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Small subspaces of L_p

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Abstract

We prove that if X is a subspace of L_p $(2 < p < \infty)$, then either X embeds isomorphically into $\ell_p \oplus \ell_2$ or X contains a subspace Y, which is isomorphic to $\ell_p(\ell_2)$. We also give an intrinsic characterization of when X embeds into $\ell_p \oplus \ell_2$ in terms of weakly null trees in X or, equivalently, in terms of the "infinite asymptotic game" played in X. This solves problems concerning small subspaces of L_p originating in the 1970's. The techniques used were developed over several decades, the most recent being that of weakly null trees developed in the 2000's.

1. **Introduction**

The study of "small subspaces" of L_p $(2 < p < \infty)$ was initiated by Kadets and Pełczyński [KP62] who proved that if X is an infinite dimensional subspace of L_p, then either X is isomorphic to ℓ_2 and the L_2-norm is equivalent to the L_p-norm on X, or for all $\varepsilon > 0$, X contains a subspace Y which is $1+\varepsilon$-isomorphic to ℓ_p. In [JO74] it was shown that if X does not contain an isomorph of ℓ_2, then X embeds isomorphically into ℓ_p. (Moreover, [KW95] showed that for all $\varepsilon > 0$, X $1+\varepsilon$-embeds into ℓ_p.) W. B. Johnson [Joh77] solved the analogous problem for $X \subseteq L_p$ (for all $1 < p < 2$) by proving that X embeds into ℓ_p if for some $K < \infty$ every weakly null sequence in S_X, the unit sphere of X, admits a subsequence K-equivalent to the unit vector basis of ℓ_p.

Using the machinery of [OS02] (see also [OS06]) and the special nature of L_p, these results were unified in [AO01] as: $X \subseteq L_p$ $(1 < p < \infty)$ embeds into ℓ_p if (and only if) every weakly null tree in S_X admits a branch equivalent to the unit vector basis of ℓ_p.

After ℓ_p and ℓ_2 the next smallest natural subspace of L_p $(2 < p < \infty)$ is $\ell_p \oplus \ell_2$. Indeed if $X \subseteq L_p$ does not embed into either ℓ_p or ℓ_2, it contains an isomorph of $\ell_p \oplus \ell_2$. The next small natural subspace after $\ell_p \oplus \ell_2$ is ℓ_p (ℓ_2) or, as it is sometimes denoted, $(\sum \ell_2)_p$. In [JO81] it was shown that if $X \subseteq L_p$ $(2 < p < \infty)$ and X is a quotient of a subspace of $\ell_p \oplus \ell_2$, then X embeds into $\ell_p \oplus \ell_2$.

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The motivating problem for this paper (and our main result) dates back to the 1970's. We prove that if $X \subseteq L_p$ $(2 < p < \infty)$ and X does not embed into $\ell_p \oplus \ell_2$, then X contains an isomorph of ℓ_p (ℓ_2). To solve this we first give an intrinsic characterization of when X embeds into $\ell_p \oplus \ell_2$. The terminology is explained in Section 3. We assume that our space L_p is defined over an atomless and separable probability space $(\Omega, \Sigma, \mathbb{P})$. We write $A \overset{K}{\sim} B$ if $A \leq KB$ and $B \leq KA$. X will always denote an infinite dimensional Banach space.

THEOREM A. *Let X be a subspace of L_p $(2 < p < \infty)$. Then the following are equivalent*:

a) *X embeds into $\ell_p \oplus \ell_2$*;
b) *every weakly null tree in S_X admits a branch (x_i) satisfying for some K and all scalars (a_i)*

$$\Big\| \sum a_i x_i \Big\| \overset{K}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big\| \sum a_i x_i \Big\|_2 \tag{1.1}$$

($\|\cdot\|_2$ denotes the L_2-norm);
c) *every weakly null tree in S_X admits a branch (x_i) satisfying, for some K, $(w_i) \subseteq [0,1]$, and all scalars (a_i)*

$$\Big\| \sum a_i x_i \Big\| \overset{K}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big(\sum |a_i|^2 w_i^2 \Big)^{1/2} . \tag{1.2}$$

Under any of these conditions the embedding of X into $\ell_p \oplus \ell_2$ is given by: producing a blocking (H_n) of the Haar basis for L_p and $1 \leq K < \infty$, so that, if $X \ni x = \sum x_n$, $x_n \in H_n$, then

$$\|x\| \overset{K}{\sim} \Big(\sum \|x_n\|_p^p \Big)^{1/p} \vee \Big(\sum \|x_n\|_2^2 \Big)^{1/2} = \Big(\sum \|x_n\|_p^p \Big)^{1/p} \vee \|x\|_2 .$$

Since $(\sum H_n)_p$ is isomorphic to ℓ_p this suffices.

The next task is to show that if X violates these conditions then X contains a complemented subspace isomorphic to ℓ_p (ℓ_2). We will present two proofs of this. The first proof will roughly show that X must contain "skinny" uniform copies of ℓ_2 and hence contain uniform ℓ_2's, $(X_n)_{n \in \mathbb{N}}$ for which if $x_n \in S_{X_n}$ then the x_n's are almost disjointly supported and hence behave like the unit vector basis of ℓ_p. Then an argument due to Schechtman will prove that a subspace of X which is isomorphic to $\ell_p(\ell_2)$ contains an isomorphic copy of $\ell_p(\ell_2)$ which is complemented in L_p. The second proof will lead to a more precise result using the random measure machinery of D. Aldous [Ald81] and the stability theory of L_p [KM81]. For easier reading we will, however, recall all relevant definitions and results concerning random measures and stability theory. We will show that the complemented copy of $\ell_p(\ell_2)$ is witnessed by *stabilized ℓ_2 sequences* living on almost disjoint supports, meaning that the joint support of the elements of the X_n's is almost disjoint, not only the support of the elements of a given sequence (x_n) with $x_n \in X_n$, for $n \in \mathbb{N}$.

This yields the following: If X is a subspace of L_p and X is not contained in $\ell_2 \oplus \ell_p$, then X must contain a complemented copy of $\ell_p(\ell_2)$. Moreover, it admits a projection onto a subspace isomorphic to $\ell_p(\ell_2)$, whose norm is arbitrarily close to that of the minimal norm projection of L_p onto any subspace isomorphic to ℓ_2.

THEOREM B. *Let $X \subseteq L_p$ $(2 < p < \infty)$. If X does not embed into $\ell_p \oplus \ell_2$, then for all $\varepsilon > 0$, X contains a subspace Y, which is $1+\varepsilon$-isomorphic to ℓ_p (ℓ_2). Furthermore, Y is complemented in L_p by a projection of norm not exceeding $(1+\varepsilon)\gamma_p$, where $\gamma_p = \|x\|_p$, x being a symmetric L_2 normalized Gaussian random variable.*

Moreover, we can write Y as the complemented sum of Y_n's, where Y_n is $(1+\varepsilon)$-isomorphic to ℓ_2 and Y is $(1+\varepsilon)$-isomorphic to the ℓ_p-sum of the Y_n's, and there exists a sequence (A_n) of disjoint measurable sets so that $\|y|_{A_n}\|_p \geq (1-\varepsilon 2^{-n})\|y\|$ for all $y \in Y_n$ and $n \in \mathbb{N}$.

The original proof of the [JO81] result about quotients of subspaces of $\ell_p \oplus \ell_2$ is quite complicated. A byproduct of our results will be to give a much easier proof (see §7). In addition, we can characterize when $X \subseteq L_p$ $(2 < p < \infty)$ embeds into $\ell_p \oplus \ell_2$ in terms of its asymptotic structure [MMTJ95]. From results in [KP62] and [JO74], we first note that $X \subseteq L_p$ $(2 < p < \infty)$ embeds into ℓ_p if and only if it is asymptotic ℓ_p, and X embeds into ℓ_2 if and only if it is asymptotic ℓ_2.

Let us say X is *asymptotic* $\ell_p \oplus \ell_2$ if for some K and all $(e_i)_1^n \in \{X\}_n$, the n^{th} asymptotic structure of X, there exists $(w_i)_1^n \subseteq [0,1]$ so that for all $(a_i)_1^n \subseteq \mathbb{R}$,

$$\Big\| \sum_1^n a_i e_i \Big\| \overset{K}{\sim} \Big(\sum_1^n |a_i|^p \Big)^{1/p} \vee \Big(\sum_1^n |a_i|^2 |w_i|^2 \Big)^{1/2} . \tag{1.3}$$

We note that the space $\ell_p \oplus \ell_2$ is itself asymptotic $\ell_p \oplus \ell_2$. Indeed, denote by (f_i) and (g_i) the unit vector bases of ℓ_p and ℓ_2, respectively, viewed as elements of $\ell_p \oplus \ell_2$. For $(x,y) \in \ell_p \oplus \ell_2$ we put $\|(x,y)\| = \|x\|_p \vee \|y\|_2$. Since (f_i) and (g_i) are 1-subsymmetric and $\ell_p \oplus \ell_2$ is reflexive, the elements of the n^{th} asymptotic structure of $\ell_p \oplus \ell_2$ are exactly the sequences $(z_i)_{i=1}^n$ in $\ell_p \oplus \ell_2$, for which there are $0 = k_0 < k_1 < k_2 < \ldots k_n$ in $\mathbb{N}$, and (a_j), (b_j) in $\mathbb{R}$ with

$$z_i = \sum_{j=k_{i-1}+1}^{k_i} (a_j f_j + b_j g_j),$$

so that $\|z_i\| = v_i \vee w_i = 1$, where

$$v_i = \Big(\sum_{j=k_i-1}^{k_i} |a_j|^p \Big)^{1/p} \text{ and } w_i = \Big(\sum_{j=k_i-1}^{k_i} |b_j|^2 \Big)^{1/2}.$$

For $(\xi_i)_{i=1}^n \subset [-1,1]$ we therefore compute

$$\Big\|\sum_{i=1}^n \xi_i z_i\Big\| = \Big(\sum_{i=1}^n |\xi_i|^p v_i^p\Big)^{1/p} \vee \Big(\sum_{i=1}^n |\xi_i|^2 w_i^2\Big)^{1/2} \le \Big(\sum_{i=1}^n |\xi_i|^p\Big)^{1/p} \vee \Big(\sum_{i=1}^n |\xi_i|^2 w_i^2\Big)^{1/2}.$$

Assuming now that (otherwise (1.3) follows immediately)

$$\Big(\sum_{i=1}^n |\xi_i|^p v_i^p\Big)^{1/p} \ge \Big(\sum_{i=1}^n |\xi_i|^2 w_i^2\Big)^{1/2},$$

we deduce that

$$\Big\|\sum_{i=1}^n \xi_i z_i\Big\|^p \ge \frac{1}{2}\Big[\sum_{i=1}^n |\xi_i|^p v_i^p + \Big(\sum_{i=1}^n |\xi_i|^2 w_i^2\Big)^{p/2}\Big] \ge \frac{1}{2}\sum_{i=1}^n |\xi_i|^p (v_i^p \vee w_i^p) = \frac{1}{2}\sum_{i=1}^n |\xi_i|^p.$$

It follows therefore that (z_i) satisfies (1.3) with $K = 2$ and we deduce that $\ell_p \oplus \ell_2$ is asymptotic $\ell_p \oplus \ell_2$.

For $n \in \mathbb{N}$, let $(e_{i,j}^{(n)} : i,j \le n)$ be the unit vector basis of $\ell_p^n(\ell_2^n)$, i.e.

$$\Big\|\sum_{i,j=1}^n a_{i,j} e_{i,j}^{(n)}\Big\| = \Big(\sum_{i=1}^n \Big(\sum_{j=1}^n |a_{i,j}|^2\Big)^{p/2}\Big)^{1/p}, \text{ for all } (a_{i,j}) \subset \mathbb{R}.$$

Note that $(e_{i,j}^{(n)})$ is, ordered lexicographically, isometrically in the $(n^2)^{th}$ asymptotic structure of $\ell_p(\ell_2)$, for all $n \in \mathbb{N}$. It is not hard to deduce from the aforementioned description of the asymptotic structure of $\ell_p \oplus \ell_2$ that $(e_{i,j}^{(n)})$ is not (uniformly in $n \in \mathbb{N}$) in the $(n^2)^{th}$ asymptotic structure of $\ell_p \oplus \ell_2$. Theorem B therefore yields the following:

Corollary C. *$X \subseteq L_p$ $(2 < p < \infty)$ embeds into $\ell_p \oplus \ell_2$ if and only if X is asymptotic $\ell_p \oplus \ell_2$.*

Indeed, if X does not embed into $\ell_p \oplus \ell_2$, then by Theorem B it contains an isomorph of ℓ_p (ℓ_2), which is not asymptotic $\ell_p \oplus \ell_2$.

Using Theorems A and B we will be able to deduce the following additional surprising characterization of subspaces of L_p which embed into $\ell_p \oplus \ell_2$. It is analogous to the characterization of subspaces of L_p which embed in ℓ_p via normalized weakly null sequences (see the aforementioned result from [Joh77]) and we thank W. B. Johnson for having pointed it out to us.

Corollary D. *$X \subseteq L_p$ $(2 < p < \infty)$ embeds into $\ell_p \oplus \ell_2$ if and only if there exists a $K \ge 1$ so that every normalized weakly null sequence in S_X admits a subsequence (x_i) satisfying for all scalars (a_i),*

$$\Big\|\sum a_i x_i\Big\| \overset{K}{\sim} \Big(\sum |a_i|^p\Big)^{1/p} \vee \Big(\sum a_i^2 \|x_i\|_2^2\Big)^{1/p}. \tag{1.4}$$

A proof of Corollary D will be given at the end of Section 5. It is worth noting that (1.4) is a reformulation of (1.1) in (b) of Theorem A. The difference here is that the constant K is uniform and not dependent on the particular sequence. Without the uniformity assumption, the Corollary would be false (see Theorem 2.4 below). In Section 2 we recall some inequalities for unconditional basic sequences and martingales in L_p. Section 3 contains the proof of Theorem A, along with the necessary preliminaries on weakly null trees, and the "infinite asymptotic game." In Section 4 we introduce a dichotomy of Kadets-Pełczynski type and apply the results of Section 2 to embed a class of subspaces of L_p into $\ell_p \oplus \ell_2$. Section 5 considers the subspaces of L_p which do not embed in $\ell_p \oplus \ell_2$; we show that such subspaces contain "thinly supported ℓ_2's". More precisely, for some $K < \infty$, we find subspaces Y_n, $n \in \mathbb{N}$, which are K-isomorphic to ℓ_2, but for which the natural equivalence of $\|\cdot\|_p$ and $\|\cdot\|_2$ on Y_n is bad. By this we mean that $\|y\|_p \geq M_n \|y\|_2$, for all $y \in Y_n$, for some sequence $(M_n) \subset \mathbb{R}$, with $M_n \nearrow \infty$, as $n \nearrow \infty$. This will enable us to argue that we can choose the Y_n's so that vectors $y_n \in S_{Y_n}$, $n \in \mathbb{N}$, are almost disjointly supported and hence the closed linear span of the Y_n's is isomorphic to $\ell_p(\ell_2)$. Section 6 refines the result of Section 5, obtaining almost disjointly supported ℓ_2's, by applying techniques from Aldous's paper [Ald81] on random measures. As well as the new proof of the result from [JO81] mentioned above, Section 7 includes a construction of subspaces of L_p, isomorphic to ℓ_2, which embed only with bad constants in spaces of the form $\ell_p \oplus \left(\bigoplus_{i=1}^m \ell_2\right)_p$. In Section 8 we recall what is known and not known about small $\mathcal{L}_p$-spaces and raise a problem about when $X \subset L_p$ embeds into $\ell_p(\ell_2)$. In light of the deep work of [BRS81] in constructing uncountably many separable $\mathcal{L}_p$ spaces, it is likely that further study of their ordinal index will be needed to make progress on classifying the next group of smaller $\mathcal{L}_p$-spaces.

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2. **Some inequalities in L_p**

We first recall the well-known fact that an unconditional basic sequence in L_p is trapped between ℓ_p and ℓ_2.

Proposition 2.1 (see e.g., [AO01]). *Let (x_i) be a normalized λ-unconditional basic sequence in L_p $(2 < p < \infty)$. Then for all $(a_i) \subseteq \mathbb{R}$,*

$$\lambda^{-1} \left(\sum |a_i|^p\right)^{1/p} \leq \left\| \sum a_i x_i \right\|_p \leq \lambda\, B_p \left(\sum |a_i|^2\right)^{1/2} .$$

In Proposition 2.1, B_p is the Khintchin constant $\|\sum a_i r_i\| \leq B_p (\sum |a_i|^2)^{1/2}$, where (r_i) is the Rademacher sequence.

