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Analyticity of periodic traveling free surface water waves with vorticity

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Abstract

We prove that the profile of a periodic traveling wave propagating at the surface of water above a flat bed in a flow with a real analytic vorticity must be real analytic, provided the wave speed exceeds the horizontal fluid velocity throughout the flow. The real analyticity of each streamline beneath the free surface holds even if the vorticity is only Hölder continuously differentiable.

1. Introduction

The theory of periodic traveling waves propagating in irrotational flow at the surface of water with a flat bed was initiated at the beginning of the 19^{th} century, with the first investigations confined to waves of small amplitude in which case linear theory provides a reasonable approximation with sinusoidal wave profiles [9]. The realization that wave trains at sea — periodic plane waves termed *swell* in oceanography, with no variation along their crests, the motion being identical in any direction parallel to the crest line and the wave profile being monotone between each crest and trough — feature flatter profiles near the trough and steeper elevations near the crest than those captured by a sinusoidal wave profile [13] prompted the development of nonlinear studies. Of great interest are wave-current interactions: wave trains which propagate steadily without change of form at the surface of a layer of water with an underlying current, over an impermeable flat bed. Vorticity is adequate for describing currents; a current which is uniform with depth is described by zero vorticity (irrotational case) [8]. Constant nonzero vorticity is appropriate for tidal flows [19] and nonconstant vorticity is the hallmark of highly irregular currents [6].

The first rigorous proofs of the existence of wave trains in irrotational flow — called *Stokes waves* — appeared at the beginning of the 20^{th} century and investigations performed over the last decades provide us with a good understanding of this phenomenon even within the context of waves of large amplitude [1], [4], [20]. The currently available existence theory for periodic traveling waves with vorticity is developed in the context of waves of small and large amplitude with profiles represented by Hölder continuously differentiable functions [6]. For irrotational flows a landmark theorem of Lewy [17] that generalizes the classical Schwarz reflection principle from complex function theory shows that such profiles must be real analytic (see [20]). Using an elegant idea pioneered in the context of parabolic problems by Angenent [2] (see the discussion in [10]) and an approach towards regularity for free boundary value problems pioneered by Kinderlehrer, Nirenberg, and Spruck [14], we extend this regularity property of Stokes waves to the case of periodic traveling water waves with a real analytic vorticity. In the particular case of zero vorticity our approach yields an alternative short proof of the real analyticity of regular Stokes waves.[1] The approach relies on use of an appropriate hodograph change of variable that transforms the free boundary value problem (corresponding in a frame moving at the constant wave speed to the governing equations for water waves with vorticity) into a nonlinear boundary problem for a quasi-linear elliptic equation in a fixed rectangular domain [6]. Subsequently we introduce an additional parameter in the problem, and then use the implicit function theorem to exploit the analytic dependence on the parameter to obtain the analyticity of all streamlines beneath the free surface if the vorticity function is Hölder continuously differentiable. For real analytic vorticity functions we prove that even the wave profile (the top streamline) is real analytic.

In Section 2 we present the governing equations for periodic traveling water waves with vorticity. We conveniently reformulate them by means of a hodograph transform into a nonlinear boundary problem for a quasi-linear elliptic equation in a fixed domain. Section 3 is devoted to the proof of the regularity results.

2. Preliminaries

It is sufficient to analyze a cross-section of the flow, orthogonal to the wave crests. We therefore choose Cartesian coordinates (X, Y) with the horizontal X-axis pointing in the direction of wave propagation, with the Y-axis pointing

[1] In the context of Stokes waves, in addition to regular profiles there are the waves of greatest height that are regular except at their crest where the wave profile has a corner (the profile admits lateral tangents at an angle of $2\pi/3$) and the flow has a stagnation point. For Stokes waves one can show that the maximal value of the horizontal fluid velocity in the flow is attained at the wave crest and for regular waves the wave speed exceeds this maximal value, while for the waves of greatest height these two values are equal (and consequently the wave crest is a stagnation point since the vertical fluid velocity there is zero)[1], [20], [21]. For rotational waves the existence of waves of this type is currently at the level of conjectures supported by formal considerations and numerical simulations (see the discussions in [7], [19], [15], [16]).

