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Abstract

We introduce spaces of exponential constructible functions in the motivic setting for which we construct direct image functors in the absolute and relative settings. This allows us to define a motivic Fourier transformation for which we get various inversion statements. We also define spaces of motivic Schwartz-Bruhat functions on which motivic Fourier transformation induces isomorphisms. Our motivic integrals specialize to nonarchimedean integrals. We give a general transfer principle comparing identities between functions defined by exponential integrals over local fields of characteristic zero, resp. of positive characteristic, having the same residue field. We also prove new results about p-adic integrals of exponential functions and stability of this class of functions under p-adic integration.

1. **Introduction**

In our previous work [11], we laid general foundations for motivic integration of constructible functions. One of the most salient features of motivic constructible functions is that they form a class which is stable under direct image and that motivic integrals of constructible functions depending on parameters are constructible as functions of the parameters. Though motivic constructible functions as defined in [11] encompass motivic analogues of many functions occurring in integrals over nonarchimedean local fields, one important class of functions was still missing in the picture, namely motivic analogues of nonarchimedean integrals of the type

$$\int_{\mathbb{Q}_p^n} f(x)\Psi(g(x))|dx|,$$

with Ψ a (nontrivial) additive character on $\mathbb{Q}_p$, f a p-adic constructible function and g a $\mathbb{Q}_p$-valued definable function on $\mathbb{Q}_p^n$, and their parametrized versions, functions of the type

$$\lambda \longmapsto \int_{\mathbb{Q}_p^n} f(x,\lambda)\Psi(g(x,\lambda))|dx|,$$

where λ runs over, say $\mathbb{Q}_p^m$, and f and g are now functions on $\mathbb{Q}_p^{m+n}$. Needless to say, integrals of this kind are ubiquitous in harmonic analysis over nonarchimedean local fields, p-adic representation theory and the Langlands program.

One of the purposes of the present paper is to fill this gap by extending the framework of [11] in order to include motivic analogues of exponential integrals of the type above. Once this is done one is able to develop a natural Fourier transform and to prove various forms of Fourier inversion. Another interesting feature of our formalism is that it makes it possible to state and prove a general transfer principle for integrals over nonarchimedean local fields, allowing one to transfer identities between functions defined by integrals over fields of characteristic zero to fields of characteristic p, when the residual characteristic is large enough, and vice versa. It should be emphasized that our statement holds for quite general functions defined by integrals depending on valued field variables. One should keep in mind that there is no meaning in comparing values of individual parameters in the integrals, or the integrals themselves, between characteristic zero and characteristic p. Our transfer principle, which can be considered as a wide generalization of the classical Ax-Kochen-Eršov result, should have a wide range of applications to p-adic representation theory and the Langlands program. It applies in particular to many forms of the Fundamental Lemma and to the integrals occurring in the Jacquet-Ye conjecture [24], which has been proved by Ngô [26] over functions fields and by Jacquet [23] in general.

Let us now review the content of the paper in more detail. In Section 3 we enlarge our Grothendieck rings in order to add exponentials. In fact it is useful to consider not only exponentials of functions with values in the valued field, but also exponentials of functions with values in the residue field. This is performed in a formal way by replacing the category RDef_S considered in [11] — consisting of certain objects $X \to S$ — by a larger category $\mathrm{RDef}_S^{\mathrm{exp}}$ consisting of the same $X \to S$ together with functions g and ξ on X with values in the valued field, resp. the residue field. We define a Grothendieck ring $K_0(\mathrm{RDef}_S^{\mathrm{exp}})$ generated by classes of objects (X, g, ξ) modulo certain relations. Here we have to add some new relations to the classical ones already considered in [11]. When $X \to S$ is the identity, the class of $(X, g, 0)$, resp. $(X, 0, \xi)$, corresponds to the exponential of g, resp. the exponential of ξ. One defines the ring $\mathscr{C}(S)^{\mathrm{exp}}$ of motivic exponential functions on S by tensoring $K_0(\mathrm{RDef}_S^{\mathrm{exp}})$ with the ring $\mathscr{P}(S)$ of constructible Presburger functions on S. We are then able to state our main results on integration of exponential functions in Section 4. In particular we show that integrals with parameters of functions in $\mathscr{C}^{\exp}$ still lie in $\mathscr{C}^{\exp}$.