H. Rosenthal proved that if the x_i's are independent and mean zero random variables in L_p, then they span a subspace of $\ell_p \oplus \ell_2$.

THEOREM 2.2 ([Ros70]). *Let $2 < p < \infty$. There exists $K_p < \infty$ so that if (x_i) is a normalized mean zero sequence of independent random variables in L_p, then for all $(a_i) \subseteq \mathbb{R}$*

$$\Big\| \sum a_i x_i \Big\|_p \overset{K_p}{\sim} \Big(\sum |a_i|^p\Big)^{1/p} \vee \Big(\sum |a_i|^2 \|x_i\|_2^2\Big)^{1/2} \ .$$

D. Burkholder extended this result to martingale difference sequences as follows.

THEOREM 2.3 ([Bur73], [BDG72], [Hit90]). *Let $2 < p < \infty$. There exists $C_p < \infty$ so that if (z_i) is a martingale difference sequence in L_p, with respect to the sequence $(\mathcal{F}_n)$ of σ-algebras, then*

$$\Big\| \sum z_i \Big\|_p \overset{C_p}{\sim} \Big(\sum \|z_i\|_p^p\Big)^{1/p} \vee \Big\| \Big(\sum \mathbb{E}[z_i^2|\mathcal{F}_{i-1}]\Big)^{1/2} \Big\|_p \ ,$$

where $\mathbb{E}(x|\mathcal{F})$ denotes the conditional expectation of an integrable random variable x with respect to a sub-σ-algebra $\mathcal{F}$.

From [KP62], it follows that every normalized weakly null sequence in L_p admits a subsequence (x_i), which is either equivalent to the unit vector basis of ℓ_p or equivalent to the unit vector basis of ℓ_2. The latter occurs if $\varepsilon = \lim_i \|x_i\|_2 > 0$ and the lower ℓ_2 estimate is (essentially)

$$\varepsilon \Big(\sum |a_i|^2\Big)^{1/2} \leq \Big\| \sum a_i x_i \Big\|_p \ .$$

By use of Theorem 2.3, W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri obtained a quantitative improvement.

THEOREM 2.4 ([JMST79, Th. 1.14]). *Let $2 < p < \infty$. There exists $D_p < \infty$ with the following property. Every normalized weakly null sequence in L_p admits a subsequence (x_i) satisfying for some $w \in [0,1]$, for all $(a_i) \subseteq \mathbb{R}$,*

$$\Big\| \sum a_i x_i \Big\|_p \overset{D_p}{\sim} \Big(\sum |a_i|^p\Big)^{1/p} \vee w \Big(\sum |a_i|^2\Big)^{1/2} \ .$$

Thus, in particular the closed linear subspace $[(x_i)]$, generated by (x_i), uniformly embeds into $\ell_p \oplus \ell_2$.

3. A criterion for embeddability in $\ell_p \oplus \ell_2$

In this section we prove Theorem A, and thus provide an intrinsic characterization of subspaces of L_p which isomorphically embed into $\ell_p \oplus \ell_2$. This characterization is based on methods developed in [OS02] and [OS06].

We will need the following notation.

Let Z be a Banach space with a finite dimensional decomposition (FDD) $E = (E_n)$. For $n \in \mathbb{N}$, we denote the n-th *coordinate projection* by P_n^E; i.e.,

$P_n^E : Z \to E_n$ with $P_n^E(z) = z_n$, for $z = \sum z_i \in Z$, with $z_i \in E_i$, for all $i \in \mathbb{N}$. For a finite $A \subset \mathbb{N}$ we put $P_A^E = \sum_{n\in A} P_n^E$.

Let c_{00} denote the vector space of sequences in $\mathbb{R}$ which are eventually 0 with unit vector basis (e_i). More generally, if (E_i) is a sequence of finite dimensional Banach spaces, we define the vector space

$$c_{00}(\oplus_{i=1}^{\infty} E_i) = \Big\{(z_i) : z_i \in E_i \text{ for } i \in \mathbb{N}, \text{ and } \{i \in \mathbb{N} : z_i \neq 0\} \text{ is finite}\Big\}.$$

The linear space $c_{00}(\oplus_{i=1}^{\infty} E_i)$ is dense in each Banach space for which (E_n) is an FDD. If $A \subset \mathbb{N}$ is finite we denote by $\oplus_{i\in A} E_i$ the linear subspace of $c_{00}(\oplus E_i)$ generated by the elements of $(E_i)_{i\in A}$. A *blocking of* (E_i) is a sequence (F_i) of finite dimensional spaces for which there is an increasing sequence (N_i) in $\mathbb{N}$ so that $(N_0 = 0)$ $F_i = \oplus_{j=N_{i-1}+1}^{N_i} E_j$, for any $i \in \mathbb{N}$.

Let V be a Banach space with a normalized 1-unconditional basis (v_i) and $E = (E_i)$ a sequence of finite dimensional spaces. Then for $\overline{x} = (x_i) \in c_{00}(\oplus_{i=1}^{\infty} E_i)$, we define

$$\|\overline{x}\|_{(E,V)} = \Big\| \sum_{i=1}^{\infty} \|x_i\| \cdot v_i \Big\|_V .$$

$\|\cdot\|_{(E,V)}$ is a norm on $c_{00}(\oplus_{i=1}^{\infty} E_i)$, and we denote the completion of $c_{00}(\oplus_{i=1}^{\infty} E_i)$, with respect to $\|\cdot\|_{(E,V)}$, by $\big(\oplus_{i=1}^{\infty} E_i\big)_V$.

For $z \in c_{00}(\oplus E_i)$ we define the *E-support of* z by $\mathrm{supp}_E(z) = \{i \in \mathbb{N} : P_i^E(z) \neq 0\}$. A nonzero sequence $(z_j) \subset c_{00}(\oplus E_i)$ is called a *block sequence of* (E_i) if $\max \mathrm{supp}_E(z_n) < \min \mathrm{supp}_E(z_{n+1})$, for all $n \in \mathbb{N}$, and it is called a *skipped block sequence of* (E_i) if $1 < \min \mathrm{supp}_E(z_1)$ and $\max \mathrm{supp}_E(z_n) < \min \mathrm{supp}_E(z_{n+1}) - 1$, for all $n \in \mathbb{N}$. Let $\bar{\delta} = (\delta_n) \subset (0,1]$. If Z is a space with an FDD (E_i), we call a sequence $(z_j) \subset S_Z = \{z \in Z : \|z\| = 1\}$ a *$\bar{\delta}$-skipped block sequence of* (E_n), if there are $1 \le k_1 < \ell_1 < k_2 < \ell_2 < \cdots$ in $\mathbb{N}$ so that $\|z_n - P_{(k_n,\ell_n]}^E(z_n)\| < \delta_n$, for all $n \in \mathbb{N}$. Of course one could generalize the notion of $\bar{\delta}$-skipped block sequences to more general sequences, but we prefer to introduce this notion only for normalized sequences. It is important to note that, in the definition of $\bar{\delta}$-skipped block sequences, $k_1 \ge 1$, and thus, that the E_1-coordinate of z_1 is small (depending on δ_1). Let

$$T_\infty = \bigcup_{\ell\in\mathbb{N}} \Big\{(n_1, n_2, \ldots, n_\ell) : n_1 < n_2 < \cdots n_\ell \text{ are in } \mathbb{N}\Big\} .$$

T_∞ is naturally partially ordered by extension; that is, $(m_1, m_2, \ldots m_k) \preceq (n_1, n_2, \ldots n_\ell)$ if $k \le \ell$ and $n_i = m_i$, for $i \le k$. We call ℓ the length of $\alpha = (n_1, n_2, \ldots n_\ell)$ and denote it by $|\alpha|$, with $|\emptyset| = 0$ In this paper *trees in a Banach space* X are families in X indexed by T_∞.

For a tree $(x_\alpha)_{\alpha\in T_\infty}$ in X, and $\alpha = (n_1, n_2, \ldots, n_\ell) \in T_\infty \cup \{\emptyset\}$, we call the sequences of the form $(x_{(\alpha,n)})_{n>n_\ell}$ *nodes of* $(x_\alpha)_{\alpha\in T_\infty}$. The sequences (y_n) with

$y_i = x_{(n_1,n_2,\dots,n_i)}$ for $i \in \mathbb{N}$, for some strictly increasing sequence $(n_i) \subset \mathbb{N}$, are called *branches of* $(x_\alpha)_{\alpha\in T_\infty}$. Thus, branches of a tree $(x_\alpha)_{\alpha\in T_\infty}$ are sequences of the form (x_{α_n}), where (α_n) is a maximal linearly ordered (with respect to extension) subset of T_∞.

If $(x_\alpha)_{\alpha\in T_\infty}$ is a tree in X and if $T' \subset T_\infty$ is closed under taking initial segments (if $(n_1, n_2, \dots, n_\ell) \in T'$ and $m < \ell$ then $(n_1, n_2, \dots, n_m) \in T'$) and has the property that for each $\alpha \in T' \cup \{\emptyset\}$ infinitely many direct successors of α are also in T', then we call $(x_\alpha)_{\alpha\in T'}$ a *full subtree of* $(x_\alpha)_{\alpha\in T_\infty}$. Note that $(x_\alpha)_{\alpha\in T'}$ could then be relabeled to a family indexed by T_∞ and note that the branches of $(x_\alpha)_{\alpha\in T'}$ are branches of $(x_\alpha)_{\alpha\in T_\infty}$ and that the nodes of $(x_\alpha)_{\alpha\in T'}$ are subsequences of certain nodes of $(x_\alpha)_{\alpha\in T_\infty}$.

We call a tree $(x_\alpha)_{\alpha\in T_\infty}$ in X *normalized* if $\|x_\alpha\| = 1$ for all $\alpha \in T_\infty$, and *weakly null* if every node is a weakly null sequence. If X has an FDD (E_i) we call $(x_\alpha)_{\alpha\in T_\infty}$ a *block tree with respect to* (E_i) if every node and every branch (y_n) is a block sequence with respect to (E_i).

Note that if (E_i) is an FDD for X and if $(\varepsilon_\alpha)_{\alpha\in T_\infty} \subset (0, 1)$,then every normalized weakly null tree $(x_\alpha)_{\alpha\in T_\infty} \subset X$ has a full subtree $(z_\alpha)_{\alpha\in T_\infty}$ which is an (ε_α)-perturbation of a block tree (y_α) with respect to (E_i); i.e., $\|z_\alpha - y_\alpha\| \le \varepsilon_\alpha$ for any $\alpha \in T_\infty$. Let us also mention that the proof of the fact, that normalized weakly null sequences have basic subsequences whose basis constants are arbitrarily close to 1, generalizes to trees. This means that for a given $\varepsilon > 0$ and for any Banach space X, every normalized weakly null tree in X has a full subtree, all of whose nodes and all of whose branches are basic, and their basis constant does not exceed $1+\varepsilon$.

Now we can state the main results of this section.

THEOREM 3.1. *Let X be a subspace of L_p, $2 < p < \infty$, and assume that there is a $C > 1$ so that every normalized weakly null tree in X admits a branch (y_i) for which*

$$\Big\|\sum_{i=1}^\infty a_i y_i\Big\|_p \overset{C}{\sim} \max\left(\Big(\sum_{i=1}^\infty |a_i|^p\Big)^{1/p}, \Big\|\sum_{i=1}^\infty a_i y_i\Big\|_2\right) \text{ for all } (a_i) \in c_{00}.$$

Then there is a blocking $H = (H_n)$ of the Haar basis (h_n) so that

$$T : X \to \ell_p \oplus L_2, \quad T(x) = \big((P_n^H(x))_{n\in\mathbb{N}}, x\big) \in \Big(\oplus_{n=1}^\infty H_n\Big)_{\ell_p} \oplus L_2 \hookrightarrow \ell_p \oplus L_2$$

is an isomorphic embedding.

Theorem 3.1 is a special case of the following result. By a 1-subsymmetric basis we mean one that is 1-unconditional and 1-spreading.

THEOREM 3.2. *Let X and Y be separable Banach spaces, with X reflexive. Let V be a Banach space with a 1-subsymmetric and normalized basis (v_i), and let $T : X \to Y$ be linear and bounded.*

Assume that for some $C \geq 1$, every normalized weakly null tree of X admits a branch (x_n) so that

$$\Big\|\sum_{i=1}^{\infty} a_n x_n\Big\|_X \overset{C}{\sim} \Big\|\sum_{i=1}^{\infty} a_n v_n\Big\|_V \vee \Big\|T\Big(\sum_{i=1}^{\infty} a_n x_n\Big)\Big\|_Y \quad \textit{for all } (a_i) \in c_{00}. \tag{3.1}$$

Then there is a sequence of finite dimensional spaces (G_i), so that X is isomorphic to a subspace of $\big(\oplus_{i=1}^{\infty} G_i\big)_V \oplus Y$.

More precisely, under the above assumptions, if Z is any reflexive space with an FDD *(E_i), and if $S : X \to Z$ is an isomorphic embedding, then there is a blocking (G_i) of (E_i) so that S is a bounded linear operator from X to $\big(\oplus_{i=1}^{\infty} G_i\big)_V$ and the operator*

$$(S, T) : X \to \big(\oplus_{i=1}^{\infty} G_i\big)_V \oplus Y, \quad x \mapsto \big(S(x), T(x)\big)$$

is an isomorphic embedding.

Remark. Theorem 3.1 can be obtained from Theorem 3.2 by letting $V = \ell_p$, $Y = L_2$, $Z = L_p$, with the FDD (E_i) given by the Haar basis, S is the inclusion map from X into L_p, and T is the formal identity map from L_p to L_2 restricted to X.

As noted in [OS06, Cor. 2, §2] (see also [OS02] for similar versions), the tree condition in Theorem 3.2 can be interpreted as follows in terms of the "infinite asymptotic game", (IAG) as it has been called by Rosendal [Ros09].

Let $C \geq 1$ and let $\mathcal{A}^{(C)}$ be the set of all sequences (x_n) in S_X which are C-basic and satisfy condition (3.1). The (IAG) is played by two players: Player I chooses a subspace X_1 of X having finite codimension, and Player II chooses $x_1 \in S_{X_1}$, then, again Player I chooses a subspace X_2 of X of finite codimension, and Player II chooses an $x_2 \in S_{X_2}$. These moves are repeated infinitely many times, and Player I is declared the winner of the game if the resulting sequence (x_n) is in $\mathcal{A}^{(C)}$.

$\mathcal{A}^{(C)}$ is closed with respect to the infinite product of (S_X, d), where d denotes the discrete topology on S_X. This implies that this game is determined [Mar75]; i.e., either Player I or Player II has a winning strategy and as noticed in [OS06, Cor. 2, §2] for all $\varepsilon > 0$, Player I has a winning strategy for $\mathcal{A}^{(C+\varepsilon)}$ if and only if for all $\varepsilon > 0$, every weakly null tree in S_X has a branch, which lies in $\mathcal{A}^{(C+\varepsilon)}$.

Proof of Theorem A *using Theorem* 3.1. In terms of the infinite asymptotic game, the interpretation of our tree condition I easily implies that the existence of a uniform $C \geq 1$, so that all weakly null trees $(x_\alpha) \subset S_X$ admit a branch in $A^{(C)}$, is equivalent to the condition, that every weakly null tree $(x_\alpha) \subset S_X$ admits a branch in $\mathcal{A}^{(C)}$, for some $C \geq 1$.

Indeed, if such a uniform C does not exist, Player II could choose a sequence (C_n) in $\mathbb{R}^+$ which increases to ∞ and could play the following strategy. First he follows his winning strategy for achieving a sequence (x_n) outside of $\mathcal{A}^{(C_1)}$ and after finitely many steps, s_1, he must have chosen a sequence $x_1, x_2, \ldots, x_{s_2}$, which is either not C_1-basic or does not satisfy (3.1) for some $a = (a_i)_{i=1}^{s_1} \in \mathbb{R}^{s_1}$. Then Player II follows his strategy for getting a sequence outside of $\mathcal{A}^{(C_2)}$, and continues that way using C_3, C_4 etc. It follows that the infinite sequence (x_n), which is obtained by Player II, cannot be in any $\mathcal{A}^{(C)}$. Therefore Player II has a winning strategy for choosing a sequence outside of $\bigcup_{C\geq 1} \mathcal{A}^{(C)}$ which means that there is a weakly null tree, (z_α), none of whose branches are in $\bigcup_{C\geq 1} \mathcal{A}^{(C)}$.

