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# Interface evolution: the Hele-Shaw and Muskat problems

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## Abstract

We study the dynamics of the interface between two incompressible 2-D flows where the evolution equation is obtained from Darcy's law. The free boundary is given by the discontinuity among the densities and viscosities of the fluids. This physical scenario is known as the two-dimensional Muskat problem or the two-phase Hele-Shaw flow. We prove local-existence in Sobolev spaces when, initially, the difference of the gradients of the pressure in the normal direction has the proper sign, an assumption which is also known as the Rayleigh-Taylor condition.


## 1. Introduction

We consider the following evolution problem for the active scalar $\rho = \rho(x,t)$, $x \in \mathbb{R}^2$, and $t \geq 0$:

$$\rho_t + v \cdot \nabla \rho = 0,$$

with a velocity $v = (v_1, v_2)$ satisfying the momentum equation

$$\frac{\mu}{\kappa} v = -\nabla p - (0, \mathrm{g}\, \rho) \tag{1.1}$$

and the incompressibility condition $\nabla \cdot v = 0$.

In the following we achieve a rather complete local existence analysis of the dynamics of the interface between two incompressible 2-D flows with different characteristics (i.e., distinct values of $\mu$ and $\rho$) which are evolving under (1.1), also known as Darcy's law [3]. This system was studied by Muskat [16] in order to model the interface between two fluids in a porous media, where $p$ is the pressure, $\mu$ is the dynamic viscosity, $\kappa$ is the permeability of the medium, $\rho$ is the liquid density and g is the acceleration due to gravity. Saffman and Taylor [17] made the observation that the one-phase version (one of the fluids has zero viscosity) was also known as the Hele-Shaw cell equation [14], which, in turn, is the zero-specific heat case of the classical one-phase Stefan problem.

There is a vast literature about those problems (see [5] and [15] for references). In order to frame our result let us point out that in [18] is treated the case where both densities are equal, showing global existence for small data in the stable case and ill-posedness in the unstable case. In [1] the well-posedness in the stable case was considered under time dependent assumption of the arc-chord condition. Finally, in the case where the viscosities are the same, the character of the interphase as the graph of a function is preserved and in [9] [10] this fact has been used to prove local existence and a maximum principle, in the stable case, together with ill-posedness in the unstable situation.

Due to the direction of gravity, the horizontal and the vertical coordinates play different roles. Here we shall assume spatial periodicity in the horizontal space variable, says $\rho(x_1+2k\pi, x_2, t) = \rho(x_1, x_2, t)$. The free boundary is given by the discontinuity on the densities and viscosities of the fluids, where $(\mu, \rho)$ are defined by

$$(\mu,\rho)(x_1,x_2,t) = \begin{cases} (\mu^1,\rho^1), & x \in \Omega^1(t) \\ (\mu^2,\rho^2), & x \in \Omega^2(t) = \mathbb{R}^2 - \Omega^1(t), \end{cases} \tag{1.2}$$

and $\mu^1 \neq \mu^2$, and $\rho^1 \neq \rho^2$ are constants.

Let the free boundary be parametrized by

$$\partial\Omega^j(t) = \{z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) : \alpha \in \mathbb{R}\}$$

such that

$$(z_1(\alpha+2k\pi,t), z_2(\alpha+2k\pi,t)) = (z_1(\alpha,t)+2k\pi, z_2(\alpha,t)),$$

with the initial data $z(\alpha,0) = z_0(\alpha)$.

Notice that each fluid is irrotational, i.e., $\omega = \nabla \times u = 0$, in the interior of each domain $\Omega^i$ $(i = 1,2)$. Therefore the vorticity $\omega$ has its support on the curve $z(\alpha,t)$ and it can be shown easily to be of the form

$$\omega(x,t) = \varpi(\alpha,t)\delta(x - z(\alpha,t)).$$

Then $z(\alpha,t)$ evolves with a velocity field coming from Biot-Savart law, which can be explicitly computed and is given by the Birkhoff-Rott integral of the amplitude $\varpi$ along the interface curve:

(1.3)

$$\begin{aligned} &BR(z,\varpi)(\alpha,t) \\ &\quad= \Bigg(-\frac{1}{4\pi} PV\int_{\mathbb{T}} \varpi(\beta,t) \frac{\tanh(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})(1+\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2}))}{\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2}) + \tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})} d\beta, \\ &\quad\frac{1}{4\pi} PV\int_{\mathbb{T}} \varpi(\beta,t) \frac{\tan(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2})(1-\tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2}))}{\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2}) + \tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})} d\beta\Bigg), \end{aligned}$$

where $PV$ denotes principal value [19]. It gives us the velocity field at the interface to which we can subtract any term in the tangential direction without modifying the geometric evolution of the curve

$$z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t). \tag{1.4}$$

A wise choice of $c(\alpha,t)$, namely

$$\begin{aligned} c(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\alpha z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\cdot \partial_\alpha BR(z,\varpi)(\alpha,t)d\alpha \\ - \int_{-\pi}^{\alpha} \frac{\partial_\alpha z(\beta,t)}{|\partial_\alpha z(\beta,t)|^2}\cdot \partial_\beta BR(z,\varpi)(\beta,t)d\beta, \end{aligned} \tag{1.5}$$

allows us to accomplish the fact that the length of the tangent vector to $z(\alpha,t)$ be just a function in the variable $t$ only:

$$A(t) = |\partial_\alpha z(\alpha,t)|^2,$$

as will be shown in Section 2 (see also [15] and [13]). Then we can close the system using Darcy's law with the equation

$$\varpi(\alpha,t) = -2A_\mu BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - 2\kappa \mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_2(\alpha,t), \tag{1.6}$$

where

$$A_\mu = \frac{\mu_1-\mu_2}{\mu_1+\mu_2}$$

is the Atwood number.

Finally we give the function which measures the arc-chord condition in the periodic case

$$\mathcal{F}(z)(\alpha,\beta,t) = \frac{\beta^2/4}{\tan^2(\frac{z_1(\alpha,t)-z_1(\alpha-\beta,t)}{2}) + \tanh^2(\frac{z_2(\alpha,t)-z_2(\alpha-\beta,t)}{2})} \tag{1.7}$$

for all $\alpha,\beta \in (-\pi,\pi)$, with

$$\mathcal{F}(z)(\alpha,0,t) = \frac{1}{|\partial_\alpha z(\alpha,t)|^2}$$

(see [13] for a closed curve).

Our main result consists of the existence of a positive time $\tau$ (depending upon the initial condition) for which we have a solution of the periodic Muskat problem (equations (1.3)–(1.6)) during the time interval $[0,\tau]$ so long as the initial data satisfy $z_0(\alpha)\in H^k(\mathbb{T})$ for $k\geq 3$, $\mathcal{F}(z_0)(\alpha,\beta)<\infty$, and

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha)) - \nabla p^1(z_0(\alpha)))\cdot\partial_\alpha^\perp z_0(\alpha) > 0,$$

where $p^j$ denote the pressure in $\Omega^j$.

It is interesting to remark that the equality of pressure at each side of the free boundary is obtained in Section 2 directly from Darcy's law without any other assumption.

THEOREM 1.1. *Let* $z_0(\alpha) \in H^k(\mathbb{T})$ *for* $k \geq 3$, $\mathcal{F}(z_0)(\alpha,\beta) < \infty$, *and*

$$\sigma_0(\alpha) = -(\nabla p^2(z_0(\alpha)) - \nabla p^1(z_0(\alpha))) \cdot \partial_\alpha^\perp z_0(\alpha) > 0.$$

*Then there exists a time* $\tau > 0$ *so that there is a solution to* (1.3)–(1.6) *in* $C^1([0,\tau];\, H^k(\mathbb{T}))$ *with* $z(\alpha,0) = z_0(\alpha)$.

We devote the rest of the paper to the proof of Theorem 1.1 which is organized as follows. In Section 2 we derive the system of equations (1.3)–(1.6) with the corresponding choice of $c(\alpha,t)$ and we also obtain the properties of the pressure. In Sections 3 and 4 we present several crucial estimates on the operator $T(u)(\alpha) = 2BR(z,u)(\alpha) \cdot \partial_\alpha z(\alpha)$ and on the inverse operator $(I - \xi T)^{-1}$, $|\xi| \leq 1$. Our proofs rely upon the boundedness properties of the Hilbert transforms associated to $C^{1,\alpha}$ curves, for which we need precise estimates obtained with arguments involving conformal mappings, Hopf maximum principle, and Harnack inequalities. We then provide upper bounds for the amplitude of the vorticity, the Birkhoff-Rott integral, the parametrization of the curve, and the arc-chord condition; namely

$$\|\varpi\|_{H^k} \leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}\right) \quad \text{(Section 5)},$$
$$\|BR(z,\varpi)\|_{H^k} \leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}\right) \quad \text{(Section 6)},$$

$$\begin{aligned}\frac{d}{dt}\|z\|^2_{H^k}(t) \leq &-\frac{\kappa}{2\pi(\mu_1+\mu_2)} \int_{\mathbb{T}} \frac{\sigma(\alpha,t)}{|\partial_\alpha z(\alpha)|^2} \partial_\alpha^k z(\alpha,t) \cdot \Lambda(\partial_\alpha^k z)(\alpha,t) d\alpha \\ &+ \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^k}\right) \qquad \text{(Section 7)},\end{aligned}$$

and

$$\frac{d}{dt}\|\mathcal{F}(z)\|^2_{L^\infty}(t) \leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right) \quad \text{(Section 8)},$$

where the operator $\Lambda$ is defined by the Fourier transform $\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi)$ and $\sigma(\alpha,t)$ is the difference of the gradients of the pressure in the normal direction. In Section 9 we study the evolution of $m(t) = \min_{\alpha\in\mathbb{T}} \sigma(\alpha,t)$, which satisfies the following lower bound

$$m(t) \geq m(0) - \int_0^t \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(s) + \|z\|^2_{H^3}(s)\right) ds.$$

Finally, in Section 10, we introduce a regularized evolution equation where we use the previous *a priori* estimates together with a pointwise inequality satisfied by the nonlocal operator $\Lambda$ [6] to show local existence.

By a similar approach, in [8] we obtain local-existence in Sobolev spaces for the full water wave problem in 2-D when the Rayleigh-Taylor condition is initially satisfied.

## 2. **The evolution equation**

Here $(\mu, \rho)$ are defined by

$$(\mu,\rho)(x_1,x_2,t) = \begin{cases} (\mu^1,\rho^1), & x \in \Omega^1(t) \\ (\mu^2,\rho^2), & x \in \Omega^2(t), \end{cases}$$

where $\mu^1 \neq \mu^2$ and $\rho^1 \neq \rho^2$. Then using the Biot-Savart law we get

$$v(x,t) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(x - z(\beta,t))^{\perp}}{|x - z(\beta,t)|^2} \varpi(\beta,t) d\beta$$

for $x \neq z(\alpha, t)$, where the principal value is taken at infinity.

It is convenient to introduce the complex notation $z = x_1 + ix_2$; then the complex conjugate $\bar{v}$ of the velocity field is given by

$$\bar{v}(z,t) = \frac{1}{2\pi i} PV \int_{\mathbb{R}} \frac{\varpi(\beta,t)}{z - z(\beta,t)} d\beta.$$

In our case of periodic interface, $z(\alpha + 2\pi k, t) = z(\alpha,t) + 2\pi k$, the following classical identity

$$\frac{1}{\pi}\left(\frac{1}{z} + \sum_{k\geq 1} \frac{2z}{z^2 - (2\pi k)^2}\right) = \frac{1}{2\pi \tan(z/2)}$$

yields

$$v(x,t) = \Bigg( -\frac{1}{4\pi} \int_{\mathbb{T}} \varpi(\beta,t) \frac{\tanh(\frac{x_2 - z_2(\beta,t)}{2})(1 + \tan^2(\frac{x_1 - z_1(\beta,t)}{2}))}{\tan^2(\frac{x_1 - z_1(\beta,t)}{2}) + \tanh^2(\frac{x_2 - z_2(\beta,t)}{2})} d\beta,$$
$$\frac{1}{4\pi} \int_{\mathbb{T}} \varpi(\beta,t) \frac{\tan(\frac{x_1 - z_1(\beta,t)}{2})(1 - \tanh^2(\frac{x_2 - z_2(\beta,t)}{2}))}{\tan^2(\frac{x_1 - z_1(\beta,t)}{2}) + \tanh^2(\frac{x_2 - z_2(\beta,t)}{2})} d\beta \Bigg)$$

for $x \neq z(\alpha, t)$.

We have that

$$v^2(z(\alpha,t),t) = BR(z,\varpi)(\alpha,t) + \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t),$$

$$v^1(z(\alpha,t),t) = BR(z,\varpi)(\alpha,t) - \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t),$$

where $v^j(z(\alpha,t),t)$ denotes the limit velocity field obtained approaching the boundary in the normal direction inside $\Omega^j$ and $BR(z,\varpi)(\alpha,t)$ is given by (1.3).

Darcy's law implies that

$$\Delta p(x,t) = -\text{div}\left(\frac{\mu(x,t)}{\kappa} v(x,t)\right) - \text{g}\, \partial_{x_2} \rho(x,t);$$

therefore

$$\Delta p(x,t) = \Pi(\alpha,t)\delta(x - z(\alpha,t)),$$

where $\Pi(\alpha,t)$ is given by

$$\Pi(\alpha,t) = \frac{\mu^2-\mu^1}{\kappa} v(z(\alpha,t),t)\cdot\partial_\alpha^\perp z(\alpha,t) + \mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t).$$

It follows that

$$p(x,t) = -\frac{1}{2\pi}\int_{\mathbb{T}} \ln\Big(\cosh(x_2 - z_2(\alpha,t)) - \cos(x_1 - z_1(\alpha,t))\Big)\Pi(\alpha,t)d\alpha$$

for $x \neq z(\alpha,t)$, implying the important identity

$$p^2(z(\alpha,t),t) = p^1(z(\alpha,t),t),$$

which is just a mathematical consequence of Darcy's law, making it unnecessary to impose it as a physical assumption.

Let us introduce the following notation:

$$[\mu v](\alpha,t) = (\mu^2 v^2(z(\alpha,t),t) - \mu^1 v^1(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t).$$

Then taking the limit in Darcy's law we obtain that

$$\begin{aligned}\frac{[\mu v](\alpha,t)}{\kappa} &= -(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z^1(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t)\\ &\quad - \mathrm{g}(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t)\\ &= -\partial_\alpha(p^2(z(\alpha,t),t) - p^1(z(\alpha,t),t)) - \mathrm{g}(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t)\\ &= -\mathrm{g}(\rho^2-\rho^1)\,\partial_\alpha z_2(\alpha,t),\end{aligned}$$

which gives us

$$\frac{\mu^2+\mu^1}{2\kappa}\varpi(\alpha,t) + \frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) = -\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t),$$

so that

$$\varpi(\alpha,t) = -A_\mu 2BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - 2\kappa\mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_2(\alpha,t).$$

Next we modify the velocity of the curve in the tangential direction

$$z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t), \tag{2.1}$$

where the scalar $c(\alpha,t)$ is chosen in such a way that the tangent vector only depends on the variable $t$ as follows:

$$|\partial_\alpha z(\alpha,t)|^2 = A(t). \tag{2.2}$$

To find such a $c(\alpha,t)$ let us differentiate the identity (2.2)

$$A'(t) = 2\partial_\alpha z(\alpha,t)\cdot\partial_\alpha z_t(\alpha,t) = 2\partial_\alpha z(\alpha,t)\cdot\partial_\alpha BR(z,\varpi)(\alpha,t) + 2\partial_\alpha c(\alpha,t)A(t),$$

so that

$$\partial_\alpha c(\alpha,t) = \frac{A'(t)}{2A(t)} - \frac{1}{A(t)}\partial_\alpha z(\alpha,t)\cdot\partial_\alpha BR(z,\varpi)(\alpha,t). \tag{2.3}$$

Because $c(\alpha,t)$ has to be periodic, we obtain

$$\frac{A'(t)}{2A(t)} = \frac{1}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z(\alpha,t)\cdot\partial_\alpha BR(z,\varpi)(\alpha,t)d\alpha. \tag{2.4}$$

Using (2.4) in (2.3) and integrating in $\alpha$, one gets

$$\begin{aligned} c(\alpha,t) = {}& \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta \\ & - \int_{-\pi}^{\alpha}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta, \end{aligned} \tag{2.5}$$

where we have chosen $c(-\pi,t) = c(\pi,t) = 0$.

Let us consider now the solutions of equation (2.1) with $c(\alpha,t)$ given by (2.5). It is easy to check that

$$\frac{d}{dt}|\partial_\alpha z(\alpha,t)|^2 = c(\alpha,t)\partial_\alpha|\partial_\alpha z(\alpha,t)|^2 + b(t)|\partial_\alpha z(\alpha,t)|^2,$$

where

$$b(t) = \frac{1}{\pi}\int_{\mathbb{T}}\frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta.$$

Next we solve this linear partial differential equation, assuming that (2.2) is satisfied initially, to find that the unique solution is given by

$$|\partial_\alpha z(\alpha,t)|^2 = |\partial_\alpha z(\alpha,0)|^2 + \frac{1}{\pi}\int_0^t\int_{\mathbb{T}}\partial_\alpha z(\alpha,t)\cdot\partial_\beta BR(z,\varpi)(\alpha,t)d\alpha ds,$$

which proves (2.2).

Our next step is to find the formula for the difference of the gradients of the pressure in the normal direction:

$$-(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z(\alpha,t),t))\cdot\partial_\alpha^\perp z(\alpha,t),$$

which we denote by $\sigma(\alpha,t)$. Approaching the boundary in Darcy's law, we get

$$\sigma(\alpha,t) = \frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t) + \mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t).$$

It is easy to check that

$$\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t) = \frac{1}{4\pi}\partial_\alpha\int_{\mathbb{T}}\varpi(\beta,t)\log G(\alpha,\beta,t)d\beta,$$

with

$$G(\alpha,\beta,t)=\sin^2\left(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2}\right)\cosh^2\left(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2}\right) + \cos^2\left(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2}\right)\sinh^2\left(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2}\right).$$

Therefore

$$\int_{\mathbb{T}}\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)d\alpha=0.$$

This shows that the condition $\rho^2\neq\rho^1$ is crucial in order to have a constant sign in the normal direction of the difference of the gradient. Furthermore, since $z_1(\alpha,t)-\alpha$ is periodic we have

$$\int_{\mathbb{T}}\partial_\alpha z_1(\alpha,t)d\alpha=2\pi.$$

*Remark* 2.1. If we consider a closed contour, then it is easy to check that

$$\int_{\mathbb{T}}\sigma(\alpha,t)d\alpha=0,$$

which makes impossible the task of prescribing a sign to $\sigma$ along a closed curve.

## 3. The basic operator

Let us consider the operator $T$ defined by the formula

$$T(u)(\alpha)=2BR(z,u)(\alpha)\cdot\partial_\alpha z(\alpha). \tag{3.1}$$

LEMMA 3.1. *Suppose that* $\|\mathcal{F}(z)\|_{L^\infty}<\infty$ *and* $z\in C^{2,\delta}$ *with* $0<\delta$. *Then* $T:L^2\to H^1$ *and*

$$\|T\|_{L^2\to H^1}\leq\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^4_{C^{2,\delta}}.$$

*Remark* 3.2. In Section 5, Lemma 5.2, there is a proof showing that $T$ also maps $H^k$ into $H^{k+1}$, $k\geq 1$.

*Proof.* Since formula (1.3) yields

$$T(u)(\alpha)=\frac{1}{\pi}\partial_\alpha\int_{\mathbb{T}}u(\beta)\arctan\left(\frac{\tanh(\frac{z_2(\alpha)-z_2(\beta)}{2})}{\tan(\frac{z_1(\alpha)-z_1(\beta)}{2})}\right)d\beta,$$

we have

$$\int_{\mathbb{T}}T(u)(\alpha)d\alpha=0$$

which implies $\|T(u)\|_{L^2}\leq\|\partial_\alpha T(u)\|_{L^2}$.

Let us denote

$$V(\alpha,\beta)=(V_1(\alpha,\beta),V_2(\alpha,\beta)) =\left(\tan\left(\frac{z_1(\alpha)-z_1(\beta)}{2}\right),\tanh\left(\frac{z_2(\alpha)-z_2(\beta)}{2}\right)\right).$$

In the following we shall refer to the Appendix for the definition of $V_j, A_j$ and their properties.

We first write

$$\partial_\alpha T(u) = 2BR(z,u)(\alpha)\cdot\partial_\alpha^2 z(\alpha) + 2\partial_\alpha z(\alpha)\cdot\partial_\alpha BR(z,u)(\alpha) = I_1 + I_2.$$

For $I_1$ we have the expression

$$\begin{aligned} I_1 &= 2\Big(BR(z,u)(\alpha) - \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^2}H(u)(\alpha)\Big)\cdot\partial_\alpha^2 z(\alpha) + 2H(u)(\alpha)\frac{\partial_\alpha^\perp z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \\ &= J_1 + J_2, \end{aligned}$$

where $H(u)$ is the (periodic) Hilbert transform of the function $u$.

Then

$$\begin{aligned} J_1 = &-\frac{1}{2\pi}\partial_\alpha^2 z_1(\alpha)\int_{\mathbb{T}} u(\beta)\frac{V_2(\alpha,\beta)V_1^2(\alpha,\beta)}{|V(\alpha,\beta)|^2}d\beta \\ &-\frac{1}{2\pi}\partial_\alpha^2 z_2(\alpha)\int_{\mathbb{T}} u(\beta)\frac{V_1(\alpha,\beta)V_2^2(\alpha,\beta)}{|V(\alpha,\beta)|^2}d\beta \\ &-\frac{1}{2\pi}\partial_\alpha^2 z_1(\alpha)\int_{\mathbb{T}} u(\alpha-\beta)\left[\frac{V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_2(\alpha)}{\tan(\beta/2)}\right]d\beta \\ &+\frac{1}{2\pi}\partial_\alpha^2 z_2(\alpha)\int_{\mathbb{T}} u(\alpha-\beta)\left[\frac{V_1(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_1(\alpha)}{\tan(\beta/2)}\right]d\beta \\ = &\; K_1 + K_2 + K_3 + K_4 \end{aligned}$$

and we may use that $|V_2(\alpha,\beta)|\le 1$ to get $|K_1|+|K_2|\le C\|z\|_{C^2}\|u\|_{L^2}$.

To estimate $K_3$ let us observe that the following term

$$A_1(\alpha,\alpha-\beta) = \frac{V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_2(\alpha)}{\tan(\frac{\beta}{2})}$$

satisfies $\|A_1\|_{L^\infty}\le\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}^2$ (see Appendix, Lemma 11.1).

In $K_4$ we have the term

$$A_2(\alpha,\alpha-\beta) = \frac{V_1(\alpha,\alpha-\beta)}{|V(\alpha,\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_1(\alpha)}{\tan(\frac{\beta}{2})},$$

which satisfies $\|A_2\|_{L^\infty}\le C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}^2$.

Then we obtain $|K_3|+|K_4|\le C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}^3\|u\|_{L^2}$; therefore

$$J_1\le C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}^3\|u\|_{L^2}.$$

Since the estimate $J_2\le C\|\mathcal{F}(z)\|_{L^\infty}^{1/2}\|z\|_{C^2}|H(u)(\alpha)|$ is immediate, we finally have

$$|I_1|\le C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}^3(\|u\|_{L^2}+|H(u)(\alpha)|). \tag{3.2}$$

Next we write $2BR(z,u)(\alpha)$ as follows:

$$2BR(z,u)(\alpha) = \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)(1-V_2^2(\alpha,\beta))\frac{V^\perp(\alpha,\beta)}{|V(\alpha,\beta)|^2}d\beta$$
$$-\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)V_2(\alpha,\beta)(1,0)d\beta = J_3(\alpha)+J_4(\alpha).$$

Easily we have $|\partial_\alpha J_4(\alpha)\cdot\partial_\alpha z(\alpha)| \le C\|z\|_{C^1}^2\|u\|_{L^2}$. Taking one derivative in $J_3(\alpha)$ and using the cancellation $\partial_\alpha z(\alpha)\cdot\partial_\alpha^\perp z(\alpha)=0$, we get

$$\partial_\alpha J_3(\alpha)\cdot\partial_\alpha z(\alpha) = K_5+K_6+K_7+K_8+K_9,$$

where

$$K_5 = -\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)(1-V_2^2(\alpha,\beta))V_2(\alpha,\beta)\partial_\alpha z_2(\alpha)\frac{V^\perp(\alpha,\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\beta)|^2}d\beta,$$
$$K_6 = \frac{1}{4\pi}\int_{\mathbb{T}} u(\beta)(1-V_2^2(\alpha,\beta))\partial_\alpha z_1(\alpha)\partial_\alpha z_2(\alpha)d\beta,$$
$$K_7 = -\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)(1-V_2^2(\alpha,\beta))(\partial_\alpha z_1(\alpha)V_1^3(\alpha,\beta)$$
$$-\partial_\alpha z_2(\alpha)V_2^3(\alpha,\beta)))\frac{V^\perp(\alpha,\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\beta)|^4}d\beta,$$
$$K_8 = \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)V_2^2(\alpha,\beta)(\partial_\alpha z_1(\alpha)V_1(\alpha,\beta)$$
$$+\partial_\alpha z_2(\alpha)V_2(\alpha,\beta))\frac{V^\perp(\alpha,\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\beta)|^4}d\beta,$$

and

$$K_9 = -\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)(\partial_\alpha z_1(\alpha)V_1(\alpha,\beta)+\partial_\alpha z_2(\alpha)V_2(\alpha,\beta))\frac{V^\perp(\alpha,\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\beta)|^4}d\beta.$$

We have $|K_5|+|K_6|+|K_7|+|K_8| \le C\|z\|_{C^1}^2\|u\|_{L^2}$.

