathcal{U}[\vec{E}]$.

Proof. By Luzin's theorem ([27, Theorem 17.12]), by possibly shrinking $\mathbf{K}$ we may assume it is compact and the restriction of Ξ to $\mathbf{K}$ is continuous. Let us first see that we may assume $\nu(\mathbf{K}) > 1/2$. Let $\mathbf{U} \subseteq \mathcal{U}[\vec{E}]$ be a basic open set such that $\nu(\mathbf{K} \cap \mathbf{U}) > \nu(\mathbf{U})/2$. Let n be large enough so that there is an open $\mathbf{U}_0 \subseteq \prod_{i<n} \mathcal{U}(E_i)$ satisfying $\mathbf{U} = \mathbf{U}_0 \times \prod_{i \geq n} \mathcal{U}(E_i)$. Fix a finite

$$F \subseteq \{a \in \mathcal{U}[\vec{E}] \mid a(i) = I_i \text{ for all } i \geq n\}$$

such that $F\mathbf{U}_0 = \mathcal{U}[\vec{E}]$. Then $\mathbf{K}' = F\mathbf{K}$ has measure $> 1/2$ and Ξ' with domain $\mathbf{K}'$ defined by $\Xi'(b) = \Xi(ab)$, where a is the first element of F such that $ab \in \mathbf{K}$ is a continuous ε-approximation of Φ on $\mathbf{K}'$.

Let $\mathcal{X} = \{(a,b) \in \mathcal{U}[\vec{E}] \times \mathbf{K}' \mid ab^* \in \mathbf{K}'\}$. This set is closed. Since $\nu_{\vec{E}}$ is invariant and unimodular, for each a there is b such that $(a,b) \in \mathcal{X}$. By Theorem 2.3 there is a Borel-measurable $f\colon \mathcal{U}[\vec{E}] \to \mathbf{K}$ such that $(a, f(a)) \in \mathcal{X}$ for all a. The map $\Xi_1(a) = \Xi(af(a)^*)\Xi(f(a))$ is clearly a 2ε-approximation to Φ and it is Borel-measurable. □

PROPOSITION 7.7. *If* M_i, $i \in \mathbb{N}$ *are pairwise disjoint infinite subsets of* $\mathbb{N}$ *and* $M = \bigcup_i M_i$ *is in* $\mathcal{J}^n_\sigma(\vec{E})$ *then there is* i *such that* $M_i \in \mathcal{J}^{n-2}(\vec{E})$.

Proof. Assume not. Write $P_i = \mathbf{P}^{\vec{E}}_{M_i}$ and $P = \mathbf{P}^{\vec{E}}_M$. Fix Borel-measurable functions Ξ_i, $i \in \mathbb{N}$, whose graphs cover a 2^{-n}-approximation to Ψ on $\mathcal{U}_M[\vec{E}]$. Let $Q_i = \bigvee_{j=i}^\infty P_j$; then $Q_0 = P$. By making unessential changes to Ψ, we may assume $\Psi(P_i)$, $i \in \mathbb{N}$, are pairwise orthogonal projections such that $\Psi(Q_i) = \bigvee_{j\geq i} \Psi(P_j)$ for all i. Let $\mathcal{V}_i = \prod_{j=i}^\infty \mathcal{U}_{M_i}[\vec{E}]$, a compact group with Haar measure μ_i. We shall find $a_i \in \mathcal{U}_{M_i}[\vec{E}]$ and a μ_i-positive compact $\mathcal{Y}_i \subseteq \mathcal{V}_{i+1}$ such that for all i and all $b \in \mathcal{Y}_i$ we have

$$\|(\Xi_i((\textstyle\sum_{j\leq i} a_j) + b) - \Psi(a_i))\Psi(P_i)\|_{\mathcal{K}} > 2^{-n}. \tag{2}$$

We shall also assume that for $j < i$ we have

$$\mathcal{Y}_i \subseteq \{b \in \mathcal{V}_{i+1} \mid b + \textstyle\sum_{k=j+1}^i a_k \in \mathcal{Y}_j\}. \tag{3}$$

Condition (3) will assure that $\widehat{\mathcal{Y}_i} \subseteq \mathcal{V}_0$ defined by $\widehat{\mathcal{Y}_i} = \{\sum_{j=0}^i a_j + b \colon b \in \mathcal{Y}_i\}$, for $i \in \mathbb{N}$, form a decreasing sequence of compact sets. Assume $a_0, a_1, \dots a_{i-1}$ and $\mathcal{Y}_{i-1}$ have been chosen to satisfy (2) and (3). Using Fubini's theorem, the Lebesgue density theorem, and the inner regularity of the Haar measure, find compact positive sets $V_i \subseteq \mathcal{U}_{M_i}[\vec{E}]$ and $W_i \subseteq \mathcal{V}_{i+1}$ such that for every $x \in V_i$ we have $\mu_{i+1}\{y \in W_i \mid (x, y) \in \mathcal{Y}_i\} > \mu_{i+1}(W_i)/2$. Let

$$\mathcal{X} = \{(a, b, c) \in V_i \times W_i \times \mathcal{U}(H) \mid \|\Xi_i((\textstyle\sum_{j<i} a_j) + a + b)\Psi(P_i) - c\|_{\mathcal{K}} \leq 2^{-n}\}.$$