Using Theorem 3.1, we deduce therefore that (b)⇒(a) in Theorem A. The implication (a)⇒(c) in Theorem A is easy, using arguments like those above establishing that $\ell_p \oplus \ell_2$ is asymptotic $\ell_p \oplus \ell_2$.

In order to show (c)⇒ (b) let (x_α) be a normalized weakly null tree in L_p. After passing to a full subtree and perturbing, we can assume that (x_α) is a block tree with respect to the Haar basis. By (c) there is branch (z_n), a sequence $(w_i) \subset [0,1]$ and $C \geq 1$ so that

$$\Big\| \sum a_i z_i \Big\|_p \overset{C}{\sim} \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big(\sum w_i^2 a_i^2 \Big)^{1/2} \text{ for all } (a_i) \in c_{00}. \tag{3.2}$$

Since (z_i) is an unconditional sequence and since $\|\cdot\|_2 \leq \|\cdot\|_p$ on L_p, it follows from Proposition 2.1 that for some constant c_p,

$$\Big\| \sum a_i z_i \Big\|_p \geq c_p \Big(\sum |a_i|^p \Big)^{1/p} \vee \Big\| \sum a_i z_i \Big\|_2. \tag{3.3}$$

We claim that our branch (z_n) satisfies (1.1) for some $K < \infty$. Assuming this were not true, then we could use (3.2), and choose a normalized block sequence (y_n) of (z_n), say

$$y_n = \sum_{i=k_{n-1}+1}^{k_n} a_i z_i, \text{ with } a_i \in \mathbb{R}, \text{ for } i \in \mathbb{N} \text{ and } 0 = k_0 < k_1 < \ldots,$$

so that for all $n \in \mathbb{N}$,

$$\sum_{i=k_{n-1}+1}^{k_n} w_i^2 a_i^2 = 1, \tag{3.4}$$

and

$$\Big(\sum_{i=k_{n-1}+1}^{k_n} |a_i|^p \Big)^{1/p} \vee \|y_n\|_2 < 2^{-n}. \tag{3.5}$$

For any $(b_i) \in c_{00}$, it follows therefore from (3.2) that

$$\Big\| \sum b_n y_n \Big\|_p \overset{C}{\sim} \Big(\sum |b_n|^2 \Big)^{1/2};$$

thus (y_n) is C-equivalent to the unit vector basis of ℓ_2. The result by Kadets and Pełczyński [KP62] yields that $\|\cdot\|_p$ and $\|\cdot\|_2$ must be equivalent on Y. But $\lim_{n\to\infty} \|y_n\|_2 = 0$ by (3.5), so we have a contradiction. $\square$

For the proof of Theorem 3.2 we need to recall some results from [OS02] and [OS06]. The following result restates Corollary 2.9 of [OS06], versions of which where already shown in [OS02].

THEOREM 3.3 ([OS06, Cor. 2.9 (c) $\Longleftrightarrow$ (d), and "Moreover"-part]). *Let X be a subspace of a reflexive space Z with an* FDD (E_i) *and let*

$$\mathcal{A} \subset \{(x_n) : x_n \in S_X \ \ \textit{for } n \in \mathbb{N}\}.$$

Then the following are equivalent:

a) *for any $\bar\varepsilon = (\varepsilon_n) \subset (0,1)$ every weakly null tree in S_X admits a branch in $\overline{\mathcal{A}_{\bar\varepsilon}}$, where*

$$\mathcal{A}_{\bar\varepsilon} = \Big\{ (x_n) \subset S_X : \exists (z_n) \in \mathcal{A} \quad \|z_n - x_n\| \le \varepsilon_n \ \textit{for } n \in \mathbb{N} \Big\},$$

and where $\overline{\mathcal{A}_{\bar\varepsilon}}$ denotes the closure in the product of the discrete topology on S_X;

b) *for any $\bar\varepsilon = (\varepsilon_n) \subset (0,1)$ there is a blocking (F_i) of (E_i) so that every $c\bar\varepsilon$-skipped block sequence $(x_n) \subset S_X$ of (F_i) lies in $\overline{\mathcal{A}_{\bar\varepsilon}}$. Here $c \in (0,1)$ is a constant which only depends on the projection constant of (E_i) in Z.*

We also need a blocking lemma which appears in various forms in [KOS99], [OS02], [OS06], and [OSZ08] and ultimately results from a blocking trick of W. B. Johnson [Joh77]. In the statement of Lemma 3.4 (and elsewhere) reference is made to the *weak*-topology of Z*, a space with a boundedly complete FDD (E_i). By this we mean the weak*-topology on Z obtained by regarding it as the dual space of the norm closure of the span of (E_i^*) in Z^*. This is then just the topology of coordinatewise convergence in Z with respect to the coordinates of (E_i).

LEMMA 3.4 ([OS06, Lemma 3, §3]). *Let X be a subspace of a space Z having a boundedly complete* FDD $E = (E_i)$ *with projection constant K with B_X being a w*-closed subset of Z. Let $\delta_i \downarrow 0$. Then there exist $0 = N_0 < N_1 < \cdots$ in $\mathbb{N}$ with the following properties. For all $x \in S_X$ there exists $(x_i)_{i=1}^\infty \subseteq X$, and for all $i \in \mathbb{N}$, there exists $t_i \in (N_{i-1}, N_i)$ satisfying ($t_0 = 0$ and $t_1 > 1$)*:

a) $x = \sum_{j=1}^\infty x_j$,

b) $\|x_i\| < \delta_i$ *or* $\|P^E_{(t_{i-1},t_i)} x_i - x_i\| < \delta_i \|x_i\|$,
c) $\|P^E_{(t_{i-1},t_i)} x - x_i\| < \delta_i$,
d) $\|x_i\| < K + 1$,
e) $\|P^E_{t_i} x\| < \delta_i$.

Proof of Theorem 3.2. Assume X embeds in a reflexive space Z with an FDD $E = (E_i)$. By Zippin's theorem [Zip88] such a space Z always exists. After renorming we can assume that the projection constant $K = \sup_{m \le n} \|P^E_{[m,n]}\| = 1$ and that X is (isometrically) a subspace of Z. We also assume without loss of generality that $\|T\| = 1$.

For a sequence $\overline{x} = (x_i) \in S_X$ and $a = \sum a_i e_i \in c_{00}$ we define

$$\Big\|\Big| \sum a_i e_i \Big\|\Big|_{\overline{x}} = \Big\| \sum a_i v_i \Big\|_V \vee \Big\| T\Big(\sum a_i x_i \Big) \Big\|_Y .$$

Then $\|\cdot\|_{\overline{x}}$ is a norm on c_{00} and we denote the completion of c_{00} with respect to $\|\cdot\|_{\overline{x}}$ by $W_{\overline{x}}$.

Define

$$\mathcal{A} = \Big\{ \overline{x} = (x_n) \subset S_X : \overline{x} \text{ is } \tfrac{3}{2}\text{-basic and } \tfrac{3}{2}C\text{-equivalent to } (e_i) \text{ in } W_{\overline{x}} \Big\} .$$

Observe that condition a) of Theorem 3.3 is satisfied for this set $\mathcal{A}$. Indeed, given any weakly null tree in S_X we may assume, as noted before the statement of Theorem 3.1, that by passing to a full subtree, the branches are basic with a constant close to 1, and thus the first requirement of the definition of $\mathcal{A}$ can be satisfied. The hypothesis from Theorem 3.2 then guarantees that $\mathcal{A}_{\bar{\varepsilon}}$ contains the required branch.

We first choose a null sequence $\bar{\varepsilon} = (\varepsilon_i) \subset (0,1)$, which decreases fast enough to 0 to ensure that every sequence $\overline{x} = (x_n)$ in $\overline{\mathcal{A}_{\bar{\varepsilon}}}$ is 2-basic and $2C$-equivalent to (e_i) in $W_{\overline{x}}$. By Theorem 3.3 applied to $\bar{\varepsilon}$ we can find a blocking $F = (F_i)$ of (E_i) and a sequence, so that every $c\bar{\varepsilon}$-skipped block sequence $(x_i) \subset S_X$ of (F_i) (c is the constant in Theorem 3.3(b)) is 2-basic and $2C$-equivalent to (e_i) in $W_{\overline{x}}$. We put $\bar{\delta} = (\delta_i) = c\bar{\varepsilon}$. Then we apply Lemma 3.4 to get a further blocking (G_i), $G_i = \oplus_{j=N_{i-1}+1}^{N_i} F_j$, for $i \in \mathbb{N}$ and some sequence $0 = N_0 < N_1 < N_2 \dots$, so that for every $x \in S_X$ there is a sequence $(t_i) \subset N$, with $t_i \in (N_{i-1}, N_i)$ for $i \in \mathbb{N}$, and $t_0 = 0$, and a sequence (x_i) satisfying (a)–(e).

We also may assume that $\sum_{i=1}^\infty \delta_i < 1/36C$ and will show that for every $x \in X$,

$$\|x\|_X \overset{36C}{\sim} \Big(\Big\| \sum_{i=1}^\infty \|P^G_i(x)\| v_i \Big\|_V \Big) \vee \|T(x)\|_Y . \tag{3.6}$$

This implies that the map $X \to (\oplus G_i)_V \oplus Y$, $x \mapsto ((P^G_i(x)), T(x))$ is an isomorphic embedding.

Let $x \in S_X$ and choose $(t_i) \subset \mathbb{N}$ and $(x_i) \subset X$ as prescribed in Lemma 3.4. Letting $B = \Big\{i \geq 2 : \|P^F_{(t_{i-1},t_i)}(x_i) - x_i\| \leq \delta_i \|x_i\|\Big\}$ it follows that $(x_i/\|x_i\|)_{i \in B}$ is a $\bar{\delta}$-skipped block sequence of (F_i) and therefore

$$\Big\|\sum_{i\in B} x_i\Big\|_X \overset{2C}{\sim} \Big\|\sum_{i\in B}\|x_i\|v_i\Big\|_V \vee \Big\|T\Big(\sum_{i\in B} x_i\Big)\Big\|. \tag{3.7}$$

We want to estimate $\Big\|\sum_{i=1}^{\infty}\|x_i\|v_i\Big\|_V \vee \|T(x)\|$. Since $1 \notin B$ (no matter how large $\|x_1\|$ is), we will distinguish between the case that $\|x_1\|$ is essential and the case that $\|x_1\|$ is small enough to be discarded.

If $\|x_1\| \geq 1/8C$, then we deduce that

(3.8)

$$\begin{aligned}
\frac{1}{8C} &\leq \|x_1\| \leq \Big\|\sum_{i=1}^{\infty}\|x_i\|v_i\Big\|_V \vee \|T(x)\|_Y \\
&\leq \Big(\Big\|\sum_{i\in B}^{\infty}\|x_i\|v_i\Big\|_V + \|x_1\| + \sum_{i\notin B}\delta_i\Big) \vee \|T(x)\|_Y \\
&\leq 2C\Big\|\sum_{i\in B}^{\infty} x_i\Big\| + 2 + \sum \delta_i \quad \text{[by (3.7), (d) of Lemma 3.4, and since } \|T\|=1] \\
&\leq 2C\|x\| + 2C\Big\|\sum_{i\notin B}^{\infty} x_i\Big\| + 2 + \sum \delta_i \\
&\leq 2C\|x\| + 2C\|x_1\| + 2C\sum\delta_i + 2 + \sum\delta_i \leq 9C.
\end{aligned}$$

If $\|x_1\| < 1/8C$, then

$$\begin{aligned}
1 = \|x\| &\leq \Big\|\sum_{i\in B} x_i\Big\| + \frac{1}{4C} \\
&\leq 2C\Big(\Big\|\sum_{i\in B}\|x_i\|v_i\Big\|_V \vee \Big\|T\Big(\sum_{i\in B} x_i\Big)\Big\|_Y\Big) + \frac{1}{4C} \qquad \text{[by (3.7)]} \\
&\leq 2C\Big(\Big\|\sum_{i=1}^{\infty}\|x_i\|v_i\Big\|_V \vee \|T(x)\|_Y\Big) + \frac{1}{2} + \frac{1}{4C} \\
&\leq 2C\Big(\Big\|\sum_{i=1}^{\infty}\|x_i\|v_i\Big\|_V \vee \|T(x)\|_Y\Big) + \frac{3}{4}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{8C} &\leq \Big\|\sum_{i=1}^{\infty}\|x_i\|v_i\Big\|_V \vee \Big\|T(x)\Big\|_Y \\
&\leq \Big(\Big\|\sum_{i\in B}\|x_i\|v_i\Big\|_V \vee \Big\|T\Big(\sum_{i\in B} x_i\Big)\Big\|_Y\Big) + \frac{1}{4C}
\end{aligned} \tag{3.9}$$

$$\leq 2C\Big\|\sum_{i\in B} x_i\Big\| + \frac{1}{4C} \quad \text{[By (3.7)]}$$

$$\leq 2C\|x\| + 2C\|x_1\| + 2C\sum \delta_i + \frac{1}{4C} \leq 8C.$$

Equations (3.8) and (3.9) imply that

$$1 \overset{9C}{\sim} \Big\|\sum_{i=1}^{\infty} \|x_i\| v_i\Big\|_V \vee \Big\|T(x)\Big\|. \tag{3.10}$$

For $n \in \mathbb{N}$, define $y_n = P^F_{(t_{n-1},t_n]}(x)$. From Lemma 3.4(c) and (e), it follows that

$$\|y_n - x_n\| \leq \|P^F_{(t_{n-1},t_n)}(x) - x_n\| + \|P^F_{t_n}(x)\| \leq 2\delta_n$$

and thus $\sum \|y_n - x_n\| \leq 1/18C$ which implies by (3.10) that

$$1 \overset{18C}{\sim} \Big\|\sum_{i=1}^{\infty} \|y_i\| v_i\Big\|_V \vee \Big\|T(x)\Big\|. \tag{3.11}$$

Since for $n \in \mathbb{N}$ we have $(N_{n-1}, N_n] \subset (t_{n-1}, t_{n+1})$ and $(t_{n-1}, t_n] \subset (N_{n-2}, N_n)$ (put $N_{-1} = N_0 = 0$ and $P^G_0 = 0$) it follows from the assumed 1-subsymmetry of (v_n) and the assumed bimonotonicity of (E_i) in Z that

$$\begin{aligned}
\frac{1}{2}\Big\|\sum_{n\in\mathbb{N}} \|y_n\| v_n\Big\|_V &\leq \frac{1}{2}\Big\|\sum_{n\in\mathbb{N}} \big(\|P^G_{n-1}(x)\| + \|P^G_n(x)\|\big) v_n\Big\|_V \\
&\leq \Big\|\sum_{n\in\mathbb{N}} \|P^G_n(x)\| v_n\Big\|_V \\
&\leq \Big\|\sum_{n\in\mathbb{N}} \Big\|P^F_{(t_{n-1},t_{n+1})}(x)\Big\| v_n\Big\|_V \\
&\leq \Big\|\sum_{n\in\mathbb{N}} \big(\|y_n\| + \|y_{n+1}\|\big) v_n\Big\|_V \leq 2\Big\|\sum_{n\in\mathbb{N}} \|y_n\| v_n\Big\|_V,
\end{aligned}$$

which implies with (3.11) that

$$1 \overset{36C}{\sim} \Big\|\sum_{i=1}^{\infty} \|P^G_i(x)\| v_i\Big\|_V \vee \Big\|T(x)\Big\|$$

and finishes the proof of our claim. □

4. Embedding small subspaces in $\ell_p \oplus \ell_2$

For a subspace X of L_p (where $p > 2$, as everywhere in this paper) we shall say that a function v in $L_{p/2}$ is a *limiting conditional variance* associated with X if there is a weakly null sequence (x_n) in X such that x_n^2 converges to v in the weak topology of $L_{p/2}$. It is equivalent to say that for all $E \in \Sigma$

(recall that L_p was defined over the atomless and separable probability space $(\Omega, \Sigma, \mathbb{P})$),

$$\mathbb{E}[1_E x_n^2] \to \mathbb{E}[1_E v]$$

as $n \to \infty$. The set of all such v will be denoted $V(X)$. Note that, since $p > 2$, every weakly null sequence (x_n) in X does of course have a subsequence (x_{n_k}) such that $x_{n_k}^2$ converges (to some $v \in V(X)$) for the weak topology of the reflexive space $L_{p/2}$.

