ANNALS OF MATHEMATICS

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SECOND SERIES, VOL. 172, NO. 3

November, 2010

ANMAAH

Annals of Mathematics, **172** (2010), 1719–1748

Boundedness of families of canonically polarized manifolds: A higher dimensional analogue of Shafarevich's conjecture

By SÁNDOR J KOVÁCS and MAX LIEBLICH

To the memory of Eckart Viehweg

Abstract

We show that the number of deformation types of canonically polarized manifolds over an arbitrary variety with proper singular locus is finite, and that this number is uniformly bounded in any finite type family of base varieties. As a corollary we show that a direct generalization of the geometric version of Shafarevich's original conjecture holds for infinitesimally rigid families of canonically polarized varieties.

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1. Introduction

Fix an algebraically closed field k of characteristic 0. Let B be a smooth projective curve of genus g over k and $\Delta \subset B$ a finite subset. A flat morphism with connected fibers will be called a *family*. For two families over the same base B

Sándor Kovács was supported in part by NSF Grants DMS-0092165, DMS-0554697, and DMS-0856185, a Sloan Research Fellowship, and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics.

Max Lieblich was supported in part by an NSF postdoctoral fellowship and by NSF Grant DMS-0758391.

a *morphism of families* is simply a morphism of B-schemes. A family $f : X \to B$ is called *isotrivial* if $X_a \simeq X_b$ for general points $a, b \in B$, and $f : X \to B$ is *admissible* (with respect to (B, Δ)) if it is not isotrivial and the map $f : X \setminus f^{-1}(\Delta) \to B \setminus \Delta$ is smooth.

At the 1962 International Congress of Mathematicians in Stockholm, Shafarevich conjectured the following

1.1. SHAFAREVICH'S CONJECTURE. *Let (B, Δ) be fixed and $q \geq 2$ an integer. Then*

(1.1.1) *there exist only finitely many isomorphism classes of admissible families of curves of genus q;*

(1.1.2) *if $2g - 2 + \#\Delta \leq 0$, then there exist no such families.*

Shafarevich showed a special case of (1.1.2): There exist no smooth families of curves of genus $q \geq 2$ over $\mathbb{P}^1$. Conjecture 1.1 was proven by Parshin [Par68] for $\Delta = \varnothing$ and by Arakelov [Ara71] in general.

This conjecture has a natural analogue for curves over number fields. For a brief discussion see [Kov03, §2] and for more details [CS86] and [Lan91]. Shafarevich's conjecture implies Mordell's conjecture in both the function field and the number field case by an argument known as Parshin's covering trick. Because of this, the proof of Shafarevich's conjecture in the number field case constitutes the lion's share [AesBC] of Faltings' celebrated proof of Mordell's conjecture [Fal83], [Fal84].

With regard to Shafarevich's conjecture, Parshin made the following observation. In order to prove that there are only finitely many admissible families, one may proceed as follows. Instead of aiming for the general statement, first prove that there are only finitely many deformation types.[1] The next step then is to prove that admissible families are rigid, that is, they do not admit nontrivial deformations over a fixed base. Now since every deformation type contains only one family, and since there are only finitely many deformation types, the original statement follows.

Based on this idea, the following reformulation of Shafarevich's conjecture was used by Parshin and Arakelov to confirm the conjecture:

1.2. SHAFAREVICH'S CONJECTURE (VERSION TWO). *Let (B, Δ) be fixed and $q \geq 2$ an integer. Then the following statements hold.*

(B) (Boundedness) *There exist only finitely many deformation types of admissible families of curves of genus q with respect to $B \setminus \Delta$.*

(R) (Rigidity) *There exist no nontrivial deformations of admissible families of curves of genus q with respect to $B \setminus \Delta$.*

(H) (Hyperbolicity) *If $2g - 2 + \#\Delta \leq 0$, then no admissible families of curves of genus q exist with respect to $B \setminus \Delta$.*

[1] See Definition 1.4.

Remark 1.3. As we discussed above, (**B**) and (**R**) together imply (1.1.1) and (**H**) is clearly equivalent to (1.1.2).

It is a natural and important question whether similar statements hold for families of higher dimensional varieties. It is easy to see that (**R**) fails [Vie01], [Kov03, 10.4] and hence Conjecture 1.1 fails in higher dimensions. This gives additional importance to the Parshin-Arakelov reformulation as it separates the clearly false part from the rest. In fact, the past decade has seen a flood of results concerning both (**B**) and (**H**). For a detailed historical overview and references to related results we refer the reader to the survey articles [Vie01], [Kov03], [MVZ06], [Kov09a], and [Kov09b].

In this article we are interested in (**B**). If there existed an algebraic stack $\mathfrak{D}$ parametrizing families of canonically polarized varieties over the base $B \setminus \Delta$, and if furthermore $\mathfrak{D}$ is of finite type, then boundedness, (**B**), would follow. Before further discussing the potential existence and properties of $\mathfrak{D}$, it behooves us to mention the following notion closely related to (**B**):

(WB) (WEAK BOUNDEDNESS) *We say that* weak boundedness *holds if for an admissible family of projective varieties, $f: X \to B$, the degree of $f_*\omega^m_{X/B}$ is bounded above in terms of $g(B), \#\Delta$, m, and $h_{X_{\text{gen}}}$, where X_{gen} denotes the general fiber of f and $h_{X_{\text{gen}}}$ the Hilbert polynomial of $\omega^m_{X_{\text{gen}}}$. In particular, the bound is independent of f.*

This was proven by Bedulev and Viehweg in 2000 [BV00]. From this they derived the consequence that as soon as a reasonable moduli theory exists for canonically polarized varieties and if a $\mathfrak{D}$ as above exists, then it is indeed of finite type. Unfortunately, such $\mathfrak{D}$ almost never exist (especially over open bases); moreover, when the base variety has dimension higher than 1, the question of how to rectify this situation (by adding elements to the family over the discriminant locus Δ) is quite subtle. The bulk of this paper is devoted to pointing out that a proxy for $\mathfrak{D}$ can be constructed by standard stack-theoretic methods, thus allowing us to show that (**WB**) implies (**B**) while skirting the difficult issues surrounding compactifications of the stack of canonically polarized manifolds.

Before stating our main result we need the following definition.

Definition 1.4. Let U be a variety over a field k and $\mathfrak{C}$ a class of schemes. A morphism $X \to U$ is a $\mathfrak{C}$-morphism if for all geometric points $u \to U$, X_u belongs to $\mathfrak{C}$. Two proper, flat $\mathfrak{C}$-morphisms $X_1 \to U$, $X_2 \to U$ are *deformation equivalent* if there is a connected scheme T with two points $t_1, t_2 \in T(k)$ and a proper, flat $\mathfrak{C}$-morphism $\mathscr{X} \to U \times T$ such that $\mathscr{X}|_{U \times t_i} \simeq_U X_i$. An equivalence class (with respect to deformation equivalence) of proper, flat $\mathfrak{C}$-morphisms $X \to U$ will be called a *deformation type*.

Remark 1.5. In the sequel, the class $\mathfrak{C}$ will be chosen to be the class of canonically polarized varieties over a field k.

The following theorem proves (**B**) in arbitrary dimension.

THEOREM 1.6. *Let U be a variety over k that is smooth at infinity* (*see Definition* 2.1). *The set* $\mathrm{Defo}_h(U)$ *of deformation types of families $\mathscr{X} \to U$ of canonically polarized manifolds with Hilbert polynomial h is finite. Furthermore, if T is a quasicompact quasiseparated $\mathbb{Q}$-scheme and $\mathcal{U} \to T$ is smooth at infinity, then there is an integer N such that for every geometric point $t \to T$, we have $|\mathrm{Defo}_h(\mathcal{U}_t)| \leq N$.*

This solves one of the open problems on the list compiled at the American Institute of Mathematics workshop "Compact moduli spaces and birational geometry" in December, 2004 [VO, Problem 2.4]. See also [Vie10].

In fact, we prove a more general result. For the relevant terminology see Section 4.A.

THEOREM 1.7. *Let $\mathcal{M}^\circ$ be a weakly bounded*[2] *compactifiable Deligne-Mumford stack over a quasicompact quasiseparated $\mathbb{Q}$-scheme T. Given a morphism $U \to T$ that is smooth at infinity, there exists an integer N such that for every geometric point $t \to T$, the number of deformation types of morphisms $U_t \to \mathcal{M}^\circ_t$ is finite and bounded above by N.*

1.8. *Definitions and notation.* For morphisms $f : X \to B$ and $\vartheta : T \to B$, the symbol X_T will denote $X \times_B T$ and $f_T : X_T \to T$ the induced morphism. Of course, by symmetry we also have the notation $\vartheta_X : T_X \simeq X_T \to X$. In particular, for $b \in B$ we write $X_b = f^{-1}(b)$. In addition, if $T = \operatorname{Spec} F$, then X_T will also be denoted by X_F. Finally, if $\mathscr{F}$ is an $\mathcal{O}_X$-module, then $\mathscr{F}_T$ will denote the $\mathcal{O}_{X_T}$-module $\vartheta_X^* \mathscr{F}$.

Given a proper scheme X over a field k, we write Pic^τ_X for the locus of numerically trivial invertible sheaves in Pic_X. This is generally larger than Pic^0_X, the connected component containing the trivial sheaf. Given a field extension L/k and an invertible sheaf $\mathcal{N}$ on X_L, we will write $[\mathcal{N}]$ for the element of $\mathrm{Pic}_X(L)$ associated to $\mathcal{N}$.

For the theory of stacks, we will use the definitions and conventions of [LMB00]. In particular, all algebraic (Deligne-Mumford or Artin) stacks are assumed to be quasiseparated. Most of the time, the stacks we use will in fact be separated; this is always indicated in the text as a hypothesis when it is used.

A quasicompact separated Deligne-Mumford stack $\mathcal{M}$ is *polarized* if there exists an invertible sheaf $\mathscr{L}$ on $\mathcal{M}$ such that the nonvanishing loci of all sections of all tensor powers of $\mathscr{L}$ generate the topology on the underlying topological space of $\mathcal{M}$. When $\mathcal{M}$ has a coarse moduli space M, this is equivalent to requiring that some tensor power of $\mathscr{L}$ is the pullback from M of an ample invertible sheaf L. A Deligne-Mumford stack is *tame* if the order of the stabilizer group of any geometric point x is invertible in $\kappa(x)$. We will only explicitly encounter tame stacks in the generalities of Section 3; otherwise, we will be working in characteristic 0, where tameness is automatic and will go unmentioned.

[2] See Definition 4.5.

2. **Coarse boundedness**

2.A. *Bounding maps to a projective scheme.* As in the introduction, k will be an algebraically closed field of characteristic 0. In what follows, *variety* will mean a k*-variety*.

Definition 2.1. For a morphism $\mathcal{U} \to T$ of algebraic spaces let $\operatorname{Sing}(\mathcal{U}/T)$ denote the smallest closed subset of $\mathcal{U}$ such that the induced morphism $\mathcal{U} \setminus \operatorname{Sing}(\mathcal{U}/T) \to T$ is smooth. The morphism $\mathcal{U} \to T$ is called *smooth at infinity* if it is of finite presentation, $\operatorname{Sing}(\mathcal{U}/T)$ is proper over T, and $\mathcal{U} \setminus \operatorname{Sing}(\mathcal{U}/T)$ is schematically dense in every geometric fiber. A variety will be called smooth at infinity if its structure morphism is smooth at infinity.

2.2. Let M be a proper k-scheme with a fixed invertible sheaf $\mathcal{N}$ and let U be an algebraic variety that is smooth at infinity. By Nagata's theorem, U embeds into a proper variety B. Blowing up and using the assumption that U is smooth at infinity, and hence $\operatorname{Sing} U$ is proper, we may assume that $B \setminus U$ is a divisor Δ (with simple normal crossings, if desired) and that B is smooth in a neighborhood of Δ. Because M is proper, it follows that given a morphism $\xi : U \to M$, there is an open subset $\iota : U' \hookrightarrow B$ containing U and every codimension 1 point of B and an extension of ξ to a morphism $\xi' : U' \to M$. Taking the reflexive hull of $\iota_* \mathcal{N}_{U'}$ yields an invertible sheaf $\mathcal{N}_\xi$ on B by [OSS80, II.1.1.15, p.154].

On the other hand, suppose C° is a smooth curve over k with smooth compactification C. Given a morphism $\nu : C \to B$ that maps C° into U and a morphism $\xi : U \to M$ as above, one obtains an extension $\xi_C : C \to M$ of $\xi_{C^\circ} \circ \nu$. It is of course not necessary for $\deg(\xi_C^* \mathcal{N})$ to equal $\deg \mathcal{N}_\xi|_C$, but this will clearly occur when C is contained in U' (in the above notation).

Definition 2.3. A (g, d)*-curve* is a smooth curve C° whose smooth compactification C has genus g and such that $C \setminus C^\circ$ consists of d closed points.

Definition 2.4. Given U and M as above, a morphism $\xi : U \to M$ is *weakly bounded with respect to* $\mathcal{N}$ if there exists a function $\mathsf{b}_\mathcal{N} : \mathbb{Z}^2_{\geq 0} \to \mathbb{Z}$ such that for every pair (g, d) of nonnegative integers, for every (g, d)-curve $C^\circ \subseteq C$, and for every morphism $C^\circ \to U$, one has that $\deg \xi_C^* \mathcal{N} \leq \mathsf{b}_\mathcal{N}(g, d)$. The function $\mathsf{b}_\mathcal{N}$ will be called a *weak bound* (with respect to $\mathcal{N}$), and we will say that ξ *is weakly bounded by* $\mathsf{b}_\mathcal{N}$.

