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Abstract

We eliminate the possibility of “escape of mass” for Hecke-Maass forms of large eigenvalue for the modular group. Combined with the work of Lindenstrauss, this establishes the Quantum Unique Ergodicity conjecture of Rudnick and Sarnak for Hecke-Maass forms on the modular surface $SL_2(\mathbb{Z}) \backslash \mathbb{H}$.

1. Introduction

An interesting problem in number theory and quantum chaos is to understand the distribution of Maass cusp forms of large Laplace eigenvalue for the modular surface $X = SL_2(\mathbb{Z}) \backslash \mathbb{H}$. Let ϕ denote a Maass form of eigenvalue λ , normalized so that its Petersson norm $\int_X |\phi(z)|^2 \frac{dx dy}{y^2}$ equals 1. Zelditch [14] has shown that as $\lambda \rightarrow \infty$, for a *typical* Maass form ϕ , the associated probability measure $\mu_\phi := |\phi(z)|^2 \frac{dx dy}{y^2}$ tends to the uniform distribution measure $\frac{3}{\pi} \frac{dx dy}{y^2}$. This result is known as “Quantum Ergodicity.” The widely studied Quantum Unique Ergodicity conjecture of Rudnick and Sarnak [10] asserts that as $\lambda \rightarrow \infty$, for *every* Maass form ϕ the measure μ_ϕ approaches the uniform distribution measure. In studying this conjecture, it is also natural to restrict to Maass forms that are eigenvalues of all the Hecke operators; it is expected that the spectrum of the Laplacian on X is simple so that this condition would automatically hold, but this is far from being proved. Using methods from ergodic theory, Lindenstrauss [7] has made great progress towards the QUE conjecture for such Hecke-Maass forms. Namely, he has shown that the only possible weak- $*$ limits of the measures μ_ϕ are of the form $c \frac{3}{\pi} \frac{dx dy}{y^2}$, where c is some constant in $[0, 1]$. In other words, Lindenstrauss establishes QUE for X except for the possibility that for some infinite subsequence of Hecke-Maass forms ϕ some of the L^2 mass of ϕ could “escape” into the cusp of X . In this paper we eliminate the possibility of escape of mass, and together with Lindenstrauss’s work this completes the proof of QUE for X .

The results of Zelditch and Lindenstrauss are in fact stronger than we have indicated above. Given a Maass form ϕ on X , Zelditch defines the “micro-local”

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lift $\tilde{\phi}$ of ϕ to $Y = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$. This micro-local lift defines a measure on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$ with two important properties: First, as the eigenvalue tends to infinity, the projection of the measure from Y to the surface X approximates the measure μ_ϕ given above. Second, as the eigenvalue tends to infinity any weak- $*$ limit of these measures on Y is invariant under the geodesic flow on Y . Zelditch's result then asserts that for a full density subsequence of eigenfunctions, the associated micro-local lifts get equidistributed on Y . Lindenstrauss's result is that any weak- $*$ limit of the lifts arising from Hecke-Maass forms is a constant c (between 0 and 1) times the normalized volume measure on Y . We remark that the analog of quantum unique ergodicity with Eisenstein series in place of cusp forms has been treated by Luo and Sarnak [8] in the modular surface version, and by Jakobson [5] for the corresponding micro-local lifts. For more complete accounts of the quantum unique ergodicity problem the reader may consult [6], [9], [11], [13] and references therein; a comprehensive introduction to the theory of Maass forms is provided in [4].

THEOREM 1. *Let ϕ be a Hecke-Maass cusp form for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, normalized to have Petersson norm 1. Let $\tilde{\phi}$ denote the micro-local lift of ϕ to $Y = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$, and let $\tilde{\mu}_\phi$ denote the corresponding measure on Y . The normalized volume measure on Y is the unique weak- $*$ limit of the measures $\tilde{\mu}_\phi$. In particular, for any compact subset C of a fundamental domain for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ we have, as $\lambda \rightarrow \infty$,*

$$\int_C |\phi(x + iy)|^2 \frac{dx dy}{y^2} = \int_C \frac{3}{\pi} \frac{dx dy}{y^2} + o(1).$$

Theorem 1 is a consequence of the following result which estimates how much of the mass of ϕ can be present high in the cusp.