vertically upwards, and with the origin lying in the mean water level. In its undisturbed state (no waves) the equation of the flat surface is $Y = 0$ and the flat impermeable bed is given by $Y = -d$ for some $d > 0$. In the presence of waves, let $Y = \eta(X - ct)$ be the free surface and let $(u(X - ct, Y),\ v(X - ct, Y))$ be the velocity field of the flow, $c > 0$ being the (constant) speed of the traveling wave. By the choice of the coordinate system we have

$$\int_0^{2\pi} \eta(X)\, dX = 0,$$

where 2π is the (normalized) wave period.

Homogeneity (constant density) implies the equation of mass conservation

$$u_X + v_Y = 0. \tag{2.1}$$

In the inviscid setting the equation of motion is Euler's equation

$$\begin{cases} (u-c)\, u_X + v u_Y &= -P_X, \\ (u-c) v_X + v v_Y &= -P_Y - g, \end{cases} \tag{2.2}$$

where $P(X - ct, Y)$ denotes the pressure and g is the gravitational constant of acceleration. Both equations (2.1) and (2.2) hold in the fluid domain

$$\Omega = \big\{ (X, Y) \in \mathbb{R}^2 :\ X \in \mathbb{R},\ -d \le Y \le \eta(X) \big\}.$$

For a justification of the assumptions of inviscid homogeneous flow in the context of waves at sea or in a channel we refer to the discussion in [18].

We also have the boundary conditions

$$P = 0 \qquad \text{on} \quad Y = \eta(X), \tag{2.3}$$

$$v = (u-c)\, \eta_X \qquad \text{on} \quad Y = \eta(X), \tag{2.4}$$

and

$$v = 0 \qquad \text{on} \quad Y = -d. \tag{2.5}$$

The dynamic boundary condition (2.3) decouples the motion of the air from that of the water in the absence of surface tension whose effects are negligible for wave lengths greater than a few centimeters [18]. The kinematic boundary conditions (2.4) and (2.5) express the fact that the same particles always form the free water surface; respectively, the fact that the horizontal bed $Y = -d$ is impermeable (see [13]).

The general description of the propagation of a wave train on a current is encompassed by equations (2.1)–(2.5), in combination with the equation

$$\omega = u_Y - v_X \tag{2.6}$$

which specifies the vorticity of the flow. We consider solutions (u, v, P, η) to (2.1)–(2.6) in the class $C^{2+\alpha}_{\text{per}}(\Omega) \times C^{2+\alpha}_{\text{per}}(\Omega) \times C^{2+\alpha}_{\text{per}}(\Omega) \times C^{3+\alpha}_{\text{per}}(\mathbb{R})$ of Hölder continuously differentiable functions with exponent $\alpha \in (0,1)$, the index "per" indicating 2π-periodicity in the X-variable. In addition, there is a single crest and trough per period, the wave profile η is decreasing from crest to trough, and u, P are symmetric while v is antisymmetric about the vertical line directly below a crest. Notice that the symmetry assumptions encompassed in the above definition of a traveling wave solution are not restrictive requirements being actually granted for wave profiles that are monotone between crest and trough [5]. We also impose the condition

$$u(X, Y) < c \tag{2.7}$$

throughout the fluid. The above requirement that the horizontal fluid velocity u is always strictly lower than the wave speed c is supported by field evidence. In swell, the particle speeds are very small compared to the wave speed unless we approach the breaking regime [18]. The assumption (2.7) expresses the fact that the waves move faster than the water (this indicates that the waves are not moving humps of water but pulses of energy moving through water) and allows us (see the discussion in [6]) to specify the vorticity ω of the flow as a function of the streamline,