Limiting conditional variances occur naturally in the context of the martingale inequalities to be used in this section, and are closely related to the random measures of Section 6. It is therefore natural to express the basic dichotomy underlying our main Theorem B in terms of $V(X)$.

Proposition 4.1. *Let X be a subspace of L_p, where $p > 2$. One of the following is true*:

(A) *there is a constant $M > 0$ such that $\|v\|_{p/2} \leq M\|v\|_1$ for all $v \in V(X)$;*
(B) *no such constant M exists, in which case there exist disjoint sets $A_i \in \Sigma$ and elements $v_i \in V(X)$ $(i \in \mathbb{N})$, such that $\|1_{A_i} v_i\|_{p/2} \to 1$ and $\|1_{\Omega\setminus A_i} v_i\|_{p/2} \to 0$ as $i \to \infty$.*

Proof. This is a consequence of the Kadets-Pełczynski dichotomy. Either there exists an $\varepsilon > 0$ so that

$$V(X) \subset \Big\{u \in L_{p/2} : \mathbb{P}[|u| \geq \varepsilon\|u\|_{p/2}] \geq \varepsilon\Big\}$$

and then

$$\|u\|_1 \geq \mathbb{E}\big[\varepsilon\|u\|_{p/2} 1_{[|u|\geq\varepsilon\|u\|_{p/2}]}\big] \geq \varepsilon^2\|u\|_{p/2} \text{ for all } u \in V(X),$$

and (A) holds for $M = \varepsilon^{-2}$. Otherwise, by the construction in Theorem 2 of [KP62], we obtain (B). □

The rest of this section will be devoted to showing that if (A) holds, then X embeds in $\ell_p \oplus \ell_2$. By Theorem 3.1, it will be enough to prove the following proposition.

Proposition 4.2. *Let X be a subspace of L_p where $p > 2$, and assume that* (A) *holds in Proposition* 4.1. *Then there is a constant K such that every weakly null tree in S_X has a branch (x_i) satisfying*

$$K^{-1}\Big\|\sum c_i x_i\Big\|_p \leq \max\Big\{\Big(\sum |c_i|^p\Big)^{1/p}, \Big\|\sum c_i x_i\Big\|_2\Big\} \leq K\Big\|\sum c_i x_i\Big\|_p,$$

for all $c_i \in \mathbb{R}$.

Proof. Our proof, using Burkholder's martingale version of Rosenthal's Inequality (Theorem 2.3), is closely modeled on Theorem 1.14 of [JMST79]. We let $(x_\alpha)_{\alpha\in T_\infty}$ be a weakly null tree in S_X. Taking small perturbations, we may suppose that we are dealing with a block tree of the Haar basis. So for each $\alpha \in T_\infty$, x_α is a finite linear combination of Haar functions, say $x_\alpha \in [h_n]_{n \le n(\alpha)}$, and for each successor (α, k) of α in T_∞, $x_{(\alpha,k)} \in [h_n]_{n(\alpha)<n\le n(\alpha,k)}$. We may then proceed to choose a full subtree T' of T_∞ having the properties (1) and (2) below, as we now describe.

First, we consider the first level of the tree, that is to say the sequence of elements $x_{(n)}$ with $n \in \mathbb{N}$. We may extract a subsequence for which $x_{(n)}^2$ converges weakly in $L_{p/2}$ to some $v_0 \in V(X)$ and then, by leaving out a finite number of terms, ensure that $|\mathbb{E}[x_{(n)}^2]^{1/2} - \mathbb{E}[v_0]^{1/2}| < \frac{1}{2}$.

We now continue by taking subsequences of the successors of each α in such a way that the following hold (for $n \in \mathbb{N}$, $\mathcal{H}_n$ denotes the σ-algebra generated by $(h_i : i \le n)$):

(1) the elements $x_{(\alpha,n)}^2$ (with $(\alpha, n) \in T'$) of $L_{p/2}$ converge weakly to some $v_\alpha \in V(X)$;

(2) for all $(\alpha, k) \in T'$ we have $\|\mathbb{E}[x_{(\alpha,k)}^2 \mid \mathcal{H}_{n(\alpha)}]^{1/2} - \mathbb{E}[v_\alpha \mid \mathcal{H}_{n(\alpha)}]^{1/2}\|_\infty < 2^{-|\alpha|-1}$.

To achieve the above, we use our earlier remark based on relexivity of $L_{p/2}$, and the fact that weak convergence implies norm convergence in the finite dimensional space $[h_n]_{n\le n(\alpha)}$.

We now take any branch (x_i) of the resulting subtree $(x_\alpha)_{\alpha \in T'}$. So $x_i = x_{\alpha_i}$ where α_i is the initial segment $(n_1, n_2, \dots, n_i)$ of some branch $(n_1, n_2, \dots)$ of T'. We consider the σ-algebras $\mathcal{F}_i$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_i = \mathcal{H}_{n(\alpha_i)}$ for $i \ge 1$ and write $\mathbb{E}_i$ for the conditional expectation relative to $\mathcal{F}_i$. Since we are dealing with a block tree, the sequence (x_i) is a block basis of the Haar basis, and hence a martingale-difference sequence with respect to $(\mathcal{F}_i)$. We may therefore apply Theorem 2.3 to conclude that the L_p-norm of a linear combination $\sum c_i x_j$ is C_p-equivalent to

$$\max\Big\{\Big(\sum |c_i|^p\Big)^{1/p}, \Big\|\sum c_i^2 \mathbb{E}_{i-1}[x_i^2]\Big\|_{p/2}^{1/2}\Big\}.$$

We shall show that provided we modify the constant of equivalence, we may replace the second term in this expression by

$$\Big\|\sum c_i^2 \mathbb{E}_{i-1}[x_i^2]\Big\|_1^{1/2},$$

which equals $\|\sum c_i x_i\|_2$. By construction, the conditional expectations $\mathbb{E}_{i-1}[x_i^2]$ are close to $\mathbb{E}_{i-1}[v_{i-1}]$, where, for $j \ge 1$, v_j denotes v_{α_j}. More precisely, we may use (2) above and the triangle inequality in $L_p(\ell_2)$ to obtain

(4.1)
$$\left|\Big\|\sum c_i^2 \mathbb{E}_{i-1}[x_i^2]\Big\|_{p/2}^{1/2} - \Big\|\sum c_i^2 \mathbb{E}_{i-1}[v_{i-1}]\Big\|_{p/2}^{1/2}\right| \le \Big\|\Big(\sum c_i^2 2^{-2i}\Big)^{1/2}\Big\|_p \le \max |c_i|.$$

We similarly get

(4.2)

$$\left|\left\|\sum c_i^2\mathbb{E}_{i-1}[x_i^2]\right\|_1^{1/2}-\left\|\sum c_i^2\mathbb{E}_{i-1}[v_{i-1}]\right\|_1^{1/2}\right|\leq\left\|(\sum c_i^2 2^{-2i})^{1/2}\right\|_2\leq\max|c_i|.$$

Using our assumption about $V(X)$, the fact that all the v_i are nonnegative, and inequalities (4.1) and (4.2), we obtain

$$\begin{aligned}\left\|\sum c_i^2\mathbb{E}_{i-1}[x_i^2]\right\|_{p/2}^{1/2}&\leq\left\|\sum c_i^2\mathbb{E}_{i-1}[v_{i-1}]\right\|_{p/2}^{1/2}+\max|c_i|\\&\leq\Big(\sum c_i^2\|\mathbb{E}_{i-1}[v_{i-1}]\|_{p/2}\Big)^{1/2}+\max|c_i|\\&\leq\Big(\sum c_i^2\left\|v_{i-1}\right\|_{p/2}\Big)^{1/2}+\max|c_i|\\&\leq\sqrt{M}\Big(\sum c_i^2\left\|v_{i-1}\right\|_1\Big)^{1/2}+\max|c_i|\\&=\sqrt{M}\left\|\sum c_i^2\mathbb{E}_{i-1}[v_{i-1}]\right\|_1^{1/2}+\max|c_i|\\&\leq\sqrt{M}\left\|\sum c_i^2\mathbb{E}_{i-1}[x_i^2]\right\|_1^{1/2}+\big(1+\sqrt{M}\big)\max|c_i|,\end{aligned}$$

which yields the left-most inequality in Proposition 4.2. The right-hand inequality is easy by Proposition 2.1 since $\|\cdot\|_p\geq\|\cdot\|_2$ and (x_i) is unconditional, being a block basis of the Haar basis. □

Corollary 4.3. *Let X be a subspace of L_p, where $p>2$, and assume that* (A) *holds in Proposition* 4.1. *Then X embeds isomorphically into $\ell_p\oplus\ell_2$.*

5. **Embedding $\ell_p(\ell_2)$ in X**

Theorem 5.1. *Let X be a subspace of L_p $(p>2)$ and suppose that* (B) *of Proposition* 4.1 *holds. Then X contains a subspace isomorphic to $\ell_p(\ell_2)$.*

The first step in the proof is to find ℓ_2-subspaces of X which have "thin support". The precise formulation of this notion that we shall use in the present section is given in the following lemma.

Lemma 5.2. *Suppose that* (B) *of Proposition* 4.1 *holds. Then, for every $M>0$ there is an infinite-dimensional subspace Y of X, on which the L_p and L_2 norms are equivalent, but in such a way that $\|y\|_p\geq M\|y\|_2$ for all $y\in Y$.*

Proof. By hypothesis, for every $M'>0$ there exists $v\in V(X)$ such that $\|v\|_1=1$ and $\|v\|_{p/2}>M'^2$. There is a weakly null sequence (x_n) in X such that x_n^2 converges weakly to v in $L_{p/2}$. By taking small perturbations of the x_n's (with respect to the L_p-norm) and by noting that the Cauchy-Schwarz inequality yields $\|x^2-y^2\|_{p/2}\le\|x-y\|_p\cdot\|x+y\|_p$, for x and $y\in L_p$, we may suppose that (x_n) is a block basis of the Haar basis. Since the sequence x_n^2 is positive and weakly convergent,

$$\|x_n^2\|_1=\mathbb{E}[x_n^2]\to\mathbb{E}[v]=\|v\|_1=1.$$

We can thus assume that $\|x_n\|_2=1$ for all n. We may choose a natural number K such that $\|\mathbb{E}[v\mid\mathcal{H}_K]\|_{p/2}>M'^2$ and by discarding the first few elements of (x_n) we have that $x_n\in[h_k]_{k>K}$, for all n. The x_n are martingale differences with respect to a subsequence $\mathcal{F}_n=\mathcal{H}_{k(n)}$ of the Haar filtration (with $k(0)=K$). Taking a further subsequence, we may suppose that

$$\Big\|\mathbb{E}[v\mid\mathcal{F}_{n-1}]^{1/2}-\mathbb{E}[x_n^2\mid\mathcal{F}_{n-1}]^{1/2}\Big\|_\infty<2^{-n}\text{ for all }n. \tag{5.1}$$

Because (x_n) is a martingale difference sequence, we can apply Theorem 2.3 to conclude that

$$\begin{aligned}\Big\|\sum c_nx_n\Big\|_p&\ge C_p^{-1}\Big\|\Big(\sum c_n^2\mathbb{E}[x_n^2|\mathcal{F}_{n-1}]\Big)^{1/2}\Big\|_p\\&=C_p^{-1}\Big\|\|(c_n\mathbb{E}^{1/2}[x_n^2|\mathcal{F}_{n-1}]:n\in\mathbb{N})\|_{\ell_2}\Big\|_p.\end{aligned}$$

If we use (5.1) and apply the triangle inequality in $L_p(\ell_2)$ we obtain

$$\begin{aligned}\Big\|\sum c_nx_n\Big\|_p&\ge C_p^{-1}\Big\|\|(c_n\mathbb{E}^{1/2}[x_n^2|\mathcal{F}_{n-1}]:n\in\mathbb{N})\|_{\ell_2}\Big\|_p\\&\ge C_p^{-1}\Big(\Big\|\|(c_n\mathbb{E}^{1/2}[v|\mathcal{F}_{n-1}]:n\in\mathbb{N})\|_{\ell_2}\Big\|_p-\|(c_n2^{-n}:n\in\mathbb{N})\|_{\ell_2}\Big)\\&=C_p^{-1}\left(\Big\|\Big(\sum c_n^2\mathbb{E}[v|\mathcal{F}_{n-1}]\Big)^{1/2}\Big\|_p-\Big(\sum c_n^22^{-2n}\Big)^{1/2}\right)\\&\ge\frac{M'-1}{C_p}\Big(\sum c_n^2\Big)^{1/2}.\end{aligned}$$

On the other hand, in L_2, the x_n are orthogonal, whence

$$\Big\|\sum c_nx_n\Big\|_2=\Big(\sum c_n^2\Big)^{1/2}.$$

Provided M' is chosen large enough, we have $\|y\|_p\ge M\|y\|_2$ for all $y\in[x_n]$ as required. □

The next step is to show that we can choose our ℓ_2-subspaces to have p-uniformly integrable unit balls. Recall that a subset A of L_p is said to be *p-uniformly integrable* if, for every $\varepsilon>0$ there exists $K>0$ such that $\|x1_{[|x|>K]}\|_p<\varepsilon$ for all $x\in A$. We shall need the following standard martingale lemma.

LEMMA 5.3. *Let (x_n) be a martingale difference sequence that is p-uniformly integrable. Then the set of linear combinations of the x_n's with ℓ_2-normalized coefficients is also p-uniformly integrable.*

Proof. We assume that $\|x_n\|_2 \le 1$ for all n and consider a vector y of the form $\sum_n c_n x_n$ with $\sum_n c_n^2 = 1$, noting that $\|y\|_2^2 = \sum c_n^2 \|x_n\|_2^2 \le 1$. Given $\varepsilon > 0$, we choose $K > \varepsilon^{-1}$ such that $\|x_j 1_E\|_2 < \varepsilon$ for all j whenever $\mathbb{P}(E) < K^{-1}$. We consider the martingale (y_n) where $y_n = \sum_{j \le n} c_j x_j$ (thus $y = y_\infty$) and introduce the stopping time

$$\tau = \inf\{n \in \mathbb{N} : |y_n| > K\}.$$

By Doob's inequality, $\mathbb{P}[\tau < \infty] \le K^{-1}\|y\|_1 \le K^{-1}$. We note that if $\tau < \infty$, then $|y_\tau| \le K + |c_\tau x_\tau|$ so that

$$|y| \le K + |y - y_\tau| + |c_\tau x_\tau 1_{[\tau<\infty]}|.$$

We shall estimate the L_p-norms of the second two terms. For the first of these, we note that $(y_k - y_{k\wedge\tau})$ is a martingale, so that (C only depends on p)

$$\begin{aligned}
\|y - y_\tau\|_p &\le C \Big\| \sum_n c_n^2 x_n^2 1_{[\tau<n]} \Big\|_{p/2}^{1/2} \quad \text{[by the square function inequality]}\\
&\le C \Big(\sum c_n^2 \|x_n^2 1_{[\tau<n]}\|_{p/2} \Big)^{1/2} \quad \text{[by the triangle inequality in } L_{p/2}]\\
&\le C \sup_n \|x_n 1_{[\tau<\infty]}\|_p \ \ \Big[\text{since } \sum c_n^2 \le 1\Big]\\
&\le C\varepsilon \qquad \text{[because } \mathbb{P}[\tau<\infty] \le K^{-1}].
\end{aligned}$$

For the second term we use the fact that the sets $[\tau = n]$ are disjoint, so that

$$\begin{aligned}
\|c_\tau x_\tau 1_{[\tau<\infty]}\|_p &= \Big\| \sum_n c_n x_n 1_{[\tau=n]} \Big\|_p\\
&= \Big(\sum_n |c_n|^p \|x_n 1_{[\tau=n]}\|_p^p \Big)^{1/p} \le \sup_n \|x_n 1_{[\tau<\infty]}\|_p \le \varepsilon
\end{aligned}$$

as before. Thus,

$$\|y 1_{[|y|>2K]}\|_p \le K \mathbb{P}^{1/p}\Big[|y - y_\tau| + |c_\tau x_\tau 1_{[\tau<\infty]}| > K\Big] + (C+1)\varepsilon \le 2(1+C)\varepsilon,$$

which implies our claim. □

LEMMA 5.4. *Let Y be a subspace of L_p $(p > 2)$, which is isomorphic to ℓ_2. There is an infinite dimensional subspace Z of Y such that the unit ball B_Z is p-uniformly integrable.*

Proof. Let (y_n) be a normalized sequence in Y equivalent to the unit vector basis of ℓ_2. By the Subsequence Splitting Lemma (see, for instance Theorem IV.2.8 of [GD92]), we can write $y_n = x_n + z_n$, where the sequence (x_n) is p-uniformly integrable, and the z_n are disjointly supported. So (x_n)

and (z_n) are weakly null. Taking a subsequence, we may suppose that the (x_n) is a martingale difference sequence, so that the set of all ℓ_2-normalized linear combinations $\sum c_n x_n$ is also p-uniformly integrable.