Notation 2.5. Given a field extension L/k, the set of morphisms $U_L \to M_L$ which are weakly bounded by $\mathsf{b}_\mathcal{N}$ will be denoted $\mathcal{W}(U, M, \mathsf{b}_\mathcal{N})(L)$. Notice that as $\mathsf{b}_\mathcal{N}$ depends on $\mathcal{N}$, so does $\mathcal{W}(U, M, \mathsf{b}_\mathcal{N})(L)$.

PROPOSITION 2.6. *Let* b *be a weak bound. Then there exists a variety* W^b *and a morphism* $\Xi : \mathsf{W}^\mathsf{b} \times U \to M$ *such that for every field extension* L/k *and for every morphism* $\xi : U_L \to M_L$ *that is weakly bounded by* b *there exists an* L*-valued point* $p : \operatorname{Spec} L \to \mathsf{W}^\mathsf{b}$ *such that* $\xi = \Xi|_{\{p\} \times U}$.

Remark 2.7. Notice that this does not necessarily mean that every point of W^{b} corresponds to a weakly bounded morphism $U \to M$. This phenomenon is common in the theory of moduli; one often produces a bounded family containing the points of interest, but possibly also containing numerous other points. In fact, this is one of the main difficulties in the present situation. It is much easier to find a bounding family than one that actually parametrizes the class in which we are interested.

The proof consists of several steps. First, we compactify $U \subseteq B$ as the complement of a divisor Δ in a proper variety as in 2.2. Then we bound the set of invertible sheaves $\mathcal{N}_\xi$. The choice of $n+1$ sections of such an $\mathcal{N}_\xi$ that simultaneously vanish only in Δ can then be parametrized by a finite type space T.

Assumption 2.8. We will assume that M is projective, fix an embedding $M \hookrightarrow \mathbb{P}^n$, and let $\mathcal{N} = \mathcal{O}_M(1) = \mathcal{O}_{\mathbb{P}^n}(1)|_M$. For simplicity we replace the phrase "weakly bounded with respect to $\mathcal{N}$" by "weakly bounded".

Let us first treat the case $M = \mathbb{P}^n$.

LEMMA 2.9. *Given a compactification $U \subseteq B$ as above, there exists a reduced subscheme of finite type $\mathsf{W}(U, \mathbb{P}^n, \mathsf{b}) \subset \mathrm{Pic}_B$ such that for all field extensions L/k and for all $\xi \in \mathcal{W}(U, \mathbb{P}^n, \mathsf{b})(L)$, we have $[\mathcal{N}_\xi] \in \mathsf{W}(U, \mathbb{P}^n, \mathsf{b})(L)$.*

Proof. We first claim that it suffices to prove the result when B is smooth and projective. Indeed, choose a projective resolution of singularities (or a projective alteration [dJ96]) $\pi : \widetilde{B} \to B$ with $\widetilde{B}$ smooth, and let $\widetilde{U}$ be the preimage of U. Now we can consider weakly bounded morphisms $\widetilde{U} \to M$ with the same weak bound b. Among these will be the compositions of π with morphisms $U \to M$ weakly bounded by b. In other words, composition with π induces a natural map $\pi^* : \mathcal{W}(U, \mathbb{P}^n, \mathsf{b}) \to \mathcal{W}(\widetilde{U}, \mathbb{P}^n, \mathsf{b})$. Observe that the pullback morphism $\pi^* : \mathrm{Pic}_B \to \mathrm{Pic}_{\widetilde{B}}$ is of finite type by [SGA6, XII.1.1], and hence if the required $\mathsf{W}(\widetilde{U}, \mathbb{P}^n, \mathsf{b}) \subset \mathrm{Pic}_{\widetilde{B}}$ exists, then $\mathsf{W}(U, \mathbb{P}^n, \mathsf{b}) := (\pi^*)^{-1}\mathsf{W}(\widetilde{U}, \mathbb{P}^n, \mathsf{b}) \subset \mathrm{Pic}_B$ satisfies the desired conditions. Therefore we may assume from now on that B is smooth and projective.

Suppose $\dim B \geq 3$ and let $Y \subset B$ be a general ample divisor. By [SGA6, XIII.3.8], the restriction morphism $\mathrm{Pic}_B \to \mathrm{Pic}_Y$ of Picard schemes is of finite type. Since the restriction of a morphism $U \to \mathbb{P}^n$ weakly bounded by b to $U \cap Y$ is also weakly bounded by b and we have $\mathcal{N}_\xi|_Y \simeq \mathcal{N}_{\xi_Y}$, we see that it suffices to prove the statement for $U \cap B \subset B$. Thus, we may assume $\dim B \leq 2$.

If $\dim B = 1$, then the inclusion $U \subset B$ is a (g, d)-curve with g the genus of B and d the number of points in $B \setminus U$. In addition, any morphism $\xi : U \to \mathbb{P}^n$ extends to a morphism $B \to \mathbb{P}^n$. By the weak boundedness assumption, $0 \leq \deg \xi_C^* \mathcal{N} \leq \mathsf{b}(g, d)$, so that $\xi_C^* \mathcal{N}$ is contained in the preimage $\mathrm{Pic}_C^{[0, \mathsf{b}(g,d)]}$ of the interval $[0, \mathsf{b}(g, d)] \subset \mathbb{Z}$ under the degree map $\mathrm{Pic}_C \to \mathbb{Z}$. Since the fiber of $\mathrm{Pic}_C \to \mathbb{Z}$

over any finite subset is of finite type, we see that setting $\mathsf{W}(U, \mathbb{P}^n, \mathsf{b}) = \mathrm{Pic}_C^{[0,\mathsf{b}(g,d)]}$ yields the result.

Hence we may assume for the rest of the proof that B is a surface. Let A be a very ample divisor on B. We will prove that for each ξ we have $0 \leq \deg_A \mathcal{N}_\xi \leq N$ and $c_1(\mathcal{N}_\xi)^2 \geq 0$. These conditions define an open subscheme $\mathsf{W}(U, \mathbb{P}^n, \mathsf{b})$ of Pic_B. Moreover, by [SGA6, Th. XIII.3.13(iii)], there exists a quasicompact scheme T and a family of invertible sheaves $\mathcal{L}$ on $B \times T$ such that every sheaf $\mathcal{N}_\xi$ appears as a fiber over a point of t. We conclude that $\mathsf{W}(U, \mathbb{P}^n, \mathsf{b})$ is quasicompact and therefore of finite type, as desired.

So it remains to verify that the above numerical conditions are satisfied. We may assume that the very ample divisor A is smooth. The definition of weak boundedness then yields a bound $\deg \xi_A^* \mathcal{O}_M(1) \leq N$ which depends only on the genus of A and on $A \cdot \Delta$. Moreover, since $\mathrm{codim}(B \setminus U', B) \geq 2$, we can choose an A such that $A \subset U'$. In this case $\mathcal{N}_\xi|_A \simeq \xi_A^* \mathcal{O}_M(1)$ and hence we conclude that $0 \leq \deg_A \mathcal{N}_\xi \leq N$.

Next consider H_1, H_2, the zero loci of two general sections of $\mathcal{O}_M(m)$ for some $m \gg 0$. Assume that ξ is nonconstant and let $\widehat{H}_i = \overline{\xi'^* H_i} \subset B$ be the closure of the pullback of H_i to U' via ξ' for $i = 1, 2$. Clearly, $\widehat{H}_1 \cdot \widehat{H}_2 \geq 0$. Notice that by definition $\mathcal{N}_\xi \simeq \mathcal{O}_B(\widehat{H}_i)$ for $i = 1, 2$ and hence $c_1(\mathcal{N}_\xi)^2 \geq 0$, as desired. □

LEMMA 2.10. *Given a finite type reduced subscheme $Y \subset \mathrm{Pic}_B$, there is a finite stratification Y_i of Y by locally closed subschemes such that the functor of tuples $(\mathcal{L}, \sigma^0, \ldots, \sigma^n)$ with $[\mathcal{L}] \in \coprod Y_i$ and $\sigma^0, \ldots, \sigma^n$ global sections of $\mathcal{L}$ at least one of which is nonzero is represented by a reduced scheme W separated and of finite type over $\coprod Y_i$.*

Proof. Let $\mathcal{P}ic_B$ be the Artin stack of invertible sheaves on B; this is a $\mathbb{G}_m$-gerbe over the Picard scheme Pic_B. Write $\mathcal{P} \to Y$ for the fiber product $\mathcal{P}ic_B \times_{\mathrm{Pic}_B} Y$. Write $\mathcal{L}_{\mathrm{univ}}$ for the universal invertible sheaf on $\mathcal{P} \times B$. The function $\mathcal{L} \mapsto \mathrm{h}^0(\mathcal{L})$ is upper semicontinuous and thus defines a stratification of $\mathcal{P}$ by reduced locally closed substacks $\mathcal{P}_i \subset \mathcal{P}$. Since $\mathcal{P} \to Y$ is a $\mathbb{G}_m$-gerbe, each $\mathcal{P}_i$ is a $\mathbb{G}_m$-gerbe over a locally closed subscheme $Y_i \subset Y$, and the Y_i form a stratification of Y.

Write $p_i : \mathcal{P}_i \times B \to \mathcal{P}_i$ for the first projection. By cohomology and base change, the sheaf $(p_i)_* \mathcal{L}_{\mathrm{univ}}|_{\mathcal{P}_i \times B}$ is a locally free $\mathcal{P}_i$-twisted sheaf (see [Lie08, §3.1.1] for the definition and basic properties of twisted sheaves). A choice of sections $\sigma^0, \ldots, \sigma^n$ such that at least one σ^i is not the zero section is a point of the stack

$$W := \mathbf{V}((((p_i)_* \mathcal{L}_{\mathrm{univ}}|_{\mathcal{P}_i \times B})^\vee)^{\oplus(n+1)}) \setminus \mathbf{0},$$

where $\mathbf{0}$ denotes the zero section of the vector bundle.

Since the inertia stack of $\mathcal{P}$ acts on $\mathcal{L}_{\mathrm{univ}}$ by scalar multiplication, the induced action on W is faithful, from which it follows that W is an algebraic space. To prove that it is a separated scheme of finite type (and to give a more concrete

description of the space), we can work étale locally on the sheafification P_i of $\mathcal{P}_i$ and thus assume that (1) $\mathcal{P}_i$ is isomorphic to $P_i \times \mathrm{B}\mathbb{G}_m$ and (2) the pullback of $(p_i)_*\mathcal{L}_{\mathrm{univ}}$ via the canonical map $P_i \to P_i \times \mathrm{B}\mathbb{G}_m$ is trivial, say of rank r. The natural left action of $\mathbb{G}_m$ on the fibers of the locally free sheaf $(((p_i)_*\mathcal{L}_{\mathrm{univ}}|_{\mathcal{P}_i \times B})^\vee)^{\oplus n}$ is via scalar multiplication. Thus, W is isomorphic to the stack-theoretic quotient of the scalar multiplication action of $\mathbb{G}_m$ on $\mathbb{A}^{nr}_{P_i} \setminus \mathbf{0}$. This is just $\mathbb{P}^{nr-1}_{P_i}$, which is certainly separated and of finite type. (Continuing along these lines shows that W is in fact isomorphic to a Brauer-Severi scheme over P_i with the same Brauer class as $[\mathcal{P}_i]$. While it may seem baffling that a projective P_i-scheme can be an open substack of a geometric vector bundle over $\mathcal{P}_i$, it arises from the fact that $\mathcal{P}_i$ — and therefore any vector bundle over $\mathcal{P}_i$ — is highly nonseparated.) □

LEMMA 2.11. *Let S be a reduced Noetherian algebraic space and $\mathcal{X} \to S$ and $\mathcal{Y} \to S$ two Artin stacks of finite presentation. Let $\mathcal{Z} \subset \mathcal{X}$ and $\mathcal{T} \subset \mathcal{Y}$ be locally closed substacks. Given an S-morphism $\varphi : \mathcal{X} \to \mathcal{Y}$, there is a monomorphism of finite type $S' \to S$ whose image contains a geometric point $s \to S$ if and only if φ_s maps $(\mathcal{Z}_s)_{\mathrm{red}}$ into $(\mathcal{T}_s)_{\mathrm{red}}$.*

Proof. Pulling back $\mathcal{T}$ to $\mathcal{X}$ and replacing the inclusion $\mathcal{T} \subset \mathcal{Y}$ by $\mathcal{T}_{\mathcal{X}} \subset \mathcal{X}$, we may assume that $\mathcal{X} \to \mathcal{Y}$ is the identity morphism. Then the set $\mathcal{Z} \setminus \mathcal{T}$ is constructible in $\mathcal{X}$, so the reduced structure on the complement of its image in S is constructible. Any constructible set admits a natural locally finite stratification by reduced algebraic spaces, yielding the desired morphism $S' \to S$. □

LEMMA 2.12. *Let W be the scheme constructed in Lemma* 2.10*. Then there is a finite type morphism $W_\Delta \to W$ such that for any $w \in W$ the reduced common zero locus of $\sigma^0_w, \cdots, \sigma^n_w$ is contained in Δ if and only if w is in the image of W_Δ. In fact, W_Δ is the union of pieces in a stratification of W.*

Proof. The sections σ^i define divisors $Z_i \subset B \times W$. Apply Lemma 2.11 with $\mathcal{X} = \mathcal{Y} = B \times W$, $S = W$, $\mathcal{Z} = Z_0 \cap \cdots \cap Z_n$, and $\mathcal{T} = \Delta \times W$. □

Proof of Proposition 2.6. Let $\mathsf{W}^{\mathsf{b}} = W_\Delta$ be the result of applying Lemma 2.10 and Lemma 2.12 to the scheme $Y = \mathsf{W}(U, \mathbb{P}^n, \mathsf{b})$ constructed in Lemma 2.9. The sections $\sigma^0, \ldots, \sigma^n$ define the required morphism $\Xi : \mathsf{W}^{\mathsf{b}} \times U \to \mathbb{P}^n$, proving the statement for $M = \mathbb{P}^n$.

