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Abstract

Let $\mathrm{Sym}_3\,\mathbf{C} \longrightarrow \mathbf{P}_*(k \oplus \mathrm{Sym}_3\,k \oplus \mathrm{Sym}_3\,k \oplus k) = \mathbf{P}^{13},\ A \mapsto (1 : A : A' : \det A)$ be the Veronese embedding of the space of symmetric matrices of degree 3, where A' is the cofactor matrix of A. The closure $\mathrm{SpG}(3,6)$ of this image is a 6-dimensional homogeneous variety of the symplectic group $\mathrm{Sp}(3)$. A canonical curve $C_{16} \subset \mathbf{P}^8$ of genus 9 over a perfect field k is isomorphic to a complete linear section of this projective variety $\mathrm{SpG}(3,6) \subset \mathbf{P}^{13}$ unless $C \otimes_k \bar{k}$, $\bar{k}$ being the algebraic closure, is a covering of degree at most 5 of the projective line. We prove this by means of linear systems of higher rank.

Introduction

Let $\mathrm{SpG}(n, 2n)$ be the *symplectic Grassmannian*, that is, the Grassmannian of Lagrangian subspaces of a $2n$-dimensional symplectic vector space, over a field k. In the case $n = 3$, $\mathrm{SpG}(3,6)$ is of dimension 6 and embedded into the projective space $\mathbf{P}^{13}$ with homogeneous coordinate $(y : X : Y : x)$, where $x, y \in k$ are scalars and $X, Y \in \mathrm{Sym}_3\,k$ are symmetric matrices. Then $\mathrm{SpG}(3,6) \subset \mathbf{P}^{13}$ is the common zero locus of the following 21 (=6+6+9) quadratic equations

$$X' = yY, \quad Y' = xX \in \mathrm{Sym}_3\,k \quad \text{and} \quad XY = xyI_3 \in \mathrm{Mat}_3\,k, \tag{0.1}$$

which will be derived in Section 2 after Proposition 2.3.

In our study of Fano 3-folds, we observed that this (symmetric) projective variety has a *canonical curve section* of genus 9, that is, a transversal intersection

$$[C \subset \mathbf{P}^8] = [\mathrm{SpG}(3,6) \subset \mathbf{P}^{13}] \cap H_1 \cap \cdots \cap H_5$$

is a curve of genus 9 embedded in $\mathbf{P}^8$ by the ratio of the differentials of the first kind. We showed that every general curve of genus 9 is obtained in this way when $k = \mathbf{C}$ ([10, Cor. 6.3]). The purpose of this article is to show the following refinement, which was partly announced in [11].

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THEOREM A. *Let C be a curve of genus* 9 *over an algebraically closed field k. Then C is isomorphic to a transversal linear section of the* 6*-dimensional symplectic Grassmannian* $\mathrm{SpG}(3,6) \subset \mathbf{P}^{13}$ *if and only if C is not pentagonal*, i.e., *C has no g^1_5.*

By Bertini's theorem we have

COROLLARY. *If C satisfies the condition given in Theorem* A *and if k is of characteristic zero*, *then C is contained in a smooth* K3 *surface as an ample divisor.*

This theorem, together with similar results [14] and [13] in genus 7 and 8, will be applied to our classification of Gorenstein-Fano 3-folds with only canonical singularities (cf. [15]).

We prove the theorem using a certain simple vector bundle of rank 3. By its uniqueness (see below) and by a standard descent argument (§7), we have the following also:

THEOREM B. *Let C be a curve of genus* 9 *defined over a perfect field k and assume that C has no g^1_5 over the algebraic closure $\bar{k}$. Then we have*

(1) *C has an embedding into the* 6*-dimensional symplectic Grassmannian* $\mathrm{SpG}(3,6) \subset \mathbf{P}^{13}$ *over k whose image is a transversal intersection with a k-linear subspace $P \subset \mathbf{P}^{13}$ of dimension* 8, *and*

(2) *such subspaces P cutting out C are unique up to the action of* $\mathrm{PGSp}(3)$. *More precisely*, *for every isomorphism $g : C = \mathrm{SpG}(3,6) \cap P \to C' = \mathrm{SpG}(3,6) \cap P'$ there exists $\gamma \in \mathrm{PGSp}(3,k)$ such that $\gamma(P) = P'$.*

Here $\mathrm{PGSp}(3)$ is the subgroup of $\mathrm{PGL}(6)$ stabilizing the 1-dimensional space generated by a symplectic form. Let $\mathrm{G}(8,\mathbf{P}^{13})$ be the Grassmannian of 8-dimensional linear subspaces P of $\mathbf{P}^{13}$ and $\mathrm{G}(8,\mathbf{P}^{13})^t$ the open subset consisting of P's such that the intersection $P \cap \mathrm{SpG}(3,6)$ is transversal.

COROLLARY. *The weighted cardinality*, *or mass*, *of the nonpentagonal curves C of genus* 9 *over the finite field $\mathbf{F}_q$ is equal to* $\#\mathrm{G}(8,\mathbf{P}^{13})^t / \#\mathrm{PGSp}(3,\mathbf{F}_q)$:

$$\sum_{\text{nonpentagonal}} \frac{1}{\#\mathrm{Aut}_{\mathbf{F}} C} = \frac{\#\mathrm{G}(8,\mathbf{P}^{13})^t(\mathbf{F}_q)}{q^9(q^6-1)(q^4-1)(q^2-1)}.$$

The key to the proof is linear systems of higher rank (§3), especially their semi-irreducibility (Definition 3.3). Let C be as in Theorem A and α a g^2_8 of C, which exists by Brill-Noether theory (cf. [1, Chap. 7]). Let β be the Serre adjoint $K_C\alpha^{-1}$ and Q_β the dual of the kernel of the evaluation homomorphism $\mathcal{O}_C^{\oplus 3} \longrightarrow \beta$. Then there exists a unique nontrivial extension of α by Q_β with $h^0(E) = 6$ (Lemma 5.2 and 5.4). Moreover, such an extension E, often denoted by $E_{\max}$, does not depend on the choice of α and is characterized by the following property (Proposition 5.6):

$$(0.2)\qquad \begin{cases} \text{i)} & \bigwedge^3 E \simeq K_C, \\ \text{ii)} & h^0(E) = 6, \text{ and} \\ \text{iii)} & |E| \text{ is free and semi-irreducible.} \end{cases}$$

Such a bundle E gives rise to a morphism $\Phi_E : C \longrightarrow \mathrm{G}(H^0(E_{\max}), 3)$ to the Grassmannian of 3-dimensional quotient spaces of $H^0(E_{\max})$ (§3). The following is the essence of Theorems A and B:

THEOREM C. *Let C be a nonhyperelliptic curve of genus 9 over an algebraically closed field and assume that a rank 3 vector bundle $E = E_{\max}$ on it satisfies the condition* (0.2). *Then the natural linear maps*

$$\lambda_2 : \bigwedge^2 H^0(E) \longrightarrow H^0\Big(\bigwedge^2 E\Big) \quad\text{and}\quad \lambda_3 : \bigwedge^3 H^0(E) \longrightarrow H^0\Big(\bigwedge^3 E\Big) \simeq H^0(K_C)$$

are surjective and $\operatorname{Ker}\lambda_2$ *is generated by a nondegenerate bivector σ. The image of Φ_E is contained in the symplectic Grassmannian* $\mathrm{G}(H^0(E), \sigma)$ (*see* §2) *and the commutative diagram*

$$(0.3)\qquad \begin{array}{ccccc} & C & \longrightarrow & \mathrm{G}(H^0(E), \sigma) & \\ \textit{canonical} & \downarrow & & \downarrow & \textit{Plücker} \\ & \mathbf{P}^8 & \underset{\mathbf{P}^*\overline{\lambda}_3}{\longrightarrow} & \mathbf{P}^* \bigwedge^3 (H^0(E), \sigma) & \end{array}$$

is cartesian, where $\overline{\lambda}_3$ is the linear map

$$(0.4)\quad \bigwedge^3 (H^0(E), \sigma) := \bigwedge^3 H^0(E)/(\sigma \wedge H^0(E)) \longrightarrow H^0\Big(\bigwedge^3 E\Big) \simeq H^0(K_C)$$

induced by λ_3.

Notation and conventions. For a vector space V, the second exterior product $\bigwedge^2 V$ is the quotient of $V \otimes V$ by the subspace generated by $v \otimes v$, $v \in V$. Similarly $S^2 V$ is the quotient generated by $u \otimes v - v \otimes u$, $u, v \in V$. An element of $\bigwedge^2 V$ is called a *bivector* of V. We denote by $\mathrm{G}(r, V)$ and $\mathrm{G}(V, r)$ the Grassmannians of r-dimensional subspaces and quotient spaces of V, respectively. Two projective spaces $\mathrm{G}(1, V)$ and $\mathrm{G}(V, 1)$ associated to V are denoted by $\mathbf{P}_*(V)$ and $\mathbf{P}^*(V)$, respectively. $\mathbf{P}_*$ is a covariant functor and $\mathbf{P}^*$ is contravariant. For a vector space or vector bundle V, its dual is denoted by $V^\vee$. The tensor product symbol $\otimes$ between a vector bundle and a line bundle is often omitted when there seems no fear of confusion.

