

# Dynamic rays of bounded-type entire functions

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## Abstract

We construct an entire function in the Eremenko-Lyubich class  $\mathcal{B}$  whose Julia set has only bounded path-components. This answers a question of Eremenko from 1989 in the negative.

On the other hand, we show that for many functions in  $\mathcal{B}$ , in particular those of finite order, every escaping point can be connected to  $\infty$  by a curve of escaping points. This gives a partial positive answer to the aforementioned question of Eremenko, and answers a question of Fatou from 1926.

## 1. Introduction

The dynamical study of transcendental entire functions was initiated by Fatou in 1926 [Fat26]. As well as being a fascinating field in its own right, the topic has recently received increasing interest partly because transcendental phenomena seem to be deeply linked with the behavior of polynomials in cases where the degree gets large. A recent example is provided by the surprising results of Avila and Lyubich [AL08], who showed that a constant-type Feigenbaum quadratic polynomial whose Julia set has positive measure would have hyperbolic dimension less than two. This phenomenon occurs naturally in transcendental dynamics; see [UZ03]. Other interesting applications of transcendental dynamics include the study of the standard family of circle maps and the use of Newton's method to study zeros of transcendental functions.

In his seminal 1926 article, Fatou observed that the Julia sets of certain explicit entire functions, such as  $z \mapsto r \sin(z)$ ,  $r \in \mathbb{R}$ , contain curves of points that escape to infinity under iteration. He then remarked

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Il serait intéressant de rechercher si cette propriété n'appartiendrait pas à des substitutions beaucoup plus générales.<sup>1</sup>

Sixty years later, Eremenko [Ere89] was the first to undertake a thorough study of the *escaping set*

$$I(f) := \{z \in \mathbb{C} : |f^{o_n}(z)| \rightarrow \infty\}$$

of an arbitrary transcendental entire function. In particular, he showed that every component of  $\bar{I}(f)$  is unbounded, and asked whether in fact each component of  $I(f)$  is unbounded. We will call this problem (the weak form of) *Eremenko's conjecture*. He also stated that

It is plausible that the set  $I(f)$  always has the following property: every point  $z \in I(f)$  can be joined with  $\infty$  by a curve in  $I(f)$ .

This can be seen as making Fatou's original question more precise, and will be referred to in the following as the *strong form* of Eremenko's conjecture.

These problems are of particular importance because the existence of such curves can be used to study entire functions using combinatorial methods. This is analogous to the notion of “dynamic rays” of polynomials introduced by Douady and Hubbard [DH], which has proved to be one of the fundamental tools for the successful study of polynomial dynamics. Consequently, Fatou's and Eremenko's questions are among the most prominent open problems in the field of transcendental dynamics.

We will show that, in general, the answer to Fatou's question (and thus also to Eremenko's conjecture in its strong form) is negative, even when restricted to the *Eremenko-Lyubich class*  $\mathcal{B}$  of entire functions with a bounded set of singular values. (For such functions, all escaping points lie in the Julia set. The class  $\mathcal{B}$  appears to be a very natural setting for this type of problem; compare also [Rem09].)

1.1. THEOREM (Entire functions without dynamic rays). *There exists a hyperbolic entire function  $f \in \mathcal{B}$  such that every path-connected component of  $J(f)$  is bounded.*

*Remark.* In fact, it is even possible to ensure that  $J(f)$  contains no non-trivial curves at all (Theorem 8.4).

On the other hand, we show that the strong form of Eremenko's conjecture does hold for a large subclass of  $\mathcal{B}$ . Recall that  $f$  has *finite order* if  $\log \log |f(z)| = O(\log |z|)$  as  $|z| \rightarrow \infty$ .

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<sup>1</sup> “It would be interesting to investigate whether this property might not hold for much more general functions.”

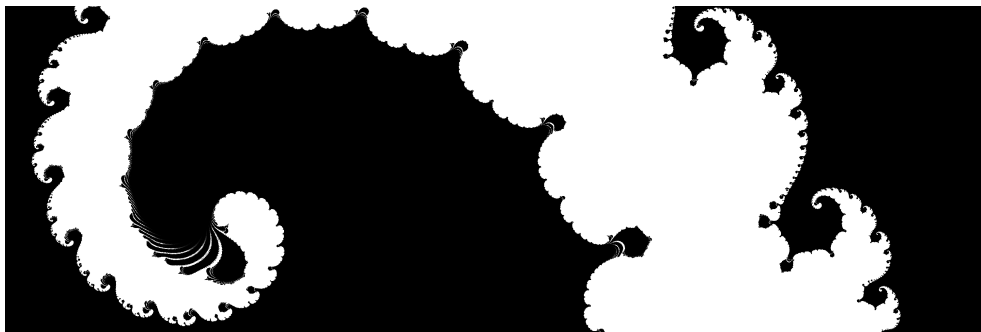


Figure 1. Julia set for the Poincaré function around a repelling fixed point of a postcritically-finite quadratic polynomial. This function belongs to  $\mathcal{B}$  and has finite order; hence [Theorem 1.2](#) applies to it. (The Julia set, plotted in black, is nowhere dense, but some details are too fine to be visible; this results in the appearance of solid black regions in the figure.)

1.2. THEOREM (Entire functions with dynamic rays). *Let  $f \in \mathcal{B}$  be a function of finite order, or more generally a finite composition of such functions. Then every point  $z \in I(f)$  can be connected to  $\infty$  by a curve  $\gamma$  such that  $f^{\text{on}}|_{\gamma} \rightarrow \infty$  uniformly.*

*Remark.* Observe that while  $\mathcal{B}$  is invariant under finite compositions, the property of having finite order is not.

[Theorem 1.2](#) applies to a large and natural class of functions, extending considerably beyond those that were previously studied. As an example, suppose that  $p$  is a polynomial with connected Julia set, and let  $\alpha$  be a repelling fixed point of  $p$ . By a classical theorem of Kœnigs,  $p$  is conformally conjugate to the linear map  $z \mapsto \mu z$  (where  $\mu = p'(\alpha)$  is the multiplier of  $\alpha$ ). The inverse of this conjugacy extends to an entire function  $\psi : \mathbb{C} \rightarrow \mathbb{C}$  with  $\psi(0) = \alpha$  and  $\psi(\mu z) = f(\psi(z))$ . This map  $\psi$ , called a *Poincaré function*, has finite order. The set of singular values of  $\psi$  is the postcritical set of  $p$ , which is bounded since  $J(p)$  is connected. So  $\psi \in \mathcal{B}$ . The properties of  $\psi$  are generally quite different from e.g. those of the commonly considered exponential and trigonometric functions; for example, if  $\mu \notin \mathbb{R}$ , then the *tracts* of  $\psi$  will spiral near infinity. (Compare [Figure 1](#).)

More generally, given any point  $z \in J(p)$ , we can find a sequence of rescalings of iterates of  $p$  that converges to an entire function  $f : \mathbb{C} \rightarrow \mathbb{C}$  of finite order by the “Zalcman lemma” (see e.g. [\[Zal98\]](#)). Again, since  $J(p)$  is connected, we have  $f \in \mathcal{B}$ . [Theorem 1.2](#) implies that, for all such functions and their finite compositions, the escaping set consists of rays.

On the other hand, given  $\varepsilon > 0$ , the counterexample from [Theorem 1.1](#) can be constructed such that  $\log \log |f(z)| = O(|\log z|^{1+\varepsilon})$  (see [Proposition 8.3](#)), so [Theorem 1.2](#) is not far from being optimal.

We remark that our methods are purely local in the sense that they apply to the dynamics of a — not necessarily globally defined — function within any number of logarithmic singularities over  $\infty$ . Roughly speaking, let  $f$  be a function defined on a union of unbounded Jordan domains such that  $f$  is a universal covering of  $\{|z| > R\}$  on each of these domains, and such that only finitely many of the domains intersect any given compact set. (In fact, our setting is even more general than this; see [Section 2](#) for the class of functions we consider.) We will provide sufficient conditions that ensure that every point  $z \in I(f)$  eventually maps into a curve in  $I(f)$  ending at  $\infty$ . In particular, we show that these conditions are satisfied if  $f$  has finite order of growth. (Our treatment also allows us to discuss under which conditions *individual* escaping components, identified by their *external addresses*, are curves to  $\infty$ .)

For meromorphic functions, we have the following corollary.

### 1.3. COROLLARY (Meromorphic functions with logarithmic singularities).

Let  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  be a meromorphic function of finite order.

- (a) Suppose that  $f$  has only finitely many poles and the set of finite singular values of  $f$  is bounded. Then every escaping point of  $f$  can be connected to  $\infty$  or to a pre-pole of  $f$  by a curve consisting of escaping points.
- (b) Suppose that  $f$  has a logarithmic singularity over  $\infty$ . Then  $J(f)$  contains uncountably many curves to  $\infty$  consisting of escaping points.

*Remark.* The second part of the corollary applies e.g. to the classical  $\Gamma$ -function, which has infinitely many poles (at the negative integers), but a logarithmic singularity to the right.

We note finally that our results also apply to the setting of “random iteration” (see [\[Com04\]](#)). For example, consider a sequence  $\mathcal{F} = (f_0, f_1, f_2, \dots)$ , where the  $f_j$  are finite-order entire functions chosen from some given finite subset of  $\mathcal{B}$ . If we consider the functions  $\mathcal{F}_n := f_n \circ \dots \circ f_0$ , and define  $I(\mathcal{F}) := \{z \in \mathbb{C} : \mathcal{F}_n(z) \rightarrow \infty\}$ , then every point of  $I(\mathcal{F})$  can be connected to infinity by a curve in  $I(\mathcal{F})$ .

*Previous results.* In the early 1980s, Devaney gave a complete description of the Julia set of any real exponential map that has an attracting fixed point; that is,  $z \mapsto \lambda \exp(z)$  with  $\lambda \in (0, 1/e)$  (see [\[DK84\]](#)). This was the first entire function for which it was discovered that the escaping set (and in fact the Julia set) consists of curves to  $\infty$ . Devaney, Goldberg and Hubbard [\[DGH86\]](#) proved the existence of certain curves to  $\infty$  in  $I(f)$  for arbitrary exponential maps  $z \mapsto \lambda \exp(z)$  and championed the idea that these should be thought of

as analogs of dynamic rays for polynomials. Devaney and Tangerman [DT86] generalized this result to a large subclass of  $\mathcal{B}$ , namely those functions whose *tracts* (see §2) are similar to those of the exponential map. (This includes virtually all functions in the Eremenko-Lyubich class that one can explicitly write down.) It seems that it was partly these developments that led Eremenko to pose the above-mentioned questions in his 1989 paper.

More recently, it was shown in [SZ03] that every escaping point of every exponential map can be connected to  $\infty$  by a curve consisting of escaping points. This was the first time that a complete classification of all escaping points, and with it a positive answer to both of Eremenko's questions, was given for a complete parameter space of transcendental functions. This result was carried over to the cosine family  $z \mapsto a \exp(z) + b \exp(-z)$  in [RS08].

After our proof of Theorem 1.2 was first announced, Barański [Bar07] independently obtained a proof of this result for hyperbolic finite-order functions  $f \in \mathcal{B}$  whose Fatou set consists of a single basin of attraction. (In fact, Barański showed that for these functions every component of the Julia set is a curve to  $\infty$ ; compare Theorem 5.10.) Together with more recent results [Rem09] on the escaping dynamics of functions in the Eremenko-Lyubich class, this provides an alternative proof of our theorem when  $f$  is of finite order.

A very interesting and surprising case in which the weak form of Eremenko's conjecture can be resolved was discovered by Rippon and Stallard [RS05]. They showed that the escaping set of a function with a multiply-connected wandering domain consists of a single, unbounded, connected component. (Such functions never belong to the Eremenko-Lyubich class  $\mathcal{B}$ .) In fact, they showed that, for any transcendental entire function, the subset  $A(f) \subset I(f)$  of "points escaping at the fastest possible rate", as introduced by Bergweiler and Hinkkanen [BH99], has only unbounded components. Also, recently [Rem07] the weak form of Eremenko's conjecture was established for functions  $f \in \mathcal{B}$  whose postsingular set is bounded (which applies, in particular, to the hyperbolic counterexample constructed in Theorem 1.1).

There has been substantial interest in the set  $I(f)$  not only with respect to its topological structure, but also because of its interesting properties from the point of view of Hausdorff dimension. The reasoning is often parallel, and progress on the topology of  $I(f)$  has entailed progress on the Hausdorff dimension. For many functions  $f$ , the set  $I(f)$  is an uncountable union of curves, each of which is the continuous injective image of either  $\mathbb{R}^+$  (a dynamic ray) or  $\mathbb{R}_0^+$  (a dynamic ray that lands at an escaping point). Often, the Hausdorff dimension of all the rays is 1, while the endpoints alone have Hausdorff dimension 2, or even infinite planar Lebesgue measure. This "dimension paradox" was discovered by Karpińska for hyperbolic exponential maps [Kar99], for which the topology of Julia sets was known. In later extensions for arbitrary exponential

maps [SZ03] and for the cosine family [RS08], the new parts were the topological classifications, while analogous results on the Hausdorff dimension followed from the methods of Karpińska and McMullen; see also [Sch07a], [Sch07b] for extreme results where every point in the complex plane is either on a dynamic ray (whose union still has dimension one) or a landing point of those rays – so the landing points of this one-dimensional set of rays is the entire complex plane with only a one-dimensional set of exceptions. Recently, it was shown by Barański [Bar08] that the dimension paradox also occurs for finite-order transcendental entire functions that are hyperbolic with a single basin of attraction. In fact, the Hausdorff dimension of  $I(f)$  is two for any entire function  $f \in \mathcal{B}$  of finite order, which was also shown independently by Schubert [Sch07]. This generalizes McMullen’s results [McM87] on the escaping sets of exponential and trigonometric functions. Further studies of the Hausdorff dimension of the escaping set for  $f \in \mathcal{B}$  can be found in [BKS09], [RS10].

*Structure of the article.* In Section 2, we define logarithmic coordinates, in which we will perform most of our constructions. Some properties of functions in logarithmic coordinates are proved in Section 3. In Section 4, we show that the escaping set of a function in logarithmic coordinates consists of arcs if the escaping points can be ordered according to their “speed” of escape. We call this property the *head-start condition*. Classes of functions that satisfy this condition, in particular logarithmic transforms of finite order functions, are discussed in Section 5.

In Section 6, we construct a function in logarithmic coordinates whose escaping set has only bounded path-components and in Section 7, we show how to translate this result into a bounded-type entire function. In an appendix, we recall several facts from hyperbolic geometry, geometric function theory and continuum theory.

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*Notation and basic definitions.* Throughout this article, we denote the Riemann sphere by  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  and the right half-plane by

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}.$$

Also, we write

$$B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$$

and

$$\mathbb{H}_R := \{z \in \mathbb{C} : \operatorname{Re} z > R\}.$$

If  $A \subset \mathbb{C}$ , the closures of  $A$  in  $\mathbb{C}$  and  $\widehat{\mathbb{C}}$  are denoted  $\overline{A}$  and  $\widehat{A}$ , respectively.

Euclidean length and distance are denoted  $\ell$  and  $\operatorname{dist}$ , respectively. If a domain  $V \subset \mathbb{C}$  omits at least two points of the plane, we similarly denote hyperbolic length and distance in  $V$  by  $\ell_V$  and  $\operatorname{dist}_V$ . We shall often use the following well-known fact [Mil06, Cor. A.8]: If  $V \subset \mathbb{C}$  is a simply connected domain, then the density  $\lambda_V$  of the hyperbolic metric on  $V$  satisfies

$$(1.1) \quad \frac{1}{2 \operatorname{dist}(z, \partial V)} \leq \lambda_V(z) \leq \frac{2}{\operatorname{dist}(z, \partial V)}$$

for all  $z \in V$ ; we shall refer to this as the *standard estimate*.

Let  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  be a meromorphic function. We call a point  $a \in \widehat{\mathbb{C}}$  a *singular value* of  $f$  if for every open neighborhood  $U$  of  $a$ , there exists a component  $V$  of  $f^{-1}(U)$  such that  $f : V \rightarrow U$  is not bijective. Denote the set of all finite singular values of  $f$  by  $S(f) \subset \mathbb{C}$ . Clearly every critical value of  $f$  belongs to  $S(f)$ .

Recall that  $a \in \widehat{\mathbb{C}}$  is an *asymptotic value* of  $f$  if there exists a curve  $\gamma : [0, \infty) \rightarrow \mathbb{C}$  with  $\lim_{t \rightarrow \infty} |\gamma(t)| = \infty$  such that  $a = \lim_{t \rightarrow \infty} f(\gamma(t))$ . (An example is given by  $f(z) = \exp(z)$ ,  $\gamma(t) = -t$ ,  $a = 0$ .) Every asymptotic value of  $f$  is a singular value; conversely,  $S(f)$  is the closure of the set  $\operatorname{sing}(f^{-1})$  of all finite critical and asymptotic values.

Let  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  be a transcendental entire or meromorphic function, and let  $a \in \widehat{\mathbb{C}}$ . Suppose there is some simply-connected open neighborhood  $D \subset \mathbb{C}$  of  $a$  and a component  $U$  of  $f^{-1}(D)$  such that  $f : U \rightarrow D \setminus \{a\}$  is a universal covering map. Then we say that  $f$  has a *logarithmic singularity* over  $a$ . In this case,  $a$  is necessarily an asymptotic value of  $f$ ; conversely  $f$  has a logarithmic singularity over every isolated asymptotic value of  $f$ . For a further discussion of types of asymptotic values, see [BE95].

As stated in the introduction, we say that a transcendental entire function  $f$  belongs to the *Eremenko-Lyubich class*  $\mathcal{B}$  if  $S(f)$  is bounded. By  $J(f)$  we denote the *Julia set* of  $f$ , i.e., the set of points at which the sequence of functions  $\{f, f \circ f, \dots, f^{\circ n}, \dots\}$  does not form a normal family in the sense of Montel. The reader is referred to [Mil06] for a general introduction to the dynamics in one complex variable, and to [Ber93], [Ber08], [Sch10] for background on transcendental dynamics.

We conclude any proof by the symbol  $\square$ . The proofs of separate claims within an argument are concluded by  $\triangle$ .

## 2. The Eremenko-Lyubich class $\mathcal{B}$ and the class $\mathcal{B}_{\log}$

*Tracts.* Let  $f \in \mathcal{B}$ , let  $D \subset \mathbb{C}$  be a bounded Jordan domain that contains  $S(f) \cup \{0, f(0)\}$  and set  $W := \mathbb{C} \setminus \overline{D}$ . It is easy to see that every component  $V$  of

$$\mathcal{V} := f^{-1}(W)$$

is an unbounded Jordan domain, and that  $f : V \rightarrow W$  is a universal covering. (In other words,  $f$  has only logarithmic singularities over  $\infty$ .) The components of  $\mathcal{V}$  are called the *tracts* of  $f$ . Observe that a given compact set  $K \subset \mathbb{C}$  will intersect at most finitely many tracts of  $f$ .

*Logarithmic coordinates.* To study logarithmic singularities, it is natural to apply a logarithmic change of coordinates (compare [EL92, §2]). More precisely, let  $\mathcal{T} := \exp^{-1}(\mathcal{V})$  and  $H := \exp^{-1}(W)$ . Then there is an analytic function  $F : \mathcal{T} \rightarrow H$  (called a *logarithmic transform* of  $f$ ) such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{F} & H \\ \exp \downarrow & & \downarrow \exp \\ \mathcal{V} & \xrightarrow{f} & W. \end{array}$$

The components of  $\mathcal{T}$  are called the *tracts* of  $F$ . Note that  $F$  is unique up to postcomposition by a map of the form  $z \mapsto z + k(z)$ , where  $k : \mathcal{T} \rightarrow 2\pi i\mathbb{Z}$  is continuous (and hence constant on every tract of  $F$ ).

By construction, the function  $F$  and its domain  $\mathcal{T}$  have the following properties (see also Figure 2):

- (a)  $H$  is a  $2\pi i$ -periodic Jordan domain that contains a right half-plane;
- (b) Every component  $T$  of  $\mathcal{T}$  is an unbounded Jordan domain with real parts bounded below, but unbounded from above;
- (c) The components of  $\mathcal{T}$  have disjoint closures and accumulate only at infinity; that is, if  $z_n \in \mathcal{T}$  is a sequence of points all belonging to different tracts, then  $z_n \rightarrow \infty$ ;
- (d) For every component  $T$  of  $\mathcal{T}$ ,  $F : T \rightarrow H$  is a conformal isomorphism. In particular,  $F$  extends continuously to the closure  $\overline{T}$  of  $T$  in  $\mathbb{C}$ ;
- (e) For every component  $T$  of  $\mathcal{T}$ ,  $\exp|_T$  is injective;
- (f)  $\mathcal{T}$  is invariant under translation by  $2\pi i$ .

