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Abstract

We prove that the variance of the current across a characteristic is of order $t^{2/3}$ in a stationary asymmetric simple exclusion process, and that the diffusivity has order $t^{1/3}$. The proof proceeds via couplings to show the corresponding moment bounds for a second class particle.

1. Introduction

The asymmetric simple exclusion process (ASEP) is a Markov process that describes the motion of particles on the one-dimensional integer lattice $\mathbb{Z}$, subject to the exclusion interaction that allows at most one particle at each site. Particles in the process jump one step to the right with rate p and one step to the left with rate $q = 1 - p$, and we assume $0 \le q < p \le 1$. Particles attempt jumps independently of each other, but any attempt to jump onto an already occupied site is suppressed. In Section 2 below we give a rigorous construction of ASEP in terms of Poisson clocks that govern the jump attempts. This process is among the interacting particle systems introduced in Spitzer's seminal paper [25]. We refer the reader to Liggett's monographs [16], [17] for coverage of most of the work on ASEP up to the late 1990's.

In 1994 Ferrari and Fontes [10] proved a central limit theorem for the net particle current seen by an observer moving at a fixed speed v. This quantity that we denote by $J^{(v)}(t)$ is the number of particles that pass the observer from left to right minus the number that pass from right to left during time interval $(0, t]$. The

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particle process is assumed to be stationary with Bernoulli-distributed occupation variables at some average density $\varrho \in (0,1)$. The result is the weak limit of the diffusively rescaled and centered current:

$$\lim_{t\to\infty} \frac{J^{(v)}(t) - \mathbf{E}[J^{(v)}(t)]}{t^{1/2}} = \chi_v.$$

The limit χ_v is a centered Gaussian random variable with variance

$$\sigma^2 = \varrho(1-\varrho)|(p-q)(1-2\varrho) - v|.$$

The interesting phenomenon is the vanishing of the variance at the characteristic speed $v = V^\varrho \equiv (p-q)(1-2\varrho)$. As we explain below, V^ϱ is the speed at which perturbations travel in the system, both at the microscopic particle level (in the expected sense as given in (2.7) below) and at the macroscopic p.d.e. level.

Physical reasoning [26] implied that the correct order of the fluctuations of the current across the characteristic should be $t^{1/3}$. This would explain the degenerate limit under $t^{1/2}$ normalization. These "$t^{1/3}$ fluctuations" remained elusive throughout the 1990's.

The seminal papers of Baik, Deift and Johansson [2] and Johansson [13] gave the first rigorous proofs of such fluctuations. The correct order was verified to be $t^{1/3}$, and the limiting fluctuations were found to obey Tracy-Widom distributions from random matrix theory. The first paper dealt with the last-passage version of Hammersley's process, and the second with the last-passage version of the *totally asymmetric* simple exclusion process (TASEP). Total asymmetry means here that particles are allowed to jump only in one direction at a constant rate, so this is the case $p = 1, q = 0$.

These papers did not study stationary particle processes, but instead processes started from special jam-type deterministic initial conditions. For TASEP this means that initially all sites to the left of the origin are occupied and all sites to the right empty. With such initial conditions the processes could be represented by last-passage percolation models, a point that had been exploited already in the past (among the seminal ones were [1], [22] and [23]). The actual analysis was then performed entirely on combinatorial descriptions of the last-passage model. Later a last-passage representation was also found for a stationary TASEP [18], and then the Tracy-Widom limit proved for the current across the characteristic in that setting [12].

The early proofs of fluctuations relied on a counting argument that utilizes the Robinson-Schensted-Knuth correspondence for Young tableaux, and Gessel's formula that converts certain Schur function sums into Toeplitz determinants. Later this step has been replaced by a more direct connection between the last-passage model and a determinantal point process. The fluctuation limits are then derived by analyzing the asymptotics of the determinant in the appropriate scaling regime. Consequently, while a genuine breakthrough has been achieved, the delicate steps of the proof restrict the reach of the results in several ways. In particular, the particles of the systems are permitted to move in only one direction and admit only the simplest type of jumps.

In the present paper we give the first proof of the accurate order of the fluctuations in systems that are only partially asymmetric. Namely, we show in the original setting of Ferrari-Fontes [10] that the variance of the current across the characteristic in (p, q) ASEP is of order $t^{2/3}$. Our arguments are entirely probabilistic and utilize couplings of several processes and bounds on second class particles. Informally speaking, *second class particles* are perturbations in the system that do not disturb the motion of the regular particles but are influenced by the ambient system. Precise definition must wait for the construction of the coupled processes in Section 2. Presently it appears that there is no way to apply the combinatorial-analytic approach pioneered in [2] to ASEP because there is no last-passage model where the analysis could begin.

We take a key insight from recent work of Cator and Groeneboom [7] and from our joint work with Cator in [4]: this is the idea of coupling processes whose densities differ by $\mathcal{O}(t^{-1/3})$ in order to bound the motion of a second class particle, whose fluctuations in turn are linked to the variance of the current. The couplings we utilize go back to Ferrari, Kipnis and Saada [11] who introduced them to study the microscopic locations of shocks in the particle system.

Fluctuation results for asymmetric exclusion processes have also been stated in terms of a quantity called the *diffusivity* $D(t)$. One way to view the link between current variance and diffusivity involves the second class particle: the variance of the current is the expected absolute deviation of the second class particle, while $tD(t)$ is the variance of the second class particle. For ASEP we also obtain the correct order $t^{1/3}$ for the diffusivity. In Section 2 we state the main result Theorem 2.2 which is a moment bound for the second class particle. Bounds for current variance and diffusivity appear as Corollaries 2.3 and 2.4.

There is also work on the diffusivity with resolvent methods. The resolvent approach can handle more general jump kernels than the nearest-neighbor type (the modifier "simple" in ASEP refers to the restriction to nearest-neighbor jumps). But so far this approach has not yielded optimal bounds. Results in one and two dimensions have appeared in [15] and [27].

The most recent work utilizing the resolvent approach is by Quastel and Valkó [19], [20]. In [19] they show that, for any two finite-range exclusion processes with nonzero-mean jump distributions, the ratio of Laplace transforms of $tD(t)$ is uniformly bounded. In [20] this comparison theorem is paired up with our Corollary 2.4 to get the correct order of the Laplace transform of $tD(t)$ for all these exclusion processes, and with an additional argument also the correct pointwise upper bound $D(t) \le Ct^{1/3}$.

A few more words about the broader context. We see this paper as an opening for a treatment of several other models, such as zero-range and bricklayer processes. For these systems diffusive current fluctuations off the characteristic were established by Balázs in [3].

The $t^{1/3}$ fluctuations with Tracy-Widom limits are universal for some class of asymmetric systems whose precise characterization is not clear at the moment. There is also another universality class among asymmetric systems in one dimension, one where fluctuations occur on the scale $t^{1/4}$ and limits are Gaussian processes related to fractional Brownian motion with Hurst parameter $H = 1/4$. Such results appear in the papers [8], [24], and [5].

Notation. We adhere to the usual notation for particle systems, except that we underline notions that pertain to the entire lattice: so $\underline{\eta}(t) = \{\eta_i(t)\}_{i\in\mathbb{Z}}$ denotes the occupation variables of an exclusion process, and $\underline{\mu} = \mu^{\otimes\mathbb{Z}}$ is the i.i.d. product measure with marginal μ. In general $\widetilde{X}$ denotes a centered random variable: $\widetilde{X} = X - \mathbf{E}X$. C and C_i for $i = 1, 2, 3, \dots$ denote positive constants that can depend on ϱ and p and can change from line to line.

2. **The exclusion process and the results**

Construction of the process and second class particles. The asymmetric simple exclusion process (ASEP) is a Markov process on the state space $\Omega = \{0,\ 1\}^{\mathbb{Z}}$. Given a state $\underline{\omega} = \{\omega_i\}_{i\in\mathbb{Z}} \in \Omega$, the following jumps can happen independently at different sites:

$$(\omega_i,\ \omega_{i+1}) \longrightarrow (\omega_i - 1,\ \omega_{i+1} + 1) \text{ with rate } p\omega_i(1-\omega_{i+1}), \tag{2.1}$$

$$(\omega_i,\ \omega_{i+1}) \longrightarrow (\omega_i + 1,\ \omega_{i+1} - 1) \text{ with rate } q(1-\omega_i)\omega_{i+1}. \tag{2.2}$$

We assume $0 \le q = 1 - p < p \le 1$. The special case $p = 1$ is called TASEP, or the totally asymmetric simple exclusion process.

We interpret the process as representing unlabeled particles that execute independent nearest-neighbor random walks on $\mathbb{Z}$, subject to the exclusion interaction that suppresses attempts to jump to an already occupied site. p is the rate of a particle to jump to the right and q is the rate to jump to the left. The value $\omega_i(t) = 0$ means that site i is vacant at time t, and $\omega_i(t) = 1$ that site i is occupied at time t. The state of the entire process at time t is then $\underline{\omega}(t)$.

A rigorous construction of this process is done by giving each site i two Poisson processes on the time line $[0, \infty)$: a rate p process $N_{i\to i+1}$ and a rate q process $N_{i\to i-1}$. The processes $\{N_{i\to i+1}, N_{i\to i-1} : i \in \mathbb{Z}\}$ are mutually independent, and also independent of the initial configuration $\underline{\omega}(0)$. The rule of evolution is that when $N_{i\to i+1}$ jumps, a particle is moved from i to $i+1$ if i is occupied and $i+1$ is vacant. And similarly with $N_{i\to i-1}$. Thus the rates (2.1)–(2.2) are realized.

Let μ_ϱ denote the measure $\mu_\varrho\{1\} = \varrho = 1-\mu_\varrho\{0\}$ on the set $\{0,1\}$, and let $\underline{\mu}_\varrho = \mu_\varrho^{\otimes\mathbb{Z}}$ be the i.i.d. Bernoulli product measure with mean density ϱ on Ω. It is known that the measures $\{\underline{\mu}_\varrho : 0 \le \varrho \le 1\}$ are the extreme points of the convex and weakly compact set of invariant distributions for the process that are also invariant under spatial translations.

It is convenient to embed the exclusion process in a height process that represents a wall of adjacent columns of bricks. On top of each interval $[i, i+1]$ sits a column of bricks with height $h_i \in \mathbb{Z}$. The entire height configuration is $\underline{h} = \{h_i\}_{i\in\mathbb{Z}}$, restricted to satisfy

$$0 \le h_{i-1} - h_i \le 1 \quad \text{for each } i \tag{2.3}$$

so that the wall slopes downward to the right. Let the Poisson processes govern the evolution of the heights: when $N_{i\to i+1}$ jumps add a brick on top of the column on $[i, i+1]$, and when $N_{i+1\to i}$ jumps remove a brick from the column on $[i, i+1]$. But suppress every step that leads to a violation of (2.3).

Given an initial particle configuration $\underline{\omega}(0)$, define an initial height configuration by

$$h_i(0) = \begin{cases} \sum_{j=i+1}^{0} \omega_i(0) & \text{for } i < 0, \\ 0 & \text{for } i = 0, \\ -\sum_{j=1}^{i} \omega_i(0) & \text{for } i > 0. \end{cases}$$

Let the heights evolve, and define

$$\omega_i(t) = h_{i-1}(t) - h_i(t).$$

Then this process $\underline{\omega}(t)$ is exactly the ASEP constructed earlier, and the height increment $h_i(t) - h_i(0)$ is the net particle current across the bond $(i, i+1)$.

The Poisson construction reveals its power when it is used to run simultaneously several processes started from different initial states. This is called the *basic coupling*. The first observation is that this coupling preserves monotonicity among both particle and height configurations. Ordering is defined sitewise: for particle configurations $\underline{\eta} \le \underline{\omega}$ means that $\eta_i \le \omega_i$ for each $i \in \mathbb{Z}$, and similarly for height configurations $\underline{g} \le \underline{h}$ if $g_i \le h_i$ for each $i \in \mathbb{Z}$. The basic coupling has this property, called attractivity:

$$\underline{\eta}(0) \le \underline{\omega}(0) \Longrightarrow \underline{\eta}(t) \le \underline{\omega}(t) \quad \text{and} \quad \underline{g}(0) \le \underline{h}(0) \Longrightarrow \underline{g}(t) \le \underline{h}(t)$$

for all $t > 0$.