All (algebraic) varieties are considered over a fixed base field k. A smooth complete geometrically irreducible curve is simply called a *curve*. By a g^r_d, we mean a line bundle L on a curve with $\deg L = d$ and $\dim H^0(L) \geq r + 1$. A *saturation* of a subsheaf $F \subset E$ is the largest subsheaf $\widetilde{F}$ between F and E such that $\widetilde{F}/F$ is torsion.

1. Preliminaries

We prove two lemmas on the number of global sections. Let ξ be a line bundle on a curve C and η the Serre adjoint $K_C\xi^{-1}$. We denote the evaluation homomorphism $H^0(\eta) \otimes_k \mathcal{O}_C \longrightarrow \eta$ by ev_η and the dual of its kernel by Q_η. We have an exact sequence

$$0 \longrightarrow Q_\eta^\vee \longrightarrow H^0(\eta) \otimes_k \mathcal{O}_C \longrightarrow \eta. \tag{1.1}$$

Its dual

$$0 \longrightarrow \eta^{-1} \longrightarrow H^0(\eta)^\vee \otimes_k \mathcal{O}_C \longrightarrow Q_\eta \longrightarrow 0 \tag{1.2}$$

is also exact if η is free. The rank of Q_η is equal to $\dim|\eta| = r-1$, where we put $r = h^0(\eta)$. The following is a variant of the so-called base point free pencil trick.

LEMMA 1.1. *For a vector bundle E of rank r on C,*

$$\dim \mathrm{Hom}\,(E, \xi) + \dim \mathrm{Hom}\,(Q_\eta, E) \geq r(h^0(E) - \deg \eta) - \chi(E).$$

Proof. Take the global section of the exact sequence (1.1) tensored with E. Then we have

$$\dim \mathrm{Hom}\,(Q_\eta, E) + h^0(E\eta) \geq r h^0(E).$$

By the Riemann-Roch theorem and Serre duality,

$$h^0(E\eta) - h^0(E^\vee\xi) = \chi(E\eta) = \chi(E) + r \deg \eta.$$

Our assertion follows immediately from these. □

If E is of canonical determinant, i.e., $\bigwedge^r E \simeq K_C$, then

$$\dim \mathrm{Hom}\,(E, \xi) + \dim \mathrm{Hom}\,(Q_\eta, E) \geq r(h^0(E) - r - s) - 2\rho + 2, \tag{1.3}$$

since $\chi(E) = (r-2)(1-g)$, where $s = h^0(\xi) = h^1(\eta)$ and $\rho := g - rs$ is the Brill-Noether number of η, or equivalently, of ξ.

The number of global sections behaves specially if a vector bundle has a nondegenerate quadratic form with values in K_C. The following is one of such phenomena clarified in Mumford [16].

PROPOSITION 1.2. *Let E and F be rank two vector bundles on a curve C such that* $(\det E) \otimes (\det F) \simeq K_C$. *Then $h^0(E \otimes F)$ is congruent to* $\deg E$ *modulo* 2.

Proof. Choose a line subbundle and express F as an extension

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0 \tag{1.4}$$

of line bundles. The alternating bihomomorphism $E \times E \to \det E$, $(s, t) \mapsto s \wedge t$, induces a bilinear map

$$\varphi : H^0(E \otimes M) \times H^1(E \otimes L) \longrightarrow H^0((\det E) \otimes (\det F)) = H^0(K_C) \simeq k,$$

which is nondegenerate by Serre duality. Let $e \in H^1(M^{-1} \otimes L)$ be the extension class of (1.4) and $\delta : H^0(E \otimes M) \longrightarrow H^1(E \otimes L)$ be the coboundary map coming from $E\otimes$ (1.4). Then $\varphi(s,\delta(s)) = s\cup(s\cup e) = (s\wedge s)\cup e = 0$ for $s\in H^0(E\otimes M)$. Therefore, the linear map δ is alternating with respect to the Serre pairing φ. Hence $h^0(E\otimes F)$ is congruent to

$$h^0(E\otimes L) + h^0(E\otimes M) = h^0(E\otimes L) + h^1(E\otimes L)$$

modulo 2. Since $h^0(E\otimes L) - h^1(E\otimes L)$ is congruent to $\deg(E\otimes L)$, we have our assertion. $\square$

2. **Symplectic Grassmannian**

Let A be a k-vector space. For a subspace $B\subset A$ the linear map $\bigwedge^2 B \to \bigwedge^2 A$ is injective.

Definition 2.1. A bivector $\sigma\in\bigwedge^2 A$ is *degenerate* if σ is contained in $\bigwedge^2 B$ for a proper subspace $B\subset A$.

A bivector σ is always degenerate if $\dim A$ is odd. In the case $\dim A$ is even, σ is degenerate if and only if the value of the Pfaffian is zero. There exists a minimal subspace $B\subset A$ such that $\sigma\in\bigwedge^2 B$. This subspace B is called the *co-radical* of σ.

Definition 2.2. A *symplectic vector space* is a pair (V,σ) of a vector space V and a nondegenerate bivector $\sigma\in\bigwedge^2 V^\vee$ of the dual vector space.

Note that $\bigwedge^2 V^\vee$ is the quotient of $V^\vee\otimes V^\vee$ by the subspace $\mathrm{SB}(V)$ of symmetric bilinear forms on V. When the characteristic of k is not 2, the equivalence class $\sigma+\mathrm{SB}(V)$ has the unique anti-symmetric representative, say σ^{AS}, in $V^\vee\otimes V^\vee$. A subspace $U\subset V$ is *Lagrangian* if $2\dim U = \dim V$ and the restriction $\sigma|_U : U\times U \longrightarrow k$ of σ to U is symmetric. If $\mathrm{char}(k)\neq 2$, then the second condition is equivalent to the usual one; that is, $\sigma^{\mathrm{AS}}|_U = 0$. We denote the set of Lagrangian subspaces of (V,σ) by $\mathrm{G}(\sigma, V)$.

Two vectors u and $v\in V$ are *perpendicular* with respect to σ if the restriction of σ to the subspace spanned by u and v is symmetric. For a nonzero vector $v\in V$, the set of vectors $u\in V$ perpendicular to v is a subspace of codimension one. We denote this subspace by $v^\perp$. σ induces a bilinear form $\bar\sigma$ on the quotient space $\bar V := v^\perp/kv$ and $(\bar V,\bar\sigma)$ becomes a symplectic vector space of dimension two less. If a Lagrangian subspace U of (V,σ) contains v, then the quotient U/kv is a Lagrangian of $(\bar V,\bar\sigma)$. Conversely, if $\bar U$ is a Lagrangian of $(\bar V,\bar\sigma)$, then its inverse image by $v^\perp\to\bar V$ is a Lagrangian of (V,σ) which contains v. By this correspondence we identify $\mathrm{G}(\bar\sigma,\bar V)$ with the subset of $\mathrm{G}(\sigma, V)$ consisting of $[U]$ with $v\in U$.

For our purpose, the Grassmannian of quotient spaces is more convenient than that of subspaces. A quotient space $A\xrightarrow{f} Q$ of A is *Lagrangian* with respect to a nondegenerate bivector σ if $2\dim W = \dim A$ and if $(\bigwedge^2 f)(\sigma) = 0$. We denote the set of Lagrangian quotient spaces of the pair (A,σ) by $\mathrm{G}(A,\sigma)$, which coincides with $\mathrm{G}(\sigma, A^\vee)$. Let $\mathcal{U}$ be the universal quotient bundle on $\mathrm{G}(A,n)$, $\dim A = 2n$. Then $\sigma \in \bigwedge^2 A$ determines a global section of $\bigwedge^2 \mathcal{U}$, which we denote by s. Then $\mathrm{G}(A,\sigma)$ coincides with the zero set of $s \in H^0(\mathrm{G}(A,n), \bigwedge^2 \mathcal{U})$. We endow $\mathrm{G}(A,\sigma)$ with a scheme structure by considering it as the zero locus of s. An element of this isomorphism class is denoted by $\mathrm{SpG}(n, 2n)$.