We will denote by  $\mathcal{B}_{\log}$  the class of all  $F : \mathcal{T} \rightarrow H$  such that  $H$ ,  $\mathcal{T}$  and  $F$  satisfy (a) to (f) regardless of whether they arise from an entire function  $f \in \mathcal{B}$  or not. In particular, if  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  is any meromorphic function that has one or



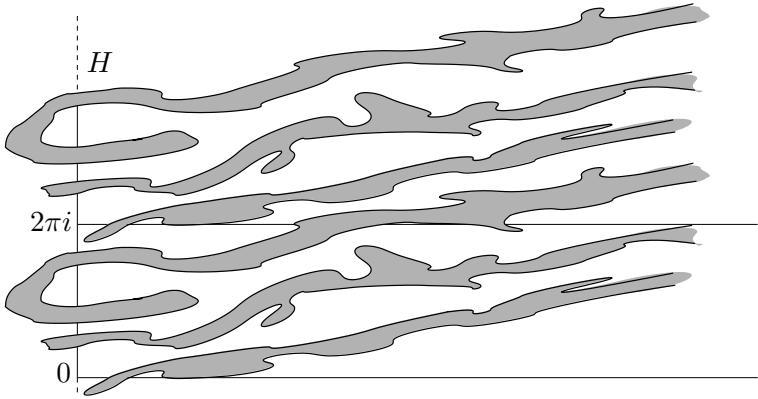


Figure 2. The domain of definition for a function  $F \in \mathcal{B}_{\log}$  is  $2\pi i$ -periodic. Note that a tract  $T$  need not be contained in  $H$ .

several logarithmic singularities over  $\infty$ , then we can associate to  $f$  a function  $F \in \mathcal{B}_{\log}$  that encodes the behavior of  $f$  near its logarithmic singularities.

If  $F \in \mathcal{B}_{\log}$  and  $T$  is a tract of  $F$ , we denote the inverse of the conformal isomorphism  $F : T \rightarrow H$  by  $F_T^{-1}$ .

*Normalized and disjoint-type functions.* Let  $F : \mathcal{T} \rightarrow H$  be a function in the class  $\mathcal{B}_{\log}$ . A simple application of Koebe’s 1/4-theorem shows that there is  $R_0 > 0$  such that

$$(2.1) \quad |F'(z)| \geq 2$$

when  $\text{Re } F(z) \geq R_0$ ; see [EL92, Lemma 2.1]. In the following, we refer to the property (2.1) as *expansivity* of  $F$ .

We shall say that  $F$  is *normalized* if  $H$  is the right half-plane  $\mathbb{H}$  and furthermore (2.1) holds for all  $z \in \mathcal{T}$ . We denote the set of all such functions by  $\mathcal{B}_{\log}^n$ .

Note that we can pass from any function  $F \in \mathcal{B}_{\log}$  to a normalized one by restricting  $F$  to  $\mathcal{T}' := F^{-1}(\mathbb{H}_{R_0})$  (where  $R_0$  is as above) and applying the change of variable  $w = z - R_0$ . For this reason, it is usually no loss of generality to assume that  $F \in \mathcal{B}_{\log}^n$ .

Let us also say that  $F$  is of *disjoint type* if  $\overline{\mathcal{T}} \subset H$ . It is easy to see that a function  $f \in \mathcal{B}$  has a disjoint-type logarithmic transform if and only if  $S(f)$  is contained in the immediate basin of an attracting fixed point of  $f$ . (This is the setting considered by Barański [Bar07].) Note that we might not be able to normalize such a function in the above-mentioned manner without losing the disjoint-type property. However, if  $F$  is of disjoint type, then  $F$  will be uniformly expanding with respect to the hyperbolic metric on  $H$ .

2.1. LEMMA (Uniform expansion for disjoint-type maps). *Suppose that  $F : \mathcal{T} \rightarrow H$  is of disjoint type.*

*Then there exists a constant  $\Lambda > 1$  such that the derivative of  $F$  with respect to the hyperbolic metric on  $H$  satisfies  $\|DF(z)\|_H = \lambda_{\mathcal{T}}(z)/\lambda_H(z) \geq \Lambda$  for all  $z \in \mathcal{T}$ .*

*In particular,  $\text{dist}_H(F(z), F(w)) \geq \Lambda \text{dist}_H(z, w)$  whenever  $z$  and  $w$  belong to the same tract of  $F$ .*

*Proof.* Equality in the first claim is satisfied because  $F : T \rightarrow H$  is a conformal isomorphism for every component  $T$  of  $\mathcal{T}$ . By Pick's theorem [Mil06, Th. 2.11], we have  $\|DF(z)\|_H > 1$  for all  $z \in \mathcal{T}$ , and we have  $\lambda_{\mathcal{T}}(z)/\lambda_H(z) \rightarrow \infty$  as  $z$  tends to the boundary of  $\mathcal{T}$  (in  $\mathbb{C}$ ). Since  $\mathcal{T}$  is  $2\pi i$ -periodic, it remains to show that  $\liminf_{\text{Re } z \rightarrow +\infty} \lambda_{\mathcal{T}}(z)/\lambda_H(z) > 1$ . By the standard estimate (1.1), we have  $\lambda_{\mathcal{T}}(z) \geq 1/2\pi$  for all  $z \in \mathcal{T}$ , while  $\lambda_H(z) \rightarrow 0$  as  $\text{Re } z \rightarrow +\infty$ . This proves the first claim. The second claim follows from the first: if  $\gamma$  is the hyperbolic geodesic of  $H$  that connects  $F(z)$  and  $F(w)$ , then  $F_T^{-1}(\gamma)$  has length at most  $\text{dist}_H(F(z), F(w))/\Lambda$ .  $\square$

*Combinatorics in  $\mathcal{B}_{\log}$ .* Let  $F \in \mathcal{B}_{\log}$ . We denote the Julia set and the set of escaping points of  $F$  by

$$J(F) := \{z \in \overline{\mathcal{T}} : F^{on}(z) \text{ is defined and in } \overline{\mathcal{T}} \text{ for all } n \geq 0\} \quad \text{and}$$

$$I(F) := \{z \in J(F) : \text{Re } F^{on}(z) \rightarrow \infty\}.$$

If  $f \in \mathcal{B}$  and  $F$  is a logarithmic transform of  $f$ , then clearly  $\exp(I(F)) \subset I(f)$ . Furthermore, if  $F$  is normalized or of disjoint type, then  $\exp(J(F)) \subset J(f)$ , respectively  $\exp(J(F)) = J(f)$ . (This follows from the uniform expansion of  $F$  in the Euclidean, respectively the hyperbolic metric, exactly as in [EL92, Th. 1].) For  $K > 0$  we also define more generally

$$J^K(F) := \{z \in J(F) : \text{Re } F^{on}(z) \geq K \text{ for all } n \geq 1\}.$$

The partition of the domain of  $F$  into tracts suggests a natural way to assign symbolic dynamics to points in  $J(F)$ . More precisely, let  $z \in J(F)$  and, for  $j \geq 0$ , let  $T_j$  be the tract of  $F$  with  $F^{oj}(z) \in \overline{T}_j$ . Then the sequence

$$\underline{s} := T_0 T_1 T_2 \dots$$

is called the *external address* of  $z$ . More generally, we refer to any sequence of tracts of  $F$  as an external address (of  $F$ ). If  $\underline{s}$  is such an external address, we define the closed set

$$J_{\underline{s}} := \{z \in J(F) : z \text{ has address } \underline{s}\};$$

we define  $I_{\underline{s}}$  and  $J_{\underline{s}}^K$  in a similar fashion (note that  $J_{\underline{s}}$ , and hence  $I_{\underline{s}}$  and  $J_{\underline{s}}^K$ , may well be empty for some addresses).

We denote the one-sided *shift operator* on external addresses by  $\sigma$ . In other words,  $\sigma(T_0T_1T_2\dots) = T_1T_2\dots$ .

*Definition 2.2* (Dynamic rays, ray tails). Let  $F \in \mathcal{B}_{\log}$ . A *ray tail* of  $F$  is a curve  $\gamma : [0, \infty) \rightarrow I(F)$  such that, for every  $n \geq 0$ ,  $F^{\circ n}|_\gamma$  is injective with  $\lim_{t \rightarrow \infty} \operatorname{Re} F^{\circ n}(\gamma(t)) = +\infty$ , and furthermore  $\operatorname{Re} F^{\circ n}(\gamma(t)) \rightarrow \infty$  uniformly in  $t$  as  $n \rightarrow \infty$ .

Likewise, we can define ray tails for an entire function  $f$ . A *dynamic ray* of  $f$  is then a maximal injective curve  $\gamma : (0, \infty) \rightarrow I(f)$  such that  $\gamma|_{[t, \infty)}$  is a ray tail for every  $t > 0$ .

*Remark.* If  $z$  is on a ray tail, then  $z$  is either on a dynamic ray or the landing point of such a ray. In particular, it is possible under our terminology for a ray tail to properly contain a dynamic ray.

In [Sections 4 and 5](#), we will construct ray tails for certain functions in the class  $\mathcal{B}_{\log}$ , and in particular for logarithmic transforms of the functions treated in [Theorem 1.2](#) and [Corollary 1.3](#). By the following fact, this will be sufficient to complete our objective.

**2.3. PROPOSITION** (Escaping points on rays). *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an entire function and let  $z \in I(f)$ . Suppose that some iterate  $f^{\circ k}(z)$  is on a ray tail  $\gamma_k$  of  $f$ . Then either  $z$  is on a ray tail, or there is some  $n \leq k$  such that  $f^{\circ n}(z)$  belongs to a ray tail that contains an asymptotic value of  $f$ .*

*In particular, there is a curve  $\gamma_0$  connecting  $z$  to  $\infty$  such that  $f^{\circ j}|_{\gamma_0}$  tends to  $\infty$  uniformly (in fact,  $f^{\circ k}(\gamma_0) \subset \gamma_k$ ).*

*Proof.* Let the ray tail be parametrized as  $\gamma_k : [0, \infty) \rightarrow \mathbb{C}$ . Let  $\gamma_{k-1} : [0, T) \rightarrow \mathbb{C}$  be a maximal lift of  $\gamma_k$  starting at  $f^{\circ(k-1)}(z)$ . That is,  $\gamma_{k-1}(0) = f^{\circ(k-1)}(z)$ ,  $f(\gamma_{k-1}(t)) = \gamma_k(t)$  and there is no extension of  $\gamma_{k-1}$  to a larger interval that has these properties. (Such a maximal lift exists, e.g., by Zorn's lemma.)

If  $T = \infty$ , then clearly  $\gamma_{k-1}(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Otherwise, the limit  $w = \lim_{t \rightarrow T} \gamma_{k-1}(t)$  exists in  $\widehat{\mathbb{C}}$ . If  $w \neq \infty$ , we could extend  $\gamma_{k-1}$  further (choosing any one of the possible branches of  $f^{-1}$  in the case where  $w$  is a critical point), contradicting maximality of  $T$ . Thus  $w = \infty$  and, in particular,  $\gamma_k(T)$  is an asymptotic value of  $f$ .

In either case, we have found a curve  $\gamma_{k-1} \subset f^{-1}(\gamma_k) \subset I(f)$  connecting  $f^{\circ(k-1)}(z)$  to infinity. This curve is a ray tail, except possibly if  $\gamma_k$  contained an asymptotic value of  $f$ . Continuing this method inductively, we are done.  $\square$

### 3. General properties of class $\mathcal{B}_{\log}$

In this section, we prove some general results for functions in class  $\mathcal{B}_{\log}$ . The first strengthens the expansion estimate of [\[EL92, Lemma 2.1\]](#) by showing that such a function expands distances like an exponential map.

3.1. LEMMA (Exponential separation of orbits). *Let  $F \in \mathcal{B}_{\log}^n$  and let  $T$  be a tract of  $F$ . If  $\omega, \zeta \in T$  are such that  $|\omega - \zeta| \geq 2$ , then*

$$|F(\omega) - F(\zeta)| \geq \exp(|\omega - \zeta|/8\pi) \cdot \min\{\operatorname{Re} F(\omega), \operatorname{Re} F(\zeta)\}.$$

*Proof.* Suppose without loss of generality that  $\operatorname{Re} F(\omega) \geq \operatorname{Re} F(\zeta)$ . Since  $T$  has height at most  $2\pi$ , it follows by the standard estimate (1.1) on hyperbolic distances that

$$|\omega - \zeta|/2\pi \leq \operatorname{dist}_T(\omega, \zeta) = \operatorname{dist}_{\mathbb{H}}(F(\omega), F(\zeta)).$$

Let  $\xi \in \mathbb{H}$  be a point that satisfies  $\operatorname{dist}_{\mathbb{H}}(F(\zeta), \xi) = \operatorname{dist}_{\mathbb{H}}(F(\omega), F(\zeta))$  and  $\operatorname{Re} F(\zeta) = \operatorname{Re} \xi$ ; see Figure 3. We will estimate the Euclidean distance  $s = |F(\zeta) - \xi|$ . Then  $|F(\omega) - F(\zeta)| \geq s$ . Let  $\gamma$  be the curve consisting of three straight line segments pictured in Figure 3, connecting  $F(\zeta)$  to  $\xi$  through  $F(\zeta) + s$  and  $\xi + s$ .

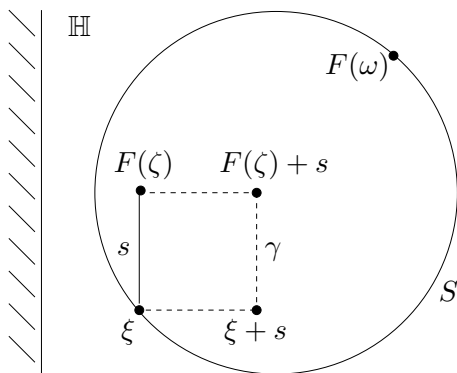


Figure 3. The set  $S$  of points of equal hyperbolic distance in  $\mathbb{H}$  to  $F(\zeta)$  is a Euclidean circle. Clearly,  $\xi$  is the Euclidean closest point to  $F(\zeta)$  on  $S$  that satisfies  $\operatorname{Re} \xi \geq \operatorname{Re} F(\zeta)$ . We use the dotted line  $\gamma$  to estimate  $s$ .

It is easy to see that the hyperbolic length of the vertical part of  $\gamma \subset \mathbb{H}$  is less than 1. On the other hand, each of the horizontal parts of  $\gamma$  has hyperbolic length precisely  $\log((\operatorname{Re} F(\zeta) + s)/\operatorname{Re} F(\zeta))$ . Hence, we get

$$\frac{|\omega - \zeta|}{2\pi} < 2 \log \left( \frac{\operatorname{Re} F(\zeta) + s}{\operatorname{Re} F(\zeta)} \right) + 1,$$

and therefore

$$|F(\omega) - F(\zeta)| \geq s \geq \left( \exp \left( \frac{|\omega - \zeta|}{4\pi} - \frac{1}{2} \right) - 1 \right) \cdot \operatorname{Re} F(\zeta).$$

Since  $e^{x-1/2} - 1 > e^{x/2}$  for  $x \geq 2$ , the claim follows.  $\square$

*Remark.* If  $w, z$  are two distinct points with the same external address, then by expansivity of  $F$  there exists  $k \in \mathbb{N}$  such that  $|F^{\circ k}(w) - F^{\circ k}(z)| \geq 2$ . Hence [Lemma 3.1](#) will apply eventually.

**3.2. LEMMA (Growth of real parts).** *Let  $F \in \mathcal{B}_{\log}^n$ . If  $\zeta, \omega \in J(F)$  are distinct points with the same external address  $\underline{s}$ , then*

$$\lim_{k \rightarrow \infty} \max\{\operatorname{Re} F^{\circ k}(\zeta), \operatorname{Re} F^{\circ k}(\omega)\} = \infty.$$

*Proof.* Suppose that  $\zeta, \omega \in J(F)$  satisfy  $\operatorname{Re} F^{\circ k}(\zeta), \operatorname{Re} F^{\circ k}(\omega) < S$  for some  $S > 0$  and infinitely many  $k \in \mathbb{N}$ . The set  $\overline{T} \cap \{z \in \mathbb{C} : \operatorname{Re} z \leq S\}$  is compact for every tract  $T$ , and thus has bounded imaginary parts. Furthermore, up to translations in  $2\pi i\mathbb{Z}$  there are only finitely many tracts of  $F$  that intersect the line  $\{\operatorname{Re} z = S\}$  at all (this follows from [property \(c\)](#) in the definition of  $\mathcal{B}_{\log}$ ). We conclude that there is  $C > 0$  such that  $|F^{\circ k}(\zeta) - F^{\circ k}(\omega)| < C$  whenever  $\operatorname{Re} F^{\circ k}(\zeta), \operatorname{Re} F^{\circ k}(\omega) < S$ . In particular,

$$|\zeta - \omega| \leq \frac{1}{2^k} \cdot |F^{\circ k}(\zeta) - F^{\circ k}(\omega)| \leq \frac{C}{2^k}$$

by expansivity of  $F$  (we have  $|(F_T^{-1})'(z)| < 1/2$  for any tract  $T$ ). Since this happens for infinitely many  $k$ , it follows that  $\zeta = \omega$ , as required.  $\square$

Note that [Lemma 3.2](#) does not imply that either  $\zeta$  or  $\omega$  escapes. Indeed, it is conceivable that both points have unbounded orbits but return to some left half-plane infinitely many times. In the next section, we introduce a property, called a *head-start condition*, which is designed precisely so that this does not occur.

As mentioned in the introduction, Rippon and Stallard [[RS05](#)] showed that the escaping set of every entire function  $f$  contains unbounded connected sets. The following theorem is a version of this result for functions in  $\mathcal{B}_{\log}$ .

**3.3. THEOREM (Existence of unbounded continua in  $J_{\underline{s}}$ ).** *For every function  $F \in \mathcal{B}_{\log}$  there exists  $K \geq 0$  with the following property. If  $z_0 \in J^K(F)$  and  $\underline{s}$  is the external address of  $z_0$ , then there exists an unbounded closed connected set  $A \subset J_{\underline{s}}$  with  $\operatorname{dist}(z_0, A) \leq 2\pi$ .*

*Proof.* We may assume without loss of generality that  $F$  is normalized. Choose  $K$  large enough so that no bounded component of  $\mathbb{H} \cap \overline{T}$  intersects the line  $\{\operatorname{Re} z = K\}$  and set  $z_k := F^{\circ k}(z_0)$ . If  $S \subset \mathbb{C}$  is an unbounded set such that  $S \setminus B_{2\pi}(z_k)$  has exactly one unbounded component, we denote this component by  $X_k(S)$ .

We claim that  $X_k(\overline{T}_k)$  is nonempty and contained in  $\mathbb{H}$  for all  $k \geq 1$ . (However, this set is *not* necessarily contained in  $\mathbb{H}_K$ .) Indeed, this is trivial if  $\overline{T}_k \subset \mathbb{H}$ . Otherwise, let  $\alpha^-$  and  $\alpha^+$  denote the two unbounded components

of  $\mathbb{H} \cap \partial T_k$ . We claim that both  $\alpha^-$  and  $\alpha^+$  must intersect the vertical line segment  $L := z_k + i[-2\pi, 2\pi]$ . Indeed, otherwise some  $2\pi i\mathbb{Z}$ -translate of  $\alpha^-$  or  $\alpha^+$  would separate  $z_k$  from  $\alpha^\pm$  in  $\mathbb{H}$ , which is not possible since  $z_k$  belongs to the unbounded component of  $T_k \cap \mathbb{H}$ . Hence it follows that the unbounded component of  $T_k \setminus L$ , which contains  $X_k(\overline{T}_k)$ , is contained in  $\mathbb{H}$ .