PROPOSITION 2. *Let ϕ denote a Hecke-Maass cusp form for $\mathrm{SL}_2(\mathbb{Z})$ with eigenvalue λ , and normalized to have Petersson norm 1. For $T \geq 1$, we have*

$$\int_{\substack{|x| \leq \frac{1}{2} \\ y \geq T}} |\phi(x + iy)|^2 \frac{dx dy}{y^2} \ll \frac{\log(eT)}{\sqrt{T}}.$$

We remark that by entirely different methods Holowinsky and Soundararajan ([2], [3], [12]) have settled the holomorphic analog of QUE for X ; it is not clear how to adapt Lindenstrauss's methods to that setting. Their methods have the advantage of yielding explicit estimates for the rate of convergence to uniform distribution; it is not clear how to obtain such a rate of convergence in Theorem 1. However the works of Holowinsky and Soundararajan use in an essential way Deligne's bounds for the Hecke eigenvalues of holomorphic modular forms; the analog of these bounds for Maass forms remains an important open problem.

While we have restricted ourselves to the full modular group, our argument would apply also to all congruence subgroups. Thus QUE for Maass forms, and its holomorphic analog, are now known for noncompact arithmetic quotients of \mathbb{H} .

In the case of compact arithmetic quotients, Lindenstrauss’s work establishes QUE for Maass forms; the analog for holomorphic forms remains open.

Our proofs of Theorem 1 and Proposition 2 exploit the particular multiplicative structure of the Hecke-operators. We say that a function f is *Hecke-multiplicative* if it satisfies the Hecke relation

$$f(m)f(n) = \sum_{d|(m,n)} f(mn/d^2),$$

and $f(1) = 1$. The key to establishing Proposition 2 is the following result on Hecke-multiplicative functions.

THEOREM 3. *Let f be a Hecke-multiplicative function. Then for all $1 \leq y \leq x$,*

$$\sum_{n \leq x/y} |f(n)|^2 \leq 10^8 \left(\frac{1 + \log y}{\sqrt{y}} \right) \sum_{n \leq x} |f(n)|^2.$$

It is noteworthy that Theorem 3 makes no assumptions on the size of the function f . Hecke-multiplicative functions satisfy $f(p^2) = f(p)^2 - 1$, so that at least one of $|f(p)|$ or $|f(p^2)|$ must be bounded away from zero; this observation plays a crucial role in our proof. We also remark that apart from the $\log y$ factor, Theorem 3 is best possible: Consider the Hecke-multiplicative function f defined by $f(p) = 0$ for all primes p . The Hecke relation then mandates that $f(p^{2k+1}) = 0$ and $f(p^{2k}) = (-1)^k$. Therefore, in this example, $\sum_{n \leq x} |f(n)|^2 = \sqrt{x} + O(1)$ and $\sum_{n \leq x/y} |f(n)|^2 = \sqrt{x/y} + O(1)$.

The argument of Theorem 3 can be generalized in several ways. For example, one could obtain an analogous result with $|f(n)|$ in place of $|f(n)|^2$. Moreover, one could consider multiplicative functions f arising from Euler products of degree d . By this we mean that for each prime p there exist complex numbers α_j ($j = 1, \dots, d$) with $|\alpha_1 \cdots \alpha_d| = 1$ and $\sum_{k=0}^{\infty} f(p^k)x^k = \prod_{j=1}^d (1 - \alpha_j x)^{-1}$; the case $d = 2$ corresponds to our Hecke-multiplicative functions above. For these functions, one of $|f(p)|, \dots, |f(p^d)|$ must be bounded away from zero, and it may be of interest to exploit this and establish an analog of Theorem 3.

2. Deducing Theorem 1 and Proposition 2 from Theorem 3

Proof of Proposition 2. Let ϕ be a Maass form of eigenvalue $\frac{1}{4} + r^2$ for the full modular group, normalized to have Petersson norm 1. We suppose that ϕ is an eigenfunction of all the Hecke operators, and let $\lambda(n)$ denote the n -th Hecke eigenvalue. Recall that ϕ has a Fourier expansion of the form

$$\phi(z) = C \sqrt{y} \sum_{n=1}^{\infty} \lambda(n) K_{ir}(2\pi ny) \cos(2\pi nx),$$

or

$$\phi(z) = C \sqrt{y} \sum_{n=1}^{\infty} \lambda(n) K_{ir}(2\pi ny) \sin(2\pi nx),$$

where C is a constant (normalizing the L^2 -norm), K denotes the usual K -Bessel function, and we have cos or sin depending on whether the form is even or odd.