For a general projective $M \hookrightarrow \mathbb{P}^n$, if we let $\mathsf{W}^{\mathsf{b}} \times U \to \mathsf{W}^{\mathsf{b}} \times \mathbb{P}^n$ be the morphism ensured by the previous case, we can take $S = \mathsf{W}^{\mathsf{b}}$, $\mathcal{Z} = \mathcal{X} = \mathsf{W}^{\mathsf{b}} \times U$, $\mathcal{Y} = \mathsf{W}^{\mathsf{b}} \times \mathbb{P}^n$, and $\mathcal{T} = \mathsf{W}^{\mathsf{b}} \times M$ in Lemma 2.11, yielding a finite type monomorphism $W' \to \mathsf{W}^{\mathsf{b}}$ and the required morphism $W' \times U \to M$. □

PROPOSITION 2.13. *Given a polarized variety $(M, \mathbb{O}_M(1))$, an open subscheme M° and a weak bound* b*, there is a k-variety $\mathsf{W}^{\mathsf{b}}_{M^\circ}$ and a morphism $\mathsf{W}^{\mathsf{b}}_{M^\circ} \times U \to M^\circ$ such that for every field extension L/k, every morphism $U_L \to M^\circ_L$ whose composition with the inclusion $M^\circ_L \hookrightarrow M_L$ is weakly bounded with respect to the polarization of M_L by* b *appears in a fiber over $\mathsf{W}^{\mathsf{b}}_{M^\circ}(L)$.*

Proof. This follows from Proposition 2.6 and Lemma 2.11. □

We briefly indicate how to extend the results above to the case of a family over a reduced base.

PROPOSITION 2.14. *Let T be a quasicompact quasiseparated reduced k-scheme and $\mathcal{U} \to T$ a separated morphism which is smooth at infinity. Given a projective T-scheme of finite presentation $(\mathcal{M}, \mathcal{O}_{\mathcal{M}}(1))$, an open subscheme $\mathcal{M}^\circ \subseteq \mathcal{M}$ of finite presentation over T, and a weak bound* b, *there exists a T-scheme of finite presentation $\mathcal{W}^{\mathsf{b}}_{\mathcal{M}^\circ}$ and a morphism $\Xi : \mathcal{W}^{\mathsf{b}}_{\mathcal{M}^\circ} \times \mathcal{U} \to \mathcal{M}^\circ$ such that for every geometric point $t \to T$ and for every morphism $\xi : \mathcal{U}_t \to \mathcal{M}^\circ_t \subseteq \mathcal{M}_t$ that is weakly bounded by* b *there exists a point $p \to \mathcal{W}^{\mathsf{b}}_{\mathcal{M}^\circ_t}$ such that $\xi = \Xi|_{\{p\}\times\mathcal{U}_t}$.*

Proof. Since T is quasicompact and quasiseparated, absolute Noetherian approximation [TT90, C.3 and Th. C.9] lets us assume that T is of finite type over k. We claim that there is a finite type morphism $T' \to T$ and a fiberwise dense open immersion $U_{T'} \hookrightarrow \mathcal{B}$ with $\mathcal{B} \to T'$ a proper scheme with geometrically integral fibers. To see this, we can first replace T with the disjoint union of its irreducible components and thus assume that T is integral. The geometric generic fiber $U_{\overline{\eta}}$ has an integral compactification $U_{\overline{\eta}} \hookrightarrow \overline{B}$ by Nagata's theorem. Since $\overline{B}$ is of finite presentation over T, there is a finite type integral T-scheme $T_1 \to T$ with a lift $\overline{\eta} \to T_1$ to a geometric generic point over the given geometric generic point of T, a proper T_1-scheme $\mathcal{B} \to T_1$, and an open immersion $U_{T_1} \hookrightarrow \mathcal{B}$ whose pullback to $\overline{\eta}$ is $U_{\overline{\eta}} \hookrightarrow \overline{B}$.

By generic flatness, there is a dense open subscheme $T_2 \subset T_1$ over which $\mathcal{B}$ and $\mathcal{B}_{T_2} \setminus U_{T_2}$ are flat. Applying [Gro67, Th. 12.2.4], there is a further open subscheme $T_3 \subset T_2$ over which the geometric fibers of $\mathcal{B}$ are integral and the geometric fibers of $\mathcal{B}_{T_3} \setminus \mathcal{U}_{T_3}$ have dimension strictly smaller than the fiber dimension of $\mathcal{B}_{T_3} \to T_3$. It follows that $U_{T_3} \subset \mathcal{B}_{T_3}$ is a fiberwise dense open immersion. Since $T_3 \to T$ is dominant and of finite type, Chevalley's theorem shows that its image contains a dense open subset. By Noetherian induction, there are morphisms $T' \to T$ and $U_{T'} \hookrightarrow \mathcal{B}$ as claimed in the first paragraph. Resolution of singularities and similar stratification and Noetherian induction argument gives a finite type morphism T'' over which there is a smooth projective morphism $\widetilde{\mathcal{B}} \to T''$ whose geometric fibers are connected and which admits a fiberwise birational morphism $\widetilde{\mathcal{B}} \to \mathcal{B}_{T''}$. We can replace T by T'' and assume that we have such a compactification and resolution.

Since the geometric fibers of $\mathcal{B} \to T$ are integral, we see that $\mathcal{B}$ is cohomologically flat in degree 0 and the Picard functor is separated. Thus, the Picard stack $\mathcal{P}ic_{\mathcal{B}/T}$ is a $\mathbb{G}_m$-gerbe over a separated algebraic space $\mathrm{Pic}_{\mathcal{B}/T}$ locally of finite type over T. Given a T-flat relatively ample smooth divisor $\mathcal{B}' \subset \mathcal{B}$, if the fiber dimension of $\mathcal{B}'$ is at least 2 then by [SGA6, Th. XIII.3.8] the restriction morphism $\mathcal{P}ic_{\mathcal{B}/T} \to \mathcal{P}ic_{\mathcal{B}'/T}$ is of finite type. Since we can always replace T by an open covering, we can always assume that such a T-smooth divisor exists.

We claim that there is an open substack $\mathcal{Y} \subset \mathcal{P}ic_{\mathcal{B}/T}$ of finite type over T such that for each geometric point $t \to T$, the invertible sheaves $\mathcal{N}_\xi$ on B_t arising from weakly bounded morphisms $U_t \to \mathcal{M}^\circ$ via the procedure of 2.2 lie in $\mathcal{Y}$. Arguing precisely as in the proof of Lemma 2.9, it suffices to prove this for the inclusion $\widetilde{U} \hookrightarrow \widetilde{\mathcal{B}}$, so that we may assume $\mathcal{B}$ is smooth over T.

First suppose $\mathcal{B}$ is a relative smooth curve. The degree map gives a finite type morphism $\mathrm{Pic}_{\mathcal{B}/T} \to \mathbb{Z}_T$. Just as in the proof of Lemma 2.9, the weak boundedness shows that the $\mathcal{N}_\xi$ are contained in the preimage of a finite interval in $\mathbb{Z}$, yielding the claim.

If the fibers of $\mathcal{B}/T$ have dimension at least 2, then arguing as in the proof of Lemma 2.9 and using the existence of smooth relatively ample divisors $\mathcal{B}' \subset \mathcal{B}$ (after possibly replacing T by an open covering), we can reduce to the case in which $\mathcal{B} \to T$ is a smooth projective relative surface. Now, again as in the proof of Lemma 2.9 we have that the sheaves $\mathcal{N}_\xi$ on the fibers over T satisfy $0 \leq \deg_A \mathcal{N}_\xi \leq N$ and $c_1(\mathcal{N}_\xi)^2 \geq 0$ (where $A \subset \mathcal{B}$ is a relatively ample smooth divisor). Invoking [SGA6, Th. XIII.3.13(iii)] again, we see that the collection of invertible sheaves on the fibers satisfying those boundedness conditions forms a finite type open substack $\mathcal{Y} \subset \mathcal{P}ic_{\mathcal{B}/T}$, as desired.

Now, to bound the map we argue as in Lemma 2.10. As written, the argument is completely general and applies in the present situation. It yields the universal collection of sections. The proofs of Lemma 2.11, Lemma 2.12, and Proposition 2.13 also carry over to yield the map Ξ. □

2.15. Next we compactify ξ in a bounded family.

Definition 2.16. Given a T-scheme $B \to T$, a *relative simple normal crossings divisor* $D \subset B$ is a divisor of the form $D = D_1 + \cdots + D_r$ such that B is flat over T in a neighborhood of D, each D_i is flat over T, and in each geometric fiber B_t the divisor $(D_1)_t + \cdots + (D_r)_t$ is a simple normal crossings divisor.

PROPOSITION 2.17. *Let T be reduced quasiseparated and quasicompact k-scheme and $\mathcal{U} \to T$ a separated morphism that is smooth at infinity. Given a proper T-scheme of finite presentation $\mathcal{M}$ and a T-morphism $\xi : \mathcal{U} \to \mathcal{M}$, there exists a finite type surjective morphism $T' \to T$, a proper scheme $\mathcal{B} \to T'$, an open immersion $\mathcal{U}_{T'} \hookrightarrow \mathcal{B}$ over T' whose complement $\mathcal{B} \setminus \mathcal{U}_{T'}$ is a relative simple normal crossings divisor, and a T'-morphism $\bar{\xi} : \mathcal{B} \to \mathcal{M}_{T'}$ such that $\bar{\xi}|_{\mathcal{U}_{T'}} = \xi_{T'}$.*

Proof. By absolute Noetherian approximation, we may assume that T is Noetherian. We may then replace T by the disjoint union of its irreducible components and assume that T is integral. Next we compactify the morphism $\mathcal{U} \to T$ to a proper scheme $\mathcal{B}' \to T$ (which is not necessarily flat!). Resolving the singularities of the generic fiber of $\mathcal{B}' \setminus \mathrm{Sing}(\mathcal{U}/T)$ yields an immersion $\mathcal{U} \to \mathcal{B}''$ into a proper scheme over the function field of T whose general fiber over T is smooth outside $\mathcal{U}$. After a birational modification of $\mathcal{B}''$, we may assume that ξ extends to $\mathcal{B}''$ and that $\mathcal{B}'' \setminus \mathcal{U}$ is a simple normal crossings divisor. This extends over an open dense

subscheme of T. By Noetherian induction, we can thus stratify T so that such compactifications exist over each stratum. Given the compactifications, we proceed as in the proof of Proposition 2.6. □

Remark 2.18. If $\mathcal{U} \to T$ is quasiprojective, then we can assume that $\mathcal{B} \to T'$ is projective, as the resolution of singularities of $\mathcal{B}'$ can be assumed to be projective.

2.B. *Bounding maps to a quasiprojective scheme.*

Definition 2.19. Given a proper T-scheme $\pi : \mathcal{M} \to T$ and an open subscheme $\mathcal{M}^\circ \subseteq \mathcal{M}$, an invertible sheaf $\mathcal{L}$ on $\mathcal{M}$ is *relatively ample with respect to* $\mathcal{M}^\circ$ if there exists an integer $m > 0$ such that

(2.19.1) $\pi^*\pi_*\mathcal{L}^m \to \mathcal{L}^m$ is surjective over $\mathcal{M}^\circ$, and

(2.19.2) the natural map $\mathcal{M}^\circ \to \mathbb{P}_T(\pi_*\mathcal{L}^m)$ is a locally closed immersion.

A *relative polarization of* $\mathcal{M}$ *with respect to* $\mathcal{M}^\circ$ is an invertible sheaf $\mathcal{L}$ that is relatively ample with respect to $\mathcal{M}^\circ$.

Note that if (2.19.2) holds for some $m > 0$, then it holds for any m sufficiently large and divisible.

Definition 2.20. Given a separated T-scheme of finite type $\mathcal{M}^\circ$, a *relative compactification of* $\mathcal{M}^\circ$ is a T-morphism $\iota : \mathcal{M}^\circ \to \mathcal{M}$ that embeds $\mathcal{M}^\circ$ as an open subscheme of the proper T-scheme $\mathcal{M}$. If there is no danger of confusion, we will abuse notation and refer to a relative compactification $\iota : \mathcal{M}^\circ \to \mathcal{M}$ simply as $\mathcal{M}$. A *morphism* between relative compactifications $\iota : \mathcal{M}^\circ \to \mathcal{M}$ and $\iota' : \mathcal{M}^\circ \to \mathcal{M}'$ is a T-morphism $\varphi : \mathcal{M} \to \mathcal{M}'$ such that $\varphi \circ \iota = \iota'$.