$$\omega = \gamma(\psi) \tag{2.8}$$

with the vorticity function $\gamma \in C^{1+\alpha}(\mathbb{R})$ if $\omega \in C^{1+\alpha}(\Omega)$. Given any $\gamma \in C^{1+\alpha}(\mathbb{R})$, an interplay between global bifurcation theory, degree theory, *a priori* estimates for nonlinear elliptic equations with nonlinear oblique boundary conditions in combination with sharp maximum principles ensures the existence of solutions of this type, representing waves of small amplitude as well as waves of large amplitude [6]. We will show that for all solutions whose existence has been rigorously established, all streamlines beneath the free surface must have maximal regularity being real analytic. Moreover, if the vorticity function is real analytic, the free surface must also be the graph of a real analytic function.

We fix the wave speed $c > 0$ and pass to the moving frame

$$x = X - ct, \qquad y = Y. \tag{2.9}$$

Define the *stream function* $\psi(x, y)$ up to a constant by

$$\psi_y = u - c, \quad \psi_x = -v, \tag{2.10}$$

so that

$$\Delta\psi = \gamma(\psi), \tag{2.11}$$

in view of (2.6), whereas (2.4) and (2.5) guarantee that ψ is constant on both components of the boundary of Ω; say $\psi = 0$ on $y = \eta(x)$ while $\psi = m$ on $y = -d$. Thus

$$\psi(x, y) = m + \int_{-d}^{y} [u(x, s) - c]\, ds$$

and ψ has period 2π in the x-variable. Notice (see [6]) that (2.2) ensures that the expression

$$\frac{(u-c)^2+v^2}{2}+g(y+d)+P-\int_0^{\psi}\gamma(s)\,ds$$

is constant throughout the fluid domain Ω. The governing equations are transformed into the equivalent free boundary value problem

$$\begin{cases}\Delta\psi=\gamma(\psi) & \text{in} \quad -d<y<\eta(x),\\ \dfrac{|\nabla\psi|^2}{2}+g(y+d)=Q & \text{on} \quad y=\eta(x),\\ \psi=0 & \text{on} \quad y=\eta(x),\\ \psi=m & \text{on} \quad y=-d,\end{cases} \tag{2.12}$$

with m and Q physical constants (related to mass flux, respectively hydraulic head). Since $\eta(x)+d>0$, we must have $Q>0$ while

$$\psi_y=u-c<0 \quad \text{in} \quad \Omega \tag{2.13}$$

ensures $m>0$ (see [6]). Assume that in the moving frame the wave crest is located at $(0,\eta(0))$ and the wave troughs at $(\pm\pi,\eta(\pm\pi))$. The hodograph change of variables

$$\begin{cases} q &= x,\\ p &= -\psi(x,y)\end{cases} \tag{2.14}$$

transforms the free boundary problem (2.11) into an elliptic boundary value problem for the function

$$h(q,p)=y+d \tag{2.15}$$

in a fixed rectangular domain. The transformed boundary value problem is

$$\begin{cases}(1+h_q^2)\,h_{pp}-2h_qh_p\,h_{pq}+h_p^2\,h_{qq}-\gamma(-p)\,h_p^3=0 & \text{in} \quad -m<p<0,\\ 1+h_q^2+(2gh-Q)h_p^2=0 & \text{on} \quad p=0,\\ h=0 & \text{on} \quad p=-m,\end{cases} \tag{2.16}$$

for $h\in C^{3+\alpha}_{\text{per}}(\mathbb{R}\times[-m,0])$, even in the q-variable (see [6]).

3. Main result

Our aim is to prove the following result.