Using Parseval we find that

$$\int_{\substack{|x| \leq \frac{1}{2} \\ y \geq T}} |\phi(x + iy)|^2 \frac{dx \, dy}{y^2} = \frac{C^2}{2} \int_T^\infty \sum_{n=1}^\infty |\lambda(n)|^2 |K_{ir}(2\pi ny)|^2 \frac{dy}{y}.$$

By a change of variables we may write this as

$$\frac{C^2}{2} \sum_{n=1}^\infty |\lambda(n)|^2 \int_{nT}^\infty |K_{ir}(2\pi t)|^2 \frac{dt}{t} = \frac{C^2}{2} \int_1^\infty |K_{ir}(2\pi t)|^2 \sum_{n \leq t/T} |\lambda(n)|^2 \frac{dt}{t}.$$

Appealing to Theorem 3 this is

$$\begin{aligned} &\ll \frac{\log eT}{\sqrt{T}} \frac{C^2}{2} \int_1^\infty |K_{ir}(2\pi t)|^2 \sum_{n \leq t} |\lambda(n)|^2 \frac{dt}{t} \\ &= \frac{\log eT}{\sqrt{T}} \int_{\substack{|x| \leq \frac{1}{2} \\ y \geq 1}} |\phi(x + iy)|^2 \frac{dx \, dy}{y^2} \ll \frac{\log eT}{\sqrt{T}}, \end{aligned}$$

since the region $|x| \leq \frac{1}{2}$, $y \geq 1$ is contained inside a fundamental domain for $SL_2(\mathbb{Z}) \backslash \mathbb{H}$. This proves Proposition 2. □

Proof of Theorem 1. As remarked in the introduction, Lindenstrauss has shown that any weak- $*$ limit of the micro-local lifts of Hecke-Maass forms is a constant c (in $[0, 1]$) times the normalized volume measure on Y . Projecting these measures down to the modular surface, we see that any weak- $*$ limit of the measures μ_ϕ associated to Hecke-Maass forms is of the shape $c \frac{3}{\pi} \frac{dx \, dy}{y^2}$. Theorem 1 claims that in fact $c = 1$, and there is no escape of mass. If on the contrary $c < 1$ for some weak- $*$ limit, then we have a sequence of Hecke-Maass forms ϕ_j with eigenvalues λ_j tending to infinity such that for any fixed $T \geq 1$ and as $j \rightarrow \infty$

$$\int_{\substack{z \in \mathcal{F} \\ y \leq T}} |\phi_j(z)|^2 \frac{dx \, dy}{y^2} = (c + o(1)) \frac{3}{\pi} \int_{\substack{z \in \mathcal{F} \\ y \leq T}} \frac{dx \, dy}{y^2} = (c + o(1)) \left(1 - \frac{3}{\pi T}\right);$$

here $\mathcal{F} = \{z = x + iy : |z| \geq 1, -1/2 \leq x \leq 1/2, y > 0\}$ denotes the usual fundamental domain for $SL_2(\mathbb{Z}) \backslash \mathbb{H}$. It follows that as $j \rightarrow \infty$

$$\int_{\substack{|x| \leq \frac{1}{2} \\ y \geq T}} |\phi_j(z)|^2 \frac{dx \, dy}{y^2} = 1 - c + \frac{3}{\pi T} c + o(1).$$

But if $c < 1$ this contradicts Proposition 2 for suitably large T , proving Theorem 1. □

3. Preliminaries for the proof of Theorem 3

Throughout the proof of Theorem 3 we shall work with a single value of x . Accordingly, we define for $1 \leq y \leq x$

$$\mathcal{F}(y) = \mathcal{F}(y; x) = \frac{\sum_{n \leq x/y} |f(n)|^2}{\sum_{n \leq x} |f(n)|^2},$$

and our goal is to show that $\mathcal{F}(y) \leq 10^8 \log(ey) / \sqrt{y}$. Note that $\mathcal{F}(y) \leq 1$ for all $y \geq 1$, and that \mathcal{F} is a decreasing function of y . Thus in proving Theorem 3 we may assume that $y \geq 10^{16}$. We adopt throughout the convention that $f(t) = 0$ when t is not a natural number, and that $\mathcal{F}(y) = 0$ if $y > x$.