Next we split $K_9 = -L_1 - L_2$, where

$$L_1 = \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\partial_\alpha z_1(\alpha)V_1(\alpha,\beta)\frac{V^\perp(\alpha,\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\beta)|^4}d\beta$$

can be rewritten as follows:

$$L_1 = \frac{1}{2\pi}\int_{\mathbb{T}} u(\alpha-\beta)\partial_\alpha z_1(\alpha)V_1(\alpha,\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}d\beta.$$

We have $L_1 = M_1 + M_2$, where

$$M_1 = \frac{(\partial_\alpha^2 z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4}|\partial_\alpha z_1(\alpha)|^2 H(u)(\alpha),$$

and

$$M_2 = \frac{1}{2\pi}\int_{\mathbb{T}} u(\alpha-\beta)\partial_\alpha z_1(\alpha)B(\alpha,\alpha-\beta)d\beta$$

for

$$B(\alpha,\alpha-\beta) = V_1(\alpha,\alpha-\beta)\frac{V(\alpha,\alpha-\beta)^{\perp}\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4} - \partial_\alpha z_1(\alpha)\frac{(\partial_\alpha^2 z(\alpha))^{\perp}\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4\tan(\beta/2)}.$$

The term $M_1$ satisfies $|M_1| \leq C\|\mathcal{F}(z)\|_{L^\infty}^{1/2}\|z\|_{C^2}|H(u)(\alpha)|$. We claim that

$$|M_2| \leq C\|\mathcal{F}(z)\|_{L^\infty}^2\|z\|_{C^{2,\delta}}^4\int_{\mathbb{T}}|\beta|^{\delta-1}|u(\alpha-\beta)|d\beta$$

(see the Appendix, Lemma 11.2 for the proof).

A similar estimate can be obtained for $L_2$. Finally we have

$$|I_2| \leq C\|\mathcal{F}(z)\|_{L^\infty}^2\|z\|_{C^{2,\delta}}^4(\|u\|_{L^2} + |H(u)(\alpha)| + \int_{\mathbb{T}}|\beta|^{\delta-1}|u(\alpha-\beta)|d\beta).$$

This inequality together with (3.2) yields

$$|\partial_\alpha T(u)(\alpha)| \leq C\|\mathcal{F}(z)\|_{L^\infty}^2\|z\|_{C^{2,\delta}}^4(\|u\|_{L^2} + |H(u)(\alpha)| + \int_{\mathbb{T}}|\beta|^{\delta-1}|u(\alpha-\beta)|d\beta).$$

To finish we use the $L^2$-boundedness of $H$ and Minkowski's inequality to obtain the estimate

$$\|\partial_\alpha T(u)\|_{L^2} \leq C\|\mathcal{F}(z)\|_{L^\infty}^2\|z\|_{C^{2,\delta}}^4\|u\|_{L^2}. \qquad \square$$

## 4. **Estimates on the inverse operator** $(I - \xi T)^{-1}$

In Lemma 3.1 we have considered the operator $T : L^2 \to H^1$

$$T(u) = 2BR(z,u)(\alpha)\cdot\partial_\alpha z(\alpha)$$

for $\mathcal{F}(z)(\alpha,\beta) < \infty$. Then $T$ is a compact operator from Sobolev space $L^2$ to itself whose adjoint is given by the formula

$$\begin{aligned} T^*(u)(\alpha) = &\frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_2(\beta)\tan(\frac{z_1(\alpha)-z_1(\beta)}{2}) - \partial_\alpha z_1(\beta)\tanh(\frac{z_2(\alpha)-z_2(\beta)}{2})}{\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2}) + \tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}d\beta \\ &- \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_2(\beta)\tan(\frac{z_1(\alpha)-z_1(\beta)}{2})\tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}{\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2}) + \tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}d\beta \\ &- \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_1(\beta)\tanh(\frac{z_2(\alpha)-z_2(\beta)}{2})\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2})}{\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2}) + \tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}d\beta. \end{aligned}$$

We will show that in $H^{\frac{1}{2}}$, $I - \xi T$ has a bounded inverse $(I - \xi T)^{-1}$ for $|\xi| \leq 1$, whose norm grows at most like $\exp(C|||z|||^2)$ with $|||z||| = \|z\|_{H^3} + \|\mathcal{F}(z)\|_{L^\infty}$.

Let $z$ be outside the curve $z(\alpha)$; then we define

$$\begin{aligned}
f(z) &= \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_2(\beta)\tan(\frac{z_1-z_1(\beta)}{2}) - \partial_\alpha z_1(\beta)\tanh(\frac{z_2-z_2(\beta)}{2})}{\tan^2(\frac{z_1-z_1(\beta)}{2}) + \tanh^2(\frac{z_2-z_2(\beta)}{2})}d\beta \\
&\quad - \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_2(\beta)\tan(\frac{z_1-z_1(\beta)}{2})\tanh^2(\frac{z_2-z_2(\beta)}{2})}{\tan^2(\frac{z_1-z_1(\beta)}{2}) + \tanh^2(\frac{z_2-z_2(\beta)}{2})}d\beta. \\
&\quad - \frac{1}{2\pi}\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_1(\beta)\tanh(\frac{z_2-z_2(\beta)}{2})\tan^2(\frac{z_1-z_1(\beta)}{2})}{\tan^2(\frac{z_1-z_1(\beta)}{2}) + \tanh^2(\frac{z_2-z_2(\beta)}{2})}d\beta. \\
&= \frac{1}{2\pi}\Im\int_{\mathbb{T}} \frac{u(\beta)\partial_\alpha z(\beta)}{\tan(\frac{z-z(\beta)}{2})}d\beta\,.
\end{aligned}$$

That is, $f$ is the real part of the Cauchy integral

$$F(z) = f(z) + ig(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{u(\beta)\partial_\alpha z(\beta)}{\tan(\frac{z-z(\beta)}{2})}d\beta,$$

which is defined in both periodic domains $\Omega_1$ and $\Omega_2$, respectively placed above and below the curve $z(\alpha)$. In the following, $\widetilde{\Omega}_j$ denotes a corresponding fundamental domain; i.e., $\Omega_j = \bigcup\{\widetilde{\Omega}_j + 2\pi n\}$.

Taking $z = z(\alpha) + \varepsilon\partial_\alpha^\perp z(\alpha)$ we obtain

$$f(z(\alpha) + \varepsilon\partial_\alpha^\perp z(\alpha)) = \frac{1}{2\pi}\Im\int_{\mathbb{T}} \frac{u(\beta)\partial_\alpha z(\beta)}{\tan(\frac{z(\alpha)-z(\beta)+\varepsilon\partial_\alpha^\perp z(\alpha)}{2})}d\beta,$$

and letting $\varepsilon \to 0$, we get

$$f(z(\alpha)) = T^*(u) - \text{sign}\,(\varepsilon)u(\alpha). \tag{4.1}$$

On the other hand we have

$$\lim_{\varepsilon\to 0} g(z(\alpha) + \varepsilon\partial_\alpha^\perp z(\alpha)) = \lim_{\varepsilon\to 0} \Im F(z(\alpha) + \varepsilon\partial_\alpha^\perp z(\alpha)) = \mathcal{G}(u)(\alpha),$$

where

$$\begin{aligned}
\mathcal{G}(u)(\alpha) &= -\frac{1}{2\pi}PV\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_1(\beta)\tan(\frac{z_1(\alpha)-z_1(\beta)}{2})(1-\tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2}))}{\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2}) + \tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}d\beta \\
&\quad - \frac{1}{2\pi}PV\int_{\mathbb{T}} u(\beta)\frac{\partial_\alpha z_2(\beta)\tanh(\frac{z_2(\alpha)-z_2(\beta)}{2})(1+\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2}))}{\tan^2(\frac{z_1(\alpha)-z_1(\beta)}{2}) + \tanh^2(\frac{z_2(\alpha)-z_2(\beta)}{2})}d\beta,
\end{aligned}$$

independent of the sign of $\varepsilon \to 0$.

First we will show that $T^*u = \lambda u \Rightarrow |\lambda| < 1$. Since $T^*$ is a compact operator (of Hilbert-Schmidt type) we can conclude the existence of $(I-\xi T^*)^{-1}$ with $|\xi| \leq 1$ (see also [2]). To do that let us compute the value of $\nabla f(z(\alpha))$.

Denoting $z = x_1 + ix_2$, the identity

$$f(z) = \frac{1}{2\pi}\Im\int_{\mathbb{T}} \frac{u(\beta)\partial_\alpha z(\beta)}{\tan(\frac{z-z(\beta)}{2})}d\beta = -\frac{1}{2\pi}\Im\int_{\mathbb{T}} u(\beta)\partial_\beta \ln\left(\sin\left(\frac{z-z(\beta)}{2}\right)\right)d\beta$$
$$= \frac{1}{2\pi}\Im\int_{\mathbb{T}} \partial_\beta u(\beta)\ln\left(\sin\left(\frac{z-z(\beta)}{2}\right)\right)d\beta$$

yields

$$\nabla f(z) = \frac{1}{2\pi}\Im\int_{\mathbb{T}} \partial_\beta u(\beta)\nabla \ln\left(\sin\left(\frac{z-z(\beta)}{2}\right)\right)d\beta.$$

That is,

$$\nabla f(x) = \Bigg(-\frac{1}{4\pi}\int_{\mathbb{T}} \partial_\beta u(\beta)(\beta,t)\frac{\tanh(\frac{x_2-z_2(\beta,t)}{2})(1+\tan^2(\frac{x_1-z_1(\beta,t)}{2}))}{\tan^2(\frac{x_1-z_1(\beta,t)}{2})+\tanh^2(\frac{x_2-z_2(\beta,t)}{2})}d\beta,$$
$$\frac{1}{4\pi}\int_{\mathbb{T}} \partial_\beta u(\beta)\frac{\tan(\frac{x_1-z_1(\beta,t)}{2})(1-\tanh^2(\frac{x_2-z_2(\beta,t)}{2}))}{\tan^2(\frac{x_1-z_1(\beta,t)}{2})+\tanh^2(\frac{x_2-z_2(\beta,t)}{2})}d\beta\Bigg).$$

Taking the limit as before we get

$$\nabla f(z(\alpha)) = 2BR(z,\partial_\alpha u)(\alpha) + \text{sign}\,(\varepsilon)\frac{\partial_\alpha u(\alpha)}{2|\partial_\alpha z(\alpha)|^2}\partial_\alpha z(\alpha). \tag{4.2}$$

Assuming now that $T^* u = \lambda u$, the analyticity of $F(z)$ allows us to obtain

$$0 < \int_{\widetilde{\Omega}_1} |F'(z)|^2 dx = -\int_{\mathbb{T}} f(z(\alpha))\nabla f(z(\alpha))\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha, \tag{4.3}$$
$$= \int_{\mathbb{T}} (1-\lambda)u(\alpha)2BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha$$

and

$$0 < \int_{\widetilde{\Omega}_2} |F'(z)|^2 dx = \int_{\mathbb{T}} f(z(\alpha))\nabla f(z(\alpha))\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha \tag{4.4}$$
$$= \int_{\mathbb{T}} (\lambda+1)u(\alpha)2BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha,$$

where we have used (4.1) and (4.2). Multiplying together both inequalities we get

$$0 < (1-\lambda^2)\left(\int_{\mathbb{T}} u(\alpha)2BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha\right)^2,$$

and therefore $|\lambda| < 1$.

PROPOSITION 4.1. *The norms* $||(I \mp T^*)^{-1}||_{L^2_0}$ *are bounded from above by* $\exp(C|||z|||^2)$ *for some universal constant* $C$ *where the space* $L^2_0$ *is the usual* $L^2$ *with the extra condition of mean value zero*, *i.e.*, *the subspace orthogonal to the constants.*

*Proof.* With the notation introduced before we have

$$\begin{aligned} F_1 &= F/\Omega_1 = f_1 + ig_1, \\ F_2 &= F/\Omega_2 = f_2 + ig_2, \\ f_1/\partial\Omega &= T^*u - u, \\ f_2/\partial\Omega &= T^*u + u, \\ g_1/\partial\Omega &= g_2/\partial\Omega = \mathcal{G}(u). \end{aligned}$$

The proof follows easily from the estimate

$$e^{-C|||z|||^2} \leq \frac{||u - T^*u||_{L^2_0}}{||u + T^*u||_{L^2_0}} \leq e^{C|||z|||^2} \tag{4.5}$$

valid for every nonzero $u \in L^2_0(\partial\Omega)$.

This is because if we assume $||u - T^*u||_{L^2_0} \leq e^{-2C|||z|||^2}$ for some $||u||_{L^2_0} = 1$, then we obtain $||u + T^*u||_{L^2_0} \geq 2||u||_{L^2_0} - e^{-2C|||z|||^2} \geq 1$ which contradicts (4.5). Therefore we must have $||u - T^*u||_{L^2_0} \geq e^{-2C|||z|||^2}$ for all $||u||_{L^2_0} = 1$, i.e., $||(I - T^*)^{-1}||_{L^2_0} \leq e^{2C|||z|||^2}$. Similarly, we also have $||(I + T^*)^{-1}||_{L^2_0} \leq e^{2C|||z|||^2}$.

Since $u - T^*u = \mathcal{H}_1(\mathcal{G}(u))$ and $u + T^*u = \mathcal{H}_2(\mathcal{G}(u))$, where $\mathcal{H}_j$ denotes the Hilbert transforms corresponding to each domain $\Omega_j$, then (4.5) is a consequence of the estimate

$$||\mathcal{H}_j||_{L^2(\partial\Omega_j)} \leq e^{C|||z|||^2}, \tag{4.6}$$

where $C$ denotes a universal constant not necessarily the same at each occurrence.

This is because the identity $\mathcal{H}_j^2 = -I$ implies

$$\begin{aligned} ||u - T^*u||_{L^2_0} = ||\mathcal{H}_1(\mathcal{G}(u))||_{L^2_0} &\leq e^{C|||z|||^2} ||\mathcal{G}(u)||_{L^2_0} \\ &\leq e^{2C|||z|||^2} ||\mathcal{H}_2(\mathcal{G}(u))||_{L^2_0} = e^{2C|||z|||^2} ||u + T^*u||_{L^2_0} \end{aligned}$$

and similarly we have $||u + T^*u||_{L^2_0} \leq e^{2C|||z|||^2} ||u - T^*u||_{L^2_0}$.

It is enough to prove (4.6) for $\Omega_1$ (the case $\Omega_2$ will follows by symmetry) and we will rely on the following geometric fact whose elementary proof is left to the reader.

LEMMA 4.2. *Let $\Omega$ be a domain in $\mathbb{R}^2$ whose boundary is a $C^{2,\delta}$ parametrized curve $z(\alpha)$ satisfying the arc-chord condition $||\mathcal{F}(z)||_{L^\infty} < \infty$. Then we have tangent balls to the boundary contained in both $\Omega$ and $\mathbb{R}^2/\Omega$. Furthermore, we can estimate from below the radius of those balls by $C|||z|||^{-1}$, for some universal constant $C > 0$.*

Let $\phi = u + iv$ be the conformal mapping from $\Omega_1$ to the upper half-plane $\mathbb{R}^2_+$. Then $v$ is a nonnegative harmonic function vanishing only on $\partial\Omega_1$. Let $\phi^{-1}$ be the inverse transformation.

LEMMA 4.3. *Since $\Omega_1$ is $2\pi$ periodic in the horizontal direction we have $\phi(z + 2\pi) = \phi(z) + \alpha$ for a certain fixed real number $\alpha$.*

*Proof.* Let us define $\psi(w) = \phi(\phi^{-1}(w) + 2\pi)$. Then $\psi$ is a conformal mapping from $\mathbb{R}^2_+$ to itself and, therefore, given by a linear fractional transformation $\psi(w) = \frac{aw+b}{cw+d}$ satisfying $ad - bc = 1$, where $a, b, c$ and $d$ are real numbers. Since $\psi$ cannot have a fixed point in $\mathbb{R}^2_+$, it follows that $c = 0$ and $a = d$. Therefore taking $z = \phi^{-1}(w)$ we get the formula $\phi(z + 2\pi) = \phi(z) + \alpha$ with $\alpha = \frac{b}{d}$, proving Lemma 4.3.

Next we observe that $\phi'(z+2k\pi) = \phi'(z)$ for every $z \in \Omega_1$ and since $\partial\Omega_1$ is smooth enough we know from general theory that $\phi$ and $\phi'$ extend continuously to $\partial\Omega_1$. Furthermore, in order to estimate the size of $\phi'|_{\partial\Omega_1}$ it will be enough to consider the compact part of that boundary corresponding to a full period.

Composing with $\phi$, $\phi^{-1}$ one easily gets the formula

$$\mathcal{H}_1 f = H(f \circ \phi^{-1}) \circ \phi, \qquad H(f \circ \phi^{-1}) = \mathcal{H}_1(f) \circ \phi^{-1}.$$

Therefore our problem is reduced to a weighted estimate for the Hilbert transform with respect to the weight $|(\phi'(x, 0))^{-1}| = w(x)$ for which we have to show that $w$ belongs to the Muckenhoupt class $A_2$ (see [12]). Now it turns out that for general $C^1$ chord-arc curves that statement is false, but we will take advantage of the fact that $\partial\Omega_1$ is of class $C^2$ (in fact $C^{1,\alpha}$ will suffice) to show that in our case $w(x)$ trivializes, i.e., it is bounded above and below. More precisely:

LEMMA 4.4. *Let $w(x) = |(\phi'(x, 0))^{-1}|$. Then we have*

$$w(x_0)e^{-C|||z|||^2} \le w(x) \le w(x_0), e^{C|||z|||^2},$$

*where $C$ is a universal constant, $|||z|||$ is our usual norm in the curve $\partial\Omega_1$ and $x_0$ is any point. Normalizing our conformal mapping $\phi$ one may take $w(x_0) = 1$.*

*Proof.* From the geometric Lemma 4.3, we know the existence of tangent balls to $\partial\Omega_1$ contained inside $\overline{\Omega}_1$ of radius $r = O(1/|||z|||)$ and such that each one of those balls touches the boundary $\partial\Omega_1$ at a single point and their centers describe a parallel curve $\Gamma$ to $\partial\Omega_1$ which is also of class $C^2$ with norm $O(|||z|||)$. In the following we shall concentrate our attention to the band $B$ of those points in $\Omega_1$ whose distance to $\partial\Omega_1$ is less that $r$. Then the boundary of $B$ consists of two parts, namely $\partial\Omega_1$ and its parallel curve $\Gamma$ at distance $r$ which can also be parametrized throughout $z(\alpha)$ in an obvious manner.

The length of the part of $\Gamma$ corresponding to a full period $0 \leq \alpha \leq 2\pi$, is clearly $O(|||z|||)$. Then, after several applications of Harnack's inequality in steps of length $O(r)$, we obtain

$$e^{-C|||z|||^2} \leq \frac{v(z_1)}{v(z_2)} \leq e^{C|||z|||^2}$$

for any $z_1, z_2 \in \Gamma$. Let us consider a point $P \in \partial\Omega_1$ and $Q \in \Gamma$ to be the center of the circle of radius $r$ tangent to $\partial\Omega_1$ at $P$; furthermore, let us denote by $\nu$ the inner normal vector. Then the nonnegative harmonic function $v$ takes its strict minimum at the point $P$ and by Hopf principle we get the estimate

$$\frac{\partial v}{\partial \nu}(P) \geq \frac{C}{r} v(Q) \tag{4.7}$$

for some absolute constant $C > 0$. On the other hand we may consider a domain $D$ contained in the band $B$ in such a way that its boundary consists of a piece of $\partial\Omega_1$ of length $2r$ containing $P$ at its medium point. Then the corresponding portion of $\Gamma$, says $L_2$, obtained by vertical translation of the points of $L_1$ and finally two arcs of $C^2$ curves smoothly connecting $L_1$ and $L_2$ in such a way that $\partial D$ becomes a $C^2$ curve with norm $O(|||z|||)$.

Let $\psi$ be conformal mapping from the unit ball $B_r$ to $D$ with standard normalization. By the Kellogg-Warschawski theorem it follows that $\psi$ extends continuously to the boundary and its derivative is bounded from above and below by universal constants. We also have the Poisson's kernel $K$ in $D$ obtained by conformal mapping of the kernel for the ball of radius $r$. Then we may represent the harmonic function $v$ as the integral of its boundary values against the Poisson kernel

$$v(x) = \int_{\partial D} K(x, y) v(y) d\sigma(y)$$

and

$$\frac{\partial v}{\partial \nu}(x) = \int_{\partial D} \frac{\partial K}{\partial \nu_x}(x, y) v(y) d\sigma(y),$$

which is a legitimate integral. We can take the limit (when $x \to P \in \partial\Omega_1$) because $v$ vanishes identically in $L_1$ and the points $y \in \partial D - L_1$ are at distance at least $r$ from P to obtain the estimate

$$\frac{\partial v}{\partial \nu}(P) \leq \frac{C}{r} \sup_{x \in D} v(x).$$

To finish we can invoke Dahlberg-Harnack principle up to the boundary for the positive harmonic function $v$ (see [4] and [11]), which gives us the inequality

$$\frac{\partial v}{\partial \nu}(P) \leq \frac{C}{r} v(Q) \tag{4.8}$$

for some fixed constant C. Then the estimates (4.7) and (4.8) yield

$$C^{-1}\frac{v(Q_1)}{v(Q_2)} \leq \frac{\frac{\partial v}{\partial \nu}(P_1)}{\frac{\partial v}{\partial \nu}(P_2)} \leq C\frac{v(Q_1)}{v(Q_2)}, \tag{4.9}$$

but we know from Harnack that

$$e^{-C|||z|||^2} \leq \frac{v(Q_1)}{v(Q_2)} \leq e^{C|||z|||^2}$$

for two arbitrary points $Q_1, Q_2$ in $\Gamma$. That ends the proofs of Lemma 4.4 and Proposition 4.1 because $|\phi'(z(\alpha))| = |\nabla v(z(\alpha))| = \frac{\partial v}{\partial \nu}(z(\alpha))$, since $\partial\Omega_1$ is the level set $v = 0$ of the positive harmonic function $v$ ($\phi = u + iv$). □

The identity $I + \xi T^* = \xi(I + T^*) + (1 - \xi)I$ allows us to conclude that

$$||(u + \xi T^* u)^{-1}||_{L^2_0} \leq e^{C|||z|||^2}$$

for $1 - e^{-C_1|||z|||^2} \leq |\xi| \leq 1$ with an appropriate constant $C_1$, but for general $\xi$ ($|\xi| \leq 1$) we have

Proposition 4.5. *For $|\xi| \leq 1$, the estimate*

$$\|(I + \xi T)^{-1}\|_{H_0^{\frac{1}{2}}} = \|(I + \xi T^*)^{-1}\|_{H_0^{\frac{1}{2}}} \leq e^{C|||z|||^2}$$

*holds for a universal constant $C$.*

*Proof.* First let us consider the inequality

$$\int_{\Omega_j} |\nabla f_j|^2 \geq e^{-C_2|||z|||^2}\|u\|^2_{H^{\frac{1}{2}}}, \tag{4.10}$$

where $F_j = f_j + ig_j$ is the Cauchy integral of $u$ in $\Omega_j$ which follows easily from estimate (4.7) for the derivative of the conformal mapping $\phi$:

$$\int_{\Omega_j} |\nabla f_j|^2 = \int_{\Omega_j} \Delta f_j^2 = \int_{\mathbb{R}^2_+} \Delta f_j^2(\phi^{-1})|(\phi^{-1})'|^2$$
$$= \int_{\mathbb{R}^2_+} \Delta(f_j \circ \phi^{-1})^2 = \int_{\partial\mathbb{R}^2_+} f_j \circ \phi^{-1}\frac{\partial}{\partial \nu} f_j \circ \phi^{-1},$$

where $\frac{\partial}{\partial \nu}$ is the derivative in the normal direction

$$\frac{\partial f_j}{\partial y}(x, 0) = \lim_{y\to 0}\frac{1}{\pi}\int \frac{u(x - t) - u(x)}{t^2 + y^2}dt = \Lambda u(x).$$

Therefore we can conclude that

$$\int_{\Omega_j} |\nabla f_j|^2 = \int_{-\infty}^{+\infty} f_j \circ \phi^{-1}\Lambda(f_j \circ \phi^{-1})$$
$$= \int_{-\infty}^{+\infty} |\Lambda^{\frac{1}{2}}(f_j \circ \phi^{-1})|^2 \geq e^{-C_2|||z|||^2}||u||^2_{H^{\frac{1}{2}}}$$

for a certain positive constant $C_2$ as a consequence of the following lemma:

Lemma 4.6. *Let $\psi$ be a diffeomorphism of the real line such that $0 < C^{-1} \leq |\psi'(x)| \leq C$. Then we have the equivalence of Sobolev norms*

$$C^{-(3+2s)}||f||_{H^s} \leq ||f \circ \psi||_{H^s} \leq C^{3+2s}||f||_{H^s}$$

*for* $0 \leq s \leq \frac{1}{2}$.