We use the following terminology: if we have two coupled exclusion processes such that $\underline{\eta}(t) \le \underline{\omega}(t)$, then the $\omega - \eta$ *second class particles* are the particles that occupy sites i at which $\omega_i(t) - \eta_i(t) = 1$. The joint process $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$ can be constructed from a two-class process: (i) The first class particles $\underline{\eta}$ obey the ASEP dynamics as described earlier. (ii) The second class particles $d_i = \omega_i - \eta_i$ also obey the Poisson clocks when they can, but they are not allowed to jump on sites occupied by first class particles, and when a first class particle jumps on a site occupied by a second class particle, they swap sites.

Let $\underline{\delta}_i \in \Omega$ denote a configuration that has only a single particle at site i. If $\underline{\eta} \in \Omega$ is such that $\eta_0 = 0$, we can legitimately define $\underline{\eta}^+ = \underline{\eta} + \underline{\delta}_0$. In this situation we say that there is a single second class particle between $\underline{\eta}^+$ and $\underline{\eta}$ at site 0. Since the basic coupling conserves the single second class particle, there is always a site $Q(t)$ such that

$$\underline{\eta}^+(t) = \underline{\eta}(t) + \underline{\delta}_{Q(t)}. \tag{2.4}$$

$Q(t)$ is the position of the second class particle at time t, which performs a nearest neighbor walk, influenced by the ambient process $\underline{\eta}(\cdot)$.

It is convenient to also have the notion of a second class *anti*particle at position $Q_a(t)$ in a process $\underline{\omega}(t)$. This means that $Q_a(t)$ is the location of the single discrepancy between two processes $\underline{\omega}(t)$ and $\underline{\omega}^-(t)$ that are started so that $\underline{\omega}^-(0) = \underline{\omega}(0) - \underline{\delta}_i$ where $i = Q_a(0)$. A moment's reflection reveals that in fact in the basic coupling of ASEP a second class particle and an antiparticle are the same thing. But in the proofs we will couple more than two processes and this extra flexibility will be convenient.

Current fluctuations and diffusivity. Let $[x]$ denote the first integer from x towards the origin, in other words $[x] = \lfloor x \rfloor$ (floor) when $x \geq 0$ and $[x] = \lceil x \rceil$ (ceiling) when $x < 0$. For a speed value $V \in \mathbb{R}$ define

$$J^{(V)}(t) = h_{[Vt]}(t), \tag{2.5}$$

the height of the column over interval $[[Vt], [Vt]+1]$ at time t. Due to the normalization $h_0(0) = 0$, $J^{(V)}(t)$ is the total net particle current seen by an observer moving at speed V during time interval $[0, t]$. Or more concretely, $J^{(V)}(t) = J_+^{(V)}(t) - J_-^{(V)}(t)$ where $J_+^{(V)}(t)$ is the number of particles that began in $(-\infty, 0]$ at time 0 but lie in $[[Vt]+1, \infty)$ at time t, and $J_-^{(V)}(t)$ is the number of particles that began in $[1, \infty)$ at time 0 but lie in $(-\infty, [Vt]]$ at time t. One can compute the density ϱ equilibrium expectation

$$\mathbf{E}J^{(V)}(t) = t(p-q)\varrho(1-\varrho) - \varrho[Vt]$$

by writing a martingale for $h_0(t)$ and then adding in $h_{[Vt]}(t) - h_0(t)$ which counts particles between sites 0 and $[Vt]$.

Our results are based on an interplay between currents and second class particles. One key fact is the next connection.

PROPOSITION 2.1. *Let $\underline{\omega}(\cdot)$ be an* ASEP *started from its stationary Bernoulli distribution $\underline{\mu}_\varrho$. Condition the origin to be empty and let $Q(\cdot)$ be a second class particle that starts at $Q(0) = 0$. Alternately condition the origin to be occupied and start a second class antiparticle $Q_a(\cdot)$ at the origin. Either situation can be used to compute the variance of the current of the stationary process for any $V \in \mathbb{R}$*:

$$\begin{aligned}\mathbf{Var}(J^{(V)}(t)) &= \varrho(1-\varrho)\mathbf{E}\big(\,|\,[Vt] - Q(t)|\,\big|\,\omega_0(0) = 0\big)\\ &= \varrho(1-\varrho)\mathbf{E}\big(\,|\,[Vt] - Q_a(t)|\,\big|\,\omega_0(0) = 1\big).\end{aligned}\tag{2.6}$$

We shall also have occasion to use the following identity (true under the assumptions of the proposition above):

$$\mathbf{E}\big(Q(t)\,\big|\,\omega_0(0) = 0\big) = \mathbf{E}\big(Q_a(t)\,\big|\,\omega_0(0) = 1\big) = t(p-q)(1-2\varrho).\tag{2.7}$$

Equations (2.6) and (2.7) have been derived and utilized earlier for TASEP, (2.6) by Ferrari and Fontes [10] and (2.7) by Prähofer and Spohn [18]. In article [6] identities (2.6) and (2.7) are proved not only for ASEP but also for zero-range and bricklayer processes.

The interesting current fluctuations occur at the *characteristic speed* $V^\varrho = (p-q)(1-2\varrho)$. From (2.7) we see that this is the average speed of the second class particle. To explain the significance of V^ϱ from another perspective, we digress briefly to discuss a large scale, deterministic view of exclusion dynamics.

On large space and time scales the macroscopic particle density $\varrho(t,x)$ of ASEP obeys the conservation law

$$\varrho_t + f(\varrho)_x = 0\tag{2.8}$$

with flux $f(\varrho) = (p-q)\varrho(1-\varrho)$. This notion is made precise through a law of large numbers called a *hydrodynamic limit*. One considers a sequence of processes $\underline{\omega}^{(n)}(t)$ such that, for each n, the initial occupation variables are independent with means $\mathbf{E}\omega_i^{(n)}(0) = \varrho_0(i/n)$ for a given continuous function ϱ_0. Then we have the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{na\le i\le nb} \omega_i^{(n)}(nt) = \int_a^b \varrho(t,x)\,\mathrm{d}x \quad \text{in probability}\tag{2.9}$$

for all intervals $[a,b]$. The limiting density $\varrho(t,x)$ is the entropy solution of (2.8) with initial condition $\varrho(0,x) = \varrho_0(x)$. Away from shocks the *characteristics* of the partial differential equation (2.8) are solutions of the ordinary differential equation $\dot{x} = f'(\varrho(t,x))$. At constant density ϱ the characteristic speed is $f'(\varrho) = V^\varrho$.

Limit (2.9) has been proved over time by a number of authors under various hypotheses. Article [21] was the first to achieve a satisfactory level of generality for a broad class of asymmetric systems, also in higher dimensions. For more information we refer to [14] for hydrodynamic limits of interacting particle systems and to [9] for the theory of scalar conservation laws.

Let us return to the stationary ASEP with Bernoulli-ϱ occupation variables at each fixed time. A basic object for understanding space-time correlations is the *two point function* $S(i,t) = \mathbf{E}[\omega_i(t)\omega_0(0)] - \varrho^2$. It can be written as the transition probability of the second class particle:

$$
\begin{aligned}
S(i,t) &= \varrho\big(\mathbf{E}[\omega_i(t) \,|\, \omega_0(0) = 1] - \mathbf{E}[\omega_i(t)]\big) \\
&= \varrho(1-\varrho)\big(\mathbf{E}[\omega_i(t) \,|\, \omega_0(0) = 1] - \mathbf{E}[\omega_i(t) \,|\, \omega_0(0) = 0]\big) \\
&= \varrho(1-\varrho)\mathbf{E}[\,\omega_i^+(t) - \omega_i(t) \,|\, \omega_0^+(0) = 1,\ \omega_0(0) = 0\,] \\
&= \varrho(1-\varrho)\mathbf{P}(Q(t) = i \mid \omega_0(0) = 0).
\end{aligned}
\tag{2.10}
$$

On the third line above we coupled $\underline{\omega}(\cdot)$ with a process $\underline{\omega}^+(\cdot)$ whose initial configurations satisfy $\underline{\omega}^+(0) = \underline{\omega}(0) + \underline{\delta}_0$. In other words, the second class particle resides initially at the origin. Note that use of the term "transition probability" is not meant to suggest that $Q(t)$ is a Markov process.

From (2.10) and (2.7) follow the properties

$$
\sum_{i\in\mathbb{Z}} S(i,t) = \varrho(1-\varrho) \quad \text{and} \quad \sum_{i\in\mathbb{Z}} iS(i,t) = \varrho(1-\varrho)V^\varrho t.
$$

The *diffusivity* is by definition a normalized second moment of the two-point function:

$$
D(t) = \frac{1}{t\varrho(1-\varrho)} \sum_{i\in\mathbb{Z}} (i - V^\varrho t)^2 S(i,t).
\tag{2.11}
$$

Consequently the diffusivity can also be expressed in terms of the variance of the second class particle:

$$
D(t) = t^{-1}\mathbf{Var}(Q(t) \,|\, \omega_0(0) = 0).
$$

If the second class particle behaved like a random walk (diffusively) then its variance would be of order t, and thereby $D(t)$ of constant order. The interesting point is that the ASEP second class particle is *superdiffusive* with variance of order $t^{4/3}$.

We can now state the main theorem, a moment bound for the second class particle in the setting where all the other occupation variables start off Bernoulli-distributed.

THEOREM 2.2. *For* ASEP $\underline{\omega}(\cdot)$ *with rates* $0 \le q = 1 - p < p \le 1$, *started in* $\underline{\mu}_\varrho$ *distribution with any* $\varrho \in (0,1)$, *and for any real* $1 \le m < 3$, *there exist constants* $0 < t_0$, $C < \infty$ *such that, for* $t \ge t_0$,

$$
C^{-1} \le \mathbf{E}\bigg\{ \bigg| \frac{Q(t) - V^\varrho t}{t^{2/3}} \bigg|^m \,\bigg|\, \omega_0(0) = 0 \bigg\} \le C.
\tag{2.12}
$$

Let us simplify notation to $J^\varrho(t) = J^{(V^\varrho)}(t)$ for the current across the characteristic. Via (2.6), taking $m = 1$ gives the variance of this current.

COROLLARY 2.3. *With assumptions as in Theorem 2.2, for $t \ge t_0$ the variance of the current across the characteristic satisfies*

$$C^{-1}t^{2/3} \le \mathbf{Var}(J^{\varrho}(t)) \le Ct^{2/3}.$$

Taking $m = 2$ identifies the order of the diffusivity.

COROLLARY 2.4. *With assumptions as in Theorem 2.2, for $t \ge t_0$*

$$C^{-1}t^{1/3} \le D(t) \le Ct^{1/3}.$$

The upper bound of (2.12) is not valid for all $t > 0$. For small t the variance of $Q(t)$ is of order t because the second class particle is likely to have experienced at most one jump.

The rest of the paper is devoted to the proof of Theorem 2.2, first the upper bound and then the lower bound.

3. **Upper bound**

Proof of the upper bound utilizes couplings of several processes and a tagged second class particle introduced by Ferrari, Kipnis and Saada [11]. We refer the reader to the exposition of these ideas in Section III.2 in Liggett's second monograph [17]. Before deriving the upper bound we introduce some preliminaries on the couplings.

Couplings. We start by describing an initial distribution on three classes of particles and with labels attached to particles of the two lower classes.