We first directly construct integrals of exponential functions in relative dimension 1 in Section 5 and then perform the general construction in Section 6. As was the case in [11], extensive use is made of the Denef-Pas cell decomposition theorem. Though some parts of our constructions and proofs are quite similar to what we performed in [11], or sometimes even follow directly from [11], others require new ideas and additional work specific to the exponential setting. As a first application, we develop in Section 7 the fundamentals of a motivic Fourier transform. More precisely, there are two Fourier transforms, the first one over residue field variables and the second one, which is more interesting, over valued field variables. Calculus with our valued field Fourier transform is completely similar to the usual one. Using convolution, we define motivic Schwartz-Bruhat functions, and we show that the valued field Fourier transform is involutive on motivic Schwartz-Bruhat functions.

We finally deduce Fourier inversion for integrable functions with the integrable Fourier transform. In the following Section 8 we move to the p-adic setting, defining the p-adic analogue of $\mathscr{C}(S)^{\exp}$ and proving stability under integration with parameters of these p-adic constructible exponential functions. Such a result is the natural extension to the exponential context of Denef's fundamental result on stability of p-adic constructible functions under integration with respect to parameters. This result of Denef greatly influenced our work [11] and the present one. It has been later generalized to the subanalytic case by the first author in [4] and [5].

In Section 9, we close the circle by showing that motivic integration of constructible exponential functions commutes with specialization to the corresponding nonarchimedean ones, when the residue characteristic is large enough. Finally, we end the paper by proving our fundamental transfer theorem, a form of which was already stated in [9] when there is no exponential. Note that in their recent paper [21] Hrushovski and Kazhdan have also considered integrals of exponentials.

Some of the results in this paper have been announced in [10].

2. **Preliminaries**

2.1. *Definable subassignments and constructible functions.* We start by recalling briefly some definitions and constructions from [11]; cf. also [7], [8]. We fix a field k of characteristic zero and we consider for any field K containing k the field of Laurent series $K((t))$ endowed with its natural valuation

$$\mathrm{ord} : K((t))^{\times} \longrightarrow \mathbb{Z} \tag{2.1.1}$$

and with the angular component mapping

$$\overline{\mathrm{ac}} : K((t)) \longrightarrow K, \tag{2.1.2}$$

defined by $\overline{\mathrm{ac}}(x) = xt^{-\mathrm{ord}(x)} \mod t$ if $x \neq 0$ and $\overline{\mathrm{ac}}(0) = 0$. We use the Denef-Pas language $\mathscr{L}_{\mathrm{DP,P}}$ which is a 3-sorted language

$$(\mathbf{L}_{\mathrm{Val}}, \mathbf{L}_{\mathrm{Res}}, \mathbf{L}_{\mathrm{Ord}}, \mathrm{ord}, \overline{\mathrm{ac}}) \tag{2.1.3}$$

with sorts corresponding respectively to valued field, residue field and value group variables. The languages $\mathbf{L}_{\mathrm{Val}}$ and $\mathbf{L}_{\mathrm{Res}}$ are equal to the ring language $\mathbf{L}_{\mathrm{Rings}} = \{+, -, \cdot, 0, 1\}$, and for $\mathbf{L}_{\mathrm{Ord}}$ we take the Presburger language

$$\mathbf{L}_{\mathrm{PR}} = \{+, -, 0, 1, \leq\} \cup \{\equiv_n \mid n \in \mathbb{N},\ n > 1\}, \tag{2.1.4}$$

with $\equiv_n$ the equivalence relation modulo n. Symbols ord and $\overline{\mathrm{ac}}$ will be interpreted respectively as valuation and angular component, so that $(K((t)), K, \mathbb{Z})$ is a structure for $\mathscr{L}_{\mathrm{DP,P}}$. We shall also add constant symbols in the Val, resp. Res, sort, for every element of $k((t))$, resp. k.