This is a Borel set, and so is

$$\mathcal{X}_1 = \{(a, c) \mid \mu_i\{b \mid (a, b, c) \in \mathcal{X}\} > \mu_i(W_i)/2\}.$$

Let $\mathcal{Z}$ be the set of all $a \in V_i$ such that $\{b \mid (a, b) \in \mathcal{X}_1\} \neq \emptyset$. This is a projection of $\mathcal{X}_1$. If $\mathcal{Z} \neq V_i$, pick $a_i \in V_i \setminus \mathcal{Z}$. Since $(a_i, \Psi(a_i)) \notin \mathcal{X}_1$, with

$$\mathcal{Y}'_{i+1} = \{b \in W_i \mid \|\Xi_i((\textstyle\sum_{j<i} a_j) + a_i + b)\Psi(P_i) - \Psi(a_i)\|_{\mathcal{K}} > 2^{-n}\}$$

we have $\mu_{i+1}(\mathcal{Y}'_{i+1}) \geq \mu_{i+1}(W_i)/2$. In this case

$$\mathcal{Y}''_{i+1} = \mathcal{Y}'_{i+1} \cap \{b \in W_i \mid (a_i, b) \in \mathcal{Y}_i\}$$

is μ_i-positive and satisfies (2) and (3). By the inner regularity of the Haar measure, find a compact positive $\mathcal{Y}_{i+1} \subseteq \mathcal{Y}''_{i+1}$ and proceed with the construction.

We may therefore without a loss of generality assume $\mathcal{Z} = V_i$. By Theorem 2.1 there is a C-measurable $\bar{f} \colon \mathcal{U}_{M_i}[\vec{E}] \to \mathcal{B}(H)$ such that $(a, \bar{f}(a)) \in \mathcal{X}_1$ for all $a \in \mathcal{Z}$. Then f defined by $f(a) = \Psi(P_i)\bar{f}\Psi(P_i)$ is also Borel. Since $\mathcal{Z} = V_i$ has positive measure, if f is a 2^{-n+1}-approximation of Φ on $\mathcal{Z}$, then Lemma 7.6 gives a Borel 2^{-n+2}-approximation of Φ on $\mathcal{U}_{M_i}[\vec{E}]$, showing that $M_i \in \mathcal{J}^{n-2}(\vec{E})$ and contradicting our assumption. Therefore we can fix $a_i \in \mathcal{Z}$ such that $\|(f(a_i) - \Psi(a_i))\Psi(P_i)\|_{\mathcal{K}} > 2^{-n+1}$. Then $\mathcal{Y}'_{i+1} = \{b \mid (a_i, b, f(a_i)) \in \mathcal{X}\}$ has a positive measure and for each $b \in \mathcal{Y}'_{i+1}$ clause (2) holds because

$$\begin{aligned}&\|(\Xi_i(\textstyle\sum_{j\leq i} a_j + b) - \Psi(a_i))\Psi(P_i)\|_{\mathcal{K}}\\ &\quad\geq \|(f(a_i) - \Psi(a_i))\Psi(P_i)\|_{\mathcal{K}} - \|(\Xi_i(\textstyle\sum_{j\leq i} a_j + b) - f(a_i))\Psi(P_i)\|_{\mathcal{K}} > 2^{-n}.\end{aligned}$$

Let $\mathcal{Y}_{i+1} \subseteq \mathcal{Y}'_{i+1}$ be a compact positive set. This describes the construction. Let $a = \sum_{i=0}^\infty a_i$. Since $a_i = P_i a P_i$ for each i and P_i are pairwise orthogonal, $\|a\| \leq \sup_i \|a_i\| = 1$. For some i we have $\|\Xi_i(a) - \Psi(a)\|_{\mathcal{K}} \leq 2^{-n}$; hence

$\|(\Xi_i(a)-\Psi(a_i))\Psi(P_i)\|_{\mathcal{K}} \leq 2^{-n}$. However, $\sum_{j=i+1}^{\infty} a_i$ is in $\mathcal{Y}_i$ by (3) and the compactness of $\mathcal{Y}_i$ in the product topology. This contradicts (2). □