Fix densities $0 \le \lambda < \varrho < 1$. Define the measure μ on $\{0, 1\} \times \{0, 1\}$ by

$$\mu(0,0) = 1-\varrho, \quad \mu(0,1) = \varrho - \lambda, \quad \mu(1,0) = 0, \quad \text{and} \quad \mu(1,1) = \lambda. \tag{3.1}$$

The first marginal of μ is μ_λ and the second is μ_ϱ. Let $\underline{\mu} = \mu^{\otimes \mathbb{Z}}$ be the i.i.d. product measure on $[\{0, 1\} \times \{0, 1\}]^{\mathbb{Z}}$ with marginals μ. (Our notation for $\underline{\mu}$ differs from that in [17] where the second variable of $\underline{\mu}$ is the process of second class particles.) We condition this measure on having an $\omega - \eta$ second class particle at the origin, and define the measure $\underline{\hat{\mu}}$ by

$$\underline{\hat{\mu}}(\mathrm{d}\underline{\eta},\ \mathrm{d}\underline{\omega}) = \underline{\mu}(\mathrm{d}\underline{\eta},\ \mathrm{d}\underline{\omega} \,|\, \eta_0 = 0,\ \omega_0 = 1).$$

Then $\underline{\hat{\mu}}$-a.s. we have the following conditions for $\underline{\eta}$ and $\underline{\omega}$:

- $\underline{\eta} \le \underline{\omega}$,
- there is a second class particle between $\underline{\eta}$ and $\underline{\omega}$ at the origin, and there are infinitely many of them on both sides of the origin.

Label the $\omega - \eta$ second class particles with integers in an increasing fashion from left to right, giving label 0 to the one initially at the origin. $X_k(0)$ denotes the position of second class particle with label k, so that

$$\cdots < X_{-1}(0) < X_0(0) = 0 < X_1(0) < \cdots . \tag{3.2}$$

Given $(\underline{\eta}, \underline{\omega})$ chosen from distribution $\underline{\bar{\mu}}$, define a third configuration $\underline{\zeta}$ as follows. Let $\zeta_i = \omega_i$ whenever $\eta_i = \omega_i$. At all other sites i we have an $\omega - \eta$ second class particle: $0 = \eta_i < \omega_i = 1$. If this second class particle has label n, then let

$$\zeta_i = \zeta_{X_n(0)} = \begin{cases} \eta_i = 0, & \text{with probability } \dfrac{1}{1+(p/q)^n}, \\[2ex] \omega_i = 1, & \text{with probability } \dfrac{(p/q)^n}{1+(p/q)^n}, \end{cases} \tag{3.3}$$

where these choices are independent from each other and everything else. For the case $p = 1$ the probabilities are defined as the limits of the above formulas as $p = 1 - q \nearrow 1$.

Let $\underline{\bar{\mu}}^*$ denote the resulting joint distribution of $(\underline{\eta}, \underline{\zeta}, \underline{\omega})$. It is clear that $\underline{\bar{\mu}}^*$-a.s. we have $\underline{\eta} \le \underline{\zeta} \le \underline{\omega}$.

Let these configurations evolve from the initial distribution $\underline{\bar{\mu}}^*$ with common Poisson clocks. This basic coupling preserves the ordering $\underline{\eta}(t) \le \underline{\zeta}(t) \le \underline{\omega}(t)$ a.s. for all times t. The effect of the coupling is that the $\zeta - \eta$ particles have priority over the $\omega - \zeta$ particles.

Consider the evolution of the the $\omega - \eta$ second class particles in this coupling. They start off labeled with integers as described above (3.2). We let each particle keep its integer label and let $X_k(t)$ denote the position of particle labeled k at time t. Since only nearest-neighbor jumps are permitted, the ordering $X_k(t) < X_{k+1}(t)$ is preserved. Additionally, particle k carries a ζ-mark $\zeta_{X_k(t)}(t) \in \{0, 1\}$, initially assigned by (3.3). $\zeta_{X_k(t)}(t) = 0$ indicates that in the further subdivision into second and third class particles, at time t position $X_k(t)$ is occupied by a third class particle. These ζ-marks are not retained but exchanged. Whenever X_k and X_{k+1} are neighbors and carry different ζ-marks $(1, 0)$ or $(0, 1)$, then $(1, 0)$ becomes $(0, 1)$ with rate p and $(0, 1)$ becomes $(1, 0)$ with rate q. This is just a restatement of the effect of the basic coupling when a $\zeta - \eta$ particle is next to an $\omega - \zeta$ particle.

The $\omega - \eta$ second class particle that starts at the origin is of special importance to us, so we set $X(t) = X_0(t)$.

PROPOSITION 3.1. *Couple the processes as described above with initial distribution $\underline{\bar{\mu}}^*$. Then for any fixed time $t \in [0, \infty)$ the marks $\{\zeta_{X_k(t)}(t) : k \in \mathbb{Z}\}$ are independent of the entire process $\{(\underline{\eta}(s), \underline{\omega}(s)) : s \ge 0\}$. Furthermore, the distribution of marks is constant in time: given the evolution $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$, for all*

$0 \le t < \infty$ *and disjoint sets* I_0 *and* I_1 *of integer labels*,

$$\mathbf{P}\{\zeta_{X_k(t)}(t) = 0 \textit{ for } k \in I_0 \textit{ and } \zeta_{X_n(t)}(t) = 1 \textit{ for } n \in I_1 \mid \underline{\eta}(\cdot), \underline{\omega}(\cdot)\}$$
$$= \prod_{k\in I_0} \frac{1}{1+(p/q)^k} \cdot \prod_{n\in I_1} \frac{(p/q)^n}{1+(p/q)^n}.$$

This statement is hinted at in a remark after Proposition III.2.13 in [17]. We summarize the argument for the reader's convenience. Observe first that we can construct the process of $(\underline{\eta}, \underline{\omega})$-particles and ζ-labels with two independent collections of Poisson clocks, one that governs the process $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$, and another one that governs the exchanges of ζ-marks among the $\omega - \eta$ second class particles. This construction is not the same as the basic coupling described above, but it leads to the same process because the infinitesimal rates are the same.

Consequently we can first construct the $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$ evolution for all time, and then superimpose on it the evolution of the ζ-marks. The claim is that, given the entire evolution $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$, the mark configuration $\{\zeta_{X_k(t)}(t) : k \in \mathbb{Z}\}$ on the $\omega - \eta$ second class particles has distribution (3.3).

The key point is reversibility. Let us abbreviate temporarily $u_k(t) = \zeta_{X_k(t)}(t)$ for the mark of second class particle X_k at time t. If the process $\underline{u}(\cdot)$ obeyed the (p, q)-ASEP dynamics, the product distribution with marginals (3.3) would be one of the well-known blocking measures that are reversible for $\underline{u}(\cdot)$. (This would in fact be the case if $\lambda = 0$ and $\varrho = 1$, for then the second class particles would occupy all sites and never move: $X_k(t) = k$ for all $0 \le t < \infty$.) The result now follows from this observation: the evolution $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$ can be thought of as a dynamical "environment" for the mark process that admits or prohibits certain exchanges at different times: X_k and X_{k+1} cannot exchange marks unless they occupy adjacent sites, and the time intervals during which this happens are determined by $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$. However, imposing such an environment on the process does not change the reversibility of the measure.

This last point can be checked rigorously by first letting only finitely many marks $U_K(t) = \{u_k(t) : -K \le k \le K\}$ evolve while the remaining marks are frozen. Then we are talking about a finite state Markov chain. Given the evolution $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$, we can partition the time axis $0 = t_0 < t_1 < t_2 < \cdots < t_i \nearrow \infty$ so that the generator of the chain $U_K(\cdot)$ does not change during (t_{i-1}, t_i). Since detailed balance is not violated by symmetrically prohibiting certain jumps, reversibility and hence invariance is preserved during each interval (t_{i-1}, t_i), and thereby for all time. Letting $K \to \infty$ extends the invariance to the infinite mark process. This argument establishes Proposition 3.1.

The purpose of the construction is to confine certain special particles to be introduced shortly. Notice that a.s. there are only finitely many positive labels

$n > 0$ for which the first event of (3.3) happens. Hence we have a rightmost position $R(0) = \sup\{i : \eta_i = \zeta_i < \omega_i\}$. This is the initial position of the rightmost second class particle between $\underline{\zeta}$ and $\underline{\omega}$. Similarly, the second event in (3.3) happens a.s. for only finitely many negative $n < 0$. Let $L(0) = \inf\{i : \eta_i < \zeta_i = \omega_i\}$ be the leftmost position where this happens, the initial position of the leftmost second class particle between $\underline{\eta}$ and $\underline{\zeta}$. Define the events

$$\mathcal{A} = \{0 \le R(0)\} \qquad \text{and} \qquad \mathcal{B} = \{L(0) \le 0 \le R(0)\}.$$

These events only depend on the marks (3.3), and hence are independent of $(\underline{\eta}, \underline{\omega})$. $\mathcal{A}$ always has positive probability, while $\mathcal{B}$ has positive probability if $p < 1$.

These positions evolve in the coupling, so at time t we let $R(t)$ be the position of the rightmost second class particle between $\underline{\zeta}(t)$ and $\underline{\omega}(t)$, and similarly $L(t)$ is the position of the leftmost second class particle between $\underline{\eta}(t)$ and $\underline{\zeta}(t)$. $L(t)$ and $R(t)$ are always among the $\omega - \eta$ second class particles $\{X_k(t)\}$. Let $n_L(t)$ and $n_R(t)$ be the random labels such that $L(t) = X_{n_L(t)}(t)$ and $R(t) = X_{n_R(t)}(t)$. These labels are functions of the marks:

$$n_L(t) = \inf\{k : \zeta_{X_k(t)}(t) = 1\} \quad \text{and} \quad n_R(t) = \sup\{k : \zeta_{X_k(t)}(t) = 0\}. \tag{3.4}$$

By the above proposition $n_L(t)$ and $n_R(t)$ are independent of $(\underline{\eta}(t), \underline{\omega}(t))$ and their distribution does not change with time.

To complete the construction we define two more initial configurations by

$$\underline{\eta}^+ = \underline{\eta} + \underline{\delta}_0 \qquad \text{and} \qquad \underline{\omega}^- = \underline{\omega} - \underline{\delta}_0$$

where $\underline{\delta}_0$ denotes a configuration with a single particle at the origin, as explained in Section 2. The basic coupling then applies to all five processes $\underline{\eta}(\cdot)$, $\underline{\eta}^+(\cdot)$, $\underline{\zeta}(\cdot)$, $\underline{\omega}^-(\cdot)$, $\underline{\omega}(\cdot)$ with the above initial data. The orderings

$$\underline{\eta}(t) \le \underline{\zeta}(t) \le \underline{\omega}(t), \qquad \underline{\eta}(t) \le \underline{\omega}^-(t) \le \underline{\omega}(t), \qquad \underline{\eta}(t) \le \underline{\eta}^+(t) \le \underline{\omega}(t) \tag{3.5}$$

are preserved by the evolution. There is always one difference between $\underline{\eta}(\cdot)$ and $\underline{\eta}^+(\cdot)$ which is thought of as a single second class particle on $\underline{\eta}(\cdot)$ in the sense (2.4). Denote its position by $Q(t)$ with $Q(0) = 0$. Also, $\underline{\omega}(\cdot)$ and $\underline{\omega}^-(\cdot)$ always have a single second class particle between them, thought of as a second class antiparticle on $\underline{\omega}(\cdot)$ as described in Section 2. Its position is $Q_a(t)$, and $Q_a(0) = 0$. See Figure 1 for an illustration of the initial configurations in this coupling.

The bounds will be proved by controlling the particles $Q(t)$ and $Q_a(t)$. The next lemma contains one of the key points.