PROPOSITION 2.3. *The symplectic Grassmannian* $\mathrm{G}(A,\sigma)$ *is a smooth variety of dimension* $n(n+1)/2$ *and the anti-canonical class is* $n+1$ *times the the hyperplane section* H *of the Plücker embedding.*

Proof. Since $\bigwedge^2 A$ generates $\bigwedge^2 \mathcal{U}$, $\mathrm{G}(A,\tilde{\sigma})$ is locally a smooth complete intersection for general $\tilde{\sigma}$ by the Bertini theorem for vector bundles ([12, Th. 1.10]). Since the $\mathrm{GL}(2n)$-orbit of nondegenerate bivectors is dense in $\bigwedge^2 A$, $\mathrm{G}(A,\sigma)$ is isomorphic to $\mathrm{G}(A,\tilde{\sigma})$. It is of dimension $n^2 - \operatorname{rank} \bigwedge^2 \mathcal{U} = n(n+1)/2$. It is irreducible since the symplectic group $\mathrm{Sp}(n)$ acts transitively. The conormal bundle $\mathcal{I}/\mathcal{I}^2$ of $\mathrm{G}(A,\sigma)$ is the restriction of $(\bigwedge^2 \mathcal{U})^\vee$, where $\mathcal{I}$ is the ideal sheaf. ($\mathcal{I}$ is the image of $(\bigwedge^2 \mathcal{U})^\vee \to \mathcal{O}_{\mathrm{G}(A,n)}$ and $[(\bigwedge^2 \mathcal{U})^\vee \to \mathcal{I}] \otimes \mathcal{O}_{\mathrm{G}(A,\sigma)}$ is an isomorphism.) Since $c_1(\mathrm{G}(A,n)) = 2nH$ and $c_1(\bigwedge^2 \mathcal{U}) = (n-1)H$, the anti-canonical class of $\mathrm{G}(A,\sigma)$ is equal to the restriction of $c_1(\mathrm{G}(A,n)) - c_1(\bigwedge^2 \mathcal{U}) = (n+1)H$. $\square$

Choose a pair of Lagrangian subspaces U_0 and U_∞ of a symplectic vector space (V,σ) with $U_0 \cap U_\infty = 0$. For a linear map $f : U_0 \to U_\infty$ the graph $\Gamma_f \subset U_0 \times U_\infty = V$ is Lagrangian if and only if $f \in \operatorname{Hom}(U_0, U_\infty) \simeq U_\infty \otimes U_\infty$ is a symmetric tensor. The Plücker coordinate of Γ_f is equal to

$$1 + f + (f \wedge f) + (f \wedge f \wedge f) + \cdots$$

(cf. [14, §1]). Hence, for example, the 9-dimensional Grassmannian $\mathrm{G}(3,6)$ is the closure of the *Veronese embedding* of the space of square matrices of degree 3,

$$\mathrm{Mat}_3\, \mathbf{C} \longrightarrow \mathbf{P}_*(k \oplus \mathrm{Mat}_3\, k \oplus \mathrm{Mat}_3\, k \oplus k),\ A \mapsto (1 : A : A' : \det A),$$

where A' is the cofactor matrix of A. It is the common zero locus of the Plücker equations

$$X' = yY, \quad Y' = xX \in \mathrm{Mat}_3\, k \quad \text{and} \quad XY = YX = xyI_3 \in \mathrm{Mat}_3\, k,$$

in the projective space $\mathbf{P}^{19}$ with homogeneous coordinate $(y : X : Y : x)$, where $x, y \in k$ are scalars and $X, Y \in \mathrm{Mat}_3\, k$ are square matrices. Restricting ourselves to symmetric matrices, we have the equations (0.1) of $\mathrm{SpG}(3,6) \subset \mathbf{P}^{13}$.

The divisor class group of the Grassmannian $\mathrm{G}(n,2n)$ is generated by the hyperplane section class H. Its Chow group of codimension 2 cycles is generated by two Schubert subvarieties:

$$(2.1) \qquad Y = \{[U] \mid U \cap W \neq 0\} \quad \text{and} \quad Y' = \{[U] \mid U + W' \neq V\}$$

for a subspace W of dimension $n-1$ and W' of codimension $n-1$. It is well known that the self intersection $H \cdot H$ is (rationally) equivalent to their sum. On the symplectic Grassmannian, obviously Y and Y' are equivalent and hence we have

$$H \cdot H \sim Y + Y' \sim 2Y. \tag{2.2}$$

Let a be a nonzero vector of A. The image $\overline{\sigma}$ of σ in $\bigwedge^2(A/ka)$ is degenerate since $\dim(A/ka)$ is odd. In fact, the co-radical $\overline{A}$ of $\overline{\sigma}$ is of codimension one. Similar to the inclusion $\mathrm{G}(\overline{\sigma}, \overline{V}) \hookrightarrow \mathrm{G}(\sigma, V)$, we have a natural inclusion $\mathrm{G}(\overline{A}, \overline{\sigma}) \hookrightarrow \mathrm{G}(A, \sigma)$. Moreover, $\mathrm{G}(\overline{A}, \overline{\sigma})$ is the scheme of zeros of the global section of $\mathscr{E} = \mathscr{U}|_{\mathrm{G}(A,\sigma)}$ corresponding to $a \in A$.

Let $\mathrm{G}(A, n) \subset \mathbf{P}^*(\bigwedge^n A)$ be the Plücker embedding of the Grassmannian $\mathrm{G}(A, n)$. The tautological line bundle $\mathscr{O}_{\mathrm{G}}(1)$ is isomorphic to $\bigwedge^n \mathscr{U}$. Since σ vanishes on $\mathrm{G}(A, \sigma)$, so do all the linear forms $\sigma \wedge (\bigwedge^{n-2} A) \subset \bigwedge^n A$. Let $\bigwedge^n(A, \sigma)$ be the quotient space of $\bigwedge^n A$ by the subspace $\sigma \wedge (\bigwedge^{n-2} A)$. Then $\mathrm{G}(A, \sigma)$ is contained in the subspace $\mathbf{P}^*(\bigwedge^n(A, \sigma))$ and we have a commutative diagram

$$\begin{array}{ccc} \mathrm{G}(A,\sigma) & \longrightarrow & \mathbf{P}^*(\bigwedge^n(A,\sigma)) \\ \cap & & \cap \\ \mathrm{G}(A,n) & \underset{\text{Plücker}}{\longrightarrow} & \mathbf{P}^*(\bigwedge^n A). \end{array} \tag{2.3}$$

$\mathrm{G}(A, \sigma)$ coincides with $\mathrm{G}(A, 1) = \mathbf{P}^1$ for $n = 1$ and is a smooth hyperplane section of the smooth 4-dimensional quadric $\mathrm{G}(A, 2) \subset \mathbf{P}^5$ for $n = 2$.

Now we set $n = 3$ and investigate the conormal space of $\mathrm{G}(A, \sigma) \subset \mathbf{P}^* \bigwedge^3(A, \sigma)$ and an important cubic cone in it. Let $A \to Q$ be a 3-dimensional quotient space and put $W = \mathrm{Ker}\,[A \to Q]$. Then we have a filtration of subspaces

$$F_0 = \bigwedge^3 W \subset F_1 = \Big(\bigwedge^2 W\Big) \wedge A \subset F_2 = W \wedge \bigwedge^2 A \subset F_3 = \bigwedge^3 A. \tag{2.4}$$

Then $\bigwedge^3 A \to F_3/F_2 \simeq \bigwedge^3 Q$ is the Plücker coordinate of Q. F_2/F_1 is isomorphic to $W \otimes (\bigwedge^2 Q)$. $(F_2/F_1) \otimes \det Q^{-1} \simeq \mathrm{Hom}\,(Q, W)$ is canonically isomorphic to the cotangent space of $\mathrm{G}(A, 3)$ at $[Q]$. $F_1 \otimes \det Q^{-1}$ is canonically isomorphic to the conormal space of $\mathrm{G}(A, 3) \subset \mathbf{P}^* \bigwedge^3 A$. Hence we have an exact sequence

$$\begin{array}{c} 0 \longrightarrow k \longrightarrow F_1 \otimes \det W^{-1} \longrightarrow \mathrm{Hom}\,(W, Q) \longrightarrow 0. \\ \| \\ N^{\vee}_{\mathrm{G}(A,3)/\mathbf{P}} \otimes \det Q \otimes \det W^{-1} \end{array}$$

Assume that $[A \to Q] \in \mathrm{G}(A, \sigma)$ is Lagrangian. Then σ belongs to $W \wedge A \subset \bigwedge^2 A$. Let

$$\overline{F}_0 \subset \overline{F}_1 \subset \overline{F}_2 \subset \overline{F}_3, \qquad \overline{F}_i = F_i/(F_i \cap \sigma \wedge A),$$

be the quotient filtration of (2.4) by $\sigma \wedge A \subset F_2$. Then $\overline{F}_3/\overline{F}_2 \simeq \bigwedge^3 Q$ is the Plücker coordinate of Q. The cotangent space of $\mathrm{G}(3, \sigma)$ at $[Q]$ is $\overline{F}_2/\overline{F}_1 \otimes \det Q^{-1} \simeq S^2 W$. The conormal space is isomorphic to $\overline{F}_1 \otimes \det Q$ and we have an exact sequence

$$(2.5)\qquad \begin{array}{c} 0 \longrightarrow k \longrightarrow \bar{F}_1 \otimes \det Q \longrightarrow S^2 Q \longrightarrow 0. \\ \| \\ N^{\vee}_{\mathrm{G}(A,\sigma)/\mathbf{P}} \otimes (\det Q)^2 \end{array}$$

Let

$$(2.6)\qquad \alpha : \mathbf{P}_*\Big(\bigwedge^3 A\Big) \cdots\longrightarrow \mathbf{P}_*\Big(\bigwedge^3 (A,\sigma)\Big)$$

be the projection with center $\mathbf{P}_*(\sigma \wedge A)$. Since σ is nondegenerate, $\mathrm{G}(3, A)$ is disjoint from the center. We consider the image of the Schubert subvariety

$$S_Q = \{[U] \mid \operatorname{rk}[U \to A \to Q] \le 1\} \subset \mathrm{G}(3, A)$$

by α for a Lagrangian quotient space $A \to Q$ (cf. (3.3) and (4.1)). S_Q is a 5-dimensional subvariety of

$$\mathbf{P}_*\Big(\Big(\bigwedge^2 W\Big) \wedge A\Big) = \mathbf{P}_*(N^{\vee}_{\mathrm{G}(A,3)/\mathbf{P},Q})$$

and $\alpha(S_Q)$ is a subvariety of

$$\mathbf{P}_*(\bar{F}_1) = \mathbf{P}_*(N^{\vee}_{\mathrm{G}(A,\sigma)/\mathbf{P},Q}) = \mathbf{P}^6.$$

By the exact sequence (2.5), $\mathbf{P}^*(N_{\mathrm{G}(A,\sigma)/\mathbf{P},[Q]})$ has the distinguished point corresponding to $\operatorname{Ker}[A \to Q]$, which we denote by κ_Q, and the special projection onto $\mathbf{P}_*(S^2 Q)$. $\alpha(S_Q)$ contains the point κ_Q.