*Proof.* Given $f$ in $H^s$ we have

$$\begin{aligned}
||\Lambda^s(f\circ\psi)||^2_{L^2} &= \int (f\circ\psi)(x)\Lambda^{2s}(f\circ\psi)(x)dx \\
&= \int f(\psi(x)) \int \frac{f(\psi(x)) - f(\psi(y))}{|x-y|^{1+2s}} dydx \\
&= \frac{1}{2}\int\int \frac{(f(\psi(x)) - f(\psi(y)))^2}{|x-y|^{1+2s}} dydx \\
&= \frac{1}{2}\int\int \frac{(f(\bar{x}) - f(\bar{y}))^2}{|(\psi^{-1})'(\bar{\bar{x}})|^{1+2s}|\bar{x}-\bar{y}|^{1+2s}} (\psi^{-1})'(\bar{x})(\psi^{-1})'(\bar{y})d\bar{y}d\bar{x},
\end{aligned}$$

where $\bar{\bar{x}}$ comes from the application of the mean value theorem. From our hypothesis we have

$$C^{-(3+2s)} \leq \frac{(\psi^{-1})'(\bar{x})(\psi^{-1})'(\bar{y})}{|(\psi^{-1})'(\bar{\bar{x}})|^{1+2s}} \leq C^{3+2s}$$

which, together with the equality

$$||\Lambda^s f||^2_{L^2} = \frac{1}{2}\int\int \frac{(f(x)-f(y))^2}{|x-y|^{1+2s}} dydx,$$

allows us to finish the proof of the lemma.

*Remark* 4.7. In our case the diffeomorphism is given by $\Psi(\alpha) = \Phi(z(\alpha))$, and we will use the periodic version of (4.10); i.e.,

$$\int_{\widetilde{\Omega}_j} |\nabla f_j|^2 \geq e^{-C_2|||z|||^2}||u||^2_{H^{\frac{1}{2}}(\mathbb{T})}.$$

To continue, let us assume that Proposition 4.5 is false; then there exist $u \in H_0^{\frac{1}{2}}$, $||u||_{H^{\frac{1}{2}}} = 1$ and $|\eta| > 1$ such that $||\eta u - T^*u||_{H^{\frac{1}{2}}} \leq e^{-C_3|||z|||^2}$, where $C_3$ will be fixed later to be big enough for our purposes.

Let us also assume that the following estimate holds:

$$\begin{aligned}
&\Big|\int_{\mathbb{T}} (\eta u - T^*u)2BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha\Big| \\
&\qquad \leq \|\eta u - T^*u\|_{H^{\frac{1}{2}}}\Big\|2BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}\Big\|_{H^{-\frac{1}{2}}} \leq e^{-50C_2|||z|||^2}.
\end{aligned}$$

Then from identities (4.3) and (4.4) we get

(4.11)
$$
\begin{aligned}
e^{-C_2|||z|||^2} &\leq \int_{\mathbb{T}} (1-\eta)u(\alpha)2BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha + e^{-50C_2|||z|||^2},\\
e^{-C_2|||z|||^2} &\leq \int_{\mathbb{T}} (1+\eta)u(\alpha)2BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha + e^{-50C_2|||z|||^2}.
\end{aligned}
$$

Adding these two inequalities together we obtain the positivity of

$$\int_{\mathbb{T}} u(\alpha)2BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}d\alpha > 0.$$

Then we get a contradiction when we substitute the value of $\eta$ in the first inequality of (4.11) if $\eta \geq 1$, or in the second one if $\eta \leq -1$. Therefore the hypothesis $\|\eta u - T^*u\|_{H^{\frac{1}{2}}} \leq e^{-C_3|||z|||^2}$ is false for every $u$ in $H_0^{\frac{1}{2}}$ and $\|u\|_{H^{\frac{1}{2}}} = 1$, and that gives us the desired estimate.

To finish the proof, we need to show that

$$\|2BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}\|_{H^{-\frac{1}{2}}} \leq e^{C|||z|||^2}\|u\|_{H^{\frac{1}{2}}}$$

for a universal constant $C$.

In order to prove it let us first observe that

$$\partial_\alpha(BR(z,u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}) = BR(z,\partial_\alpha u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} + O_p(z)u,$$

where $O_p(z)$ is a bounded operator in $L^2$ whose norm is controlled by $e^{C|||z|||^2}$ for a convenient value of $C$. Therefore our task is equivalent to show the estimate

$$\|2BR(z,u)(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|}\|_{H^{\frac{1}{2}}} \leq e^{C|||z|||^2}\|u\|_{H^{\frac{1}{2}}}.$$

Decoding the notation we have to consider the operators $\partial_\alpha^\perp z_k(\alpha)\cdot T_j u(\alpha)$, where

$$T_j u(\alpha) = PV\int \frac{z_j(\alpha)-z_j(\alpha-\beta)}{|z(\alpha)-z(\alpha-\beta)|^2}u(\alpha-\beta)d\beta.$$

Let $\phi$ be a $C^\infty$ cut-off function supported on $|x| \leq r$ such that $\phi \equiv 1$ on $|x| \leq \frac{r}{2}$ where $r = \frac{|||z|||}{2}$. Then $T_j u(\alpha) = T_j^1 u + T_j^2 u$ for

$$
\begin{aligned}
T_j^1 u &= PV\int \phi(\beta)\frac{z_j(\alpha)-z_j(\alpha-\beta)}{|z(\alpha)-z(\alpha-\beta)|^2}u(\alpha-\beta)d\beta,\\
T_j^2 u &= PV\int (1-\phi(\beta))\frac{z_j(\alpha)-z_j(\alpha-\beta)}{|z(\alpha)-z(\alpha-\beta)|^2}u(\alpha-\beta)d\beta.
\end{aligned}
$$

It is straightforward to check that $T_j^2$ is a smoothing operator for which the desired estimate trivializes. Furthermore, a convenient Taylor expansion allows us to write $T_j^1 u(\alpha) = m(\alpha)Hu(\alpha) + R(u)$ where $R$ is a smoothing operator, $H$ is the Hilbert transform, and the bounded smooth function $m$ depends upon the curve $z$ in such a way that $\|\frac{\partial m}{\partial \alpha}\|_{L^\infty} \leq e^{C|||z|||^2}$. Finally we may invoke the following commutator estimate

$$\|\Lambda^{\frac{1}{2}}(bv) - b\Lambda^{\frac{1}{2}}v\|_{L^2(\mathbb{T})} \leq C\|\nabla b\|_{L^\infty}\|v\|_{L^2(\mathbb{T})}$$

to complete our task. □

*Remark* 4.8. Although it will not be needed to establish our main theorem, we will improve the estimate on the eigenvalues of $T^*, T$ by showing the existence of a constant $C_0 = C_0(z)$ whose inverse $C_0^{-1}$ grows at most as a polynomial in $|||z|||$ and such that the eigenvalues of $T^*$ must satisfy the estimate $|\lambda| \leq 1 - C_0$. To see this let us consider the identities

$$\int_{\Omega_1} |\nabla f_1|^2 dx + \int_{\Omega_2} |\nabla f_2|^2 dx = 2\int_{\mathbb{T}} u(\alpha) 2BR(z, \partial_\alpha u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha,$$

$$\left|\int_{\Omega_1} |\nabla f_1|^2 dx - \int_{\Omega_2} |\nabla f_2|^2 dx\right| = 2|\lambda| \int_{\mathbb{T}} u(\alpha) 2BR(z, \partial_\alpha u)(\alpha) \cdot \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|} d\alpha.$$

Then it will be enough to show that both integrals $\int_{\Omega_j} |\nabla f|^2 dx$ are comparable; i.e., there exists a constant $1 \leq C = C(|||z|||) < \infty$ such that

$$\frac{1}{C}\int_{\Omega_1} |\nabla f_1|^2 \leq \int_{\Omega_2} |\nabla f_2|^2 \leq C\int_{\Omega_1} |\nabla f_1|^2.$$

Observe that the Cauchy-Riemann equations imply that this is equivalent to showing the analogous estimate for $g$ in place of $f$.

The existence of such $C$ depending continuously upon $|||z|||$ follows easily by a standard compactness argument. Nevertheless it is convenient to have a control of the growth of the constants. In the following we present an argument to show that $C(|||z|||)$ grows polynomially with $|||z|||$.

Proposition 4.9. *We shall consider periodic curves $z(\alpha)$ (period $2\pi$). Because of the smoothness and arc-chord conditions, such a curve divides the cylinder $\mathbb{R}/2\pi\mathbb{Z} \times (-\infty, \infty)$ in two regions $\Omega_j$ ($j = 1, 2$, above and below the curve respectively) containing tangent balls as in the previous lemma. Then there exist a constant $C = P(||z||_{C^{k,\delta}}, ||\mathcal{F}(z)||_{L^\infty})$, polynomial in $|||z|||$, such that*

$$\frac{1}{C}\int_{\Omega_1} |F_1'|^2 \leq \int_{\Omega_2} |F_2'|^2 \leq C\int_{\Omega_1} |F_1'|^2$$

*for any pair of periodic (in $x_1$) holomorphic functions $F_j = f_j + ig_j$ ($j = 1, 2$), with $f_j, g_j$ in Sobolev space $H^1(\Omega_j)$ and such that the imaginary parts $g_j$, $j = 1, 2$ (or respectively the real part $f_j$, $j = 1, 2$) take the same boundary values.*

*Proof.* In the following we shall use the expression $P(\gamma)$ for different constants to denote that they grow at most polynomially with $\gamma$.

For $\frac{1}{r} = P(|||z|||)$ there exist two tangent circles to the curve $z$ of radius $r$ and contained respectively in $\Omega_1$ and $\Omega_2$. Therefore we can foliate the plane near $z$ by parallel curves $z^j_\epsilon$ ($z^j_0 = Z$); these curves are the locus of points in $\Omega_j$ whose distance to $z$ is $\epsilon$, in such a way that $|||z^j_\epsilon||| \leq C|||z|||$ uniformly on $0 \leq \epsilon \leq \frac{1}{10}r$ for some universal finite constant $C$.

The Cauchy-Riemann equations for the holomorphic functions $F_j = f_j + ig_j$ yield

$$\int_{\Omega_j} |F_j'|^2 = \int_{\Omega_j} |\nabla f_j|^2 = \int_{\Omega_j} |\nabla g_j|^2.$$

Let us assume (without loss of generality) that

$$\int_{\Omega_1} |\nabla g_1|^2 \geq \int_{\Omega_2} |\nabla g_2|^2;$$

then we want to show the estimate

$$\int_{\Omega_1} |\nabla g_1|^2 \leq P(|||z|||) \int_{\Omega_2} |\nabla g_2|^2$$

and that will finish the proof.

Let $\phi$ be a $C^\infty$ cut-off function such that $\phi(t) \equiv 1$ when $|t| \leq \frac{1}{20}r$ and $\phi \equiv 0$ when $|t| \geq \frac{1}{10}r$; then we reflect the values of $g_1$ near $z(\alpha)$ by the formula

$$\tilde{g}_1(P) = g_2(Q)\phi(\mathrm{dist}(P, z)),$$

where $Q \in \Omega_2$ is obtained reflecting $P \in \Omega_1$ with respect to $z$, that is $\mathrm{dist}(P, z) = \mathrm{dist}(Q, z)$, and the line segment connecting them is normal to $z$ at its medium point.

By the Dirichlet principle

$$\int_{\Omega_1} |\nabla g_1|^2 \leq \int_{\Omega_1} |\nabla \tilde{g}_1|^2 \leq P(|||z|||) \left( \int_{\Omega_2} |\nabla g_2|^2 |\phi|^2 + \int_{\Omega_2} |g_2|^2 |\nabla \phi|^2 \right).$$

Since $F_2$ is holomorphic, we have the equalities

$$\int_0^{2\pi} F_2(x, y_1)dx = \int_0^{2\pi} F_2(x, y_2)dx$$

for $|y_j|$ big enough so that the horizontal lines $(x, y_j)$ do not meet the curve $z$. The hypothesis that $f_j \in L^2(\Omega_j)$ implies that

$$\int_0^{2\pi} g_2(x, y)dx = 0$$

for those $y$ which can be taken at distance $P(|||z|||)$ of the curve $z$. For such a $y$ we get the estimate

$$|g_2(x, y)| \leq \int_0^{2\pi} |\nabla g_2(t, y)|dt,$$

which implies

$$\int_0^{2\pi}\int_{-y-1}^{-y}|g_2(x,s)|^2dsdx \le (2\pi)^2\int_0^{2\pi}\int_{-y-1}^{-y}|\nabla g_2(t,s)|^2dsdt;$$

therefore

$$m\{|g_2(x,s)| \ge 10\cdot 2\pi||\nabla g_2||_{L^2(\Omega_2)}|0\le x\le 2\pi, -y-1\le s\le -y\} \le \frac{1}{100},$$

where $m$ denotes the Lebesgue measure.

Let $(x_m, y_m)$ be in the curve $z$ such that $y_m$ has a minimum value. Then for all points $Q$ in $\Omega_2$ inside the band $1/(20r) \le \text{dist}(Q,z) \le 1/(20r)$ whose distance to $(x_m,y_m)$ is less than $1/P(|||z|||)$ (we shall denote by $N$ the set of such $Q$) the segments connecting its points to those of $\{(x,t), -y\le t\le -y-1\}$ are completely contained in $\Omega_2$. For each $(x_0,y_0)\in N$ let us consider the line segment connecting $(x_0,y_0)$ with the set

$$E = \{(x,s)||g_2(x,s)| < 10\cdot 2\pi||\nabla g_2||_{L^2(\Omega_2)}|0\le x\le 2\pi, -y-1\le s\le -y\};$$

then given $(x,s)\in E$ we have the estimate

$$|g_2(x_0,y_0)| \le 10\cdot 2\pi||\nabla g_2||_{L^2(\Omega_2)} + \int_0^L|\nabla g_2((x_0,y_0)+t\omega)|dt,$$

where $\omega = \frac{(x-x_0,s-y_0)}{((x-x_0)^2+(s-y_0)^2)^{\frac{1}{2}}}$ and $0\le L\le P(|||z|||)$.

Since the measure of E is big enough ($\ge \pi$) the measure of the region described in the unit circle by those $\omega$'s is also big enough ($\ge 1/P(|||z|||)$). Therefore,

$$|g_2(x_0,y_0)| \le P(|||z|||)\left(||\nabla g_2||_{L^2(\Omega_2)} + \int\int\frac{|\nabla g_2((x_0,y_0)-(x,y))|}{||(x,y)||}dxdy\right).$$

This inequality implies that

$$\int_N|g_2(x_0,y_0)|^2dx_0dy_0 \le P(|||z|||)\int_{\Omega_2}|\nabla g_2(x,y)|^2dxdy.$$

To conclude the argument we observe that the integral $\int_{\Omega_2}|g_2|^2|\nabla\phi|^2$ is bounded by $P(|||z|||)(\int_N|g_2(x_0,y_0)|^2dx_0dy_0 + \int_{\Omega_2}|\nabla g_2(x,y)|^2dxdy)$ because the parallel curves have tangent vectors whose lengths are uniformly bounded by $P(|||z|||)$. □

## 5. Estimates on $\varpi$

In this section we show that the amplitude of the vorticity $\varpi$ is at the same level than $\partial_\alpha z$. We prove the following result:

LEMMA 5.1. *Let $\varpi$ be a function given by*

$$\varpi(\alpha) = -\frac{\mu^2-\mu^1}{\mu^2+\mu^1}2BR(z,\varpi)(\alpha)\cdot\partial_\alpha z(\alpha) - 2\kappa\text{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_2(\alpha). \quad (5.1)$$

*Then*

$$\|\varpi\|_{H^k} \le \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}\right) \tag{5.2}$$

*for* $k \ge 2$.

*Proof.* We have $|A_\mu| \le 1$. Then the formula (5.1) is equivalent to

$$\varpi(\alpha) + A_\mu T(\varpi)(\alpha) = -2\kappa \mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_2(\alpha) \tag{5.3}$$

or

$$\varpi(\alpha) = -2\kappa \mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}(I + A_\mu T)^{-1}(\partial_\alpha z_2)(\alpha).$$

It yields

$$\|\varpi\|_{H^{\frac12}} \le C\|(I+A_\mu T)^{-1}\|_{H^{\frac12}}\|\partial_\alpha z_2\|_{H^{\frac12}},$$

and Proposition 4.1 gives

$$\|\varpi\|_{H^{\frac12}} \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}). \tag{5.4}$$

Taking the $k$-th derivative of (5.3) we get

$$\partial^k_\alpha\varpi(\alpha) + A_\mu T(\partial^k_\alpha \varpi)(\alpha) = \Omega_k(\varpi) + C\partial^{k+1}_\alpha z_2(\alpha), \quad C = -2\kappa \mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}.$$

Using Leibniz's rule we can write

$$\Omega_k(\varpi)(\alpha)$$
$$= \sum_{j=1}^k C_j \int \Phi(\beta)\partial^j_\alpha\left(\frac{(z(\alpha)-z(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\right)\partial^{k-j}_\alpha\varpi(\alpha-\beta)d\beta + S(\varpi)(\alpha),$$

where $S$ is a smoothing operator, $C_j$ are suitable constants, and $\Phi$ is a $C^\infty$ cut-off such that $\Phi \equiv 0$ outside the ball $B(0,r)$ of radius $r = \frac{1}{2|||z|||}$ and $\Phi \equiv 1$ in $B(0, \frac{r}{2})$.

Next let us consider

$$\Lambda^{\frac12}\partial^k_\alpha\varpi(\alpha) + A_\mu T(\Lambda^{\frac12}\partial^k_\alpha\varpi)(\alpha) = A_\mu T(\Lambda^{\frac12}\partial^k_\alpha\varpi)(\alpha) - A_\mu\Lambda^{\frac12}T(\partial^k_\alpha\varpi)(\alpha)$$
$$+ \Lambda^{\frac12}\Omega_k(\varpi) + C\Lambda^{\frac12}\partial^{k+1}_\alpha z_2(\alpha).$$

Using the estimate for the inverse $(I + A_\mu T)^{-1}$ in the space $H^{\frac12}$ we get

$$\|\varpi\|_{H^{k+1}} = \|\Lambda^{\frac12}\partial^k_\alpha\varpi\|_{H^{\frac12}} \le \exp\left(C|||z|||^2\right)$$
$$\cdot\left(A_\mu\|T\Lambda^{\frac12}\partial^k_\alpha\varpi\|_{H^{\frac12}} + A_\mu\|T\partial^k_\alpha\varpi\|_{H^1} + \|\Omega_k\|_{H^1} + \|z\|_{H^{k+2}}\right).$$

Then we have

$$\|T\partial^k_\alpha\varpi\|_{H^1} \le C|||z|||^4\|\varpi\|_{H^k}$$

by Lemma 3.1 and

$$||\Omega_k||_{H^1} \le C\|\mathcal{F}(z)\|^2_{L^\infty}||\varpi||_{H^k}||z||^2_{H^{k+2}}$$

by Lemma 5.2 (see below). Finally,

$$\begin{aligned}\|T\Lambda^{\frac{1}{2}}\partial_\alpha^k\varpi\|_{H^{\frac{1}{2}}} &\leq \|T\Lambda^{\frac{1}{2}}\partial_\alpha^k\varpi\|_{H^1} \leq C|||z|||^4\|\varpi\|_{H^{k+\frac{1}{2}}} \\ &\leq e^{C|||z|||^2}(\|\Omega_k\|_{H^{\frac{1}{2}}} + \|z\|_{H^{k+\frac{3}{2}}}) \\ &\leq e^{C|||z|||^2}(||\mathcal{F}(z)||_{L^\infty}||\varpi||_{H^k}||z||_{H^{k+2}} + ||z||_{H^{k+\frac{3}{2}}}),\end{aligned}$$

where we have used $\partial_\alpha^k\varpi(\alpha) = (I - A_\mu T)^{-1}(\Omega_k + C\partial_\alpha^{k+1}z_2)$ and the estimate of the norm in $H^{\frac{1}{2}}$ of the inverse operator $(I - A_\mu T)^{-1}$.

A straightforward induction on $k \geq 2$ allows us to finish the proof. The estimates for $k = 1, \frac{3}{2}$ are obtained similarly, but in all of them the norm $||z||_{H^3}$ has to appear; i.e., we have

$$\|\varpi\|_{H^{\frac{3}{2}}} \leq \exp C\big(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}\big).$$

LEMMA 5.2. *The operator $T$ maps Sobolev space $H^k$, $k \geq 1$, into $H^{k+1}$ (so long as $\|z\|_{H^{k+2}} < \infty$) and satisfies the estimate*

$$\|T\|_{H^k \to H^{k+1}} \leq C|||z|||^2\|z\|^2_{H^{k+2}}.$$

*Proof.* We have that

$$\begin{aligned}T(u)(\alpha) &= 2BR(z,u)(\alpha)\cdot\partial_\alpha z(\alpha) \\ &= \frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{(z(\alpha)-z(\alpha-\beta))^\perp\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}u(\alpha-\beta)d\beta,\end{aligned}$$

where, as usual and to simplify notation, we have dropped the time dependence of all functions.

Let $\psi$ be a $C^\infty$ cut-off such that $\psi \equiv 0$ outside the ball $B(0,r)$ of radius $r = \frac{1}{2|||z|||}$ and $\psi \equiv 1$ in $B(0, \frac{r}{2})$. Then

$$\begin{aligned}T(u)(\alpha) &= \frac{1}{\pi}PV\int_{-\infty}^{\infty}\psi(\beta)\frac{(z(\alpha)-z(\alpha-\beta))^\perp\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}u(\alpha-\beta)d\beta \\ &\quad + \frac{1}{\pi}PV\int_{-\infty}^{\infty}(1-\psi(\beta))\frac{(z(\alpha)-z(\alpha-\beta))^\perp\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}u(\alpha-\beta)d\beta \\ &= T_1u(\alpha) + T_2u(\alpha).\end{aligned}$$

i) Estimate of $T_2u(\alpha)$: Leibniz's rule gives us

$$\begin{aligned}\partial_\alpha^{k+1}T_2u(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty}(1-\psi(\beta))\frac{(z(\alpha)-z(\alpha-\beta))^\perp\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\partial_\alpha^{k+1}u(\alpha-\beta)d\beta \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty}(1-\psi(\beta))\frac{(z(\alpha)-z(\alpha-\beta))^\perp\partial_\alpha^{k+2}z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}u(\alpha-\beta)d\beta \\ &\quad + \text{“other terms”} \\ &= I_1 + I_2 + \text{“other terms”}.\end{aligned}$$

The estimate for "other terms" is straightforward. For $I_1$ we integrate by parts:

$$\begin{aligned} I_1 = &\frac{1}{\pi}\int_{-\infty}^{\infty}\psi'(\beta)\frac{(z(\alpha)-z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\partial_\alpha^k u(\alpha-\beta)d\beta \\ &-\frac{1}{\pi}\int_{-\infty}^{\infty}(1-\psi(\beta))\partial_\beta(\frac{(z(\alpha)-z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2})\partial_\alpha^k u(\alpha-\beta)d\beta. \end{aligned}$$

Then clearly we have that

$$||I_1||_{L^2}\le C|||z|||^2||z||^2_{H^{k+2}}||u||_{H^k}.$$

Regarding $I_2$ we have that

$$I_2(\alpha)=\sum_{j=1}^{2}\partial_\alpha^{k+2}z_j(\alpha)\cdot L_j u(\alpha)$$

and clearly $||L_j u||_{L^\infty}\le C|||z|||^2||u||_{H^k}$. Therefore,

$$||I_2||_{L^2}\le C|||z|||^2||z||_{H^{k+2}}||u||_{H^k}.$$

ii) Estimate of $T_1 u(\alpha)$: We have that

$$\begin{aligned} \partial_\alpha^{k+1}T_1 u(\alpha) = &\frac{1}{\pi}\int_{-\infty}^{\infty}\psi(\beta)\frac{(z(\alpha)-z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\partial_\alpha^{k+1}u(\alpha-\beta)d\beta \\ &+\frac{1}{\pi}\int_{-\infty}^{\infty}\psi(\beta)\frac{(z(\alpha)-z(\alpha-\beta))^{\perp}\partial_\alpha^{k+2}z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}u(\alpha-\beta)d\beta \\ &+\frac{1}{\pi}\int_{-\infty}^{\infty}\psi(\beta)\frac{(\partial_\alpha^{k+1}z(\alpha)-\partial_\alpha^{k+1}z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}u(\alpha-\beta)d\beta \\ &+\text{“other terms”} \\ = &J_1+J_2+J_3+\text{“other terms”}. \end{aligned}$$

As in the previous case the "other terms" are easy to handle and we shall show how to estimate the remaining three cases.