LEMMA 3.2. *In the coupling of five processes we have these implications*:

$$\mathcal{A} \subset \{Q_a(t) \le R(t) \textit{ for all } t \ge 0\} \tag{3.6}$$

and

$$\mathcal{B} \subset \{Q_a(t) \le R(t) \textit{ and } L(t) \le Q(t) \textit{ for all } t \ge 0\}. \tag{3.7}$$

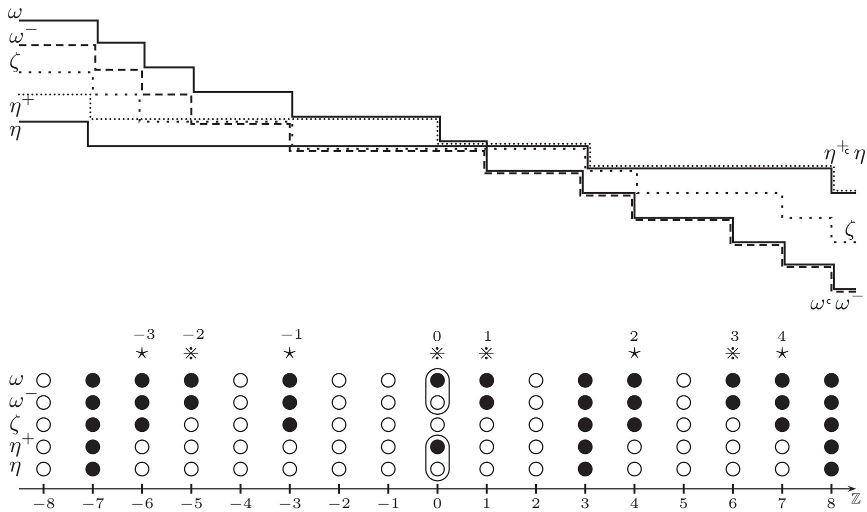

Figure 1. A realization of the initial state of the five coupled processes. The top portion of the figure shows the height functions and the bottom portion the corresponding particle configurations. The lattice $\mathbb{Z}$ runs at the bottom. Solid disks denote particles and open circles denote vacant sites. Above the particles, $\divideontimes$ marks locations of $\omega - \zeta$ particles (first case in (3.3)) and $\star$ marks locations of $\zeta - \eta$ particles (second case in (3.3)). Together these make up the $\omega - \eta$ second class particles and the numbers above these symbols represent the labeling (3.2). Both events $\mathscr{A}$ and $\mathscr{B}$ happen in this picture. The particle-hole pairs inside the ovals represent the initial locations of Q_a and Q.

Proof. We prove (3.6) and leave the similar argument for (3.7) to the reader. Under the event $\mathscr{A}$, $Q_a(0) = 0 \le R(0)$ holds initially. That is, we have a $\omega - \zeta$ second class particle at or to the right of the second class antiparticle Q_a. We show that the coupling preserves the inequality $Q_a(t) \le R(t)$. We have two cases to consider that could potentially allow $Q_a > R$ to happen.

- Suppose the ordering $\underline{\zeta}(s) \le \underline{\omega}^-(s) \le \underline{\omega}(s)$ holds at some time s. Since the ordering is then preserved for all later times $t \ge s$, $\zeta_{Q_a(t)}(t) \le \omega^-_{Q_a(t)}(t) < \omega_{Q_a(t)}(t)$, which implies an $\omega - \zeta$ second class particle at location $Q_a(t)$. As $R(t)$ is the position of the rightmost such second class particle, $Q_a(t) \le R(t)$ holds.
- Consequently the only possibility for producing $Q_a > R$ is to have a jump in a situation of this type: $Q_a(t) = i$, $R(t) = i + 1$ for some site i, and $\omega^-_i(t) = 0$,

$\zeta_i(t) = \omega_i(t) = 1$, $\zeta_{i+1}(t) = 0$, $\omega^-_{i+1}(t) = \omega_{i+1}(t) = 1$. In this case column i of $\underline{\zeta}$ increases by one with rate p, or column i of $\underline{\omega}^-$ decreases by one with rate q. Neither one of these steps interchanges Q_a and R.

This proves (3.6). □

For applications of this lemma it is crucial that the events $\mathscr{A}$ and $\mathscr{B}$ depend only on the initial marks (3.3) which were independent of the initial configuration $(\underline{\eta}, \underline{\omega})$. Consequently $\mathscr{A}$ and $\mathscr{B}$ are independent of the joint evolution $(\underline{\eta}(\cdot), \underline{\omega}(\cdot), X(\cdot), Q(\cdot), Q_a(\cdot))$. When we condition the processes on $\mathscr{A}$ or $\mathscr{B}$ we call this construction a *conditional coupling*.

Proof of the upper bound. Throughout this section the probability measure $\mathbf{P}$ represents the five-process coupling constructed in the previous section. We begin with the proof of the upper bound in Theorem 2.2. C and its variants $C_1, C_2, \dots$ denote positive constants that possibly depend on p and ϱ and whose values can change from line to line. We first prove a lower bound on the density of the $\omega - \eta$ second class particles. For integers $j \in \mathbb{Z}$ and $u > 0$ let

$$N_j(t) = \sum_{i=j+1}^{j+2u} (\omega_i(t) - \eta_i(t)). \tag{3.8}$$

LEMMA 3.3. *Let $\lambda, \varrho \in (0, 1)$ and $d \geq 0$ be an integer. Then there are strictly positive finite constants $\gamma = \gamma(\varrho)$, $C_1 = C_1(\varrho, d)$ and $C_2 = C_2(\varrho)$ such that the following holds: if $0 < \varrho - \lambda < \gamma$, then for all integers $j \in \mathbb{Z}$ and $u > 0$ and any time $t \geq 0$,*

$$\mathbf{P}\Big\{N_j(t) < u(\varrho - \lambda) + d\Big\} \leq C_1 \exp\{-C_2 u(\varrho - \lambda)^2\}.$$

Proof. For the moment, denote by $\underline{y}(\cdot)$ a (p, q) exclusion process such that $y_i(0) = \omega_i(0)$ at all sites i except for $i = 0$, and $\underline{z}(\cdot)$ is a (p, q) exclusion process such that $z_i(0) = \eta_i(0)$ at all sites i except for $i = 0$. For $i = 0$, we pick the pair $(z_0(0), y_0(0))$ in distribution μ (3.1), independently of the configuration on other sites. Apply the basic coupling to ensure $\underline{\eta}(t) \leq \underline{z}(t) \leq \underline{y}(t) \leq \underline{\omega}(t)$ for all $t \geq 0$ (notice that this holds initially). $\underline{y}(t)$ and $\underline{z}(t)$ are marginally time-stationary processes, hence we can omit the notation for their time dependence in our arguments. However, the pair $(\underline{z}(t), \underline{y}(t))$ is *not* in product distribution for $t > 0$. Define

$$Y = \sum_{i=j+1}^{j+2u} y_i \qquad \text{and} \qquad Z = \sum_{i=j+1}^{j+2u} z_i,$$

so that $N_j(t) \geq Y - Z$. Then for any $\alpha > 0$,

$$
\begin{aligned}
\mathbf{P}\Big\{N_j(t) < u(\varrho-\lambda)+d\Big\} &\le \mathbf{P}\Big\{\mathrm{e}^{-\alpha(Y-Z)} > \mathrm{e}^{-\alpha u(\varrho-\lambda)-d\alpha}\Big\}\\
&\le \mathrm{e}^{\alpha u(\varrho-\lambda)+d\alpha}\cdot \mathbf{E}\big(\mathrm{e}^{-\alpha Y}\mathrm{e}^{\alpha Z}\big)\\
&\le \mathrm{e}^{\alpha u(\varrho-\lambda)+d\alpha}\cdot \big[\mathbf{E}\big(\mathrm{e}^{-2\alpha Y}\big)\big]^{1/2}\cdot\big[\mathbf{E}\big(\mathrm{e}^{2\alpha Z}\big)\big]^{1/2}\\
&= \mathrm{e}^{\alpha u(\varrho-\lambda)+d\alpha}\cdot \big[\mathbf{E}\big(\mathrm{e}^{-2\alpha y_0}\big)\big]^{u}\cdot\big[\mathbf{E}\big(\mathrm{e}^{2\alpha z_0}\big)\big]^{u}\\
= \exp\big\{\alpha u(\varrho-\lambda)+d\alpha + u\log\big[1+\varrho(\mathrm{e}^{-2\alpha}-1)\big] &+ u\log\big[1+\lambda(\mathrm{e}^{2\alpha}-1)\big]\big\}\\
\le \exp\big\{-\alpha u(\varrho-\lambda)+d\alpha+2\alpha^2 u(\varrho+\lambda-\varrho^2-\lambda^2)+uC\alpha^3\big\}.&
\end{aligned}
$$

Here we used the marginal $\underline{\mu}_\lambda$ and $\underline{\mu}_\varrho$ product distributions of $\underline{z}$ and $\underline{y}$, while the last inequality comes from Taylor expansion w.r.t. α. The $\mathcal{O}(\alpha^3)$ term is uniform over $\lambda < \varrho$ for a fixed ϱ. Next we pick

$$
\alpha = \frac{\varrho-\lambda}{4(\varrho+\lambda-\varrho^2-\lambda^2)},
$$

this optimizes the ϱ and λ-dependent terms, and finishes the proof. $\square$

We turn to develop the main estimate for the upper bound. The objective is to bound the deviation $\mathbf{P}\{Q_a(t) \ge u + [V^\varrho t]\}$ with a suitable expression that involves the moment $\mathbf{E}|Q_a(t) - [V^\varrho t]|$. This is reached in (3.16) below and then the upper bound comes from an elementary integration step.

Along the way we compare currents in two processes that we abbreviate as follows:

$$
\begin{aligned}
J^\varrho(t) &= J^{((p-q)(1-2\varrho))}(t) \quad \text{for current in the } \underline{\omega}(\cdot) \text{ process, and}\\
J^{V^\varrho,\lambda}(t) &= J^{((p-q)(1-2\varrho))}(t) \quad \text{for current in the } \underline{\eta}(\cdot) \text{ process.}
\end{aligned}
$$

Notice that both use the same speed $V^\varrho = f'(\varrho) = (p-q)(1-2\varrho)$ for the observer. As already defined in the Introduction, this is the characteristic speed of the $\underline{\omega}(\cdot)$ process with density ϱ. Let $A = (\mathbf{P}\{\mathcal{A}\})^{-1} < \infty$ from the conditional coupling, and recall that a tilde centers a random variable.

LEMMA 3.4. *Suppose $\varrho - \lambda < \gamma$ with γ from Lemma 3.3. Then for positive integers u and times $t \in [0,\infty)$,*

$$
\begin{aligned}
&\mathbf{P}\{Q_a(t) \ge 4u + [V^\varrho t]\}\\
&\qquad \le A\mathbf{P}\{\widetilde{J}^\varrho(t) - \widetilde{J}^{V^\varrho,\lambda}(t) \ge u(\varrho-\lambda) - t(p-q)(\varrho-\lambda)^2\}\\
&\qquad\qquad + C_1\exp\{-C_2 u(\varrho-\lambda)^2\}.
\end{aligned} \tag{3.9}
$$

Proof. By the independence observed after the proof of Lemma 3.2, conditioning on $\mathscr{A}$ does not affect the probability of the event of interest. By (3.6),

$$\begin{aligned}
&\mathbf{P}\{Q_a(t) \geq 4u + [V^\varrho t]\} = \mathbf{P}\{Q_a(t) \geq 4u + [V^\varrho t] \mid \mathscr{A}\} \\
&\quad \leq \mathbf{P}\{R(t) \geq 4u + [V^\varrho t] \mid \mathscr{A}\} \leq A\mathbf{P}\{R(t) \geq 4u + [V^\varrho t]\} \\
&\quad \leq A\mathbf{P}\{X(t) \geq 2u + [V^\varrho t]\} + A\mathbf{P}\{R(t) \geq 4u + [V^\varrho t],\ X(t) < 2u + [V^\varrho t]\}.
\end{aligned}$$