THEOREM. *Consider a Hölder continuously differentiable classical solution of the governing equations, representing a periodic traveling water wave in a flow with a Hölder continuously differentiable vorticity function and such that the wave speed exceeds the horizontal fluid velocity throughout the flow. Then each streamline beneath the wave profile is a real-analytic curve. If, in* *addition, the vorticity function is real analytic, then the free surface is the graph of a real-analytic function.*

Remark. Reformulated by use of the notation of Section 2, the first statement above means that for a given $\gamma \in C^{1+\alpha}(\mathbb{R})$, if $h \in C^{3+\alpha}_{\text{per}}(\mathbb{R} \times [-m, 0])$ is a solution to (2.16) satisfying

$$\inf_{(q,p)\in\mathbb{R}\times[-m,0]} h_p(q,p) > 0, \tag{3.1}$$

then the map $q \mapsto h(q,p)$ must be real-analytic for all $p \in [-m, 0)$. Indeed, recall (2.15) and notice that the change of variables (2.14) yields

$$h_p = \frac{1}{c-u}. \tag{3.2}$$

If, in addition, γ is real analytic, then we claim that $q \mapsto h(q, 0)$ is also real analytic.

Before providing the proof, we introduce some useful notation. Let us denote by D the strip $\{(q,p) \in \mathbb{R}^2 : \; q \in \mathbb{R}, \; -m \le p \le 0\}$. We consider the Banach spaces

$$X = \{h \in C^{3+\alpha}_{\text{per}}(D) : \; h(q,-m) = 0\}, \qquad Y = C^{1+\alpha}_{\text{per}}(D), \qquad Z = C^{2+\alpha}_{\text{per}}(\mathbb{R}),$$

of differentiable functions with Hölder continuous derivatives of exponent $\alpha \in (0,1)$, 2π-periodic in the q-variable. We also introduce the open set

$$\mathcal{O} = \{h \in X : \; \inf_{(q,p)\in D} h_p(q,p) > 0\} \subset X.$$

For $h \in \mathcal{O}$ we define $F(h) \in Y \times Z$ by

$$F(h) := \Big((1+h_q^2)h_{pp} - 2h_p h_q\, h_{pq} + h_p^2\, h_{qq} - \gamma(-p)\, h_p^3\,, \; [1+h_q^2+(2gh-Q)h_p^2]\Big|_{p=0}\Big). \tag{3.3}$$

Denoting by $\mathcal{A}$ a real-analytic dependence, we see that

$$F \in \mathcal{A}(\mathcal{O}, Y \times Z). \tag{3.4}$$

Furthermore, if we denote by τ_a the translation by the amount ap in the q-variable, that is,

$$\tau_a\, f(q,p) := f(q+ap, p), \qquad (q,p) \in D,$$

then

$$\partial_q(\tau_a f) = \tau_a(\partial_q f), \qquad \partial_p(\tau_a f) = \tau_a(\partial_p f + a\partial_q f).$$

Using these relations, a direct calculation yields

$$F(\tau_a h) - \tau_a F(h) = aK(\tau_a h, a), \tag{3.5}$$

where the operator $K = (K_1, K_2)$ is given by

$$\begin{aligned} K_1(h,a) &= (2h_{qp} - ah_{qq}) - \gamma h_q[3h_p^2 - 3ah_ph_q + a^2h_q^2], \\ K_2(h,a) &= (2gh - Q)(2h_qh_p - ah_q^2)\Big|_{p=0}. \end{aligned}$$

Note that (3.5) holds true for any $h \in \mathcal{O}$ and any $a \in \mathbb{R}$ with $|a|$ sufficiently small to ensure that $\tau_a h \in \mathcal{O}$. Note also that

$$K \in \mathcal{A}(\mathcal{O} \times \mathbb{R}, Y \times Z). \tag{3.6}$$

We close our preliminary considerations with the following remark. Let $h^0 \in \mathcal{O}$ be a solution to (2.16) and denote by τ_a^0 the standard translation in the q-variable, i.e. $\tau_a^0 f(q,p) = f(a+q,p)$. Then $F(\tau_a^0 h^0) = \tau_a^0 F(h^0) = 0$. Hence, writing $\mathrm{DF}(h^0)$ for the Fréchet derivative of F at h_0, this equivariance implies that $\mathrm{DF}(h^0)[h_q^0] = 0$. In particular, we note that the linear operator $\mathrm{DF}(h^0)$ has a nontrivial kernel if h^0 is not constant in the q-variable.