Remark 2.21. These notions seem most natural if $\mathcal{M}_t^\circ$ is dense in $\mathcal{M}_t$ for all $t \in T$, but we do not need to make this assumption here.

The next statement allows us to replace a polarization with respect to an open subscheme with a polarization everywhere.

PROPOSITION 2.22. *Let T be a Noetherian scheme, $\mathcal{M}^\circ$ a separated T-scheme of finite type, $\iota : \mathcal{M}^\circ \to \mathcal{M}$ a relative compactification, and $\mathcal{L}$ a relative polarization of $\mathcal{M}$ with respect to $\mathcal{M}^\circ$. Then there exists a diagram of relative compactifications of $\mathcal{M}^\circ$*

$$\begin{array}{ccccc} & & \widetilde{\mathcal{M}} & & \\ & \swarrow^{\sigma} & \uparrow{\scriptstyle \tilde{\iota}} & \searrow^{\tau} & \\ \mathcal{M} & \xleftarrow[\iota]{} & \mathcal{M}^\circ & \xrightarrow[\iota']{} & \mathcal{M}' \end{array}$$

and a T-ample invertible sheaf $\mathcal{A}$ on $\mathcal{M}'$ such that:

(2.22.1) *There exists an inclusion of invertible sheaves $\tau^*\mathcal{A} \subseteq \sigma^*\mathcal{L}$ which is an isomorphism on $\mathcal{M}^\circ$.*

(2.22.2) *In particular, given a weak bound* b, *a geometric point* $t \to T$, *and a morphism* $\xi : U \to \mathcal{M}^\circ_t$, *if* $\iota \circ \xi$ *is weakly bounded with respect to* $\mathcal{L}_t$ *by* b, *then* $\iota' \circ \xi$ *is weakly bounded with respect to* $\mathcal{A}_t$ *by* b.

Proof. Let $m > 0$ be the integer given in Definition 2.19 and $\mathcal{E} = \pi_* \mathcal{L}^m$. Consider the natural map $\nu : \pi^* \mathcal{E} = \pi^* \pi_* \mathcal{L}^m \to \mathcal{L}^m$ which is surjective on $\mathcal{M}^\circ$. Let $\mathcal{I} = \nu(\pi^* \mathcal{E}) \otimes \mathcal{L}^{-m} \subseteq \mathcal{O}_{\mathcal{M}}$ and let $\sigma : \widetilde{\mathcal{M}} \to \mathcal{M}$ be the blowing up of the ideal sheaf $\mathcal{I}$.

Since the support of $\mathcal{O}_{\mathcal{M}} / \mathcal{I}$ is disjoint from $\mathcal{M}^\circ$, $\tilde{\iota} = \sigma^{-1} \circ \iota : \mathcal{M}^\circ \to \widetilde{\mathcal{M}}$ is a relative compactification of $\mathcal{M}^\circ$ and $\mathcal{N} = \sigma^* \mathcal{L}^m \otimes \sigma^{-1} \mathcal{I} \cdot \mathcal{O}_{\widetilde{\mathcal{M}}}$ is relatively ample with respect to $\tilde{\iota}(\mathcal{M}^\circ)$. The surjective morphism $\sigma^* \pi^* \mathcal{E} \twoheadrightarrow \mathcal{N}$ induces a T-morphism $\tau : \widetilde{\mathcal{M}} \to \mathbb{P}_T(\mathcal{E})$ which is an embedding on $\mathcal{M}^\circ$. Letting $\mathcal{M}'$ be the scheme-theoretic image of τ and $\mathcal{A}$ the restriction of $\mathcal{O}_{\mathbb{P}_T(\mathcal{E})}(1)$ to $\mathcal{M}'$ yields (2.22.1).

Given a curve C and a morphism $\gamma : C \to \widetilde{\mathcal{M}}_t$ such that $\gamma(C) \cap \mathcal{M}^\circ_t \neq \varnothing$, the natural map $\gamma^* \tau^* \mathcal{A} \to \gamma^* \sigma^* \mathcal{L}$ remains injective. Therefore (2.22.1) implies (2.22.2). □

COROLLARY 2.23. *Let* T *be a Noetherian scheme,* $\mathcal{M}^\circ$ *a separated* T*-scheme of finite type,* $\iota : \mathcal{M}^\circ \to \mathcal{M}$ *a relative compactification,* $\mathcal{L}$ *a relative polarization of* $\mathcal{M}$ *with respect to* $\mathcal{M}^\circ$, *and* b *a weak bound. Then there exists a* T*-scheme of finite type* $\mathcal{W}^{\mathsf{b}}_{\mathcal{M}^\circ}$ *and a morphism* $\mathcal{W}^{\mathsf{b}}_{\mathcal{M}^\circ} \times \mathcal{U} \to \mathcal{M}^\circ$ *such that for every geometric point* $t \to T$, *every morphism* $\mathcal{U}_t \to \mathcal{M}^\circ_t \subseteq \mathcal{M}_t$ *which is weakly bounded with respect to* $\mathcal{L}_t$ *by* b *appears in a fiber over* $\mathcal{W}^{\mathsf{b}}_{\mathcal{M}^\circ_t}$.

Proof. This follows directly from Propositions 2.14 and 2.22. □

3. **Weak stacky stable reduction**

3.A. *Groupoid-equivariant objects in a stack.* We start with a few basic results about equivariant objects and their liftings. While the main result of this section can be stated in purely stack-theoretic language (as we indicate in the alternative proof of Corollary 3.5), the formalism we briefly sketch here is useful for clarifying the proof of Proposition 3.12.

Let (R, Z) be a groupoid object in the category of algebraic spaces with big fppf stack quotient $[Z/R]$. Write $\sigma, \tau : R \to Z$ for the two structural morphisms. At the moment, we make no (e.g., flatness or finiteness) assumptions about σ and τ. One way to understand the stack $[Z/R]$ is as the stackification (see Lemma 3.2 of [LMB00]) of an intermediate prestack, which we will denote $\{Z/R\}$. The objects of $\{Z/R\}$ over T are given by the elements of $Z(T)$. Given two such objects $a, b \in Z(T)$, we define the sheaf of isomorphisms $\mathrm{Isom}_T(a, b)$ to be the fiber of $R \to Z \times Z$ over (a, b). Using the groupoid structure on (R, Z), one can check that this defines a prestack, and that the natural map to the 2-categorical fiber product $R \to Z \times_{\{Z/R\}} Z$ is an isomorphism (see Paragraph 2.4.3 of [LMB00]). For any

stack $\mathcal{Y}$, the universal property of stackification says that the restriction functor

$$\mathrm{Hom}([Z/R], \mathcal{Y}) \to \mathrm{Hom}(\{Z/R\}, \mathcal{Y})$$

is an equivalence of groupoids.

Let $R^{(2)}$ denote the fiber product $R \times_Z R$. The groupoid structure yields three morphisms $R^{(2)} \to R$: the two projections pr_1 and pr_2, and the composition map m. Given a prestack $\mathcal{Y}$, an object $\varphi : Z \to \mathcal{Y}$, and an isomorphism $\eta : \varphi\sigma \xrightarrow{\sim} \varphi\tau$, the *coboundary* of η is defined to be the element $\partial\eta := (\eta\,\mathrm{pr}_1)(\eta m)^{-1}(\eta\,\mathrm{pr}_2)$ of $\mathrm{Aut}(\varphi\sigma\,\mathrm{pr}_2)$.

Definition 3.1. Let $\mathcal{Y}$ be a prestack. Then an *R-equivariant object of $\mathcal{Y}$ over Z* is an object $\varphi : Z \to \mathcal{Y}$ and an isomorphism $\eta : \varphi\sigma \xrightarrow{\sim} \varphi\tau$ of morphisms $R \to \mathcal{Y}$ such that the coboundary $\partial\eta$ is trivial.

The R-equivariant objects of $\mathcal{Y}$ naturally form a groupoid, which we will denote by $\mathcal{Y}_{(R,Z)}$. A 1-morphism of groupoids $\mathcal{Y} \to \mathcal{Y}'$ induces a functor $\mathcal{Y}_{(R,Z)} \to \mathcal{Y}'_{(R,Z)}$.

A basic example of an equivariant object of a prestack comes from the morphism $\varphi : Z \to \{Z/R\}$ induced by the point $\mathrm{id} \in Z(Z)$. The isomorphism $\eta : \varphi\sigma \xrightarrow{\sim} \varphi\tau$ arises as follows: by definition, we have that $\mathrm{Isom}_R(\varphi\sigma, \varphi\tau)$ is the fiber product sheaf

$$\begin{array}{ccc} \mathrm{Isom}_R(\varphi\sigma, \varphi\tau) & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & Z \times Z, \end{array}$$

where both maps $R \to Z \times Z$ are the pair (σ, τ). The diagonal of $R \times R$ yields a canonical section of $\mathrm{Isom}_R(\varphi\sigma, \varphi\tau)$, giving rise to an equivariant object.

In fact, this is the universal equivariant object, as we now make precise. Given a prestack $\mathcal{Y}$, the constructions of the two previous paragraphs yield a functor

$$\varepsilon : \mathrm{Hom}(\{Z/R\}, \mathcal{Y}) \to \mathcal{Y}_{(R,Z)}$$

between groupoids.

PROPOSITION 3.2. *The functor ε is an equivalence for any stack $\mathcal{Y}$.*

Proof. We first describe the groupoid $\mathrm{Hom}(\{Z/R\}, \mathcal{Y})$. Let $\mathcal{P}$ be the groupoid of pairs (φ, ι) consisting of a 1-morphism $\varphi : Z \to \mathcal{Y}$ and a morphism $\iota : (R, Z) \to (Z \times_{\mathcal{Y}} Z, Z)$ of groupoids. The isomorphisms in $\mathcal{P}$ are given by isomorphisms between the maps φ which are compatible with the maps ι.

Given a morphism $\varphi : \{Z/R\} \to \mathcal{Y}$, composition with the natural morphism $Z \to \{Z/R\}$ defined above yields a diagram

$$\begin{array}{ccc} R = Z \times_{\{Z/R\}} Z & \longrightarrow & Z \times_{\mathcal{Y}} Z \\ \downarrow\downarrow & & \downarrow\downarrow \\ Z & \xrightarrow{\mathrm{id}_Z} & Z \\ \downarrow & & \downarrow \\ \{Z/R\} & \longrightarrow & \mathcal{Y}. \end{array}$$

The diagram induces a morphism of groupoids $\iota : (R, Z) \to (Z \times_{\mathcal{Y}} Z, Z)$, yielding a functor from $\mathrm{Hom}(\{X/R\}, \mathcal{Y})$ to $\mathcal{P}$.

We can produce a functor $\mathcal{P} \to \mathrm{Hom}(\{Z/R\}, \mathcal{Y})$ in the opposite direction. Given a 1-morphism $\varphi : Z \to \mathcal{Y}$ and a morphism $\iota : (R, Z) \to (Z \times_{\mathcal{Y}} Z, Z)$ of groupoids, we make a 1-morphism of prestacks $\{Z/R\} \to \mathcal{Y}$ as follows. An object $\alpha \in \{Z/R\}(T) = Z(T)$ gets sent to $\varphi\alpha \in \mathcal{Y}_T$, and an arrow $r \in \mathrm{Isom}_{\{Z/R\}(T)}(\alpha, \beta) = R(T)$ gets sent to the arrow $\psi : \varphi\alpha \xrightarrow{\sim} \varphi\beta$ determined by the image of r in $Z \times_{\mathcal{Y}} Z$ (i.e., r maps to the triple (α, β, ψ) in the functorial construction of the 2-fiber product).

The result is an equivalence of groupoids $\mathrm{Hom}(\{X/R\}, \mathcal{Y}) \to \mathcal{P}$. Sending a pair (φ, ι) to the pair $(\varphi, \iota(\sigma, \tau))$ gives a functor $e : \mathcal{P} \to \mathcal{Y}_{(R,Z)}$ which factorizes ε. The result is thus proven if we show that e is an equivalence of groupoids. A morphism $(R, Z) \to (Z \times_{\mathcal{Y}} Z, Z)$ extending φ is given by a morphism $y : R \to Z \times_{\mathcal{Y}} Z$ with image (σ, τ) in $Z(R) \times Z(R)$ such that the composition arrow $R^{(2)} \to R$ is compatible via y with the canonical morphism

$$(Z \times_{\mathcal{Y}} Z) \times_Z (Z \times_{\mathcal{Y}} Z) \xrightarrow{\sim} Z \times_{\mathcal{Y}} (Z \times_Z Z) \times_{\mathcal{Y}} Z \xrightarrow{\sim} Z \times_{\mathcal{Y}} Z.$$

The arrow $y : R \to Z \times_{\mathcal{Y}} Z$ gives a triple (σ, τ, η) with $\eta : \varphi\sigma \xrightarrow{\sim} \varphi\tau$. The coboundary condition on η is precisely the condition that a triple (σ, τ, η) give rise to a morphism of groupoids over φ, as desired. □

Remark 3.3. A similar result is proven in Section 3.8 of [Vis05], where equivariant objects of stacks are treated. There, the groupoid (R, Z) is given by a group action $G \times Z \to Z$.