In particular, we can pull back the set  $X_k(T_k)$  into  $T_{k-1}$  using  $F_{T_{k-1}}^{-1}$ . By expansivity of  $F$ ,  $F_{T_{k-1}}^{-1}(X_k(T_k))$  has distance at most  $\pi$  from  $z_{k-1}$ . Continuing inductively, we obtain the sets

$$A_k := X_0(F_{T_0}^{-1}(X_1(F_{T_1}^{-1}(\dots(X_{k-1}(F_{T_{k-1}}^{-1}(X_k(\overline{T}_k))))\dots))))$$

for  $k \geq 1$ . Let  $A_0 = X_0(\overline{T}_0)$ . Each  $\widehat{A}_k \subset \widehat{\mathbb{C}}$  is a continuum, has distance  $2\pi$  from  $z_0$  and contains  $\widehat{A}_{k+1}$ . (Recall that  $\widehat{A}_k$  denotes the closure of  $A_k$  in  $\widehat{\mathbb{C}}$ .)

Hence the set  $A' = \bigcap_{k \geq 0} \widehat{A}_k$  has the same properties and there exists a component  $A$  of  $A' \setminus \{\infty\}$  with  $\text{dist}(A, z_0) \leq 2\pi$ . By definition,  $A$  is closed and connected, and it is unbounded by the Boundary Bumping Theorem ([Theorem A.4](#) in the appendix).  $\square$

#### 4. Functions satisfying a head-start condition

Throughout most of this section, we will fix some function  $F \in \mathcal{B}_{\log}$ . Fix an external address  $\underline{s}$ , and suppose that the set  $J_{\underline{s}}$  is a ray tail. Then  $J_{\underline{s}} \cup \{\infty\}$  is homeomorphic to  $[0, \infty]$ , and as such possesses a natural total ordering. In this section, we use a converse idea: we introduce a “head-start condition”, which implies that the points in  $J_{\underline{s}}$  are totally ordered by their speed of escape, and deduce from this that  $J_{\underline{s}}$  is (essentially) a ray tail. In the next section, we develop sufficient conditions on  $F$  under which a head-start condition is satisfied.

*Definition 4.1* (Head-start condition). Let  $T$  and  $T'$  be tracts of  $F$  and let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a (not necessarily strictly) monotonically increasing continuous function with  $\varphi(x) > x$  for all  $x \in \mathbb{R}$ . We say that the pair  $(T, T')$  satisfies the *head-start condition for  $\varphi$*  if, for all  $z, w \in \overline{T}$  with  $F(z), F(w) \in \overline{T'}$ ,

$$\text{Re } w > \varphi(\text{Re } z) \implies \text{Re } F(w) > \varphi(\text{Re } F(z)).$$

An external address  $\underline{s}$  satisfies the *head-start condition for  $\varphi$*  if all consecutive pairs of tracts  $(T_k, T_{k+1})$  satisfy the head-start condition for  $\varphi$ , and if for all distinct  $z, w \in J_{\underline{s}}$ , there is  $M \in \mathbb{N}$  such that  $\text{Re } F^{\circ M}(z) > \varphi(\text{Re } F^{\circ M}(w))$  or  $\text{Re } F^{\circ M}(w) > \varphi(\text{Re } F^{\circ M}(z))$ .

We say that  $F$  satisfies a *head-start condition* if every external address of  $F$  satisfies the head-start condition for some  $\varphi$ . If the same function  $\varphi$  can be chosen for all external addresses, we say that  $F$  satisfies the *uniform head-start condition for  $\varphi$* .

4.2. THEOREM (Ray tails). *Suppose that  $F \in \mathcal{B}_{\log}$  satisfies a head-start condition. Then for every escaping point  $z$ , there exists  $k \in \mathbb{N}$  such that  $F^{\circ k}(z)$  is on a ray tail  $\gamma$ . This ray tail is the unique arc in  $J(F)$  connecting  $F^{\circ k}(z)$  to  $\infty$  (up to reparametrization).*

We devote the remainder of this section to the proof of [Theorem 4.2](#).

If  $\underline{s}$  satisfies any head-start condition, the points in  $J_{\underline{s}}$  are eventually ordered by their real parts: for any two points  $z, w \in J_{\underline{s}}$ ,  $F^{\circ k}(z)$  is to the right of  $F^{\circ k}(w)$  for all sufficiently large  $k$ , or vice versa.

4.3. DEFINITION AND LEMMA (Speed ordering). *Let  $\underline{s}$  be an external address satisfying the head-start condition for  $\varphi$ . For  $z, w \in J_{\underline{s}}$ , we say that  $z \succ w$  if there exists  $K \in \mathbb{N}$  such that  $\operatorname{Re} F^{\circ K}(z) > \varphi(\operatorname{Re} F^{\circ K}(w))$ . We extend this order to the closure  $\widehat{J}_{\underline{s}}$  in  $\widehat{\mathbb{C}}$  by the convention that  $\infty \succ z$  for all  $z \in J_{\underline{s}}$ .*

*With this definition,  $(\widehat{J}_{\underline{s}}, \succ)$  is a totally ordered space. Moreover, the order does not depend on  $\varphi$ .*

Note that if  $z \succ w$ , then  $\operatorname{Re} F^{\circ k}(z) > \varphi(\operatorname{Re} F^{\circ k}(w))$  for all  $k \geq K$ .

*Proof.* By definition,  $\operatorname{Re} F^{\circ k}(z) < \varphi(\operatorname{Re} F^{\circ k}(z))$  for all  $k \in \mathbb{N}$  and  $z \in J_{\underline{s}}$ . Hence “ $\succ$ ” is nonreflexive.

Let  $a, b, c \in J_{\underline{s}}$  such that  $a \succ b$  and  $b \succ c$ . Then, there exist  $k, l \in \mathbb{N}$  such that  $\operatorname{Re} F^{\circ k}(a) > \varphi(\operatorname{Re} F^{\circ k}(b))$  and  $\operatorname{Re} F^{\circ l}(b) > \varphi(\operatorname{Re} F^{\circ l}(c))$ . Setting  $m := \max\{k, l\}$ , we obtain from the head-start condition that

$$\operatorname{Re} F^{\circ m}(a) > \varphi(\operatorname{Re} F^{\circ m}(b)) > \operatorname{Re} F^{\circ m}(b) > \varphi(\operatorname{Re} F^{\circ m}(c)).$$

Hence  $a \succ c$  and “ $\succ$ ” is transitive.

By assumption, for any distinct  $z, w \in J_{\underline{s}}$  there exists  $k \in \mathbb{N}$  such that  $\operatorname{Re} F^{\circ k}(w) > \varphi(\operatorname{Re} F^{\circ k}(z))$  or  $\operatorname{Re} F^{\circ k}(z) > \varphi(\operatorname{Re} F^{\circ k}(w))$ . It follows that any two distinct points are comparable under “ $\succ$ ”.

Furthermore, note that  $w \succ z$  if and only if  $\operatorname{Re} F^n(w) > \operatorname{Re} F^n(z)$  for all sufficiently large  $n$ . This formulation is independent of  $\varphi$ , proving the final claim.  $\square$

4.4. PROPOSITION (Arcs in  $J_{\underline{s}}$ ). *Let  $\underline{s}$  be an external address satisfying the head-start condition for  $\varphi$ . Then the topology of  $\widehat{J}_{\underline{s}}$  as a subset of the Riemann sphere  $\widehat{\mathbb{C}}$  agrees with the order topology induced by  $\succ$ . In particular,*

- (a) *every component of  $\widehat{J}_{\underline{s}}$  is homeomorphic to a (possibly degenerate) compact interval, and*
- (b) *if  $J_{\underline{s}}^K \neq \emptyset$  for  $K$  as in [Theorem 3.3](#), then  $J_{\underline{s}}$  has a unique unbounded component, which is a closed arc to infinity.*

*Proof.* Let us first show that  $\operatorname{id} : \widehat{J}_{\underline{s}} \rightarrow (\widehat{J}_{\underline{s}}, \succ)$  is continuous. Since  $\widehat{J}_{\underline{s}}$  is compact and the order topology on  $\widehat{J}_{\underline{s}}$  is Hausdorff, this will imply that  $\operatorname{id}$  is

a homeomorphism and that both topologies agree. It suffices to show that the sets  $U_a^- := \{w \in J_{\underline{s}} : a \succ w\}$  and  $U_a^+ := \{w \in \widehat{J}_{\underline{s}} : w \succ a\}$ , which form a subbasis for the order topology, are open in  $\widehat{J}_{\underline{s}}$  for every  $a \in \widehat{J}_{\underline{s}}$ . We will only give a proof for the sets  $U_a^-$ ; the proof for  $U_a^+$  is analogous.

Let  $w \in U_a^-$ . Choose  $k \in \mathbb{N}$  minimal such that  $\operatorname{Re} F^{\circ k}(a) > \varphi(\operatorname{Re} F^{\circ k}(w))$ . Since  $\varphi$ ,  $\operatorname{Re}$ , and  $F^{\circ k}$  are continuous, this is true for a neighborhood  $V$  of  $w$ . It follows that  $V \cap \widehat{J}_{\underline{s}} \subset U_a^-$ ; hence  $U_a^-$  is a neighborhood of  $w$  in  $\widehat{J}_{\underline{s}}$ .

Thus the topology of  $\widehat{J}_{\underline{s}}$  agrees with the order topology. Every connected component  $C$  of  $\widehat{J}_{\underline{s}}$  is compact; it follows from a well-known characterization of the arc ([Theorem A.5](#) in the appendix) that  $C$  is either a point or an arc. This proves (a). To prove (b), observe that existence follows from [Theorem 3.3](#), while uniqueness follows because  $\infty$  is the largest element of  $(\widehat{J}_{\underline{s}}, \succ)$ .  $\square$

4.5. COROLLARY (Growth of real parts). *Let  $\underline{s}$  be an external address that satisfies the head-start condition for  $\varphi$  and let  $z$  and  $w$  belong to the unbounded component of  $J_{\underline{s}}$ . If  $w \succ z$ , then  $w \in I(F)$ .*

*Proof.* If  $F$  is normalized (or of disjoint type), this is immediate from [Lemmas 3.2 and 4.3](#). Since these are the main cases of interest, we shall only sketch the proof in the general setting.

Set  $R_0 := \inf_{\zeta \in \mathcal{T}} \operatorname{Re} \zeta$ . For every  $M \geq 1$ , we can choose  $M$  points  $z_0, \dots, z_M \in J_{\underline{s}}$  such that  $z_0 = z$ ,  $z_M = w$  and  $z_{j+1} \succ z_j$  for all  $j$ . If  $n$  is sufficiently large, then we will have  $\operatorname{Re} F^{\circ n}(z_{j+1}) > \varphi(\operatorname{Re} F^{\circ n}(z_j))$  for all  $j$  (by the head-start condition for  $\underline{s}$ ). Hence  $\operatorname{Re} F^{\circ n}(w) > \varphi^{\circ M}(R_0)$ . The latter term tends to  $\infty$  with  $M$ , by the assumptions on  $\varphi$ , and hence  $w \in I(F)$ , as required.  $\square$

4.6. PROPOSITION (Points in the unbounded component of  $J_{\underline{s}}$ ). *Let  $\underline{s}$  be an external address that satisfies the head-start condition for  $\varphi$ . Then there exists  $K' \geq 0$  such that  $J_{\underline{s}}^{K'}$  is either empty or contained in the unbounded component of  $J_{\underline{s}}$  (and this component is a closed arc). The value  $K'$  depends on  $F$  and  $\varphi$ , but not on  $\underline{s}$ .*

*Proof.* We may assume without loss of generality that  $F$  is normalized, i.e.  $F \in \mathcal{B}_{\log}^n$ . Let  $K$  be the constant from [Theorem 3.3](#), set  $K' := \max\{\varphi(0)+1, K\}$  and let  $z_0 \in J_{\underline{s}}^{K'}$ . For each  $k \geq 0$ , we let  $z_k := F^{\circ k}(z_0)$  and consider the set

$$S_k := \{w \in J_{\sigma^k(\underline{s})} : w \succeq z_k\}.$$

By [Proposition 4.4](#), each  $S_k$  has a unique unbounded component  $A_k$  that is a closed arc. By [Theorem 3.3](#),  $A_k$  satisfies  $\operatorname{dist}(z_k, A_k) \leq 2\pi$ .

Let us show  $A_k \subset \mathbb{H}$  for  $k \geq 1$ , so that we may apply  $F^{-1}$  to it. Indeed, if  $w \in J_{\sigma^k(\underline{s})}$  with  $\operatorname{Re} w \leq 0$ , then the choice of  $K'$  and monotonicity of  $\varphi$  yield that  $\operatorname{Re} z_k > \varphi(0) \geq \varphi(\operatorname{Re} w)$ , and therefore  $z_k \succ w$ . Thus  $w \notin S_k$ . We



conclude that  $F_{T_{k-1}}^{-1}(A_k) \subset A_{k-1}$ , because it is unbounded and contained in  $S_{k-1}$ . Since  $F$  is expanding, this means that

$$\text{dist}(A_0, z_0) \leq 2^{-k} \text{dist}(z_k, A_k) \leq 2^{-(k-1)}\pi$$

for all  $k \geq 0$ . Thus  $z_0 \in A_0$ , as required. That  $A_0$  is an arc follows from [Proposition 4.4](#). □

*Proof of Theorem 4.2.* Let  $z$  be an escaping point for  $F$  and  $\underline{s}$  its external address. By hypothesis, there exists  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\underline{s}$  satisfies the head-start condition for  $\varphi$ . If  $K'$  is the constant from [Proposition 4.6](#), then there exists  $k \geq 0$  such that  $F^{\circ k}(z) \in J_{\underline{s}}^{K'}$  and  $\gamma_k := \{w \in I_{\sigma^k(\underline{s})} : w \succeq F^{\circ k}(z)\}$  is an injective curve connecting  $F^{\circ k}(z)$  to  $\infty$ . Furthermore, since the order topology agrees with the usual topology on  $J_{\underline{s}} \cup \infty$ ,  $\gamma_k$  is unique with this property.

To show that  $\gamma_k$  is a ray tail, we still need to show that

$$\lim_{m \rightarrow \infty} \inf_{w \in \gamma_k} \text{Re } F^{\circ m}(w) = \infty.$$

This follows from the head-start condition. Indeed, for  $w \in \gamma_k$  and  $m \in \mathbb{N}$ , we have  $\text{Re } F^{\circ m}(w) \geq \inf\{\varphi^{-1}(\text{Re } F^{\circ(k+m)}(z))\}$ , because  $w \succ z$  or  $w = z$  (we have to take the infimum because  $\varphi$  need not be invertible). This lower bound tends to infinity as  $m \rightarrow \infty$ . □

**4.7. THEOREM (Existence of absorbing brush).** *Suppose that  $F \in \mathcal{B}_{\log}$  satisfies a head-start condition. Then there exists a closed subset  $X \subset J(F)$  with the following properties:*

- (a)  $F(X) \subset X$ .
- (b) Each connected component  $C$  of  $X$  is a closed arc to infinity all of whose points except possibly the finite endpoint escape.
- (c) Every escaping point of  $F$  enters  $X$  after finitely many iterations. If  $F$  satisfies the uniform head-start condition for some function, then there exists  $K' > 0$  such that  $J^{K'}(F) \subset X$ .

*If, additionally,  $F$  is of disjoint type, then we may choose  $X = J(F)$ . If  $F$  is  $2\pi i$ -periodic, then  $X$  can also be chosen to be  $2\pi i$ -periodic.*

*Remark.* If  $F$  is  $2\pi i$ -periodic, one can show furthermore that the set  $X$  can be chosen to be a *Cantor Bouquet*; i.e., ambiently homeomorphic to a “straight brush” in the sense of Aarts and Oversteegen [[AO93](#)]; see [[BJR10](#)].

*Proof.* Let  $X$  denote the union of all the unbounded components of  $J(F)$ . By the Boundary Bumping [Theorem A.4](#),  $\widehat{X}$  is the connected component of the compact set  $J(F) \cup \{\infty\}$  that contains  $\infty$ ; hence  $X \subset \mathbb{C}$  is a closed set. Clearly  $X$  is  $F$ -invariant, and satisfies (b) and (c) by [Propositions 4.4 and 4.6](#) (recall that the choice of  $K'$  did not depend on the external address). If  $F$  is  $2\pi i$ -periodic, then the set  $X$  is  $2\pi i$ -periodic by definition.

Recall that  $F : \mathcal{T} \rightarrow H$  is of disjoint type if  $\overline{\mathcal{T}} \subset H$ . In this case, the set  $J(F) \cup \{\infty\}$  is connected, since it is the nested intersection of the unbounded compact connected sets  $F^{-n}(\overline{H}) \cup \{\infty\}$ . It follows from the above that  $X = J(F)$ .  $\square$

## 5. Geometry, growth and head-start

This section discusses geometric properties of tracts that imply a head-start condition. Moreover, we show that (compositions of) functions of finite order satisfy these properties, which completes the proof of [Theorem 1.2](#).

Let  $K > 1$  and  $M > 0$ . We say that  $\underline{s}$  satisfies the *linear head-start condition* with constants  $K$  and  $M$  if it satisfies the head-start condition for

$$\varphi(t) := K \cdot t^+ + M,$$

where  $t^+ = \max\{t, 0\}$ .

We will restrict our attention to functions whose tracts do not grow too quickly in the imaginary direction.

*Definition 5.1* (Bounded slope). Let  $F \in \mathcal{B}_{\log}$ . We say that the tracts of  $F$  have *bounded slope* (with constants  $\alpha, \beta > 0$ ) if

$$|\operatorname{Im} z - \operatorname{Im} w| \leq \alpha \max\{\operatorname{Re} z, \operatorname{Re} w, 0\} + \beta$$

whenever  $z$  and  $w$  belong to a common tract of  $F$ . We denote the class of all functions with this property by  $\mathcal{B}_{\log}(\alpha, \beta)$ , and use  $\mathcal{B}_{\log}^n(\alpha, \beta)$  to denote those that are also normalized.

*Remark.* By [property \(e\)](#) in the definition of  $\mathcal{B}_{\log}$ , this condition is equivalent to the existence of a curve  $\gamma : [0, \infty) \rightarrow \mathcal{T}$  with  $|F(\gamma(t))| \rightarrow \infty$  and

$$\limsup |\operatorname{Im} \gamma(t)| / \operatorname{Re} \gamma(t) < \infty.$$

Hence if one tract of  $F$  has bounded slope, then all tracts do.

The bounded slope condition means that the absolute value of a point is proportional to its real part. As we see in the next lemma, this easily implies that the second requirement of a linear head-start condition, that any two orbits eventually separate far enough for one to have a head-start over the other, is automatically satisfied when the tracts have bounded slope.

**5.2. LEMMA** (Linear separation of orbits). *Let  $F \in \mathcal{B}_{\log}^n$ , and let  $\alpha, \beta > 0$ . Let  $T$  be a tract of  $F$ , and suppose that  $z, w \in \overline{T}$  satisfy  $\operatorname{Re} F(w) \geq \operatorname{Re} F(z)$  and  $|\operatorname{Im} F(w) - \operatorname{Im} F(z)| \leq \alpha \operatorname{Re} F(w) + \beta$ .*

- (a) *There exists a constant  $\delta = \delta(\alpha, \beta)$ , depending only on  $\alpha$  and  $\beta$ , with the following property: if  $|z - w| \geq \delta$ , then*

$$\operatorname{Re} F(w) > e^{|z-w|/16\pi} \operatorname{Re} F(z).$$

(b) Let  $K \geq 1$  and  $Q \geq 0$ . Then there is a constant  $\delta = \delta(\alpha, \beta, K, Q)$  with the following property: if  $|z - w| \geq \delta$ , then

$$\operatorname{Re} F(w) > K \operatorname{Re} F(z) + |z - w| + Q.$$

In particular, suppose that  $F \in \mathcal{B}_{\log}^n(\alpha, \beta)$ , and let  $\underline{s}$  be an external address. If  $z, w \in J_{\underline{s}}$  with  $|z - w| \geq \delta(\alpha, \beta, K, 0)$ , then

$$\operatorname{Re} F^{\circ k}(z) > K \operatorname{Re} F^{\circ k}(w) + |z - w| \quad \text{or} \quad \operatorname{Re} F^{\circ k}(w) > K \operatorname{Re} F^{\circ k}(z) + |z - w|$$

for all  $k \geq 1$ .