PROPOSITION 2.4. *The image $\alpha(S_Q)$ is a cubic hypersurface of*

$$\mathbf{P}^*(N_{\mathrm{G}(A,\sigma)/\mathbf{P},[Q]}).$$

More precisely, it is the cone over the discriminant hypersurface of $\mathbf{P}_*(S^2 Q)$ with vertex κ_Q.

Proof. Choose a basis $\{v_1, v_2, v_3, v_{-1}, v_{-2}, v_{-3}\}$ of A such that $\{v_1, v_2, v_3\}$ is a basis of $\operatorname{Ker}[A \to Q]$ and $\sigma = v_1 \wedge v_{-1} + v_2 \wedge v_{-2} + v_3 \wedge v_{-3}$. Let $\{u_1, u_2, u_3\}$ be a basis of $U \in S_Q$ such that $u_1, u_2 \in \operatorname{Ker}[U \to Q]$. The exterior product $u_1 \wedge u_2$ is equal to

$$a_1 v_2 \wedge v_3 + a_2 v_3 \wedge v_1 + a_3 v_1 \wedge v_2 \in \bigwedge^2 \operatorname{Ker}[A \to Q]$$

for a_1, a_2 and $a_3 \in k$. Put $u_3 = a_4 v_1 + a_5 v_2 + a_6 v_3 + b_1 v_{-1} + b_2 v_{-2} + b_3 v_{-3}$. Then the Plücker coordinate $u_1 \wedge u_2 \wedge u_3$ of U is

$$\begin{aligned} &a_0 v_1 \wedge v_2 \wedge v_3 + (a_1 v_2 \wedge v_3 + a_2 v_3 \wedge v_1 + a_3 v_1 \wedge v_2) \wedge (b_1 v_{-1} + b_2 v_{-2} + b_3 v_{-3}) \\ &\quad = a_0 v_1 \wedge v_2 \wedge v_3 + (a_2 b_1 v_{12} - a_1 b_2 v_{21}) + (a_1 b_3 v_{31} - a_3 b_1 v_{13}) \\ &\qquad\qquad + (a_3 b_2 v_{23} - a_2 b_3 v_{32}) + \sum_{i=1}^{3} a_i b_i v_{ii}, \end{aligned}$$

where we put $a_0 = a_1 a_4 + a_2 a_5 + a_3 a_6$,

$$v_{11} = v_{-1} \wedge v_2 \wedge v_3, \quad v_{22} = v_1 \wedge v_{-2} \wedge v_3, \quad v_{33} = v_1 \wedge v_2 \wedge v_{-3}$$

and $v_{jk} = v_i \wedge v_j \wedge v_{-j}$ for every $\{i, j, k\} = \{1, 2, 3\}$. Since $v_{jk} + v_{kj} \in A \wedge \sigma$ for every $j \neq k$, $u_1 \wedge u_2 \wedge u_3$ is congruent to

$$a_0 v_1 \wedge v_2 \wedge v_3 - (a_1 b_2 + a_2 b_1) v_{12} \\ - (a_1 b_3 + a_3 b_1) v_{13} + (a_2 b_3 + a_3 b_2) v_{23} + \sum_{i=1}^{3} a_i b_i v_{ii}$$

modulo $A \wedge \sigma$. Hence $\alpha(S_Q)$ consists of those $\gamma_0 v_1 \wedge v_2 \wedge v_3 + \sum_{1 \leq i \leq j \leq 3} \gamma_{ij} v_{ij}$ such that the quadratic form $\sum_{1 \leq i \leq j \leq 3} \gamma_{ij} X_i X_j$ is of rank ≤ 2 (or equivalently, $4\gamma_{11}\gamma_{22}\gamma_{33} - \gamma_{11}\gamma_{23}^2 - \gamma_{22}\gamma_{13}^2 - \gamma_{33}\gamma_{12}^2 + \gamma_{12}\gamma_{13}\gamma_{23} = 0$). □

3. Linear systems of higher rank

A *linear system of rank* r is a pair (E, A) of a vector bundle E of rank r and a space of global sections $A \subset H^0(E)$. The special one with $A = H^0(E)$ is called a *complete* linear system and denoted by $|E|$. A linear system (E, A) on an algebraic variety C is *free* if the evaluation homomorphism $\mathrm{ev}_{E,A} : A \otimes_k \mathcal{O}_C \longrightarrow E$ is surjective. If this holds, we obtain a morphism $\Phi_{E,A}$ of C to the Grassmannian $\mathrm{G}(A, r)$ of r-dimensional quotient spaces. It is characterized by the property that $\Phi_{E,A}^*(\mathcal{U}, A) = (E, A)$, where $\mathcal{U}$ is the universal quotient bundle on $\mathrm{G}(A, r)$ and $\Phi_{|E|}$ is abbreviated to Φ_E.

Let

$$\bigwedge^m \mathrm{ev}_{E,A} : \bigwedge^m A \otimes_k \mathcal{O}_C \longrightarrow \bigwedge^m E$$

be the exterior product of the evaluation homomorphism $\mathrm{ev}_{E,A}$. It induces the linear map

$$\bigwedge^m A \longrightarrow H^0\Big(\bigwedge^m E\Big),$$

which we denote by λ_m. The image $\lambda_m(s_1 \wedge \cdots \wedge s_m)$ of a simple m-vector $s_1 \wedge \cdots \wedge s_m$ is zero if and only if m global sections $s_1, \ldots, s_m \in A \subset H^0(E)$ are linearly dependent at the generic point of C, that is, they generate a subsheaf of rank less than m. This linear map is most important when $m = r$. Assume that $\lambda_r : \bigwedge^r A \longrightarrow H^0(\det E)$ is surjective. Then the map

$$\Psi : \mathbf{P}^*(H^0(\det E)) \to \mathbf{P}^*\Big(\bigwedge^r A\Big) \tag{3.1}$$

induced by λ_r is a linear embedding and the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{\Phi_E} & \mathrm{G}(A, r) \\ \cap & & \cap \quad \text{Plücker} \\ \mathbf{P}^*(H^0(\det E)) & \xrightarrow{\Psi} & \mathbf{P}^*(\bigwedge^r A). \end{array} \tag{3.2}$$

Even when λ_r is not surjective, the above is still commutative though $\Psi = \mathbf{P}^*\lambda_r$ is only a rational map. The linear map λ_r is important in analyzing E itself also.

Now we assume that the base field k is algebraically closed (until the end of §6). The dual Grassmannian $\mathrm{G}(r, A) \subset \mathbf{P}_*(\bigwedge^r A)$ is also important for understanding (E, A).

Definition 3.1. A linear system (E, A) of rank r is *irreducible* if it satisfies the following equivalent conditions:

i) For every r-dimensional linear subspace U of A the image of $U \otimes_k \mathcal{O}_C \longrightarrow E$ is of rank r, and

ii) The kernel of the natural linear map $\lambda_r : \bigwedge^r A \longrightarrow H^0(C, \det E)$ contains no nonzero simple r-vectors; that is, $\mathrm{G}(r, A) \cap \mathbf{P}_*(\operatorname{Ker} \lambda_r) = \varnothing$.

The following is known as Castelnuovo's trick (cf. [2, Chap. 10]):

PROPOSITION 3.2. *If* $r(\dim A - r) \geq h^0(\det E)$, *then* (E, A) *is reducible,*

Proof. The left-hand side of the inequality is the dimension of $\mathrm{G}(r, A)$. The codimension of $\mathbf{P}_*(\operatorname{Ker} \lambda_r) \subset \mathbf{P}_*(\bigwedge^r H^0(E))$ is at most $h^0(\det E)$. Hence, if the inequality holds, then the intersection $\mathrm{G}(r, A) \cap \mathbf{P}_* \operatorname{Ker} \lambda_r$ is not empty. □

A line bundle is irreducible. But irreducibility seems a strong condition in general. Irreducible bundles of rank ≥ 2 will not appear in the sequel. Instead the following concept plays a crucial role in our proof.