We now consider ℓ_2-normalized blocks of the form

$$y_k' = (N_k - N_{k-1})^{-1/2} \sum_{N_{k-1}<n\le N_k} y_n = x_k' + z_k',$$

where

$$x_k' = (N_k - N_{k-1})^{-1/2} \sum_{N_{k-1}<n\le N_k} x_n \text{ and } z_k' = (N_k - N_{k-1})^{-1/2} \sum_{N_{k-1}<n\le N_k} z_n.$$

Since the z_n are disjointly supported in L_p we have $\|z_k'\|_p \le (N_k - N_{k-1})^{1/p-1/2}$, so we can choose the N_k such that $\|z_k'\|_p < 2^{-k}$. The sequence (x_k'), being ℓ_2 normalized linear combinations of the x_n, are p-uniformly integrable. Hence the y_k', which are small perturbations of the x_k', are also p-uniformly integrable. Another application of Lemma 5.3 yields the result. $\square$

We are now ready for the proof of Theorem 5.1.

Proof of Theorem 5.1. By Lemmas 5.2 and 5.4 there exists, for each $M>0$, a subspace Z_M of X, isomorphic to ℓ_2 with p-uniformly integrable unit ball such that

$$\|y\|_p \ge M\|y\|_2$$

for all $y \in Z_M$. For a specified $\varepsilon > 0$, we shall choose inductively $M_1 < M_2 < \cdots$ and define $Y_n = Z_{M_n}$, such that

$$\||y_m| \wedge |y_n|\|_p \le \varepsilon/n2^n, \tag{5.2}$$

whenever $y_m \in B_{Y_m}$, $y_n \in B_{Y_n}$ and $m < n$.

To achieve this, we start by taking an arbitrary value for M_1, say $M_1 = 1$. Recursively, if $M_1, \ldots, M_n$ have been chosen, we use the p-uniform integrability of $\bigcup_{m\le n} B_{Y_m}$ to find K_n such that $\Big\||y| - |y| \wedge K_n\Big\|_p < \varepsilon/(n+1)2^{n+2}$ whenever $y \in B_{Y_m}$ and $m \le n$.

We now choose M_{n+1} such that $M_{n+1}^2 > K_n^{p-2}(n+1)^p 2^{p(n+2)}\varepsilon^{-p}$. We need to check that (5.2) is satisfied, so let $y_{n+1} \in B_{Y_{n+1}}$ and let $y_m \in B_{Y_m}$ with $m \le n$. We have that

$$|y_m| \wedge |y_{n+1}| \le K_n \wedge |y_{n+1}| + (|y_m| - |y_m| \wedge K_n)$$

and we have chosen K_n in such a way as to ensure that

$$\Big\||y_m| - |y_m| \wedge K_n\Big\|_p < \varepsilon/(n+1)2^{n+2}.$$

For the first term, we note that

$$\mathbb{E}[(K_n \wedge |y_{n+1}|)^p] \le \mathbb{E}[K_n^{p-2}|y_{n+1}|^2] = K_n^{p-2}\|y_{n+1}\|_2^2 \le K_n^{p-2}M_{n+1}^{-2},$$

which is smaller than $\varepsilon^p(n+1)^{-p}2^{-p(n+2)}$, by our choice of M_{n+1}.

Now let $y_n \in S_{Y_n}$ for all $n \in \mathbb{N}$. We shall show that the y_n's are small perturbations of elements that are disjoint in L_p. Indeed, let us set

$$y'_n = \operatorname{sign}(y_n)\Big(|y_n| - |y_n| \wedge \bigvee_{m\neq n} |y_m|\Big).$$

Then the y'_n are disjointly supported and from (5.2),

$$\begin{aligned}\|y_n - y'_n\|_p = \Big\| |y_n| \wedge \bigvee_{m\neq n} |y_m| \Big\|_p &\le \sum_{m\neq n} \Big\| |y_n| \wedge |y_m| \Big\|_p \\ &\le (n-1)\varepsilon/n2^n + \sum_{m>n} \varepsilon/m2^m < \varepsilon/2^n.\end{aligned}$$

Standard manipulation of inequalities now shows us that the closure of the sum $\sum_n Y_n$ in L_p is almost an ℓ_p-sum. Indeed,

$$\begin{aligned}(1-2\varepsilon)\Big(\sum |c_n|^p\Big)^{1/p} &\le \Big(\sum |c_n|^p \|y'_n\|_p^p\Big)^{1/p} - \varepsilon\Big(\sum |c_n|^p\Big)^{1/p} \\ &= \Big\|\sum c_n y'_n\Big\|_p - \varepsilon\Big(\sum |c_n|^p\Big)^{1/p} \\ &\le \Big\|\sum c_n y_n\Big\|_p \\ &\le \Big\|\sum c_n y'_n\Big\|_p + \varepsilon\Big(\sum |c_n|^p\Big)^{1/p} \le (1+\varepsilon)\Big(\sum |c_n|^p\Big)^{1/p}.\end{aligned}$$

At this point in the proof, we have obtained subspaces Y_n of X, each isomorphic to ℓ_2 such that the closed linear span $\overline{\sum_n Y_n}$ is almost isometric to $(\bigoplus Y_n)_p$. By stability (see [KM81] or [AO01]) we can take, for each n, a subspace X_n of Y_n which is $(1+\varepsilon)$-isomorphic to ℓ_2. In this way we obtain a subspace of X which is almost isometric to $\ell_p(\ell_2)$. ☐

The last part of the claim of Theorem B — namely that we can pass to a further subspace of X which is still $(1+\theta)$-isomorphic to $\ell_p(\ell_2)$ and, moreover, complemented in L_p — follows from our results in the next section. G. Schechtman [Sch] showed us that if one is not concerned with minimizing the norm of the projection, then there is a short argument that gives a complemented copy of $\ell_p(\ell_2)$. We thank him for allowing us to present it here.

Proposition 5.5. *Let $X \subset L_p$ be isomorphically equivalent to $\ell_p(\ell_2)$. Then there is a subspace Y of X which is isomorphic to $\ell_p(\ell_2)$ and complemented in L_p.*

Proof. Let $\{x(m,n) : m,n \in \mathbb{N}\} \subset X$ be a normalized basis of X equivalent to the usual unconditional basis of $\ell_p(\ell_2)$; i.e., there is a constant $C \ge 1$

so that

$$\Big\|\sum_{m,n\in\mathbb{N}} a(m,n)x(m,n)\Big\| \overset{C}{\sim} \Big(\sum_{m\in\mathbb{N}}\Big(\sum_{n\in\mathbb{N}} a(m,n)^2\Big)^{p/2}\Big)^{1/p}$$

for all $(a(m,n))\in c_{00}(\mathbb{N}^2)$.

In [PR75] it was shown that for any $C>1$ there is a $g_p(C)<\infty$ so that every subspace E of L_p, which is C isomorphic to ℓ_2, is $g_p(C)$ complemented in L_p. For $m\in\mathbb{N}$ let $P_m : L_p\to[(x(m,n):n\in\mathbb{N}]$ be a projection of norm at most $g_p(C)$. We can write

$$P_m(x)=\sum_{n\in\mathbb{N}} x^*(m,n)(x)x(m,n) \text{ for } x\in L_p,$$

where $(x^*(m,n):n\in\mathbb{N})$ is a weakly null sequence in L_q, $\frac{1}{p}+\frac{1}{q}=1$, and biorthogonal to $x(m,n):n\in\mathbb{N})$. By passing to subsequences, using a diagonal argument, and perturbing we may assume that there is a blocking $(H(m,n):m,n\in\mathbb{N})$ of the Haar basis of L_p, in some order, so that $x(m,n)\in H(m,n)$ and $x^*(m,n)\in H^*(m,n)$, for $m,n\in\mathbb{N}$, where $(H^*(m,n))$ denotes the blocking of the Haar basis in L_q which corresponds to $(H(m,n))$.

We will show that the operator

$$P: L_p\to L_p,\quad x\mapsto \sum_{m,n\in\mathbb{N}} x^*(m,n)(x)x(m,n)$$

is bounded, and thus it is a bounded projection onto $[x(m,n):m,n\in\mathbb{N}]$.

For $y=\sum_{m,n\in\mathbb{N}} y(m,n)$, with $y(m,n)\in H(m,n)$, if $m,n\in\mathbb{N}$, we deduce that

$$\begin{aligned}
\|P(y)\| &= \Big\|\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} x^*(m,n)(y(m,n))x(m,n)\Big\| \\
&\le C\Big(\sum_{m\in\mathbb{N}}\Big(\sum_{n\in\mathbb{N}}(x^*(m,n)(y(m,n)))^2\Big)^{p/2}\Big)^{1/p} \\
&\le C^2\Big(\sum_{m\in\mathbb{N}}\|P_m(y_m)\|^p\Big)^{1/p} \le C^2 g_p(C)\Big(\sum_{m\in\mathbb{N}}\|y_m\|^p\Big)^{1/p},
\end{aligned}$$

where $y_m=\sum_{n\in\mathbb{N}} y(m,n)$ for $m\in\mathbb{N}$.

The Haar basis is unconditional in L_p. If we denote the unconditional constant in L_p by U_p, we deduce from Proposition 2.1 that

$$\|y\|\ge U_p^{-1}(\sum_{m\in\mathbb{N}}\|y_m\|^p)^{1/p},$$

which implies our claim. □

Remark. G. Schechtman [Sch] has also proved, by a more complicated argument, that if $X\subset L_p$, $1<p<2$ is an isomorph of $\ell_p(\ell_2)$, then X contains a copy of $\ell_p(\ell_2)$ which is complemented in L_p.

Let us now deduce the statement of Corollary D.

Proof of Corollary D. First assume that X embeds into $\ell_p \oplus \ell_2$. Note that every weakly null sequence (x_n) can be turned into a weakly null tree (x_α), whose branches are exactly the subsequences of (x_n) (put $x_{(n_1,n_2,\ldots,n_\ell)} = x_{n_\ell}$ for $(n_1, n_2, \ldots, n_\ell) \in T_\infty$). This fact, together with the remarks at the beginning of the proof of Theorem A (about the existence of K), shows that condition (b) of Theorem A for a subspace X of L_p implies that there exists a $K \geq 1$, so that every weakly null sequence in S_X admits a subsequence (x_i) satisfying condition (1.1) in (b) of Theorem A for all scalars (a_i).

Conversely, assume that X does not embed into $\ell_p \oplus \ell_2$. Then Propositions 4.1 and 4.2 together with Theorem A imply that condition (B) of Proposition 4.1 is satisfied. Now, using Lemma 5.2, we can find for every $M < \infty$ a subspace Y of X which is isomorphic to ℓ_2, so that $\|\cdot\|_p \geq M\|\cdot\|_2$ on Y. This implies that there cannot be a $K \geq 1$, so that every weakly null sequence in S_X admits a subsequence (x_i) satisfying (1.4). $\square$

6. Improving the embedding via random measures

We shall give a quick review of what we need from the theory of stable spaces and random measures. We shall then obtain the optimally complemented embeddings of $\ell_p(\ell_2)$.

We start this section by recalling some facts about random measures and their relation to types on L_p. The introductory part is valid for $1 < p < \infty$. Later we will restrict ourselves again to the case $p > 2$. As far as possible, we shall follow the notation and terminology of [Ald81]; for the theory of types and stability we refer the reader to [KM81] (or [AO01]). The lecture notes of Garling [Gar82] is one of the few works where the connection between random measures and types on function spaces is explicitly considered.

We shall denote by $\mathcal{P}$ the set of probability measures on $\mathbb{R}$ which is a Polish space for its usual topology. This topology, often called the "narrow topology", can be thought of as the topology induced by the weak* topology $\sigma(\mathcal{C}_b(\mathbb{R})^*, \mathcal{C}_b(\mathbb{R}))$.

A *random measure* on $(\Omega, \Sigma, \mathbb{P})$ is a mapping $\xi : \omega \mapsto \xi_\omega; \Omega \to \mathcal{P}$ which is measurable from Σ to the Borel σ-algebra of $\mathcal{P}$. The set of all such random measures is denoted by $\mathcal{M}$ and is a Polish space when equipped with what Aldous calls the wm-*topology*. Sequential convergence for this topology can be characterized by saying that $\xi^{(n)} \xrightarrow{\text{wm}} \xi$ if and only if

$$\mathbb{E}\left[1_F \int_{\mathbb{R}} f(t)\mathrm{d}\xi^{(n)}(t)\right] \to \mathbb{E}\left[1_F \int_{\mathbb{R}} f(t)\mathrm{d}\xi(t)\right],$$

for all $F \in \Sigma$ and all $f \in \mathcal{C}^{\mathrm{b}}(\mathbb{R})$. In interpreting the expectation operator in the above formula (and in similar expressions involving "implicit" ω's) the reader should bear in mind that ξ is random. If we translate the expectation into integral notation,

$$\mathbb{E}\left[1_F \int_{\mathbb{R}} f(t)\mathrm{d}\xi(t)\right] \text{ becomes } \int_F \int_{\mathbb{R}} f(t)\,\mathrm{d}\xi_\omega(t)\,\mathrm{d}\mathbb{P}(\omega).$$

It is sometimes useful to use the notation ξ_F, when F is a nonnull set in Σ for the probability measure given by

$$\int_{\mathbb{R}} f(t)\,\mathrm{d}\xi_F(t) = \mathbb{P}(F)^{-1}\mathbb{E}[1_F \int_{\mathbb{R}} f(t)\,\mathrm{d}\xi(t)] \quad (f \in \mathcal{C}_0(\mathbb{R})).$$

The usual convolution operation on $\mathcal{P}$ may be extended to an operation on $\mathcal{M}$ by defining $\xi * \eta$ to be the random measure with $(\xi * \eta)_\omega = \xi_\omega * \eta_\omega$. Garling (Proposition 8 of [Gar82]) observes that this operation is separately continuous for the wm topology. This result is also implicit in Lemma 3.14 of [Ald81]. We may also introduce a "scalar multiplication": when $\xi \in \mathcal{M}$ and α is a random variable, we define the random measure $\alpha.\xi$ by setting

$$\int f(t)\,\mathrm{d}(\alpha.\xi)(t) = \int_{\mathbb{R}} f(\alpha t)\,\mathrm{d}\xi(t) \quad (f \in \mathcal{C}^{\mathrm{b}}(\mathbb{R})).$$

Every random variable x on $(\Omega, \Sigma, \mathbb{P})$ defines a random (Dirac) measure $\omega \mapsto \delta_{x(\omega)}$. Aldous [Ald81, after Lemma 2.14] has remarked that (provided that the probability space $(\Omega, \Sigma, \mathbb{P})$ is atomless) these δ_x form a wm-dense subset of $\mathcal{M}$. While we do not need this fact here, it may be helpful to note that the definition given above of $\alpha.\xi$ is so chosen that $\delta_{\alpha x_n} \xrightarrow{\mathrm{wm}} \alpha.\xi$ whenever $\delta_{x_n} \xrightarrow{\mathrm{wm}} \xi$. The L_p-norms extend to wm-lower semicontinuous $[0,\infty]$-valued functions $|\cdot|_p$ on $\mathcal{M}$, defined by

$$|\xi|_p = \mathbb{E}\left[\int_{\mathbb{R}} |t|^p\,\mathrm{d}\xi(t)\right]^{1/p}.$$

We shall write $\mathcal{M}_p$ for the set of all ξ for which $|\xi|_p$ is finite.