We can write $J_2$ in the form

$$J_2(\alpha)=\sum\partial_\alpha^{k+2}z_j(\alpha)\int\psi(\beta)K_j(\alpha,\alpha-\beta)u(\alpha-\beta)d\beta=\sum_{j=1}^{2}\partial_\alpha^{k+2}z_j(\alpha)\cdot L_j u(\alpha)$$

and observe that

$$||L_j u||_{L^\infty}\le||L_j u||_{H^1}\le C|||z|||^2||z||_{H^2}||u||_{H^1},$$

which yields

$$||J_2||_{L^2}\le C|||z|||^2||z||^2_{H^{k+2}}||u||_{H^k}.$$

To estimate $J_1$ we integrate by parts:

$$\begin{aligned}J_1(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\psi'(\beta)\frac{(z(\alpha)-z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\partial_\alpha^k u(\alpha-\beta)d\beta\\ &\quad+\frac{1}{\pi}\int_{-\infty}^{\infty}\psi(\beta)\partial_\beta\Big(\frac{(z(\alpha)-z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\Big)\partial_\alpha^k u(\alpha-\beta)d\beta\\ &= J_1^1+J_1^2.\end{aligned}$$

For the first part $J_1^1$ we have that

$$||J_1^1||_{L^2}\le C|||z|||^2||z||_{H^3}^2||u||_{H^k}.$$

We also have that

$$\begin{aligned}J_1^2(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\psi(\beta)\frac{(\partial_\alpha z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}\partial_\alpha^k u(\alpha-\beta)d\beta\\ &+\frac{2}{\pi}\int_{-\infty}^{\infty}\psi(\beta)\frac{[(z(\alpha)-z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha)][\partial_\alpha z(\alpha-\beta)(z(\alpha)-z(\alpha-\beta))]}{|z(\alpha)-z(\alpha-\beta)|^4}\partial_\alpha^k u(\alpha-\beta)d\beta\\ &= J_1^{2,1}+J_1^{2,2}.\end{aligned}$$

For $J_1^{2,1}$,

$$\begin{aligned}(\partial_\alpha z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha) &= (\partial_\alpha z(\alpha-\beta)-\partial_\alpha z(\alpha))^{\perp}\partial_\alpha z(\alpha)\\ &= -\partial_\alpha^2 z^{\perp}(\alpha)\partial_\alpha z(\alpha)\beta+O(\beta^2)\end{aligned}$$

and

$$|z(\alpha)-z(\alpha-\beta)|^2=|\partial_\alpha z|^2\beta^2+O(\beta^3),$$

where the constants in the "$O$" terms (and in theirs first derivatives) are properly bounded in terms of $||z||_{H^3}$. That is,

$$J_1^{2,1}(\alpha)=-\frac{\partial_\alpha^2 z^{\perp}(\alpha)\partial_\alpha z(\alpha)}{|z_\alpha(\alpha)|^2}H\partial_\alpha^k u(\alpha)+\text{“bounded terms”},$$

where $H$ denotes the Hilbert transform. Therefore for the first integral we get

$$||J_1^{2,1}||_{L^2}\le C|||z|||^2||z||_{H^3}^2||u||_{H^k}.$$

Finally for $J_1^{2,2}$ we have

$$\begin{aligned}&[(z(\alpha)-z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha)][\partial_\alpha z(\alpha-\beta)(z(\alpha)-z(\alpha-\beta))]\\ &\qquad=\frac{1}{2}\partial_\alpha^2 z^{\perp}(\alpha)\partial_\alpha z(\alpha)|\partial_\alpha z(\alpha)|^2\beta^3+O(\beta^4)\end{aligned}$$

and

$$|z(\alpha)-z(\alpha-\beta)|^4=|\partial_\alpha z|^4\beta^4+O(\beta^4).$$

By a similar approach we obtain

$$J_1^{2,1}(\alpha)=\frac{\partial_\alpha^2 z^{\perp}(\alpha)\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2}H\partial_\alpha^k u(\alpha)+\text{“bounded terms”},$$

and it yields

$$||J_1^{2,2}||_{L^2}\le C|||z|||^2||z||_{H^3}^2||u||_{H^k}.$$

To estimate $J_3$ we observe first that the substitution of $u(\alpha-\beta)$ by $u(\alpha)-\partial_\alpha u(\alpha)\beta$ produces an error bounded by $||z||^2_{H^{k+2}}|||z|||^2||u||_{H^k}$.

Using the expansions

$$\frac{\psi(\beta)}{|z(\alpha)-z(\alpha-\beta)|^2} = \frac{\psi(\beta)}{|\partial_\alpha z(\alpha)|^2}\frac{1}{|\beta|^2} + O(1)\psi(\beta)$$
$$\partial_\alpha^{k+1} z(\alpha) - \partial_\alpha^{k+1} z(\alpha-\beta) = \beta\int_0^1 \partial_\alpha^{k+2} z(\alpha - t\beta)dt$$

and since the term corresponding to $\partial_\alpha u(\alpha)\beta$ can be handled very easily, it remains to estimate

$$\frac{u(\alpha)}{\pi}\int_{-\infty}^{\infty}\psi(\beta)\frac{(\partial_\alpha^{k+1} z(\alpha)-\partial_\alpha^{k+1} z(\alpha-\beta))^{\perp}\partial_\alpha z(\alpha)}{|z(\alpha)-z(\alpha-\beta)|^2}d\beta = \int_0^1 K_t(\alpha)dt,$$

where

$$K_t(\alpha) = \frac{u(\alpha)}{\pi|\partial_\alpha z(\alpha)|^2}\int_{-\infty}^{\infty}\psi(\frac{\beta}{t})\frac{\partial_\alpha^{k+2} z^{\perp}(\alpha-\beta)\cdot\partial_\alpha z(\alpha)}{\beta}d\beta.$$

Finally, the $L^2$-boundedness of the Hilbert transform yields

$$||K_t||_{L^2} \leq ||z||^2_{H^{k+2}}||u||_{H^k}|||z|||$$

uniformly on $t$, allowing us to finish the proof. □

## 6. **Estimates on** $BR(z,\varpi)$

This section is devoted to show that the Birkhoff-Rott integral is as regular as $\partial_\alpha z$.

LEMMA 6.1. *The estimate*

$$\|BR(z,\varpi)\|_{H^k} \leq \exp(C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}) \tag{6.1}$$

*holds for* $k \geq 2$.

*Remark* 6.2. Using this estimate for $k=2$ we easily find that

$$\|\partial_\alpha BR(z,\varpi)\|_{L^\infty} \leq \exp(C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}), \tag{6.2}$$

which shall be used throughout the paper.

*Proof.* We show the proof for $k=2$, with the rest of the cases being analogous. We have

$$BR(z,\varpi)(\alpha) = \frac{1}{4\pi}PV\!\!\int_{\mathbb{T}}\varpi(\alpha-\beta)\Big(-\frac{V_2(\alpha,\beta)(1+V_1^2(\alpha,\beta))}{|V(\alpha,\beta)|^2}, \frac{V_1(\alpha,\beta)(1-V_2^2(\alpha,\beta))}{|V(\alpha,\beta)|^2}\Big)\,d\beta$$

which is decomposed as follows:

$$
\begin{aligned}
(6.3) \qquad BR(z,\varpi)(\alpha) &= \frac{1}{4\pi} PV \int_{\mathbb{T}} \varpi(\alpha-\beta) \frac{V^{\perp}(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} d\beta \\
&\quad - \frac{1}{4\pi} \int_{\mathbb{T}} \varpi(\beta) V_2^2(\alpha,\beta) \frac{V^{\perp}(\alpha,\beta)}{|V(\alpha,\beta)|^2} d\beta \\
&\quad - \frac{1}{4\pi} \int_{\mathbb{T}} \varpi(\beta) V_2(\alpha,\beta) d\beta (1,0) \\
&= P_1(\alpha) + P_2(\alpha) + P_3(\alpha).
\end{aligned}
$$

Using that $|V_2(\alpha,\beta)| \leq 1$, we get $|P_2(\alpha)| + |P_3(\alpha)| \leq C\|\varpi\|_{L^2}$. Thus Lemma 5.1 yields $\|P_2\|_{L^2} + \|P_3\|_{L^2} \leq \exp(C(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2))$.

Let us write

$$
\begin{aligned}
P_1(\alpha) = \frac{1}{4\pi} \int_{\mathbb{T}} \Big(-A_1(\alpha,\alpha-\beta), A_2(\alpha,\alpha-\beta)\Big) \varpi(\alpha-\beta) d\beta \\
+ \frac{\partial_\alpha^{\perp} z(\alpha)}{|\partial_\alpha z(\alpha)|^2} H\varpi(\alpha) d\alpha = J_1 + J_2,
\end{aligned}
$$

where, as before,

$$
A_1(\alpha,\alpha-\beta) = \frac{V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2} \frac{\partial_\alpha z_2(\alpha)}{\tan(\frac{\beta}{2})}
$$

and

$$
A_2(\alpha,\alpha-\beta) = \frac{V_1(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2} \frac{\partial_\alpha z_1(\alpha)}{\tan(\frac{\beta}{2})}.
$$

For $J_1$, since $\|A_1\|_{L^\infty} \leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$ and $\|A_2\|_{L^\infty} \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}^2$ (see the Appendix), one gets $\|J_1\|_{L^2} \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|z\|_{L^2}\|\varpi\|_{L^2}$. The inequality $|\partial_\alpha z(\alpha)|^{-1} \leq \|\mathcal{F}(z)\|_{L^\infty}^{1/2}$ gives us $\|J_2\|_{L^2} \leq C\|\mathcal{F}(z)\|_{L^\infty}^{1/2}\|z\|_{L^2}\|\varpi\|_{L^2}$.

Next it is easy to check that $|\partial_\alpha^2 P_3(\alpha)| \leq C\|\varpi\|_{L^2}(|\partial_\alpha^2 z(\alpha)| + \|z\|_{C^2}^2)$ and to estimate $\|\partial_\alpha^2 P_3\|_{L^2}$. The kernel in the integral $P_2(\alpha)$ has order 1 in $\beta$, and taking two derivatives in $\alpha$ we get integrals as in $P_3$ and kernels of degree $-1$ which can be estimated as before. Similar terms of lower order are obtained in $\partial_\alpha^2 P_1(\alpha)$ which are controlled analogously. The most singular terms are given by

$$
\begin{aligned}
Q_1(\alpha) &= \frac{1}{4\pi} PV \int_{\mathbb{T}} \partial_\alpha^2 \varpi(\alpha-\beta) \frac{V^{\perp}(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} d\beta, \\
Q_2(\alpha) &= \frac{1}{8\pi} PV \int_{\mathbb{T}} \varpi(\alpha-\beta) \frac{\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} d\beta, \\
Q_3(\alpha) &= -\frac{1}{4\pi} PV \int_{\mathbb{T}} \varpi(\alpha-\beta) \\
&\qquad \cdot \frac{V^{\perp}(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4} \Big( V(\alpha,\alpha-\beta) \cdot \big(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)\big)\Big) d\beta.
\end{aligned}
$$

We have

$$Q_1 = \frac{1}{4\pi} PV \int_{\mathbb{T}} \partial_\alpha^2 \varpi(\alpha-\beta) \left( \frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^2 \tan(\beta/2)} \right) d\beta + \frac{\partial_\alpha^\perp z(\alpha)}{2|\partial_\alpha z(\alpha)|^2} H(\partial_\alpha^2 \varpi)(\alpha),$$

which gives us

$$|Q_1(\alpha)| \leq C \|\mathcal{F}(z)\|_{L^\infty}^k \|z\|_{C^2}^k \|\partial_\alpha^2 \varpi\|_{L^2} + \|\mathcal{F}(z)\|_{L^\infty}^{1/2} |H(\partial_\alpha^2 \varpi)(\alpha)| \leq \left(1 + |H(\partial_\alpha^2 \varpi)(\alpha)|\right) \exp C \left( \|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t) \right). \tag{6.4}$$

Next we write $Q_2 = R_1 + R_2 + R_3$, where

$$R_1(\alpha) = \frac{1}{8\pi} \int_{\mathbb{T}} \big(\varpi(\alpha-\beta) - \varpi(\alpha)\big) \frac{\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} d\beta,$$

$$R_2(\alpha) = \frac{\varpi(\alpha)}{8\pi} \int_{\mathbb{T}} \big(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)\big) \cdot \left( \frac{1}{|V(\alpha,\alpha-\beta)|^2} - \frac{4}{|\partial_\alpha z(\alpha)|^2 |\beta|^2} \right) d\beta,$$

$$R_3(\alpha) = \frac{1}{8\pi} \frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2} \int_{\mathbb{T}} \big(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)\big) \left( \frac{4}{|\beta|^2} - \frac{1}{\sin^2(\beta/2)} \right) d\beta + \frac{1}{2} \frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2} \Lambda(\partial_\alpha^2 z)(\alpha).$$

Using that

$$|\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)| \leq |\beta|^\delta \|z\|_{C^{2,\delta}},$$

we get

$$|R_1(\alpha)| + |R_2(\alpha)| \leq \|\varpi\|_{C^1} \|\mathcal{F}(z)\|^k \|z\|_{C^{2,\delta}}^k \leq \exp C \left( \|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2 \right).$$

Meanwhile, for $R_3$ we have

$$|R_3(\alpha)| \leq C \|\varpi\|_{L^\infty} \|\mathcal{F}(z)\|_{L^\infty} (\|z\|_{C^2} + |\Lambda(\partial_\alpha^2 z)(\alpha)|) \leq \left(1 + |\Lambda(\partial_\alpha^2 z)(\alpha)|\right) \exp C \left( \|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2 \right);$$

that is,

$$|Q_2(\alpha)| \leq \left(1 + |\Lambda(\partial_\alpha^2 z)(\alpha)|\right) \exp C \left( \|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2 \right). \tag{6.5}$$

Let us consider $Q_3 = R_4 + R_5 + R_6 + R_7 + R_8 + R_9$, where

$$R_4 = -\frac{1}{4\pi}\int_{\mathbb{T}}\big(\varpi(\alpha-\beta)-\varpi(\alpha)\big)$$
$$\cdot\frac{V^{\perp}(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}\big(V(\alpha,\alpha-\beta)\cdot(\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta))\big)d\beta,$$

$$R_5 = -\frac{\varpi(\alpha)}{4\pi}\int_{\mathbb{T}}\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^{\perp}}{|V(\alpha,\alpha-\beta)|^4}\big(V(\alpha,\alpha-\beta)\cdot(\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta))\big)d\beta,$$

$$R_6 = -\frac{\varpi(\alpha)(\partial_\alpha z(\alpha))^{\perp}}{8\pi}\int_{\mathbb{T}}\frac{\beta(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)\cdot(\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta))}{|V(\alpha,\alpha-\beta)|^4}d\beta,$$

$$R_7 = -\frac{\varpi(\alpha)(\partial_\alpha z(\alpha))^{\perp}}{16\pi}\partial_\alpha z(\alpha)\int_{\mathbb{T}}\beta^2\big(\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta)\big)$$
$$\cdot\left(\frac{1}{|V(\alpha,\alpha-\beta)|^4}-\frac{16}{|\partial_\alpha z(\alpha)|^4|\beta|^4}\right)d\beta,$$

$$R_8 = -\frac{\varpi(\alpha)(\partial_\alpha z(\alpha))^{\perp}}{4\pi|\partial_\alpha z(\alpha)|^4}\partial_\alpha z(\alpha\cdot\int_{\mathbb{T}}\big(\partial_\alpha^2 z(\alpha)-\partial_\alpha^2 z(\alpha-\beta)\big)\left(\frac{4}{|\beta|^2}-\frac{1}{\sin^2(\beta/2)}\right)d\beta,$$

and

$$R_9 = -\frac{\varpi(\alpha)(\partial_\alpha z(\alpha))^{\perp}}{|\partial_\alpha z(\alpha)|^4}\partial_\alpha z(\alpha)\cdot\Lambda\big(\partial_\alpha^2 z(\alpha)\big).$$

Proceeding as before we get

$$|Q_3(\alpha)| \le (1+|\Lambda(\partial_\alpha^2 z)(\alpha)|)\exp\left(\|\mathcal{F}(z)\|_{L^\infty}^2+\|z\|_{H^3}^2\right),$$

which together with (6.4) and (6.5) gives us the estimate

$$|\partial_\alpha^2 P_1(\alpha)| \le (1+|\Lambda(\partial_\alpha^2 z)(\alpha)|+|H(\partial_\alpha^2\varpi)(\alpha)|)\exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2+\|z\|_{H^3}^2\right).$$

Then $\|\partial_\alpha^2 P_1\|_{L^2} \le \exp C(\|\mathcal{F}(z)\|_{L^\infty}^2+\|z\|_{H^3}^2$.

Finally we get

$$\|\partial_\alpha^2 BR(z,\varpi)\|_{L^2} \le \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2+\|z\|_{H^3}^2\right). \tag{6.6}$$

## 7. Estimates on $z(\alpha, t)$

In this section we give the proof of the lemma below when $k=3$. The case $k>3$ is left to the reader.

Lemma 7.1. *Let $z(\alpha,t)$ be a solution of 2DM. Then, the following a priori estimate holds*:

$$\frac{d}{dt}\|z\|_{H^k}^2(t) \le -\frac{\kappa}{2\pi(\mu_1+\mu_2)}\int_{\mathbb{T}}\frac{\sigma(\alpha,t)}{|\partial_\alpha z(\alpha)|^2}\partial_\alpha^k z(\alpha,t)\cdot\Lambda(\partial_\alpha^k z)(\alpha,t)d\alpha$$
$$+\exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2(t)+\|z\|_{H^k}^2\right) \tag{7.1}$$

*for $k\ge 3$.*

We split the proof in the following four parts.

7.1. *Estimates for the $L^2$ norm of the curve.* We have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|z(\alpha)|^2d\alpha=\int_{\mathbb{T}}z(\alpha)\cdot z_t(\alpha)d\alpha=\int_{\mathbb{T}}z(\alpha)\cdot BR(z,\varpi)(\alpha)d\alpha$$
$$+\int_{\mathbb{T}}c(\alpha)z(\alpha)\cdot\partial_\alpha z(\alpha)d\alpha=I_1+I_2.$$

Taking $I_1\leq\|z\|_{L^2}\|BR(z,\varpi)\|_{L^2}$ and inequality (6.1), let us estimate $I_1$.

Next we get

$$I_2\leq A^{1/2}(t)\|c\|_{L^\infty}\int_{\mathbb{T}}|z(\alpha)|d\alpha\leq 2\int_{\mathbb{T}}|\partial_\alpha BR(z,\varpi)(\alpha)|d\alpha\int_{\mathbb{T}}|z(\alpha)|d\alpha$$

which yields

$$I_2\leq\exp(C\|F(z)\|^2_{L^\infty}+\|z\|^2_{H^3})$$

using estimate (6.2). Then we may conclude that

$$\frac{d}{dt}\|z\|^2_{L^2}(t)\leq\exp\left(C|||z|||^2\right) \tag{7.2}$$

for an appropriate finite constant $C$, where, as before, $|||z|||^2=\|F(z)\|^2_{L^\infty}+\|z\|^2_{H^3}$.

7.2. *The integrable terms in $\partial^3_\alpha BR(z,\varpi)$.* Since $z_t(\alpha)=BR(z,\varpi)(\alpha)+c(\alpha)\cdot\partial_\alpha z(\alpha)$, we have

$$\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^3_\alpha z_t(\alpha)d\alpha=\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^3_\alpha BR(z,\varpi)(\alpha)d\alpha$$
$$+\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^3_\alpha\Big(c(\alpha)\partial_\alpha z(\alpha)\Big)d\alpha=I_1+I_2.$$

Here and in Section 7.3 we study $I_1$. We shall estimate $I_2$ in Section 7.4.

Let us write $BR(z,\varpi)(\alpha)=P_1(\alpha)+P_2(\alpha)+P_3(\alpha)$ as in (6.3). Then it is easy to check that

$$|\partial^3_\alpha P_3(\alpha)|\leq C\|\varpi\|_{L^2}\Big(|\partial^3_\alpha z_2(\alpha)|+\|z\|^3_{C^2}\Big),$$

giving us a term controlled by the energy estimate. The kernel in the integral $P_2(\alpha)$ has order 1 in $\beta$, therefore taking two derivatives in $\alpha$ produces regular integrals as in $P_3$ and kernels of degree $-1$ in $\beta$, for which we first exchange $\beta$ by $\alpha-\beta$ and then take one more derivative. We obtain kernels of grade $-1$ in $\beta$ acting in $\varpi$ or $\varpi_\alpha$ which can be estimated as before. For the most singular term $P_1(\alpha)$, we have

$$\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^3_\alpha P_1(\alpha)d\alpha=I_3+I_4+I_5+I_6,$$

where

$$I_3=\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\partial^3_\alpha\left(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}\right)\varpi(\alpha-\beta)d\beta,$$

$$I_4 = \frac{3}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\partial_\alpha^2\left(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}\right)\partial_\alpha\varpi(\alpha-\beta)d\beta,$$

$$I_5 = \frac{3}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\partial_\alpha\left(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}\right)\partial_\alpha^2\varpi(\alpha-\beta)d\beta,$$

$$I_6 = \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\left(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}\right)\partial_\alpha^3\varpi(\alpha-\beta)d\beta.$$

The most singular terms for $I_3$ are those in which three derivatives appear and the kernels have degree $-1$. The rest of the terms have kernels with degree $k > -1$ and can be estimated as before. One of the two singular terms of $I_3$ is given by

$$J_1 = \frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\alpha-\beta))^\perp}{|V(\alpha,\alpha-\beta)|^2}\varpi(\alpha-\beta)d\beta d\alpha,$$

which we decompose as follows:

$$\begin{aligned}
J_1 &= \frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))^\perp}{|V(\alpha,\beta)|^2}\varpi(\beta)d\beta d\alpha\\
&= \frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))^\perp}{|V(\alpha,\beta)|^2}\frac{\varpi(\beta)+\varpi(\alpha)}{2}d\beta d\alpha\\
&\quad+\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))^\perp}{|V(\alpha,\beta)|^2}\frac{\varpi(\beta)-\varpi(\alpha)}{2}d\beta d\alpha\\
&= K_1 + K_2.
\end{aligned}$$

That is, we have made a kind of integration by parts in $J_1$, allowing us to show that the most singular term $K_1$ vanishes:

$$\begin{aligned}
K_1 &= -\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\beta)\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))^\perp}{|V(\alpha,\beta)|^2}\frac{\varpi(\beta)+\varpi(\alpha)}{2}d\beta d\alpha\\
&= \frac{1}{16\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))\cdot\frac{(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta))^\perp}{|V(\alpha,\beta)|^2}\frac{\varpi(\beta)+\varpi(\alpha)}{2}d\beta d\alpha\\
&= 0,
\end{aligned}$$

while for $K_2$ we have

$$\begin{aligned}
K_2 &= \frac{1}{16\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\left(\partial_\alpha^3 z(\beta)\right)^\perp\frac{\varpi(\alpha)-\varpi(\beta)}{|V(\alpha,\beta)|^2}d\beta d\alpha\\
&= \int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\left(\partial_\alpha^3 Hz(\alpha)\right)^\perp\frac{\partial_\alpha\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2}d\alpha\\
&\quad+\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\left(\partial_\alpha^3 z(\alpha-\beta)\right)^\perp B_1(\alpha,\beta)d\beta d\alpha,
\end{aligned}$$

where $|B_1(\alpha,\beta)| \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|\varpi\|_{C^{1,\delta}}|\beta|^{\delta-1}$. The other singular term with three derivatives in $z(\alpha)$ and kernel of degree $-1$ inside $I_3$ is given by

$$J_2 = -\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \cdot \frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4} V(\alpha,\alpha-\beta)\cdot\big(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\alpha-\beta)\big)\varpi(\alpha-\beta)d\beta d\alpha.$$

Here we take $J_2 = K_3 + K_4 + K_5$, where

$$K_3 = -\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \cdot \frac{V^\perp(\alpha,\beta)}{|V(\alpha,\beta)|^4}(V(\alpha,\beta)-W(\alpha,\beta))\cdot\big(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\beta)\big)\varpi(\beta)d\beta d\alpha,$$

$$K_4 = -\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \cdot \frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4} B_2(\alpha,\alpha-\beta)\cdot\big(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\alpha-\beta)\big)\varpi(\alpha-\beta)d\beta d\alpha$$

with

$$B_2(\alpha,\alpha-\beta) = W(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta/2.$$

Thus

$$W(\alpha,\beta) = \left(\left(\frac{z_1(\alpha)-z_1(\beta)}{2}\right)_p, \left(\frac{z_2(\alpha)-z_2(\beta)}{2}\right)_p\right)$$

is defined in the Appendix. Finally we have

$$K_5 = -\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \cdot \frac{V^\perp(\alpha,\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4} \partial_\alpha z(\alpha)\cdot\big(\partial_\alpha^3 z(\alpha)-\partial_\alpha^3 z(\alpha-\beta)\big)\varpi(\alpha-\beta)d\beta d\alpha.$$

The $L^\infty$ norm of

$$\frac{V^\perp(\alpha,\beta)}{|V(\alpha,\beta)|^4}\big(V(\alpha,\beta)-W(\alpha,\beta)\big)$$

is given in the Appendix, allowing us to estimate the term $K_3$ as before.

Next we split $K_4 = L_1 + L_2$, where

$$L_1 = -\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4} B_2(\alpha,\alpha-\beta)\cdot\partial_\alpha^3 z(\alpha)\varpi(\alpha-\beta)d\beta d\alpha$$

and

$$L_2 = \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4} B_2(\alpha,\alpha-\beta)\cdot\partial_\alpha^3 z(\alpha-\beta)\varpi(\alpha-\beta)d\beta d\alpha.$$

We have

$$|L_1| \leq C\Big|\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^4}\partial_\alpha^2 z(\alpha)\cdot\partial_\alpha^3 z(\alpha) H\varpi(\alpha)d\beta d\alpha\Big| \\ + \|\partial_\alpha^3 z\|_{L^2}^2\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^{2,\delta}}\|\varpi\|_{L^\infty},$$

$$|L_2| \leq C\Big|\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^4}\varpi(\alpha)\partial_\alpha^2 z(\alpha)\cdot H(\partial_\alpha^3 z)(\alpha)d\beta d\alpha\Big| \\ + \|\partial_\alpha^3 z\|_{L^2}^2\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^{2,\delta}}\|\varpi\|_{C^1};$$

and the term $K_4$ is controlled.