Proof of Proposition 7.1. Enumerate $\mathbb{N}$ as n_s ($s \in 2^{<\mathbb{N}}$) and write $M_x = \{n_{x\restriction j} \mid j \in \mathbb{N}\}$. By Lemma 7.2, for every m the set $\{x \mid M_x \notin \mathcal{J}_\sigma^n(\vec{E})\}$ is at most countable. We may therefore fix x_0 such that $M_0 = M_{x_0}$ belongs to $\mathcal{J}_\sigma^n(\vec{E})$ for each n. Partition M_0 into infinitely many infinite pieces. By Lemma 7.7 at least one of these pieces, call it M_1, belongs to $\mathcal{J}^1(\vec{E})$. By successively applying this argument we find a decreasing sequence M_j of infinite subsets of M_0 such that $M_j \in \mathcal{J}^j(\vec{E})$ for each j. Fix an infinite M such that $M \setminus M_j$ is finite for all j. Then $M \in \bigcap_j \mathcal{J}^j(\vec{E})$ and on $\mathcal{D}_M[\vec{E}]$ there is a Borel-measurable 2^{-j}-approximation to Φ for each j. By Lemma 4.4 there is a C-measurable representation of Φ on $\mathcal{D}_M[\vec{E}]$. By Theorem 6.3, Φ is inner on $\mathcal{D}_M[\vec{E}]$ and by Lemma 6.2, Φ is inner on $\mathcal{D}[\vec{E}]$. □

Proof of Theorem 1. Fix an automorphism Φ of $\mathcal{C}(H)$ and an orthonormal basis (e_n) for H. For every partition $\vec{E}$ of $\mathbb{N}$ into finite intervals such that $\#E_n$ is nondecreasing, Proposition 7.1 implies there is a partial isomorphism $u = u(\vec{E})$ between cofinite-dimensional subspaces of H such that Ψ_u is a representation of Φ on $\mathcal{C}[\vec{E}]$. Therefore $\{(\vec{E}, u(\vec{E}))\}$ is a coherent family of unitaries and Theorem 3.2 implies Φ is inner. □

8. **Concluding remarks**

Let S denote the unilateral shift operator. The following problem of Brown-Douglas-Fillmore is well-known.

Problem 8.1. Is it consistent with the usual axioms of mathematics that some automorphism of the Calkin algebra sends $\pi(S)$ to its adjoint?

Ilan Hirshberg pointed out that there are essentially normal operators a and b with the same essential spectrum such that $\Phi(\pi(a)) \neq \pi(b)$ for all inner automorphisms Φ of the Calkin algebra. This is because for a fixed Φ either $\operatorname{index}(\Phi(a)) = \operatorname{index}(a)$ for all Fredholm operators a, or $\operatorname{index}(\Phi(a)) = -\operatorname{index}(a)$ for all Fredholm operators a. Together with the Brown-Douglas-Fillmore characterization of unitary equivalence modulo compact perturbation of essentially normal operators, this implies that a positive answer to Problem 8.1 is equivalent to the consistency of the existence of normal operators a and b in $\mathcal{C}(H)$ and an automorphism Φ of $\mathcal{C}(H)$ such that $\Phi(a) = b$ but for every inner automorphism Ψ of $\mathcal{C}(H)$ we have $\Psi(a) \neq b$. An argument using [2] shows that if an automorphism Φ sends the standard atomic masa to itself then Φ cannot send $\dot{S}$ to $\dot{S}^*$ (see [18, Prop. 7.7]).

Recall that for a C*-algebra A its *multiplier algebra*, the quantized analogue of the Čech-Stone compactification, is denoted by $M(A)$ (see [5, 1.7.3]). For example, $M(\mathcal{K}(H)) = \mathcal{B}(H)$, $M(C_0(X)) = C(\beta X)$ for a locally compact

Hausdorff space X, and $M(A) = A$ for every unital C*-algebra A. George Elliott suggested investigating when all automorphisms of $M(A)/A$ are trivial and Ping Wong Ng suggested investigating when isomorphism of the corona algebras $M(A)/A$ and $M(B)/B$ implies isomorphism of A and B. The following is the set-theoretic core of both of these problems and it is very close to [16] and [13] in spirit.

Problem 8.2. Assume A and B are separable nonunital C*-algebras. When does every isomorphism between the corona algebras $M(A)/A$ and $M(B)/B$ lift to a *-homomorphism Φ of $M(A)$ into $M(B)$, so that the diagram

$$\begin{tikzcd} M(A) \arrow[r, "\Phi"] \arrow[d, "\pi"] & M(B) \arrow[d, "\pi"] \\ M(A)/A \arrow[r, "\Psi"'] & M(B)/B \end{tikzcd}$$

commutes?

TA implies the positive answer when $A = B = \mathcal{K}(H)$ (Theorem 1) and TA+MA implies the positive answer when both A and B are of the form $C_0(X)$ for a countable locally compact space X ([13, Ch. 4]). One could also ask analogous questions for *-homomorphisms instead of isomorphisms or, as suggested by Ping Wong Ng, for $\ell^\infty(A)/c_0(A)$ instead of the corona algebra. A number of analogous lifting results for quotient Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ were proved in [13] (see also [15], [16]).

It was recently proved by the author [17] that the Proper Forcing Axiom, PFA, implies all automorphisms of the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ are inner, even for nonseparable Hilbert space.

An analogous result for automorphisms of the Boolean algebra $\mathcal{P}(\kappa)/\operatorname{Fin}$, where κ is arbitrary, was proved in [42].

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