As a special case of the characterization of wm-compactness by the condition of "tightness" we note that a subset of $\mathcal{M}_p$ which is bounded for $|\cdot|_p$ is wm-relatively compact. In particular, if (x_n) is a sequence that is bounded in L_p, then there is a subsequence (x_{n_k}) such that $\delta_{x_{n_k}} \xrightarrow{\mathrm{wm}} \xi$ for some $\xi \in \mathcal{M}_p$. Moreover, if (x_n) is p-uniformly integrable, an easy truncation argument shows that

$$\lim_{n\to\infty} \|x_n\|_p = \lim_{n\to\infty} \mathbb{E}\bigg(\int |t|^p d\delta_{x_n}(t)\bigg) = \mathbb{E}\bigg(\int |t|^p d\xi(t)\bigg).$$

For a subspace X of L_p we write $\mathcal{M}_p(X)$ for the set of all ξ that arise as wm-limits of sequences (δ_{x_n}) with (x_n) an L_p-bounded sequence in X. It is an easy consequence of separate continuity that $\mathcal{M}_p(X)$ is closed under the convolution operation $*$ (cf. the proof of [Ald81, Prop. 3.9]).

We recall that a function $\tau : X \to \mathbb{R}$ on a (separable) Banach space X is called a *type* if there is a sequence (x_n) in X such that for all $y \in X$,

$$\|x_n + y\| \to \tau(y) \quad \text{as } n \to \infty.$$

The set of all types on X is denoted by $\mathcal{T}_X$ and is a locally compact Polish space for the *weak* topology; this topology may be characterized by saying that $\tau_n \xrightarrow{\text{w}} \tau$ if $\tau_n(y) \to \tau(y)$ for all $y \in X$. If we introduce, for each $x \in X$, the *degenerate type* τ_x defined by

$$\tau_x(y) = \|x + y\|,$$

then $\mathcal{T}_X$ is the w-closure of the set of all τ_x. We introduce a "scalar multiplication" of types, defining $\alpha.\tau$, for $\alpha \in \mathbb{R}$ and $\tau \in \mathcal{T}_X$, by setting

$$\alpha.\tau = \text{w-lim}\, \tau_{\alpha x_n} \quad \text{when} \quad \tau = \text{w-lim}\, \tau_{x_n}.$$

A Banach space X is *stable* if, for x_m and y_n in X,

$$\lim_{m\to\infty} \lim_{n\to\infty} \|x_m + y_n\| = \lim_{n\to\infty} \lim_{m\to\infty} \|x_m + y_n\|,$$

whenever the relevant limits exist. All L_p-spaces $(1 \le p < \infty)$ are stable [KM81].

Stability of a Banach space X permits the introduction of a (commutative) binary operation $*$ on $\mathcal{T}_X$, defined by

$$\tau * \upsilon(z) = \lim_{m\to\infty} \lim_{n\to\infty} \|x_m + y_n + z\|$$

when $\tau = \text{w-lim}\, \tau_{x_m}$ and $\upsilon = \text{w-lim}\, \tau_{y_n}$.

A type $\tau \in \mathcal{T}_X$ is said to be an ℓ_q-type if

$$(\alpha.\tau) * (\beta.\tau) = (|\alpha|^q + |\beta|^q)^{1/q}.\tau$$

for all real α, β. The big theorem of [KM81] shows first that on every stable space there are ℓ_q-types for some value(s) of q, and secondly that the existence of an ℓ_q type implies that the space has subspaces almost isometric to ℓ_q. In fact the proof of Théorème III.1 in [KM81] proves something slightly more than the existence of such a subspace. We now record the statement we shall need.

Proposition 6.1. *Let X be a stable Banach space, let $1 \le q < \infty$, and let (x_n) be a sequence in X such that τ_{x_n} converges to an ℓ_q-type τ on X. Then there is a subsequence (x_{n_k}) such that τ_{z_n} converges to τ for every ℓ_q-normalized block subsequence (z_n) of (x_{n_k}).*

The results of [KM81] were extended, and gave an alternative approach to the theorem of [Ald81], which obtained ℓ_q's in subspaces of L_1 using random measures. We shall need elements from both approaches. The link is provided by the following lemma, for which we refer the reader to the final paragraphs of [Gar82]. We shall write $\mathcal{T}_p$ for $\mathcal{T}_{L_p}$ and, when X is a subspace of L_p, we shall write $\mathcal{T}_p(X)$ for the weak closure in $\mathcal{T}_p$ of the set of all τ_x with $x \in X$.

LEMMA 6.2. *Let (x_n) be a bounded sequence in L_p, and suppose that $\delta_{x_n} \xrightarrow{wm} \xi$ in $\mathcal{M}$. Suppose further that $\|x_n\|_p \to \alpha$ as $n \to \infty$. Then, for all $y \in L_p$,*

$$\|x_n + y\|_p^p \to \mathbb{E}\left[\int_{\mathbb{R}} |y+t|^p \, d\xi(t)\right] + \beta^p,$$

where the nonnegative constant β is given by

$$\alpha^p = \|\xi\|_p^p + \beta^p.$$

The sequence (x_n) is p-uniformly integrable if and only if $\beta = 0$.

We thus have the following formula showing how the type $\tau = \lim \tau_{x_n} \in \mathcal{T}_p$ is related to the random measure $\xi = \text{wm-}\lim \delta_{x_n} \in \mathcal{M}_p$ and the index of p-uniform integrability β:

$$\tau(y)^p = \mathbb{E}\left[\int_{\mathbb{R}} |y+t|^p \mathrm{d}\xi(t)\right] + \beta^p. \tag{6.1}$$

If $q < p$ then a sequence (x_n) as above in L_p can be thought of as a sequence in L_q. If we wish to distinguish the type determined on L_q from the type on L_p, we use superscripts. Of course,

$$\tau^{(q)}(y)^q = \mathbb{E}\left[\int_{\mathbb{R}} |y+t|^q \mathrm{d}\xi(t)\right]$$

with no "β" term, because an L_p-bounded sequence is q-uniformly integrable.

The * operations on $\mathcal{T}_p$ and on $\mathcal{M}_p$ are related by the following lemma, also to be found in [Gar82].

LEMMA 6.3. *Let τ_1 and τ_2 be types on L_p represented as*

$$\tau_1(y)^p = \mathbb{E}\left[\int_{\mathbb{R}} |y+t|^p \, d\xi_1(t)\right] + \beta_1^p \text{ and } \tau_2(y)^p = \mathbb{E}\left[\int_{\mathbb{R}} |y+t|^p \, d\xi_2(t)\right] + \beta_2^p.$$

Then

$$(\tau_1 * \tau_2)(y)^p = \mathbb{E}\left[\int_{\mathbb{R}} |y+t|^p \, d(\xi_1 * \xi_2)(t)\right] + \beta_1^p + \beta_2^p.$$

It has been noted already in the literature (e.g., [Gar82]) that the representation given in (6.1) is not in general unique. However, for most values of p, it is, as we now show.

PROPOSITION 6.4. *Let $1 \le p < \infty$ and assume that p is not an even integer. In the representation of a type τ on L_p by the formula (6.1) the random measure ξ and the constant β are uniquely determined by τ. If (x_n) is any sequence in L_p with $\tau_{x_n} \xrightarrow{w} \tau$ we have $\delta_{x_n} \xrightarrow{wm} \xi$ and*

$$\inf_M \lim_{n\to\infty} \|x_n 1_{[|x_n| \ge M]}\|_p = \beta.$$

Proof. Suppose that ξ, β and ξ', β' yield the same type τ. For any nonnull $E \in \Sigma$ and any real number u, we consider $\tau(y)$ where $y = u1_E \in L_p$ to obtain

$$\mathbb{E}\left[\int_{\mathbb{R}} |t + u1_E|^p \mathrm{d}\xi(t)\right] + \beta^p = \mathbb{E}\left[\int_{\mathbb{R}} |t + u1_E|^p \mathrm{d}\xi'(t)\right] + \beta'^p,$$

or, equivalently,

$$\int_{\mathbb{R}} |t + u|^p \mathrm{d}\xi_E(t) = \int_{\mathbb{R}} |t + u|^p \mathrm{d}\xi'_E(t) + \alpha^p,$$

where

$$\mathbb{P}(E)\alpha^p = \beta'^p - \beta^p + \mathbb{E}\left[1_{\Omega\setminus E}\int |t|^p \mathrm{d}\xi'(t) - 1_{\Omega\setminus E}\int |t|^p \mathrm{d}\xi(t)\right].$$

By the Equimeasurability Theorem (cf. [KK01, p. 903]), $\alpha = 0$ and the measures ξ_E and ξ'_E are equal. Since this is true for all E, $\xi = \xi'$.

Now let (x_n) be any sequence with $\tau_{x_n} \xrightarrow{\mathrm{w}} \tau$. By the uniqueness that we have just proved, the only cluster point of the sequence δ_{x_n} in $\mathcal{M}$ is ξ. Since (by L_1-boundedness) $\{\delta_{x_n} : n \in \mathbb{N}\}$ is relatively wm-compact in $\mathcal{M}$, it must be that $\delta_{x_n} \xrightarrow{\mathrm{wm}} \xi$. □

We have already noted that $\mathcal{M}_p(X)$ is closed under $*$ when X is a subspace of L_p. The next proposition, which is closely related to that of [Ald81, Prop. 3.9], shows that under appropriate conditions, $\mathcal{M}_p(X)$ is wm-closed.

PROPOSITION 6.5. *Let $1 \le p < \infty$ and let X be a subspace of L_p with no subspace isomorphic to ℓ_p. Then $\mathcal{M}_p(X)$ is* wm-*closed in $\mathcal{M}$.*

Proof. The hypothesis implies that the L_p-norm is equivalent to the L_1-norm on X, so that we may regard X as a (reflexive) subspace of L_1. Aldous [Ald81, Lemma 3.12] shows (by a straightforward uniform integrability argument) that $\xi \mapsto |\xi|_1$ is wm-continuous and finite on $\mathcal{D}$, where $\mathcal{D}$ is the wm-closure of $\{\delta_x : x \in X\}$. Thus every ξ in $\mathcal{D}$ is in the wm-closure of an L_1-bounded subset of X, and hence by equivalence of norms, in $\mathcal{M}_p(X)$. □

To finish this round-up of types and random measures, we need to mention the connection between ℓ_2-types and the normal distribution (a special case of the connection between ℓ_q-types and symmetric stable laws). We write γ for the probability measure (or *law*) of a standard $\mathcal{N}(0,1)$ random variable. If σ is a nonnegative random variable, then $\sigma.\gamma$ is a random measure (a normal distribution with random variance). Provided $\sigma \in L_p$, this random measure defines a type on L_p by

$$\tau(y)^p = \mathbb{E}\left[\int_{\mathbb{R}} |y + t|^p \, \mathrm{d}(\sigma.\gamma)(t)\right] = \mathbb{E}\left[\int_{\mathbb{R}} |y + \sigma t|^p \, \mathrm{d}\gamma(t)\right].$$

It is a property of the normal distribution that $(\alpha.\gamma) * (\beta.\gamma) = (\alpha^2 + \beta^2)^{1/2}.\gamma$ for real α, β. By Lemma 6.3, this allows us to see that τ is an ℓ_2-type on L_p.

We are finally ready to return to the main subject matter of this paper.

Lemma 6.6. *Let X be a subspace of L_p, with $p > 2$, and let v be a nonzero element of $L_{p/2}$. The following are equivalent*:

(1) $v \in V(X)$,

(2) *there exists $\xi \in \mathcal{M}_p(X)$ such that $\int_{\mathbb{R}} t\, d\xi = 0$ and $\int_{\mathbb{R}} t^2\, d\xi = v$ almost surely*,

(3) $\sqrt{v}.\gamma \in \mathcal{M}_p(X)$.

Proof. We start by assuming (1). Let (x_n) be a weakly null sequence in X such that (x_n^2) converges weakly to v in $L_{p/2}$. Replacing (x_n) with a subsequence, we may suppose that $\delta_{x_n} \xrightarrow{\text{wm}} \xi$ for some $\xi \in \mathcal{M}_p(X)$. Since the sequence (x_n) is L_p-bounded, it is 2-uniformly integrable and so

$$\mathbb{E}\left[1_E \int_{\mathbb{R}} t \mathrm{d}\xi(t)\right] = \lim \mathbb{E}\left[1_E x_n\right] = 0 \tag{6.2}$$

and

$$\mathbb{E}\left[1_E \int_{\mathbb{R}} t^2 \mathrm{d}\xi(t)\right] = \lim \mathbb{E}\left[1_E x_n^2\right] = \mathbb{E}\left[1_E v\right], \tag{6.3}$$

for all $E \in \Sigma$. This yields (2).

We now assume (2). Let (x_n) be an L_p-bounded sequence in X such that δ_{x_n} is wm-convergent to ξ. Since $\int_{\mathbb{R}} d\xi(t) = 0$ a.s. it follows that (x_n) is weakly null and since $\xi \neq \delta_0$, $\|x_n\|_2$ does not tend to zero. By [KP62], it follows that X_0, the closed linear span of a subsequence of (x_i), is isomorphic to ℓ_2. The assumption about ξ is that, for almost all ω, the probability measure ξ_ω is the law of a random variable with mean 0 and variance $v(\omega)$.

By the Central Limit Theorem

$$n^{-1/2}.\underbrace{(\xi_\omega * \xi_\omega * \cdots * \xi_\omega)}_{n \text{ terms}}$$

tends to $\sqrt{v(\omega)}.\gamma$ for all such ω. So in $\mathcal{M}$ we have

$$n^{-1/2}.(\xi * \xi * \cdots * \xi) \xrightarrow{\text{wm}} \sqrt{v}.\gamma.$$

Since $\mathcal{M}_p(X_0)$ is closed under convolution and is closed in the wm-topology (by Proposition 6.5), we see that $\sqrt{v}.\gamma \in \mathcal{M}_p(X_0) \subseteq \mathcal{M}_p(X)$.

Finally, if we assume (3) we may take (x_n) to be an L_p-bounded sequence in X such that $\delta_{x_n} \xrightarrow{\text{wm}} \sqrt{v}.\gamma$. Calculations like those used in the proof of (1) $\implies$ (2), justified by 2-uniform integrability, show that (x_n) is weakly null and that x_n^2 tends weakly to v. $\square$

We shall say that a sequence (y_n) in L_p is a *stabilized ℓ_2 sequence* with limiting conditional variance v if, for every ℓ_2 normalized block subsequence (z_n) of (y_n), the following are true:

$$\delta_{z_n} \xrightarrow{\text{wm}} \sqrt{v}.\gamma \text{ as } n \to \infty; \tag{6.4}$$

$$\|z_n\|_p \to \gamma_p \|\sqrt{v}\|_p \text{ as } n \to \infty. \tag{6.5}$$

(Recall that $\gamma_p = \|x\|_p$, where x is a symmetric L_2 normalized Gaussian random variable.) For p not an even integer, it is not hard to establish the existence of such sequences using Propositions 6.1 and 6.4. The proof of the next proposition avoids the irritating problem posed by nonunique representations, by switching briefly to the L_1-norm.

Proposition 6.7. *Let X be a closed subspace of L_p $(p>2)$ and let v be a nonzero element of $V(X)$. Then there exists a stabilized ℓ_2 sequence in X with limiting conditional variance v.*

Proof. By Lemma 6.6 the random measure $\sqrt{v}.\gamma$ is in $\mathcal{M}_p(X)$. Let (x_n) be a bounded sequence in X with $\delta_{x_n} \xrightarrow{\text{wm}} \sqrt{v}.\gamma$. For the moment, think of the x_n as elements of L_1 and consider the types $\tau^{(1)}_{x_n}$ defined on L_1. By L_p-boundedness, the sequence (x_n) is uniformly integrable, so the sequence $(\tau^{(1)}_{x_n})$ converges weakly to the ℓ_2-type $\tau^{(1)}$, where

$$\tau^{(1)}(y) = \mathbb{E}\left[\int |y + \sqrt{v}t| \mathrm{d}\gamma(t)\right].$$

By Proposition 6.1 we may replace (x_n) by a subsequence in such a way that $\tau^{(1)}_{z_n} \xrightarrow{\text{w}} \tau^{(1)}$ for every ℓ_2-normalized block subsequence (z_n). By Proposition 6.4 we have $\delta_{z_n} \xrightarrow{\text{wm}} \sqrt{v}.\gamma$ for all such (z_n).