LEMMA 3.1. *For any integer $m \leq x$ we have*

$$|f(m)| \leq \frac{\tau(m)}{\mathcal{F}(m)^{\frac{1}{2}}},$$

where $\tau(m)$ denotes the number of divisors of m . Consequently, for any $p \leq \sqrt{x}$ we have

$$|f(p)| \leq \frac{2}{\mathcal{F}(p^2)^{\frac{1}{4}}}.$$

Proof. We start with the relation

$$(3.1) \quad |f(m)|^2 \sum_{n \leq x/m} |f(n)|^2 = |f(m)|^2 \mathcal{F}(m) \sum_{n \leq x} |f(n)|^2.$$

The Hecke relation gives

$$|f(m)f(n)| = \left| \sum_{d|(m,n)} f(mn/d^2) \right| \leq \sum_{d|m} |f(mn/d^2)|,$$

so that by Cauchy-Schwarz

$$|f(m)f(n)|^2 \leq \tau(m) \sum_{d|m} |f(mn/d^2)|^2.$$

Hence the left-hand side of (3.1) is

$$\leq \tau(m) \sum_{d|m} \sum_{n \leq x/m} |f(mn/d^2)|^2 \leq \tau(m)^2 \sum_{n \leq x} |f(n)|^2.$$

Our bound on $|f(m)|$ follows. To obtain the bound for $|f(p)|$ we take $m = p^2$ getting $|f(p^2)| \leq 3/\mathcal{F}(p^2)^{\frac{1}{2}}$. Since $|f(p)|^2 \leq |f(p^2)| + 1$ the claimed estimate for $|f(p)|$ follows. \square

PROPOSITION 3.2. *Let d be a square-free number. Then*

$$\sum_{\substack{n \leq x/y \\ d|n}} |f(n)|^2 \leq \tau(d) \prod_{p|d} (1 + |f(p)|^2) \mathcal{F}(yd) \sum_{n \leq x} |f(n)|^2.$$

Moreover,

$$\sum_{\substack{n \leq x/y \\ d^2 | n}} |f(n)|^2 \leq \tau_3(d) \prod_{p|d} (2 + |f(p^2)|^2) \mathfrak{F}(yd^2) \sum_{n \leq x} |f(n)|^2,$$

where τ_3 denotes the 3-divisor function (being the number of ways of writing d as abc).

Proof. The Hecke relations give that $f(p)f(m) = f(pm) + f(m/p)$, and so $|f(pm)| \leq |f(p)f(m)| + |f(m/p)|$. By induction we find that

$$|f(md)| \leq \sum_{ab=d} |f(a)||f(m/b)|,$$

so that

$$|f(md)|^2 \leq \tau(d) \sum_{ab=d} |f(a)|^2 |f(m/b)|^2.$$

Summing over all $m \leq x/(yd)$ we obtain that

$$\begin{aligned} \sum_{\substack{n \leq x/y \\ d|n}} |f(n)|^2 &\leq \tau(d) \sum_{ab=d} |f(a)|^2 \sum_{m \leq x/(yd)} |f(m/b)|^2 \\ &\leq \tau(d) \sum_{ab=d} |f(a)|^2 \mathfrak{F}(yd) \sum_{n \leq x} |f(n)|^2. \end{aligned}$$

The first statement follows.

To prove the second assertion, note the fact that the Hecke relations give that $f(m)f(p^2)$ equals $f(mp^2)$ if $p \nmid m$, $f(mp^2) + f(m)$ if $p \parallel m$, and $f(mp^2) + f(m) + f(m/p^2)$ if $p^2 \mid m$. In all cases we find that $|f(mp^2)| \leq |f(p^2)f(m)| + |f(m)| + |f(m/p^2)|$. By induction we may see that

$$|f(md^2)| \leq \sum_{abc=d} |f(a^2)||f(m/c^2)|.$$

Therefore,

$$|f(md^2)|^2 \leq \tau_3(d) \sum_{abc=d} |f(a^2)|^2 |f(m/c^2)|^2.$$

Summing over all $m \leq x/(yd^2)$ we obtain that

$$\begin{aligned} \sum_{n=md^2 \leq x/y} |f(n)|^2 &\leq \tau_3(d) \sum_{abc=d} |f(a^2)|^2 \sum_{m \leq x/(yd^2)} |f(m/c^2)|^2 \\ &\leq \tau_3(d) \mathfrak{F}(yd^2) \left(\sum_{n \leq x} |f(n)|^2 \right) \sum_{a|d} |f(a^2)|^2 \tau(d/a), \end{aligned}$$