*Proof.* Set  $\delta' := \alpha + \beta + 2$  and  $\delta := \max\{\delta', 16\pi \log \delta'\}$ . The hypotheses on  $z$  and  $w$  imply that  $|F(w) - F(z)| \leq (\alpha + 1) \operatorname{Re}(F(w)) + \beta$ . By expansivity of  $F$ , we have  $|F(w) - F(z)| \geq 2\delta' > \alpha + 1 + \beta$ , and thus  $\operatorname{Re} F(w) > 1$ . We hence conclude that  $|F(w) - F(z)| \leq \delta' \operatorname{Re} F(w)$ . Because  $|z - w| \geq 2$ , [Lemma 3.1](#) now yields

$$(5.1) \quad \begin{aligned} \operatorname{Re} F(w) &\geq \frac{|F(w) - F(z)|}{\delta'} \\ &\geq \frac{\exp(|w - z|/8\pi)}{\delta'} \cdot \operatorname{Re} F(z) > e^{\frac{|w-z|}{16\pi}} \cdot \operatorname{Re} F(z), \end{aligned}$$

because  $\exp(x/8\pi)/\delta' > \exp(x/16\pi)$  for all  $x > 16\pi \log \delta'$ . This proves part (a).

To prove part (b), we now choose  $\delta \geq \delta(\alpha, \beta) + 1/2$  sufficiently large that all  $x \geq \delta - 1/2$  satisfy  $e^{x/16\pi} > x + K + Q + 1/2$ .

Let  $z' \in T$  be the point with  $\operatorname{Re} F(z') = \max\{1, \operatorname{Re} F(z)\}$  and  $\operatorname{Im} F(z') = \operatorname{Im} F(z)$ . Then  $|z - z'| \leq 1/2$  by expansivity of  $F$ , so we can apply (5.1) to  $z'$  and  $w$ :

$$\begin{aligned} \operatorname{Re} F(w) &> e^{\frac{|w-z'|}{16\pi}} \cdot \operatorname{Re} F(z') > (|w - z'| + K + Q + 1/2) \cdot \operatorname{Re} F(z') \\ &\geq K \operatorname{Re} F(z') + |w - z'| + Q + 1/2 \geq K \operatorname{Re} F(z) + |w - z| + Q. \end{aligned}$$

The final claim follows from (b) by induction. □

*Remark 1.* The lemma shows that, if  $F \in \mathcal{B}_{\log}^n(\alpha, \beta)$  satisfies the linear head-start condition for some  $K$  and  $M$ , then  $F$  satisfies the linear head-start condition for all  $\widetilde{K} \geq K$  and  $\widetilde{M} \geq \max\{M, \delta(\alpha, \beta, K, 0)\}$ .

*Remark 2.* By the final claim of the lemma, if  $F \in \mathcal{B}_{\log}^n(\alpha, \beta)$ , then we only need to verify the first requirement of a linear head-start condition: if  $w$  is ahead of  $z$  in terms of real parts, the same should be true for  $F(w)$  and  $F(z)$ . Note that this condition is not dynamical in nature; rather, it concerns the mapping behavior of the conformal map  $F : T \rightarrow \mathbb{H}$ . As such, it is not too difficult to translate it into a geometrical condition. Roughly speaking, the tract should not “wiggle” in the sense of first growing in real parts to reach the larger point  $w$ , then turning around to return to  $z$ , until finally starting to grow again. (We exploit this idea in §6 to construct a counterexample; compare also

Figure 4 in the proof of Theorem 6.1.) The precise geometric condition is as follows.

*Definition 5.3* (Bounded wiggling). Let  $F \in \mathcal{B}_{\log}$ , and let  $T$  be a tract of  $F$ . We say that  $T$  has *bounded wiggling* if there exist  $K > 1$  and  $\mu > 0$  such that for every  $z_0 \in \bar{T}$ , every point  $z$  on the hyperbolic geodesic of  $T$  that connects  $z_0$  to  $\infty$  satisfies

$$(\operatorname{Re} z)^+ > \frac{1}{K} \operatorname{Re} z_0 - \mu.$$

We say that  $F \in \mathcal{B}_{\log}$  has *uniformly bounded wiggling* if the wiggling of all tracts of  $F$  is bounded by the same constants  $K, \mu$ .

5.4. PROPOSITION (Head-start and wiggling for bounded slope). *Let  $K > 1$  and  $F \in \mathcal{B}_{\log}^n(\alpha, \beta)$ . Then the following are equivalent:*

- (a) *For some  $M > 0$ ,  $F$  satisfies the uniform linear head-start condition with constants  $K$  and  $M$ .*
- (b) *For some  $\mu > 0$ , the tracts of  $F$  have uniformly bounded wiggling with constants  $K$  and  $\mu$ .*
- (c) *For some  $M' > 0$ , the following holds. If  $T$  is a tract of  $F$  and  $z, w \in \bar{T}$  with  $\operatorname{Re} w > K(\operatorname{Re} z)^+ + M$  and  $|\operatorname{Im} F(z) - \operatorname{Im} F(w)| \leq \alpha \max\{\operatorname{Re} F(z), \operatorname{Re} F(w)\} + \beta$ , then  $\operatorname{Re} F(w) > K \operatorname{Re} F(z) + M$ .*

*Proof.* Condition (c) implies (a) by definition. To show that (b) implies (c), let us set  $\widetilde{M} := K \cdot (\mu + 2\pi(\alpha + \beta))$  and define  $M := \max\{\delta, \widetilde{M}, 1\}$ , where  $\delta = \delta(\alpha, \beta, K, 0)$  is the constant from Lemma 5.2. Let  $T$  be a tract of  $F$  and let  $z, w \in \bar{T}$  be as in (c). Then  $|z - w| > M$  and, by Lemma 5.2 (b), it suffices to show that  $\operatorname{Re} F(w) \geq \operatorname{Re} F(z)$ .

Thus, suppose by way of contradiction that  $\operatorname{Re} F(z) > \operatorname{Re} F(w)$ . Since  $M \geq 1$ , we see from Lemma 5.2 that  $\operatorname{Re} F(z) \geq |z - w| > M \geq 1$ . Set  $\Gamma := \{F(w) + t : t \geq 0\}$  and  $\gamma := F_T^{-1}(\Gamma)$ ; in other words,  $\gamma$  is the geodesic of  $T$  connecting  $w$  to  $\infty$ . The assumption on  $F(z)$  and  $F(w)$  ensures that

$$\operatorname{dist}_T(z, \gamma) = \operatorname{dist}_{\mathbb{H}}(F(z), \Gamma) \leq \frac{|\operatorname{Im} F(z) - \operatorname{Im} F(w)|}{\operatorname{Re} F(z)} \leq \alpha + \beta.$$

Therefore,  $\operatorname{dist}(z, \gamma) \leq 2\pi(\alpha + \beta)$  by the standard estimate (1.1), and consequently  $\operatorname{Re} z + 2\pi(\alpha + \beta) \geq \min_{\zeta \in \gamma} \operatorname{Re} \zeta$ . By the bounded wiggling condition, we also have  $(\operatorname{Re} \zeta)^+ \geq \frac{1}{K} \operatorname{Re} w - \mu$  for all  $\zeta \in \gamma$ . Thus

$$\operatorname{Re} w \leq K((\operatorname{Re} z)^+ + \mu + 2\pi(\alpha + \beta)) < K(\operatorname{Re} z)^+ + \widetilde{M} \leq K(\operatorname{Re} z)^+ + M,$$

a contradiction.

Now suppose that (a) holds. Let  $T$  be a tract and  $z \in \bar{T}$ . Because the domain  $\mathcal{T}$  of  $F$  is  $2\pi i$ -periodic, we can pick a curve  $\Gamma \subset \mathcal{T} \cap \mathbb{H}$ , connecting some point  $\omega$  to  $\infty$  such that  $\operatorname{Re} w \geq \operatorname{Re} \omega$  for all  $w \in \Gamma$  and such that  $|F(z) - \omega| < \kappa$ ,

where  $\kappa > 0$  is a constant that is independent of  $T$  and  $z$ . Pulling back, we obtain a point  $\zeta := F_T^{-1}(\omega)$  that can be connected to  $z$  by a curve of Euclidean length at most  $\kappa/2$ , and to  $\infty$  by the curve  $\gamma := F_T^{-1}(\Gamma) \subset T$ . By the uniform head-start condition, we have

$$(\operatorname{Re} w)^+ \geq \frac{\operatorname{Re} \zeta}{K} - \frac{M}{K}$$

for all  $w \in \gamma$ . Hence there exists a curve  $\gamma' \subset T$  connecting  $z$  to  $\infty$  such that for every  $w \in \gamma'$ ,

$$(\operatorname{Re} w)^+ \geq \frac{\operatorname{Re} \zeta}{K} - \frac{M}{K} - \kappa/2 \geq \frac{\operatorname{Re} z}{K} - \frac{\kappa}{K} - \frac{M}{K} - \kappa/2.$$

Now (b) follows easily by general principles of hyperbolic geometry (see Lemma A.2 in the appendix).  $\square$

We now consider functions of finite order.

*Definition 5.5* (Finite order). We say that  $F \in \mathcal{B}_{\log}$  has *finite order* if

$$\log \operatorname{Re} F(w) = O(\operatorname{Re} w)$$

as  $\operatorname{Re} w \rightarrow \infty$  in  $\mathcal{T}$ .

Note that this definition ensures that  $f \in \mathcal{B}$  has finite order, i.e.

$$\limsup_{r \rightarrow \infty} \sup_{|z|=r} \frac{\log \log |f(z)|}{\log |z|} < \infty,$$

if and only if any logarithmic transform  $F \in \mathcal{B}_{\log}$  of  $f$  has finite order in the sense of Definition 5.5.

5.6. THEOREM (Finite order functions have good geometry). *Suppose that  $F \in \mathcal{B}_{\log}^n$  has finite order. Then the tracts of  $F$  have bounded slope and (uniformly) bounded wiggling.*

*Proof.* By the Ahlfors Spiral Theorem (Theorem A.1),  $F \in \mathcal{B}_{\log}^n(\alpha, \beta)$  for some constants  $\alpha, \beta$ . By the finite-order condition, there are  $\varrho$  and  $M$  such that  $\log \operatorname{Re} F(z) \leq \varrho \operatorname{Re} z + M$  for all  $z \in \mathcal{T}$ . Let  $T$  be a tract of  $F$  and  $z \in \bar{T}$ .

Suppose first that  $\operatorname{Re} F(z) \geq 1$ . Consider the geodesic  $\gamma(t) := F_T^{-1}(F(z) + t)$  (for  $t \geq 0$ ). Since the hyperbolic distance between  $z$  and  $\gamma(t)$  is at most  $\log(1+t)$ , we have

$$\operatorname{Re} z - \operatorname{Re} \gamma(t) \leq 2\pi \log(1+t) \leq 2\pi \log \operatorname{Re} F(\gamma(t)) \leq 2\pi(\varrho \operatorname{Re} \gamma(t) + M).$$

In other words,  $\operatorname{Re} z \leq (1 + 2\pi\varrho) \operatorname{Re} \gamma(t) + 2\pi M$ , i.e.

$$\operatorname{Re} \gamma(t) \geq \frac{1}{1 + 2\pi\varrho} \operatorname{Re} z - \frac{2\pi M}{1 + 2\pi\varrho}.$$

Since  $z$  was chosen arbitrarily, we have established the uniformly bounded wiggling condition for all points with  $\operatorname{Re} z \geq 1$ , with constants  $1/(1+2\pi\rho)$  and  $2\pi M/(1+2\pi\rho)$ .

If  $\operatorname{Re} F(z) < 1$ , then by expansivity of  $F$  we can connect  $z$  to a point  $w \in T$  with  $\operatorname{Re} F(w) \geq 1$  by a curve of bounded Euclidean diameter.  $\square$

To complete the proof of [Theorem 1.2](#), it only remains to show that linear head-start conditions are preserved under composition. In logarithmic coordinates, this is given by the following statement: let  $\tau_a(z) = z - a$  for  $a \geq 0$  and  $\mathbb{H}_a := \{z \in \mathbb{C} : \operatorname{Re}(z) > a\}$ .

**5.7. LEMMA** (Linear head-start is preserved by composition). *Let  $n \geq 1$  and let  $F_i : \mathcal{T}_{F_i} \rightarrow \mathbb{H} \in \mathcal{B}_{\log}^n$  for  $i = 1, 2, \dots, n$  such that  $F_i$  is  $2\pi i$ -periodic for  $i \geq 2$ . Then there is an  $a \geq 0$  such that  $G_a := \tau_a \circ F_n \circ \dots \circ F_1$  belongs to  $\mathcal{B}_{\log}^n$  when restricted to appropriate tracts  $\mathcal{T}_a \subset \mathcal{T}_{F_1}$ ; in particular,  $G_a$  is a conformal isomorphism from each component of  $\mathcal{T}_a$  onto  $\mathbb{H}$ . If  $F_1$  has bounded slope and all  $F_i$  satisfy uniform linear head-start conditions, then  $G_a$  also has bounded slope and satisfies a uniform linear head-start condition.*

*Proof.* There is an  $a_2 \geq 0$  so that  $F_2^{-1}(\mathbb{H}_{a_2}) \subset \mathbb{H}$ ; there is an  $a_3 \geq 0$  so that  $F_3^{-1}(\mathbb{H}_{a_3}) \subset \mathbb{H}_{a_2}$ , etc.. Finally, there is an  $a = a_n \geq 0$  such that every branch of  $(F_n \circ \dots \circ F_1)^{-1}$  is defined on all of  $\mathbb{H}_a$ . Let  $\mathcal{T}_a := (F_n \circ \dots \circ F_1)^{-1}(\mathbb{H}_a) \subset \mathcal{T}_{F_1}$ . Then  $F_n \circ \dots \circ F_1$  is a conformal isomorphism from each component of  $\mathcal{T}_a$  onto  $\mathbb{H}_a$ , and the first claim follows. In particular, the tracts of  $G_a$  have bounded slope.

For  $i = 1, \dots, n$ , let  $K_i$  and  $M_i$  be the constants for the linear head-start condition of  $F_i$ , and set  $K := \max_i K_i$  and  $M := \max\{\delta, \max_i M_i\}$ , where  $\delta = \delta(\alpha, \beta, K, 0)$  is the constant from [Lemma 5.2](#). Let  $T$  be a tract of  $F_i$  and  $w, z \in T$  such that  $\operatorname{Re} w > K \operatorname{Re} z + M$ . Then  $|w - z| \geq \operatorname{Re} w - \operatorname{Re} z > M \geq \delta$ , and [Lemma 5.2](#) gives that

$$\operatorname{Re} F_i(w) > K \operatorname{Re} F_i(z) + M \quad \text{or} \quad \operatorname{Re} F_i(z) > K \operatorname{Re} F_i(w) + M.$$

Since  $F_i$  satisfies a head-start condition, the first inequality must hold. Hence, all  $F_i$  satisfy a linear head-start condition with constants  $K, M$ , and it is now easy to see that  $G_a$  does, too.  $\square$

*Proof of [Theorem 1.2](#) and [Corollary 1.3](#).* Let  $f_1, \dots, f_n \in \mathcal{B}$  be functions of finite order. By applying a suitable affine change of variable, to all  $f_i$ , we may assume without loss of generality that each  $f_i$  has a *normalized* logarithmic transform  $F_j \in \mathcal{B}_{\log}^n$ ; furthermore  $F_j$  may be chosen to be  $2\pi i$ -periodic. By [Theorem 5.6](#), each  $F_j$  satisfies a linear head-start condition, and by [Lemma 5.7](#),  $G_a := \tau_a \circ F_n \circ \dots \circ F_1 \in \mathcal{B}_{\log}$  satisfies a linear head-start condition. (The purpose of  $\tau_a$  is only to arrange the maps so that their image is all of  $\mathbb{H}$ .) Now, on a sufficiently restricted domain,  $F := G_a \circ \tau_a^{-1}$  is a logarithmic transform

of  $f = f_n \circ \cdots \circ f_1$  and satisfies a linear head-start condition. Thus every escaping point of  $F$ , and hence of  $f$ , is eventually mapped into some ray tail. By [Proposition 2.3](#), this completes the proof of [Theorem 1.2](#).

The proof of [Corollary 1.3](#) is analogous. (Recall that the order of a meromorphic function is defined in terms of its Nevanlinna characteristic. However, if  $f$  has finite order, then it is well-known that the restriction of  $f$  to its logarithmic tracts will also have finite order in the previously defined sense.)  $\square$

*Remark.* If our goal was only to prove [Theorem 1.2](#) and [Corollary 1.3](#), a somewhat faster route would be possible (compare [[Rot05](#), Ch. 3]). For example, the linear head-start condition can be verified directly for functions of finite order, without explicitly considering the geometry of their tracts. Also, the bounded slope condition can be used to simplify the proof of [Theorem 4.2](#) in this context, eliminating e.g. the need for [Theorem 3.3](#). We have chosen the current approach because it provides both additional information and a clear conceptual picture of the proof.

Let us collect together some of the results obtained in this and the previous section for future reference.

5.8. COROLLARY (Linear head-start conditions). *Let  $\mathcal{H}_{\log}$  consist of all functions  $F \in \mathcal{B}_{\log}$  that satisfy a uniform linear head-start condition and have tracts of bounded slope.*

- (a) *The class  $\mathcal{H}_{\log}$  contains all function  $F \in \mathcal{B}_{\log}^n$  of finite order.*
- (b) *The class of  $2\pi i$ -periodic functions in  $\mathcal{H}_{\log}$  is closed under composition.*
- (c) *If  $F \in \mathcal{H}_{\log}$ , then there is some  $K > 0$  such that every point of  $J^K(F)$  can be connected to infinity by a curve in  $I(F)$ .*
- (d) *If  $F \in \mathcal{H}_{\log}$  is of disjoint type, then every component of  $J(F)$  consists of a dynamic ray together with a unique landing point.*

*Remark 1.* Here closure under composition should be understood in the sense of [Lemma 5.7](#). That is, given  $2\pi i$ -periodic functions  $F_1, \dots, F_n \in \mathcal{H}_{\log}$ , the function  $F_1 \circ \cdots \circ F_n$  belongs to  $\mathcal{B}_{\log}$  after a suitable restriction and conjugation with a translation; this map then also belongs to  $\mathcal{H}_{\log}$ .

*Remark 2.* It is easy to see that the class  $\mathcal{H}_{\log}$  is also closed under *quasi-conformal equivalence near infinity* in the sense of [[Rem09](#)].

*Proof.* The first claim is a combination of [Theorem 5.6](#) and [Proposition 5.4](#). The second follows from [Lemma 5.7](#), the third from [Proposition 4.6](#), and the final claim from [Theorem 4.7](#).  $\square$

In order to apply our results to functions in  $\mathcal{B}_{\log}$  that are of disjoint type but not necessarily normalized, we need to be able to verify linear head-start

conditions for these functions. The following lemma allows us to do this using the results we proved for normalized functions.

**5.9. PROPOSITION** (Disjoint-type maps and linear head-start). *Let  $F$  be a disjoint-type map in  $\mathcal{B}_{\log}(\alpha, \beta)$ , say  $F : \mathcal{T} \rightarrow H$ , and let  $R > 0$  such that  $\mathbb{H}_R \subset H$ . Then  $F$  satisfies a uniform linear head-start condition if and only if the map  $\widetilde{F} := F|_{F^{-1}(\mathbb{H}_R)}$  satisfies a uniform linear head-start condition.*

*Proof.* The “only if” direction is trivial, so suppose that  $\widetilde{F}$  satisfies a uniform linear head-start condition. We may assume that  $H \subset \mathbb{H}$  and that  $R$  is sufficiently large that (2.1) holds whenever  $\operatorname{Re} F(z) \geq R$ ; set  $\widetilde{\mathcal{T}} := F^{-1}(\mathbb{H}_R)$ . Then the map  $G := \widetilde{F}(z + R) - R$  is an element of  $\mathcal{B}_{\log}^n(\alpha, \beta + R\alpha)$  and satisfies a uniform linear head-start condition.

Define  $C$  to be the maximal hyperbolic diameter, in  $H$ , of a component of  $\widetilde{\mathcal{T}} \setminus \mathbb{H}_R$ . (This is a finite number because  $F$  is of disjoint type.) Applying Lemma 3.2 and Proposition 5.4 (c) to  $G$ , and translating the results back to  $\widetilde{F}$ , we see that there are constants  $K$  and  $M$  with the following property. Suppose that  $z, w$  belong to a component  $\widetilde{T}$  of  $\widetilde{\mathcal{T}}$  and  $F(z), F(w)$  belong to a component  $T'$  of  $\mathcal{T}$ . If  $\operatorname{Re} w > (\operatorname{Re} z)^+ + M$ , then

$$(5.2) \quad \operatorname{Re} F(w) > K \operatorname{Re} F(z) + M + R + 4\pi C.$$

We shall show that  $F$  satisfies the uniform head-start condition for  $\varphi(t) = Kt^+ + M + 4\pi C$ . Let  $T$  and  $T'$  be tracts of  $F$ , and suppose that  $z, w \in T$  with  $F(z), F(w) \in T'$  and  $\operatorname{Re} w > \varphi(\operatorname{Re} z)$ .