We now return to the L_p-norm, for which we can assume, after passing to a subsequence, if necessary, that (x_n) is equivalent to the unit vector basis of ℓ_2. By stability of L_p there is an ℓ_2-normalized block subsequence (y_n) such that $\tau^{(p)}_{y_n} \xrightarrow{\text{w}} \tau^{(p)}$ for some ℓ_2-type $\tau^{(p)}$ on L_p. Moreover, by Proposition 6.1 we can arrange that $\tau^{(p)}_{z_n} \xrightarrow{\text{w}} \tau^{(p)}$ for every further such ℓ_2-normalized block subsequence (z_n). By (6.1),

$$\tau^{(p)}(y)^p = \mathbb{E}\left[\int_{\mathbb{R}} |y + \sqrt{v}t|^p \mathrm{d}\gamma(t)\right] + \beta^p$$

for some nonnegative constant β. Now $\tau^{(p)}$ is an ℓ_2-type, so $\tau^{(p)} * \tau^{(p)} = \sqrt{2}.\tau^{(p)}$. That is to say

$$(\tau^{(p)} * \tau^{(p)})(y)^p = \mathbb{E}\left[\int_{\mathbb{R}} |y + \sqrt{2v}t|^p \mathrm{d}\gamma(t)\right] + (\sqrt{2}\beta)^p.$$

On the other hand, by Lemma 6.3,

$$\begin{aligned}(\tau^{(p)} * \tau^{(p)})(y)^p &= \mathbb{E}\left[\int_{\mathbb{R}} |y + \sqrt{v}t|^p \mathrm{d}(\gamma * \gamma)(t)\right] + 2\beta^p \\ &= \mathbb{E}\left[\int_{\mathbb{R}} |y + \sqrt{2v}t|^p \mathrm{d}\gamma(t)\right] + 2\beta^p.\end{aligned}$$

Since $p \neq 2$, we are forced to conclude that $\beta = 0$.

To sum up, for every ℓ_2-normalized block subsequence (z_n) of (y_n) we have, first of all, that $\delta_{z_n} \overset{\text{wm}}{\longrightarrow} \sqrt{v}\gamma$, since the z_n are normalized blocks of (x_n). But also

$$\|z_n\|_p \to \tau^{(p)}(0) = \mathbb{E}\left[\int_{\mathbb{R}} |\sqrt{v}t|^p \mathrm{d}\gamma\right]^{1/p} = \gamma_p \|\sqrt{v}\|_p. \qquad \square$$

THEOREM 6.8. *Let X be a subspace of L_p $(p > 2)$ and assume that* (B) *of Proposition* 4.1 *holds. Then, for every $\theta > 0$, there is a subspace Y of X which is $(1+\theta)$-isomorphic to $\ell_p(\ell_2)$ and a projection P from L_p onto Y with $\|P\| \le (1+\theta)\gamma_p$.*

Remark. The fact that Theorem 6.8 is the optimal result concerning the norm of a projection onto a copy of $\ell_p(\ell_2)$ follows from [GLR73, Th. 5.12], where it was shown that L_p contains subspaces isometric to ℓ_2 which are γ_p complemented.

Proof. Let $\varepsilon \in (0,1)$ be fixed and, for $m \in \mathbb{N}$, let $v_m \in V(X)$, together with disjoint sets $A_m \in \Sigma$, $A_m \subset \operatorname{supp}(v_m)$, be chosen so that $\|v_m^{1/2} 1_{A_m}\|_p = 1$ and $\|v_m^{1/2}\|_p^p < 1 + \varepsilon^p 2^{-(m+2)p}$. Using Proposition 6.7 choose for each m a stabilized ℓ_2-sequence $(x_n^{(m)})_{n\in\mathbb{N}}$ in X with limiting conditional variance v_m. By (6.4), we have that

$$\liminf_{n\to\infty} \mathbb{E}[|y_n|^p 1_{A_m}] \ge \gamma_p^p \quad \text{and} \quad \liminf_{n\to\infty} \mathbb{E}[y_n^2 v_m^{\frac{p}{2}-1} 1_{A_m}] \ge 1$$

and by (6.5),

$$\lim_{n\to\infty} \mathbb{E}[|y_n|^p] = \gamma_p^p \|\sqrt{v}_m^{1/2}\|_p^p < \gamma_p^p (1 + \varepsilon^p 2^{-(m+2)p})$$

for all ℓ_2-normalized block subsequences (y_n) of $(x_n^{(m)})$. By relabeling the sequence $(x_n^{(m)})$, starting at a suitably large value of n, we may suppose that the following hold for all ℓ_2-normalized linear combinations y of the $x_n^{(m)}$:

$$\|y 1_{A_m}\|_p^p \ge (1 - \varepsilon 2^{-(m+2)p})\gamma_p^p, \tag{6.6}$$

$$\mathbb{E}\left[y^2 v_m^{\frac{p}{2}-1} 1_{A_m}\right] \ge 1 - \varepsilon 2^{-m-1}, \tag{6.7}$$

$$\|y\|_p^p \le (1 + \varepsilon 2^{-(m+2)p})\gamma_p^p. \tag{6.8}$$

Of course, (6.6) and (6.8) imply that the closed linear span $Y_m = [x_n^{(m)}]_{n\in\mathbb{N}}$ is almost isometric to ℓ_2; indeed, by homogeneity, they yield

$$(1 - \varepsilon 2^{-(m+2)p})^{1/p} \gamma_p \left(\sum c_n^2\right)^{1/2} \le \|y\|_p \le (1 + \varepsilon 2^{-(m+2)p})^{1/p} \gamma_p \left(\sum c_n^2\right)^{1/2},$$

when $y = \sum c_n x_n^{(m)} \in Y_m$.

Moreover, from the same inequalities we obtain

$$\|y - y 1_{A_m}\|_p \le \varepsilon 2^{-m} \|y\|_p \ \text{ for all } y \in Y_m. \tag{6.9}$$

If y_m is an element of S_{Y_m} for each $m \in \mathbb{N}$, then $y'_m = y_m 1_{A_m}$ are disjointly supported and are small perturbations of the y_m. As in the proof of Theorem 5.1, we see that, by an appropriate choice of ε, we can arrange for the closure of $\sum_m Y_m$ in X to be $(1+\theta)$-isomorphic to $\ell_p(\ell_2)$. We are now ready to show that the subspace $Y = \overline{\sum_m Y_m}$ is complemented in L_p. We shall do this by combining the disjoint perturbation procedure used above with a standard "change-of-density" argument.

For each m let $\phi_m = v_m^{p/2} 1_{A_m}$; thus $\|\phi_m\|_1 = 1$. Let $\Phi_m : L_p \to L_p(\phi_m)$ be defined by

$$\Phi_m(f) = 1_{A_m} \phi_m^{-1/p} f,$$

which is well-defined since $A_m \subset \operatorname{supp}(v_m)$, and observe that

$$\|\Phi_m(f)\|_{L_p(\phi_m)} = \|f 1_{A_m}\|_p.$$

Let $J_m : L_p(\phi_m) \to L_2(\phi_m)$ be the standard inclusion and let $I_m : Y_m \to L_p$ be the natural embedding. We note that for $y \in Y_m$,

$$\begin{aligned}
\|J_m \Phi_m I_m y\|^2_{L_2(\phi_m)} &= \mathbb{E}[y^2 \phi_m^{-2/p} \phi_m 1_{A_m}] \\
&= \mathbb{E}\left[y^2 v_m^{\frac{p}{2}-1} 1_{A_m}\right] \geq (1 - \varepsilon 2^{-m})^2 \gamma_p^{-2} \|y\|_p^2
\end{aligned}$$

by (6.7), (6.8), and homogeneity. So if W_m is the image

$$W_m = J_m \Phi_m I_m [Y_m],$$

then W_m is closed in $L_2(\phi_m)$ and the inverse mapping

$$R_m = (J_m \Phi_m I_m)^{-1} : W_m \to Y_m$$

satisfies $\|R_m\| \leq (1 - \varepsilon 2^{-m})^{-1} \gamma_p$.

We now introduce the orthogonal projections

$$P_m : L_2(\phi_m) \to W_m$$

and consider $Q_m : L_p \to Y_m$ defined to be $Q_m = R_m P_m J_m \Phi_m$. For $f \in L_p$, we have

$$\begin{aligned}
\sum \|Q_m f\|_p^p \leq \sum \|R_m\|^p \cdot \|\Phi_m f\|^p_{L_p(\phi_m)} &\leq (1-\varepsilon)^{-p} \gamma_p^p \sum \|f 1_{A_m}\|_p^p \\
&\leq (1-\varepsilon)^{-p} \gamma_p^p \|f\|_p^p,
\end{aligned}$$

the last inequality following by disjointness of the sets A_m. Since we already know that $Y = \overline{\sum Y_m}$ is naturally isomorphic to $(\bigoplus Y_m)_p$, we see that the series $\sum Q_m f$ converges to an element Qf of Y. Moreover, the operator Q thus defined satisfies $\|Q\| \leq \gamma_p / (1-\varepsilon)$.

To finish, we investigate $\|Q(y) - y\|_p$, when $y = \sum y_k$ with $y_k \in Y_k$. If, as before, we write $y'_k = y_k 1_{A_k}$, we may note that $Q_k(y_k) = Q_k(y'_k)$ and

$Q_m(y'_k) = 0$ for $m \neq k$. Thus

$$\begin{aligned}
\|Q(y) - y\|_p &= \Big\| \sum_k \Big(\sum_m Q_m y_k - y_k \Big) \Big\|_p \\
&= \Big\| \sum_k \sum_{m \neq k} Q_m y_k \Big\|_p \qquad [\text{since } Q_k y_k = y_k] \\
&= \Big\| \sum_k \sum_m Q_m (y_k - y'_k) \Big\|_p \\
&= \Big\| Q\Big(\sum_k y_k - y'_k \Big) \Big\|_p \\
&\leq \|Q\| \sum_k \|y_k - y'_k\|_p \leq \gamma_p (1-\varepsilon)^{-1} \sum 2^{-k} \varepsilon \|y_k\|_p,
\end{aligned}$$

using our estimate for $\|Q\|$ and (6.9) at the last stage. We can now see that for suitable chosen ε, Q may be modified to give a projection $\tilde{Q} : L_p \to Y$ with $\|\tilde{Q}\| \leq (1+\theta)\gamma_p$. □

7. Quotients and embeddings

7.1. *Subspaces of L_p that are quotients of $\ell_p \oplus \ell_2$.* It was shown in [JO81] that a subspace of L_p ($p > 2$) that is isomorphic to a quotient of a subspace of $\ell_p \oplus \ell_2$ is in fact isomorphic to a subspace of $\ell_p \oplus \ell_2$. We can give an alternative proof of this result by applying the main theorem of this paper. Clearly all that is needed is to show that $\ell_p(\ell_2)$ is not a quotient of a subspace of $\ell_p \oplus \ell_2$.

We shall prove something more general, namely that $\ell_p(\ell_q)$ is not a quotient of a subspace of $\ell_p \oplus \ell_q$ when $p, q > 1$, and $p \neq q$. By duality it will be enough to consider the case $p > q$. For elements $w = (w_1, w_2)$ of $\ell_p \oplus \ell_q$, we shall write $\|w\|_p = \|w_1\|_p$, $\|w\|_q = \|w_2\|_q$, and $\|w\| = \|w\|_p \vee \|w\|_q$.

LEMMA 7.1. *Let $1 < q < p < \infty$ and let W be a subspace of $\ell_p \oplus \ell_q$. Let $X = \ell_q$, let $Q : W \to X$ be a quotient mapping, and let λ be a constant with $0 < \lambda < \|Q\|^{-1}$. For every $M > 0$ there is a finite-codimensional subspace Y of X such that for $w \in W$,*

$$\|w\| \leq M,\ Q(w) \in Y,\ \|Q(w)\| = 1 \implies \|w\|_q > \lambda.$$

Proof. Suppose otherwise. We can find a normalized block basis (x_n) in X and elements w_n of W with $\|w_n\| \leq M$, $Q(w_n) = x_n$, and $\|w_n\|_q \leq \lambda$. Taking a subsequence and perturbing slightly, we may suppose that $w_n = w + w'_n$, where (w'_n) is a block basis in $\ell_p \oplus \ell_q$ satisfying $\|w'_n\| \leq M$, $\|w'_n\|_q \leq \lambda$.

Since $Q(w) = \text{w-lim}\, Q(w_n) = 0$, we see that $Q(w'_n) = x_n$. We may now estimate as follows using the fact that the w'_n are disjointly supported:

$$\Big\| \sum_{n=1}^N w'_n \Big\| = \Big(\sum_{n=1}^N \|w'_n\|_p^p \Big)^{1/p} \vee \Big(\sum_{n=1}^N \|w'_n\|_q^q \Big)^{1/q} \leq N^{1/p} M \vee N^{1/q} \lambda.$$

Since the x_n are normalized blocks in $X = \ell_q$, we have

$$N^{1/q} = \Big\| \sum_{n=1}^{N} x_n \Big\| \leq \|Q\| \, \Big\| \sum_{n=1}^{N} w'_n \Big\| \leq M\|Q\| N^{1/p} \vee \lambda \|Q\| N^{1/q}.$$

Since $\lambda\|Q\| < 1$, this is impossible once N is large enough. $\square$

PROPOSITION 7.2. *If $1 < q < p < \infty$, then $\ell_p(\ell_q)$ is not a quotient of a subspace of $\ell_p \oplus \ell_q$.*

Proof. Suppose, if possible, that there exists a quotient operator

$$\ell_p \oplus \ell_q \supseteq Z \xrightarrow{\mathrm{Q}} X = \Big(\bigoplus_{n\in\mathbb{N}} X_n \Big)_p,$$

where $X_n = \ell_q$ for all n. Let K be a constant such that $Q[KB_Z] \supseteq B_X$, let λ be fixed with $0 < \lambda < \|Q\|^{-1}$, choose a natural number m with $m^{1/q-1/p} > K\lambda^{-1}$, and set $M = 2Km^{1/p}$.

Applying the lemma, we find, for each n, a finite-codimensional subspace Y_n of X_n such that

$$z \in MB_Z,\ Q(z) \in Y_n,\ \|Q(z)\| = 1 \implies \|z\|_q > \lambda. \tag{7.1}$$

For each n, let $(e_i^{(n)})$ be a sequence in Y_n, 1-equivalent to the unit vector basis of ℓ_q. For each m-tuple $\mathbf{i} = (i_1, i_2, \ldots, i_m) \in \mathbb{N}^m$, let $z(\mathbf{i}) \in Z$ be chosen with

$$Q(z(\mathbf{i})) = e_{i_1}^{(1)} + e_{i_2}^{(2)} + \cdots + e_{i_m}^{(m)}$$

and $\|z(\mathbf{i})\| \leq Km^{1/p}$.

Taking subsequences in each coordinate, we may suppose that the following weak limits exist in Z:

$$\begin{aligned} z(i_1, i_2, \ldots, i_{m-1}) &= \text{w-lim}_{i_m\to\infty}\, z(i_1, i_2, \ldots, i_m) \\ &\vdots \\ z(i_1, i_2, \ldots, i_j) &= \text{w-lim}_{i_{j+1}\to\infty}\, z(i_1, i_2, \ldots, i_{j+1}) \\ &\vdots \\ z(i_1) &= \text{w-lim}_{i_2\to\infty} z(i_1, i_2). \end{aligned}$$

Notice that for all j and all $i_1, i_2, \ldots, i_j$, the following hold:

$$\begin{aligned} Q(z(i_1, \ldots, i_j) &= e_{i_1}^{(1)} + \cdots + e_{i_j}^{(j)}, \\ \|z(i_1, \ldots, i_j)\| &\leq Km^{1/p}, \\ \|z(i_1, \ldots, i_j) - z(i_1, \ldots, i_{j-1})\| &\leq 2Km^{1/p} = M. \end{aligned}$$

Since $Q(z(i_1, \ldots, i_j) - z(i_1, \ldots, i_{j-1})) = e_{i_j}^{(j)} \in S_{Y_j}$, it must be that

$$\|z(i_1, \ldots, i_j) - z(i_1, \ldots, i_{j-1})\|_q > \lambda, \quad \text{[by (7.1)]}. \tag{7.2}$$

We now choose recursively some special i_j, in such a way that for all j, $\|z(i_1,\dots,i_j)\|_q > \lambda j^{1/q}$. Start with $i_1 = 1$; since $\|z(i_1)\| \le M$ and $Q(z(i_1)) = e^{(1)}_{i_1}$ we certainly have $\|z(i_1)\|_q > \lambda$ by 7.1. Since $z(i_1,k) - z(i_1) \to 0$ weakly we can choose i_2 such that $z(i_1,i_2) - z(i_1)$ is essentially disjoint from $z(i_1)$. More precisely, because of (7.2), we can ensure that

$$\|z(i_1,i_2)\|_q = \|z(i_1) + (z(i_1,i_2) - z(i_1))\|_q > (\lambda^q + \lambda^q)^{1/q} = \lambda 2^{1/q}.$$

Continuing in this way, we can indeed choose $i_3,\dots,i_m$ in such a way that

$$\|z(i_1,\dots,i_j)\|_q \ge \lambda j^{1/q}.$$

However, for $j = m$ this yields $\lambda m^{1/q} \le K m^{1/p}$, contradicting our initial choice of m. □

Remark. The proof we have just given actually establishes the following quantitative result: if Y is a quotient of a subspace of $\ell_p \oplus \ell_q$, then the Banach-Mazur distance $d\big(Y, \big(\bigoplus_{j=1}^m \ell_q\big)_p\big)$ is at least $m^{|1/q-1/p|}$.