and the second statement follows. □

Let $\mathcal{P} = \mathcal{P}(y)$ denote the set of primes in $[\sqrt{y}/2, \sqrt{y}]$. The prime number theorem gives for large y that $|\mathcal{P}(y)| \sim \sqrt{y}/\log y$. In fact, using only a classical result of Chebyshev we find that for $y \geq 10^{16}$ we have $|\mathcal{P}(y)| \geq \sqrt{y}/(2 \log y)$ (see,

for example, Dusart’s thesis [1] which gives more precise estimates). Lemma 3.1 gives that $|f(p)| \leq 2/\mathcal{F}(y)^{1/4}$ for $p \in \mathcal{P}$. Therefore, we select

$$J = \left\lceil \frac{1}{4 \log 2} \log(1/\mathcal{F}(y)) \right\rceil + 3,$$

and partition \mathcal{P} into sets $\mathcal{P}_0, \dots, \mathcal{P}_J$ where \mathcal{P}_0 contains those primes in \mathcal{P} with $|f(p)| \leq 1/2$, and for $1 \leq j \leq J$ the set \mathcal{P}_j contains those primes in \mathcal{P} with $2^{j-2} < |f(p)| \leq 2^{j-1}$.

Let $k \geq 1$ be a natural number. Define $\mathcal{N}_0(k)$ to be the set of integers divisible by at most k distinct squares of primes in \mathcal{P}_0 . For $1 \leq j \leq J$ we define $\mathcal{N}_j(k)$ to be the set of integers divisible by at most k distinct primes in \mathcal{P}_j .

PROPOSITION 3.3. *Keep the notation above. For $2 \leq k \leq |\mathcal{P}_0|/4$ we have*

$$\sum_{\substack{n \leq x/y \\ n \in \mathcal{N}_0(k)}} |f(n)|^2 \leq \frac{4k}{|\mathcal{P}_0|} \sum_{n \leq x} |f(n)|^2.$$

Further, if $1 \leq j \leq J$ and $1 \leq k \leq |\mathcal{P}_j|/4 - 1$ we have

$$\sum_{\substack{n \leq x/y \\ n \in \mathcal{N}_j(k)}} |f(n)|^2 \leq \frac{2^{12}k^2}{2^{4j}|\mathcal{P}_j|^2} \sum_{n \leq x} |f(n)|^2.$$

Proof. Note that if $p \in \mathcal{P}_0$, then $|f(p)| \leq 1/2$, and so $|f(p^2)| = |f(p)^2 - 1| \geq 3/4$. Therefore,

$$\begin{aligned} (3.2) \quad \sum_{\substack{n \leq x/y \\ n \in \mathcal{N}_0(k)}} |f(n)|^2 \left(\sum_{\substack{p \in \mathcal{P}_0 \\ p^2 \nmid n}} |f(p^2)|^2 \right) &\geq \frac{9}{16} (|\mathcal{P}_0| - k) \sum_{\substack{n \leq x/y \\ n \in \mathcal{N}_0(k)}} |f(n)|^2 \\ &\geq \frac{27}{64} |\mathcal{P}_0| \sum_{\substack{n \leq x/y \\ n \in \mathcal{N}_0(k)}} |f(n)|^2. \end{aligned}$$

If $p \in \mathcal{P}_0$ and $p^2 \nmid n$, then we claim that $|f(n)f(p^2)| \leq |f(p^2n)|$. If $p \nmid n$, then equality holds in this claim. If p exactly divides n then the claim amounts to $|f(p^3)| \geq |f(p)f(p^2)|$, and to see this note that $f(p^3) = f(p)(f(p)^2 - 2)$ and $f(p^2) = f(p)^2 - 1$, and the estimate $|f(p)^2 - 2| \geq |f(p)^2 - 1|$ holds since $|f(p)| \leq 1/2$. Therefore the left-hand side of (3.2) is

$$\leq \sum_{m \leq x} |f(m)|^2 \left(\sum_{\substack{m=np^2 \\ n \leq x/y, n \in \mathcal{N}_0(k) \\ p \in \mathcal{P}_0, p^2 \nmid n}} 1 \right) \leq (k + 1) \sum_{m \leq x} |f(m)|^2,$$

since the sum over n and p above is zero unless m is divisible by at most $k + 1$ squares of primes in \mathcal{P}_0 and in this case the number of choices for p in that sum is at most $k + 1$. Since $k \geq 2$ the stated bound follows.