In particular, we may apply Proposition 3.2 to the case of a group G acting on an algebraic space Z, yielding an equivalence between G-equivariant maps $Z \to \mathcal{Y}$ and morphisms $[Z/G] \to \mathcal{Y}$. We can use this to prove a purity theorem for maps $[Z/G] \to \mathcal{Y}$.

PROPOSITION 3.4. *Suppose $\mathcal{M}$ is a separated Deligne-Mumford stack with coarse moduli space M. Let (R, Z) be a groupoid of algebraic spaces with Z regular and R normal and with flat structural morphisms. Suppose $\psi : Z \to M$ is an R-invariant morphism, $U \subset Z$ is a dense R-invariant open subspace, and $\varphi : U \to \mathcal{M}$ is an R-equivariant object covering $\psi|_U$. If $Z \setminus U$ has codimension at least 2 in Z then φ extends to an R-equivariant object of $\mathcal{M}$ over all of Z which covers ψ.*

Proof. By the Purity Lemma [AV02, 2.4.1 and 2.4.2], φ lifts to $\overline{\varphi} : Z \to \mathcal{M}$. It remains to show equivariance. We are given an isomorphism $\alpha : \varphi\sigma \xrightarrow{\sim} \varphi\tau$. As $\mathcal{M}$ is separated, $\mathrm{Isom}_R(\overline{\varphi}\sigma, \overline{\varphi}\tau)$ is finite over R (via either projection). Furthermore, since R is normal, any finite birational morphism $Y \to R$ is an isomorphism. It follows that taking the closure of α in $\mathrm{Isom}_R(\overline{\varphi}\sigma, \overline{\varphi}\tau)$ yields a global section over R. Moreover, we know that the coboundary of α is trivial over the preimage of U. Since U is schematically dense in Z and the structural morphisms of the groupoid are flat, it follows that the preimage of U in $R^{(2)}$ is schematically dense. (To prove this, first note that it suffices to prove that some open subspace of U has schematically dense preimage in $R^{(2)}$. Since Z is regular and quasiseparated, by working with one component at a time we can choose a dense open subspace U' of U whose inclusion $i : U' \hookrightarrow Z$ is a quasicompact morphism, so that $\mathcal{O}_Z \to i_*\mathcal{O}_{U'}$ is an injective map of quasicoherent $\mathcal{O}_Z$-algebras. Since pushforward and flat base change commute for quasicoherent sheaves, we see that the induced map $\mathcal{O}_{R^{(2)}} \to (i \times \mathrm{id}_{R^{(2)}}) * \mathcal{O}_{U' \times_Z R^{(2)}}$ is injective. This shows that $U' \times_Z R^{(2)}$ is schematically dense in $R^{(2)}$, as desired.) Using the fact that $\mathrm{Isom}_R(\overline{\varphi}\sigma, \overline{\varphi}\tau)$ is separated over R, we see that the coboundary of α is trivial over all of $R^{(2)}$. □

COROLLARY 3.5. *Let $\mathcal{X}$ be a smooth Deligne-Mumford stack and $\mathcal{U} \subset \mathcal{X}$ an open substack of complementary codimension at least 2. Let $\mathcal{M}$ be a separated Deligne-Mumford stack with coarse moduli space M. Given a morphism $\psi : Z \to M$ and a lift $\varphi_{\mathcal{U}} : \mathcal{U} \to \mathcal{M}$, there is a unique extension $\varphi : \mathcal{X} \to \mathcal{M}$ up to unique isomorphism.*

Proof. We include an alternative, purely stack-theoretic proof (without invoking groupoids). This proof has the advantage of greater intrinsic clarity, although we find the groupoid formalism helpful in the proof of Proposition 3.12 below.

Consider the morphism $\varphi : \mathcal{X} \times_M \mathcal{M} \to \mathcal{X}$. By assumption, there is a section σ over $\mathcal{U}$. Let $\mathcal{Y} = \overline{\sigma(\mathcal{U})}$ be the stack-theoretic closure. The projection $\mathcal{Y} \to \mathcal{X}$ is proper, quasifinite, and an isomorphism in codimension 1. This persists after any étale base change $Z \to \mathcal{X}$, whence, since $\mathcal{X}$ is smooth, we see that $\rho : \mathcal{Y}' := \mathcal{Y} \times_{\mathcal{X}} Z \to Z$ must be an isomorphism. Indeed, it immediately follows that (via ρ) Z is the coarse moduli space of $\mathcal{Y}'$. On the other hand, over the strict localizations of Z, $\mathcal{Y}'$ is a finite group quotient $[\mathrm{Spec}\, R/G]$ with coarse space $\mathrm{Spec}\, S$. By assumption, S is regular and $S \subset R$ is finite and unramified in codimension 1, hence is finite étale by purity. It follows that $\mathrm{Spec}\, S \simeq [\mathrm{Spec}\, R/G]$. We conclude that $\mathcal{Y} \to \mathcal{X}$ is an isomorphism, and thus that there is a lift $\mathcal{X} \to \mathcal{M}$ over M. □

3.B. *Stacky branched covers.* We briefly recall the basic facts concerning stacky branched covers. Let $D \subset Z$ be an effective Cartier divisor in an algebraic space, corresponding to a pair $(\mathcal{L}, s)$ with $\mathcal{L}$ an invertible sheaf on Z and $s \in \Gamma(Z, \mathcal{L})$ a regular global section (i.e., s is not a zero divisor). Let $\mathbb{L}$ be the Artin stack $[\mathbb{A}^1/\mathbb{G}_m]$; $\mathbb{L}$ represents the stack (on the category of algebraic spaces)

of pairs $(\mathscr{L}, s)$ consisting of an invertible sheaf and a global (not necessarily regular) section. The map $x \mapsto x^n$ defines a morphism $\nu_n : \mathbb{L} \to \mathbb{L}$.

PROPOSITION 3.6. *Let $Z, D, \mathscr{L}, s$ be as above. Define $Z[D^{1/n}]$ to be*

$$Z \times_{(\mathscr{L},s),\mathbb{L},\nu_n} \mathbb{L}.$$

(3.6.1) *$\pi : Z[D^{1/n}] \to Z$ is a tame Artin stack with coarse moduli space Z; the natural morphism $Z[D^{1/n}] \times_Z (Z \setminus D) \to Z \setminus D$ is an isomorphism.*

(3.6.2) *$(Z[D^{1/n}] \times_Z D)_{\mathrm{red}} \to D_{\mathrm{red}}$ is the $\boldsymbol{\mu}_n$-gerbe of n^{th} roots of the invertible sheaf $\mathscr{L}|_D$.*

(3.6.3) *There exists a pair $(\mathscr{L}, \sigma)$ of an invertible sheaf and a global section on $Z[D^{1/n}]$ with an isomorphism $\mathscr{L}^{\otimes n} \xrightarrow{\sim} \pi^*\mathscr{L}$ sending $\sigma^{\otimes n}$ to $\pi^* s$. The section σ is regular. Moreover, the pair $(\mathscr{L}, \sigma)$ is universal: $Z[D^{1/n}]$ represents the stack of such pairs of n^{th} roots.*

(3.6.4) *Zariski locally on Z, $Z[D^{1/n}]$ has the form $[\mathbf{Spec}(\mathcal{O}_Z[z]/(z^n - t))/\boldsymbol{\mu}_n]$, where $t = 0$ is a local equation for D.*

(3.6.5) *The stack $Z[D^{1/n}]$ is a global quotient of the form $[Q/\mathbb{G}_m]$. If Z and D are regular, then so is Q.*

(3.6.6) *If Z is projective over a field and D is smooth then there is a finite flat morphism $Y \to Z[D^{1/n}]$ with Y a projective scheme.*

Proof. The proof of (3.6.1) through (3.6.4) has been treated numerous times in the literature (see for example [MO05, 4.1] and [Cad07, §2]). The penultimate statement may be proven as follows: given the universal pair $(\mathscr{L}, \sigma)$, let $Q \to Z[D^{1/n}]$ be the total space of the $\mathbb{G}_m$-torsor associated to $\mathscr{L}$. Since the stabilizer action on each geometric fiber of $\mathscr{L}$ is faithful, it is a standard result that Q (which is the bundle of frames of the line bundle associated to $\mathscr{L}$) is an algebraic space. It immediately follows that $Z[D^{1/n}] \simeq [Q/\mathbb{G}_m]$.

To prove the final statement, we recall Viehweg's formulation of the Kawamata covering trick [Vie95, Lemma 2.5] and point out a slight modification. Write Z^{sm} for the smooth locus of Z; this is an open subscheme containing D. Let $d = \dim Z$. Let H be an ample divisor on Z. For sufficiently large m, the divisor $nmH - D$ is very ample. Choose general members $E_1, \ldots, E_d$ such that $(E_1 + E_2 + \cdots + E_d + D)|_{Z^{\mathrm{sm}}}$ is a simple normal crossings divisor. Each $E_i + D$ is in n Pic, so we can construct the usual cyclic cover branched over $E_i + D$ (see for example [KM98, Def. 2.49(3)]), say $C_i \to Z$. By construction, $C_i \to Z$ is a finite flat morphism. Let $\overline{Y} := C_1 \times_Z C_2 \times_Z \cdots \times_Z C_d$. Over $Z^{\mathrm{sm}} \setminus D$, the transversality of $E_1, \ldots, E_d$ ensures that $\overline{Y}$ is smooth. On the other hand, one can check that the normalization Y^ν of $\overline{Y}|_{Z^{\mathrm{sm}}}$ is smooth. Moreover, the reduced structure on the preimage of D in Y^ν gives an effective Cartier divisor D' such that $nD' = D|_{Y^\nu}$. Gluing $\overline{Y}|_{Z \setminus D}$ to Y^ν yields a finite flat morphism $f : Y \to Z$ such that $f^{-1}(Z^{\mathrm{sm}})$ is smooth and there is an effective Cartier divisor $D' \in Y$ such that $nD' = f^*D$.

By the universal property of $Z[D^{1/n}]$, there is a Z-morphism $f' : Y \to Z[D^{1/n}]$. Over the complement of D, f' and f are naturally isomorphic. On the other hand, in a neighborhood of D, both $Z[D^{1/n}]$ and Y are regular and equidimensional of the same dimension. Applying [Mat89, Cor. to Th. 32.1] to the pullback of f' over an affine étale neighborhood of D, we see that f' is finite and flat, as desired. □

Given a simple normal crossings divisor $D = D_1 + \cdots + D_\ell$ in Z (which implies that the strict local rings of Z are regular at each point in the support of D), define

$$Z\langle D^{1/n}\rangle := Z[D_1^{1/n}, \ldots, D_\ell^{1/n}] := Z[D_1^{1/n}] \times_Z \cdots \times_Z Z[D_\ell^{1/n}].$$

In Cadman's notation [Cad07], $X[D^{1/n}]$ is written as $X_{\mathscr{L},s,n}$ and $X\langle D^{1/n}\rangle$ is written as $X_{(D_1,\ldots,D_n),(n,\ldots,n)}$.

We assume in what follows that Z is excellent. (By definition, an algebraic space is excellent if *every* étale cover by a scheme is excellent. Simply requiring it for one cover is not sufficient, as shown in [Gro67, 18.7.7].)

LEMMA 3.7. *$Z\langle D^{1/n}\rangle$ is regular in a neighborhood of $Z\langle D^{1/n}\rangle \times_Z D$.*

Proof. It is clear that the formation of $Z\langle D^{1/n}\rangle$ and the statement of the lemma are compatible with étale base change, so we may assume that $Z = \operatorname{Spec} R$ is an affine scheme. Since D is a simple normal crossings divisor and Z is excellent, Z is regular in a Zariski neighborhood of D. Upon replacing Z by this neighborhood, we may assume that Z is regular. Shrinking Z further if necessary, we may also assume that $\mathcal{O}(D_1), \ldots, \mathcal{O}(D_\ell)$ are trivial. Let $t_i = 0$ be an equation for D_i. In this case, $Z\langle D^{1/n}\rangle$ is locally isomorphic to the quotient stack for the action of $\boldsymbol{\mu}_n^{\oplus \ell}$ on $Y = \operatorname{Spec} R[z_1, \ldots, z_\ell]/[(z_1^n - t_1, \ldots, z_\ell^n - t_\ell)]$. Since D is a simple normal crossings divisor, it is easy to see that Y is regular (and excellent). □

LEMMA 3.8. *Let $D \subset Z \to S$ be a flat relative simple normal crossings divisor. The formation of $Z\langle D^{1/n}\rangle$ is compatible with base change.*

Proof. This follows immediately from the definition. □

3.C. *Applications to lifting problems.* In this section we fix a discrete valuation ring R with uniformizer t, fraction field K, and residue field κ. Let $\mathscr{X}$ be a tame separated Deligne-Mumford stack of finite type over R with coarse moduli space Spec R and trivial generic stabilizer.