7.2. *Uniform bounds for isomorphic embeddings.* As we remarked in the introduction, the Kalton-Werner refinement [KW95] of the result of [JO74] gives an almost isometric embedding of X into ℓ_p when X is a subspace of L_p $(p > 2)$, not containing ℓ_2. By contrast, the main result of the present paper does not have an almost isometric version, and indeed it is easy to see that there is no constant K (let alone $K = 1 + \varepsilon$) such that every subspace of L_p not containing $\ell_p(\ell_2)$ K-embeds in $\ell_p \oplus \ell_2$. It is enough to consider spaces X of the form $X = \big(\bigoplus_{j=1}^m \ell_2\big)_p$. A straightforward argument, or an application of the more general result mentioned in the remark above, shows that the Banach-Mazur distance from X to a subspace of $\ell_p \oplus \ell_2$ is at least $m^{1/2-1/p}$.

If we are looking for a "uniform" version of our Main Theorem, perhaps it is not unreasonable to conjecture the existence of a constant K such that every subspace of L_p not containing $\ell_p(\ell_2)$ K-embeds in some space of the form $\ell_p \oplus_p \big(\bigoplus_{j=1}^m \ell_2\big)_p$. However, no such constant M exists, as is shown by the following proposition. The structure of the space X considered below suggests that if there is some uniform version of our main result, then it will involve independent sums (see [Als99]), rather than, or as well as, ℓ_p sums. The proof of the next result follows a construction due to Alspach and could be compiled from arguments in [Als99, Ch. 2]. The following is a self-contained proof.

Proposition 7.3. *Let $p > 2$. For every $K > 0$ there is a subspace X of L_p, isomorphic to ℓ_2, such that for all $m \in \mathbb{N}$, X is not K-isomorphic to a subspace of $\ell_p \oplus_p \big(\bigoplus_{l=1}^m \ell_2\big)_p$.*

Proof. Fix a constant $M > 1$. Let $\{v_i, z_{j,k} : i, j, k \in \mathbb{N}\}$ be a family of independent random variables in $L_p[0,1]$ with distributions defined as follows: for $i, j \in \mathbb{N}$, $z_{i,j}$ is $\mathcal{N}(0,1)$, while v_i is $\{0, M\}$-valued with $\mathbb{P}[v_i = M] = 1 - \mathbb{P}[v_i = 0] = M^{-p/2}$. We set $x_{i,j} = z_{i,j}\sqrt{v_i}$, noting that

$$\|x_{i,j}\|_p^p = \mathbb{E}[v_i^{p/2}|z_{i,j}|^p] = \mathbb{E}[v_i^{p/2}]\mathbb{E}[|z_{i,j}|^p] = \gamma_p^p.$$

We now define $X_i = [x_{i,j}]_{j\in\mathbb{N}}$ and $X = [x_{i,j}]_{i,j\in\mathbb{N}}$. We start by calculating the norm of a general element of X.

Let $x = \sum_{i,j} c_{i,j} x_{i,j}$. By independence and properties of the normal distribution, the distribution of x *conditional on* $v_1, v_2, v_3, \ldots$ is $\mathcal{N}(0, w)$, where $w = \sum_{i,j} c_{i,j}^2 v_i$. So
(7.3)

$$\|x\|_p^p = \mathbb{E}\left[\mathbb{E}[|x|^p \mid v_1, v_2, \ldots]\right] = \gamma_p^p \mathbb{E}\left[\Big(\sum_i \Big(\sum_j c_{i,j}^2\Big) v_i\Big)^{p/2}\right] = \gamma_p^p \Big\|\sum a_i v_i\Big\|_{p/2}^{p/2},$$

where $a_i = \sum_j c_{i,j}^2$, for $i \in \mathbb{N}$. Let us first note that (7.3) implies that $(x_{i,j})$ is equivalent to the unit vector basis of ℓ_2. Indeed, Jensen's inequality yields

$$\Big\|\sum a_i v_i\Big\|_{p/2}^{p/2} \geq \mathbb{E}^{p/2}\Big[\sum a_i v_i\Big] = \Big(\sum a_i M^{1-p/2}\Big)^{p/2} = \left(M^{1/2-p/4}\Big(\sum_{i,j} c_{i,j}^2\Big)^{1/2}\right)^p.$$

On the other hand, letting $\tilde{v}_i = v_i - \mathbb{E}(v_i) = v_i - M^{1-p/2}$, the triangle inequality in $L_{p/2}$ and the fact that for some $C < \infty$ (depending on M and p) the sequence $(\tilde{v}_i)$, as sequence in $L_{p/2}$, is C-equivalent to the unit vector basis in ℓ_2, imply that

$$\begin{aligned}\Big\|\sum a_i v_i\Big\|_{p/2} &\leq M^{1-p/2}\sum a_i + \Big\|\sum a_i \tilde{v}_i\Big\|_{p/2}\\ &\leq M^{1-p/2}\sum a_i + C\Big(\sum a_i^2\Big)^{1/2} \leq (M^{1-p/2} + C)\sum a_i\end{aligned}$$

and thus

$$\Big\|\sum a_i v_i\Big\|_{p/2}^{p/2} \leq \Big((M^{1-p/2} + C)^{1/2}\Big(\sum_{i,j} c_{i,j}^2\Big)^{1/2}\Big)^p,$$

which finishes the proof of our claim that $(x_{i,j})$ is equivalent to the unit basis of ℓ_2.

We note two special cases of (7.3). First, if $x = x_i \in X_i$ for some i (thus $c_{i',j} = 0$ for all $i' \neq i$ and all j),

$$\|x_i\|_p = \gamma_p\Big(\sum_j c_{i,j}^2\Big)^{1/2}.$$

In particular, $\|x_i\|_p = 1$ if and only if $\left(\sum_j c_{i,j}^2\right)^{1/2} = \gamma_p^{-1}$. Secondly, if $x = n^{-1/2}\sum_{i=1}^n x_i$, where the x_i are normalized elements of X_i,

$$\begin{aligned}\|x\|_p &= n^{-1/2}\gamma_p \mathbb{E}\left[\left(\sum_{i=1}^n \Big(\sum_j c_{i,j}^2\Big) v_i\right)^{p/2}\right]^{1/p} \\ &= n^{-1/2}\mathbb{E}\left[\left(\sum_{i=1}^n v_i\right)^{p/2}\right]^{1/p} = \left\|n^{-1}\sum_{i=1}^n v_i\right\|_{p/2}^{1/2}.\end{aligned}$$

Now, by the weak law of large numbers, $n^{-1}\sum_{i=1}^n v_i$ converges in probability to the constant $\mathbb{E}[v_1] = M^{1-p/2}$. Because these averages are uniformly bounded (by M), the convergence holds also for the $L_{p/2}$-norm. So as $n \to \infty$,

$$\left\|n^{-1}\sum_{i=1}^n v_i\right\|_{p/2} \to M^{1-p/2}.$$

Summarizing, we can say that if x_i are L_p-normalized elements of X_i, then

$$\left\|n^{-1/2}\sum_{i=1}^n x_i\right\|_p = \left\|n^{-1}\sum_{i=1}^n v_i\right\|_{p/2}^{1/2} \to M^{(2-p)/4} \text{ as } n \to \infty. \tag{7.4}$$

Let $T = (T_\ell)_{\ell=0}^m : X \to Y = \ell_p \oplus_p \left(\bigoplus_{\ell=1}^m \ell_2\right)_p$, with $T_0 : X \to \ell_p$ and $T_i : X \to \ell_2$, for $\ell = 1, 2 \ldots, m$, be an isomorphic embedding. We assume that $\|T(x)\| \ge \|x\|$ for all x and shall show that $\|T\| \ge M^{(p-2)/4}$.

We note that for each i, the sequence $\left(T_0(x_{i,j})\right)_{j=1}^\infty$ is a weakly null sequence in ℓ_p. So by taking vectors of the form

$$x'_{i,k} = \gamma_p^{-1} k^{-1/2} \sum_{r=1}^k x_{i,j_r(k)},$$

with $j_{k-1}(k-1) < j_1(k) < j_2(k) < \cdots < j_k(k)$, we construct an L_p-normalized, weakly null sequence $(x'_{i,k})_{k=1}^\infty$ in X_i with $\|T_0(x'_{i,k})\|_p \to 0$ as $k \to \infty$.

Passing to a subsequence, we may assume that for all $i \in \mathbb{N}$ and all $\ell = 1, 2 \ldots m$ the sequence $T_\ell(x'_{i,k}))$ tends to a limit $\mu_{i,\ell}$ as $k \to \infty$. Since $\|T(x'_{i,k})\| \ge 1$ and $\|T_0(x'_{i,k}))\|_p \to 0$, it must be that $\|\mu_i\|_p \ge 1$, where $\mu_i = (\mu_{i,\ell})_{\ell=1}^m$. Passing to a subsequence in i, we may assume that μ_i converges to some $\mu \in \mathbb{R}^m$, as $i \to \infty$, with $\|\mu\|_p \ge 1$.

For $\ell = 1, 2 \dots m$ and $n \in \mathbb{N}$ we observe that

$$\begin{aligned}
&\lim_{k_1\to\infty}\lim_{k_2\to\infty}\dots\lim_{k_n\to\infty}\|n^{-1/2}T_\ell\Big(\sum_{i=1}^n x'_{i,k_i}\Big)\|_2 \\
&\qquad= \lim_{k_1\to\infty}\lim_{k_2\to\infty}\dots\lim_{k_{n-1}\to\infty} n^{-1/2}\Big(\Big\|T_\ell\Big(\sum_{i=1}^{n-1} x'_{i,k_i}\Big)\Big\|^2 + \mu_{i,\ell}^2\Big)^{1/2} \\
&\qquad= \dots = n^{-1/2}\Big(\sum_{i=1}^n \mu_{i,\ell}^2\Big)^{1/2} \equiv \tilde{\mu}_{n,\ell}.
\end{aligned}$$

Since $\tilde{\mu}_n \to \mu$, as $n \to \infty$, where $\tilde{\mu}_n = (\tilde{\mu}_{n,\ell})_{\ell=1}^m$, we deduce that

(7.5)

$$\lim_{n\to\infty}\lim_{k_1\to\infty}\lim_{k_2\to\infty}\dots\lim_{k_n\to\infty}\Big\|n^{-1/2}T\Big(\sum_{i=1}^n x'_{i,k_i}\Big)\Big\|_Y = \lim_{n\to\infty}\|\tilde{\mu}_n\|_p = \|\mu\|_p \geq 1.$$

On the other hand, as we have already noted above (7.4),

$$\Big\|n^{-1/2}\sum_{i=1}^n x'_{i,k_i}\Big\| = \Big\|n^{-1}\sum_{i=1}^n v_i\Big\|_{p/2}^{1/2} \to M^{(2-p)/4}, \text{ as } n \to \infty,$$

Comparing this with (7.5), we conclude that $\|T\| \geq M^{(p-2)/4}$ as claimed. $\square$

8. **Concluding remarks**

A natural question remains, namely to characterize when a subspace $X \subseteq L_p$ $(2 < p < \infty)$ embeds into $\ell_p(\ell_2)$. We do not know the answer. In light of the [JO81] $\ell_p \oplus \ell_2$ quotient result (see paragraph 7.1 above) we ask the following.

Problem 8.1. Let $X \subseteq L_p$ $(2<p<\infty)$. If X is a quotient of $\ell_p(\ell_2)$, does X embed into $\ell_p(\ell_2)$?

Extensive study has been made of the $\mathcal{L}_p$ spaces, i.e., the complemented subspaces of L_p which are not isomorphic to ℓ_2 (see e.g. [LP68] and [LR69]). In particular there are uncountably many such spaces [BRS81] and even infinitely many which embed into ℓ_p (ℓ_2) [Sch75]. Thus it seems that a deeper study of the index in [BRS81] will be needed for further progress. However some things, which we now recall, are known.

Theorem 8.2 ([Pełczyński60]). *If Y is complemented in ℓ_p, then Y is isomorphic to ℓ_p.*

Theorem 8.3 ([JZ72]). *If Y is a $\mathcal{L}_p$ subspace of ℓ_p, then Y is isomorphic to ℓ_p.*

Theorem 8.4 ([ÈW76]). *If Y is complemented in $\ell_p \oplus \ell_2$, then Y is isomorphic to ℓ_p, ℓ_2 or $\ell_p \oplus \ell_2$.*

Theorem 8.5 ([Ode76]). *If Y is complemented in $\ell_p(\ell_2)$, then Y is isomorphic to ℓ_p, ℓ_2, $\ell_p \oplus \ell_2$ or ℓ_p (ℓ_2).*

We recall that X_p is the $\mathcal{L}_p$ discovered by H. Rosenthal [Ros70]. For $p > 2$, X_p may be defined to be the subspace of $\ell_p \oplus \ell_2$ spanned by $(e_i + w_i f_i)$, where (e_i) and (f_i) are the unit vector bases of ℓ_p and ℓ_2, respectively, and where $w_i \to 0$ with $\sum w_i^{2p/p-2} = \infty$. Since $\ell_p \oplus \ell_2$ embeds into X_p, the subspaces of X_p and of $\ell_p \oplus \ell_2$ are (up to isomorphism) the same. For $1 < p < 2$ the space X_p is defined to be the dual of $X_{p'}$ where $1/p + 1/p' = 1$. When restricted to $\mathcal{L}_p$-spaces, the results of this paper lead to a dichotomy valid for $1 < p < \infty$.

Proposition 8.6. *Let Y be a $\mathcal{L}_p$-space* $(1 < p < \infty)$. *Either Y is isomorphic to a complemented subspace of X_p or Y has a complemented subspace isomorphic to $\ell_p(\ell_2)$.*

Proof. For $p > 2$ it is shown in [JO81] that a $\mathcal{L}_p$-space which embeds in $\ell_p \oplus \ell_2$ embeds complementedly in X_p. Combining this with the main theorem of the present paper gives what we want for $p > 2$. When $1 < p < 2$, the space X_p is defined to be the dual of $X_{p'}$ and so a simple duality argument extends the result to the full range $1 < p < \infty$. □

It remains a challenging problem to understand more deeply the structure of the $\mathcal{L}_p$-subspaces of X_p and $\ell_p \oplus \ell_2$.

Theorem 8.7 ([JO81]). *If Y is a $\mathcal{L}_p$ subspace of $\ell_p \oplus \ell_2$ (or X_p), $2 < p < \infty$, and Y has an unconditional basis, then Y is isomorphic to ℓ_p, $\ell_p \oplus \ell_2$, or X_p.*

It is known [JRZ71] that every $\mathcal{L}_p$ space has a basis but it remains open if it has an unconditional basis.

Theorem 8.8 ([JO81]). *If Y is a $\mathcal{L}_p$ subspace of $\ell_p \oplus \ell_2$* $(1 < p < 2)$ *with an unconditional basis, then Y is isomorphic to ℓ_p or $\ell_p \oplus \ell_2$.*

So the main open problem for small $\mathcal{L}_p$ spaces is to overcome the unconditional basis requirement of Theorems 8.7 and 8.8.

Problem 8.9. (a) Let X be a $\mathcal{L}_p$ subspace of $\ell_p \oplus \ell_2$ $(2 < p < \infty)$. Is X isomorphic to ℓ_p, $\ell_p \oplus \ell_2$, or X_p?

(b) Let X be a $\mathcal{L}_p$ subspace of $\ell_p \oplus \ell_2$ $(1 < p < 2)$. Is X isomorphic to ℓ_p or $\ell_p \oplus \ell_2$?

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