The second assertion is similar. If $p \in \mathcal{P}_j$, then $|f(p)| \geq 2^{j-2}$. Therefore,

$$\begin{aligned} \sum_{\substack{n \leq x/y \\ n \in \mathcal{N}_j(k)}} |f(n)|^2 \left(\sum_{\substack{p_1, p_2 \in \mathcal{P}_j \\ p_1 < p_2 \\ p_i \nmid n}} |f(p_1 p_2)|^2 \right) &\geq 2^{4(j-2)} \binom{|\mathcal{P}_j| - k}{2} \sum_{\substack{n \leq x/y \\ n \in \mathcal{N}_j(k)}} |f(n)|^2 \\ &\geq 2^{4j-8} \frac{9|\mathcal{P}_j|^2}{32} \sum_{\substack{n \leq x/y \\ n \in \mathcal{N}_j(k)}} |f(n)|^2. \end{aligned}$$

But the left-hand side sums terms of the form $|f(m)|^2$ where $m = np_1 p_2 \leq x$ and m is divisible by at most $k + 2$ distinct primes in \mathcal{P}_j ; moreover each such term appears at most $\binom{k+2}{2}$ times on the left-hand side. Therefore the left-hand side above is

$$\leq \binom{k+2}{2} \sum_{n \leq x} |f(n)|^2 \leq 3k^2 \sum_{n \leq x} |f(n)|^2,$$

and the proposition follows in this case. □

4. Proof of Theorem 3

Consider the set of values y with $\mathcal{F}(y) \geq 10^8 \log(ey)/\sqrt{y}$. Pick a “maximal” element from this set; precisely, a value y belonging to the exceptional set, but such that no value larger than $y + 1$ is in this set. We shall use the work in Section 3 with this maximal value of y in mind, and employ the notation introduced there. The argument splits into two cases: since $|\mathcal{P}| \geq \sqrt{y}/(2 \log y)$ we must have either $|\mathcal{P}_0| \geq \sqrt{y}/(4 \log y)$, or that $|\mathcal{P}_j| \geq \sqrt{y}/(4J \log y)$ for some $1 \leq j \leq J$. In either case we shall derive a contradiction, proving Theorem 3.

4.1. *Case 1:* $|\mathcal{P}_0| \geq \sqrt{y}/(4 \log y)$. Take $K = \lceil |\mathcal{P}_0| \mathcal{F}(y)/8 \rceil$, so that $10^4 \leq K \leq |\mathcal{P}_0|/4$. Proposition 3.3 gives that

$$\sum_{\substack{n \leq x/y \\ n \in \mathcal{N}_0(K)}} |f(n)|^2 \leq \frac{1}{2} \mathcal{F}(y) \sum_{n \leq x} |f(n)|^2,$$

so that

$$(4.1) \quad \sum_{\substack{n \leq x/y \\ n \notin \mathcal{N}_0(K)}} |f(n)|^2 \geq \frac{1}{2} \mathcal{F}(y) \sum_{n \leq x} |f(n)|^2.$$

If $n \notin \mathcal{N}_0(K)$ then n must be divisible by at least $K + 1$ squares of primes in \mathcal{P}_0 . There are $\binom{|\mathcal{P}_0|}{K+1}$ integers that are products of exactly $K + 1$ primes from \mathcal{P}_0 . Each of these integers exceeds $(\sqrt{y}/2)^{K+1}$, and a number $n \notin \mathcal{N}_0(K)$ must be divisible by the square of one of these integers. Thus, using the second bound

in Proposition 3.2 we find that the left-hand side of (4.1) is

$$(4.2) \quad \leq \binom{|\mathcal{P}_0|}{K+1} 3^{K+1} \cdot 3^{K+1} \mathcal{F}(y(y/4)^{K+1}) \sum_{n \leq x} |f(n)|^2.$$