For $K_5$ we split

$$\begin{aligned} K_5 = &\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \\ &\cdot\frac{V^\perp(\alpha,\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4}\big(\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta)\big)\cdot\partial_\alpha^3 z(\alpha-\beta)\varpi(\alpha-\beta)d\beta d\alpha \\ &-\frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot B_3\big(\alpha,\alpha-\beta)(\partial_\alpha z(\alpha)\cdot\partial_\alpha^3 z(\alpha) \\ &\qquad\qquad -\partial_\alpha z(\alpha-\beta)\cdot\partial_\alpha^3 z(\alpha-\beta)\big)d\beta d\alpha \\ = &\, L_3+L_4, \end{aligned}$$

where

$$B_3(\alpha,\alpha-\beta)=\frac{V^\perp(\alpha,\alpha-\beta)\varpi(\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4}.$$

Then we have

$$|L_3| \leq C\Big|\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\frac{\partial_\alpha^\perp z(\alpha)}{|\partial_\alpha z(\alpha)|^4}\partial_\alpha^2 z(\alpha)\cdot H(\partial_\alpha^3 z\,\varpi)(\alpha)d\alpha| \\ + \|\partial_\alpha^3 z\|_{L^2}^2\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^{2,\delta}}\|\varpi\|_{L^\infty}.$$

For $L_4$ we use an appropriated integration by part:

$$\partial_\alpha z(\alpha)\cdot\partial_\alpha^3 z(\alpha) = -|\partial_\alpha^2 z(\alpha)|^2$$

to obtain

$$L_4 = \frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot B_3(\alpha,\alpha-\beta)\big(|\partial_\alpha^2 z(\alpha)|^2-|\partial_\alpha^2 z(\alpha-\beta)|^2\big)d\beta d\alpha.$$

Next we write $L_4 = M_1 + M_2$, with

$$M_1 = \frac{1}{8\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot C(\alpha,\alpha-\beta)\big(|\partial_\alpha^2 z(\alpha)|^2-|\partial_\alpha^2 z(\alpha-\beta)|^2\big)d\beta d\alpha$$

for

$$C(\alpha,\alpha-\beta) = B_3(\alpha,\alpha-\beta) - \frac{2\partial_\alpha^\perp z(\alpha)\varpi(\alpha)}{|\partial_\alpha^2 z(\alpha)|^4\sin^2(\beta/2)}$$
$$= \frac{V^\perp(\alpha,\alpha-\beta)\varpi(\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4} - \frac{2\partial_\alpha^\perp z(\alpha)\varpi(\alpha)}{|\partial_\alpha^2 z(\alpha)|^4\sin^2(\beta/2)},$$

and

$$M_2 = C\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^\perp z(\alpha)\frac{\varpi(\alpha)}{|\partial_\alpha^2 z(\alpha)|^4}\Lambda(|\partial_\alpha^2 z|^2)d\alpha.$$

Since

$$|C(\alpha,\alpha-\beta)| \leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|\varpi\|_{C^1}\frac{1}{|\beta|}$$

(see Lemma 11.3 in the Appendix for more details) and

$$\Big||\partial_\alpha^2 z(\alpha)|^2 - |\partial_\alpha^2 z(\alpha-\beta)|^2\Big| \leq 2\|z\|_{C^1}|\beta|\int_0^1\Big|\partial_\alpha^3 z(\alpha+(s-1)\beta)\Big|ds,$$

we get $|M_1| \leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|\varpi\|_{C^1}\|\partial_\alpha^3 z\|_{L^2}^2$.

For the term $M_2$, we use the estimate

$$\|\Lambda(|\partial_\alpha^2 z|^2)\|_{L^2} = \|\partial_\alpha(|\partial_\alpha^2 z|^2)\|_{L^2} \leq 2\|\partial_\alpha^2 z\|_{L^\infty}\|\partial_\alpha^3 z\|_{L^2}$$

to obtain $|M_2| \leq C\|\mathcal{F}(z)\|_{L^\infty}\|\varpi\|_{L^\infty}\|\partial_\alpha^2 z\|_{L^\infty}\|\partial_\alpha^3 z\|_{L^2}^2$.

For $I_4$, the most singular terms are those for which two derivatives are applied to $z(\alpha)$. One of those is $J_3$:

$$J_3 = C\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\big(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)\big)\frac{\partial_\alpha\varpi(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta.$$

We split $J_3 = K_6 + K_7 + K_8$ and obtain

$$K_6 = C\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\big(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)\big)\frac{\partial_\alpha\varpi(\alpha-\beta)-\partial_\alpha\varpi(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$
$$K_7 = C\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha\varpi(\alpha)\partial_\alpha^3 z(\alpha)$$
$$\cdot\big(\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)\big)\left(\frac{1}{|V(\alpha,\alpha-\beta)|^2} - \frac{4}{|\partial_\alpha z(\alpha)|^2\beta^2}\right)d\beta,$$
$$K_8 = C\int_{\mathbb{T}} \partial_\alpha\varpi(\alpha)\partial_\alpha^3 z(\alpha)\cdot\int_{\mathbb{T}}\frac{\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta)}{\beta^2}d\beta.$$

Using that

$$\partial_\alpha^2 z(\alpha) - \partial_\alpha^2 z(\alpha-\beta) = \beta\int_0^1 \partial_\alpha^3 z\big(\alpha+(s-1)\beta\big)ds \tag{7.3}$$

and $|\partial_\alpha\varpi(\alpha-\beta)-\partial_\alpha\varpi(\alpha)|\leq\|w\|_{C^{1,\delta}}|\beta|^\delta$, we have

$$
\begin{aligned}
|K_6| &\leq C\|\mathcal{F}(z)\|_{L^\infty}\|w\|_{C^{1,\delta}}\int_0^1\int_{\mathbb{T}}|\beta|^{\delta-1}\int_{\mathbb{T}}|\partial_\alpha^3 z(\alpha)|\Big|\partial_\alpha^3 z\big(\alpha+(s-1)\beta\big)\Big|d\alpha d\beta ds\\
&\leq C\|\mathcal{F}(z)\|_{L^\infty}\|w\|_{C^{1,\delta}}\\
&\quad\cdot\int_0^1\int_{\mathbb{T}}|\beta|^{\delta-1}\int_{\mathbb{T}}\big(|\partial_\alpha^3 z(\alpha)|^2+|\partial_\alpha^3 z\,(\alpha+(s-1)\beta)\,|^2\big)d\alpha d\beta ds\\
&\leq C\|\mathcal{F}(z)\|_{L^\infty}\|w\|_{C^{1,\delta}}\|\partial_\alpha^3 z\|_{L^3}^2.
\end{aligned}
$$

Due to (7.3) and the estimates obtained in the Appendix, we have

$$|K_7|\leq C\|\mathcal{F}(z)\|_{L^\infty}\|w\|_{C^1}\|z\|_{C^2}\|\partial_\alpha^3 z\|_{L^2}^2.$$

Then using that $1/\beta-1/2\sin(\beta/2)$ is bounded, we get

$$K_8\leq C\|\mathcal{F}(z)\|_{L^\infty}\|w\|_{C^1}\|\partial_\alpha^3 z\|_{L^3}^2.$$

Regarding $I_5$, its most singular term is given by

$$J_4=C\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\big(\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta)\big)\frac{\partial_\alpha^2\varpi(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

which, after being decomposed in the form

$$
\begin{aligned}
J_4 &= C\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\\
&\quad\cdot\left(\frac{\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}-\frac{2\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2\tan(\beta/2)}\right)\partial_\alpha^2\varpi(\alpha-\beta)d\beta\\
&\quad+C\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\frac{\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2}H(\partial_\alpha^2\varpi)(\alpha)d\alpha,
\end{aligned}
$$

can be estimated as before: $|J_4|\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^{2,\delta}}\|\partial_\alpha^2 w\|_{L^2}\|\partial_\alpha^3 z\|_{L^2}$.

7.3. *Looking for $\sigma(\alpha)$.* The term $I_6$ will gives us the proper sign (Rayleigh-Taylor condition) that has to be imposed upon $\sigma(\alpha)$. Let us recall the formula

$$\sigma(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)+\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t).$$

We write $I_6$ in the form $I_6=J_1+J_2$, where

$$J_1=\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\left(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}-\frac{(\partial_\alpha z(\alpha))^\perp}{|\partial_\alpha z(\alpha)|^2\tan(\beta/2)}\right)\partial_\alpha^3\varpi(\alpha-\beta)d\beta d\alpha$$

and

$$J_2=\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\alpha^3 z(\alpha)\cdot\frac{(\partial_\alpha z(\alpha))^\perp}{|\partial_\alpha z(\alpha)|^2}H(\partial_\alpha^3\varpi)(\alpha)d\beta d\alpha.$$

Let us denote the kernel of $J_1$ by $\Sigma(\alpha, \alpha-\beta)$, which is of degree 0 in $\beta$. After an integration by parts we obtain

$$\begin{aligned} J_1 &= -\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\Sigma(\alpha,\alpha-\beta)\partial_\beta\big(\partial_\alpha^2\varpi(\alpha-\beta)\big)d\beta d\alpha \\ &= \frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\partial_\beta\Sigma(\alpha,\alpha-\beta)\partial_\alpha^2\varpi(\alpha-\beta)d\beta d\alpha. \end{aligned}$$

Then $\partial_\beta\Sigma(\alpha,\alpha-\beta)$ has terms of degree 0 which are estimated easily. The term with degree $-1$ is given by

$$\frac{(\partial_\alpha z(\alpha-\beta))^\perp}{2|V(\alpha,\alpha-\beta)|^2}+\frac{(\partial_\alpha z(\alpha))^\perp}{2|\partial_\alpha z(\alpha)|^2\sin^2(\beta/2)}-\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}V(\alpha,\alpha-\beta)\cdot\partial_\alpha z(\alpha-\beta),$$

and we decompose it as a sum of a kernel of degree 0 (easy to estimate)

$$-\frac{(\partial_\alpha z(\alpha))^\perp}{|\partial_\alpha z(\alpha)|^2}\left(\frac{2}{|\beta|^2}-\frac{1}{2\sin^2(\beta/2)}\right)$$

and six kernels of degree $-1$, $(P_1,\dots,P_6)$ given by

$$\begin{aligned} P_1(\alpha,\alpha-\beta) &= \frac{(\partial_\alpha z(\alpha-\beta)-\partial_\alpha z(\alpha))^\perp}{2|V(\alpha,\alpha-\beta)|^2}, \\ P_2(\alpha,\alpha-\beta) &= -\frac{(\partial_\alpha z(\alpha))^\perp}{2}\left(\frac{1}{|V(\alpha,\alpha-\beta)|^2}-\frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\right), \\ P_3(\alpha,\alpha-\beta) &= \frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}V(\alpha,\alpha-\beta)\cdot\big(\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta)\big), \\ P_4(\alpha,\alpha-\beta) &= -\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}\big(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2\big)\cdot\partial_\alpha z(\alpha), \\ P_5(\alpha,\alpha-\beta) &= -\frac{|\partial_\alpha z(\alpha)|^2\beta}{2}\frac{V^\perp(\alpha,\alpha-\beta)-\partial_\alpha^\perp z(\alpha)\beta/2}{|V(\alpha,\alpha-\beta)|^4}, \\ P_6(\alpha,\alpha-\beta) &= -\frac{\partial_\alpha^\perp z(\alpha)|\partial_\alpha z(\alpha)|^2|\beta|^2}{4|V(\alpha,\alpha-\beta)|^2}\left(\frac{1}{|V(\alpha,\alpha-\beta)|^2}-\frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\right). \end{aligned}$$

To control the term with kernel $P_2$ we consider $P_2 = Q_1 + Q_2$:

$$\begin{aligned} Q_1(\alpha,\alpha-\beta) &= P_2(\alpha,\alpha-\beta)-\frac{(\partial_\alpha z(\alpha))^\perp}{2}\frac{2\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4\beta}, \\ Q_2(\alpha,\alpha-\beta) &= -\partial_\alpha^\perp z(\alpha)\left(\frac{\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4}\left(\frac{1}{\beta}-\frac{1}{2\tan(\beta/2)}\right)\right. \\ &\qquad\left.+\frac{\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4 2\tan(\beta/2)}\right). \end{aligned}$$

In the Appendix, we show that $\|Q_1\|_{L^\infty} \leq \|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^{2,\delta}}|\beta|^{\delta-1}$ (see Lemma 11.4) giving us

$$\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot Q_1(\alpha,\alpha-\beta)\partial^2_\alpha\varpi(\alpha-\beta)d\beta d\alpha \leq C\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^{2,\delta}}\|\partial^3_\alpha z\|_{L^2}\|\partial^2_\alpha\varpi\|_{L^2}.$$

The integral

$$K_1=\frac{1}{4\pi}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot Q_2(\alpha,\alpha-\beta)\partial^2_\alpha\varpi(\alpha-\beta)d\beta d\alpha$$

is bounded by

$$\begin{aligned}|K_1| &\leq C\|\mathcal{F}(z)\|^{3/2}_{L^\infty}\|z\|_{C^2}\int_{\mathbb{T}}\Big|\partial^3_\alpha z(\alpha)\Big|\Big(\|\partial^2_\alpha\varpi\|_{L^2}+|H(\partial^2_\alpha\varpi)(\alpha)|\Big)d\alpha\\ &\leq C\|\mathcal{F}(z)\|^{3/2}_{L^\infty}\|z\|_{C^2}\|\partial^3_\alpha z\|_{L^2}\|\partial^2_\alpha\varpi\|_{L^2}.\end{aligned}$$

It is now very clear that the other $P_i$ terms can be estimated as above or as before; i.e., we finally have

$$J_1 \leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}\right).$$

Now we consider the $J_2$ term which can be written as follows:

$$\begin{aligned}J_2&=\frac{1}{4\pi A(t)}\int_{\mathbb{T}}\partial^3_\alpha z(\alpha)\cdot\partial^\perp_\alpha z(\alpha)\Lambda(\partial^2_\alpha\varpi)(\alpha)d\alpha\\ &=\frac{1}{4\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\partial^2_\alpha\varpi(\alpha)d\alpha.\end{aligned}$$

Using formula (5.3) we separate $J_2$ as a sum of two parts, $K_2$ and $K_3$, where

$$K_2=-\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\partial^3_\alpha z_2(\alpha)d\alpha$$

and

$$K_3=-\frac{A_\mu}{4\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\partial^2_\alpha T(\varpi)(\alpha)d\alpha.$$

For $K_2$, we further decompose $K_2=L_1+L_2$, where

$$L_1=\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z_1\partial_\alpha z_2)(\alpha)\partial^3_\alpha z_2(\alpha)d\alpha$$

and

$$L_2=-\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z_2\partial_\alpha z_1)(\alpha)\partial^3_\alpha z_2(\alpha)d\alpha.$$

Then $L_1$ is written as $L_1=M_1+M_2$ with

$$M_1=\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\Big(\Lambda(\partial^3_\alpha z_1\partial_\alpha z_2)(\alpha)-\Lambda(\partial^3_\alpha z_1)(\alpha)\partial_\alpha z_2(\alpha)\Big)\partial^3_\alpha z_2(\alpha)d\alpha$$

and

$$M_2 = \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z_1)(\alpha)\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha)d\alpha.$$

Using the commutator estimate, we get

$$M_1 \le C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^{2,\delta}}\|\partial_\alpha^3 z\|_{L^2}^2 \le \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2+\|z\|_{H^3}^2\right).$$

The identity

$$\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha) = -\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha) - |\partial_\alpha^2 z(\alpha)|^2$$

allows us to write $M_2$ as the sum of $N_1$ and $N_2$, where

$$N_1 = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z_1)(\alpha)|\partial_\alpha^2 z(\alpha)|^2 d\alpha$$

and

$$N_2 = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha.$$

Integration by parts shows that

$$N_1 \le C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|\partial_\alpha^3 z\|_{L^2}^2 \le \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2+\|z\|_{H^3}^2\right).$$

Writing $L_2$ in the form

$$L_2 = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_2(\alpha)\Lambda(\partial_\alpha^3 z_2)(\alpha)d\alpha,$$

we finally obtain

$$\begin{aligned} K_2 \le{}& \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2+\|z\|_{H^3}^2\right) \\ &- \frac{2\kappa \mathrm{g}(\rho^2-\rho^1)}{4\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}}\partial_\alpha z_1(\alpha)\partial_\alpha^3 z(\alpha)\cdot\Lambda(\partial_\alpha^3 z)(\alpha)d\alpha. \end{aligned}$$

In the estimate above we can observe how a part of $\sigma(\alpha)$ appears in the non-integrable terms. Let us now return to $K_3 = L_3 + L_4 + L_5$, where

$$L_3 = -\frac{A_\mu}{4\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)(2\partial_\alpha^2 BR(z,\varpi)(\alpha))\cdot\partial_\alpha z(\alpha)d\alpha,$$

$$L_4 = -\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)\partial_\alpha BR(z,\varpi)(\alpha))\cdot\partial_\alpha^2 z(\alpha)d\alpha,$$

and

$$L_5 = -\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)(\alpha)BR(z,\varpi)(\alpha)\cdot\partial_\alpha^3 z(\alpha)d\alpha.$$

We shall first control the terms $L_3$ and $L_4$ and then we shall show how the rest of $\sigma(\alpha)$ appears in $L_5$. Integrating by parts in $L_4$ we obtain

$$\begin{aligned} L_4 \le C\|\mathcal{F}(z)\|_{L^\infty}\|H(\partial_\alpha^3 z\cdot\partial_\alpha^\perp z)\|_{L^2}\Big(&\|\partial_\alpha^2 BR(z,\varpi)\|_{L^2}\|\partial_\alpha^2 z\|_{L^\infty} \\ &+\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}\|\partial_\alpha^3 z\|_{L^2}\Big), \end{aligned}$$

and using the estimates for $\|\partial^2_\alpha BR(z,\varpi)\|_{L^2}$, we get

$$L_4 \leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}\right).$$

With $L_3$ we also integrate by parts to obtain $L_3 = M_3 + M_4$, where

$$M_3 = -\frac{A_\mu}{4\pi A(t)}\int_{\mathbb{T}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\big(2\partial^2_\alpha BR(z,\varpi)(\alpha)\big)\cdot\partial^2_\alpha z(\alpha)d\alpha$$

and

$$M_4 = -\frac{A_\mu}{4\pi A(t)}\int_{\mathbb{T}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\big(2\partial^3_\alpha BR(z,\varpi)(\alpha)\big)\cdot\partial_\alpha z(\alpha)d\alpha.$$

Easily we have

$$\begin{aligned} M_3 &\leq C\|\mathcal{F}(z)\|_{L^\infty}\|H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)\|_{L^2}\|\partial^2_\alpha BR(z,\varpi)\|_{L^2}\|\partial^2_\alpha z\|_{L^\infty}\\ &\leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}\right). \end{aligned}$$

In $M_4$ the application of Leibniz's rule to $\partial^3_\alpha BR(z,\varpi)$ produces many terms which can be estimated with the same tools used before with $I_4$ and $I_5$. For the most singular terms we have the expressions

$$\begin{aligned} N_3 &= -\frac{A_\mu}{4\pi A(t)}\int_{\mathbb{T}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)2\partial_\alpha\big(BR(z,\partial^2_\alpha\varpi)(\alpha)\big)\cdot\partial_\alpha z(\alpha)d\alpha,\\ N_4 &= -\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\\ &\quad\cdot\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{(\partial^3_\alpha z(\alpha)-\partial^3_\alpha z(\alpha-\beta))^\perp}{|V(\alpha,\alpha-\beta)|^2}d\beta\cdot\partial_\alpha z(\alpha)d\alpha,\\ N_5 &= \frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)(\alpha)\\ &\quad\cdot\int_{\mathbb{T}}\varpi(\alpha-\beta)B(\alpha,\alpha-\beta)\cdot\big(\partial^3_\alpha z(\alpha)-\partial^3_\alpha z(\alpha-\beta)\big)d\beta d\alpha, \end{aligned}$$

where

$$B(\alpha,\alpha-\beta) = \frac{V^\perp(\alpha,\alpha-\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}V(\alpha,\alpha-\beta).$$

Let us consider

$$\begin{aligned} &\partial_\alpha\big(BR(z,\partial^2_\alpha\varpi)(\alpha)\big)\cdot\partial_\alpha z(\alpha)\\ &\quad= \partial_\alpha\big(BR(z,\partial^2_\alpha\varpi)(\alpha)\cdot\partial_\alpha z(\alpha)\big) - BR(z,\partial^2_\alpha\varpi)(\alpha)\cdot\partial^2_\alpha z(\alpha)\\ &\quad= \frac{1}{2}\partial_\alpha\big(T(\partial^2_\alpha\varpi)(\alpha)\big) - BR(z,\partial^2_\alpha\varpi)(\alpha)\cdot\partial^2_\alpha z(\alpha), \end{aligned}$$

which yields

$$N_3 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|H(\partial^3_\alpha z\cdot\partial^\perp_\alpha z)\|_{L^2}\big(\|T(\partial^2_\alpha\varpi)\|_{H^1}+\|BR(z,\partial^2_\alpha\varpi)\|_{L^2}\|\partial^2_\alpha z\|_{L^\infty}\big);$$

therefore,

$$N_3 \leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}\right).$$

Next we write $N_4 = O_1 + O_2 + O_3 + O_4 + O_5$,

$$O_1 = -\frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} H\big(\partial_\alpha^3 z \partial_\alpha^\perp z)(\alpha)(\partial_\alpha^3 z(\alpha)\big)^\perp \cdot \partial_\alpha z(\alpha) \int_{\mathbb{T}} \left( \frac{\varpi(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{4\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2|\beta|^2} \right) d\beta d\alpha,$$

$$O_2 = \frac{A_\mu}{2\pi A(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha) \int_{\mathbb{T}} \big(\partial_\alpha^3 z(\alpha-\beta)\big)^\perp \cdot \partial_\alpha z(\alpha) \left( \frac{\varpi(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{4\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2|\beta|^2} \right) d\beta d\alpha,$$

$$O_3 = -\frac{A_\mu}{2\pi A^2(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha)\varpi(\alpha)\big(\partial_\alpha^3 z(\alpha)\big)^\perp \cdot \partial_\alpha z(\alpha) \int_{\mathbb{T}} \left( \frac{4}{|\beta|^2} - \frac{1}{\sin^2(\beta/2)} \right) d\beta d\alpha,$$

$$O_4 = \frac{A_\mu}{2\pi A^2(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha)\varpi(\alpha) \int_{\mathbb{T}} \big(\partial_\alpha^3 z(\alpha-\beta)\big)^\perp \cdot \partial_\alpha z(\alpha) \left( \frac{4}{|\beta|^2} - \frac{1}{\sin^2(\beta/2)} \right) d\beta d\alpha,$$

$$O_5 = -\frac{A_\mu}{2\pi A^2(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha)\varpi(\alpha)\partial_\alpha z(\alpha) \cdot \Lambda\big((\partial_\alpha^3 z)^\perp\big)(\alpha) d\alpha.$$

The terms $O_1$, $O_2$, $O_3$, and $O_4$ can be estimated as before. Then we split $O_5 = R_1 + R_2$, where

$$R_1 = \frac{A_\mu}{2\pi A^2(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha)\big(\Lambda(\varpi\partial_\alpha z \cdot (\partial_\alpha^3 z)^\perp)(\alpha) - \varpi(\alpha)\partial_\alpha z(\alpha) \cdot \Lambda((\partial_\alpha^3 z)^\perp)(\alpha)\big) d\alpha$$

and

$$R_2 = -\frac{A_\mu}{2\pi A^2(t)} \int_{\mathbb{T}} H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha)\Lambda(\varpi\partial_\alpha z \cdot (\partial_\alpha^3 z)^\perp)(\alpha) d\alpha.$$

Using the commutator estimate, we obtain

$$\begin{aligned} R_1 &\le C\|\mathcal{F}(z)\|_{L^\infty}^2 \|H(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)\|_{L^2} \|\varpi\partial_\alpha z\|_{C^{1,\delta}} \|\partial_\alpha^3 z\|_{L^2} \\ &\le \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2\right). \end{aligned}$$

The identity $\Lambda(H) = -\partial_\alpha$ gives

$$\begin{aligned} R_2 &= \frac{A_\mu}{2\pi A^2(t)} \int_{\mathbb{T}} \partial_\alpha(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha)\varpi(\alpha)\partial_\alpha z(\alpha) \cdot (\partial_\alpha^3 z(\alpha))^\perp d\alpha \\ &= -\frac{A_\mu}{2\pi A^2(t)} \int_{\mathbb{T}} \partial_\alpha(\partial_\alpha^3 z \cdot \partial_\alpha^\perp z)(\alpha)\varpi(\alpha)\partial_\alpha^3 z(\alpha) \cdot \partial_\alpha^\perp z(\alpha) d\alpha, \end{aligned}$$

and integrating by parts we get

$$R_2 \le C\|\mathcal{F}(z)\|_{L^\infty}^2 \|\partial_\alpha^3 z \cdot \partial_\alpha^\perp z\|_{L^2}^2 \|\partial_\alpha \varpi\|_{L^\infty} \le \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2\right).$$

Regarding the term $N_5$ we have the expression

$$B(\alpha,\alpha-\beta)=\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}V(\alpha,\alpha-\beta),$$

which shows that $B(\alpha,\alpha-\beta)$ has order $-1$ and, therefore, the term $N_5$ can be estimated as before.

Finally we have to find $\sigma(\alpha)$ in $L_5$ to finish the proof of the lemma. To do that let us split $L_5=M_5+M_6+M_7+M_8$, where

$$M_5=\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z_1\partial_\alpha z_2)(\alpha)BR_1(z,\varpi)(\alpha)\partial^3_\alpha z_1(\alpha)d\alpha,$$
$$M_6=\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z_1\partial_\alpha z_2)(\alpha)BR_2(z,\varpi)(\alpha)\partial^3_\alpha z_2(\alpha)d\alpha,$$
$$M_7=-\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z_2\partial_\alpha z_1)(\alpha)BR_1(z,\varpi)(\alpha)\partial^3_\alpha z_1(\alpha)d\alpha,$$
$$M_8=-\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}\Lambda(\partial^3_\alpha z_2\partial_\alpha z_1)(\alpha)BR_2(z,\varpi)(\alpha)\partial^3_\alpha z_2(\alpha)d\alpha,$$

and $BR_j$, $j=1,2$, is the $j$-th component of the Birkhoff-Rott integral.