LEMMA 3.9. *With the above notation, there exists a positive integer n_0 such that for all n divisible by n_0, there is a unique R-morphism* Spec $R[t^{1/n}] \to \mathscr{X}$ *up to unique isomorphism.*

Proof. Let R' be the strict Henselization of R. By the local structure theory of Deligne-Mumford stacks, there exists a finite generically Galois extension S/R' with Galois group G and an R'-isomorphism $\mathscr{X} \otimes R' \simeq [S/G]$. Since $\mathscr{X}$ is tame, we may assume that the order of G is invertible in R. Let $\widetilde{S}$ denote the normalization

of S. Since $\widetilde{S}/R'$ is generically Galois of degree invertible in R, it follows from Abhyankar's Lemma and the structure of finite unramified extensions of Henselian local rings that $\widetilde{S}$ is a finite product of rings of the form $R'[t^{1/n_0}]$ for a fixed n_0. (A treatment of the case of discrete valuation rings, which is all we use here, may be found in [Ser79, Ch. IV, §§1–2].) It follows that the stabilizer of the factor $R'[t^{1/n_0}]$ is isomorphic to $\boldsymbol{\mu}_{n_0}(\bar{\kappa})$. Let n be any integer divisible by n_0. Then there exists a natural $\boldsymbol{\mu}_n$-equivariant map $\operatorname{Spec} R[t^{1/n}] \to \operatorname{Spec} R[t^{1/n_0}]$. Since $R'[t^{1/n}] = R' \otimes_R R[t^{1/n}]$, it follows that there is an étale surjection $U \to \operatorname{Spec} R[t^{1/n}]$ and an R-morphism $\varphi : U \to \mathcal{X}$. Let $Y = \operatorname{Spec} R[t^{1/n}]$. Over the generic fiber of $U \times_Y U$ there is a descent datum $\psi : \mathrm{pr}_1^* \varphi \xrightarrow{\sim} \mathrm{pr}_2^* \varphi$ arising from the fact that $\mathcal{X}$ is generically isomorphic to R. Since $U \to Y$ is unramified, $U \times_Y U$ is a Dedekind scheme, and since $\mathcal{X}$ is separated, it follows that ψ extends to a descent datum for the covering $U \to Y$, yielding an R-morphism $Y \to \mathcal{X}$. Uniqueness follows from separatedness and the fact that $\mathcal{X} \to \operatorname{Spec} R$ is a generic isomorphism. □

LEMMA 3.10. *In the situation of the previous lemma, let* $Y = \operatorname{Spec} R[t^{1/n_0}]$. *There is an induced morphism of* R*-stacks* $[Y/\boldsymbol{\mu}_{n_0}] \to \mathcal{X}$ *which identifies* $[Y/\boldsymbol{\mu}_{n_0}]$ *with the normalization of* $\mathcal{X}$.

Proof. The scheme $Y \times \boldsymbol{\mu}_{n_0}$ is Dedekind, and the generic morphism $Y \otimes K \to \mathcal{X} \otimes K$ is clearly equivariant. Arguing as in the proof of Lemma 3.9, the induced generic isomorphism between the two maps $Y \times \boldsymbol{\mu}_{n_0} \to \mathcal{X}$ extends (uniquely) over all of $Y \times \boldsymbol{\mu}_{n_0}$. It follows that $Y \to \mathcal{X}$ is equivariant, yielding a morphism $\rho : [Y/\boldsymbol{\mu}_{n_0}] \to \mathcal{X}$ by Proposition 3.2. Since, in the notation of Lemma 3.9, $\boldsymbol{\mu}_{n_0}(\bar{\kappa})$ is a subgroup of G (namely, the subgroup fixing the closed point of $\widetilde{S}$), it follows that the morphism of stabilizers induced by $[Y/\boldsymbol{\mu}_{n_0}] \to \mathcal{X}$ is injective. Thus, $[Y/\boldsymbol{\mu}_{n_0}] \to \mathcal{X}$ is proper, quasifinite, birational, and injective on geometric stabilizers, which implies that it is a finite (affine) morphism. The result follows from the uniqueness of normalization. □

Remark 3.11. It is an amusing exercise to understand how Lemma 3.10 applies to the case of the stack $\mathcal{X}$ given by the quotient of a wedge of n lines (in the sense of topology) by one of the natural actions of $\mathbb{Z}/n\mathbb{Z}$. It is easy to see that the coarse moduli space is a line, and that there is a single stacky point (corresponding to the point at which all of the lines are wedged, which is fixed by $\mathbb{Z}/n\mathbb{Z}$). In particular, $\mathcal{X}$ is integral (but admits a finite étale cover by a connected reducible scheme). In this case $n_0 = 1$ and the normalization is simply a (nonstacky) line.

PROPOSITION 3.12. *Let* $\mathcal{M}$ *be a proper tame Deligne-Mumford stack with coarse moduli space* M. *Let* $U \subset Z$ *be the complement of a simple normal crossings divisor* D *in an excellent scheme. Suppose* $\varphi : U \to \mathcal{M}$ *is a morphism such that* $\pi \circ \varphi : U \to M$ *extends to a morphism* $\psi : Z \to M$. *Then there is an extension of* φ *to a morphism* $\widetilde{\psi} : Z\langle D^{1/N}\rangle \to \mathcal{M}$ *lifting* ψ, *where* N *is the least common multiple of the orders of the geometric stabilizers of* $\mathcal{M}$.

Proof. Let the generic points of $Z \setminus U$ be $p_1, \dots, p_r$. We claim that it is sufficient to extend φ across the Zariski localizations $Z_{p_i}\langle D_i^{1/N}\rangle$ at each p_i. Indeed, let $Y \to Z\langle D^{1/N}\rangle$ be the $\mathbb{G}_m^{\oplus r}$-torsor defined in (3.6.5). Giving an extension of φ to $Z\langle D^{1/N}\rangle$ is the same as giving an equivariant extension of $\varphi|_{U_Y}$ to a morphism $Y \to \mathcal{M}$. Such an extension is unique up to unique isomorphism, so it immediately follows that once one has an extension over each p_i, one gets an extension over an open subspace $V \subset Z$ whose complement has codimension at least 2 and is contained entirely in the regular locus of Z. Applying Proposition 3.4 yields the result.

Thus, let R be the local ring at some p_i. Consider $\mathcal{Y} := \operatorname{Spec} R \times_{\mathcal{M}} \mathcal{M} \to \operatorname{Spec} R$. Since R is normal, we may apply Lemma 3.10 to conclude that $\mathcal{Y} = \operatorname{Spec} R[p_i^{1/m}]$, where m is the order of the geometric stabilizer over p_i. Since $\operatorname{Spec} R[p_i^{1/N}]$ naturally dominates $\operatorname{Spec} R[p_i^{1/m}]$ over R for any multiple N of m, we see that we can extend φ across the preimage of p_i in $Z\langle D^{1/N}\rangle$, as required. □

Remark 3.13. The reason we call this section "weak stacky stable reduction" is the following: given a discrete valuation ring R and a family over its generic point, the methods of this section produce a family over a stack with coarse moduli space $\operatorname{Spec} R$, *as long as we already have the extension of the coarse moduli map*. This makes the statement easier to prove but far weaker than stable reduction, even in a stacky form (cf. [Ols04]).

4. **Proof of the main theorems**

4.A. *Terminology.* Let $\mathcal{M}^\circ$ be a separated Deligne-Mumford stack of finite type over T with coarse moduli space $\mathcal{M}^\circ$.

Definition 4.1. The stack $\mathcal{M}^\circ$ is *compactifiable* if there is an open immersion $\mathcal{M}^\circ \hookrightarrow \mathcal{M}$ into a proper Deligne-Mumford stack. If $\mathcal{M}^\circ$ is provided with a compactification, we will say it is *compactified*.

LEMMA 4.2. *Any separated Deligne-Mumford stack arising as the quotient of an action by a linear group on a quasiprojective scheme over a field is compactifiable.*

Proof. By Theorem 5.3 of [Kre09], such a stack admits a locally closed immersion into a smooth proper Deligne-Mumford stack with projective coarse moduli space. Taking the stack-theoretic closure yields the result. □

Definition 4.3. A *coarse compactification* of $\mathcal{M}^\circ$ is a compactification of the coarse moduli space $\mathcal{M}^\circ$. If $\mathcal{M}^\circ$ is provided with a coarse compactification, we will say it is *coarsely compactified*.

Let $\mathcal{M}^\circ$ be a coarsely compactified separated Deligne-Mumford stack of finite type over T. Suppose the coarse compactification $\mathcal{M}^\circ \hookrightarrow \mathcal{M}$ is relatively polarized by $\mathcal{L}$.

Definition 4.4. Given a function $\mathsf{b} : \mathbb{Z}^2_{\geq 0} \to \mathbb{Z}$, we will say that $\mathcal{M}^\circ$ is *weakly bounded with respect to* $\mathcal{M}$ *and* $\mathcal{L}$ *by* b if for every geometric point $t \to T$ and every (g,d)-curve $C^\circ \subseteq C$ over $\kappa(t)$, every morphism $\xi : C^\circ \to \mathcal{M}_t$ factoring through $\mathcal{M}^\circ_t$ satisfies $\deg \xi^*_C \mathcal{L} \leq \mathsf{b}(g,d)$, where ξ_C is the extension of ξ to a morphism $C \to \mathcal{M}_t$. Cf. Definition 2.4.

Definition 4.5. The stack $\mathcal{M}^\circ$ is *weakly bounded* if there exists a coarse compactification $\mathcal{M}^\circ \hookrightarrow \mathcal{M}$, a relative polarization $\mathcal{L}$ of $\mathcal{M}$ with respect to $\mathcal{M}^\circ$, and a function $\mathsf{b} : \mathbb{Z}^2_{\geq 0} \to \mathbb{Z}$ such that $\mathcal{M}^\circ$ is weakly bounded with respect to $\mathcal{M}$ and $\mathcal{L}$ by b.

Given a scheme U, define a relation on the set of isomorphism classes of morphisms $\varphi : U \to \mathcal{M}^\circ$ as follows: $\varphi_1 \sim \varphi_2$ if and only if there exists a connected k-scheme T, two points $t_1, t_2 \in T(k)$, and a morphism $\psi : U \times T \to \mathcal{M}^\circ$ such that $\psi|_{U \times t_i} \simeq \varphi_i$ for $i = 1, 2$. This generates an equivalence relation $\equiv$.

Definition 4.6. The equivalence classes for the equivalence relation $\equiv$ are called *deformation types*.

It is clear that this notion agrees with Definition 1.4 when $\mathcal{M}^\circ$ is the moduli stack of canonically polarized manifolds.

4.B. *The main theorem.*

Proof of Theorem 1.7. Observe that by Corollary 2.23 there is a finite type extension $\widetilde{T} \to T$ and a morphism $U_{\widetilde{T}} \to \mathcal{M}^\circ$ with the following property: for a geometric point $t \to T$, every morphism $U_t \to \mathcal{M}^\circ_t$ that arises by composition $U_t \to \mathcal{M}^\circ \to \mathcal{M}^\circ$ is parametrized by a point of $\widetilde{T}$. Let $\mathcal{M}^\circ \hookrightarrow \mathcal{M}$ be a compactification of $\mathcal{M}^\circ$, and let $\mathcal{M}$ be the coarse moduli space of $\mathcal{M}$. Applying Proposition 2.17, there is a further finite type extension $\sigma : T' \to \widetilde{T}$, a proper morphism $\mathcal{B} \to T'$, a relative simple normal crossings divisor $D \subset \mathcal{B}$, an isomorphism $\mathcal{B} \setminus D \simeq U_{T'}$, and a morphism $\mathcal{B} \to \mathcal{M}$ such that for every fiber U_t and every morphism $\varphi : U_t \to \mathcal{M}^\circ$, there exists a point $t' \to T'_t$ such that $\sigma(t') = t$ and the restriction of the induced morphism $\mathcal{B}_{t'} \to \mathcal{M}_t$ to $U_{t'}$ is the coarse morphism associated to φ.

By Proposition 3.12 and the fact that T has characteristic 0, the morphism $U_{t'} \to \mathcal{M}^\circ_{t'}$ extends to a morphism $\mathcal{B}\langle D^{1/N}\rangle_{t'} \to \mathcal{M}_{t'}$ over the coarse moduli map $\mathcal{B}_{t'} \to \mathcal{M}_{t'}$ for any geometric point $t' \to T'$. Consider the morphism of stacks

$$\mu : \underline{\mathrm{Hom}}_{T'}(\mathcal{B}\langle D^{1/N}\rangle, \mathcal{M}_{T'}) \to \underline{\mathrm{Hom}}_{T'}(\mathcal{B}, \mathcal{M}_{T'}).$$

We know that μ is of finite type: if U is quasiprojective, then this follows from [AOV08, Th. C.4], as all of the stacks involved are tame (the characteristic of k being 0) and separated, and $\mathcal{B}\langle D^{1/N}\rangle$ is proper and flat over the base. Furthermore, by Lemma 2.11 there is a finite type monomorphism

$$\mathcal{S} \to \underline{\mathrm{Hom}}_{T'}(\mathcal{B}\langle D^{1/N}\rangle, \mathcal{M}_{T'})$$

parametrizing morphisms that pull back the boundary $\mathcal{M} \setminus \mathcal{M}^\circ$ into D. The given "universal" coarse moduli map $\mathcal{B} \to \mathcal{M}$ determines a section of $\underline{\mathrm{Hom}}_{T'}(\mathcal{B}, \mathcal{M}_{T'})$ over T'. Pulling this back to $\mathcal{S}$ yields a finite type T'-stack $\mathcal{H} \to T'$ such that for any geometric point $t \to T$, the set of deformation types of morphisms $U_t \to \mathcal{M}^\circ$ is a quotient of the set of connected components of $\mathcal{H}_t$ (the fiber of the morphism $\mathcal{H} \to T' \to T$). Indeed, any point of $\mathcal{H}_t$ parametrizes a morphism sending U_t to $\mathcal{M}^\circ$ and any such morphism occurs as such a point, so any two points in the same connected component represent deformation equivalent morphisms $U_t \to \mathcal{M}^\circ$, and any deformation type is represented by a point of $\mathcal{H}_t$. (There could conceivably be deformation equivalent morphisms which lie in different components of $\mathcal{H}_t$, as our construction makes frequent use of stratification.) Since $\mathcal{H} \to \mathcal{B}$ is of finite type, the number of connected components is bounded above for all points t, giving a bound on the number of deformation types. □

COROLLARY 4.7. *If $\mathcal{M}^\circ$ is weakly bounded then there exists a function $\mathsf{b}_{\mathcal{M}} : \mathbb{Z}^2_{\geq 0} \to \mathbb{Z}$ such that for every smooth curve C of genus g with d marked points $p_1, \ldots, p_d$, the number of deformation types of morphisms $C \setminus \{p_1, \ldots, p_d\} \to \mathcal{M}^\circ$ is finite and bounded above by $\mathsf{b}_{\mathcal{M}}(g, d)$.*

Proof. Choosing an affine cover of $\mathcal{M}_{g,d}$ and pulling back the universal curve yields a quasicompact family containing all d-pointed smooth curves of genus g. The result thus follows immediately from Theorem 1.7. □

Remark 4.8. The uniformity result of Corollary 4.7 was first proven by Caporaso for families of curves (i.e., for $\mathcal{M}^\circ = \mathcal{M}_g$) in [Cap02], using methods specific to the stack of curves. In [Hei04], Heier refined Caporaso's results to produce an effective uniform bound. It would perhaps be interesting to determine what auxiliary data about the stack $\mathcal{M}^\circ$ are necessary to prove an abstract effective form of Corollary 4.7.