By definition of  $C$ , we can find a point  $z' \in T$  such that

$$\operatorname{Re} F(z') = \max\{\operatorname{Re} F(z), R\}, \quad F(z') \in T', \quad \text{and} \quad \operatorname{dist}_H(F(z'), F(z)) \leq C.$$

Because  $F$  is a conformal isomorphism, we have  $\operatorname{dist}_T(z', z) \leq C$ , and by the standard estimate (1.1),  $|z' - z| \leq 2\pi C$ . There is also a point  $w'$  with the corresponding properties for  $w$ . We now apply (5.2) to  $z'$  and  $w'$  to see that

$$\begin{aligned} \operatorname{Re} F(w) &\geq \operatorname{Re} F(w') - R > K \cdot \operatorname{Re} F(z') + M + 4\pi C \\ &\geq K \cdot \operatorname{Re} F(z) + M + 4\pi C = \varphi(\operatorname{Re} F(z)). \end{aligned}$$

This proves the first requirement of the head-start condition. The second follows easily from the fact that  $F$  uniformly expands the hyperbolic metric (Lemma 2.1); we leave the details to the reader.  $\square$

We can use the above lemma to describe the Julia sets of certain hyperbolic functions that are compositions of finite-order functions in class  $\mathcal{B}$ . As mentioned in the introduction, this has been proved by Barański [Bar07] when  $f$  is of finite order.



5.10. THEOREM (Disjoint-type maps). *Let  $f = f_1 \circ f_2 \circ \cdots \circ f_n$ , where  $f_i \in \mathcal{B}$  for all  $i$ , and all  $f_i$  have finite order. Suppose that  $S(f) \subset F(f)$  and that  $F(f)$  consists only of the immediate basin of an attracting fixed point of  $f$ . Then every component of  $J(f)$  is a dynamic ray together with a single landing point; in particular, every point of  $I(f)$  is on a ray tail of  $f$ .*

*Proof.* The assumptions imply that there is a bounded Jordan domain  $D$  such that  $S(f) \subset D$  and  $f(\overline{D}) \subset D$ . (This is a simple exercise; compare also [MB09, Prop. 2.7].) Using this domain in the definition of a logarithmic transform  $F$  of  $f$ , we see that  $F$  is of disjoint type, and that  $\exp(J(F)) = J(f)$ . As in the proof of Theorem 1.2, a suitable restriction of  $F$  satisfies a uniform linear head-start condition. The claim now follows from Proposition 5.9 and Theorem 4.7.  $\square$

To conclude this section, we comment on the issue of “random iteration”, where we are considering a sequence  $\mathcal{F} = (F_0, F_1, F_2, \dots)$  of functions, and study the corresponding “escaping set” set  $I(\mathcal{F}) = \{z \in \mathbb{C} : \mathcal{F}_n(z) \rightarrow \infty\}$ , where  $\mathcal{F}_n = F_n \circ F_{n-1} \circ \cdots \circ F_0$ . (Now the pairs  $(T, T')$  will consist of a tract  $T$  of  $F_k$  and a tract  $T'$  of  $F_{k+1}$ , etc.) Our proofs carry through analogously in this setting. In particular, if all tracts of all  $F_j$  have uniformly bounded wiggling and uniformly bounded slope, then again for every  $z \in I(\mathcal{F})$ , there is some iterate  $\mathcal{F}_n(z)$  that can be connected to infinity by a curve in the escaping set  $I(F_n, F_{n+1}, \dots)$ .

## 6. Counterexamples

This section is devoted to the proof of Theorem 1.1; that is, the construction of a counterexample to the strong form of Eremenko’s Conjecture. As mentioned in the previous section, such an example will be provided by a function with a tract that has sufficiently large “wiggles”.

We begin by formulating the exact properties that our counterexample should have. Then we construct a tract (and hence a function  $F \in \mathcal{B}_{\log}$ ) with the required properties. Finally, we show how to realize such a tract as that of an entire function  $f \in \mathcal{B}$ , using a function-theoretic principle.

To facilitate discussion in this and the next section, let us call an unbounded Jordan domain  $T$  a *tract* if the real parts of  $T$  are unbounded from above and the translates  $T+2\pi in$  (for  $n \in \mathbb{Z}$ ) have pairwise disjoint closures in  $\mathbb{C}$ .

6.1. THEOREM (No curve to infinity). *Let  $T \subset \mathbb{H}$  be a tract, and let  $F_0: T \rightarrow \mathbb{H}$  be a Riemann map, with continuous extension  $F_0: \widehat{T} \rightarrow \widehat{\mathbb{H}}$  given by Carathéodory’s Theorem. Suppose that the following hold:*

- (a)  $F_0(\infty) = \infty$ ;
- (b)  $|\operatorname{Im} z - \operatorname{Im} z'| < H$  for some  $H < 2\pi$  and all  $z, z' \in T$ ;

- (c) there are countably many disjoint hyperbolic geodesics  $C_k, \dot{C}_k \subset T$ , for  $k = 0, 1, \dots$ , so that all  $F_0(C_k)$  and  $F_0(\dot{C}_k)$  are semi-circles in  $\mathbb{H}$  centered at 0 with radii  $\varrho_{k+1}$  and  $\dot{\varrho}_{k+1}$  so that  $\varrho_1 < \dot{\varrho}_1 < \varrho_2 < \dots$ ;
- (d) all  $\varrho_k + H < \dot{\varrho}_k/2$  and all  $\dot{\varrho}_k + H < \varrho_{k+1}/2$ ;
- (e) all  $C_k$  and  $\dot{C}_k$  have real parts strictly between  $\dot{\varrho}_k + H$  and  $\varrho_{k+1}/2$ ;
- (f) all points in the unbounded component of  $T \setminus \dot{C}_k$  have real parts greater than  $\dot{\varrho}_k$ ;
- (g) every curve in  $T$  that connects  $C_k$  to  $\dot{C}_k$  intersects the vertical line  $\{z \in \mathbb{C} : \operatorname{Re} z = \varrho_k/2\}$ .

Define  $T_n := T + 2\pi in$  for  $n \in \mathbb{Z}$  and  $\mathcal{T} := \bigcup_n T_n$ , define Riemann maps  $F_n : T_n \rightarrow \mathbb{H}$  via  $F_n(z) := F_0(z - 2\pi in)$ , and define a map  $F : \mathcal{T} \rightarrow \mathbb{H}$  that coincides on  $T_n$  with  $F_n$  for each  $n$ .

Then the set  $J := J(F) = \{z \in T : F^{\circ k}(z) \in \mathcal{T} \text{ for all } k\}$  contains no curve to  $\infty$ .

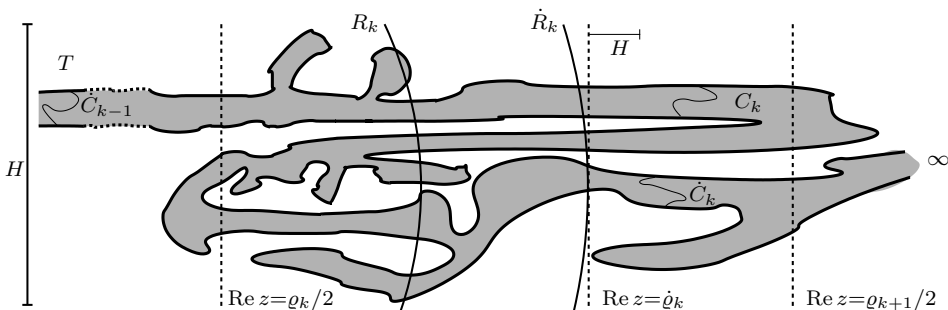


Figure 4. A tract that satisfies the conditions of [Theorem 6.1](#). The figure is not to scale, as can be seen from the horizontal and vertical dimensions of the length  $H$ .

*Proof.* Since the  $T_n$  have disjoint closures,  $F$  extends continuously to  $\overline{\mathcal{T}}$ . Let  $R_k$  and  $\dot{R}_k$  be semicircles in  $\mathbb{H}$  centered at 0 with radii  $\varrho_k$  and  $\dot{\varrho}_k$ , respectively.

Every  $z \in J$  has an external address  $\underline{s} = T_{s_0}T_{s_1}T_{s_2} \dots$  so that  $F^{\circ k}(z) \in T_{s_k}$  for all  $k$ . Clearly, all points within any connected component of  $J$  have the same external address, so we may fix an external address  $\underline{s}$  and show that the set  $J_{\underline{s}}$  (i.e., the points in  $J$  with address  $\underline{s}$ ) contains no curve to  $\infty$ . We may assume that there is an orbit  $(w_k)$  with external address  $\underline{s}$ . (If not, then we have nothing to show.)

For simplicity, we write  $C_m^k$  for  $C_m + 2\pi is_k$  and  $\dot{C}_m^k$  for  $\dot{C}_m + 2\pi is_k$ .

*Claim 1.* There is an  $m \geq 0$  so that for all  $k \geq 0$ ,  $\dot{C}_{m+k}^k$  separates  $w_k$  from  $\infty$  within  $T_{s_k}$ , and  $|w_{k+1}| < \dot{\varrho}_{m+k+1}$ .

*Proof.* We prove this claim by induction, based on [Condition \(f\)](#): some  $\dot{C}_m^0$  separates  $w_0$  from  $\infty$ . For the inductive step, suppose that  $\dot{C}_{m+k}^k$  separates  $w_k$  from  $\infty$  within  $T_{s_k}$ . Then  $\dot{R}_{m+k+1}$  separates  $w_{k+1}$  from  $\infty$  within  $\mathbb{H}$ ; i.e.,  $\operatorname{Re} w_{k+1} \leq |w_{k+1}| < \varrho_{m+k+1}$  ([Condition \(c\)](#)). By [Condition \(f\)](#), it follows that  $w_{k+1}$  is in the bounded component of  $T_{s_{k+1}} \setminus \dot{C}_{m+k+1}^{k+1}$ , so that  $\dot{C}_{m+k+1}^{k+1}$  separates  $w_{k+1}$  from  $\infty$  within  $T_{s_{k+1}}$ , and this keeps the induction going and proves the claim.  $\triangle$

*Claim 2.* For all  $k \geq 1$ , the semicircle  $R_{m+k+1}$  surrounds  $C_{m+k}^k$ ,  $\dot{C}_{m+k}^k$  and all points in  $T_{s_k}$  with real parts at most  $\varrho_{m+k+1}/2$ .

*Proof.* Recall that  $C_{m+k}^k$  and  $\dot{C}_{m+k}^k$  have real parts at most  $\varrho_{m+k+1}/2$  by [Condition \(e\)](#). So suppose that  $z \in T_{s_k}$  has  $\operatorname{Re} z \leq \varrho_{m+k+1}/2$ . We have  $|\operatorname{Im} w_k| \leq |w_k| < \varrho_{m+k}$  by the first claim, and since  $T_{s_k}$  contains  $w_k$  as well as  $z$  and has height at most  $H$  ([Condition \(b\)](#)), it follows that  $|\operatorname{Im} z| < \varrho_{m+k} + H$ . So, by [Condition \(d\)](#),

$$|z| \leq \operatorname{Re} z + |\operatorname{Im} z| < \varrho_{m+k+1}/2 + \varrho_{m+k} + H < \varrho_{m+k+1}. \quad \triangle$$

Now suppose there is a curve  $\gamma \subset J_{\underline{s}}$  that converges to  $\infty$ , and suppose that  $w_0$  was chosen with  $w_0 \in \gamma$ . For every  $k \geq 0$ , the curve  $F^{o^k}(\gamma)$  connects  $w_k$  to  $\infty$  ([Condition \(a\)](#)). The point  $w_k$  is surrounded by both  $R_{m+k+1}$  and  $\dot{R}_{m+k+1}$ ; by the first claim, we have  $|w_k| < \varrho_{m+k} < \varrho_{m+k+1} < \varrho_{m+k+1}$ . As a result,  $F^{o^k}(\gamma)$  must contain a subcurve  $\gamma_k$  connecting  $R_{m+k+1}$  with  $\dot{R}_{m+k+1}$ . But this implies that  $F^{o^{(k-1)}}(\gamma)$  contains a subcurve  $\gamma_{k-1}$  connecting  $C_{m+k}^{k-1}$  with  $\dot{C}_{m+k}^{k-1}$  ([Condition c](#)). Since  $C_{m+k}^{k-1}$  and  $\dot{C}_{m+k}^{k-1}$  have real parts greater than  $\varrho_{m+k} + H$  by [Condition \(e\)](#), it follows that both endpoints of  $\gamma_{k-1}$  are outside of  $\dot{R}_{m+k}$ . But  $\gamma_{k-1}$  must be contained within  $T_{s_{k-1}}$ , so [Condition \(g\)](#) implies that  $\gamma_{k-1}$  must contain a point  $z_{k-1} \in T_{s_{k-1}}$  with real part  $\varrho_{m+k}/2$ . Now the last claim shows that  $z_{k-1}$  is surrounded by  $R_{m+k}$ . As a result,  $\gamma_{k-1}$  must contain two disjoint subcurves that connect  $R_{m+k}$  with  $\dot{R}_{m+k}$ .

Continuing the argument inductively, it follows that  $\gamma$  contains  $2^k$  disjoint subcurves that connect  $C_{m+1}^0$  with  $\dot{C}_{m+1}^0$ . Since this is true for every  $k \geq 0$ , we obtain a contradiction.  $\square$

Now we give conditions under which the set  $J$  not only contains no curve to  $\infty$ , but in fact no unbounded curve at all. In many cases these conditions are satisfied automatically, such as in the example constructed below (see [Theorem 6.3](#)).

**6.2. COROLLARY (Bounded path components).** *Suppose that, in addition to the conditions of [Theorem 6.1](#), there are countably many disjoint hyperbolic geodesics  $\ddot{C}_k \subset T$  so that all  $F_0(\ddot{C}_k)$  are semi-circles in  $\mathbb{H}$  centered at 0 with*

radii  $\ddot{\varrho}_{k+1} > \dot{\varrho}_{k+1}$  so that the bounded component of  $T \setminus \check{C}_{k+1}$  has real parts at most  $\ddot{\varrho}_{k+1}/2$ . Then every path component of  $J$  is bounded.

*Proof.* We continue the proof of the previous theorem. Suppose there is an unbounded curve  $\gamma \subset J_{\underline{s}}$  with  $w_0 \in \gamma$ . For every  $k \geq 0$  the curve  $F^{\circ k}(\gamma)$  connects  $w_k$  to points at arbitrarily large real parts. We will show that there must be a point  $z_0 \in \gamma$  so that for every  $k$  the subcurve of  $\gamma$  between  $w_0$  and  $z_0$  contains  $2^k$  disjoint subcurves that connect  $C_{m+1}^0$  with  $\check{C}_{m+1}^0$ ; this is a contradiction.

Let  $\ddot{R}_k$  be semi-circles centered at 0 with radii  $\ddot{\varrho}_k$ . Since  $\ddot{\varrho}_{k+1} > \dot{\varrho}_{k+1}$ , it follows that every  $\check{C}_k$  is in the unbounded component of  $T \setminus \check{C}_k$ . Define vertical translates  $\check{C}_m^k = \check{C}_m + 2\pi i s_k$  in analogy to the  $C_m^k$  and  $\check{C}_m^k$ . As in the second claim in the proof above, it follows that the bounded component of  $T_{\underline{s}_k} \setminus \check{C}_{m+k+1}^k$  is surrounded by  $\ddot{R}_{m+k+1}$ .

By the first claim in the proof above,  $\check{C}_m^0$  separates  $w_0$  from  $\infty$  within  $T_{s_0}$ , so that  $\check{C}_m^0$  and also  $\check{C}_{m+1}^0$  must do the same. Let  $z_0$  be a point in the intersection of  $\gamma$  with  $\check{C}_{m+1}^0$  and denote by  $[w_0, z_0]_\gamma$  the subcurve of  $\gamma$  connecting  $w_0$  with  $z_0$ . Then  $F([w_0, z_0]_\gamma)$  connects  $w_1$  with  $F(z_0) \in \ddot{R}_{m+2}$ . So  $F(z_0)$  belongs to the unbounded component of  $T_{s_1} \setminus \check{C}_{m+2}^1$ , and there is thus a point  $z_1 \in [w_0, z_0]_\gamma$  with  $F(z_1) \in \check{C}_{m+2}^1$ , and  $F^{\circ 2}([w_0, z_1]_\gamma)$  connects  $w_2$  with  $\ddot{R}_{m+3}$ . By induction, for any  $k \geq 0$ , the curve  $F^{\circ k}([w_0, z_{k-1}]_\gamma)$  connects  $w_k$  with  $\ddot{R}_{m+k+1}$ , and hence it connects  $R_{m+k+1}$  with  $\check{R}_{m+k+1}$ .

The same arguments from the proof of the theorem now show that in fact  $[w_0, z_k]_\gamma \subset [w_0, z_0]_\gamma$  must contain  $2^k$  subcurves connecting  $C_{m+1}^0$  with  $\check{C}_{m+1}^0$  for every  $k \geq 0$ , and this is the desired contradiction.  $\square$

*Remark.* We stated the results in the form above in order to minimize the order of growth of the resulting entire function, and to show that the entire functions we construct can be rather close to finite order; see [Section 8](#). If we were only interested in the nonexistence of unbounded path components in  $I$ , we could have formulated conditions that are somewhat simpler than those in the preceding theorem and its corollary. For instance, the necessity for introducing a third geodesic  $\check{C}_k$  would have been removed if we had placed  $C_k$  at real parts at most  $\varrho_k/2$ , and required that the entire bounded component of  $T \setminus C_k$  has real parts less than  $\varrho_k/2$  (the image of any curve in  $T$  connecting  $\check{C}_{m+k}^k$  with  $C_{m+k+2}^k$  would then connect  $\check{C}_{m+k+1}^{k+1}$  and  $C_{m+k+3}^{k+1}$ ; this keeps the induction going as before.)

**6.3. THEOREM** (Tract with bounded path components). *There exist a tract  $T$  with  $\bar{T} \subset \mathbb{H}$  and a conformal isomorphism  $F_0 : T \rightarrow \mathbb{H}$  fixing  $\infty$  that satisfies the conditions of [Theorem 6.1](#) and [Corollary 6.2](#), so that every path component of  $J$  is bounded.*

In fact,  $T$  and  $F_0$  can be chosen so as to satisfy the following conditions for an arbitrary  $M \in (1, 1.75)$  (with the same notation as in [Theorem 6.1](#)):

- (d')  $\varrho_k^M < \dot{\varrho}_k$  and  $\dot{\varrho}_k^M < \varrho_{k+1}$ ;
- (e') the geodesics  $C_k$  and  $\dot{C}_k$  have real parts strictly between  $\dot{\varrho}_k^M$  and  $\varrho_{k+1}/3$ ;
- (f') all points in the unbounded component of  $T \setminus \dot{C}_k$  have real parts greater than  $\dot{\varrho}_k^M$ ;
- (g') every curve in  $T$  that connects  $C_k$  to  $\dot{C}_k$  intersects the vertical line  $\{z \in \mathbb{C} : \operatorname{Re}(z) = \varrho_k^{1/M}\}$ ;
- (h') the geodesics  $\dot{C}_{k+1}$  from [Corollary 6.2](#) have the property that the bounded component of  $T \setminus \dot{C}_{k+1}$  has real parts at most  $(\dot{\varrho}_{k+1})^{1/M}$ .

*Remark.* The modified conditions as written in this theorem are needed in order to show that this tract is “approximately” realized by an entire function: they are adapted to the quality of the approximation that we get later in this section.

*Proof.* Our domain  $T$  will be a countable union of long horizontal tubes of unit thickness, together with countably many vertical tubes and countably many turns made of quarter and half annuli, all of unit thickness as well (see [Figure 5\(a\)](#)). The domain  $T$  terminates at the far left with a semidisk at center  $P$ . The lengths of the various tubes are labelled as in [Figure 5\(b\)](#).