Recall that $N_{[V^\varrho t]}(t)$ of (3.8) counts the number of $\omega - \eta$ second class particles at time t in the interval $\{[V^\varrho t] + 1, \dots, [V^\varrho t] + 2u\}$. Since the second class particles stay ordered and $X(t)$ started at the origin, the event $\{X(t) \geq 2u + [V^\varrho t]\}$ implies that all these second class particles crossed the path $s \mapsto [V^\varrho s] + 1/2$ by time t. Each such second class particle crossing increases $J^\varrho(t) - J^{V^\varrho,\lambda}(t)$ by one. Therefore

$$\begin{aligned}
&\mathbf{P}\{X(t) \geq 2u + [V^\varrho t]\} \leq \mathbf{P}\{J^\varrho(t) - J^{V^\varrho,\lambda}(t) \geq N_{[V^\varrho t]}(t)\} \\
&\qquad \leq \mathbf{P}\Big\{J^\varrho(t) - J^{V^\varrho,\lambda}(t) \geq u(\varrho - \lambda) + 3\Big\} + \mathbf{P}\Big\{N_{[V^\varrho t]}(t) < u(\varrho - \lambda) + 3\Big\}.
\end{aligned}$$

Combine the previous displays to get

$$\mathbf{P}\{Q_a(t) \geq 4u + [V^\varrho t]\} \leq A\mathbf{P}\Big\{J^\varrho(t) - J^{V^\varrho,\lambda}(t) \geq u(\varrho - \lambda) + 3\Big\} \tag{3.10}$$

$$+ A\mathbf{P}\Big\{N_{[V^\varrho t]}(t) < u(\varrho - \lambda) + 3\Big\} \tag{3.11}$$

$$+ A\mathbf{P}\{R(t) \geq 4u + [V^\varrho t],\ X(t) < 2u + [V^\varrho t]\}. \tag{3.12}$$

To line (3.11) apply Lemma 3.3 with $d = 3$. The event in (3.12) can happen in two ways:

- The label $n_R(t)$ of $R(t)$ is larger than $u(\varrho - \lambda)$. The probability of this is, according to (3.3) and its time-invariance, bounded by

$$\sum_{n=\lceil u(\varrho-\lambda)\rceil}^{\infty} \frac{1}{1 + (p/q)^n} \leq \sum_{n=\lceil u(\varrho-\lambda)\rceil}^{\infty} (q/p)^n \leq \frac{p}{p-q} \cdot (q/p)^{u(\varrho-\lambda)}.$$

- There are fewer than $u(\varrho - \lambda)$ $\omega - \zeta$ second class particles in the discrete interval $\{[V^\varrho t] + 2u, \dots, [V^\varrho t] + 4u - 1\}$. The probability of this event is bounded in Lemma 3.3.

Hence line (3.12) is bounded by

$$C_1 \exp\{-uC_2(\varrho - \lambda)^2\} + Ap(p-q)^{-1} \exp\{-uC_3(\varrho - \lambda)\}.$$

By modifying the constants, we see that lines (3.11) and (3.12) are bounded by the last exponential term in (3.9).

If $\underline{\omega}$ and $\underline{\eta}$ start from their respective $\underline{\mu}_{\varrho}$ and $\underline{\mu}_{\lambda}$ equilibria, then we would have

$$
\begin{aligned}
\mathbf{E}(J^{\varrho}(t)) - \mathbf{E}(J^{V^{\varrho},\lambda}(t)) &= t\big(f(\varrho) - f(\lambda) - f'(\varrho)(\varrho-\lambda)\big) \\
&= -\tfrac{1}{2} t f''(\varrho)(\varrho-\lambda)^2 \\
&= t(p-q)(\varrho-\lambda)^2,
\end{aligned}
\tag{3.13}
$$

where we utilized the precise form $f(\varrho) = (p-q)\varrho(1-\varrho)$ of the flux and ignored the error coming from the integer part of $V^{\varrho}t$. This last error is at most one. Our processes are also perturbed initially at the origin by the conditioning in $\underline{\mu}$, which gives an error not larger than 2 in the above quantity. The term $+3$ inside the probability on line (3.10) makes up for these errors. $\square$

Remark. We spelled out calculation (3.13) because the nonvanishing of f'' is a crucial factor that produces the order $t^{2/3}$ for the fluctuations of the second class particle. In the end the variable u is of order $t^{2/3}$ and $\varrho - \lambda$ of order $t^{-1/3}$.

LEMMA 3.5. *There is a constant $C > 0$ such that for any time $t \geq 0$,*

$$
\mathbf{E}(|X(t) - L(t)|) \leq C(\varrho-\lambda)^{-1} \quad \textit{and} \quad \mathbf{E}(|R(t) - X(t)|) \leq C(\varrho-\lambda)^{-1}.
$$

Proof. By Proposition III.2.10 of Liggett [17], the joint distribution $\underline{\bar{\mu}}\overline{S(t)}$ of the pair $(\underline{\eta}(t), \underline{\omega}(t))$, as seen from $X(t)$, can also be obtained by conditioning the distribution $\underline{\mu}S(t)$ of the pair $(\underline{\eta}(t), \underline{\omega}(t))$ at time t on having a second class particle at the origin. (The result is denoted by $\overline{\underline{\mu}S(t)}$ in [17].) The measure $\underline{\mu}S(t)$ is translation-invariant, and gives probability $\varrho - \lambda$ for having a second class particle at the origin. If $X_k(t)$ denotes the position of the k^{th} second class particle at time t, then by Theorem B47 in [17], we have

$$
\begin{aligned}
\mathbf{E}(X_{k+1}(t) - X_k(t)) &= \mathbf{E}^{\underline{\bar{\mu}}\overline{S(t)}}(X_{k+1} - X_k) = \mathbf{E}^{\overline{\underline{\mu}S(t)}}(X_{k+1} - X_k) \\
&= \mathbf{E}^{\underline{\mu}S(t)}(X_{k+1} - X_k \,|\, \omega_0 - \eta_0 = 1) \\
&= (\varrho-\lambda)^{-1} \mathbf{E}^{\underline{\mu}S(t)}\big((X_{k+1} - X_k) \cdot \mathbf{1}\{\omega_0 - \eta_0 = 1\}\big) \\
&= \frac{1}{\varrho - \lambda}.
\end{aligned}
$$

Recall that $R(t) = X_{n_R(t)}(t)$ with $n_R(t)$ defined by (3.4) and $X(t) = X_0(t)$. Use the independence and the time-invariance of the distribution of marks from Proposition 3.1:

$$\begin{aligned}\mathbf{E}(|R(t)-X(t)|) &= \mathbf{E}\Big(\sum_{k=-\infty}^{-1}(X_{k+1}(t)-X_k(t))\cdot\mathbf{1}\{n_R(t)\le k\}\Big)\\&\qquad+\mathbf{E}\Big(\sum_{k=0}^{\infty}(X_{k+1}(t)-X_k(t))\cdot\mathbf{1}\{k<n_R(t)\}\Big)\\&=\sum_{k=-\infty}^{-1}\mathbf{E}(X_{k+1}(t)-X_k(t))\cdot\mathbf{P}\{n_R(t)\le k\}\\&\qquad+\sum_{k=0}^{\infty}\mathbf{E}(X_{k+1}(t)-X_k(t))\cdot\mathbf{P}\{k<n_R(t)\}\\&=\frac{1}{\varrho-\lambda}\cdot\mathbf{E}(|n_R(t)|)=\frac{C}{\varrho-\lambda}.\end{aligned}$$

Similar considerations give the result for $L(t)$. □

LEMMA 3.6.

$$\mathbf{Var}(J^{V^\varrho,\lambda}(t))\le\lambda\mathbf{E}\big(|[V^\varrho t]-Q_a(t)|\big)+2t(p-q)\lambda(\varrho-\lambda)+C(\varrho-\lambda)^{-1}.$$

Proof. The variance **Var** in the statement is taken in the five-process coupling where the Bernoulli distribution $\underline{\mu}_\lambda$ is initially perturbed at the origin. Denote by $\mathbf{Var}^\lambda$ variance in the stationary process with initial invariant distribution $\underline{\mu}_\lambda$. By the conditional variance formula

$$\begin{aligned}\mathbf{Var}^\lambda(J^{V^\varrho,\lambda}(t)) &= \mathbf{E}^\lambda\mathbf{Var}^\lambda(J^{V^\varrho,\lambda}(t)\mid\eta_0(0))\\&\qquad+\mathbf{Var}^\lambda\mathbf{E}^\lambda(J^{V^\varrho,\lambda}(t)\mid\eta_0(0))\\&\ge(1-\lambda)\cdot\mathbf{Var}^\lambda(J^{V^\varrho,\lambda}(t)\mid\eta_0(0)=0)\\&=(1-\lambda)\cdot\mathbf{Var}(J^{V^\varrho,\lambda}(t)).\end{aligned}\tag{3.14}$$

Apply Proposition 2.1 to the first term $\mathbf{Var}^\lambda(J^{V^\varrho,\lambda}(t))$. The conditional expectations on the right-hand side of (2.6) match exactly the marginals of the five-process coupling constructed earlier in this Section, and so we come back to the present setting:

$$\begin{aligned}\mathbf{Var}(J^{V^\varrho,\lambda}(t)) &\le\frac{1}{1-\lambda}\mathbf{Var}^\lambda(J^{V^\varrho,\lambda}(t))=\frac{1}{1-\lambda}\lambda(1-\lambda)\mathbf{E}\big(|[V^\varrho t]-Q(t)|\big)\\&\le\lambda\mathbf{E}\big(|[V^\varrho t]-Q_a(t)|\big)+\lambda\mathbf{E}\big(|Q(t)-Q_a(t)|\big).\end{aligned}$$

Write the last term above as

$$\mathbf{E}\big(|Q(t)-Q_a(t)|\big)=\mathbf{E}(Q(t))-\mathbf{E}(Q_a(t))+2\mathbf{E}([Q(t)-Q_a(t)]^-).\tag{3.15}$$

Proposition 2.1 gives

$$\mathbf{E}(Q(t))-\mathbf{E}(Q_a(t))=2t(p-q)(\varrho-\lambda).$$

The last term in (3.15) is treated separately for TASEP ($p = 1$) and ASEP ($0 < q = 1 - p < p < 1$).

Consider first the totally asymmetric case $p = 1$. Initially $Q(0) = Q_a(0) = 0$, and we show $Q(t) \geq Q_a(t)$ for all $t \geq 0$. The following situations cover all cases where Q and Q_a could interchange positions:

- If $Q(t) = Q_a(t) = i$, then at site i (3.5) has the unique solution $\eta_i(t) = \omega_i^-(t) = 0$ and $\eta_i^+(t) = \omega_i(t) = 1$. A right step of Q_a without Q would require a brick on column i of $\underline{\omega}$, but not of $\underline{\eta}^+$. This is impossible by the basic coupling and $\eta_{i+1}^+(t) = \eta_{i+1}(t) \leq \omega_{i+1}(t)$. A left step of Q without Q_a would require a brick on column $i - 1$ of $\underline{\eta}$, but not of $\underline{\omega}^-$. This is again impossible by $\eta_{i-1}(t) \leq \omega_{i-1}(t) = \omega_{i-1}^-(t)$.
- If $Q_a(t) = i$ and $Q(t) = i + 1$, then (3.5) has the unique solution $\eta_i(t) = \eta_i^+(t) = \omega_i^-(t) = 0$, $\omega_i(t) = 1$, $\eta_{i+1}(t) = 0$, and $\eta_{i+1}^+(t) = \omega_{i+1}^-(t) = \omega_{i+1}(t) = 1$. None of the processes can have column i grow in this situation, hence Q_a and Q cannot interchange positions.

We conclude that the second term on the right of (3.15) is zero in the totally asymmetric case.

For ASEP we use conditional coupling with event $\mathcal{B}$ of (3.7). Let $B = (\mathbf{P}\{\mathcal{B}\})^{-1} < \infty$.

$$\begin{aligned}\mathbf{E}([Q(t) - Q_a(t)]^-) &= \mathbf{E}([Q(t) - Q_a(t)]^- \mid \mathcal{B}) \leq \mathbf{E}([L(t) - R(t)]^- \mid \mathcal{B}) \\ &\leq B\mathbf{E}([L(t) - R(t)]^-) \leq B\mathbf{E}(|R(t) - X(t)|) + B\mathbf{E}(|X(t) - L(t)|) \\ &\leq C(\varrho - \lambda)^{-1}\end{aligned}$$

by the previous lemma.