Definition 3.3. A linear system (E, A) of rank r on a (smooth complete) curve C is *semi-irreducible* if the evaluation homomorphism $\mathrm{ev}_U : U \otimes_k \mathcal{O}_C \longrightarrow E$ is either injective or everywhere of rank $r-1$ for every r-dimensional subspace U of A.

For an r-dimensional quotient space $A \to Q$, we denote by S_Q the Schubert subvariety

$$\{[U] \mid \operatorname{rk}[U \to A \to Q] \leq r-2\} \subset \mathrm{G}(r, A) \tag{3.3}$$

associated to Q. Also, S_Q is contained in the projective space $\mathbf{P}_*((\bigwedge^2 W) \wedge (\bigwedge^{r-2} A))$, which is the projectivisation $\mathbf{P}_*(N^\vee_{\mathrm{G}(A,r)/\mathbf{P},[Q]})$ of the conormal space of $\mathrm{G}(A, r) \subset \mathbf{P}_*(\bigwedge^r A)$ at $[Q]$. The following is obvious:

LEMMA 3.4. (E, A) *is semi-irreducible if and only if* $S_{E_p} \cap \mathbf{P}_* \operatorname{Ker} \lambda_r = \varnothing$ *for every fiber* E_p *of* E, $p \in C$.

Now we restrict ourselves to complete linear systems for simplicity.

PROPOSITION 3.5. *Assume that a complete linear system* $|E|$ *of rank r is free and semi-irreducible.*

(1) *If* F *is a proper nonzero subbundle, then* $h^0(F) \leq r(F) + 1$, *where* $r(F)$ *is the rank of* F.

(2) *If* $h^0(E) \geq r+2$ *and if* F *is a subbundle of rank* $\leq r-2$, *then* $h^0(F) \leq r(F)$.

(3) *If* $h^0(E) \geq r+3$, *then* E *is simple*, i.e., $\operatorname{End} E = k$.

Proof. (1) Assume that F is of rank $r-1$ and $h^0(F) \geq r$. Then the evaluation homomorphism $B \otimes_k \mathcal{O}_C \to F$ is surjective for every r-dimensional subspace $B \subset H^0(F)$ by semi-irreducibility. Hence we have $h^0(F) \leq r$. The general case follows from this since, for every proper subbundle F, there exists a subsheaf $F' \subset E$ of rank $r-1$ which contains F and $h^0(F') \geq h^0(F) + r(F') - r(F)$.

(2) By the same reason as above, we may assume that F is of rank $r-2$. We prove $h^0(F) \neq r(F)+1$ by contradiction. Assume that $h^0(F) = r(F)+1$ and put $G = E/F$. We regard the quotient space $H^0(E)/H^0(F)$ as a subspace of $H^0(G)$. Since $\dim H^0(E)/H^0(F) \geq h^0(E) - (r-1) \geq 3$ and since G is of rank 2, there exists a global section $s \in H^0(E) \setminus H^0(F)$ such that $\bar{s} \in H^0(G)$ vanishes at a point on C. Then $H^0(F)$ and s do not generate a subsheaf of rank r or a subbundle of rank $r-1$, which contradicts the semi-irreducibility of $|E|$. Therefore, we have $h^0(F) \leq r(F)$ by (1).

(3) It suffices to show that every endomorphism $\phi : E \longrightarrow E$ is either zero or an isomorphism. Assume that ϕ is neither. Then both the kernel and the image are proper subsheaves and we have

$$h^0(E) \leq h^0(\operatorname{Ker}\phi) + h^0(\operatorname{Im}\phi) \leq r(\operatorname{Ker}\phi) + 1 + r(\operatorname{Im}\phi) + 1 = r+2$$

by (1), which is a contradiction. $\square$

The following is proved similarly.

LEMMA 3.6. *Assume that two complete linear systems $|E|$ and $|E'|$ are free, semi-irreducible and of the same rank r and assume further that $h^0(E) \geq r+3$. Then every nonzero homomorphism $E \to E'$ is injective.*

4. Linear sections of the symplectic Grassmannian

Throughout this section $C \subset \mathbf{P}^8$ is a transversal linear section $\mathrm{SpG}(3,6) \cap H_1 \cap \cdots \cap H_5$ of the 6-dimensional symplectic Grassmannian.

LEMMA 4.1. *C is of genus 9 and the restriction of tautological line bundle $\mathcal{O}(1)$ is isomorphic to the canonical bundle K_C of C.*

Proof. By Proposition 2.3 and by adjunction, we have $K_C \simeq \mathcal{O}_C(K_{\mathrm{SpG}} + H_1 + \cdots + H_5) \simeq \mathcal{O}_C(1)$. The Chern class of the universal quotient bundle $\mathcal{U}$ on $\mathrm{G}(3,6)$ is the sum $1 + \sigma_1 + \sigma_2 + \sigma_3$ of the special Schubert cycles ([8, Chap. 1]). By Pieri's formula, we have

$$\begin{aligned} 2g(C) - 2 = \deg[\mathrm{SpG}(3,6) \subset \mathbf{P}^{13}] &= \Big(c_3\Big(\bigwedge^2 \mathcal{U}\Big).c_1(\mathcal{U})^6\Big) \\ &= (\sigma_1\sigma_2 - \sigma_3.\sigma_1^6) = 21 - 5 = 16, \end{aligned}$$

since $\mathrm{SpG}(3,6)$ is the zero locus of a global section of $\bigwedge^2 \mathcal{U}$. Hence C is of genus 9. $\square$

Let $\mathrm{G}(A,\sigma)$, $\dim A = 6$, be a representative of $\mathrm{SpG}(3,6)$.

LEMMA 4.2. *The linear map* $\bigwedge^3(A,\sigma) \to H^0(K_C)$ *is surjective and its kernel is generated by the linear forms* $f_1,\dots,f_5 \in \bigwedge^3(A,\sigma)$ *defining the five hyperplanes* $H_1,\dots,H_5$.

Proof. Let X_i be the common zero locus of the first i linear forms $f_1,\dots,f_i$ for $1 \le i \le 5$. Then we obtain a ladder

$$C = X_5 \subset X_4 \subset X_3 \subset X_2 \subset X_1 \subset X_0 := \mathrm{G}(A,\sigma).$$

Since C is irreducible, so is each X_i. Hence the kernel of the restriction map $H^0(X_i, \mathcal{O}_X(1)) \longrightarrow H^0(X_{i+1}, \mathcal{O}_X(1))$ is generated by f_{i+1}, for every $1 \le i \le 4$. Hence $\bigwedge^3(A,\sigma)/\langle f_1,\dots,f_5\rangle \longrightarrow H^0(K_C)$ is injective. This map is also surjective because the source and the target have the same dimension. □

Let $\mathcal{E}$ be the restriction of $\mathcal{U}$ to $\mathrm{G}(A,\sigma)$ and E the restriction to C.

LEMMA 4.3. *The restriction map* $A \to H^0(E)$ *is injective.*

Proof. Assume the contrary. Then for each of the Lagrangian quotient spaces $A \to Q$ parametrized by C, $\mathrm{Ker}\,[A \to Q]$ contains a nonzero common vector a. Hence C is contained in the symplectic Grassmannian $\mathrm{G}(\bar{A},\bar{\sigma})$, where $\bar{A}$ is the co-radical of A/ka. This contradicts the preceding lemma since $\mathrm{G}(\bar{A},\bar{\sigma})$ lies in a 4-dimensional linear subspace. □

By this lemma we identify A with its image in $H^0(E)$.

LEMMA 4.4. (1) *A nonzero global section* $s \in A$ *of* E *has at most two zeros* (*counted with multiplicity*); *that is,* $A \cap H^0(E(-D)) = 0$ *for every effective divisor* D *of degree* 3 *on* C.

(2) *If* $A' \subset A$ *is a subspace of codimension one*, *then the cokernel of the evaluation homomorphism* $A' \otimes_k \mathcal{O}_C \longrightarrow E$ *is of length* ≤ 2.

Proof. Assume that s has at least three zeros. Then we have an exact sequence $E^\vee \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \longrightarrow 0$ for an effective divisor D of degree ≥ 3. Let $\mathrm{G}(\bar{A},\bar{\sigma}) \subset \mathrm{G}(A,\sigma)$ be the 3-dimensional symplectic Grassmannian determined by $s \in A$. Then the intersection $\mathrm{G}(\bar{A},\bar{\sigma}) \cap C$ contains D. Since $\mathrm{G}(\bar{A},\bar{\sigma})$ is a quadric, its intersection with the linear span $\langle D\rangle$ is of positive dimension, which is a contradiction. This shows (1). The proof of (2) is similar. □

Let $U \subset A$ be a 3-dimensional subspace and $H_U \subset \mathbf{P}^* \bigwedge^3 A$ the hyperplane corresponding to it. Then the intersection $H_U \cap \mathrm{G}(A,r)$ consists of the r-dimensional quotient spaces $A \to Q$ such that the composite $U \hookrightarrow A \to Q$ is not an isomorphism. It is singular along the Schubert subvariety

$$\{[A \to Q] \mid \mathrm{rank}\,[U \hookrightarrow A \to Q] \le 1\}. \tag{4.1}$$

If $H_U \not\supset C$, then the evaluation homomorphism $\mathrm{ev}_U : U \otimes \mathcal{O}_C \longrightarrow E$ is of rank 3 at the generic point. Hence it is injective. If $H_U \supset C$, then H_U belongs to $\langle [H_1],\dots,[H_5]\rangle$. Since the intersection $C = H_1 \cap \cdots \cap H_5 \cap \mathrm{G}(A,\sigma)$ is transversal, $H_U \cap \mathrm{G}(A,\sigma)$ must be smooth along C. Hence ev_U is of rank 2 everywhere. So we have proved the following, which indicates that the semi-irreducibility is a key concept for canonical curves of genus 9.