5. **Finiteness of infinitesimally rigid families**

Let $\mathcal{M}^\circ$ be a Deligne-Mumford stack and let N be the least common multiple of the orders of the stabilizers of geometric points of $\mathcal{M}$. Suppose U is a k-scheme.

Definition 5.1. A morphism $\chi : U \to \mathcal{M}^\circ$ is *infinitesimally rigid* if for every $n \geq 0$, any two extensions of χ to $U \otimes_k k[t]/(t^n)$ are isomorphic.

Since the diagonal of $\mathcal{M}^\circ$ is unramified, there is at most one isomorphism between two extensions of χ.

THEOREM 5.2. *Let U be a smooth variety. If $\mathcal{M}^\circ$ is a weakly bounded compactifiable Deligne-Mumford stack then the set of isomorphism classes of infinitesimally rigid morphisms $U \to \mathcal{M}^\circ$ is finite. Moreover, the number of isomorphism classes is bounded in a manner which is uniform in any finite type family of bases U.*

In the standard terminology, this theorem says that "infinitesimal rigidity implies rigidity." For applications of this result to families of canonically polarized manifolds; see Section 6.

We start with two lemmas.

LEMMA 5.3. *Let S be a reduced locally Noetherian scheme, $\pi : Z \to S$ and $P \to S$ two S-schemes of finite type with P separated over S. Further let $V \subset Z$ be an open subscheme that is dense in every fiber Z_s. Assume that $V \to S$ has a section and the geometric fibers of π are reduced. Then*

(5.3.1) *any S-morphism $\zeta : Z \to P$ such that the restriction of ζ to each geometric fiber V_s is constant factors through a section $S \to P$;*

(5.3.2) *if in addition S is of finite type over a field, it is sufficient for ζ to be constant on geometric fibers over closed points of S.*

Proof. The statement is local on S, so we may assume that S is Noetherian. Since $V \to S$ has a section, it is a universal effective epimorphism [SGA3, IV.1.12]. Since S is reduced and Noetherian, it has a dense subscheme consisting of finitely many reduced points $t_1, \ldots, t_n$ (the generic points of the irreducible components of S). Extending the residue field of t_i is also a universal effective epimorphism, so if ζ is constant on the geometric fiber of V over t_i, it must be constant on $V \otimes \kappa(t_i)$ for each i. Write p and q for the two projections $V \times_S V \to V$. In the exact diagram

$$\mathrm{Hom}(S, P) \to \mathrm{Hom}(V, P) \rightrightarrows \mathrm{Hom}(V \times_S V, P),$$

we have that the two compositions ζp and ζq agree on the fibers over each t_i. Since these fibers are dense in V and P is separated, the two maps agree on all of $V \times_S V$, whence there is a morphism $\gamma : S \to P$ such that $\zeta|_V = \gamma \circ \pi|_V$. Since V is everywhere dense in Z and P is separated, it follows that ζ factors through S, as required.

The second statement works precisely the same way, using the fact that for any closed set F containing all of the closed fibers of π we have $F = Z$. □

LEMMA 5.4. *Let R be a ring and $y \in R$ a regular element. Let $R[\varepsilon] := R[x]/(x^n)$. Let A be a finite $R[\varepsilon]$-algebra such that the natural maps $R \to A/\varepsilon A$ and $R[\varepsilon][1/y] \to A[1/y]$ are isomorphisms. Then $R[\varepsilon] \to A$ is an isomorphism.*

Proof. In the diagram

$$\begin{array}{ccc} R[\varepsilon] & \longrightarrow & A \\ {\scriptstyle \iota}\downarrow & & \downarrow \\ R[\varepsilon][1/y] & \xrightarrow[\rho]{} & A[1/y] \end{array}$$

the natural maps ι and ρ are injective by the hypotheses and hence $R[\varepsilon] \to A$ is injective as well. On the other hand, $R[\varepsilon] \to A$ is surjective modulo the nilpotent ε, which implies that $R[\varepsilon] \to A$ is itself surjective. □

Proof of Theorem 5.2. We already know from Theorem 1.7 that the set of deformation types of infinitesimally rigid morphisms $U \to \mathcal{M}^\circ$ is finite and of cardinality bounded above in a finite type family of bases U. To show finiteness of the set of infinitesimally rigid morphisms, it thus suffices to show the following: *if* (T,t) *is a pointed smooth connected curve over* k *and* $\Xi^\circ : U \times T \to \mathcal{M}^\circ$ *is a morphism whose restriction to* U_t *is infinitesimally rigid then there is a finite base change* $T' \to T$ *such that* $\Xi^\circ|_{T'} \cong \Xi_t^\circ \times \mathrm{id}_{T'}$. We will refer to this statement as (†). If (†) holds then any two deformation equivalent infinitesimally rigid morphisms are in fact isomorphic, as desired.

To show (†), we first note that by Lemma 5.3 with $U = S$, $V = Z = U \times T$, and $P = U \times M^\circ$, the coarse morphism $U \times T \to M^\circ$ factors through a morphism $\chi : U \to M^\circ$. Indeed, it suffices to show that for each closed point $u \in U$, the induced map $\widetilde{\Xi}_u^\circ : T_u \to M^\circ$ is constant. Since Ξ_t° is infinitesimally rigid, for every $n \geq 0$, the map $(\widetilde{\Xi}_u^\circ)|_{\operatorname{Spec} \mathbb{O}_{T_u,t}/\mathfrak{m}_t^n}$ factors through the natural map $\operatorname{Spec} \mathbb{O}_{T_u,t}/\mathfrak{m}_t^n \to \operatorname{Spec} \mathbb{O}_{T_u,t}/\mathfrak{m}_t$. It follows that the induced map $\operatorname{Spec} \widehat{\mathbb{O}}_{T_u,t} \to M^\circ$ factors through the section $t_u \to T_u$, so that $\widetilde{\Xi}_u^\circ$ satisfies the hypotheses of Lemma 5.3.

Choose a compactification of $U \subset B$ by a simple normal crossings divisor D over which there is an extension of χ to a morphism $B \to M$. By Proposition 3.12 (using the fact that T is regular), we can extend Ξ° to a morphism $\Xi : B\langle D^{1/N}\rangle \times T \to \mathcal{M}$, corresponding to a homomorphism $T \to \underline{\mathrm{Hom}}(B\langle D^{1/N}\rangle, \mathcal{M})$. (Note that $\underline{\mathrm{Hom}}(B\langle D^{1/N}\rangle, \mathcal{M})$ is a separated Deligne-Mumford stack by [AOV08, Th. C.4] and the fact that $B\langle D^{1/N}\rangle$ is smooth.) We claim that any morphism $B\langle D^{1/N}\rangle \to \mathcal{M}$ whose restriction to U is infinitesimally rigid is infinitesimally rigid.

Granting this claim, let us demonstrate that (†) follows. An infinitesimally rigid point $\xi : \operatorname{Spec} k \to \underline{\mathrm{Hom}}(B\langle D^{1/N}\rangle, \mathcal{M})$ has the property that any extension of ξ to $\operatorname{Spec} k[\varepsilon]$ is isomorphic to $\xi \times \operatorname{Spec} k[\varepsilon]$, and there is a unique such isomorphism extending the identity over the closed point $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]$. In particular, the miniversal deformation of ξ is isomorphic to $\operatorname{Spec} k$, showing that ξ is a smooth morphism. It follows that the residual gerbe of ξ is a connected component of $\underline{\mathrm{Hom}}(B\langle D^{1/N}\rangle, \mathcal{M})$. Applied to the morphism Ξ, this implies that $\Xi|_{\operatorname{Spec} \widehat{\mathbb{O}}_{T,t}}$ is isomorphic to $\Xi_t \times \mathrm{id}_{\operatorname{Spec} \widehat{\mathbb{O}}_{T,t}}$.

Now consider the finite scheme $I := \mathrm{Isom}_T(\Xi, \Xi_t \times \mathrm{id}_T) \to T$. By assumption, $I(\widehat{\mathbb{O}}_{T,t}) \neq \varnothing$. Applying Popescu's theorem (see, e.g., [Spi99]), the excellence of $\mathbb{O}_{T,t}$ implies that $\widehat{\mathbb{O}}_{T,t}$ is a filtering colimit of smooth $\mathbb{O}_{T,t}$-algebras, and since I is locally of finite presentation there is thus a smooth T-scheme $\widetilde{T} \to T$ such that $I(\widetilde{T}) \neq \varnothing$. Since any smooth T-scheme has étale-local sections around any point, we find a quasifinite generically étale morphism $T'' \to T$ whose image contains t such that $I(T'') \neq \varnothing$. Letting T' equal the normalization of T in the function field of T'', we find a finite morphism $T' \to T$ such that there is a generic isomorphism between $\Xi|'_T$ and $\Xi_t \times \mathrm{id}_{T'}$. Since $\mathrm{Isom}(\Xi|_{T'}, \Xi_t \times \mathrm{id}_{T'}) \to T'$ is finite and T' is a Dedekind scheme, any generic section extends to a global section by the valuative

criterion of properness. Restricting to $U \times T' \subset B\langle D^{1/N}\rangle \times_T T'$, we see that (†) is established.

Thus, it remains to show that if $\xi : B\langle D^{1/N}\rangle \to \mathcal{M}$ maps U to $\mathcal{M}^\circ$ and ξ_U is infinitesimally rigid then ξ itself is infinitesimally rigid. Let ξ_1 and ξ_2 be two infinitesimal deformations of ξ over $k[\varepsilon] := k[x]/(x^n)$. Consider the sheaf $I := \mathrm{Isom}_{B\langle D^{1/N}\rangle[\varepsilon]}(\xi_1, \xi_2)$ on the étale site of $B\langle D^{1/N}\rangle$. Since $\mathcal{M}$ is separated, $\pi : I \to B\langle D^{1/N}\rangle[\varepsilon]$ is a finite representable morphism of stacks. By the definition of infinitesimal rigidity, we have that there is a section $\sigma : U[\varepsilon] \to I$ of π.

Let $J \subset I$ be the stack-theoretic closure of $\sigma(U[\varepsilon])$, so that $J \to B\langle D^{1/N}\rangle[\varepsilon]$ is finite, representable, and an isomorphism over a dense open subscheme. We claim that $J \to B\langle D^{1/N}\rangle[\varepsilon]$ is an isomorphism. To show this, it suffices to work étale-locally on $B\langle D^{1/N}\rangle$. Indeed, since σ is a quasicompact morphism we have that J is defined by the quasicoherent kernel of the natural morphism $\mathcal{O}_I \to \sigma_* \mathcal{O}_{U[\varepsilon]}$ of quasicoherent sheaves. Since the formation of this kernel commutes with étale base change on $B\langle D^{1/n}\rangle$, we see that the formation of J commutes with étale base change on $B\langle D^{1/n}\rangle$.

Let R be an étale local ring of $B\langle D^{1/N}\rangle$, so that J is locally represented by an $R[\varepsilon]$-algebra A. Since R is regular and σ is defined over a dense open substack U, it follows that A satisfies the conditions of Lemma 5.4. Indeed, if $R \to A/\varepsilon A$ is not an isomorphism, then $A/\varepsilon A$ cannot be irreducible (as R is normal and $R \to A/\varepsilon A$ is finite and birational). But Spec A is irreducible, being the scheme-theoretic closure of an irreducible scheme. On the other hand, if f is any element of R vanishing on the complement of U, we see that Spec $A[1/f]$ is contained in the open subscheme $\sigma(U) \subset J$, which is isomorphic to U by definition; thus, $R[\varepsilon][1/f] \to A[1/f]$ is an isomorphism. We conclude that J is an isomorphism, and thus that $\xi_1 \simeq \xi_2$ via an isomorphism extending the given isomorphism over $U[\varepsilon]$, as required. □

5.5. It may seem that the condition of infinitesimal rigidity is unnatural, especially for families over nonproper base varieties U. For families over curves this is true (in fact, infinitesimal rigidity almost never holds for families of canonically polarized manifolds over an affine curve). However, for bases U such that the boundary divisor in a compactification B is nonample, there are many examples of infinitesimally rigid families. This is captured in the following proposition (which is far from optimal, but serves to illustrate the point).