Since

$$\binom{|\mathcal{P}_0|}{K+1} \leq \frac{|\mathcal{P}_0|^{K+1}}{(K+1)!} < \left(\frac{e|\mathcal{P}_0|}{K+1}\right)^{K+1} < \left(\frac{24}{\mathcal{F}(y)}\right)^{K+1},$$

and, by the maximality of y ,

$$\mathcal{F}(y(y/4)^{K+1}) \leq 10^8 \cdot 2^{K+1} \frac{1 + (K+2) \log y}{y^{(K+2)/2}} < 10^8 \cdot 2^{K+1} \cdot (10^{-8} \mathcal{F}(y))^{K+2},$$

we deduce that the quantity in (4.2) is

$$\leq \left(\frac{432}{10^8}\right)^{K+1} \mathcal{F}(y) \sum_{n \leq x} |f(n)|^2 < \frac{1}{2} \mathcal{F}(y) \sum_{n \leq x} |f(n)|^2,$$

which contradicts (4.1). This completes our argument for the first case.

4.2. *Case 2:* $|\mathcal{P}_j| \geq \sqrt{y}/(4J \log y)$ for some $1 \leq j \leq J$. Here we take $K = \lfloor 2^{2j-9} |\mathcal{P}_j| \mathcal{F}(y)^{\frac{1}{2}} \rfloor$. Using that $J \leq 3 + (\log(1/\mathcal{F}(y)))/(4 \log 2) \leq (\log y)/4$, and $y \geq 10^{16}$ we may check that $K \geq 10$. Moreover, for any prime p in \mathcal{P}_j we have $2^{2j-4} \leq |f(p)|^2 \leq 4/\mathcal{F}(y)^{\frac{1}{2}}$ by the second part of Lemma 3.1, and so $K \leq |\mathcal{P}_j|/8$. Thus the second part of Proposition 3.3 applies, and it shows that

$$\sum_{\substack{n \leq x/y \\ n \in \mathcal{N}_j(K)}} |f(n)|^2 \leq \frac{1}{2} \mathcal{F}(y) \sum_{n \leq x} |f(n)|^2.$$

Therefore,

$$(4.3) \quad \sum_{\substack{n \leq x/y \\ n \notin \mathcal{N}_j(K)}} |f(n)|^2 \geq \frac{1}{2} \mathcal{F}(y) \sum_{n \leq x} |f(n)|^2.$$

If $n \notin \mathcal{N}_j(K)$, then n must be divisible by one of the $\binom{|\mathcal{P}_j|}{K+1}$ integers composed of exactly $K+1$ primes in \mathcal{P} . Each of those numbers exceeds $(\sqrt{y}/2)^{K+1}$. Appealing to the first part of Proposition 3.2 we find that the left-hand side of (4.3) is

$$(4.4) \quad \leq \binom{|\mathcal{P}_j|}{K+1} 2^{2j(K+1)} \mathcal{F}(y(\sqrt{y}/2)^{K+1}) \sum_{n \leq x} |f(n)|^2.$$

Since

$$\binom{|\mathcal{P}_j|}{K+1} \leq \frac{|\mathcal{P}_j|^{K+1}}{(K+1)!} \leq \left(\frac{e|\mathcal{P}_j|}{K+1}\right)^{K+1} < \left(\frac{2^{11}}{2^{2j} \mathcal{F}(y)^{\frac{1}{2}}}\right)^{K+1},$$

and, by the maximality of y ,

$$\mathcal{F}(y(\sqrt{y}/2)^{K+1}) \leq 10^8 \cdot 2^{\frac{K+1}{2}} \frac{1 + \frac{K+3}{2} \log y}{y^{\frac{K+3}{4}}} < 10^8 \cdot 2^{\frac{K+1}{2}} (10^{-8} \mathcal{F}(y))^{\frac{K+3}{2}},$$

we deduce that the quantity in (4.4) is

$$\leq \left(\frac{2^{23}}{10^8}\right)^{\frac{K+1}{2}} \mathcal{F}(y) \sum_{n \leq x} |f(n)|^2 < \frac{1}{2} \mathcal{F}(y) \sum_{n \leq x} |f(n)|^2,$$

which contradicts (4.3). This completes our argument in the second case, and hence also the proof of Theorem 3.

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