PROPOSITION 4.5. *The induced rank three linear system* (A, E) *on* $C = \mathrm{G}(A, \sigma) \cap H_1 \cap \cdots \cap H_5$ *is semi-irreducible.*

By Proposition 3.2, there exists a 3-dimensional subspace U of A such that $H_U \supset C$. Let F and α be the image and the cokernel of ev_U. Then α is a line bundle, $\det F$ is isomorphic to $\beta := K_C \alpha^{-1}$ and we have exact sequences

$$(4.2) \quad 0 \longrightarrow \beta^{-1} \longrightarrow \mathcal{O}_C^{\oplus 3} \longrightarrow F \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F \longrightarrow E \longrightarrow \alpha \longrightarrow 0.$$

By (2.2), the line bundles α and β are both of degree 8.

PROPOSITION 4.6. *C is nonpentagonal.*

Proof. It is obvious that C is nonhyperelliptic. Since $\mathrm{SpG}(3, 6) \subset \mathbf{P}^{13}$ is an intersection of quadrics (see (0.1)), so is $C \subset \mathbf{P}^8$. In particular, $C \subset \mathbf{P}^8$ has no tri-secant lines. By the geometric version of the Riemann-Roch theorem ([1, Chap I, §2]), C has no g^1_3. Also C has no g^2_5 either, since the (geometric) genus of a plane quintic is at most 6. Let ξ be a g^1_5 on C. Then we have $h^0(\xi) = 2$. Let U and F be as above. Taking the global section of the exact sequence

$$[0 \longrightarrow F^\vee \longrightarrow \mathcal{O}_C^{\oplus 3} \longrightarrow \beta \longrightarrow 0] \otimes \xi,$$

we have

$$6 \le 3h^0(\xi) \le \dim \mathrm{Hom}\,(F, \xi) + h^0(\xi\beta) = \dim \mathrm{Hom}\,(F, \xi) + 5 + h^1(\xi\beta).$$

Hence we have

$$(4.3) \qquad \dim \mathrm{Hom}\,(F, \xi) + \dim \mathrm{Hom}\,(\xi, \alpha) \ge 1.$$

Assume that there exists a nonzero homomorphism $F \to \xi$ and let s be a nonzero global section in the kernel of $U \hookrightarrow H^0(F) \to H^0(\xi)$. Then s has at least three zeros since $\deg F - \deg \xi = 3$. If $\mathrm{Hom}\,(F, \xi)$ is zero, then $\mathrm{Hom}\,(\xi, \alpha)$ is not by (4.3). Hence α contains a subsheaf isomorphic to ξ. Let A' be the inverse image of $H^0(\xi)$ by $A \to H^0(\alpha)$. Then the cokernel of the evaluation homomorphism $A' \otimes_k \mathcal{O}_C \to E$ is of length 3. Both contradict Lemma 4.4. □

Remark 4.7. (1) For a curve of genus 9, the nonexistence of g^1_5 is equivalent to its Clifford index which equals 4 (Martens [9, Beispiel 9]).

(2) Green's property (N_p) ([6]) gives another proof of the proposition: First a general curve of genus 9 satisfies (N_3) by Ein [3]. Hence $\mathrm{SpG}(3, 6) \subset \mathbf{P}^{13}$ and its complete linear section do so. By the converse of Green's conjecture (Green-Lazarsfeld [7]), C is nonpentagonal.

By the proposition and (1) of the remark, C has no g^3_8. Hence we have $h^0(\alpha) = h^0(\beta) = 3$. By Lemma 5.1 below, we have $h^0(E) \le h^0(\alpha) + H^0(Q_\beta) \le 6$. Combining this with Lemma 4.3, we have

PROPOSITION 4.8. *The restriction map $A \to H^0(E)$ is an isomorphism.*

In the following sections we aim at a kind of converse of Proposition 4.5.

5. Rank 3 linear systems on a nonpentagonal curve

Throughout this section we assume that C is a nonpentagonal curve of genus 9. In particular, C has no g^2_7. Let α be a g^2_8, β its Serre adjoint and Q_β the cokernel of ev_β as in the introduction and in (1.1). The image of $\Phi_\beta : C \longrightarrow \mathbf{P}^2$ is a singular plane curve of degree 8. Hence there exists a pair (p, q) of points (not necessarily distinct) such that $h^0(\beta(-p-q)) = 2$. By assumption $\xi := \beta(-p-q)$ is a free g^1_6. Hence we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \beta^{-1} & \longrightarrow & \mathcal{O}_C^{\oplus 3} & \longrightarrow & Q_\beta & \longrightarrow & 0 \\ & & \cap & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \xi^{-1} & \longrightarrow & \mathcal{O}_C^{\oplus 2} & \longrightarrow & \xi & \longrightarrow & 0 \end{array} \tag{5.1}$$

and an exact sequence

$$0 \longrightarrow \mathcal{O}_C(p+q) \longrightarrow Q_\beta \longrightarrow \xi \longrightarrow 0. \tag{5.2}$$

LEMMA 5.1. (1) $h^0(Q_\beta) = 3$;
(2) $H^0(\alpha^{-1} Q_\beta) = 0$.

Proof. (1) $h^0(Q_\beta) \geq 3$ is obvious from the defining exact sequence of Q_β. The opposite inequality $h^0(Q_\beta) \leq 3$ follows from (5.2).

(2) Q_β is isomorphic to $\beta Q_\beta^\vee$ since it is of rank 2. Hence Q_β is a subbundle of $\beta^{\oplus 3}$. If $\alpha \not\simeq \beta$, then $H^0(\alpha^{-1}\beta) = 0$ and hence $H^0(\alpha^{-1} Q_\beta) = 0$. If $\alpha \simeq \beta$, then $H^0(\alpha^{-1} Q_\beta) \simeq H^0(Q_\beta^\vee) = 0$ by the exact sequence (1.1). ☐

We consider extensions

$$0 \longrightarrow Q_\beta \longrightarrow E \longrightarrow \alpha \longrightarrow 0 \tag{5.3}$$

which are Γ-*split*; that is, $H^0(E) \to H^0(\alpha)$ is surjective.

LEMMA 5.2. *There exists a nontrivial extension E of α by Q_β with $h^0(E) = 6$.*

Proof. The extensions with $h^0(E) = 6$ are parametrized by the kernel of the natural linear map $\varphi : \mathrm{Ext}^1(\alpha, Q_\beta) \longrightarrow H^0(\alpha)^\vee \otimes H^1(Q_\beta)$, which is equal to the first cohomology H^1 of the homomorphism

$$[\alpha^{-1} \xrightarrow{\mathrm{ev}^\vee} H^0(\alpha)^\vee \otimes \mathcal{O}_C] \otimes Q_\beta.$$

Since its cokernel is $Q_\alpha \otimes Q_\beta$, we have an exact sequence

(5.4)

$$H^0(\alpha)^\vee \otimes H^0(Q_\beta) \xrightarrow{\psi} H^0(Q_\alpha \otimes Q_\beta) \longrightarrow H^1(\alpha^{-1} Q_\beta) \xrightarrow{\varphi} H^0(\alpha)^\vee \otimes H^1(Q_\beta)$$

The first map ψ is injective by (2) of Lemma 5.1 and $h^0(Q_\alpha \otimes Q_\beta)$ is even by Proposition 1.2. Since $h^0(\alpha) h^0(Q_\beta) = 9$, φ is not injective. ☐

PROPOSITION 5.3. *Let E be as in the preceding lemma. Then the complete linear system $|E|$ is free and semi-irreducible.*

Proof. $|E|$ is free since both $|Q_\beta|$ and $|\alpha|$ are. Let $U \subset H^0(E)$ be a 3-dimensional subspace and $F \subset E$ the saturation of the subsheaf F' generated by U. Obviously $h^0(F) \geq 3$. If F is of rank one, then $\deg F \geq 8$ by our assumption. Since $F \not\subset Q_\beta$, the extension (5.3) splits, which is a contradiction. Hence F is of rank two. Let ξ be the quotient line bundle E/F. Since $|E|$ is free, so is ξ. Since $\mathrm{Hom}(E, \mathcal{O}_C) = 0$, we have $h^0(\xi) \geq 2$. By duality and (1) of Remark 4.7, $h^0(\det F) = h^1(\xi) \leq g + 1 - 4 - h^0(\xi) \leq 4$.