Then

$$\begin{aligned}M_5=&\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}\big(\Lambda(\partial_\alpha z_2\partial^3_\alpha z_1)(\alpha)\\&\qquad-\partial_\alpha z_2(\alpha)\Lambda(\partial^3_\alpha z_1)(\alpha)\big)BR_1(z,\varpi)(\alpha)\partial^3_\alpha z_1(\alpha)d\alpha\\&+\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}\partial_\alpha z_2(\alpha)BR_1(z,\varpi)(\alpha)\partial^3_\alpha z_1(\alpha)\Lambda(\partial^3_\alpha z_1)(\alpha)d\alpha,\end{aligned}$$

and the commutator estimates yields

$$\begin{aligned}M_5\le&\exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}\right)\\&+\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}BR_1(z,\varpi)(\alpha)\partial_\alpha z_2(\alpha)\partial^3_\alpha z_1(\alpha)\Lambda(\partial^3_\alpha z_1)(\alpha)d\alpha.\end{aligned}\tag{7.4}$$

In a similar way we have

$$\begin{aligned}M_6\le&\exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}\right)\\&+\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}BR_2(z,\varpi)(\alpha)\partial_\alpha z_2(\alpha)\partial^3_\alpha z_2(\alpha)\Lambda(\partial^3_\alpha z_1)(\alpha)d\alpha.\end{aligned}$$

Let us introduce the notation

$$N_4=\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}BR_2(z,\varpi)(\alpha)\partial_\alpha z_2(\alpha)\partial^3_\alpha z_2(\alpha)\Lambda(\partial^3_\alpha z_1)(\alpha)d\alpha.$$

The equality $\partial_\alpha z_2(\alpha)\partial^3_\alpha z_2(\alpha)=-\partial_\alpha z_1(\alpha)\partial^3_\alpha z_1(\alpha)-|\partial^2_\alpha z(\alpha)|^2$ gives $N_4=O_6+O_7$, where

$$O_6=-\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}}BR_2(z,\varpi)(\alpha)|\partial^2_\alpha z(\alpha)|^2\Lambda(\partial^3_\alpha z_1)(\alpha)d\alpha,$$

$$O_7 = -\frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} BR_2(z,\varpi)(\alpha)\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha.$$

Integrating by parts in $O_6$ we get

$$\begin{aligned}O_6 &\le C\|\mathcal{F}(z)\|_{L^\infty}\Big(\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}\|\partial_\alpha^2 z\|_{L^\infty}^2 \\ &\qquad + \|BR(z,\varpi)\|_{L^\infty}\|\partial_\alpha^2 z\|_{L^\infty}\|\partial_\alpha^3 z\|_{L^2}\Big)\|H(\partial_\alpha^3 z_1)\|_{L^2} \\ &\le \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2\right).\end{aligned}$$

Finally we get the estimate

$$\begin{aligned}M_6 &\le \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2\right) \\ &\quad - \frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} BR_2(z,\varpi)(\alpha)\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha,\end{aligned}$$

which together with (7.4) yields

$$\begin{aligned}M_5 + M_6 &\le \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2\right) \\ &\quad - \frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} BR(z,\varpi)(\alpha)\cdot\partial_\alpha^\perp z(\alpha)\,\partial_\alpha^3 z_1(\alpha)\Lambda(\partial_\alpha^3 z_1)(\alpha)d\alpha.\end{aligned}$$

With $M_7$ and $M_8$ we use the equality $\partial_\alpha z_1(\alpha)\partial_\alpha^3 z_1(\alpha) = -\partial_\alpha z_2(\alpha)\partial_\alpha^3 z_2(\alpha) - |\partial_\alpha^2 z(\alpha)|^2$. Then operating similarly as we did with $M_5$ and $M_6$, we get

$$\begin{aligned}M_7 + M_8 &\le \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2\right) \\ &\quad - \frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} BR(z,\varpi)(\alpha)\cdot\partial_\alpha^\perp z(\alpha)\,\partial_\alpha^3 z_2(\alpha)\Lambda(\partial_\alpha^3 z_2)(\alpha)d\alpha.\end{aligned}$$

The addition of both inequalities produces

$$\begin{aligned}L_5 &\le \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2 + \|z\|_{H^3}^2\right) \\ &\quad - \frac{A_\mu}{2\pi A(t)}\int_{\mathbb{T}} BR(z,\varpi)(\alpha)\cdot\partial_\alpha^\perp z(\alpha)\,\partial_\alpha^3 z(\alpha)\cdot\Lambda(\partial_\alpha^3 z)(\alpha)d\alpha,\end{aligned}$$

and all the previous discussion shows that $I_5$ satisfies estimates identical to $L_5$.

7.4. *Estimates on* $\partial_\alpha^3(c(\alpha,t)\partial_\alpha z(\alpha,t))$. In the evolution of the $L^2$ norm of $\partial_\alpha^3 z(\alpha)$ it remains to control the term

$$I_2 = \int_{\mathbb{T}} \partial_\alpha^3 z(\alpha)\cdot\partial_\alpha^3\Big(c(\alpha)\partial_\alpha z(\alpha)\Big)d\alpha.$$

Let us recall the formula

$$\begin{aligned}c(\alpha,t) &= \frac{\alpha+\pi}{2\pi A(t)}\int_{\mathbb{T}}\partial_\beta z(\beta,t)\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta \\ &\quad - \frac{1}{A(t)}\int_{-\pi}^{\alpha}\partial_\beta z(\beta,t)\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta.\end{aligned}\tag{7.5}$$

We take $I_2 = J_1 + J_2 + J_3 + J_4$, where

$$J_1 = \int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \cdot \partial_\alpha^4 z(\alpha)\, c(\alpha) d\alpha, \qquad J_2 = 3\int_{\mathbb{T}} |\partial_\alpha^3 z(\alpha)|^2 \partial_\alpha c(\alpha) d\alpha,$$
$$J_3 = 3\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \cdot \partial_\alpha^2 z(\alpha)\, \partial_\alpha^2 c(\alpha) d\alpha, \qquad J_4 = \int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \cdot \partial_\alpha z(\alpha)\, \partial_\alpha^3 c(\alpha) d\alpha.$$

An integration by parts in $J_1$ yields

$$\begin{aligned} J_1 = -\frac{1}{2}\int_{\mathbb{T}} |\partial_\alpha^3 z(\alpha)|^2 \partial_\alpha c(\alpha) d\alpha &\leq \|\partial_\alpha c\|_{L^\infty} \|\partial_\alpha^3 z\|_{L^2}^2 \\ &\leq 2\|\mathcal{F}(z)\|_{L^\infty}^{1/2} \|\partial_\alpha BR(z,\varpi)\|_{L^\infty} \|\partial_\alpha^3 z\|_{L^2}^2, \end{aligned}$$

and the estimate for $\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}$ obtained before gives us

$$J_1 \leq \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t)\right).$$

The term $J_2$ satisfies that $J_2 = -6J_1$; therefore,

$$J_2 \leq \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t)\right).$$

Next we split $J_3 = K_1 + K_2$, where

$$K_1 = -3\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \cdot \partial_\alpha^2 z(\alpha) \frac{\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \cdot \partial_\alpha BR(z,\varpi) d\alpha,$$

$$K_2 = -3\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \cdot \partial_\alpha^2 z(\alpha) \frac{\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2} \cdot \partial_\alpha^2 BR(z,\varpi) d\alpha.$$

We have

$$\begin{aligned} K_1 &\leq 3\|\mathcal{F}(z)\|_{L^\infty} \|z\|_{C^2}^2 \|\partial_\alpha BR(z,\varpi)\|_{L^\infty} \|\partial_\alpha^3 z\|_{L^2} \\ &\leq \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t)\right), \\ K_2 &\leq \|\mathcal{F}(z)\|_{L^\infty}^{1/2} \|z\|_{C^2} \|\partial_\alpha^3 z\|_{L^2} \|\partial_\alpha^2 BR(z,\varpi)\|_{L^2}, \end{aligned}$$

and therefore

$$I_3 \leq \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t)\right).$$

The equality $\partial_\alpha^3 z(\alpha) \cdot \partial_\alpha z(\alpha) = -|\partial_\alpha^2 z(\alpha)|^2$ yields

$$I_4 = -\int_{\mathbb{T}} |\partial_\alpha^2 z(\alpha)|^2 \partial_\alpha^3 c(\alpha) d\alpha = 2\int_{\mathbb{T}} \partial_\alpha^3 z(\alpha) \cdot \partial_\alpha^2 z(\alpha) \partial_\alpha^2 c(\alpha) d\alpha = \frac{2}{3} I_3,$$

and finally

$$I_4 \leq \exp C\left(\|\mathcal{F}(z)\|_{L^\infty}^2(t) + \|z\|_{H^3}^2(t)\right).$$

## 8. **The arc-chord condition**

In this section we analyze the evolution of the quantity $\|\mathcal{F}(z)\|_{L^\infty}(t)$, which gives the local control of the arc-chord condition.

LEMMA 8.1. *The following estimate holds*:

$$\frac{d}{dt}\|\mathcal{F}(z)\|^2_{L^\infty}(t) \le \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right). \tag{8.1}$$

*Proof.* Let us take $p > 1$. It follows that

$$\begin{aligned}\frac{d}{dt}\|\mathcal{F}(z)\|^p_{L^p}(t) &= \frac{d}{dt}\int_{\mathbb{T}}\int_{\mathbb{T}}\left(\frac{|\beta|^2/4}{|V(\alpha,\alpha-\beta,t)|^2}\right)^p d\beta d\alpha\\ &= I_1 + I_2 + I_3,\end{aligned}$$

where

$$I_1 = -p\int_{\mathbb{T}}\int_{\mathbb{T}}(|\beta|/2)^{2p}\frac{V(\alpha,\alpha-\beta,t)\cdot(z_t(\alpha,t)-z_t(\alpha-\beta,t))}{|V(\alpha,\alpha-\beta,t)|^{2p+2}}d\beta d\alpha,$$

$$I_2 = -p\int_{\mathbb{T}}\int_{\mathbb{T}}(|\beta|/2)^{2p}\frac{V_1^3(\alpha,\alpha-\beta,t)(z_{1\,t}(\alpha,t)-z_{1\,t}(\alpha-\beta,t))}{|V(\alpha,\alpha-\beta,t)|^{2p+2}}d\beta d\alpha,$$

and

$$I_3 = p\int_{\mathbb{T}}\int_{\mathbb{T}}(|\beta|/2)^{2p}\frac{V_2^3(\alpha,\alpha-\beta,t)(z_{2\,t}(\alpha,t)-z_{2\,t}(\alpha-\beta,t))}{|V(\alpha,\alpha-\beta,t)|^{2p+2}}d\beta d\alpha.$$

For $I_1$, we have

$$I_1 \le p\int_{\mathbb{T}}\int_{\mathbb{T}}\left(\frac{|\beta|/2}{|V(\alpha,\alpha-\beta,t)|}\right)^{2p+1}\frac{|z_t(\alpha,t)-z_t(\alpha-\beta,t)|}{|\beta|}d\beta d\alpha.$$

Let us consider

$$\begin{aligned}z_t(\alpha)-z_t(\alpha-\beta) &= (BR(z,\varpi)(\alpha)-BR(z,\varpi)(\alpha-\beta))\\ &\quad + \big(c(\alpha)-c(\alpha-\beta)\big)\partial_\alpha z(\alpha)\\ &\quad + c(\alpha-\beta)\big(\partial_\alpha z(\alpha)-\partial_\alpha z(\alpha-\beta)\big)\\ &= J_1 + J_2 + J_3.\end{aligned}$$

Then for $J_1$ we get $|J_1| \le \|\partial_\alpha BR(z,\varpi)\|_{L^\infty}|\beta|$, and the estimate (6.2) gives

$$|J_1| \le \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right)|\beta|.$$

Using the definition of $c(\alpha)$ we easily obtain that

$$|c(\alpha)-c(\alpha-\beta)| \le \frac{\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}}{A(t)^{1/2}}|\beta|,$$

and again using (6.2) we get

$$|J_2| \le \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right)|\beta|.$$

For $J_3$ we have $|J_3| \leq \|c\|_{L^\infty}\|z\|_{C^2}|\beta|$; that is,

$$|J_3| \leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right)|\beta|.$$

Those estimates obtained for $J_i$ allow us to write

$$\begin{aligned}I_1 &\leq p\exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right)\int_{\mathbb{T}}\int_{\mathbb{T}}\Big(\frac{|\beta|/2}{|V(\alpha,\alpha-\beta,t)|}\Big)^{2p+1}d\beta d\alpha\\ &\leq p\|\mathcal{F}(z)\|^{1/2}_{L^\infty}(t)\left(\exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right)\right)\|\mathcal{F}(z)\|^p_{L^p}(t).\end{aligned}$$

The Hölder inequality implies

$$\begin{aligned}|I_2| &\leq pC\|z_{1\,t}\|_{L^\infty}\int_{\mathbb{T}}\int_{\mathbb{T}}\left(\frac{|\beta|/2}{|V(\alpha,\alpha-\beta,t)|}\right)^{2p-1}d\beta d\alpha\\ &\leq pC\|z_{1\,t}\|_{L^\infty}(1 + \|\mathcal{F}(z)\|^p_{L^p})\\ &\leq p\left(\exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right)\right)\|\mathcal{F}(z)\|^p_{L^p}.\end{aligned}$$

Since $|V_2(\alpha,\alpha-\beta)| \leq 1$, we have

$$\begin{aligned}|I_3| &\leq 2p\|z_{2\,t}\|_{L^\infty}\int_{\mathbb{T}}\int_{\mathbb{T}}\left(\frac{|\beta|/2}{|V(\alpha,\alpha-\beta,t)|}\right)^{2p}d\beta d\alpha\\ &\leq 2p\left(\exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right)\right)\|\mathcal{F}(z)\|^p_{L^p}.\end{aligned}$$

Those estimates for $I_1$,$I_2$, and $I_3$ give us

$$\frac{d}{dt}\|\mathcal{F}(z)\|^p_{L^p}(t) \leq p\left(\exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right)\right)\|\mathcal{F}(z)\|^p_{L^p}(t);$$

therefore,

$$\frac{d}{dt}\|\mathcal{F}(z)\|_{L^p}(t) \leq \left(\exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|z\|^2_{H^3}(t)\right)\right)\|\mathcal{F}(z)\|_{L^p}(t).$$

After an integration in the time variable $t$ we get

$$\|\mathcal{F}(z)\|_{L^p}(t+h) \leq \|\mathcal{F}(z)\|_{L^p}(t)\exp\left(\int_t^{t+h}e^{C(\|\mathcal{F}(z)\|^2_{L^\infty}(s)+\|z\|^2_{H^3}(s))}ds\right),$$

and letting $p \to \infty$ we obtain

$$\|\mathcal{F}(z)\|_{L^\infty}(t+h) \leq \|\mathcal{F}(z)\|_{L^\infty}(t)\exp\left(\int_t^{t+h}e^{C(\|\mathcal{F}(z)\|^2_{L^\infty}(s)+\|z\|^2_{H^3}(s))}ds\right).$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\|\mathcal{F}(z)\|_{L^\infty}(t) &= \lim_{h\to 0}(\|\mathcal{F}(z)\|_{L^\infty}(t+h) - \|\mathcal{F}(z)\|_{L^\infty}(t))h^{-1}\\ &\leq \|\mathcal{F}(z)\|_{L^\infty}(t)\lim_{h\to 0}\left(\exp\left(\int_t^{t+h}e^{C(\|\mathcal{F}(z)\|^2_{L^\infty}(s)+\|z\|^2_{H^3}(s))}ds\right) - 1\right)h^{-1}\\ &\leq \|\mathcal{F}(z)\|_{L^\infty}(t)e^{C(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t))}.\end{aligned}$$

With this we finish the proof of Lemma 8.1. □

## 9. **The evolution of the minimum of** $\sigma(\alpha, t)$

In this section we get an *a priori* estimate for the evolution of the minimum of the difference of the gradients of the pressure in the normal direction to the interface. This quantity is given by

$$\sigma(\alpha,t)=\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)+\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_1(\alpha,t). \tag{9.1}$$

LEMMA 9.1. *Let us consider a solution* $z(\alpha,t)$ *of the system with* $z(\alpha,t)\in C^1([0,\tau];H^3)$, *and*

$$m(t)=\min_{\alpha\in\mathbb{T}}\sigma(\alpha,t).$$

*Then*

$$m(t)\geq m(0)-\int_0^t \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(s)+\|z\|^2_{H^3}(s)\right)ds.$$

*Proof.* Suppose that $z(\alpha,t)\in C^1([0,\tau];H^3)$ is a solution of the system. Then the result obtained in the preceding sections together with Sobolev inequalities shows that $\sigma(\alpha,t)\in C^1([0,\tau]\times\mathbb{T})$. Therefore we may consider $\alpha_t\in\mathbb{T}$ such that

$$m(t)=\min_{\alpha\in\mathbb{T}}\sigma(\alpha,t)=\sigma(\alpha_t,t),$$

which is a Lipschitz function differentiable almost everywhere. With an analogous argument to the one used in [7] and [10], we may calculate the derivative of $m(t)$, to obtain

$$m'(\alpha_t,t)=\sigma_t(\alpha_t,t).$$

The identity (9.1) yields

$$\begin{aligned}\sigma_t(\alpha,t)&=\frac{\mu^2-\mu^1}{\kappa}\partial_t\big(BR(z,\varpi)\big)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t)\\&\quad+\left(\frac{\mu^2-\mu^1}{k}\,BR(z,\varpi)(\alpha)\cdot\partial_\alpha^\perp z_t(\alpha,t)+\mathrm{g}(\rho^2-\rho^1)\partial_\alpha z_{1\,t}(\alpha,t)\right)\\&=I_1+I_2,\end{aligned}$$

and we have

$$|I_2|\leq C\big(\|BR(z,\varpi)\|_{L^\infty}+1\big)\|\partial_\alpha z_t\|_{L^\infty}.$$

We can easily estimate $\|BR(z,\varpi)\|_{L^\infty}$, obtaining

$$|I_2|\leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}+\|z\|^2_{H^3}\right)\|\partial_\alpha z_t\|_{L^\infty}.$$

Next we use equation (1.4) to get

$$\begin{aligned}\|\partial_\alpha z_t\|_{L^\infty}&\leq\big(\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}+\|\partial_\alpha c\|_{L^\infty}\|\partial_\alpha z\|_{L^\infty}+\|c\|_{L^\infty}\|\partial^2_\alpha z\|_{L^\infty}\big)\\&\leq C\|\partial_\alpha BR\left(z,\varpi)\|_{L^\infty}(1+\|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|z\|_{C^2}\right).\end{aligned}$$

With the bound obtained before for $\|\partial_\alpha BR(z,\varpi)\|_{L^\infty}$ (6.2), we have

$$|I_2| \leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^3}\right).$$

Let us write $BR(z,\varpi)(\alpha,t) = P_1(\alpha,t) + P_2(\alpha,t) + P_3(\alpha,t)$, where the $P_j$ were defined in (6.3). We have

$$|\partial_t P_2(\alpha)| + |\partial_t P_3(\alpha)| \leq C\Big(\|\varpi_t\|_{L^2} + \|\varpi\|_{L^2}\|z_t\|_{L^\infty}\Big).$$

The norm $\|z_t\|_{L^\infty}$ is bounded by (1.4) and the adequate estimates for $\|\varpi_t\|_{L^2}$ which will be introduced later. In $\partial_t P_1$ there are terms of lower order which can be estimated as $\partial_t P_2$ and $\partial_t P_3$, but the most singular ones are given by

$$J_1 = \frac{1}{4\pi} PV \int_{\mathbb{T}} \varpi_t(\alpha-\beta)\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$
$$J_2 = \frac{1}{8\pi} PV \int_{\mathbb{T}} \varpi(\alpha-\beta)\frac{z_t(\alpha)-z_t(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$
$$J_3 = -\frac{1}{4\pi} PV \int_{\mathbb{T}} \varpi(\alpha-\beta)\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^4}\Big(V(\alpha,\alpha-\beta)\cdot(z_t(\alpha)-z_t(\alpha-\beta))\Big)d\beta.$$

Let us now split $J_1$ in a similar way as before, to obtain

$$J_1 = \frac{1}{4\pi} PV \int_{\mathbb{T}} \varpi_t(\alpha-\beta)\left(\frac{V^\perp(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{\partial^\perp_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^2\tan(\beta/2)}\right)d\beta$$
$$+ \frac{\partial^\perp_\alpha z(\alpha)}{2|\partial_\alpha z(\alpha)|^2}H(\varpi_t)(\alpha)$$

and

$$|J_1| \leq C\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|\varpi_t\|_{L^2} + \|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|\varpi_t\|_{C^\delta}.$$

Next we divide $J_2 = K_1 + K_2 + K_3$, where

$$K_1 = \frac{1}{8\pi}\int_{\mathbb{T}}\big(\varpi(\alpha-\beta)-\varpi(\alpha)\big)\frac{z_t(\alpha)-z_t(\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$
$$K_2 = \frac{\varpi(\alpha)}{8\pi}\int_{\mathbb{T}}\big(z_t(\alpha)-z_t\big(\alpha-\beta\big)\big)\left(\frac{1}{|V(\alpha,\alpha-\beta)|^2} - \frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\right)d\beta,$$
$$K_3 = \frac{1}{8\pi}\frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2}\int_{\mathbb{T}}\big(z_t(\alpha)-z_t\big(\alpha-\beta\big)\big)\left(\frac{4}{|\beta|^2} - \frac{1}{\sin^2(\beta/2)}\right)d\beta$$
$$+ \frac{1}{2}\frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2}\Lambda(z_t)(\alpha).$$

The identity

$$z_t(\alpha) - z_t(\alpha-\beta) = \beta\int_0^1 \partial_\alpha z_t\Big(\alpha + (s-1)\beta\Big)ds$$

gives us

$$|K_1| + |K_2| \leq \|\varpi\|_{C^1}\|\mathcal{F}(z)\|^k\|z\|^k_{C^2}\|\partial_\alpha z_t\|_{L^\infty}.$$

For $K_3$ we have

$$|K_3| \leq C\|\varpi\|_{L^\infty}\|\mathcal{F}(z)\|_{L^\infty}\|z_t\|_{C^{1,\delta}}.$$

In order to control $\|\varpi_t\|_{C^\delta}$, we will use the inequality

$$\|f\|_{C^\delta} \leq C(\|f\|_{L^2} + \|f\|_{\overline{C}^\delta})$$

with

$$\|f\|_{\overline{C}^\delta} = \sup_{\alpha\neq\beta}\frac{|f(\alpha)-f(\beta)|}{|\alpha-\beta|^\delta}.$$

Let us now take the time derivative of the identity (5.1). We get

$$\varpi_t(\alpha) + A_\mu T(\varpi_t)(\alpha) = -\frac{A_\mu}{2\pi}R(\alpha) - 2\kappa g\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\alpha z_{2\,t}(\alpha),$$

which yields

$$\|\varpi_t\|_{H^{\frac{1}{2}}} \leq C\|(I+A_\mu T)^{-1}\|_{H^{\frac{1}{2}}}(\|R\|_{H^{\frac{1}{2}}} + \|\partial_\alpha z_t\|_{H^{\frac{1}{2}}}),$$

and since we control $\|(I+A_\mu T)^{-1}\|_{H^{\frac{1}{2}}}$ it remains to estimate $\|R\|_{H^{\frac{1}{2}}}$.