Let φ be a formula in the language $\mathscr{L}_{\mathrm{DP,P}}$ with respectively m, n and r free variables in the various sorts. For every K in Field_k, the category of fields containing k, we denote by $h_\varphi(K)$ the subset of

$$h[m, n, r](K) := K((t))^m \times K^n \times \mathbb{Z}^r \tag{2.1.5}$$

consisting of points satisfying φ. We call the assignment $K \mapsto h_\varphi(K)$ a definable subassignment and we define a category Def_k whose objects are definable subassignments. For Z in Def_k, a point x of Z is by definition a tuple $x = (x_0, K)$ such that x_0 is in $Z(K)$ and K is in Field_k. For a point $x = (x_0, K)$ of Z, we write $k(x) = K$ and we call $k(x)$ the residue field of x.

More generally for S in Def_k, we denote by Def_S the category of objects of Def_k over S. We denote by RDef_S the subcategory of Def_S consisting of definable subassignments of $S \times h[0, n, 0]$, for variable n, and by $K_0(\mathrm{RDef}_S)$ the corresponding Grothendieck ring.

We consider the ring

$$\mathbb{A} = \mathbb{Z}\Big[\mathbb{L}, \mathbb{L}^{-1}, \Big(\frac{1}{1 - \mathbb{L}^{-i}}\Big)_{i>0}\Big], \tag{2.1.6}$$

where $\mathbb{L}$ is a symbol, and the subring $\mathscr{P}(S)$ of the ring of functions from the set of points of S to $\mathbb{A}$ generated by constant functions, definable functions $S \to \mathbb{Z}$ and functions of the form $\mathbb{L}^\beta$ with β definable $S \to \mathbb{Z}$. If Y is a definable subassignment of S, we denote by $\mathbf{1}_Y$ the function in $\mathscr{P}(S)$ with value 1 on Y and 0 outside. We denote by $\mathscr{P}^0(S)$ the subring of $\mathscr{P}(S)$ generated by such functions and by the constant function $\mathbb{L}$. There is a morphism $\mathscr{P}^0(S) \to K_0(\mathrm{RDef}_S)$ sending $\mathbf{1}_Y$ to the class of Y and sending $\mathbb{L}$ to the class of $h[0, 1, 0]$. Finally we define the ring of constructible motivic functions on S by

$$\mathscr{C}(S) := K_0(\mathrm{RDef}_S) \otimes_{\mathscr{P}^0(S)} \mathscr{P}(S). \tag{2.1.7}$$

To any algebraic subvariety Z of $\mathbb{A}^m_{k((t))}$ we assign the definable subassignment h_Z of $h[m,0,0]$ given by $h_Z(K) = Z(K((t)))$. The Zariski closure of a subassignment S of $h[m,0,0]$ is the intersection W of all algebraic subvarieties Z of $\mathbb{A}^m_{k((t))}$ such that $S \subset h_Z$. We set $\dim S := \dim W$. More generally, if S is a subassignment of $h[m,n,r]$, we define $\dim S$ to be $\dim p(S)$ with p the projection $h[m,n,r] \to h[m,0,0]$. One proves, using results of [29] and [32], that two isomorphic objects in Def_k have the same dimension. For every nonnegative integer d, we denote by $\mathscr{C}^{\leq d}(S)$ the ideal of $\mathscr{C}(S)$ generated by the characteristic functions $\mathbf{1}_Z$ of definable subassignments Z of S with $\dim Z \leq d$. We set $C(S) = \oplus_d C^d(S)$ with $C^d(S) := \mathscr{C}^{\leq d}(S)/\mathscr{C}^{\leq d-1}(S)$.

In [11], we defined, for k a field of characteristic zero, S in Def_k, and Z in Def_S, a graded subgroup $\mathrm{I}_S C(Z)$ of $C(Z)$ together with pushforward morphisms

$$f_! : \mathrm{I}_S C(Z) \longrightarrow \mathrm{I}_S C(Y) \tag{2.1.8}$$

for every morphism $f : Z \to Y$ in Def_S. When S is the final object $h[0,0,0]$ and f is the morphism $Z \to S$, the morphism $f_!$ corresponds to motivic integration and we denote it by μ.