Assume that $h^0(F) \geq 4$. Then F contains a line subbundle ζ with $h^0(\zeta) \geq 2$ by Proposition 3.2. Since $\zeta \not\subset Q_\beta$, ζ is isomorphic to a proper subsheaf of α. Hence we have $h^0(\zeta) = 2$. Let η be the quotient line bundle F/ζ. Then we have $h^0(\eta) \geq h^0(F) - h^0(\zeta) = 2$. Since $\deg \zeta + \deg \eta + \deg \xi = 16$, one of the three line bundles is of degree ≤ 5, which is a contradiction. Hence we have $h^0(F) = 3$ and $h^0(\xi) \geq h^0(E) - h^0(F) = 3$. Since $h^1(\xi) = h^0(\det F) \geq 3$, ξ is a g^2_8 and F' is isomorphic to Q_ξ. In particular, $F' = F$ and F' is a subbundle. $\square$

Now conversely we study a uniqueness.

LEMMA 5.4. *Nontrivial extensions E of α by Q_β with $h^0(E) = 6$ are unique.*

Proof. The assertion is equivalent to $h^0(Q_\alpha \otimes Q_\beta) \leq 10$ by the exact sequence (5.4). Take the global section of the exact sequence

$$(5.2) \otimes Q_\alpha : 0 \longrightarrow Q_\alpha(p+q) \longrightarrow Q_\alpha \otimes Q_\beta \longrightarrow Q_\alpha \xi \longrightarrow 0.$$

Now

$$\begin{aligned} h^0(Q_\alpha \otimes Q_\beta) &\leq h^0(Q_\alpha(p+q)) + h^0(Q_\alpha \xi) \\ &= h^0(Q_\alpha(p+q)) + h^1(Q_\alpha(p+q)) \\ &= 2h^0(Q_\alpha(p+q)) - \chi(Q_\alpha(p+q)). \end{aligned}$$

Since $\chi(Q_\alpha(p+q)) = -4$, it suffices to show $h^0(Q_\alpha(p+q)) \leq 3$. Assume the contrary:

The case where $h^0(Q_\alpha(p+q)) = 4$. Let $\{s_1, s_2, s_3, s_4\}$ be a basis of the vector space $H^0(Q_\alpha(p+q))$ such that $s_1, s_2, s_3 \in H^0(Q_\alpha)$ and F is the image of the evaluation homomorphism $H^0(Q_\alpha(p+q)) \otimes_k \mathcal{O}_C \longrightarrow Q_\alpha(p+q)$. Then the quotient F/Q_α is generated by the image of s_4. Hence, $\deg F \leq \deg Q_\alpha + 2 = 10$. We have $h^0(\det F) \leq 4$ by the nonexistence of g^2_6. Since $h^0(F) \geq 4$, there exists a 2-dimensional subspace of $H^0(F)$ which generates a rank one subsheaf by Proposition 3.2. This contradicts the nonexistence of g^1_5.

The case where $h^0(Q_\alpha(p+q)) \geq 5$. Since $\deg Q_\alpha(p+q) = 12$ and since C has no g^1_4, we have $h^0(\det(Q_\alpha(p+q))) \leq 5$. By Proposition 3.2, there exists an exact sequence

$$0 \longrightarrow \zeta \longrightarrow Q_\alpha(p+q) \longrightarrow \eta \longrightarrow 0$$

such that $h^0(\zeta) \geq 2$. Since $\eta(-p-q)$ is a quotient of Q_α, we have $h^0(\eta(-p-q)) \geq 2$ and $\deg \eta(-p-q) \geq 6$, which implies $\deg \zeta \leq 4$. This is a contradiction. □

We strengthen this lemma.

LEMMA 5.5. *A rank* 3 *vector bundle* E *on* C *which satisfies*

i) $\bigwedge^3 E \simeq K_C$,

ii) $h^0(E) \geq 6$, *and*

iii) $|E|$ *is semi-irreducible*

is an extension of α *by* Q_β.

Proof. By Lemma 1.1, or by (1.3), we have

$$\dim \mathrm{Hom}\,(Q_\beta, E) + \dim \mathrm{Hom}\,(E, \alpha) \geq 2.$$

($h^0(E) = r + s$ and the Brill-Noether number ρ is equal to 0.) Hence there exists a nonzero homomorphism either $f : Q_\beta \longrightarrow E$ or $g : E \longrightarrow \alpha$.

If the image of f is a line bundle L, then $h^0(L) \geq 2$ since $\mathrm{Hom}\,(Q_\beta, \mathcal{O}_C) = 0$. This contradicts (1) of Proposition 3.5. Hence f is injective. By semi-irreducibility, the cokernel is a line bundle and is isomorphic to α.

If $g : E \longrightarrow \alpha$ is not surjective, then the kernel of $H^0(E) \longrightarrow H^0(\alpha)$ is of dimension ≥ 4, which contradicts semi-irreducibility. Hence g is surjective and its kernel is isomorphic to Q_β. □

By the two lemmas above, we have the following:

PROPOSITION 5.6. *Vector bundles* E *on* C *which satisfy the condition of the lemma are unique up to isomorphism.*

This vector bundle is denoted by $E_{\max}$.

COROLLARY. *If* E *is a rank* 3 *vector bundle of canonical determinant on* C *and if* $|E|$ *is semi-irreducible, then* $h^0(E) \leq 6$.

Remark 5.7. (1) By the proposition and its proof, we obtain an explicit bijection between two sets: $W^2_8(C)$, the set of g^2_8's of C, and the intersection

$$\mathrm{G}(3, H^0(E_{\max})) \cap \mathbf{P}^{10}.$$

It is known that the cardinality of $W^{r-1}_d(C)$ of a general curve C of genus g is equal to the degree of a g-dimensional Grassmannian when the Brill-Noether number ρ is zero (cf. [1, Chap. VII, Th. (4.4)] and [4, Ex. 14.4.5]).

(2) By (1) of Proposition 3.5, it is easy to show that $E_{\max}$ is stable. It is also easy to show a converse: if E is stable, $\bigwedge^3 E \simeq K_C$ and $h^0(E) \geq 6$, then $|E|$ is semi-irreducible.

6. **Linear section theorems**

We prove Theorem C in several steps. Assume that $E = E_{\max}$ satisfies the condition (0.2). Since E is a rank 3 vector bundle of canonical determinant, $K_C E^\vee$ is isomorphic to $\bigwedge^2 E$. Hence, by the Riemann-Roch theorem, we have

$$h^0(E) - h^0\Big(\bigwedge^2 E\Big) = \deg E + 3(1-9) = -8.$$

and $h^0(\bigwedge^2 E) = 14$. Since $\dim \bigwedge^2 H^0(E) = 15$, the linear map

$$\lambda_2 : \bigwedge^2 H^0(E) \longrightarrow H^0\Big(\bigwedge^2 E\Big)$$

is not injective.

Step 1. Every nonzero bivector σ in $\operatorname{Ker}\lambda_2$ is nondegenerate.

Proof. The rank of σ is either 2, 4 or 6. If σ is of rank 2, then σ is equal to $s_1 \wedge s_2$ for a pair of global sections s_1 and s_2 which are linearly independent in $H^0(E)$ and generate a rank-one subsheaf in E. This contradicts (2) of Proposition 3.5. Assume that σ is of rank 4. Then σ is equal to $s_1 \wedge s_2 - s_3 \wedge s_4$ for s_1, s_2, s_3 and $s_4 \in H^0(E)$. By semi-irreducibility, s_1 and s_2 generate a rank two subsheaf in E. Let F be its saturation. Since $\lambda_2(s_1 \wedge s_2) = \lambda_2(s_3 \wedge s_4)$, we have $\lambda_3(s_1 \wedge s_2 \wedge s_i) = \lambda_3(s_3 \wedge s_4 \wedge s_i) = 0$ for $i = 3, 4$. Hence s_3 and s_4 are contained in $H^0(F)$ and we have $h^0(F) \geq 4$. This contradicts the semi-irreducibility of $|E|$ by Proposition 3.5. □

The nondegeneracy of σ is equivalent to the nonvanishing of the Pfaffian. Hence $\operatorname{Ker}\lambda_2$ is of dimension one and λ_2 is surjective. Since $|E|$ is free, we obtain a morphism $\Phi_E : C \longrightarrow \mathrm{G}(A, 3)$ to the Grassmannian of 3-dimensional quotient spaces of $A := H^0(E)$. Its image is contained in the symplectic Grassmannian $\mathrm{G}(A, \sigma)$ and we obtain the commutative diagram (0.3), where σ is a generator of $\operatorname{Ker}\lambda_2$. Since $\bigwedge^3(A, \sigma)$ is of dimension 14, the kernel of $\overline{\lambda}_3 : \bigwedge^3(A, \sigma) \longrightarrow H^0(K_C)$ is of dimension $\geq 14 - 9 = 5$. Let $f_1, \ldots, f_k$, $k \geq 5$, be its basis and $H_1, \ldots, H_k$ the hyperplanes corresponding to them. Since $|E|$ is semi-irreducible, the intersection $S_{E_p} \cap \mathbf{P}_*\operatorname{Ker}\lambda_3$ is empty for every $p \in C$ by Lemma 3.4. Hence so is $\alpha(S_{E_p}) \cap \mathbf{P}_*\operatorname{Ker}\overline{\lambda}_3$ for the projection α in (2.6).