Collecting terms completes the proof of the lemma. □

Remark. The constant $B = (\mathbf{P}\{\mathcal{B}\})^{-1}$ diverges as $p \nearrow 1$ and hence the same is true of C in the last term of the bound in Lemma 3.6. This suggests that there should be a more effective way to bound the variance of $J^{V^\varrho,\lambda}(t)$ of the λ-system in terms of quantities computed in the ϱ-system.

LEMMA 3.7.

$$\mathbf{Var}(J^\varrho(t) - J^{V^\varrho,\lambda}(t)) \leq 2\mathbf{E}\big(|\lfloor V^\varrho t\rfloor - Q_a(t)|\big) + 4t(p-q)\lambda(\varrho - \lambda) + C(\varrho - \lambda)^{-1}.$$

Proof. Similarly to the previous proof, we write

$$\mathbf{Var}^\varrho(J^\varrho(t)) \geq \varrho \cdot \mathbf{Var}(J^\varrho(t)).$$

Then we proceed by

$$\begin{aligned}\mathbf{Var}(J^{\varrho}(t)-J^{V^{\varrho},\lambda}(t)) &\le 2\mathbf{Var}(J^{\varrho}(t))+2\mathbf{Var}(J^{V^{\varrho},\lambda}(t))\\ &\le 2\varrho^{-1}\mathbf{Var}^{\varrho}(J^{\varrho}(t))+2\mathbf{Var}(J^{V^{\varrho},\lambda}(t))\\ &\le 2(1-\varrho+\lambda)\mathbf{E}\big(|[V^{\varrho}t]-Q_a(t)|\big)+4t(p-q)\lambda(\varrho-\lambda)+C(\varrho-\lambda)^{-1}\end{aligned}$$

utilizing the previous lemma. □

We come to the lemma that summarizes all the previous estimation.

LEMMA 3.8. *For any real $u\ge 1$ and time $t>0$,*

$$\mathbf{P}\{Q_a(t)\ge 4u+[V^{\varrho}t]\}\le C_3\frac{t^2}{u^4}\cdot\mathbf{E}\big(|Q_a(t)-[V^{\varrho}t]|\big)+C_4\frac{t^2}{u^3}+C_5\frac{t^3}{u^5}+C_1\exp\Big\{-C_2\frac{u^3}{t^2}\Big\}+\mathrm{e}^{-2u}. \tag{3.16}$$

Proof. Set $b=2(\gamma\wedge\varrho)(p-q)$ where γ is the constant from Lemma 3.3. We proceed by cases.

Case 1. $1\le u<bt$. Suppose first u is an integer as it was in the proof of (3.9). Throughout density ϱ has been fixed, and now we also fix

$$\lambda=\varrho-\frac{u}{2t(p-q)}.$$

The constraint on u guarantees that $\lambda>0$ and $\varrho-\lambda<\gamma$ which was required for (3.9). The point of this choice of λ is to maximize the lower bound inside the probability on the right-hand side of (3.9). So, continuing from (3.9) with Chebyshev's inequality and the previous lemma,

$$\begin{aligned}&\mathbf{P}\{Q_a(t)\ge 4u+[V^{\varrho}t]\}\\ &\quad\le A\mathbf{P}\Big\{\tilde J^{\varrho}(t)-\tilde J^{V^{\varrho},\lambda}(t)\ge\frac{u^2}{4t(p-q)}\Big\}+C_1\exp\Big\{-C_2\frac{u^3}{t^2}\Big\}\\ &\quad\le C_3\frac{t^2}{u^4}\mathbf{Var}(J^{\varrho}(t)-J^{V^{\varrho},\lambda}(t))+C_1\exp\Big\{-C_2\frac{u^3}{t^2}\Big\}\\ &\quad\le C_3\frac{t^2}{u^4}\cdot\mathbf{E}\big(|Q_a(t)-[V^{\varrho}t]|\big)+C_4\frac{t^2}{u^3}+C_5\frac{t^3}{u^5}+C_1\exp\Big\{-C_2\frac{u^3}{t^2}\Big\}.\end{aligned}$$

Extension from integral u to real u is achieved by adjusting constants on the last line above.

Case 2. $bt\le u<2t$. Then $bu/2<bt$ and

$$\mathbf{P}\{Q_a(t)\ge 4u+[V^{\varrho}t]\}\le\mathbf{P}\{Q_a(t)\ge 4\cdot bu/2+[V^{\varrho}t]\}.$$

Case 1 can be applied with u replaced by $bu/2$, at the price of adjusting some constants with powers of b.

Case 3. $u \geq 2t$. Since $Q_a(t)$ is bounded above by a rate one Poisson process,

$$\mathbf{P}\{Q_a(t) \geq 4u + [V^\varrho t]\} \leq \mathbf{P}\{Q_a(t) \geq 3u\} \leq \mathrm{e}^{-2u}$$

for all t. Combining the bounds from the cases proves the lemma. $\square$

We are ready to complete the proof of the upper bound of Theorem 2.2. Abbreviate

$$\Psi(t) = \mathbf{E}\big(\,|Q_a(t) - [V^\varrho t]|\,\big).$$

First we need to complement (3.16) with a matching lower tail bound for $Q_a(t)$. This can be derived by arguments analogous to the ones we have pursued throughout Section 3.

Alternately, we can derive the lower tail bound by a particle-hole interchange followed by reflection of the lattice. Define $\widehat{\omega}_i(t) = 1 - \omega_i(t)$, an ASEP with density $\widehat{\varrho} = 1 - \varrho$ and rightward jump rate $\widehat{p} = 1 - p$. Its second class particle has position $\widehat{Q}(t) = Q_a(t)$. Let $\underline{\widehat{\omega}}^R(\cdot)$ be the process obtained from $\underline{\widehat{\omega}}(\cdot)$ through a reflection about the origin. Then $\underline{\widehat{\omega}}^R(\cdot)$ is ASEP with the original parameters (p, q). The second class particle of this process is at position $\widehat{Q}(t)^R = -\widehat{Q}(t) = -Q_a(t)$, and the characteristic speed is $\widehat{V}^{\varrho,R} = (p - q)(1 - 2(1 - \varrho)) = -V^\varrho$. Since $[-V^\varrho t] = -[V^\varrho t]$, the expectation

$$\Psi(t) = \mathbf{E}\big(|Q_a(t) - [V^\varrho t]|\big) = \mathbf{E}\big(|\widehat{Q}(t)^R - [\widehat{V}^{\varrho,R} t]|\big)$$

is the same for $\underline{\omega}(\cdot)$ and $\underline{\widehat{\omega}}^R(\cdot)$. Consequently

$$\mathbf{P}\{Q_a(t) - [V^\varrho t] \leq -4u\} = \mathbf{P}\{\widehat{Q}(t)^R - [V^{\varrho,R} t] \geq 4u\}$$

which is again bounded by the right-hand side of (3.16).

Introduce a large constant $4 < r < \infty$. Consider $t \geq 1$ and $u \geq rt^{2/3}$. Then we can combine the upper and lower tail bounds whose common right-hand side is given in (3.16), replace $4u$ by u (we made sure $u/4 \geq 1$), and simplify a little to arrive at

$$\begin{aligned} \mathbf{P}\{\,|Q_a(t) - [V^\varrho t]| \geq u\} \\ \leq C_1 \frac{t^2}{u^4} \Psi(t) + C_2(r)\Big(\frac{t^2}{u^3} + \exp\Big\{-C_3(r)\frac{u}{t^{2/3}}\Big\}\Big). \end{aligned} \tag{3.17}$$

Constants were renamed and their dependence on r expressed in the notation because now it is of importance that C_1 in front of $\Psi(t)$ does not depend on r. In particular, the exponentials in (3.16) were combined via $e^{-2u} \leq \exp(-2ut^{-2/3})$ and $\exp(-C_2 u^3 t^{-2}) \leq \exp(-C_2 r^2 u t^{-2/3})$.

Let $1 \le m < 3$. Integrate bound (3.17) over $u \in [rt^{2/3}, \infty)$:

$$\begin{aligned}(3.18)\qquad &\mathbf{E}\big(\,|Q_a(t) - [V^\varrho t]|^m\,\big)\\ &\quad\le r^m t^{2m/3} + m\int_{rt^{2/3}}^{\infty} \mathbf{P}\{\,|Q_a(t) - [V^\varrho t]| \ge u\} u^{m-1}\,\mathrm{d}u\\ &\quad\le C_1 r^{m-4}\Psi(t) t^{2+(2/3)(m-4)} + C_4(r) t^{2m/3}.\end{aligned}$$

Constants were renamed again and their dependence on m ignored. To get the final bounds, take first $m = 1$ in (3.18) to get

$$\Psi(t) \le C_1 r^{-3}\Psi(t) + C_4(r) t^{2/3}.$$

Since C_1 is independent of r, fixing r large enough gives $\Psi(t) \le C_5(r) t^{2/3}$. Putting this bound back on the last line of (3.18) then gives

$$\mathbf{E}\big(\,|Q_a(t) - [V^\varrho t]|^m\,\big) \le C_6(r) t^{2m/3}$$

for $1 < m < 3$. The upper bound of Theorem 2.2 has been proved.

4. Lower bound

The lower bound is proved by perturbing an initial equilibrium on a segment of the lattice. Again we begin with a description and some properties of the coupling.

Perturbing a segment initially. Recall again the characteristic speeds $V^\varrho = (p-q)(1-2\varrho)$ and $V^\lambda = (p-q)(1-2\lambda)$. We assume $\varrho > \lambda$, hence $V^\varrho < V^\lambda$. Throughout this section $u > 0$ denotes a fixed positive integer, and

$$n = [V^\lambda t] - [V^\varrho t] + u.$$

To begin define an initial product distribution on two configurations $(\underline{\eta}(0), \underline{\zeta}(0))$ by describing the marginals on each lattice site i:

$$\begin{cases} (\eta_i(0),\ \zeta_i(0)) \sim \mu \text{ of (3.1)} & \text{if } i < -n,\\ \eta_i(0) = 0,\ \zeta_i(0) \sim \mu_\varrho & \text{if } i = -n,\\ \eta_i(0) = \zeta_i(0) \sim \mu_\lambda & \text{if } -n < i \le 0,\\ (\eta_i(0),\ \zeta_i(0)) \sim \mu \text{ of (3.1)} & \text{if } i > 0.\end{cases}$$

Let $Q^{(-n)}(t)$ be the position at time t of a second class particle in the process $\underline{\eta}(\cdot)$, initially at site $Q^{(-n)}(0) = -n$. Note that site $-n$ was set vacant for $\underline{\eta}(0)$. As before $\underline{\eta}(t)$ together with the particle at $Q^{(-n)}(t)$ make up the process $\underline{\eta}^+(t)$. Except for this perturbation $\underline{\eta}(0)$ starts in the Bernoulli $\underline{\mu}_\lambda$ distribution. The process $\underline{\zeta}(\cdot)$ initially has distribution $\underline{\mu}_\varrho$, except at sites $\{-n+1, \ldots, 0\}$ where the parameter ϱ has been replaced by λ.

Define a third initial configuration by

$$\xi_i(0) = \begin{cases} \zeta_i(0) & \text{if } i \le -n, \\ \eta_i(0) & \text{if } i > -n. \end{cases}$$

We let all these processes evolve jointly in the basic coupling. The following majorizations are true initially and are preserved by the evolution:

$$\underline{\eta}(t) \le \underline{\xi}(t) \le \underline{\zeta}(t) \qquad \text{and} \qquad \underline{h}^{\zeta}(t) \le \underline{h}^{\xi}(t)$$

where the last inequality is for column heights.