Finally, fix Λ in Def_k. Replacing dimension by relative dimension, we define relative analogues $C(Z \to \Lambda)$ of $C(Z)$ for $Z \to \Lambda$ in Def_Λ and extend the above constructions to this relative setting. In particular we construct a morphism

$$\mu_\Lambda : \mathrm{I}_\Lambda C(Z \to \Lambda) \longrightarrow \mathscr{C}(\Lambda) = \mathrm{I}_\Lambda C(\Lambda \to \Lambda) \tag{2.1.9}$$

which corresponds to motivic integration along the fibers of the morphism $Z \to \Lambda$.

2.2. *Cell decomposition.* We now recall the definition of cells given in [11], which is a slight generalization of the one in [29].

Let C be a definable subassignment in Def_k. Let α, ξ, and c be definable morphisms $\alpha : C \to \mathbb{Z}$, $\xi : C \to h_{\mathbb{G}_{m,k}}$, and $c : C \to h[1,0,0]$. The cell $Z_{C,\alpha,\xi,c}$ with basis C, order α, center c, and angular component ξ is the definable subassignment of $C[1,0,0]$ defined by $\mathrm{ord}(z - c(y)) = \alpha(y)$, and $\overline{\mathrm{ac}}(z - c(y)) = \xi(y)$, where y lies in C and z in $h[1,0,0]$. Similarly, if c is a definable morphism $c : C \to h[1,0,0]$, we define the cell $Z_{C,c}$ with center c and basis C as the definable subassignment of $C[1,0,0]$ defined by $y \in C$ and $z = c(y)$.

More generally, a definable subassignment Z of $S[1,0,0]$ for some S in Def_k will be called a 1-cell, resp. a 0-cell, if there exists a definable isomorphism

$$\lambda : Z \to Z_C = Z_{C,\alpha,\xi,c} \subset S[1,s,r], \tag{2.2.1}$$

resp. a definable isomorphism

$$\lambda : Z \to Z_C = Z_{C,c} \subset S[1,s,0], \tag{2.2.2}$$

for some $s, r \geq 0$, some $C \subset S[0,s,r]$, and some 1-cell $Z_{C,\alpha,\xi,c}$, resp. 0-cell $Z_{C,c}$, such that the morphism $\pi \circ \lambda$, with π the projection on the $S[1,0,0]$-factor, is the identity on Z.

We shall call the data $(\lambda, Z_{C,\alpha,\xi,c})$, resp. $(\lambda, Z_{C,c})$, sometimes written for short (λ, Z_C), a presentation of the cell Z.

One should note that λ^* induces a canonical bijection between $\mathscr{C}(Z_C)$ and $\mathscr{C}(Z)$.

In [11], we proved the following variant of the Denef-Pas Cell Decomposition Theorem [29]:

THEOREM 2.2.1. *Let X be a definable subassignment of $S[1,0,0]$ with S in* Def_k.

(1) *The subassignment X is a finite disjoint union of cells.*

(2) *For every φ in $\mathscr{C}(X)$ there exists a finite partition of X into cells Z_i with presentation (λ_i, Z_{C_i}), such that $\varphi_{|Z_i} = \lambda_i^* p_i^*(\psi_i)$, with ψ_i in $\mathscr{C}(C_i)$ and $p_i : Z_{C_i} \to C_i$ the projection. Similar statements hold for φ in $\mathscr{P}(X)$, and in* $K_0(\mathrm{RDef}_X)$.

We shall call a finite partition of X into cells Z_i as in Theorem 2.2.1 (1), resp. Theorem 2.2.1 (2) for a function φ, a cell decomposition of X, resp. a cell decomposition of X adapted to φ.

The following result is in [11, Th. 7.5.3], except for (6) which is new.

THEOREM 2.2.2. *Let X be in* Def_k, *Z be a definable subassignment of $X[1,0,0]$, and let $f : Z \to h[1,0,0]$ be a definable morphism. There exists a cell decomposition of Z into cells Z_i such that the following conditions hold for every y in C_i, for every K in* $\mathrm{Field}_{k(y)}$, *and for every* 1*-cell Z_i with presentation $\lambda_i : Z_i \to Z_{C_i} = Z_{C_i,\alpha_i,\xi_i,c_i}$ and with projections $p_i : Z_{C_i} \to C_i$, $\pi_i : Z_{C_i} \to h[1,0,0]$:*

(1) *The set $\pi_i(p_i^{-1}(y))(K)$ is either empty or a ball of volume $\mathbb{L}^{-\alpha_i(y)-1}$.*

(2) *When $\pi_i(p_i^{-1}(y))(K)$ is nonempty, the function*

$$g_{y,K} : \begin{cases} \pi_i(p_i^{-1}(y))(K) \to K((t)) \\ x \mapsto f \circ \lambda_i^{-1}(y,x) \end{cases}$$

is strictly analytic.