Step 2. There exists a point $p \in C$ such that the intersection $\mathrm{G}(A, \sigma) \cap H_1 \cap \cdots \cap H_k$ is transversal at $\Phi_E(p)$.

Proof. Assume the contrary. Then, for every $p \in C$, there exists a member H_p of $\langle [H_1], \ldots, [H_k] \rangle = \mathbf{P}_*\operatorname{Ker}\overline{\lambda}_3$ such that the intersection $\mathrm{G}(A, \sigma) \cap H_p$ is singular at $\Phi_E(p)$. The intersection $\mathbf{P}_*(N^\vee_{\mathrm{G}(A,\sigma)/\mathbf{P},[E_p]}) \cap \mathbf{P}_*\operatorname{Ker}\overline{\lambda}_3$ is a point for every p by Proposition 2.4. Therefore, we obtain a section of the $\mathbf{P}^6$-bundle $\mathbf{P}^*(\Phi_E^* N_{\mathrm{G}(A,\sigma)/\mathbf{P}})$ over C which is disjoint from $\coprod_{p \in C} \alpha(S_{E_p})$. By projecting from $\coprod_{p \in C} \kappa_p$, we obtain a section of $\mathbf{P}_*(S^2 E)$ over which the discriminant form

$\delta \in H^0(S^3(S^2E)^\vee \otimes (\det E)^{\otimes 2})$ has no zeros. Let $\xi \subset S^2E$ be the line subbundle corresponding to the section. Then δ induces a nowhere-vanishing global section of $\xi^{-3} \otimes (\det E)^{\otimes 2}$. This implies $3\deg\xi = 2\deg E = 32$, which is absurd. □

In particular, we have $k = 5$ and hence the linear map $\overline{\lambda}_3$ is surjective. Therefore, $\mathbf{P}^*\overline{\lambda}_3$ is a linear embedding. Since the canonical morphism Φ_K is an embedding, so is Φ_E by the commutative diagram (0.3). We identify C with its image $\Phi_E(C)$.

By Step 2, the intersection $\mathrm{G}(A,\sigma)\cap H_1 \cap\cdots\cap H_5$ is complete on a nonempty open subset C_0 of C. Hence the twisted normal bundle $N_{C/\mathrm{G}(A,\sigma)}(-1)$ is generated by the five global sections induced from $f_1,\ldots,f_5$ over C_0. It is generated over C, since $N_{C/\mathrm{G}(A,\sigma)}(-1)$ is of trivial determinant. Therefore, the intersection is complete along C and contains it as a connected component. By the connectedness of linear sections (Fulton-Lazarsfeld [5, Th. 2.1]), the intersection coincides with C, which completes the proof of Theorem C. (If we use the refined Bézout theorem (Fulton[4, Th. 12.3]), the proof finishes at the last paragraph.)

Theorem A is an immediate consequence of Theorem C, Proposition 5.3 and Proposition 4.6.

7. **Proof of Theorem** B

We do not assume that k is algebraically closed anymore. Let $C \simeq \mathrm{G}(A',\sigma') \cap P'$ be another expression of $C = \mathrm{G}(A,\sigma)\cap P$ as a complete linear section of a 6-dimensional symplectic Grassmannian and $\mathscr{E}'|_C$ the restriction of the universal quotient bundle. Both $|\mathscr{E}|_C|$ and $|\mathscr{E}'|_C|$ are semi-irreducible (over $\bar{k}$) by Proposition 4.5. Hence they are isomorphic to each other over $\bar{k}$ by Proposition 5.6 and there exists a nonzero homomorphism $f : \mathscr{E}|_C \longrightarrow \mathscr{E}'|_C$ over k. This is an isomorphism by Lemma 3.6. Since the diagram

$$\begin{array}{ccc} \bigwedge^2 A = \bigwedge^2 H^0(\mathscr{E}|_C) & \xrightarrow{\bigwedge^2 H^0(f)} & \bigwedge^2 H^0(\mathscr{E}'|_C) = \bigwedge^2 A' \\ \downarrow & & \downarrow \\ H^0(\bigwedge^2 \mathscr{E}|_C) & \xrightarrow[H^0(\bigwedge^2 f)]{} & H^0(\bigwedge^2 \mathscr{E}'|_C) \end{array}$$

is commutative, the isomorphism $H^0(f)$ maps $k\sigma$ onto $k\sigma'$. Thus we have proved (2) of Theorem B.

Assume that k is perfect and let $\overline{E}$ be a vector bundle on $\overline{C} = C\otimes_k \bar{k}$. We consider a descent problem of $\overline{E}$ under the following condition:

(∗) $\overline{E}$ is simple and $\sigma^*\overline{E} \simeq \overline{E}$ for every element σ of the Galois group $\mathrm{Gal}\,k$ of $\bar{k}/k$.

As is well known, the obstruction $\mathrm{ob}(\overline{E})$ for $\overline{E}$ to descend to C is an element of the second Galois cohomology group $H^2(\mathrm{Gal}\,k, \mathrm{Aut}\,\overline{E})$. Choose an isomorphism

$f_\sigma : \bar{E} \xrightarrow{\sim} \sigma^* \bar{E}$ for each $\sigma \in \operatorname{Gal} k$. Then $\operatorname{ob}(\bar{E})$ is the cohomology class of the cocycle $\{c_{\sigma,\tau}\}_{\sigma,\tau \in \operatorname{Gal} k}$ defined by $c_{\sigma,\tau} = f_{\sigma\tau}^{-1} \circ \tau^*(f_\sigma) \circ f_\tau \in \operatorname{Aut}_{\bar{k}} \bar{E}$. In other words, $\operatorname{ob}(\bar{E})$ is the factor set of the extension

$$1 \longrightarrow \operatorname{Aut}_{\bar{k}} \bar{E} \longrightarrow \operatorname{Aut}_k \bar{E} \longrightarrow \operatorname{Gal} k \longrightarrow 1.$$

LEMMA 7.1. *If* $\dim H^i(\bar{C}, \bar{E}) = n > 0$, *then the obstruction* $\operatorname{ob}(\bar{E})$ *is n-torsion.*

Proof. Let $\{s_1, \ldots, s_n\}$ be a basis of $H^i(\bar{C}, \bar{E})$ and $A_\sigma \in M_n(\bar{k})$ the matrix representing

$$H^i(f_\sigma) : H^i(\bar{C}, \bar{E}) \longrightarrow H^i(\bar{C}, \sigma^* \bar{E})$$

with respect to the bases $\{s_1, \ldots, s_n\}$ and $\{\sigma^* s_1, \ldots, \sigma^* s_n\}$. Then

$$\det H^i(c_{\sigma,\tau}) = (\det A_{\sigma\tau})^{-1} \tau(\det A_\sigma) \det A_\tau$$

in $\bar{k}^\times$. Therefore, $\{\det H^i(c_{\sigma,\tau})\}_{\sigma,\tau \in \operatorname{Gal} k}$ is cohomologous to zero. Since $c_{\sigma,\tau}$ are all constant multiplications, $\det H^i(c_{\sigma,\tau})$ are equal to $c_{\sigma,\tau}^n$. Hence $\operatorname{ob}(\bar{E})$ is n-torsion. □

Now we prove (1) of Theorem B. Let C be a nonpentagonal curve of genus 9 defined over k. It suffices to show the following:

PROPOSITION 7.2. *Assume that C has no g_5^1 over $\bar{k}$. Then there exists a vector bundle E on C such that $E \otimes_k \bar{k}$ is isomorphic to the vector bundle $E_{\max}$ on $C \otimes_k \bar{k}$.*

Proof. By (3)of Proposition 3.5 and Proposition 5.6, $E_{\max}$ satisfies $(*)$. Hence the obstruction $\operatorname{ob}(E_{\max})$ belongs to $H^2(\operatorname{Gal} k, \operatorname{Aut}_{\bar{k}} E_{\max}) = H^2(\operatorname{Gal} k, \bar{k}^\times)$. Let

$$\operatorname{Det} : H^2(\operatorname{Gal} k, \operatorname{Aut}_{\bar{k}} E_{\max}) \longrightarrow H^2(\operatorname{Gal} k, \operatorname{Aut}_{\bar{k}} \det E_{\max})$$

be the determinant homomorphism. Since $\det E_{\max}$ is the canonical bundle, it descends to C. Hence $\operatorname{ob}(E_{\max})$ belongs to the kernel and is 3-torsion. On the other hand, $\operatorname{ob}(E_{\max})$ is 14-torsion by the preceding lemma since $\dim H^1(E_{\max}) = 14$. Therefore, $\operatorname{ob}(E_{\max})$ vanishes and $E_{\max}$ descends to C. (This is a Galois group variant of an argument of Mumford-Newstead [17].) □

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