Instead we will estimate $\|R\|_{H^1}$, and to do that we consider the splitting $R = S_1 + S_2 + S_3$, where

$$S_1(\alpha) = \int_{\mathbb{T}}\varpi(\alpha-\beta)\partial_t\left(\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2}\right)d\beta,$$

$$S_2(\alpha) = -\int_{\mathbb{T}}\varpi(\alpha-\beta)\partial_t\left(V_2^2(\alpha,\alpha-\beta)\frac{V^\perp(\alpha,\alpha-\beta)\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2}\right)d\beta,$$

$$S_3(\alpha) = -\int_{\mathbb{T}}\varpi(\alpha-\beta)\partial_t\Big(V_2(\alpha,\alpha-\beta)\partial_\alpha z_1(\alpha)\Big)d\beta.$$

The terms $S_2(\alpha)$ and $S_3(\alpha)$ are controlled as follows:

$$|S_2(\alpha)| + |S_3(\alpha)| \leq C\|z_t\|_{C^1}\|\varpi\|_{L^2}.$$

For $S_1$ we split $S_1(\alpha) = Q_1(\alpha)+Q_2(\alpha)+Q_3(\alpha)+Q_4(\alpha)+Q_5(\alpha)$, where

$$Q_1(\alpha) = \int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp\cdot\partial_\alpha z_t(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$Q_2(\alpha) = \frac{1}{2}\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{(z_t(\alpha)-z_t(\alpha-\beta)-\partial_\alpha z_t(\alpha)\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$Q_3(\alpha) = \frac{1}{2}\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}B(\alpha,\alpha-\beta)d\beta,$$

with

$$B(\alpha,\alpha-\beta) = V(\alpha,\alpha-\beta)\cdot\Big(z_t(\alpha)-z_t(\alpha-\beta)\Big), \tag{9.2}$$

$$Q_4(\alpha) = \frac{1}{2}\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{C(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$C(\alpha,\alpha-\beta)=V_1^2(\alpha,\alpha-\beta)\Big(z_{1\,t}(\alpha)-z_{1\,t}(\alpha-\beta)\Big)\partial_\alpha z_2(\alpha)$$
$$+V_2^2(\alpha,\alpha-\beta)\Big(z_{2\,t}(\alpha)-z_{2\,t}(\alpha-\beta)\Big)\partial_\alpha z_1(\alpha),$$

$$Q_5(\alpha)=-\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}D(\alpha,\alpha-\beta)d\beta,$$

$$D(\alpha,\alpha-\beta)=V_1^3(\alpha,\alpha-\beta)(z_{1\,t}(\alpha)-z_{1\,t}(\alpha-\beta))-V_2^3(\alpha,\alpha-\beta)(z_{2\,t}(\alpha)-z_{2\,t}(\alpha-\beta)).$$

We have $|Q_4(\alpha)|\leq C\|z\|_{C^1}\|\varpi\|_{L^2}\|z_t\|_{L^\infty}$. In a similar way this estimate follows for $Q_5$:

$$|Q_5(\alpha)|\leq C\left(\|z\|_{C^1}+\|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|z\|^2_{C^1}\right)\|\varpi\|_{L^2}\|z_t\|_{L^\infty}.$$

For $Q_1$ we proceed as before to obtain

$$|Q_1(\alpha)|\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|\varpi\|_{L^2}\|z_t\|_{C^1}.$$

The inequality

$$|z_t(\alpha)-z_t(\alpha-\beta)-\partial_\alpha z_t(\alpha)\beta|\leq\|z_t\|_{C^{1,\delta}}|\beta|^{1+\delta}$$

gives

$$|Q_2(\alpha)|\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^1}\|\varpi\|_{L^\infty}\|z_t\|_{C^{1,\delta}},$$

and

$$|z_t(\alpha)-z_t(\alpha-\beta)|\leq\|z_t\|_{C^1}|\beta|$$

yields

$$|Q_3(\alpha)|\leq\|\mathcal{F}(z)\|^{3/2}_{L^\infty}\|z\|^2_{C^2}\|\varpi\|_{L^2}\|z_t\|_{C^1}.$$

Finally we have

$$|R_1(\alpha)|\leq\|\mathcal{F}(z)\|^{3/2}_{L^\infty}\|z\|^2_{C^2}\|\varpi\|_{H^1}\|z_t\|_{C^{1,\delta}}.$$

Using (5.1) we obtain

$$\|\varpi_t\|_{\overline{C}^\delta}\leq C\left(\|T(\varpi_t)\|_{\overline{C}^\delta}+\|R\|_{\overline{C}^\delta}+\|\partial_\alpha z_t\|_{\overline{C}^\delta}\right).$$

For $\delta\leq 1/2$ we have

$$\|T(\varpi_t)\|_{\overline{C}^\delta}\leq\|T(\varpi_t)\|_{H^1}\leq 2\|\partial_\alpha T(\varpi_t)\|_{L^2}\leq\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^4_{C^{2,\delta}}\|w_t\|_{L^2}.$$

Now to estimate $\|R\|_{\overline{C}^\delta}\leq\|R\|_{H^1}$ we consider $\|\partial_\alpha R\|_{L^2}$. The most singular terms for this quantity are those with two derivatives in $\alpha$ and one in time, or with one derivative in $\alpha$, one in time and a principal value. Let us write

$$Q_6(\alpha)=\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp\cdot\partial^2_\alpha z_t(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$Q_7(\alpha)=\frac{1}{2}\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta)-\partial^2_\alpha z_t(\alpha)\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$Q_8(\alpha)=\frac{1}{2}\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}D(\alpha,\alpha-\beta)d\beta,$$

with

$$D(\alpha,\alpha-\beta)=V(\alpha,\alpha-\beta)\cdot(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta)). \tag{9.3}$$

We have

$$|Q_6(\alpha)|\leq\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|w\|_{L^2}|\partial^2_\alpha z_t(\alpha)|$$

and

$$|Q_8(\alpha)|\leq\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|w\|_{L^\infty}\|z_t(\alpha)\|_{C^{1,\delta}}.$$

Let us split $Q_7(\alpha)=J_4+J_5$, where

$$J_4=\frac{1}{2}PV\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta$$

and

$$J_5=-\frac{1}{2}\big(\partial^2_\alpha z_t(\alpha)\big)^\perp\cdot\partial_\alpha z(\alpha)PV\int_{\mathbb{T}}\varpi(\alpha-\beta)\frac{\beta}{|V(\alpha,\alpha-\beta)|^2}d\beta.$$

For $J_5$ we have $|J_5|\leq\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|w\|_{H^1}|\partial^2_\alpha z_t(\alpha)|$. Next we divide $J_4=K_4+K_5+K_6+K_7$, where

$$K_4=\frac{1}{2}\int_{\mathbb{T}}\big(\varpi(\alpha-\beta)-\varpi(\alpha)\big)\frac{(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^2}d\beta,$$

$$K_5=\frac{\varpi(\alpha)}{2}\int_{\mathbb{T}}\big(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta)\big)^\perp \cdot\partial_\alpha z(\alpha)\left(\frac{1}{|V(\alpha,\alpha-\beta)|^2}-\frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\right)d\beta,$$

$$K_6=\frac{2\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2}\int_{\mathbb{T}}\big(\partial_\alpha z_t(\alpha)-\partial_\alpha z_t(\alpha-\beta)\big)^\perp\cdot\partial_\alpha z(\alpha)\left(\frac{1}{|\beta|^2}-\frac{1}{4\sin^2(\beta/2)}\right)d\beta,$$

$$K_7=2\pi\frac{\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^2}\big(\Lambda(\partial_\alpha z_t)(\alpha)\big)^\perp\cdot\partial_\alpha z(\alpha).$$

We have

$$|K_4|+|K_5|\leq C\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\|w\|_{H^1}\|z_t\|_{C^{1,\delta}},$$
$$|K_6|\leq\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|w\|_{H^1}\|z_t\|_{C^1}$$

and

$$|K_7|\leq\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}\|w\|_{H^1}|\Lambda(\partial_\alpha z_t)(\alpha)|.$$

Finally let us observe that $\|z_t\|_{C^{1,\delta}}\leq\|z_t\|_{H^2}$, which provides us the control of $\|\partial^2_\alpha z_t\|_{L^2}$. We consider now the terms of $\partial^2_\alpha z_t(\alpha)$ given by

$$I_3=\partial^2_\alpha BR(z,\varpi)(\alpha),\quad I_4=\partial^2_\alpha\big(c(\alpha)\partial_\alpha z(\alpha)\big).$$

Easily we get

$$|I_4|\leq\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\left(1+|\partial^3_\alpha z(\alpha)|+|\partial^2_\alpha BR(z,\varpi)(\alpha)|\right)$$

which yields

$$\|I_4\|_{L^2} \leq \|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^2}\left(1+\|z\|_{H^3}+\|\partial^2_\alpha BR(z,\varpi)\|_{L^2}\right),$$

so that we can control $\|\partial^2_\alpha BR(z,\varpi)\|_{L^2}$ as in (6.6), and finish the estimate of $I_3$.

The upper bound

$$|\sigma_t(\alpha,t)| \leq \exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t)\right)$$

gives us

$$m'(t) \geq -\exp C\left(\|\mathcal{F}(z)\|^2_{L^\infty}(t)+\|z\|^2_{H^3}(t)\right)$$

for almost every $t$, and a further integration yields Lemma 8.1:

$$m(t) \geq m(0) - \int_0^t \exp(C|||z|||^2)ds.$$

## 10. **Regularization and approximation**

Our next step is to use the *a priori* estimates to get local-existence. For that purpose we introduce a regularized evolution equation having local-existence independently of the sign condition on $\sigma(\alpha,t)$ at $t=0$. But for $\sigma(\alpha,0)>0$, we find a time of existence for the Muskat problem uniformly in the regularization, allowing us to take the limit.

Let $z^\varepsilon(\alpha,t)$ be a solution of the following system:

$$\begin{aligned} z^{\varepsilon,\delta}_t(\alpha,t) &= BR^\delta(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})(\alpha,t)+c^{\varepsilon,\delta}(\alpha,t)\partial_\alpha z^{\varepsilon,\delta}(\alpha,t),\\ z^{\varepsilon,\delta}(\alpha,0) &= z_0(\alpha), \end{aligned}$$

where

$$\begin{aligned} &BR^\delta(z,\varpi)(\alpha,t)\\ &= \Bigg(-\frac{1}{4\pi}\int_{\mathbb{T}}\varpi(\beta,t)\frac{\tanh(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})(1+\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2}))}{\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2})+\tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})+\delta}d\beta,\\ &\qquad \frac{1}{4\pi}\int_{\mathbb{T}}\varpi(\beta,t)\frac{\tan(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2})(1-\tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2}))}{\tan^2(\frac{z_1(\alpha,t)-z_1(\beta,t)}{2})+\tanh^2(\frac{z_2(\alpha,t)-z_2(\beta,t)}{2})+\delta}d\beta\Bigg), \end{aligned}$$

$$\begin{aligned} \varpi^{\varepsilon,\delta}(\alpha,t) &= -A_\mu\phi_\varepsilon*\phi_\varepsilon*\left(2BR(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})\cdot\partial_\alpha z^{\varepsilon,\delta}\right)(\alpha)\\ &\quad -2\kappa \mathrm{g}\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\phi_\varepsilon*\phi_\varepsilon*(\partial_\alpha z^{\varepsilon,\delta}_2)(\alpha), \end{aligned}$$

$$\begin{aligned} c^{\varepsilon,\delta}(\alpha,t) &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\alpha z^{\varepsilon,\delta}(\alpha,t)}{|\partial_\alpha z^{\varepsilon,\delta}(\alpha,t)|^2}\cdot\partial_\alpha BR^\delta(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})(\alpha,t)d\alpha\\ &\quad -\int_{-\pi}^{\alpha}\frac{\partial_\alpha z^{\varepsilon,\delta}(\beta,t)}{|\partial_\alpha z^{\varepsilon,\delta}(\alpha,t)|^2}\cdot\partial_\beta BR^\delta(z^{\varepsilon,\delta},\varpi^{\varepsilon,\delta})(\beta,t)d\beta, \end{aligned}$$

$$\phi \in C_c^\infty(\mathbb{R}), \quad \phi(\alpha) \geq 0, \quad \phi(-\alpha) = \phi(\alpha),$$
$$\int_{\mathbb{R}} \phi(\alpha) d\alpha = 1, \quad \phi_\varepsilon(\alpha) = \phi(\alpha/\varepsilon)/\varepsilon$$

for $\varepsilon > 0$ and $\delta > 0$.

Then the operator $I + A_\mu \phi_\varepsilon * \phi_\varepsilon * T$ has a bounded inverse in $H^{\frac{1}{2}}$, for $\varepsilon$ small enough, with a norm bounded independently of $\varepsilon > 0$. For this system there is local-existence for initial data with $\mathcal{F}(z_0)(\alpha, \beta) < \infty$ even if $\sigma(\alpha, 0)$ does not have the proper sign (see [13]), so that there exists a time $\tau^{\varepsilon,\delta}$ and a solution of the system $z^{\varepsilon,\delta} \in C^1([0, \tau^{\varepsilon,\delta}], H^k)$ for $k \leq 3$, and as long as the solution exists, we have $|\partial_\alpha z^{\varepsilon,\delta}(\alpha, t)|^2 = A^{\varepsilon,\delta}(t)$. Taking advantage of this property and using that $\varpi^{\varepsilon,\delta}$ is regular, we obtain estimates which are independent of $\delta$. Now letting $\delta \to 0$, we get local-existence for the following system:

$$z_t^\varepsilon(\alpha, t) = BR(z^\varepsilon, \varpi^\varepsilon)(\alpha, t) + c^\varepsilon(\alpha, t)\partial_\alpha z^\varepsilon(\alpha, t),$$
$$z^\varepsilon(\alpha, 0) = z_0(\alpha),$$

where

$$\begin{aligned} c^\varepsilon(\alpha, t) = {}& \frac{\alpha + \pi}{2\pi} \int_{\mathbb{T}} \frac{\partial_\alpha z^\varepsilon(\alpha, t)}{|\partial_\alpha z^\varepsilon(\alpha, t)|^2} \cdot \partial_\alpha BR(z^\varepsilon, \varpi^\varepsilon)(\alpha, t) d\alpha \\ & - \int_{-\pi}^{\alpha} \frac{\partial_\alpha z^\varepsilon(\beta, t)}{|\partial_\alpha z^\varepsilon(\alpha, t)|^2} \cdot \partial_\beta BR(z^\varepsilon, \varpi^\varepsilon)(\beta, t) d\beta, \\ \varpi^\varepsilon(\alpha, t) = {}& -A_\mu \phi_\varepsilon * \phi_\varepsilon * (2BR(z^\varepsilon, \varpi^\varepsilon) \cdot \partial_\alpha z^\varepsilon)(\alpha) \\ & - 2\kappa \mathrm{g} \frac{\rho^2 - \rho^1}{\mu^2 + \mu^1} \phi_\varepsilon * \phi_\varepsilon * (\partial_\alpha z_2^\varepsilon)(\alpha). \end{aligned}$$

Next we shall show that for this system we have

(10.1)
$$\begin{aligned} \frac{d}{dt}\|z^\varepsilon\|_{H^k}^2(t) \leq {}& -\frac{\kappa}{2\pi(\mu_1 + \mu_2)} \int_{\mathbb{T}} \frac{\sigma^\varepsilon(\alpha, t)}{|\partial_\alpha z^\varepsilon(\alpha, t)|^2} \phi_\varepsilon * (\partial_\alpha^k z^\varepsilon)(\alpha, t) \\ & \cdot \Lambda\Big(\phi_\varepsilon * (\partial_\alpha^k z^\varepsilon)\Big)(\alpha, t) d\alpha + \exp C \left( \|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2(t) + \|z^\varepsilon\|_{H^k}^2(t) \right), \end{aligned}$$

where

$$\sigma^\varepsilon(\alpha, t) = \frac{\mu^2 - \mu^1}{\kappa} BR(z^\varepsilon, \varpi^\varepsilon)(\alpha, t) \cdot \partial_\alpha^\perp z^\varepsilon(\alpha, t) + \mathrm{g}(\rho^2 - \rho^1)\partial_\alpha z_1^\varepsilon(\alpha, t).$$

To do the task we have to repeat the arguments in our previous sections, with the exception of Section 7.3. (looking for $\sigma^\varepsilon(\alpha)$) where we proceed differently using the following well-known estimate for the commutator of the convolution:

$$\|\phi_\varepsilon * (gf) - g\phi_\varepsilon * (f)\|_{H^1} \leq C\|g\|_{C^1}\|f\|_{L^2}, \tag{10.2}$$

where the constant $C$ is independent of $\varepsilon$.

In the following we will present the details of the evolution of the $L^2$ norm of the third derivatives, being the case of the $k$-th derivative ($k > 3$) completely analogous. Furthermore, with regard to the different decompositions introduced in the previous sections, in the following we shall select only the more singular terms, showing for them the corresponding uniform estimates and leaving to the reader the remaining easier cases.

If we consider the term corresponding to $K_2$ in Section 7.3, we have

$$K_2^\varepsilon = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)} \int_{\mathbb{T}} \Lambda\Big(\partial_\alpha^3 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon\Big)(\alpha)\phi_\varepsilon * \phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)(\alpha) d\alpha,$$

which we write in the following manner:

$$K_2^\varepsilon = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)} \int_{\mathbb{T}} \Lambda\Big(\phi_\varepsilon * (\partial_\alpha^3 z^\varepsilon \cdot \partial_\alpha^\perp z^\varepsilon)\Big)(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)(\alpha) d\alpha.$$

Then we have $K_2^\varepsilon = L_1^\varepsilon + L_2^\varepsilon$, where

$$L_1^\varepsilon = \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)} \int_{\mathbb{T}} \Lambda\Big(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon \partial_\alpha z_2^\varepsilon)\Big)(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)(\alpha) d\alpha,$$

$$L_2^\varepsilon = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)} \int_{\mathbb{T}} \Lambda\Big(\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon \partial_\alpha z_1^\varepsilon)\Big)(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)(\alpha) d\alpha.$$

Next we write $L_1^\varepsilon = M_1^\varepsilon + M_2^\varepsilon + M_3^\varepsilon + M_4^\varepsilon$, where

$$\begin{aligned} M_1^\varepsilon = {} & \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)} \\ & \cdot \int_{\mathbb{T}} \Lambda\Big(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon \partial_\alpha z_2^\varepsilon) - \phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\partial_\alpha z_2^\varepsilon\Big)(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)(\alpha) d\alpha, \\ M_2^\varepsilon = {} & \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)} \int_{\mathbb{T}} \Big[\Lambda\Big(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\partial_\alpha z_2^\varepsilon\Big)(\alpha) \\ & - \Lambda\Big(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\Big)(\alpha)\partial_\alpha z_2^\varepsilon(\alpha)\Big]\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)(\alpha) d\alpha, \\ M_3^\varepsilon = {} & \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)} \int_{\mathbb{T}} \Lambda\Big(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\Big)(\alpha)\Big[\partial_\alpha z_2^\varepsilon(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)(\alpha) \\ & - \phi_\varepsilon * (\partial_\alpha z_2^\varepsilon \partial_\alpha^3 z_2^\varepsilon)(\alpha)\Big] d\alpha, \\ M_4^\varepsilon = {} & \frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)} \int_{\mathbb{T}} \Lambda\Big(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\Big)(\alpha)\phi_\varepsilon * (\partial_\alpha z_2^\varepsilon \partial_\alpha^3 z_2^\varepsilon)(\alpha) d\alpha. \end{aligned}$$

Using (10.2), we get

$$\begin{aligned} M_1^\varepsilon & \leq C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty} \left\|\Lambda\left(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon \partial_\alpha z_2^\varepsilon) - \phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\partial_\alpha z_2^\varepsilon\right)\right\|_{L^2} \|\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)\|_{L^2}^2 \\ & \leq C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty} \left\|\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon \partial_\alpha z_2^\varepsilon) - \phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\partial_\alpha z_2^\varepsilon\right\|_{H^1} \|\partial_\alpha^3 z_2^\varepsilon\|_{L^2}^2 \\ & \leq C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty} \|\partial_\alpha^3 z_1^\varepsilon\|_{L^2} \|\partial_\alpha z_2^\varepsilon\|_{C^1} \|\partial_\alpha^3 z_2^\varepsilon\|_{L^2}^2, \end{aligned}$$

and therefore

$$M_1^\varepsilon \leq \exp C\left(\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2 + \|z^\varepsilon\|_{H^3}^2\right).$$

With $M_2^\varepsilon$ we use the commutator estimate for the operator $\Lambda$ to obtain

$$\begin{aligned} M_2^\varepsilon &\leq C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\left\|\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\right\|_{L^2}\|\partial_\alpha z_2^\varepsilon\|_{C^{1,\delta}}\left\|\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)\right\|_{L^2} \\ &\leq \exp C\left(\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2 + \|z^\varepsilon\|_{H^3}^2\right). \end{aligned}$$

Regarding $M_3^\varepsilon$ we have

$$\begin{aligned} M_3^\varepsilon = &\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)} \\ &\cdot \int_{\mathbb{T}} \phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)(\alpha)\Lambda\left(\partial_\alpha z_2^\varepsilon \phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon) - \phi_\varepsilon * (\partial_\alpha z_2^\varepsilon \partial_\alpha^3 z_2^\varepsilon)\right)(\alpha)d\alpha, \end{aligned}$$

showing that it can be estimated as $M_1^\varepsilon$.

The identity

$$\partial_\alpha z_2^\varepsilon(\alpha)\partial_\alpha^3 z_2^\varepsilon(\alpha) = -\partial_\alpha z_1^\varepsilon(\alpha)\partial_\alpha^3 z_1^\varepsilon(\alpha) - |\partial_\alpha^2 z^\varepsilon(\alpha)|^2$$

allows us to write $M_4^\varepsilon$ as the sum of $N_1^\varepsilon$ and $N_2^\varepsilon$, where

$$N_1^\varepsilon = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \Lambda\Big(\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon\Big)(\alpha)\phi_\varepsilon * (|\partial_\alpha^2 z^\varepsilon|^2)(\alpha)d\alpha$$

and

$$N_2^\varepsilon = -\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \phi_\varepsilon * (\partial_\alpha z_1^\varepsilon \partial_\alpha^3 z_1^\varepsilon)(\alpha)\Lambda\Big(\phi_\varepsilon * \partial_\alpha^3 z_1^\varepsilon\Big)(\alpha)d\alpha.$$

Then an integration by parts shows that

$$N_1^\varepsilon \leq C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|z^\varepsilon\|_{C^2}\|\partial_\alpha^3 z^\varepsilon\|_{L^2}^2 \leq \exp C\left(\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2 + \|z^\varepsilon\|_{H^3}^2\right).$$

Using again the identity (10.2) in $N_2^\varepsilon$, we finally obtain

$$\begin{aligned} L_1^\varepsilon \leq &-\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \partial_\alpha z_1^\varepsilon(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)(\alpha)\Lambda\left(\phi_\varepsilon * (\partial_\alpha^3 z_1^\varepsilon)\right)(\alpha)d\alpha \\ &+ \exp C\left(\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2 + \|z^\varepsilon\|_{H^3}^2\right). \end{aligned}$$

In a similar way for $L_2^\varepsilon$ we get

$$\begin{aligned} L_2^\varepsilon \leq &-\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A^\varepsilon(t)}\int_{\mathbb{T}} \partial_\alpha z_1^\varepsilon(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)(\alpha)\Lambda\left(\phi_\varepsilon * (\partial_\alpha^3 z_2^\varepsilon)\right)(\alpha)d\alpha \\ &+ \exp C\left(\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2 + \|z^\varepsilon\|_{H^3}^2\right), \end{aligned}$$

giving us

$$\begin{aligned} K_2^\varepsilon \leq &-\frac{\kappa \mathrm{g}(\rho^2-\rho^1)}{2\pi(\mu^2+\mu^1)A(t)}\int_{\mathbb{T}} \partial_\alpha z_1^\varepsilon(\alpha)\phi_\varepsilon * (\partial_\alpha^3 z)(\alpha)\cdot\Lambda\left(\phi_\varepsilon * (\partial_\alpha^3 z)\right)(\alpha)d\alpha \\ &+ \exp C\left(\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}^2 + \|z^\varepsilon\|_{H^3}^2\right). \end{aligned}$$

The formula for $\sigma^\varepsilon(\alpha,t)$ begins to appear in the nonintegrable terms. Using a similar method for the rest of the nonintegrable terms we obtain inequality (10.1) for $k=3$.

The next step is to integrate the system during a time $\tau$ independent of $\varepsilon$. First let us observe that if $z_0(\alpha)\in H^k$, then we have the solution $z^\varepsilon\in C^1([0,\tau^\varepsilon];H^k)$, and if initially $\sigma(\alpha,0)>0$, then there is a time depending on $\varepsilon$, denoted by $\tau^\varepsilon$ again, in which $\sigma^\varepsilon(\alpha,t)>0$. Now, for $t\le\tau^\varepsilon$ we have (10.1), and then we use the following pointwise inequality (see [6]):

$$f(\alpha)\Lambda f(\alpha)-\frac{1}{2}\Lambda(f^2)(\alpha)\ge 0$$

to obtain

$$\frac{d}{dt}\|z^\varepsilon\|^2_{H^k}(t)\le I+\exp C\left(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)+\|z^\varepsilon\|^2_{H^k}(t)\right),$$

where

$$I=-\frac{\kappa}{2\pi(\mu_1+\mu_2)A^\varepsilon(t)}\int_{\mathbb{T}}\sigma^\varepsilon(\alpha,t)\frac{1}{2}\Lambda\left(\left|\phi_\varepsilon*(\partial^k_\alpha z^\varepsilon)\right|^2\right)(\alpha,t)d\alpha.$$

We have

$$\begin{aligned}\|\Lambda(\sigma^\varepsilon)\|_{L^\infty}(t)&\le C\|\sigma^\varepsilon\|_{H^2}(t)\\&\le C\left(\|BR(z^\varepsilon,\varpi^\varepsilon)\|_{L^2}(t)+\|\partial^2_\alpha BR(z^\varepsilon,\varpi^\varepsilon)\|_{L^2}(t)+1\right)\|z\|_{H^3}(t).\end{aligned}$$

Writing

$$I=-\frac{\kappa}{2\pi(\mu_1+\mu_2)A^\varepsilon(t)}\int_{\mathbb{T}}\Lambda(\sigma^\varepsilon)(\alpha,t)\frac{1}{2}\left|\phi_\varepsilon*(\partial^k_\alpha z^\varepsilon)\right|^2(\alpha,t)d\alpha,$$

we obtain

$$I\le C\|\mathcal{F}(z^\varepsilon)\|_{L^\infty}\|\Lambda(\sigma^\varepsilon)\|_{L^\infty}\|\partial^k_\alpha z^\varepsilon\|^2_{L^2}\le\exp C\left(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}+\|z^\varepsilon\|^2_{H^k}\right).$$

Finally, for $t\le\tau^\varepsilon$ we have

$$\frac{d}{dt}\|z^\varepsilon\|^2_{H^k}(t)\le C\exp C\left(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)+\|z^\varepsilon\|^2_{H^k}(t)\right).\tag{10.3}$$

We also have (see §8) that

$$\frac{d}{dt}\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)\le C\exp C\left(\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)+\|z^\varepsilon\|^2_{H^3}(t)\right),$$

and from (10.3) it follows that

$$\frac{d}{dt}\left(\|z^\varepsilon\|^2_{H^k}(t)+\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)\right)\le C\exp C\left(\|z^\varepsilon\|^2_{H^k}(t)+\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)\right)$$

for $t\le\tau^\varepsilon$. Integrating

(10.4)

$$\|z^\varepsilon\|^2_{H^k}(t)+\|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t)\le-\frac{1}{C}\ln\left(-t+\exp\left(-C\left(\|z_0\|^2_{H^k}+\|\mathcal{F}(z_0)\|^2_{L^\infty}\right)\right)\right)$$

for $t \leq \tau^\varepsilon$. Let us mention that at this point of the proof we cannot assume local-existence, because we have the above estimate for $t \leq \tau^\varepsilon$, and if we let $\varepsilon \to 0$, it could be possible that $\tau^\varepsilon \to 0$; i.e., we cannot assume that if the initial data satisfy $\sigma(\alpha, 0) > 0$, then there must be a time $\tau$, independent of $\varepsilon$, in which (10.4) is satisfied. In other words, at this stage of the proof we do not have local-existence when $\varepsilon \to 0$. But since in the evolution equation everything depends upon the sign of $\sigma^\varepsilon(\alpha, t)$, the following argument will allow us to continue the proof. First let us observe that as in Section 9 we have

$$m^\varepsilon(t) \geq m(0) - \int_0^t \exp C \left( \|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(s) + \|z^\varepsilon\|^2_{H^3}(s) \right) ds, \tag{10.5}$$

where

$$m^\varepsilon(t) = \min_{\alpha \in \mathbb{T}} \sigma^\varepsilon(\alpha, t)$$

and $t \leq \tau^\varepsilon$. Using (10.4) in (10.5) we get

$$\begin{aligned} m^\varepsilon(t) \geq m(0) &+ C \left( \|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty} \right) \\ &+ \ln \Big( -t + \exp\Big( -C\Big( \|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty} \Big)\Big)\Big) \end{aligned} \tag{10.6}$$

for $t \leq \tau^\varepsilon$. Using (10.6) and (10.4), we find that if $\varepsilon \to 0$, then $\tau^\varepsilon \nrightarrow 0$, because if we take $\tau = \min(\tau_1, \tau_2)$, where $\tau_1$ satisfies

$$\begin{aligned} &m(0) + C \left( \|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty} \right) \\ &\qquad + \ln \Big( -\tau_1 + \exp\Big( -C\Big( \|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty} \Big)\Big)\Big) > 0 \end{aligned}$$

and $\tau_2$ satisfies

$$-\frac{1}{C} \ln \Big( -\tau_2 + \exp\Big( -C\Big( \|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty} \Big)\Big)\Big) < \infty$$

for $t \leq \tau$, we have $m^\varepsilon(t) > 0$ and

$$\begin{aligned} &\|z^\varepsilon\|^2_{H^k}(t) + \|\mathcal{F}(z^\varepsilon)\|^2_{L^\infty}(t) \\ &\qquad \leq -\frac{1}{C} \ln \Big( -\tau + \exp\Big( -C\Big( \|z_0\|^2_{H^k} + \|\mathcal{F}(z_0)\|^2_{L^\infty} \Big)\Big)\Big) < \infty, \end{aligned}$$

and $\tau$ only depends on the initial data $z_0$. Now we let $\varepsilon$ tends to 0, to conclude the existence result.