For each i we can furthermore ensure that either $g_{y,K}$ is constant or (3) *up to* (6) *hold.*

(3) *There exists a definable morphism* $\beta_i : C_i \to h[0,0,1]$ *such that*

$$\mathrm{ord}\,\frac{\partial}{\partial x} g_{y,K}(x) = \beta_i(y)$$

for every x *in* $\pi_i(p_i^{-1}(y))(K)$.

(4) *When* $\pi_i(p_i^{-1}(y))(K)$ *is nonempty, the map* $g_{y,K}$ *is a bijection onto a ball of volume* $\mathbb{L}^{-\alpha_i(y)-1-\beta_i(y)}$.

(5) *For every* x, x' *in* $\pi_i(p_i^{-1}(y))(K)$,

$$\mathrm{ord}(g_{y,K}(x) - g_{y,K}(x')) = \beta_i(y) + \mathrm{ord}(x - x').$$

(6) *There exists a morphism* $r_i : C_i \to h[1,0,0]$ *s.t. for every* x *in* $\pi_i(p_i^{-1}(y))(K)$

$$g_{y,K}(x) = r_i(y) \;\text{ or }\; \mathrm{ord}(g_{y,K}(x) - r_i(y)) \geq \alpha_i(y) + \beta_i(y).$$

Proof. We only have to prove (6). First take a cell decomposition with properties (1) up to (5). By replacing X we may suppose that the identity maps are presentations of the occurring cells. We consider the graph of f in $Z[1,0,0]$ and its image $W \subset X[1,0,0]$ under the morphism $Z[1,0,0] \to X[1,0,0]$ sending (z,x) to $(p(z),x)$, with $p : Z \to X$ the morphism induced by the projection $X[1,0,0] \to X$. If one takes a cell decomposition of $W \subset X[1,0,0]$, the centers of the cells we obtain that way are approximations of f as required in (6), and again by replacing X one can assume that the identity maps are presentations of the occurring cells. Now take again a cell decomposition of X such that properties (1) up to (5) are fulfilled. Then automatically (6) is fulfilled as well. □

3. Constructible exponential functions

3.1. *Adding exponentials to Grothendieck rings.* Let Z be in Def_k. We consider the category $\mathrm{RDef}_Z^{\exp}$ whose objects are triples $(Y \to Z, \xi, g)$ with Y in RDef_Z and $\xi : Y \to h[0,1,0]$ and $g : Y \to h[1,0,0]$ morphisms in Def_k. A morphism $(Y' \to Z, \xi', g') \to (Y \to Z, \xi, g)$ in $\mathrm{RDef}_Z^{\exp}$ is a morphism $h : Y' \to Y$ in Def_Z such that $\xi' = \xi \circ h$ and $g' = g \circ h$. The functor sending Y in RDef_Z to $(Y,0,0)$, with 0 denoting the constant morphism with value 0 in $h[0,1,0]$, resp. $h[1,0,0]$ being fully faithful, we may consider RDef_Z as a full subcategory of $\mathrm{RDef}_Z^{\exp}$. We shall also consider the intermediate full subcategory $\mathrm{RDef}_Z^{\mathrm{e}}$ consisting of objects $(Y, \xi, 0)$ with $\xi : Y \to h[0,1,0]$ a morphism in Def_k.

To the category $\mathrm{RDef}_Z^{\exp}$ one assigns a Grothendieck ring $K_0(\mathrm{RDef}_Z^{\exp})$ defined as follows. As an abelian group it is the quotient of the free abelian group over symbols $[Y \to Z, \xi, g]$ with $(Y \to Z, \xi, g)$ in $\mathrm{RDef}_Z^{\exp}$ by the following four relations (R1)–(R4). The first two are natural analogues of the ones occurring in the definition of $K_0(\mathrm{RDef}_Z)$ while the last two are specific to the exponential setting.