Proof of the theorem. Let $h^0 \in \mathcal{O}$ be a solution to (2.16) and assume first that $\gamma \in C^{1+\alpha}(\mathbb{R}, \mathbb{R})$. Then $\Phi(h^0, 0) = F(h^0) = 0$, where

$$\Phi : \mathcal{O} \times \mathbb{R} \to Y \times Z, \quad \Phi(h,a) = F(h) - aK(h,a) + \Big(0, \lambda\big(h_p - h_p^0 - ah_q^0\big)\Big|_{p=0}\Big), \tag{3.7}$$

with $\lambda > 0$ being a positive number chosen so that (3.9) below holds true. Since $F(h^0) = 0$ as h^0 is a solution to (2.16), and

$$\Big(\partial_p(\tau_a h^0)\Big)(q,0) = h_p^0(q,0) \,+\, a\, h_q^0(q,0),$$

we deduce in view of (3.5) that

$$\Phi(\tau_a h^0, a) = \tau_a F(h^0) = 0. \tag{3.8}$$

Using the implicit function theorem for real-analytic maps we now show that $a \mapsto \tau_a h^0$ is the unique solution of $\Phi(h,a) = 0$ near $(h^0, 0)$. Indeed, by (3.4) and (3.6), Φ is real analytic in both variables in a neighbourhood of $(h^0, 0)$ in $\mathcal{O} \times \mathbb{R}$. Denoting the Fréchet derivative

$$\mathrm{DF}(h_0) =: (L, T) : X \to Y \times Z,$$

we observe that (3.1) ensures that L is a uniformly elliptic operator. Moreover L satisfies the weak maximum principle, since it has no zero order term [12]. The boundary operator T is of uniform oblique type and has the form

$$Th = [2h_q^0 h_q + 2h_p^0(2gh^0 - Q)h_p + 2g(h_p^0)^2 h]\Big|_{p=0}.$$

We claim that $D_1\Phi(h^0, 0)$ defined by

$$\Big(D_1\Phi(h^0,0)\Big)h := \frac{d}{d\varepsilon}\,\Phi(h^0 + \varepsilon h, 0)\Big|_{\varepsilon=0} = (L, T + \lambda\partial_p)\,h$$

is an isomorphism from X onto $Y \times Z$, provided $\lambda > 0$ is such that

$$\lambda > \sup_{q\in\mathbb{R}} \left\{ 2h_p^0 \,(Q - 2gh^0)\Big|_{p=0} \right\}. \tag{3.9}$$

Indeed, in view of (3.1) and (3.2), for such $\lambda > 0$ the boundary operator $T+\lambda\partial_p$ is uniformly oblique, while L is uniformly elliptic. The approach pursued in Section 4 of [6] shows then that $(L, T + \lambda\partial_p)$ is a Fredholm operator of index zero. Hence it suffices to show that it is injective. Assume, in contrast to our claim, that there is a nonzero $h \in X$ such that $(L, T + \lambda\partial_p)h = 0$. We may assume that h has a positive maximum (otherwise consider $-h$). By the weak maximum principle [12] and the boundary condition $h(\cdot, -m) = 0$ we conclude that h takes its positive maximum on $p = 0$, say at $(q_0, 0)$. Then $h_q(q_0, 0) = 0$, whereas $h_p(q_0, 0) > 0$ by Hopf's maximum principle [11]. This contradicts our assumption $(Th + \lambda h_p)|_{p=0} = 0$ since $h_q(q_0, 0) = 0$ yields that $(Th + \lambda h_p)$ evaluated at $(q_0, 0)$ equals

$$h_p(q_0,0)\left[\lambda - 2\,h_p^0(q_0,0)\left(Q - 2gh^0(q_0,0)\right)\right] + 2g\left[h_p^0(q_0,0)\right]^2 h(q_0,0) > 0$$

in view of (3.9) and the fact that $h(q_0, 0) > 0$. Thus for $\lambda > 0$ satisfying (3.9), we have