More precisely, our tract  $T$  is specified by the length  $h' > 1$  (fixed below) and sequences  $(\xi_k)_{k \geq 0}$  and  $(\dot{\xi}_k)_{k \geq 0}$ , with  $\xi_0 > 2$  and  $\xi_k < \dot{\xi}_k < \xi_{k+1} - 4 - 2h'$  for all  $k$ . Let us set  $P := 1$ ,  $P_k := \dot{\xi}_k + h'$ , and  $\dot{P}_k := P_k - 4i$ . We define a curve

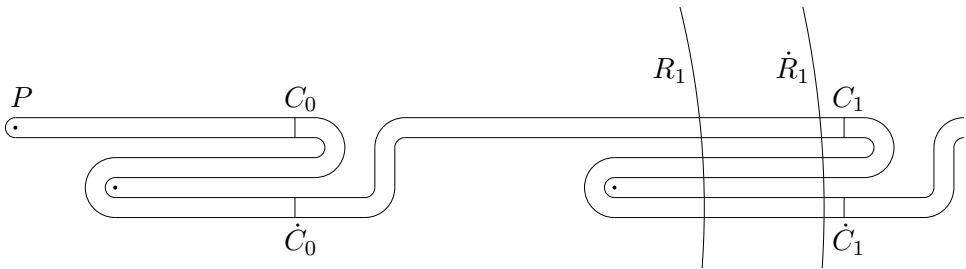
$$\Gamma = \bigcup_{k \geq 0} \gamma_k \cup \dot{\gamma}_k,$$

where  $\gamma_0$  is the straight line segment  $[P, P_0]$  and

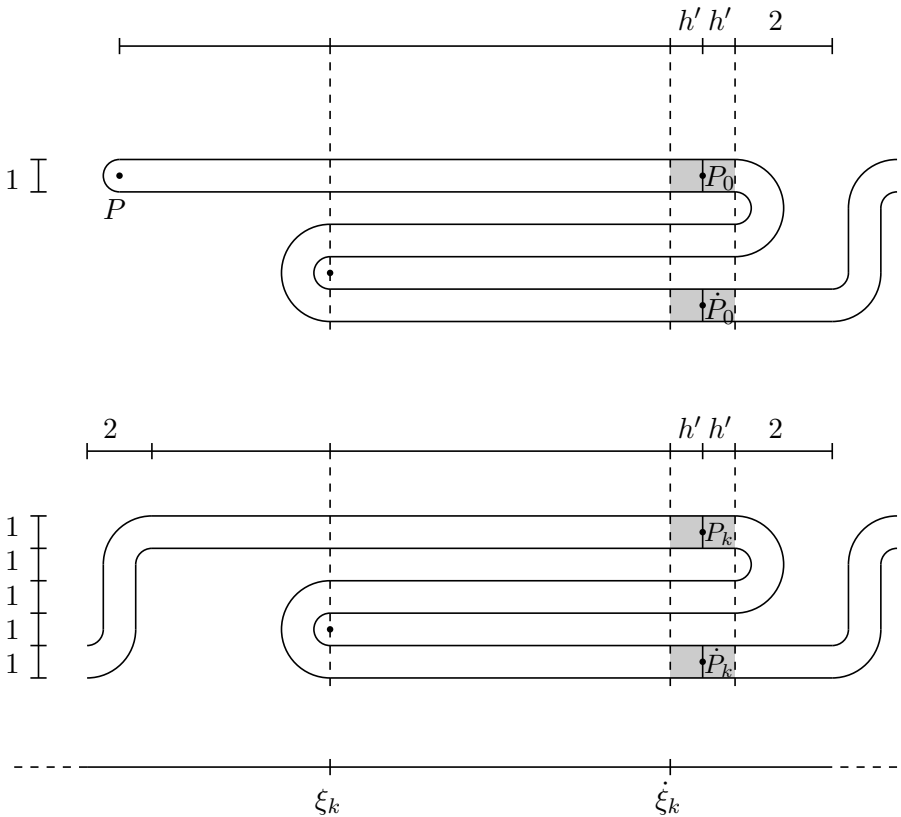
$$\begin{aligned} \dot{\gamma}_k &= [P_k, P_k + h'] \cup \{P_k + h' - i + e^{2\pi i \theta} : |\theta| < \pi/2\} \\ &\quad \cup [P_k + h' - 2i, \xi_k - 2i] \cup \{\xi_k - 3i + e^{2\pi i \theta} : |\theta - \pi| < \pi/2\} \cup [\xi_k - 4i, \dot{P}_k], \\ \gamma_{k+1} &= [\dot{P}_k, \dot{P}_k + h' + 2] \cup \{\dot{P}_k + h' + 2 + i + e^{2\pi i \theta} : \theta \in (-\pi/2, 0)\} \\ &\quad \cup [\dot{\xi}_k + 2h' + 3 - 3i, \dot{\xi}_k + 2h' + 3 - i] \\ &\quad \cup \{\dot{\xi}_k + 2h' + 4 - i + e^{2\pi i \theta} : \theta \in (\pi/2, \pi)\} \cup [\dot{\xi}_k + 2h' + 4, P_{k+1}]. \end{aligned}$$

The tract  $T$  then consists of all points that have distance at most  $1/2$  from  $\gamma$ . The map  $F_0$  is chosen such that  $F_0(1) = 1$  and  $F_0(\infty) = \infty$ ; this determines  $F_0$  completely.

Note that  $T$  and  $F_0$  (regardless of the choices of  $\xi_k$ ,  $\dot{\xi}_k$  and  $h'$ ) satisfy conditions (a) and (b), where  $H = 5 < 2\pi$ . We set  $\varrho_{k+1} := |F_0(P_k)|$  and



(a) The tract  $T$ .



(b) Length scales in the construction.

Figure 5. Construction of an example for [Theorem 6.3](#). The length  $h'$  is independent of  $k$ . In (b), the boxes  $Q_k$  and  $\dot{Q}_k$  are shaded.

$\dot{\varrho}_{k+1} := |F_0(\dot{P}_k)|$ . Let  $R_{k+1}$  and  $\dot{R}_{k+1}$  be the semicircles around 0 with radii  $\varrho_{k+1}$  and  $\dot{\varrho}_{k+1}$  and let  $C_k := F_0^{-1}(R_{k+1})$  and  $\dot{C}_k := F_0^{-1}(\dot{R}_{k+1})$ .

Then  $C_k$  and  $\dot{C}_k$  are hyperbolic geodesics of  $T$ . If  $h'$  is sufficiently large — in fact,  $h' := 2$  is sufficient (see [Lemma A.3](#) in the appendix) — then  $C_k$  and  $\dot{C}_k$  will be contained in the boxes

$$Q_k := \{z \in \mathbb{C} : \operatorname{Re}(z) \in (\dot{\xi}_k, \dot{\xi}_k + 2h'), |\operatorname{Im} z| < 1/2\},$$

and  $\dot{Q}_k := Q_k - 4hi$ , and they connect the upper with the lower boundaries of their boxes. In particular,  $\dot{C}_k$  separates  $C_k$  from  $C_{k+1}$ , so [condition \(c\)](#) is also satisfied.

We now define the sequences  $\xi_k$  and  $\dot{\xi}_k$ . Begin by choosing  $\xi_0 > 2$  sufficiently large (see below) and setting  $\dot{\xi}_0 := \xi_0^{12M^2}$ . We then proceed inductively by setting

$$\xi_{k+1} := \exp(\dot{\xi}_k/M) \quad \text{and} \quad \dot{\xi}_{k+1} := \exp(12M\dot{\xi}_k) = \xi_{k+1}^{12M^2}.$$

To see that these indeed give rise to a well-defined tract  $T$  as above, we need to verify that

$$(6.1) \quad \xi_{k+1} = \exp(\dot{\xi}_k/M) > \dot{\xi}_k + 4 + 2h'.$$

If  $\xi_0$  — and hence all  $\xi_k$  — was chosen sufficiently large, then this inequality will certainly hold. We will use other, similar, elementary inequalities below that may hold only if  $\xi_0$  is sufficiently large. We use the symbol “ $\star$ ” to mark such inequalities (e.g., “ $\exp(\dot{\xi}_k/M) \star \dot{\xi}_k + 4 + 2h'$ ”).

It is easy to see that the remaining conditions from [Theorem 6.1](#) and [Corollary 6.2](#) follow from the modified ones in the statement of the theorem, provided that  $\xi_0$  was chosen sufficiently large. So it remains to verify (d')–(h'); we begin by estimating  $\varrho_{k+1}$  and  $\dot{\varrho}_{k+1}$  for  $k \geq 0$ . We claim that

$$(6.2) \quad \xi_{k+1}^M = \exp(\dot{\xi}_k) < \varrho_{k+1} < \exp((4\pi/3)\dot{\xi}_k)$$

and

$$(6.3) \quad \dot{\varrho}_{k+1} < \exp(12\dot{\xi}_k) = \xi_{k+1}^{1/M}.$$

We prove [inequalities \(6.2\)](#) and [\(6.3\)](#) using the hyperbolic metric in the domain  $T$ . Indeed, we have  $\log \varrho_{k+1} = \operatorname{dist}_{\mathbb{H}}(P, R_{k+1}) = \operatorname{dist}_T(P, C_k)$ , and similarly for  $\dot{\varrho}_{k+1}$ . Hence it suffices to estimate the hyperbolic distance between  $P$  and  $C_k$ , which we can easily do using the standard estimate [\(1.1\)](#).

Let  $\gamma$  be the piece of  $\Gamma$  that connects  $P$  to  $P_k$ ; i.e.

$$\gamma = \bigcup_{j \leq k} \gamma_j \cup \bigcup_{j < k} \dot{\gamma}_j.$$

If  $k \geq 1$ , we have

$$\ell(\gamma) < \dot{\xi}_k + h' + 2(\dot{\xi}_{k-1} + 2h') + 4k\pi \stackrel{\star}{<} \dot{\xi}_k + 3\dot{\xi}_{k-1} = \dot{\xi}_k + (\log \dot{\xi}_k)/4M \stackrel{\star}{<} (\pi/3)\dot{\xi}_k.$$

For  $k = 0$ , we also have  $\ell(\gamma) = \dot{\xi}_0 - 1 + h' \stackrel{*}{<} (\pi/3)\dot{\xi}_0$ . So

$$\log \varrho_{k+1} = \text{dist}_T(P, C_k) \leq \ell_T(\gamma) \leq 4\ell(\gamma) < (4\pi/3)\dot{\xi}_k.$$

The upper bound for  $\dot{\varrho}_{k+1}$  is proved analogously. Let  $\dot{\gamma}$  be the piece of  $\Gamma$  connecting  $P$  to  $\dot{P}_k$ . If  $k \geq 1$ , then

$$\ell(\dot{\gamma}) < 3(\dot{\xi}_k + 2h') + 3(k+1)\pi - (\xi_k - \dot{\xi}_{k-1} - 2h').$$

Note that

$$\xi_k = \exp(\dot{\xi}_{k-1}/M) \stackrel{*}{>} 2\dot{\xi}_{k-1} + 8h' \stackrel{*}{>} \dot{\xi}_{k-1} + 8h' + 3(k+1)\pi,$$

so we have  $\ell(\dot{\gamma}) < 3\dot{\xi}_k$ . If  $k = 0$ , also

$$\ell(\dot{\gamma}) < 3(\dot{\xi}_0 + 2h') + \pi - 2\xi_0 \stackrel{*}{<} 3\dot{\xi}_0.$$

Hence  $\log \dot{\varrho}_{k+1} \leq 4\ell(\dot{\gamma}) < 12\dot{\xi}_k$ .

To prove the lower bound for  $\varrho_{k+1}$ , note that *any* curve  $\alpha$  connecting  $P$  to  $C_k$  must have  $\ell(\alpha) \geq \dot{\xi}_k + h' - 1 > \dot{\xi}_k$ , and every point of  $\alpha$  has distance at most  $1/2$  from  $\partial T$ . Therefore

$$\log \varrho_{k+1} \geq \inf_{\alpha} \ell_T(\alpha) \geq \inf_{\alpha} \ell(\alpha) > \dot{\xi}_k,$$

as claimed.

Now we show that  $\varrho_{k+1}$  and  $\dot{\varrho}_{k+1}$  satisfy condition (d'). The second inequality follows from (6.1), (6.2) and (6.3):

$$\dot{\varrho}_k^M < \dot{\xi}_k < \xi_{k+1} < \varrho_{k+1}^{1/M} < \varrho_{k+1}.$$

To prove the first inequality, note that  $F_0$  maps the subdomain of  $T$  bounded by  $C_k$  and  $\dot{C}_k$  conformally onto the semi-annulus in  $\mathbb{H}$  between the semicircles  $R_{k+1}$  and  $\dot{R}_{k+1}$ . So the moduli are equal. Since the above-mentioned subdomain contains two rectangles of modulus  $\dot{\xi}_k - \xi_k$ , we see by the Grötzsch inequality that

$$\frac{1}{\pi} \log(\dot{\varrho}_{k+1}/\varrho_{k+1}) > 2(\dot{\xi}_k - \xi_k) = 2\left(\dot{\xi}_k - \dot{\xi}_k^{1/(12M^2)}\right) \stackrel{*}{>} \dot{\xi}_k.$$

Thus, using (6.2),

$$\dot{\varrho}_{k+1} > \varrho_{k+1} \exp(\pi\dot{\xi}_k) = \varrho_{k+1} \left(\exp((4\pi/3)\dot{\xi}_k)\right)^{3/4} > \varrho_{k+1}^{1.75} > \varrho_{k+1}^M.$$

The construction of  $C_{k+1}$  and  $\dot{C}_{k+1}$  is such that their real parts are at least  $\dot{\xi}_{k+1}$ , which is larger than  $\dot{\varrho}_{k+1}^M$  by (6.3), and at most

$$\dot{\xi}_{k+1} + 2h' = M \cdot (\log \xi_{k+2}) + 2h' \stackrel{*}{<} \xi_{k+2}/3 < \varrho_{k+2}/3$$

(by (6.2)), so that condition (e') is satisfied.

Condition (f') is obvious: the construction ensures that all points in the unbounded component of  $T \setminus \dot{C}_{k+1}$  have real parts at least  $\dot{\xi}_{k+1}$ , which is greater than  $\dot{\varrho}_{k+1}^M$  by (6.3).



Furthermore, every curve in  $T$  that connects  $C_{k+1}$  with  $\dot{C}_{k+1}$  must reach real parts less than  $\xi_k < \varrho_k^{1/M}$ , and this is condition (g').

To conclude, we show that (h') is satisfied, so that [Corollary 6.2](#) applies and all path components of  $J$  are bounded. We define  $\ddot{P}_k := M(12M + 1)\dot{\xi}_k$ . Since  $M(12M + 1)\dot{\xi}_k \stackrel{*}{<} \exp(\dot{\xi}_k/M) = \xi_{k+1}$ , we have  $P_k \in T$ , so we can set  $\ddot{\varrho}_k := |F_0(\ddot{P}_k)|$ . Let  $\ddot{R}_k$  be the semicircles in  $\mathbb{H}$  centered at 0 with radii  $\ddot{\varrho}_k$ , and let  $\ddot{C}_k \ni \ddot{P}_k$  be the  $F_0$ -preimage of  $\ddot{R}_{k+1}$  within  $T$ . Then, by [Lemma A.3](#) as above, all points in the bounded component of  $T \setminus \ddot{C}_{k+1}$  have real parts at most  $\text{Re } \ddot{P}_{k+1} + h'$ .

We can again estimate  $\log \ddot{\varrho}_{k+1} = \text{dist}_T(P, \ddot{C}_k) > \text{Re } \ddot{P}_k$ . Hence

$$\begin{aligned} \ddot{\varrho}_{k+1}^{1/M} &> \exp((12M + 1)\dot{\xi}_k) = \exp(\dot{\xi}_k) \cdot \dot{\xi}_{k+1} \\ &\stackrel{*}{>} M(12M + 1)\dot{\xi}_{k+1} + h' = \text{Re } \ddot{P}_{k+1} + h'. \quad \square \end{aligned}$$

In order to complete the proof of [Theorem 1.1](#), we need to show that there is an entire function that suitably approximates the previously constructed map. To this end, we will use the following fact on the existence of entire functions with a prescribed tract; a proof can be found in the next section. Let us introduce the following notation: If  $F : T \rightarrow \mathbb{H}$  is a conformal isomorphism, then a geodesic of  $T$  that is mapped by  $F$  to a semicircle centered at 0 is called a *vertical geodesic* (of  $F$ ).

**6.4. PROPOSITION** (Approximation by entire functions). *Let  $T$  be a tract, and let  $F : T \rightarrow \mathbb{H}$  be a conformal isomorphism fixing  $\infty$ . Let  $\theta > 1$ .*

*Then there is an entire function  $g \in \mathcal{B}$  with  $S(f) \subset B_1(0)$  and a single tract  $W = g^{-1}(\{|z| > 1\})$ , and a  $2\pi i$ -periodic logarithmic transform  $G : \log W \rightarrow \mathbb{H}$  of  $g$  with the following properties:*

- (a)  $\log W$  has a component  $\widetilde{T}$  satisfying  $\widetilde{T} \subset T$ ;
- (b) the vertical geodesics of  $G$  have uniformly bounded diameters;
- (c)  $|F(z)| \leq |G(z)| \leq |F(z)|^\theta$  when  $z \in \widetilde{T}$ , with  $\text{Re } z$  sufficiently large.

*Remark.* If we apply the above proposition to a tract  $T$  with  $\overline{T} \subset \mathbb{H}$  (such as the one from [Theorem 6.3](#)), then the resulting function  $g$  satisfies  $f(B_1(0)) \Subset B_1(0)$ . It follows that the postsingular set is compactly contained in the Fatou set of  $g$ , and hence that  $g$  is hyperbolic.

*Proof of Theorem 1.1 (using Proposition 6.4).* Let  $F_0 \in \mathcal{B}_{\log}$  be the function constructed in [Theorem 6.3](#), and let  $T$  be its single tract. Choose a constant  $1 < \theta < M$  (where  $M$  is the constant from [Theorem 6.3](#)). Let  $g$  be a function as in [Proposition 6.4](#), with logarithmic transform  $G : \widetilde{T} \rightarrow \mathbb{H}$ . (Recall that  $G$  extends continuously to the closure  $\text{cl}(\widetilde{T})$ .)

The vertical geodesics  $C_k$  and  $\dot{C}_k$  of  $T$  intersect  $\widetilde{T}$  for sufficiently large  $k$ . Let  $\dot{\sigma}_{k+1}$  be maximal with the property that the geodesic

$$\dot{D}_k := \{z \in \text{cl}(\widetilde{T}) : |G(z)| = \dot{\sigma}_{k+1}\}$$

intersects  $\dot{C}_k$ , and define  $\sigma_{k+1}$  and  $D_k$  similarly. We claim that, with this choice of geodesics, the function  $G$  also satisfies the conclusions of [Theorem 6.3](#) (for a constant  $M' < M/\theta$ ).

Indeed, by (c) of [Proposition 6.4](#), we have  $\varrho_k \leq \sigma_k \leq \varrho_k^\theta$  and  $\dot{\varrho}_k \leq \dot{\sigma}_k \leq \dot{\varrho}_k^\theta$ . Thus

$$\sigma_k^{M'} \leq \varrho_k^M < \dot{\varrho}_k \leq \dot{\sigma}_k \quad \text{and} \quad \dot{\sigma}_k^{M'} \leq \dot{\varrho}_k^M < \varrho_{k+1} \leq \sigma_k.$$

Thus (d') holds. (e') and (g') follow similarly, using the fact that the geodesics  $D_k$  and  $\dot{D}_k$  have uniformly bounded diameters.

To prove (f'), note that the unbounded component of  $\widetilde{T} \setminus \dot{D}_k$  does not intersect  $\dot{C}_k$  since we chose  $\dot{\sigma}_{k+1}$  to be maximal. Hence it follows from condition (f') for  $F_0$  that this component has real parts at least  $\dot{\varrho}_k^M \geq \dot{\sigma}_k^{M'}$ . Property (h') follows analogously.  $\square$

## 7. Entire functions with prescribed tracts

We will now prove [Proposition 6.4](#). Eremenko and Lyubich [[EL87](#)] were the first to introduce methods of approximation theory into holomorphic dynamics; more precisely they used Arakelian's approximation theorem to construct various entire functions with "pathological" dynamics. It is possible likewise to use this theorem to approximate any given tract by a logarithmic tract of an entire function; this would be enough to give a counterexample to the strong form of Eremenko's conjecture. However, Arakelian's theorem provides no information on the singular values of the approximating map, so a function obtained in this manner might not belong to the Eremenko-Lyubich class.