Isomorphism: For $(Y \to Z, \xi, g)$ isomorphic to $(Y' \to Z, \xi', g')$,

$$[Y \to Z, \xi, g] = [Y' \to Z, \xi', g']. \tag{R1}$$

Additivity: For Y and Y' definable subassignments of some X in RDef_Z and ξ, g defined on $Y \cup Y'$,

$$\begin{aligned}[(Y \cup Y') \to Z, \xi, g] + [(Y \cap Y') \to Z, \xi_{|Y\cap Y'}, g_{|Y\cap Y'}]\\ = [Y \to Z, \xi_{|Y}, g_{|Y}] + [Y' \to Z, \xi_{|Y'}, g_{|Y'}].\end{aligned} \tag{R2}$$

Compatibility with reduction: For $h : Y \to h[1,0,0]$ a definable morphism with $\mathrm{ord}(h(y)) \geq 0$ for all y in Y and $\bar{h}$ the reduction of h modulo (t),

$$[Y \to Z, \xi, g + h] = [Y \to Z, \xi + \bar{h}, g]. \tag{R3}$$

Sum over the line: When $p : Y[0,1,0] \to h[0,1,0]$ is the projection and when the morphisms $Y[0,1,0] \to Z$, g, and ξ all factorize through the projection $Y[0,1,0] \to Y$,

$$[Y[0,1,0] \to Z, \xi + p, g] = 0. \tag{R4}$$

Relation (R3) expresses compatibility under reduction modulo the uniformizing parameter between the exponential over the valued field and over the residue field. It can be considered as an analogue of the relation $\exp((2\pi i/p)x) = \exp((2\pi i/p)\bar{x})$ for x in $\mathbb{Z}_p$ reducing mod p to $\bar{x}$ in $\mathbb{Z}/p\mathbb{Z}$. Relation (R4) expresses abstractly the familiar fact that the sum of the values of a nontrivial character over all points in a finite field is zero. Indeed, in the special case where $Y = Z$ and ξ and g are the constant morphisms with value zero, the relation reduces to

$$[Y[0,1,0] \to Y, p, 0] = 0 \tag{R4$'$}$$

which one can understand as expressing that the integral of a nontrivial character over a residue field line is zero.

The following lemma allows us to endow $K_0(\mathrm{RDef}_Z^{\exp})$ with a ring structure.

LEMMA 3.1.1. *We may endow $K_0(\mathrm{RDef}_Z^{\exp})$ with a ring structure by setting*

$$[Y \to Z, \xi, g]\cdot[Y' \to Z, \xi', g'] = [Y \otimes_Z Y' \to Z, \xi \circ p_Y + \xi' \circ p_{Y'}, g \circ p_Y + g' \circ p_{Y'}],$$

where $Y \otimes_Z Y'$ is the fiber product of Y and Y', p_Y the projection to Y, and $p_{Y'}$ the projection to Y'.

Proof. Clearly the fiber product induces a commutative ring structure on the free group on symbols $[Y \to Z, \xi, g]$ with $(Y \to Z, \xi, g)$ in $\mathrm{RDef}_Z^{\exp}$. The subgroup generated by the four relations (R1) up to (R4) is an ideal of this ring, hence, the quotient by this subgroup is a ring. ☐

Similarly, using relations (R1), (R2), (R4), and the subcategory $\mathrm{RDef}^{\mathrm{e}}_Z$, one may define the subring $K_0(\mathrm{RDef}^{\mathrm{e}}_Z)$ of $K_0(\mathrm{RDef}^{\exp}_Z)$.