$$D_1\Phi(h^0, 0) \in \mathrm{Isom}(X, Y \times Z).$$

By the implicit function theorem for real-analytic maps [3] we now conclude the existence of some $\varepsilon > 0$ and some $\varphi \in \mathcal{A}((-\varepsilon, \varepsilon), \mathcal{O})$ such that in a sufficiently small neighbourhood of $(h^0, 0) \in X \times \mathbb{R}$ all solutions of $\Phi(h, a) = 0$ are given by $(h, a) = (\varphi(a), a)$. Taking into account (3.8), by uniqueness we deduce that $\tau_a h^0 = \varphi(a)$ for $a \in (-\varepsilon, \varepsilon)$. In particular, given $(q, p) \in \mathbb{R} \times [-m, 0)$, we have

$$[a \mapsto h^0(ap + q, p)] \in \mathcal{A}((-\varepsilon, \varepsilon), \mathbb{R}).$$

Real-analyticity being a local property, we conclude that $[q \mapsto h^0(q, p)] \in \mathcal{A}(\mathbb{R}, \mathbb{R})$ for any $p \in [-m, 0)$. The first part of the proof is thus completed.

Assume now that $\gamma \in \mathcal{A}(\mathbb{R}, \mathbb{R})$. We prove the analyticity of the wave profile η by applying the approach developed in [14] to the problem (2.12). Using the notation in [14], we choose some $d^+ > \sup_{x\in\mathbb{R}} \{\eta(x)\}$ and we set $\Gamma = \mathrm{graph}\,\eta$,

$$\Omega_- := \left\{(x, y) \in \mathbb{R}^2\,;\, -d < y < \eta(x)\right\},\;\; \Omega_+ := \left\{(x, y) \in \mathbb{R}^2\,;\, \eta(x) < y < d^+\right\},$$

as well as $u_- := \psi$ and $u_+ \equiv 0$. Recall that

$$\nu(x, \eta(x)) = \frac{(-\eta'(x), 1)}{\sqrt{1 + \eta'(x)^2}}, \qquad x \in \mathbb{R}$$

is a unit normal vector to Γ. Differentiation of the relation $\psi(x, \eta(x)) \equiv 0$ yields

$$\psi_x(x, \eta(x)) = -\eta'(x)\psi_y(x, \eta(x)), \quad x \in \mathbb{R}. \tag{3.10}$$

This implies the following expression for the normal derivative:

$$\partial_\nu \psi(x,\eta(x)) = \sqrt{1+\eta'(x)^2}\,\psi_y(x,\eta(x)), \quad x \in \mathbb{R}. \tag{3.11}$$

On the other hand, using (3.10), the first boundary condition on Γ in (2.12) takes the form

$$\Big(1+\eta'(x)^2\Big)\,\psi_y^2(x,\eta(x)) + 2g\,\Big(\eta(x)+d\Big) - 2Q = 0, \quad x \in \mathbb{R}. \tag{3.12}$$

Now introduce the function $f : \mathbb{R}^4 \to \mathbb{R}$ by

$$f(x,y,n_-,n_+) := n_-^2 + 2g(y+d) - 2Q.$$

Invoking (3.11), relation (3.12) can be expressed as $f(x,y,\partial_\nu u_-,\partial_\nu u_+) = 0$. Clearly f is real analytic in all its arguments and $\partial_3 f \cdot \partial_\nu u_- = 2\,(\partial_\nu\psi)^2$ on Γ. But (2.13) and (3.11) imply that $(\partial_\nu\psi)^2 \neq 0$ on Γ. Since $\partial_\nu u_+ \equiv 0$, we see that all assumptions of Theorem 3.2 and the Remark following it in [14] are satisfied. Hence Γ is real analytic. □

Remark. Notice that $h \notin \mathcal{A}(D)$ if $\gamma \in C^{1+\alpha}(\mathbb{R}) \setminus \mathcal{A}(\mathbb{R})$.

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