PROPOSITION 5.6. *Let B be a proper smooth k-variety and $D \subset B$ a smooth irreducible divisor. Assume that $\xi : B \to \mathcal{M}$ is an infinitesimally rigid morphism to a Deligne-Mumford k-stack such that $\xi|_D$ is also infinitesimally rigid and that $\xi^*\Omega^1_{\mathcal{M}/k}$ is locally free in a neighborhood of D. If $\Gamma(D, \mathcal{O}(-D)|_D) \neq 0$ then $\xi|_U$ is infinitesimally rigid.*

Proof. It is a standard fact [Ill71, III.2.2.2] that the first-order infinitesimal deformations of ξ form a torsor under $\mathrm{Hom}(\xi^* \mathrm{L}_{\mathcal{M}/k}, \mathcal{O}_B)$, where $\mathrm{L}_{\mathcal{M}/k}$ is the cotangent complex of $\mathcal{M}$ over k. Since $L_{\mathcal{M}/k}$ is bounded above at 0 and $\mathscr{H}^0(L_{\mathcal{M}}) \cong \Omega^1_{\mathcal{M}/k}$,

this space is just $\operatorname{Hom}(\xi^*\Omega^1_{\mathcal{M}/k}, \mathcal{O}_B) = \Gamma(B, \mathcal{H})$, where $\mathcal{H} = \mathcal{H}om(\xi^*\Omega^1_{\mathcal{M}/k}, \mathcal{O}_B)$. We know that $\Gamma(B, \mathcal{H}) = 0 = \Gamma(D, \mathcal{H}|_D)$ and we wish to conclude that $\Gamma(U, \mathcal{H}) = 0$. Any section of $\mathcal{H}|_U$ extends to a section of $\mathcal{H}(nD)$ for some $n \in \mathbb{N}$, so it suffices to show that $\Gamma(B, \mathcal{H}(nD)) = 0$ for all $n \in \mathbb{N}$.

Since $\mathcal{H}|_D$ is locally free, any nonzero section of $\mathcal{O}(-D)|_D$ is $\mathcal{H}|_D$-regular. Thus, since $\Gamma(D, \mathcal{H}|_D) = 0$, it follows that $\Gamma(D, \mathcal{H}(nD)|_D) = 0$ for all $n \in \mathbb{N}$. Consider the sequence

$$0 \to \mathcal{H}((n-1)D) \to \mathcal{H}(nD) \to \mathcal{H}(nD)|_D \to 0.$$

Since $\Gamma(D, \mathcal{H}(nD)|_D) = 0$, it follows that $\Gamma(X, \mathcal{H}((n-1)D)) \to \Gamma(X, \mathcal{H}(nD))$ is an isomorphism for all $n \in \mathbb{N}$, and since $\Gamma(X, \mathcal{H}) = 0$, it follows that $\Gamma(X, \mathcal{H}(nD)) = 0$ for all $n \in \mathbb{N}$. □

An example of this phenomenon arises by considering families over $C \times C$, where C is a curve of high genus. If $D \subset C \times C$ is the diagonal, it follows from the adjunction formula that $\mathcal{O}(-D)|_D \simeq \Omega^1_D$, which is globally generated. If $X \to C$ and $Y \to C$ are two infinitesimally rigid families of smooth canonically polarized varieties (e.g., two nonisotrivial families of smooth curves) then the fiber product $X \times_C Y$ is also infinitesimally rigid. Similarly, by the Künneth formula, it is easy to see that the family $Z := X \times Y \to C \times C$ is infinitesimally rigid. Applying the proposition, it follows that the restricted family $Z|_{C \times C \setminus D}$ is infinitesimally rigid.

6. **Applications to canonically polarized varieties**

Write $\mathcal{M}^\circ_h$ for the (Deligne-Mumford) stack of canonically polarized manifolds with Hilbert polynomial h and M°_h for its coarse moduli space. If $\mathfrak{f} : \mathcal{X} \to \mathcal{M}^\circ_h$ is the universal family, then the invertible sheaf

$$\lambda^{(p)}_m := \left(\det \mathfrak{f}_* \omega^{\otimes m}_{\mathcal{X}/\mathcal{M}^\circ_h}\right)^p$$

is the pullback of an ample invertible sheaf on M°_h [Vie95, Th. 1.11].

We recall a well-known fact about $\mathcal{M}^\circ_h$. (A similar statement probably first appeared in a lecture of M. Artin [Kol90, 2.8].)

LEMMA 6.1. *The stack $\mathcal{M}^\circ_h$ is isomorphic to a separated stack of the form $[U/\operatorname{PGL}_r]$ (for $r = h(m)$ with m sufficiently large), where U is a quasiprojective k-scheme.*

Proof. By Matsusaka's Big Theorem ([Mat70, Th. 2], [Mat72, Th. 4.2]), there is a positive integer m such that for any canonically polarized manifold X with Hilbert polynomial h, the global sections of $\omega_X^{\otimes m}$ give a nondegenerate embedding into $\mathbb{P}^{h(m)-1}$. Let H be the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^{h(m)-1}$ with Hilbert polynomial h; it is well-known that H is projective. There is an open subscheme $V \subset H$ parametrizing closed subschemes which are smooth and geometrically connected. Let $\mathcal{X} \to V$ be the universal family with universal

embedding $\Upsilon : \mathscr{X} \hookrightarrow \mathbb{P}_V^{h(m)-1}$. Consider the invertible sheaf $\mathscr{L} := \Upsilon^*\mathcal{O}(1) \otimes (\omega_{\mathscr{X}/V}^{\otimes m})^\vee$. It follows from cohomology and base change that there is a closed subscheme $U \subset V$ parametrizing the locus over which $\mathscr{L}$ is isomorphic to the trivial invertible sheaf (see, e.g., the proof of [Mum70, Cor. 6 of §II.5]). It is easy to see that U is $\mathrm{PGL}_{h(m)}$-invariant, and it follows from standard methods that $\mathcal{M}_h^\circ \simeq [U/\mathrm{PGL}_{h(m)}]$.

That $\mathcal{M}_h^\circ$ is separated follows easily from the fact that any family of canonically polarized manifolds is its own relative canonical model. Indeed, using the valuative criterion of separatedness the question reduces to the following statement: if two families of canonically polarized manifolds are given over the same smooth curve such that they agree over an open set, then they agree everywhere. However, this follows from the fact that within a fixed birational class the relative canonical model over a fixed base is unique. To see this, let $f : X \to C$ be one of the families. Then the relative canonical model is $\mathrm{Proj}_C\left(\sum_{m\geq 0} f_*\omega_{X/C}^m\right) \to C$. Since $\omega_{X/C}$ is relatively ample, this is actually isomorphic to f. On the other hand, the sheaves $f_*\omega_X^m$ are birational invariants and since C is fixed, this means that so is $\mathrm{Proj}_C\left(\sum_{m\geq 0} f_*\omega_{X/C}^m\right) \to C$. □

LEMMA 6.2. *The stack $\mathcal{M}_h^\circ$ is weakly bounded and compactifiable.*

Proof. The compactifiability of $\mathcal{M}_h^\circ$ follows from Lemma 6.1 and Lemma 4.2. Weak boundedness is much more subtle. Given $m > 0$, Viehweg [Vie10, Th. 3] produced a projective compactification M_h of $(M_h^\circ)_{\mathrm{red}}$ and an invertible sheaf $\lambda_m^{(p)} \in \mathrm{Pic}(M_h)$, nef and ample with respect to $(M_h^\circ)_{\mathrm{red}}$, such that for any morphism $\xi : C \to M_h$ induced by a semistable family $f : X \to C$, we have that $\xi^*\lambda_m^{(p)} = \det(f_*\omega_{X/C}^m)^p$.

We claim that $\mathcal{M}_h^\circ$ is weakly bounded with respect to M_h and $\lambda_m^{(p)}$, as in Definition 4.4. The proof is similar to the proof of Corollary 4.1 and Addendum 4.2 of [BV00]. Let $C^\circ \subset C$ be a (g,d)-curve and let $f^\circ : X^\circ \to C^\circ$ be a family of canonically polarized manifolds with Hilbert polynomial h. There exists a morphism of smooth projective varieties $f : X \to C$ including $f^\circ : X^\circ \to C^\circ$ as an "open subdiagram." By the semistable reduction theorem [KKMSD73, Chapter II], there is a finite morphism $\gamma : D \to C$ and a diagram

$$\begin{array}{ccccc} X & \longleftarrow & X_D \simeq X \times_C D & \longleftarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow & \swarrow_{f'} & \\ C & \xleftarrow[\gamma]{} & D & & \end{array}$$

with Y semistable over D and $Y \to X_D$ a resolution of singularities. By [Vie83, Lemma 3.2], there is an inclusion

$$f'_*\omega_{Y/D}^m \hookrightarrow \gamma^* f_*\omega_{X/C}^m.$$

Thus,

$$\deg\left(\det(f'_*\omega^m_{Y/D})\right) \leq (\deg\gamma)\deg\left(\det(f_*\omega^m_{X/C})\right).$$

The composed map $D \to M_h$ comes from a semistable family, so that (by the result of Viehweg quoted in the previous paragraph)

$$\deg(\gamma\circ\xi)^*\lambda^{(p)}_m = \deg\left(\det(f'_*\omega^m_{Y/D})^p\right).$$

It follows that

$$\deg\xi^*\lambda^{(p)}_m \leq \deg\left(\det(f_*\omega^m_{X/C})^p\right).$$

By [BV00, Th. 1.4(c)], for m sufficiently large and divisible the right-hand side of the last equation is bounded above by an explicit polynomial in g, d, n and some constants depending upon m (which are fixed once h is fixed). □

Theorem 1.6 is now an immediate corollary of Theorem 1.7.

Remark 6.3. At the time of this writing, the finite generation of the canonical ring has apparently just been proven [BCHM10]. It has been claimed that under the assumption of the minimal model program in dimension $\deg h + 1$ (in fact, one seemingly needs only the existence of relative canonical models), one knows that there are a compactification $\mathcal{M}^\circ_h \subset \mathcal{M}_h$ and an invertible sheaf $\mathcal{L}$ on $\mathcal{M}_h$ such that (1) $\mathcal{L}|_{(\mathcal{M}^\circ_h)_{\mathrm{red}}} \simeq \lambda^{(p)}_m$ for fixed sufficiently large and divisible m and p, and (2) $\mathcal{L}$ is the pullback of an invertible sheaf from the coarse moduli space M_h of $\mathcal{M}_h$. Using these results would give a more natural proof of 6.2. Unfortunately, at the present time a proper explanation of this implication is not in the literature, so we find it prudent to include an alternative proof.

Remark 6.4. Because of the terminology that has been used in studying this problem, it behooves us to point out that the powerful results of Viehweg and Zuo [VZ01], [VZ02], [VZ03], concerning the boundedness problem for families of varieties, fall short of addressing the entire question. In particular, without the use of stack-theoretic methods, the numerical boundedness results (usually referred to as "weak boundedness") are not enough in themselves to show constructibility of the locus of coarse moduli maps arising from families. It is only by combining the numerical results with a study of lifts of coarse maps into stacks that one can prove the concrete boundedness results of Theorem 1.6 and Theorem 1.7. This fact is implicit in the work of Caporaso [Cap02], but rather than lifting to the stack, she lifted to a level cover of the stack of curves. This allowed her to avoid the use of stack-theoretic constructions but limited the argument to handle only families of curves.

COROLLARY 6.5 (cf. [Cap02], [Hei04] for families of curves). *There exists a function* $\mathsf{b}_h(g,d)$ *such that for any* d*-pointed smooth projective curve of genus* g, $(C, p_1, \ldots, p_d)$, *the number of deformation types of families of canonically polarized manifolds* X *over* $C \setminus \{p_1, \ldots, p_d\}$ *with Hilbert polynomial* h *is bounded above by* $\mathsf{b}_h(g,d)$.

Proof. This is an application of Corollary 4.7. □

Acknowledgments. Our work started after an inspiring conversation with Eckart Viehweg at the AIM workshop mentioned above. We have also benefited from discussions with Brian Conrad, Brendan Hassett, János Kollár, Martin Olsson, and Paul Smith. We would like to thank them all for their helpful comments.

We are grateful to AIM for catalyzing our research collaboration and would like to thank the referees for several lists of useful comments, corrections, and suggestions.

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(Received December 7, 2006)
(Revised May 11, 2009)

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ISSN 0003-486X
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This periodical is published bimonthly by the Department of Mathematics at Princeton University with the cooperation of the Institute for Advanced Study. Annals is typeset in TEX by Sarah R. Warren and produced by Mathematical Sciences Publishers. The six numbers each year are divided into two volumes of three numbers each.

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Printed in U.S.A. by Sheridan Printing Company, Inc., Alpha, NJ

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