## 11. **Appendix**

Let us denote

$$V(\alpha,\beta) = (V_1(\alpha,\beta), V_2(\alpha,\beta)) = \left( \tan\left( \frac{z_1(\alpha) - z_1(\beta)}{2} \right), \tanh\left( \frac{z_2(\alpha) - z_2(\beta)}{2} \right) \right)$$

and

$$W(\alpha,\beta) = (W_1(\alpha,\beta), W_2(\alpha,\beta)) = \left( \left( \frac{z_1(\alpha) - z_1(\beta)}{2} \right)_p, \left( \frac{z_2(\alpha) - z_2(\beta)}{2} \right)_p \right),$$

where $(\alpha)_p$ is the periodic extension of the function $\alpha$ in $\mathbb{T}$. We give the following equalities for the hyperbolic tangent function:

$$(\tanh(\alpha) - (\alpha)_p)/\tanh^2(\alpha) = (\alpha)_p f(\alpha) \quad \text{with} \quad f \in L^\infty(\mathbb{R}), \tag{11.1}$$

$$(\tanh(\alpha) - (\alpha)_p)/\tanh^3(\alpha) = g(\alpha) \quad \text{with} \quad g \in L^\infty(\mathbb{R}). \tag{11.2}$$

For the tangent function we have

$$(\tan(\alpha/2) - (\alpha/2)_p)/\tan(\alpha/2) = (\alpha/2)_p h(\alpha) \quad \text{with} \quad h \in L^\infty(\mathbb{R}), \tag{11.3}$$

$$(\tan(\alpha/2) - (\alpha/2)_p)/\tan^2(\alpha/2) = (\alpha/2)_p j(\alpha) \quad \text{with} \quad j \in L^\infty(\mathbb{R}), \tag{11.4}$$

$$(\tan(\alpha/2) - (\alpha/2)_p)/|\tan^3(\alpha/2)| = k(\alpha) \quad \text{with} \quad k \in L^\infty(\mathbb{R}). \tag{11.5}$$

Also we shall use that the functions below are bounded on $[-\pi,\pi]$

$$2/\alpha - 1/\tan(\alpha/2),\ 4/\alpha^2 - 1/\sin^2(\alpha/2) \in L^\infty(\mathbb{T}) \tag{11.6}$$

and the following estimates:

$$|W(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta/2| \leq \frac{1}{2}\|z\|_{C^2}|\beta|^2, \tag{11.7}$$

$$|W(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta/2 - \partial_\alpha^2 z(\alpha)\beta^2/4| \leq \frac{1}{2}\|z\|_{C^{2,\delta}}|\beta|^{2+\delta}. \tag{11.8}$$

Lemma 11.1. *Given*

$$A_1(\alpha,\alpha-\beta) = \frac{V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_2(\alpha)}{\tan(\frac{\beta}{2})},$$

$$A_2(\alpha,\alpha-\beta) = \frac{V_1(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2} - \frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_1(\alpha)}{\tan(\frac{\beta}{2})},$$

*we have*

$$\|A_1(\alpha,\alpha-\beta)\|_{L^\infty} \leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$$

*and*

$$\|A_2(\alpha,\alpha-\beta)\|_{L^\infty} \leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}^2.$$

*Proof.* We introduce the splitting $A_1(\alpha,\alpha-\beta) = I_1 + I_2 + I_3 + I_4$ where

$$I_1 = \frac{\tanh(\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2}) - (\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2})_p}{V(\alpha,\alpha-\beta)},$$

$$I_2 = \mathcal{F}(z)(\alpha,\beta)\frac{((z_2(\alpha)-z_2(\alpha-\beta))/2)_p - \partial_\alpha z_2(\alpha)\beta/2}{\beta^2/4},$$

$$I_3 = \frac{\partial_\alpha z_2(\alpha)}{\beta/2}\left(\mathcal{F}(z)(\alpha,\beta) - \frac{1}{|\partial_\alpha z(\alpha)|^2}\right),$$

$$I_4 = \frac{\partial_\alpha z_2(\alpha)}{|\partial_\alpha z(\alpha)|^2}\left(\frac{2}{\beta} - \frac{1}{\tan(\frac{\beta}{2})}\right)$$

and $\mathcal{F}(z)(\alpha,\beta)$ was defined in (1.7).

Since

$$I_1 = \frac{1}{1+\frac{V_1^2(\alpha,\alpha-\beta)}{V_2^2(\alpha,\alpha-\beta)}} f\left(\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2}\right)\left(\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2}\right)_p$$

by (11.1), we get $I_1 \leq C$. Also $I_2 \leq \|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$ using (11.7), and $I_4 \leq C\|\mathcal{F}(z)\|_{L^\infty}^{1/2}$. We rewrite

$$I_3 = \frac{\partial_\alpha z_2(\alpha)}{\beta/2}\frac{(\partial_\alpha z(\alpha)\beta/2 + V(\alpha,\alpha-\beta))\cdot(\partial_\alpha z(\alpha)\beta/2 - V(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2|V(\alpha,\alpha-\beta)|^2}$$

and split further

$$I_3 = J_1 + J_2,$$

where

$$J_1 = \frac{\partial_\alpha z_2(\alpha)}{\beta/2}\frac{(\partial_\alpha z_1(\alpha)\beta/2 + V_1(\alpha,\alpha-\beta))(\partial_\alpha z_1(\alpha)\beta/2 - V_1(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2|V(\alpha,\alpha-\beta)|^2},$$

$$J_2 = \frac{\partial_\alpha z_2(\alpha)}{\beta/2}\frac{(\partial_\alpha z_2(\alpha)\beta/2 + V_2(\alpha,\alpha-\beta))(\partial_\alpha z_2(\alpha)\beta/2 - V_2(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2|V(\alpha,\alpha-\beta)|^2}.$$

We continue as follows:

$$J_1 = K_1 + K_2$$

with

$$K_1 = \frac{\partial_\alpha z_2(\alpha)\partial_\alpha z_1(\alpha)(\partial_\alpha z_1(\alpha)\beta/2 - V_1(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2|V(\alpha,\alpha-\beta)|^2},$$

$$K_2 = \frac{\partial_\alpha z_2(\alpha)V_1(\alpha,\alpha-\beta)(\partial_\alpha z_1(\alpha)\beta/2 - V_1(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2|V(\alpha,\alpha-\beta)|^2\beta/2}.$$

Next we take $K_1 = L_1 + L_2$,

$$L_1 = \frac{\partial_\alpha z_2(\alpha)\partial_\alpha z_1(\alpha)((\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p - V_1(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2|V(\alpha,\alpha-\beta)|^2},$$

$$L_2 = \frac{\partial_\alpha z_2(\alpha)\partial_\alpha z_1(\alpha)(\partial_\alpha z_1(\alpha)\beta/2 - (\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p)}{|\partial_\alpha z(\alpha)|^2|V(\alpha,\alpha-\beta)|^2}.$$

We find that

$$L_1 = \frac{\partial_\alpha z_2(\alpha)\partial_\alpha z_1(\alpha)}{|\partial_\alpha z(\alpha)|^2}\frac{1}{1+\frac{V_2^2(\alpha,\alpha-\beta)}{V_1^2(\alpha,\alpha-\beta)}}\frac{(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p - V_1(\alpha,\alpha-\beta)}{V_1^2(\alpha,\alpha-\beta)},$$

and using (11.4), we obtain $L_1 \leq C$.

Since

$$L_2 = \frac{\partial_\alpha z_2(\alpha)\partial_\alpha z_1(\alpha)}{|\partial_\alpha z(\alpha)|^2}\mathcal{F}(z)(\alpha,\beta)\frac{(\partial_\alpha z_1(\alpha)\beta/2 - (\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p)}{\beta^2/4},$$

we have $L_2 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$. Next let us write $K_2 = L_3 + L_4$ for

$$L_3 = \frac{\partial_\alpha z_2(\alpha)V_1(\alpha,\alpha-\beta)((\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p - V_1(\alpha,\alpha-\beta))}{|\partial_\alpha z(\alpha)|^2|V(\alpha,\alpha-\beta)|^2\beta/2},$$

$$L_4 = \frac{\partial_\alpha z_2(\alpha)V_1(\alpha,\alpha-\beta)(\partial_\alpha z_1(\alpha)\beta/2 - (\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p)}{|\partial_\alpha z(\alpha)|^2|V(\alpha,\alpha-\beta)|^2\beta/2}.$$

In a similar way we find that

$$L_3 = \frac{\partial_\alpha z_2(\alpha)}{|\partial_\alpha z(\alpha)|^2}\frac{1}{1+\frac{V_2^2(\alpha,\alpha-\beta)}{V_1^2(\alpha,\alpha-\beta)}}\frac{(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p - V_1(\alpha,\alpha-\beta)}{V_1(\alpha,\alpha-\beta)}\frac{2}{\beta}.$$

By (11.3) one gets

$$L_3 \leq C\frac{|\partial_\alpha z_2(\alpha)|}{|\partial_\alpha z(\alpha)|^2}\frac{|(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p|}{|\beta|/2} \leq C.$$

As before we conclude that

$$L_4 \leq C\|\mathcal{F}(z)\|_{L^\infty}\frac{|\partial_\alpha z_1(\alpha)\beta/2 - (\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p|}{|\beta|^2} \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}.$$

Now we consider $J_2 = K_3 + K_4$, where

$$K_3 = \frac{|\partial_\alpha z_2(\alpha)|^2}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_2(\alpha)\beta/2 - V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}$$

and

$$K_4 = \frac{\partial_\alpha z_2(\alpha)}{|\partial_\alpha z(\alpha)|^2}\frac{V_2(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}\frac{\partial_\alpha z_2(\alpha)\beta/2 - V_2(\alpha,\alpha-\beta)}{\beta/2}.$$

Using (11.1), we have

$$K_3 \leq C + \frac{|\partial_\alpha z_2(\alpha)|^2}{|\partial_\alpha z(\alpha)|^2}\frac{|\partial_\alpha z_2(\alpha)\beta/2 - (\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2})_p|}{|V(\alpha,\alpha-\beta)|^2} \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$$

and

$$\begin{aligned} K_4 \leq& \frac{|\partial_\alpha z_2(\alpha)|}{|\partial_\alpha z(\alpha)|^2}\Bigg(\frac{|V_2(\alpha,\alpha-\beta) - W_2(\alpha,\alpha-\beta)|}{(1+\frac{(V_1(\alpha,\alpha-\beta))^2}{(V_2(\alpha,\alpha-\beta))^2})|V_2(\alpha,\alpha-\beta)||\beta/2|} \\ &+ \frac{|\partial_\alpha z_2(\alpha)\beta/2 - W_2(\alpha,\alpha-\beta)|}{|V(\alpha,\alpha-\beta)||\beta/2|}\Bigg) \\ &\leq C\|\mathcal{F}(z)\|_{L^\infty}^{1/2}\frac{|(\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2})_p|}{|\beta/2|} + \|\mathcal{F}(z)\|_{L^\infty}\frac{|\partial_\alpha z_2(\alpha)\beta/2 - (\frac{z_2(\alpha)-z_2(\alpha-\beta)}{2})_p|}{(\beta/2)^2} \\ &\leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}; \end{aligned}$$

that is, $K_3 + K_4 \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$.

Putting all the previous estimates together we get

$$|A_1(\alpha,\alpha-\beta)| \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}.$$

Regarding

$$A_2(\alpha,\alpha-\beta)=\frac{V_1(\alpha,\alpha-\beta)}{|V(\alpha,\alpha-\beta)|^2}-\frac{1}{|\partial_\alpha z(\alpha)|^2}\frac{\partial_\alpha z_1(\alpha)}{\tan(\beta/2)},$$

we have the splitting $A_2=I_5+I_6+I_7+I_8$, where

$$I_5=\frac{V_1(\alpha,\alpha-\beta)-(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p}{|V(\alpha,\alpha-\beta)|^2},$$

$$I_6=\mathcal{F}(z)(\alpha,\beta)\frac{(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2})_p-\partial_\alpha z_1(\alpha)\beta/2}{\beta^2/4},$$

$$I_7=\frac{\partial_\alpha z_1(\alpha)}{\beta/2}\left(\mathcal{F}(z)(\alpha,\beta)-\frac{1}{|\partial_\alpha z(\alpha)|^2}\right),$$

$$I_8=\frac{\partial_\alpha z_1(\alpha)}{|\partial_\alpha z(\alpha)|^2}\left(\frac{2}{\beta}-\frac{1}{\tan(\beta/2)}\right).$$

Then by the same arguments used above, we obtain $|A_2|\le C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$.

LEMMA 11.2. *Let $B(\alpha,\beta)$ be defined by*

$$B(\alpha,\alpha-\beta)=V_1(\alpha,\alpha-\beta)\frac{V(\alpha,\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}-\partial_\alpha z_1(\alpha)\frac{(\partial_\alpha^2 z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4\tan(\beta/2)}.$$

*Then it satisfies the inequality*

$$|B(\alpha,\alpha-\beta)|\le C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^3_{C^{2,\delta}}|\beta|^{\delta-1}.$$

*Proof.* Let us decompose $B(\alpha,\beta)=I_1+I_2$, where

$$I_1=\left(V_1(\alpha,\alpha-\beta)-\left(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2}\right)_p\right)\frac{V(\alpha,\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}$$

and

$$I_2=\left(\frac{z_1(\alpha)-z_1(\alpha-\beta)}{2}\right)_p\frac{V(\alpha,\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}-\partial_\alpha z_1(\alpha)\frac{(\partial_\alpha^2 z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4\tan(\beta/2)}.$$

Using the identity (11.5), we can rewrite $I_1$ as follows:

$$I_1=k(z_1(\alpha)-z_1(\alpha-\beta))\frac{1}{\left(1+\frac{V_2^2(\alpha,\beta)}{V_1^2(\alpha,\beta)}\right)^{3/2}}\frac{V(\alpha,\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|}$$

to get $|I_1|\le C\|z\|_{C^1}$.

Next we consider $I_2=J_1+J_2$, where

$$J_1=W_1(\alpha,\alpha-\beta)\frac{(V(\alpha,\alpha-\beta)-W(\alpha,\alpha-\beta))^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}$$

and

$$J_2=W_1(\alpha,\alpha-\beta)\frac{W(\alpha,\alpha-\beta)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4}-\partial_\alpha z_1(\alpha)\frac{(\partial_\alpha^2 z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4\tan(\beta/2)}.$$

Using (11.2), (11.5), and the fact that $(\beta/2)_p/\tan(\beta/2)$ is bounded, we obtain $|J_1| \leq C\|z\|_{C^1}$. To continue we can rewrite $J_2$ as follows:

$$J_2 = W_1(\alpha,\alpha-\beta)\frac{(W(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4} - \partial_\alpha z_1(\alpha)\frac{(\partial_\alpha^2 z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4\tan(\beta/2)}$$

and $J_2 = K_1 + K_2 + K_3 + K_4$, where

$$K_1 = (W_1(\alpha,\alpha-\beta)-\partial_\alpha z_1(\alpha)\beta/2)\frac{(W(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4},$$

$$K_2 = \partial_\alpha z_1(\alpha)\beta/2\frac{(W(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2-\partial_\alpha^2 z(\alpha)\beta^2/4)^\perp\cdot\partial_\alpha z(\alpha)}{|V(\alpha,\alpha-\beta)|^4},$$

$$K_3 = 2\partial_\alpha z_1(\alpha)(\partial_\alpha^2 z(\alpha)^\perp\cdot\partial_\alpha z(\alpha))(\mathcal{F}(z)(\alpha,\beta)^2 - \frac{1}{|\partial_\alpha z(\alpha)|^4})/\beta,$$

$$K_4 = \frac{\partial_\alpha z_1(\alpha)(\partial_\alpha^2 z(\alpha))^\perp\cdot\partial_\alpha z(\alpha)}{|\partial_\alpha z(\alpha)|^4}\left(\frac{2}{\beta}-\frac{1}{\tan(\beta/2)}\right).$$

Clearly we have $|K_4| \leq C\|\mathcal{F}(z)\|_{L^\infty}\|z\|_{C^2}$, and using (11.7) we obtain

$$|K_1| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^3_{C^2}.$$

Also, estimate (11.8) allows us to obtain $|K_2| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^3_{C^{2,\delta}}|\beta|^{\delta-1}$. Next we consider in $K_3$ the factor $L(\alpha,\beta)$ given by

$$L(\alpha,\beta) = \left(\mathcal{F}(z)(\alpha,\beta)^2 - \frac{1}{|\partial_\alpha z(\alpha)|^4}\right)/\beta.$$

We can write $L(\alpha,\beta)$ as follows:

$$\frac{(|\partial_\alpha z(\alpha)|^2\beta^2/4+|V(\alpha,\alpha-\beta)|^2)}{|\partial_\alpha z(\alpha)|^4|V(\alpha,\alpha-\beta)|^2} \cdot \frac{(\partial_\alpha z(\alpha)\beta/2+V(\alpha,\alpha-\beta))\cdot(\partial_\alpha z(\alpha)\beta/2-V(\alpha,\alpha-\beta))}{|V(\alpha,\alpha-\beta)|^2\beta}. \tag{11.9}$$

Then proceeding as in the previous lemma we get $|K_3| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^3_{C^2}$. This ends the proof. □

LEMMA 11.3. *Given $C(\alpha,\beta)$ by the equality*

$$C(\alpha,\alpha-\beta) = \frac{V^\perp(\alpha,\alpha-\beta)\varpi(\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4} - \frac{2\partial_\alpha^\perp z(\alpha)\varpi(\alpha)}{|\partial_\alpha^2 z(\alpha)|^4,\sin^2(\beta/2)},$$

*we obtain*

$$|C(\alpha,\alpha-\beta)| \leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{C^2}\|\varpi\|_{C^1}\frac{1}{|\beta|}.$$

*Proof.* We decompose $C(\alpha,\alpha-\beta)=I_1+I_2+I_3+I_4+I_5$, where

$$I_1=\frac{(V(\alpha,\alpha-\beta)-W(\alpha,\alpha-\beta))^{\perp}\varpi(\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4},$$

$$I_2=\frac{(W(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta/2)^{\perp}\varpi(\alpha-\beta)\beta}{|V(\alpha,\alpha-\beta)|^4},$$

$$I_3=\frac{\partial_\alpha^{\perp} z(\alpha)\beta^2(\varpi(\alpha-\beta)-\varpi(\alpha))}{2|V(\alpha,\alpha-\beta)|^4},$$

$$I_4=8\partial_\alpha^{\perp} z(\alpha)\varpi(\alpha)\left(\mathcal{F}(z)(\alpha,\beta)^2-\frac{1}{|\partial_\alpha z(\alpha)|^4}\right)/\beta^2,$$

$$I_5=2\frac{\partial_\alpha^{\perp} z(\alpha)\varpi(\alpha)}{|\partial_\alpha z(\alpha)|^4}(4/\beta^2-\sin^2(\beta/2)).$$

Using (11.2) and (11.5) we get $|I_1|\leq C\|\mathcal{F}(z)\|^{1/2}_{L^\infty}\|\varpi\|_{L^\infty}$. Using (11.7) clearly we obtain $|I_2|\leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{C^2}\|\varpi\|_{L^\infty}/|\beta|$. For the next term it holds that

$$|I_3|\leq C\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{C^1}\|\varpi\|_{C^1}/|\beta|.$$

The reference (11.6) gives $|I_5|\leq C\|\mathcal{F}(z)\|^{3/2}_{L^\infty}\|\varpi\|_{L^\infty}$. Finally, the estimate given in the previous lemma for the term

$$\left(\mathcal{F}(z)(\alpha,\beta)^2-\frac{1}{|\partial_\alpha z(\alpha)|^4}\right)/\beta,$$

written in (11.9), allows us to conclude $|I_4|\leq\|\mathcal{F}(z)\|^2_{L^\infty}\|z\|_{C^1}\|\varpi\|_{L^\infty}/|\beta|$.

LEMMA 11.4. *Let $Q_1(\alpha,\beta)$ be given by*

$$Q_1(\alpha,\alpha-\beta)=-\frac{(\partial_\alpha z(\alpha))^{\perp}}{2}\left(\frac{1}{|V(\alpha,\alpha-\beta)|^2}-\frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}+\frac{2\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4\beta}\right).$$

*Then it satisfies the estimate $\|Q_1\|_{L^\infty}\leq\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^{2,\delta}}|\beta|^{\delta-1}$.*

*Proof.* To simplify we will consider

$$C(\alpha,\alpha-\beta)=\frac{1}{|V(\alpha,\alpha-\beta)|^2}-\frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}+\frac{4\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4\beta},$$

and we will show that $\|C\|_{L^\infty}\leq\|\mathcal{F}(z)\|^k_{L^\infty}\|z\|^k_{C^{2,\delta}}|\beta|^{\delta-1}$. We can rewrite

$$C(\alpha,\alpha-\beta)=\frac{(\partial_\alpha z(\alpha)\beta+2V(\alpha,\alpha-\beta))\cdot(\partial_\alpha z(\alpha)\beta-2V(\alpha,\alpha-\beta))}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2|\beta|^2}+\frac{4\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^4\beta}$$

and then take $C(\alpha,\alpha-\beta)=I_1+I_2+I_3$, where

$$I_1=-\frac{|2V(\alpha,\alpha-\beta)-\partial_\alpha z(\alpha)\beta|^2}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2|\beta|^2},$$

$$I_2 = -\frac{2\partial_\alpha z(\alpha)\beta \cdot (2V(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta - \partial_\alpha^2 z(\alpha)\beta^2/2)}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2|\beta|^2},$$

$$I_3 = -\frac{\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)\beta}{|\partial_\alpha z(\alpha)|^2}\left(\frac{1}{|V(\alpha,\alpha-\beta)|^2} - \frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\right).$$

Since

$$|I_1| \le \frac{|2V(\alpha,\alpha-\beta) - 2W(\alpha,\alpha-\beta)|^2}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2|\beta|^2} + \frac{|2W(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta|^2}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)|^2|\beta|^2},$$

using (11.1), (11.3), and the inequality (11.7), we control the term $I_1$. For $I_2$ it holds that

$$|I_2| \le \frac{4|V(\alpha,\alpha-\beta) - W(\alpha,\alpha-\beta)|}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)||\beta|} + \frac{4|W(\alpha,\alpha-\beta) - \partial_\alpha z(\alpha)\beta - \partial_\alpha^2 z(\alpha)\beta^2/2|}{|V(\alpha,\alpha-\beta)|^2|\partial_\alpha z(\alpha)||\beta|}.$$

Using (11.1), (11.4), and (11.8) we get the appropriate inequality. For $I_3$ we write

$$I_3 = -\frac{4\partial_\alpha z(\alpha)\cdot\partial_\alpha^2 z(\alpha)}{|\partial_\alpha z(\alpha)|^2}\left(\mathcal{F}(z)(\alpha,\beta) - \frac{4}{|\partial_\alpha z(\alpha)|^2|\beta|^2}\right)/\beta$$

and proceed as before.

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