Let us denote the net particle currents (2.5) by $J^{V,\eta}$ and $J^{V,\zeta}$ in the respective processes $\underline{\eta}(\cdot)$ and $\underline{\zeta}(\cdot)$. The first observation is that $Q^{(-n)}$ gives one-sided control over the difference of these currents.

LEMMA 4.1. *There is a nonnegative process $N(t)$ with constant Geometric(q/p) time marginals $\mathbf{P}[N(t) = k] = (1-q/p)(q/p)^k$ and such that for any $V \in \mathbb{R}$*

$$Q^{(-n)}(t) \le [Vt] \quad \textit{implies} \quad J^{V,\zeta}(t) - J^{V,\eta}(t) \le N(t).$$

Proof. Denote the positions of the $\xi - \eta$ second class particles at time t by

$$\cdots < Y_k(t) < \cdots < Y_{-2}(t) < Y_{-1}(t) < Y_0(t).$$

As argued before, we can arrange for these particles to keep their labels, and then the ordering is preserved. Initially $Y_0(0) \le -n = Q^{(-n)}(0)$.

Let $m_Q(t) = \max\{k : Y_k(t) \le Q^{(-n)}(t)\}$ be the label of the $\xi - \eta$ particle at or closest to the left of $Q^{(-n)}(t)$. Initially $m_Q(0) = 0$. Once $Q^{(-n)}(t) \in \{Y_k(t)\}$, this containment property will hold forever because the basic coupling will preserve the ordering $\underline{\eta}^+ \le \underline{\xi}$ if this is ever established.

We claim that m_Q remains zero while $Q^{(-n)}$ is disjoint from the second class particles $\{Y_k\}$. This follows from showing that there is no jump which swaps the ordering $Y_0 < Q^{(-n)}$. If $Y_0 = i$ and $Q^{(-n)} = i+1$, then we have $0 = \eta_i = \eta_i^+ < \xi_i = 1$ and $0 = \eta_{i+1} = \xi_{i+1} < \eta_{i+1}^+ = 1$. In this situation column i of ξ can increase, or the same column of η^+ can decrease. Neither of these steps can swap the positions of Y_0 and $Q^{(-n)}$, instead they make $Y_0 = Q^{(-n)}$.

Once $Q^{(-n)}$ is riding on the $\{Y_k\}$ particles so that $Q^{(-n)}(t)$ actually equals $Y_{m_Q(t)}(t)$, its label m_Q evolves in the basic coupling as follows.

- Suppose $Q^{(-n)} = Y_k = i$ and $Y_{k+1} = i+1$. A right Poisson arrow $(i \to i+1)$ appears at rate p and increases m_Q from k to $k+1$.
- Suppose $Q^{(-n)} = Y_k = i$ and $Y_{k-1} = i-1$. A left Poisson arrow $(i \to i-1)$ appears at rate q and decreases m_Q from k to $k-1$.

When $\xi - \eta$ particle $Y_{m_Q(t)}$ itself jumps, $Q^{(-n)}$ jumps with it.

To get bounds on $Q^{(-n)}(t)$ we introduce a suitable steady-state object. As we did with the ζ-marks (3.3) for the upper bound, we introduce a further classification among the $\xi - \eta$ second class particles so that exactly one of them has priority over all the others. This special particle is marked by the label $m(t)$. The Poisson arrows move $m(t)$ exactly the same way as $m_Q(t)$ on the labels $\{-\infty < k \le 0\}$:

- If $Y_{m(t)}(t) = i$ and $Y_{m(t)+1}(t) = i+1$ and there is a right Poisson arrow $(i \to i+1)$ then $m(t)$ increases by 1. This happens at rate p.
- If $Y_{m(t)-1}(t) = i-1$ and $Y_{m(t)}(t) = i$ and there is a left Poisson arrow $(i \to i-1)$ then $m(t)$ decreases by 1. This happens at rate q.

From these rates we see that $m(t)$ behaves like a birth and death chain on $\mathbb{Z}_-$ whenever adjacency of $\xi - \eta$ second class particles permits $m(t)$ to jump. Without obstruction this birth and death chain would have reversible measure $\pi(k) = (1-q/p)(q/p)^{|k|}$ for $k \le 0$. We give $m(0)$ initial distribution π. Then the argument given for Proposition 3.1 implies that for each fixed time t we have $\mathbf{P}[m(t) = k] = \pi(k)$.

We have arranged $m(0) \le m_Q(0)$ initially. Since $Q^{(-n)}$ cannot hop over Y_0 without joining it, the identical responses of m and m_Q to the Poisson arrows implies that $m(t) \le m_Q(t)$ for all time.

To connect with currents, note that $-m_Q(t)$ equals the number of $\xi - \eta$ second class particles strictly to the right of $Q^{(-n)}(t)$ at time t. Since these particles started off in $(-\infty, 0]$ and they account for all the discrepancies between the processes $\underline{\xi}$ and $\underline{\eta}$, this number equals the height difference $h^{\xi}_{Q^{(-n)}(t)}(t) - h^{\eta}_{Q^{(-n)}(t)}(t)$. Thus under $\{Q^{(-n)}(t) \le [Vt]\}$ we have

$$\begin{aligned}
-m(t) \ge -m_Q(t) &= h^{\xi}_{Q^{(-n)}(t)}(t) - h^{\eta}_{Q^{(-n)}(t)}(t) \\
&= h^{\xi}_{[Vt]}(t) - h^{\eta}_{[Vt]}(t) + \sum_{i=Q^{(-n)}(t)+1}^{[Vt]} (\xi_i(t) - \eta_i(t)) \\
&\ge h^{\xi}_{[Vt]}(t) - h^{\eta}_{[Vt]}(t) \\
&\ge h^{\zeta}_{[Vt]}(t) - h^{\eta}_{[Vt]}(t) = J^{V,\zeta}(t) - J^{V,\eta}(t).
\end{aligned}$$

To obtain the statement of the lemma take $N(t) = -m(t)$. □

Let $\underline{\omega}(\cdot)$ be a stationary ASEP started from the $\underline{\mu}_{\varrho}$ Bernoulli distribution. The next lemma gives a way to compare the distributions of $\underline{\zeta}$ and $\underline{\omega}$.

LEMMA 4.2. *Denote by $\mathbf{P}^{\omega}$ and $\mathbf{P}^{\zeta}$ the probability of events that depend only on the respective processes $\underline{\omega}(\cdot)$ and $\underline{\zeta}(\cdot)$. Then*

$$\mathbf{P}^{\zeta}(\cdot) \le \mathbf{P}^{\omega}(\cdot)^{\frac{1}{2}} \cdot \exp\left[\frac{n(\varrho - \lambda)^2}{2\varrho(1-\varrho)}\right].$$

Proof. Let

$$Z = \sum_{i=-n+1}^{0} \zeta_i(0).$$

Z has a Binomial(n, λ) distribution $\nu^\lambda(z) = \binom{n}{z}\lambda^z(1-\lambda)^{n-z}$ for $z = 0, \dots, n$. We use the Cauchy-Schwarz inequality below to perform a change of measure on this binomial distribution. The binomial mass functions in the second line are easily added up.

$$\begin{aligned}
\mathbf{P}^\zeta(\cdot) &= \sum_{z=0}^{n} \mathbf{P}^\zeta(\cdot \,|\, Z = z)\, [\nu^\varrho(z)]^{\frac{1}{2}} \cdot \frac{\nu^\lambda(z)}{[\nu^\varrho(z)]^{\frac{1}{2}}} \\
&\le \Big[\sum_{z=0}^{n} [\mathbf{P}^\zeta(\cdot \,|\, Z = z)]^2\, \nu^\varrho(z)\Big]^{\frac{1}{2}} \cdot \Big[\sum_{z=0}^{n} \frac{[\nu^\lambda(z)]^2}{\nu^\varrho(z)}\Big]^{\frac{1}{2}} \\
&= \Big[\sum_{z=0}^{n} [\mathbf{P}^\zeta(\cdot \,|\, Z = z)]^2\, \nu^\varrho(z)\Big]^{\frac{1}{2}} \cdot \Big[1 + \frac{(\varrho-\lambda)^2}{\varrho(1-\varrho)}\Big]^{\frac{n}{2}} \\
&\le \Big[\sum_{z=0}^{n} \mathbf{P}^\zeta(\cdot \,|\, Z = z)\, \nu^\varrho(z)\Big]^{\frac{1}{2}} \cdot \exp\Big[\frac{n(\varrho-\lambda)^2}{2\varrho(1-\varrho)}\Big].
\end{aligned}$$

All that is left is to recognize that $\mathbf{P}^\zeta(\cdot \,|\, Z = z)$ is the probability of a process $\underline{\zeta}(\cdot)$ whose initial distribution is Bernoulli(ϱ) outside $\{-n+1 \dots 0\}$, with z particles distributed in that interval with each configuration equally likely. Summing these with the Binomial(n, ϱ) coefficients $\nu^\varrho(z)$ gives the Bernoulli $\underline{\mu}_\varrho$ initial distribution of the process $\underline{\omega}(\cdot)$. □

Proof of the lower bound. With these preparations we are ready to prove the lower bound. As for the upper bound, $Q_a(t)$ is the position of a second class antiparticle started from the origin on a $\underline{\mu}_\varrho$-equilibrium process $\underline{\omega}(\cdot)$ initially perturbed by setting $\omega_0(0) = 1$. Our target quantity is abbreviated as before by $\Psi(t) = \mathbf{E}(|Q_a(t) - [V^\varrho t]|)$. We start with bounding the probability of the complement of the event in Lemma 4.1. As before u is an arbitrary but fixed positive integer and $n = [V^\lambda t] - [V^\varrho t] + u$.

LEMMA 4.3.

$$\mathbf{P}\{Q^{(-n)}(t) > [V^\varrho t]\} \le \frac{\Psi(t)}{u} + \frac{4t(p-q)(\varrho-\lambda)+2}{u} + \frac{C}{u(\varrho-\lambda)}.$$

Proof. For this proof, $Q(t)$ will refer to a second class particle started from the origin on a process $\widehat{\underline{\eta}}(\cdot)$ in $\underline{\mu}_\lambda$ distribution, except for $\widehat{\eta}_0(0) = 0$. Translation invariance implies

$$\mathbf{P}\{Q^{(-n)}(t) > [V^\varrho t]\} = \mathbf{P}\{Q(t) - [V^\lambda t] > u\} \le \frac{\mathbf{E}(|Q(t) - [V^\lambda t]|)}{u}$$
$$\le \frac{\mathbf{E}(|Q(t) - Q_a(t)|)}{u} + \frac{\mathbf{E}(|Q_a(t) - [V^\varrho t]|)}{u} + \frac{[V^\lambda t] - [V^\varrho t]}{u}$$

where we introduced the couplings of Section 3. As in Lemma 3.6, the first term is bounded from above by $2t(p-q)(\varrho-\lambda)/u + C/[u(\varrho-\lambda)]$. The second term is $\Psi(t)/u$, and the third term is bounded from above by $2t(p-q)(\varrho-\lambda)/u + 2/u$. □

LEMMA 4.4. *For any* $0 < K < (p-q)t(\varrho-\lambda)^2$,

$$\mathbf{P}\{Q^{(-n)}(t) \le [V^\varrho t]\} \le \frac{\varrho^{1/2}(1-\varrho)^{1/2}\Psi(t)^{1/2}}{(p-q)t(\varrho-\lambda)^2 - K} \cdot \exp\Big[\frac{n(\varrho-\lambda)^2}{2\varrho(1-\varrho)}\Big]$$
$$+ \frac{4\lambda\Psi(t)}{K^2} + \frac{8t(p-q)\lambda(\varrho-\lambda)}{K^2} + \frac{C}{K^2(\varrho-\lambda)} + \frac{C}{K-8}.$$