Hence, instead we use the method of approximating a given tract using Cauchy integrals, which is also well-established. (Compare e.g. [[GE79](#)] for a similar construction.) There appears to be no result stated in the literature that is immediately applicable to our situation. We will therefore first provide a proof of the following, more classical-looking statement, and then proceed to indicate how it implies [Proposition 6.4](#).

7.1. PROPOSITION (Existence of functions with prescribed tracts). *Let  $V \subset \mathbb{C}$  be an unbounded Jordan domain and let  $\Psi : V \rightarrow \mathbb{H}$  be a conformal isomorphism with  $\Psi(\infty) = \infty$ . Let  $\varrho$  be arbitrary with  $1 < \varrho < 2$  and define*

$$f : V \rightarrow \mathbb{C}; z \mapsto \exp((\Psi(z))^\varrho).$$

*Then there exists an entire function  $g \in \mathcal{B}$  and a number  $K > 0$  such that the following hold:*

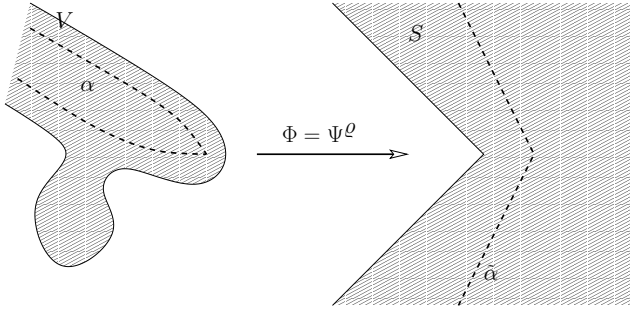


Figure 6. Definition of  $\alpha$  and  $\tilde{\alpha}$  in the proof of Proposition 7.1

- (a)  $W := \{z : |g(z)| > K\}$  is a simply connected domain that is contained in  $V$ , and  $g|_W$  is a universal covering;
- (b)  $|g(z) - f(z)| = O(1)$  on  $W$ , and  $g(z) = O(1)$  outside  $W$ .

*Remark.* In particular, the tract  $W$  of  $g$  satisfies

$$V \supset W \supset \{z : \operatorname{Re} \Psi(z) > C \text{ and } |\arg \Psi(z)| < \theta\}$$

(where  $\theta$  can be chosen arbitrarily close to  $\pi/2\rho$  if  $C$  is sufficiently large). So this proposition really does present a result on the realization of a prescribed tract (up to a certain “pruning” of the edges) by an entire function.

*Proof.* The idea of the proof is simple. We define a function  $h$ , using an integral along the boundary  $\alpha$  of the desired tract, which changes by  $f(z)$  as  $z$  crosses the curve  $\alpha$ . That is, we set

$$h(z) := \frac{1}{2\pi i} \int_{\alpha} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We are using  $\Psi^\rho$ , rather than  $\Psi$  itself, in the definition of  $f$  to ensure that this integral converges uniformly and that  $h$  is bounded. Then the function  $g$  that agrees with  $h$  on the outside of  $\alpha$  and with  $h + f$  on the inside will be entire, and it follows easily that it is in class  $\mathcal{B}$ .

Let us now provide the details of this argument. We define  $\Phi := \Psi^\rho$  and let  $S$  denote the sector  $S := \Phi(V) = \{z : |\arg z| < \frac{\rho\pi}{2}\}$ . Also fix some  $\eta \in (\pi/2, \rho\pi/2)$  and set  $\nu := \exp(i\eta)$ . We define

$$\tilde{\alpha} : (-\infty, \infty) \rightarrow S; \quad t \mapsto \begin{cases} 1 + \nu t & t \geq 0 \\ 1 + \bar{\nu}|t| & t < 0 \end{cases} \quad \text{and} \quad \alpha := \Phi^{-1} \circ \tilde{\alpha}.$$

Let  $V'$  denote the component of  $\mathbb{C} \setminus \alpha$  with  $V' \subset V$ .

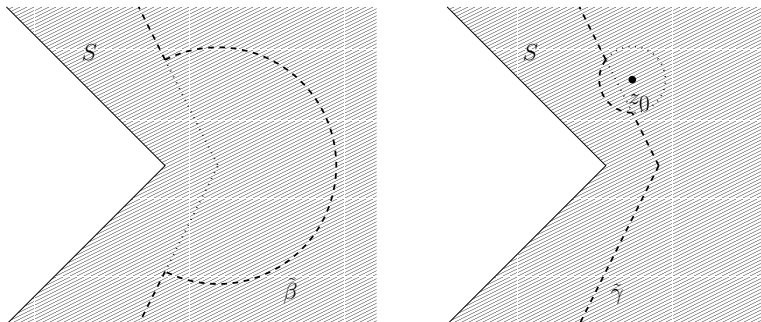


Figure 7. Definition of  $\tilde{\beta}$  and  $\tilde{\gamma}$  as in Claims 2 and 3

*Claim 1.* The integral  $\int_{\alpha} f(\zeta)d\zeta$  converges absolutely. In particular,

$$h(z) := \frac{1}{2\pi i} \int_{\alpha} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

defines a holomorphic function  $h : \mathbb{C} \setminus \alpha \rightarrow \mathbb{C}$ .

*Proof.* Note that  $|\alpha'(t)| = |1/\Phi'(\alpha(t))|$  for  $t \neq 0$ . By the Schwarz lemma and Koebe’s theorem,

$$|\Phi'(\alpha(t))| \geq \frac{\text{dist}(\tilde{\alpha}(t), \partial S)}{4 \text{dist}(\alpha(t), \partial V)}.$$

Clearly  $\text{dist}(\tilde{\alpha}(t), \partial S) \geq C_1(1 + |t|)$  for some  $C_1 > 0$ . So the hyperbolic length of  $\alpha|_{[0,t]}$  satisfies  $\ell_S(\alpha|_{[0,t]}) = O(\log(|t| + 1))$ . On the other hand, by the standard estimate (1.1), the density  $\lambda_V$  of the hyperbolic metric on  $V$  satisfies  $\lambda_V(z) \geq 1/(2(|z| + |z_0|))$  for all points  $z \in V$  (where  $z_0$  is an arbitrary point of  $\partial V$ ), which means that  $\log |z| = O(\text{dist}_V(z, \alpha(0)))$  as  $z \rightarrow \infty$ . Combining these estimates, we see that  $|\alpha(t)|$  grows at most polynomially in  $|t|$ ; in particular,  $\text{dist}(\alpha(t), \partial V) \leq C_2(1 + |t|)^c$  for some  $c, C_2 > 0$ .

Together, these estimates imply by the Koebe theorem that  $|\Phi'(\alpha(t))| \geq C/4(1 + |t|)^{c-1}$  for  $C := C_1/C_2$ . In particular,

$$\begin{aligned} \int_{\alpha} |f(\zeta)||d\zeta| &= \int_{-\infty}^{+\infty} \exp(\text{Re } \tilde{\alpha}(t))|\alpha'(t)|dt \\ &= \int_{-\infty}^{+\infty} \frac{\exp(1 - |\text{Re}(\nu)t|)}{|\Phi'(\alpha(t))|} dt \leq \frac{e}{C} \int_{-\infty}^{+\infty} (1 + |t|)^{c-1} e^{-|\text{Re}(\nu)t|} dt < \infty. \end{aligned}$$

This completes the proof. △

*Claim 2.* The function

$$g(z) := \begin{cases} h(z) & z \notin \overline{V'} \\ h(z) + f(z) & z \in V' \end{cases}$$

extends to an entire function  $g : \mathbb{C} \rightarrow \mathbb{C}$ .

*Proof.* Let  $R \gg 1$  be arbitrary, and modify  $\tilde{\alpha}$  to obtain a curve

$$\tilde{\beta} := (\tilde{\alpha} \cap \{|\zeta| > R\}) \cup \{1 + Re^{2\pi i\theta} : \theta \in [-\eta, \eta]\}.$$

Set  $\beta := \Phi^{-1} \circ \tilde{\beta}$  and let  $W$  be the unbounded component of  $\mathbb{C} \setminus \beta$  that contains  $\mathbb{C} \setminus \overline{V'}$ . Then

$$\tilde{g} : W \rightarrow \mathbb{C}; z \mapsto \frac{1}{2\pi i} \int_{\beta} \frac{f(\zeta)}{\zeta - z} d\zeta$$

defines an analytic function on  $W$ . By the Cauchy integral theorem,  $\tilde{g}$  agrees with  $g$  on  $\mathbb{C} \setminus \overline{V'}$ .

Furthermore, for  $z \in W \cap V'$ , we have by the residue theorem that

$$\tilde{g}(z) - h(z) = \text{res}_z \left( \frac{f(\zeta)}{\zeta - z} \right) = f(z).$$

In particular,  $\tilde{g} = g|_W$ . Since  $R$  was arbitrary, the claim follows. △

*Claim 3.* The function  $h$  is uniformly bounded.

*Proof.* Let  $z_0 \in \mathbb{C} \setminus \alpha$ . We set  $\delta := \sin(\eta)/2 = \text{dist}(\tilde{\alpha}, \partial S)/2$  and define a curve  $\tilde{\gamma}$  (depending on  $z_0$ ) as follows. If  $z_0 \notin V$  or if  $z_0 \in V$  and  $\text{dist}(\Phi(z_0), \tilde{\alpha}) \geq \delta/2$ , we simply set  $\tilde{\gamma} := \tilde{\alpha}$ . Otherwise, we set

$$\tilde{\gamma} := (\tilde{\alpha} \setminus \{z : |z - \Phi(z_0)| < \delta/2\}) \cup C,$$

where  $C$  is the arc of the circle  $\{|z - \Phi(z_0)| = \delta/2\}$  for which  $\tilde{\alpha} \cup C$  does not separate  $\Phi(z_0)$  from  $\infty$ .

We also set  $\gamma := \Phi^{-1} \circ \tilde{\gamma}$ . By Cauchy's integral theorem,

$$h(z_0) = \frac{1}{2\pi i} \int_{\alpha} \frac{\exp(\Phi(\zeta))}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\gamma} \frac{\exp(\Phi(\zeta))}{\zeta - z} d\zeta.$$

Thus it is sufficient to show that the second integral is bounded independently of  $z_0$ . By the Koebe 1/4-theorem and the definition of  $\gamma$ , we have  $|\gamma(t) - z_0| \geq \delta/8|\Phi'(\gamma(t))|$  for all  $t$ . If we parametrize  $\tilde{\gamma}$  by arc length, then clearly

$$\text{Re } \tilde{\gamma}(t) \leq C - K|t|,$$

where the constants  $K = |\text{Re } \nu|$  and  $C$  are independent of  $z_0$ . We thus have

$$\begin{aligned} 2\pi|h(z_0)| &= \left| \int_{\gamma} \frac{\exp(\Phi(\zeta))}{\zeta - z_0} d\zeta \right| \leq \int_{-\infty}^{+\infty} \frac{|\exp(\tilde{\gamma}(t))|}{|\gamma(t) - z_0|} |\tilde{\gamma}'(t)| dt \\ &= \int_{-\infty}^{+\infty} \frac{\exp(\text{Re } \tilde{\gamma}(t))}{|\Phi'(\gamma(t))| \cdot |\gamma(t) - z_0|} dt \leq \int_{-\infty}^{+\infty} \exp(C - K|t|) \frac{8|\Phi'(\gamma(t))|}{\delta|\Phi'(\gamma(t))|} dt \\ &= \frac{8}{\delta} \int_{-\infty}^{+\infty} \exp(C - K|t|) dt < \infty. \end{aligned}$$

So  $h$  is uniformly bounded, as required. △

To complete the proof, let  $M > 0$  be such that  $|h(z)| < M$  for all  $z$ . Set  $K := 2M$  and  $W := \{z \in \mathbb{C} : |g(z)| > K\}$ . If  $\beta$  is a simple closed curve in  $W$ , then  $|\Phi(z)| > M$  on  $\beta$ . By the minimum principle, we also have  $|\Phi(z)| > M$  on the region  $U$  surrounded by  $\beta$ . It follows that  $g$  has no zeros in  $U$ , and by the minimum principle,  $U \subset W$ . Thus  $W$  is simply connected.

We can therefore define a function

$$G := \log g : W \rightarrow \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta > \log(2M)\}.$$

It is easy to see that  $G$  is proper. Since there is exactly one homotopy class of curves in  $W$  along which  $G(z) \rightarrow \infty$ , the degree of  $G$  is 1. In other words,  $G$  is a conformal isomorphism, and  $f|_W = \exp \circ G$  is a universal cover, as required.  $\square$

*Proof of Proposition 6.4.* Let  $V := \exp(T)$ , and let  $\Psi : V \rightarrow \mathbb{H}$  be the conformal isomorphism with  $\Psi \circ \exp = F$ . Let  $1 < \varrho < \min(\theta, 2)$ , let  $f$  be as in Proposition 7.1, and let  $\tilde{g}$  be the entire function constructed there. Recall that this function satisfies  $|\tilde{g}(z) - f(z)| = O(1)$  on its tract  $W = \tilde{g}^{-1}(\{|z| > K\})$ . It follows that the logarithmic transform  $\tilde{G}$  also satisfies  $|\tilde{G}(z) - F(z)^\varrho| \leq C_1$  for some  $C_1 > 0$ .

Now set  $g(z) := \tilde{g}(z)/K$ , and let  $G : \tilde{T} \rightarrow \mathbb{H}$  be its logarithmic transform; i.e.,  $G(z) = \tilde{G}(z) - \log K$ . We claim that  $g$  is the desired entire function. Indeed, by choice of  $\tilde{g}$ , we have

$$|(F(z))^\varrho - G(z)| \leq C_1 + \log K =: C,$$

which proves (c).

To complete the proof, let  $\gamma = \{z \in \tilde{T} : |G(z)| = R\}$  be a vertical geodesic (where  $R$  is sufficiently large; say  $R \geq C + 1$ ). We need to prove that the diameter of  $\gamma$  is bounded independently of  $R$ . So let  $z \in \gamma$ . Then  $|F(z)^\varrho - G(z)| \leq C$ , which implies that

$$\left| |F(z)| - R^{1/\varrho} \right| \leq C \quad \text{and} \quad |\arg F(z)| < \pi/(2 + \varepsilon)$$

(where  $\varrho = 1 + \varepsilon$ ), provided  $R$  was chosen large enough.

The hyperbolic diameter of the subset of  $\mathbb{H}$  described by these inequalities — and hence that of  $F(\gamma)$  — is uniformly bounded. Since  $F : T \rightarrow \mathbb{H}$  is a conformal isomorphism, the standard estimate (1.1) on the hyperbolic metric on  $T$ , together with the fact that  $T$  does not intersect its  $2\pi i\mathbb{Z}$ -translates, implies that the Euclidean diameter of  $\gamma$  is uniformly bounded as well.  $\square$

## 8. Properties of the counterexample

The goal of this section is to indicate how the counterexample  $f$  from Theorem 1.1 (constructed in §6) can be strengthened in various ways. We begin by discussing the growth behavior of the function  $f$ , and how to modify

the construction to reduce this growth further. The section concludes with a sketch of the construction of a hyperbolic entire function whose Julia set contains no nontrivial curves at all.

*Order of growth.* By [Theorem 1.2](#), we know that the counterexample  $f$  from [Theorem 1.1](#) does not have finite order; that is, we cannot have  $\log \log |f(z)| = O(\log |z|)$ . We now see that its growth is not all that much faster than this.

8.1. PROPOSITION (Growth of counterexample). *The function  $f$  constructed in the proof of [Theorem 1.1](#) satisfies*

$$\log \log |f(z)| = O\left((\log |z|)^{12M^2}\right).$$

*Proof.* We verify that the function  $F : T \rightarrow \mathbb{H}$  from [Theorem 6.3](#) satisfies

$$(8.1) \quad \log \operatorname{Re} F(z) = O\left((\operatorname{Re} z)^{12M^2}\right).$$

(The claim then follows immediately from the fact that  $f$  is obtained from  $F$  by applying [Proposition 6.4](#).)

We use the notation of the proof of [Theorem 6.3](#) (recall [Figure 5\(b\)](#)). Pick points  $p_k$  with real parts  $\operatorname{Re}(p_k) = \xi_k$  and satisfying  $F(p_k) \in (\varrho_{k+1}, \infty)$  (that is,  $p_k$  lies half-way between the geodesics  $C_k$  and  $\dot{C}_k$ , in the place where  $T$  “turns around”; it is here that the values of  $\operatorname{Re} F(z)$  are largest in terms of  $\operatorname{Re} z$ ). It is not difficult to see that it is sufficient to verify (8.1) when  $z = p_k$ . (In other parts of the tract,  $\log \operatorname{Re} F(z)$  increases at most linearly with  $\operatorname{Re} z$ .)

We have

$$\log \operatorname{Re} F(p_k) \leq \log \dot{\varrho}_{k+1} < \log \dot{\xi}_{k+1}^{1/M} = 12\dot{\xi}_k.$$

It remains to estimate  $\dot{\xi}_k$  in terms of  $\operatorname{Re}(p_k) = \xi_k$ , which we can do because

$$(8.2) \quad \dot{\xi}_k = \xi_k^{12M^2}$$

by definition. So

$$\log \operatorname{Re} F(p_k) \leq 12\dot{\xi}_k = 12\xi_k^{12M^2} = 12 \operatorname{Re}(p_k)^{12M^2},$$

as required. □

We are now going to discuss how to improve the growth behavior of  $f$ . Recall that  $M > 1$  was arbitrary. We will show how to reduce the constant 12 in the growth estimate to any number greater than 1. Note that the main fact that influenced the growth of  $f$  in the previous proof was (8.2). We can improve the growth behavior of our counterexample by making the part of the tract leading up to  $C_{k-1}$  thinner: this will increase  $R_k$  and hence  $\xi_k$ , while keeping  $\dot{\xi}_k/\xi_k$  essentially the same.

More precisely, consider a tract described by a variation of [Figure 5\(b\)](#), where the upper of the three horizontal tubes connecting real parts  $\xi_k$  and

$\dot{\xi}_k$  has small height  $\delta > 0$ , while the other two tubes remain at unit height. In order for the proof to go through as before,  $\xi_{k+1}$  will be roughly of size  $\exp(\dot{\xi}_k/(M\delta))$ , while  $\dot{\xi}_{k+1}$  should be chosen of size

$$\dot{\xi}_{k+1} \sim \xi_{k+1}^{M^2} \cdot \exp(CM\dot{\xi}_k).$$

In other words, we will have

$$\dot{\xi}_{k+1} \lesssim \xi_{k+1}^{M^2(1+\delta C)}.$$

(With such choices of  $\xi_{k+1}$  and  $\dot{\xi}_{k+1}$ , better estimates on  $\varrho_{k+1}$  and  $\dot{\varrho}_{k+1}$  are required in the proof of [Theorem 6.3](#). These are not difficult to furnish, but we do not give the details here.)

Hence we see, as in [Proposition 8.1](#), that

$$\log \operatorname{Re} F(p_k) \leq C \operatorname{Re}(p_k)^{M^2(1+\delta C)}.$$

By letting  $\delta > 0$  and  $M > 1$  be sufficiently small, we have obtained the following result.

**8.2. PROPOSITION (Counterexamples of mild growth).** *For every  $\varepsilon > 0$ , there is a hyperbolic function  $f \in \mathcal{B}$  such that  $J(f)$  has no unbounded path-connected components, and such that*

$$\log \log |f(z)| = O((\log |z|)^{1+\varepsilon}). \quad \square$$

Finally, we do not need to fix the height  $\delta$ , but rather can let it tend to 0 in a controlled fashion, so that wiggles at large real parts have values of  $\delta$  close to 0.

Also note that, in all our examples,  $\log \operatorname{Re} F(z)$  grows at most linearly with  $\operatorname{Re}(z)$  within the long horizontal tubes *between* two consecutive “wiggles” (i.e., between  $\dot{\xi}_k + 2h' + 4$  and  $\xi_{k+1}$  in [Figure 5\(b\)](#)). We claim that this means that the *lower order* of  $F$ ; i.e., the number

$$\liminf_{r \rightarrow \infty} \sup_{\operatorname{Re} z=r} \frac{\log \operatorname{Re} F(z)}{r}$$

is finite.