Proof. Lemma 4.1 leads to

$$\mathbf{P}\{Q^{(-n)}(t) \le [V^\varrho t]\} \le \mathbf{P}\{J^{V^\varrho,\zeta}(t) - J^{V^\varrho,\eta}(t) \le N(t)\}$$
$$\le \mathbf{P}\{J^{V^\varrho,\zeta}(t) \le K + (p-q)t(2\varrho\lambda - \lambda^2) - 1\} \tag{4.1}$$
$$+ \mathbf{P}\Big\{J^{V^\varrho,\eta}(t) - (p-q)t(2\varrho\lambda - \lambda^2) > \frac{K}{2} + 3\Big\} \tag{4.2}$$
$$+ \mathbf{P}\Big\{N(t) > \frac{K}{2} - 4\Big\}. \tag{4.3}$$

We apply Lemma 4.2 to line (4.1) to bound it by the equilibrium $\underline{\mu}_\varrho$-probability:

$$\big[\mathbf{P}^\varrho\{J^\varrho(t) \le K + (p-q)t(2\varrho\lambda - \lambda^2) - 1\}\big]^{\frac{1}{2}} \cdot \exp\Big[\frac{n(\varrho-\lambda)^2}{2\varrho(1-\varrho)}\Big]$$
$$\le \big[\mathbf{P}^\varrho\{\widetilde{J}^\varrho(t) \le K - (p-q)t(\varrho-\lambda)^2\}\big]^{\frac{1}{2}} \cdot \exp\Big[\frac{n(\varrho-\lambda)^2}{2\varrho(1-\varrho)}\Big]$$
$$\le \frac{[\mathbf{Var}(J^\varrho(t))]^{1/2}}{(p-q)t(\varrho-\lambda)^2 - K} \cdot \exp\Big[\frac{n(\varrho-\lambda)^2}{2\varrho(1-\varrho)}\Big]$$
$$= \frac{\varrho^{1/2}(1-\varrho)^{1/2}\Psi(t)^{1/2}}{(p-q)t(\varrho-\lambda)^2 - K} \cdot \exp\Big[\frac{n(\varrho-\lambda)^2}{2\varrho(1-\varrho)}\Big].$$

The term -1 was subsumed in errors caused by integer parts when we centered $J^\varrho(t)$.

A simple coupling consideration shows that $\mathbf{E}(J^{V^\varrho,\eta}(t))$ differs by at most one from the same expectation taken under an unperturbed $\underline{\mu}_\lambda$ initial condition. Thus taking integer parts again into account, line (4.2) is bounded by

$$\mathbf{P}\left\{\tilde{J}^{V^\varrho,\eta}(t) > \frac{K}{2}\right\} \le \frac{4\mathbf{Var}(J^{V^\varrho,\eta})}{K^2}.$$

Lemma 3.6 can be applied to bound this variance even though the second class particle now starts at $-n$ rather than at the origin. The only change needed in the proof of Lemma 3.6 is in the calculation (3.14) where one must condition on $\eta_{-n}(0)$ instead of on $\eta_0(0)$. Hence we can continue from above to bound line (4.2) with

$$\frac{4\lambda\Psi(t)}{K^2} + \frac{8t(p-q)\lambda(\varrho-\lambda)}{K^2} + \frac{C}{K^2(\varrho-\lambda)}.$$

Lastly, the geometric probability (4.3) is bounded by $C/(K-8)$ by Chebyshev's inequality. $\square$

Now the last step of the lower bound of Theorem 2.2. By Hölder's or Jensen's inequality it suffices to prove the case $m=1$, in other words that

$$\liminf_{t\to\infty} t^{-2/3}\Psi(t) > 0.$$

In the two last lemmas take

$$u = \lceil ht^{2/3}\rceil, \quad \varrho-\lambda = bt^{-1/3}, \quad \text{and} \quad K = bt^{1/3},$$

where h and b are large, in particular b large enough to have $b < (p-q)b^2$ so that K satisfies Lemma 4.4. Then

$$n = [V^\lambda t] - [V^\varrho t] + u \le [2(p-q)b + h]t^{2/3} + 3 \le Ct^{2/3}$$

for large enough t. We can simplify the outcomes of Lemma 4.3 and Lemma 4.4 to the inequalities

$$\mathbf{P}\{Q^{(-n)}(t) > [V^\varrho t]\} \le C\frac{\Psi(t)}{t^{2/3}} + \frac{4b}{h} + \frac{C}{t^{1/3}} \tag{4.4}$$

and

$$\mathbf{P}\{Q^{(-n)}(t) \le [V^\varrho t]\} \le C\left(\frac{\Psi(t)}{t^{2/3}}\right)^{1/2} + C\frac{\Psi(t)}{t^{2/3}} + \frac{8}{b} + \frac{C}{t^{1/3}}. \tag{4.5}$$

The new constant C depends on b and h.

The lower bound now follows because the left-hand sides of (4.4)–(4.5) add up to one for each fixed t, while we can fix b large enough and then h large enough so that $4b/h + 8/b < 1$. Then $t^{-2/3}\Psi(t)$ must have a positive lower bound for all large enough t. This completes the proof of Theorem 2.2.

Remark. The key properties, namely $f'' \ne 0$ for the flux and asymmetry of the jump kernel, were used somewhat surreptitiously in the above proof. As in calculation (3.13), the nonvanishing of f'' renders $\mathbf{E}[J^{\varrho}(t) - J^{V^{\varrho},\eta}(t)]$ of order $t^{1/3}$. This allowed us to take K of order $t^{1/3}$, then $\varrho - \lambda$ of order $t^{-1/3}$, and u of order $t^{2/3}$. This way the final bounds (4.4)–(4.5) have $\Psi(t)$ divided by the correct order $t^{2/3}$.

Asymmetry also came in: if $p = q = 1/2$ then the means of the equilibrium currents are zero and there would be no deviations to take advantage of in the proof of Lemma 4.4.

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References

[1] D. ALDOUS and P. DIACONIS, Hammersley's interacting particle process and longest increasing subsequences, *Probab. Theory Related Fields* **103** (1995), 199–213. MR 96k:60017 Zbl 0836.60107

[2] J. BAIK, P. DEIFT, and K. JOHANSSON, On the distribution of the length of the longest increasing subsequence of random permutations, *J. Amer. Math. Soc.* **12** (1999), 1119–1178. MR 2000e:05006 Zbl 0932.05001

[3] M. BALÁZS, Growth fluctuations in a class of deposition models, *Ann. Inst. H. Poincaré Probab. Statist.* **39** (2003), 639–685. MR 2005j:60178a Zbl 1029.60075

[4] M. BALÁZS, E. CATOR, and T. SEPPÄLÄINEN, Cube root fluctuations for the corner growth model associated to the exclusion process, *Electron. J. Probab.* **11** (2006), no. 42, 1094–1132. MR 2007k:60312 Zbl 1139.60046

[5] M. BALÁZS, F. RASSOUL-AGHA, and T. SEPPÄLÄINEN, The random average process and random walk in a space-time random environment in one dimension, *Comm. Math. Phys.* **266** (2006), 499–545. MR 2007k:60334 Zbl 1129.60097

[6] M. BALÁZS and T. SEPPÄLÄINEN, Exact connections between current fluctuations and the second class particle in a class of deposition models, *J. Stat. Phys.* **127** (2007), 431–455. MR 2008e:82056 Zbl 1147.82348

[7] E. CATOR and P. GROENEBOOM, Second class particles and cube root asymptotics for Hammersley's process, *Ann. Probab.* **34** (2006), 1273–1295. MR 2008e:60021 Zbl 1101.60076

[8] D. DÜRR, S. GOLDSTEIN, and J. L. LEBOWITZ, Asymptotics of particle trajectories in infinite one-dimensional systems with collisions, *Comm. Pure Appl. Math.* **38** (1985), 573–597. MR 87e:60166 Zbl 0578.60094

[9] L. C. EVANS, *Partial Differential Equations*, *Grad. Stud. Math.* **19**, Amer. Math. Soc., Providence, RI, 1998. MR 99e:35001 Zbl 0902.35002

[10] P. A. FERRARI and L. R. G. FONTES, Current fluctuations for the asymmetric simple exclusion process, *Ann. Probab.* **22** (1994), 820–832. MR 95j:60162 Zbl 0806.60099

[11] P. A. FERRARI, C. KIPNIS, and E. SAADA, Microscopic structure of travelling waves in the asymmetric simple exclusion process, *Ann. Probab.* **19** (1991), 226–244. MR 92b:60099 Zbl 0725.60113

[12] P. L. FERRARI and H. SPOHN, Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process, *Comm. Math. Phys.* **265** (2006), 1–44. MR 2007g:82038a Zbl 1118.82032

[13] K. JOHANSSON, Shape fluctuations and random matrices, *Comm. Math. Phys.* **209** (2000), 437–476. MR 2001h:60177 Zbl 0969.15008

[14] C. KIPNIS and C. LANDIM, *Scaling Limits of Interacting Particle Systems*, *Grundl. Math. Wissen.* **320**, Springer-Verlag, New York, 1999. MR 2000i:60001 Zbl 0927.60002

[15] C. LANDIM, J. QUASTEL, M. SALMHOFER, and H.-T. YAU, Superdiffusivity of asymmetric exclusion process in dimensions one and two, *Comm. Math. Phys.* **244** (2004), 455–481. MR 2005a:60118 Zbl 1064.60164

[16] T. M. LIGGETT, *Interacting Particle Systems*, *Grundl. Math. Wissen.* **276**, Springer-Verlag, New York, 1985. MR 86e:60089 Zbl 1103.82016

[17] ———, *Stochastic Interacting Systems*: *Contact*, *Voter and Exclusion Processes*, *Grundl. Math. Wissen.* **324**, Springer-Verlag, New York, 1999. MR 2001g:60247 Zbl 0949.60006

[18] M. PRÄHOFER and H. SPOHN, Current fluctuations for the totally asymmetric simple exclusion process, in *In and Out of Equilibrium* (Mambucaba, 2000), *Progr. Probab.* **51**, Birkhäuser, Boston, MA, 2002, pp. 185–204. MR 2003e:60224 Zbl 1015.60093

[19] J. QUASTEL and B. VALKÓ, $t^{1/3}$ Superdiffusivity of finite-range asymmetric exclusion processes on $\mathbb{Z}$, *Comm. Math. Phys.* **273** (2007), 379–394. MR 2008h:60414 Zbl 1127.60091

[20] ———, A note on the diffusivity of finite-range asymmetric exclusion processes on $\mathbb{Z}$, in *In and Out of Equilibrium.* 2, *Progr. Probab.* **60**, Birkhäuser, Basel, 2008, pp. 543–549. MR 2477398 Zbl 05363779

[21] F. REZAKHANLOU, Hydrodynamic limit for attractive particle systems on $\mathbf{Z}^d$, *Comm. Math. Phys.* **140** (1991), 417–448. MR 93f:82035 Zbl 0738.60098

[22] T. SEPPÄLÄINEN, A microscopic model for the Burgers equation and longest increasing subsequences, *Electron. J. Probab.* **1** (1996), 51 pp. MR 97d:60162 Zbl 0891.60093

[23] ———, Existence of hydrodynamics for the totally asymmetric simple K-exclusion process, *Ann. Probab.* **27** (1999), 361–415. MR 2000i:60116 Zbl 0947.60088

[24] ———, Second-order fluctuations and current across characteristic for a one-dimensional growth model of independent random walks, *Ann. Probab.* **33** (2005), 759–797. MR 2006d:60153 Zbl 1108.60083

[25] F. SPITZER, Interaction of Markov processes, *Advances in Math.* **5** (1970), 246–290 (1970). MR 42 #3856 Zbl 0312.60060

[26] H. VAN BEIJEREN, R. KUTNER, and H. SPOHN, Excess noise for driven diffusive systems, *Phys. Rev. Lett.* **54** (1985), 2026–2029. MR 86e:82031

[27] H.-T. YAU, $(\log t)^{2/3}$ law of the two dimensional asymmetric simple exclusion process, *Ann. of Math.* **159** (2004), 377–405. MR 2005d:60156 Zbl 1060.60099

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