Indeed, let  $w_n$  and  $\dot{w}_n$  be points at the beginning and the end of this tube, respectively. That is,  $w_n$  is at real parts slightly larger than  $\dot{\xi}_k$  and  $\dot{w}_n$  is at real parts slightly below  $\xi_{k+1}$ . We then have

$$|F(\dot{w}_n)| \leq |F(w_n)| \cdot \exp(C \cdot \operatorname{Re} \dot{w}_n),$$

where essentially  $C = \pi$ . It follows from the construction that  $|F(w_n)|$  grows at most like  $\dot{\xi}_{k+1}$ , and hence by [\(8.2\)](#) is bounded by  $\xi_{k+1}^A$  for some  $A > 1$ . So overall

$$\log \operatorname{Re} F(\dot{w}_n) \leq A \log \xi_{k+1} + C \operatorname{Re} \dot{w}_n \leq A \log \operatorname{Re} \dot{w}_n + C \operatorname{Re} \dot{w}_n,$$

and the lower order is at most  $C$ .



Since there are no other parts of the tract  $T$  between the real parts of  $w_n$  and  $\dot{w}_n$ , we can actually modify  $T$  so that these tubes have the maximal possible height  $2\pi$ . Then the lower order of the resulting function  $F$  will be  $C = 1/2$ , which is the minimal possible value for a function in  $\mathcal{B}_{\log}$  by the Ahlfors distortion theorem [Ahl73, §4.12].

Altogether, this yields the following.

**8.3. PROPOSITION** (More counterexamples of mild growth). *There exists a function  $F \in \mathcal{B}_{\log}$  such that:*

- (1)  $\log \operatorname{Re} F(z) = (\operatorname{Re} z)^{1+o(1)}$  as  $\operatorname{Re} z \rightarrow \infty$ ,
- (2)  $F$  has lower order  $1/2$ , and
- (3)  $J(F)$  has no unbounded path-connected components.

Note that this function will not satisfy the stronger requirements in [Theorem 6.3](#) for a fixed  $M$  (we need to let  $M$  tend to 1 as  $k \rightarrow \infty$ ). So we will not be able to use [Proposition 6.4](#) to obtain an entire function from  $F$ . (Also, an application of [Proposition 6.4](#) would slightly increase the lower order.) It is, however, possible to modify [Proposition 6.4](#) so as to allow the construction of an entire function of class  $\mathcal{B}$  with these properties; see [[Rem10](#)].

*No nontrivial path components.* To conclude, we would like to note that our construction can also be adapted to yield a topologically stronger form of the counterexample. We content ourselves with giving a sketch of the proof, which involves a nontrivial amount of bookkeeping but is not conceptionally more involved than the previous arguments.

**8.4. THEOREM** (No nontrivial paths in the Julia set). *There exists a (hyperbolic) function  $f \in \mathcal{B}$  such that  $J(f) \cup \{\infty\}$  is a compact connected set that contains no nontrivial curve.*

*Sketch of proof.* Again, the result will be established by designing a function  $F \in \mathcal{B}_{\log}$  with a single tract  $T \subset \mathbb{H}$  whose Julia set contains no nontrivial curve; the existence of an entire function with the same property is easily obtained using [Proposition 6.4](#). (Recall that  $J(f) \cup \{\infty\}$  is always a compact connected set when  $f \in \mathcal{B}$ , so only the second part of the claim needs to be established.)

Let us say that a tract  $T$  has a *wiggle over*  $(r, R)$  if any curve in  $T$  that connects a point at real part  $r/2$  to one at real part at least  $2R$  contains at least three disjoint subcurves connecting the real parts  $r$  and  $R$ .

Our aim is now to construct a tract  $T$ , a conformal map  $F : T \rightarrow \mathbb{H}$ , and an associated set  $\mathcal{W}$  of wiggles  $(r, R)$  such that

- (1)  $T$  has a wiggle over  $(r, R)$  for every  $(r, R) \in \mathcal{W}$ .
- (2) For every sufficiently large  $\eta$ , there is some wiggle  $(r, R) \in \mathcal{W}$  with  $\eta < r < R < 10\eta$ .

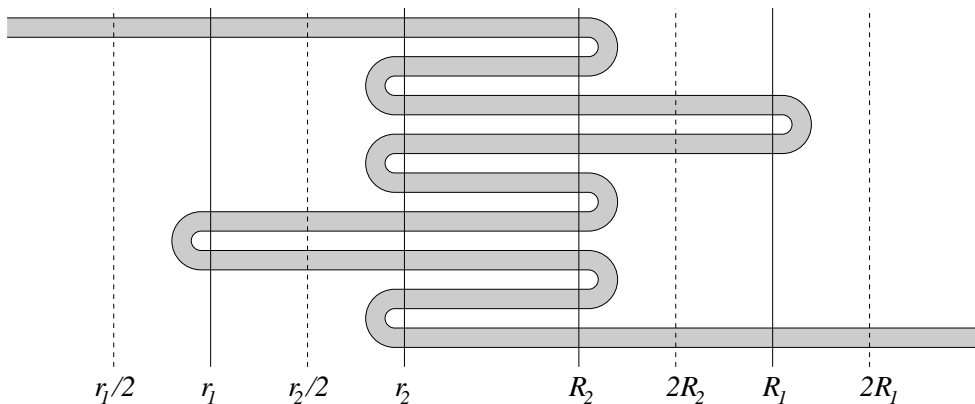


Figure 8. Illustration of the proof of [Theorem 8.4](#). The tract pictured here has a wiggle over  $(r_1, R_1)$  and over  $(r_2, R_2)$ .

- (3) Every wiggle  $(r, R) \in \mathcal{W}$  “propagates”, roughly in the sense that curves connecting real parts  $r$  and  $R$  are going to map to an “image wiggle”  $(r', R') \in \mathcal{W}$ .

More precisely, suppose that  $\gamma : [0, 1] \rightarrow T$  connects real parts  $r/3$  and  $3R$ , and  $\text{Re}(\gamma(t)) \in (r/3, 3R)$  for all  $t \in (0, 1)$ . Let us suppose without loss of generality that  $|F(\gamma(0))| < |F(\gamma(1))|$ . Then there should be  $(r', R') \in \mathcal{W}$  with the following properties:

- $|F(\gamma(0))| < r'/3$ .
- $|F(\gamma(1))| > 4R'$ .
- Suppose that  $t_1, t_2 \in [0, 1]$  are such that  $r' \leq |F(\gamma(t_1))| \leq 2r'$  and  $R' \leq |F(\gamma(t_2))| \leq 2R'$ . Then  $\min(\text{Re } \gamma(t_1), \text{Re } \gamma(t_2)) < r/2$  and  $\max(\text{Re } \gamma(t_1), \text{Re } \gamma(t_2)) > 2R$ .

By linear separation of real parts ([Lemma 3.2](#)), for any two points  $z, w \in J(F)$  with the same external address, there is an iterate  $F^{\circ k}$  so that

$$\text{Re } F^{\circ k}(z) / \text{Re } F^{\circ k}(w) > 90$$

(assuming without loss of generality that  $\text{Re } F^{\circ k}(z) > \text{Re } F^{\circ k}(w)$ ). So, by (2), there is a wiggle  $(r, R) \in \mathcal{W}$  such that  $\text{Re } F^{\circ k}(w) < r/3 < 3R < \text{Re } F^{\circ k}(z)$ . The condition in (3) will then guarantee, by an inductive argument as in [Theorem 6.1](#), that any curve in  $J(F)$  connecting  $F^{\circ k}(z)$  and  $F^{\circ k}(w)$  would need to connect real parts  $r$  and  $R$  at least  $3^n$  times for every  $n$ , which is impossible.

To complete our sketch, we now indicate how to construct such a tract  $T$ , which will be a winding strip contained in  $\{|\text{Im } z| < \pi\}$ , similarly as before. However, the number of times that  $T$  crosses the line  $\{\text{Re } z = R\}$  will tend to infinity as  $R$  does, so the width of  $T$  will necessarily tend to 0 as real

parts increase. Similarly as in [Theorem 6.3](#), the tract will be constructed by inductively defining pieces  $T_1, T_2, \dots$ , in the following fashion:

- (a)  $T_j$  is the piece of  $T$  between real parts  $\eta_{j-1}$  and  $\eta_j$ , where  $\eta_0 < \eta_1 < \eta_2 < \dots$  is a sequence tending to infinity.
- (b) At each step in the construction, there is a set

$$\mathcal{W}_k = \{(r_1^k, R_1^k), \dots, (r_{m_k}^k, R_{m_k}^k)\}$$

of “wiggles”, with  $r_j^k/3 \geq \eta_{k-1}$  and  $3R_j^k \leq \eta_k$ . Also,  $T_k$  is constructed to have a wiggle over each  $(r, R) \in \mathcal{W}_k$  (see [Figure 8](#)).

- (c) The next set of wiggles  $\mathcal{W}_{k+1}$  is determined by the construction of  $T_k$ .

More precisely, we begin by setting  $\eta_0 := 1$ ,  $\mathcal{W}_1 := \{(r_1, r_1 + A)\}$ , where  $A$  is a sufficiently large number (fixed for the whole construction), and  $r_1$  is large enough. We also set  $\eta_1 := 3(r_1 + A)$ .

Given  $\mathcal{W}_k$ , we construct a piece  $T_k$ , connecting real parts  $\eta_{k-1}$  and  $\eta_k$ , by first constructing a “central curve” that has a wiggle over every  $(r, R) \in \mathcal{W}_k$  (this is easy to achieve; compare [Figure 8](#)), and then thickening this curve slightly (see below) to obtain  $T_k$ .

We then construct the set  $\mathcal{W}_{k+1}$  as follows. Suppose that  $(r, R) \in \mathcal{W}_k$ , and that  $\gamma : [0, 1] \rightarrow T_k$  is a minimal piece of the central curve of  $T_k$  that connects real parts  $r/3$  and  $3R$ . (Note that there may be several such pieces; we will add a wiggle to  $\mathcal{W}_{k+1}$  for each of them.) To fix our ideas, let us assume that  $\gamma(0)$  and  $\gamma(1)$  occur in this order on the central curve, so that  $|F(\gamma(0))| < |F(\gamma(1))|$ ; the other case is analogous.

Let  $z$  be the first point on  $\gamma$  that has real part  $r/2$ , and let  $Z$  be the last point on  $\gamma$  that has real part  $2R$ . Using the semi-hyperbolic metric, i.e. the reciprocal of the distance to  $\partial T_k$ , we can estimate  $|F(z)|$  and  $|F(Z)|$  (up to an exponent of 2), independently of the construction of  $T_K$  for  $K > k$ . Hence we can add a new wiggle  $(r_j^{k+1}, R_j^{k+1})$  to  $\mathcal{W}_{k+1}$  such that  $|F(z)| \geq 2r_j^{k+1}$  and  $|F(Z)| \leq 2R_j^{k+1}$ .

If the width of  $T_j$  along  $\gamma$  was chosen sufficiently thin, we can easily ensure that  $r_j^{k+1}/3 > |F(\gamma(0))| > \eta_k$ , and that  $4R_j^{k+1} < |F(\gamma(1))|$ .

Having added these finitely many wiggles to  $\mathcal{W}_{k+1}$ , we set

$$\eta_{k+1} := \max_{(r,R) \in \mathcal{W}_{k+1}} 3R.$$

Finally, we add sufficiently many wiggles of the form  $(t, t+A)$  to  $\mathcal{W}_{k+1}$  to ensure that, for every  $\eta \in [3\eta_k, \eta_{k+1}/3 - A]$ , there is some wiggle between real parts  $\eta$  and  $3\eta$ . This completes the description of the inductive construction.  $\square$

## Appendix A. Some geometric and topological facts

In this section, we collect some of the simple geometrical and topological results that we required in the course of the article. The first is a version of the *Ahlfors spiral theorem* [Hay89, Th. 8.21] (which states that any entire function of finite order has controlled spiralling). We give a simple proof of this fact for functions in class  $\mathcal{B}_{\log}$  below. In [Section 5](#), we also required a characterization of domains with bounded wiggling, which we prove here for completeness. [Lemma A.3](#) below was used in [Theorem 6.3](#).

Finally, the Boundary Bumping [Theorem A.4](#) was used a number of times in topological considerations, and [Theorem A.5](#) was instrumental in the proof of [Theorem 1.2](#).

**A.1. THEOREM (Spiral Theorem).** *Suppose that  $F \in \mathcal{B}_{\log}$  has finite order. Then the tracts of  $F$  have bounded slope.*

*Proof.* Let  $T$  be a tract of  $F$ , set

$$\varrho := \sup \left\{ \frac{\log \operatorname{Re} F(z)}{\operatorname{Re} z} : z \in \mathcal{T}, \operatorname{Re} z, \operatorname{Re} F(z) \geq 1 \right\} < \infty,$$

and consider the central geodesic  $\gamma : [1, \infty) \rightarrow T; t \mapsto F_T^{-1}(t)$ . Then for every sufficiently large  $t \geq 1$ ,

$$|\gamma(t)| - |\gamma(1)| \leq |\gamma(t) - \gamma(1)| \leq 2\pi \ell_T(\gamma([1, t])) = 2\pi \log t \leq 2\pi \varrho \operatorname{Re} \gamma(t).$$

Thus we have proved the existence of an asymptotic curve  $\gamma$  satisfying  $|\operatorname{Im} \gamma(t)| \leq |\gamma(t)| \leq K \operatorname{Re} \gamma(t) + M$ , for  $K = 2\pi \varrho$  and  $M = |\gamma(1)|$ , which is equivalent to the bounded slope condition.  $\square$

**A.2. LEMMA (Domains with bounded wiggling).** *Let  $V$  be an unbounded Jordan domain such that  $\exp|_V$  is injective. Suppose that there are  $K, M > 0$  such that every  $z_0 \in V$  can be connected to  $\infty$  by a curve  $\gamma \subset V$  satisfying*

$$\operatorname{Re} z \geq \frac{\operatorname{Re} z_0}{K} - M$$

*for all  $z \in \gamma$ . Then there is  $M' > 0$  that depends only on  $M$  such that, for every  $z_0 \in V$ ,*

$$\operatorname{Re} z \geq \frac{\operatorname{Re} z_0}{K} - M'$$

*for all  $z$  on the geodesic connecting  $z_0$  to  $\infty$ .*

*Proof.* Let  $z_0 \in V$ , let  $\gamma$  be a curve as in the statement of the theorem, and let  $F : V \rightarrow \mathbb{H}$  be a conformal isomorphism with  $F(\infty) = \infty$  and  $F(z_0) = 1$ . Then

$$\gamma' := F^{-1}([1, \infty))$$

is the horizontal geodesic connecting  $z_0$  to  $\infty$ .

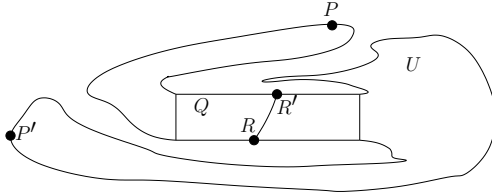


Figure 9. Illustration of Lemma A.3.

Let  $z \in \gamma'$ . By [Pom92, Cor. 4.18], we can find geodesics  $\alpha^+$  and  $\alpha^-$  of  $\mathbb{H}$ , connecting  $F(z)$  to the positive, resp. negative, imaginary axis, such that the geodesics  $F^{-1}(\alpha^\pm)$  of  $V$  have diameter at most  $C \operatorname{dist}(z, \partial V)$ . (Here  $C$  is a universal constant.) Hence the crosscut  $\alpha := F^{-1}(\alpha^+) \cup F^{-1}(\alpha^-)$ , which separates  $z_0$  from  $\infty$  in  $V$ , has diameter at most  $2C \operatorname{dist}(z, \partial V) \leq 4C\pi$ .

The curve  $\gamma$  must intersect  $\alpha$  in some point  $w$ . We thus have

$$\operatorname{Re} z \geq \operatorname{Re} w - 4C\pi \geq \operatorname{Re} z_0/K - M - 4C\pi. \quad \square$$

A.3. LEMMA (Geometry of geodesics). *Consider the rectangle*

$$Q = \{z \in \mathbb{C} : |\operatorname{Re} z| < 4, |\operatorname{Im} z| < 1\}$$

and let  $U \subset \widehat{\mathbb{C}}$  be a simply connected Jordan domain with  $U \supsetneq Q$  such that  $\partial Q \cap \partial U$  consists exactly of the two horizontal boundary sides of  $Q$ . Let  $P, R, P', R' \in \partial U$  be four distinct boundary points in this cyclic order, subject to the condition that  $P$  and  $P'$  are in the boundary of different components of  $U \setminus Q$ , and so that the quadrilateral  $U$  with the marked points  $P, R, P', R'$  has modulus 1.

Let  $\gamma$  be the hyperbolic geodesic in  $U$  connecting  $R$  with  $R'$ . If  $0 \in \gamma$ , then the two endpoints of  $\gamma$  are on the horizontal boundaries of  $Q$ , one endpoint each on the upper and lower boundary.

*Remark.* This is essentially a simple version of the well-known Ahlfors distortion theorem (see e.g. [Ahl73, §4.12] or [Pom92, §11.5]). However, in the way it is usually stated, this theorem cannot be applied directly to obtain our lemma. Hence we provide the proof for completeness, following [Ahl73, §4.12].

*Proof.* We need to show that  $\gamma$  does not cross the left side  $L$  or the right side  $R$  of the rectangle  $Q$ . We show this for the left side; the statement for the right side follows by symmetry. Let  $M$  be the vertical crosscut of  $Q$  that passes through 0, and let  $Q'$  be the square bounded by  $L, M$  and the horizontal boundaries of  $Q$ . (That is,  $Q'$  is the “left half” of  $Q$ .)

Let  $\varphi$  be the conformal isomorphism that takes  $U$  to the bi-infinite strip  $\{0 < \operatorname{Im} z < \pi\}$  in such a way that  $P$  and  $P'$  are mapped to  $-\infty$  and  $+\infty$  and

$R$  and  $R'$  are mapped to  $\pi i$  and  $0$ . Set

$$l := \sup_{z \in L} \operatorname{Re} \varphi(z) \quad \text{and} \quad m := \inf_{z \in M} \operatorname{Re} \varphi(z) \leq 0.$$

We need to show that  $l \leq 0$ .

Let  $\widetilde{Q}'$  be the quadrilateral obtained by joining  $\varphi(Q')$  with its reflection in the real axis; then  $\widetilde{Q}'$  has modulus equal to  $1/2$ . The exponential map takes  $\widetilde{Q}'$  together with its upper and lower boundaries to an annulus surrounding  $0$  that also has modulus  $1/2$ .

The following is a consequence of the modulus theorem of Teichmüller (see e.g. [Ahl73, Th. 4.7 and the subsequent remark]): *If the annulus  $A$  separates the points  $0$  and  $z$  from the points  $\infty$  and  $w$  where  $|w| < |z|$ , then  $\operatorname{mod}(A) < 1/2$ .*

This implies that  $l \leq m$ , as required.  $\square$

We conclude by stating two results of continuum theory that are used in this article.

**A.4. THEOREM (Boundary Bumping Theorem [Nad92, Th. 5.6]).** *Let  $X$  be a nonempty, compact, connected metric space, and let  $E \subsetneq X$  be nonempty. If  $C$  is a connected component of  $E$ , then  $\partial C \cap \partial E \neq \emptyset$  (where boundaries are taken relative to  $X$ ).*

*Remark.* We apply this theorem only in the case where  $X \subset \widehat{\mathbb{C}}$  is a compact connected set containing  $\infty$ , and  $E = X \cap \mathbb{C}$ . In this case, the theorem states that every component of  $E$  is unbounded.

**A.5. THEOREM (Order characterization of arcs [Nad92, Ths. 6.16, 6.17]).** *Let  $X$  be a nonempty, compact, connected metric space. Suppose that there is a total ordering  $\prec$  on  $X$  such that the order topology of  $(X, \prec)$  agrees with the metric topology of  $X$ . Then either  $X$  consists of a single point or there is an order-preserving homeomorphism from  $X$  to the unit interval  $[0, 1]$ .*

*Remark.* This result follows from the perhaps better-known *non-cut-point characterization* of the arc: a compact, connected metric space is homeomorphic to an arc if and only if it has exactly two non-cut-points. Conversely, this characterization also follows from [Theorem A.5](#); see [Nad92, Th. 6.16] for details.

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