3.1.2. *Notation and abbreviations.* We write $\mathbf{e}^{\xi}E(g)[Y\to Z]$ for $[Y\to Z,\xi,g]$. We abbreviate $\mathbf{e}^{0}E(g)[Y\to Z]$ by $E(g)[Y\to Z]$, $\mathbf{e}^{\xi}E(0)[Y\to Z]$ by $\mathbf{e}^{\xi}[Y\to Z]$, and $\mathbf{e}^{0}E(0)[Y\to Z]$ by $[Y\to Z]$. Likewise we write $\mathbf{e}^{\xi}E(g)$ for $\mathbf{e}^{\xi}E(g)[Z\to Z]$, $E(g)$ for $\mathbf{e}^{0}E(g)[Z\to Z]$ and $\mathbf{e}^{\xi}$ for $\mathbf{e}^{\xi}E(0)[Z\to Z]$. Note that the element $[Z\to Z]$ is the multiplicative unit of $K_0(\mathrm{RDef}^{\exp}_Z)$.

LEMMA 3.1.3. *There are natural injections of rings*

$$K_0(\mathrm{RDef}_Z)\to K_0(\mathrm{RDef}^{\mathrm{e}}_Z)\to K_0(\mathrm{RDef}^{\exp}_Z)$$

sending $[Y\to Z]$ *to* $[Y\to Z]$ *and* $\mathbf{e}^{\xi}[Y\to Z]$ *to* $\mathbf{e}^{\xi}[Y\to Z]$.

Proof. Since both statements are similar, we will prove that $i: K_0(\mathrm{RDef}_Z)\to K_0(\mathrm{RDef}^{\mathrm{e}}_Z)$ is an injection. Suppose that there are a_1,a_2 in $K_0(\mathrm{RDef}_Z)$ having the same image in $K_0(\mathrm{RDef}^{\mathrm{e}}_Z)$. The equality $i(a_1)=i(a_2)$ in $K_0(\mathrm{RDef}^{\mathrm{e}}_Z)$ induces an equality in the free group on symbols $[Y\to Z,\xi,0]$ of the form

$$\sum_i [Y_i\to Z,\xi_i,0]=\sum_j [Y_j\to Z,\xi_j,0], \tag{3.1.1}$$

by adding up relations. For each i,j, let $Y_i'\subset Y_i$ and $Y_j'\subset Y_j$ be the subassignments defined by $\xi_i=0$, resp. $\xi_j=0$. Then

$$\sum_i [Y_i'\to Z,0,0]=\sum_j [Y_j'\to Z,0,0] \tag{3.1.2}$$

holds in the free group. Hence, $a_1=a_2$ in $K_0(\mathrm{RDef}_Z)$. □

3.2. *Pull-back.* For $f: Z\to Z'$ in Def_k we have a natural pull-back morphism $f^*: K_0(\mathrm{RDef}^{\exp}_{Z'})\to K_0(\mathrm{RDef}^{\exp}_Z)$, induced by the fiber product.

If $f: Z\to Z'$ is a morphism in $\mathrm{RDef}_{Z'}$, composition with f induces a morphism $f_!: K_0(\mathrm{RDef}^{\exp}_Z)\to K_0(\mathrm{RDef}^{\exp}_{Z'})$.

We have similar morphisms when we replace $\mathrm{RDef}^{\exp}$ by $\mathrm{RDef}^{\mathrm{e}}$.

3.3. *Constructible exponential functions.* For Z in Def_k we define the ring $\mathscr{C}(Z)^{\exp}$ of constructible exponential functions by

$$\mathscr{C}(Z)^{\exp}:=\mathscr{C}(Z)\otimes_{K_0(\mathrm{RDef}_Z)}K_0(\mathrm{RDef}^{\exp}_Z). \tag{3.3.1}$$

Note that the element $E(\mathrm{id})$ of $\mathscr{C}(h[1,0,0])^{\exp}$, with id the identity map on $h[1,0,0]$, can be seen as an abstract additive character, taking nontrivial values at elements of order ≤ 0 only; see (R3) and the explanation of axiom (R3). Likewise, the element $\mathbf{e}^{\mathrm{id}}$ of $\mathscr{C}(h[0,1,0])^{\exp}$, with id the identity map on $h[0,1,0]$, can be seen as an additive character on the residue field. These two meanings of abstract additive characters underlie, of course, the definition of the ring structure in Lemma 3.1.1.

For every $d \geq 0$ we define $\mathscr{C}^{\leq d}(Z)^{\exp}$ as the ideal of $\mathscr{C}(Z)^{\exp}$ generated by the characteristic functions $\mathbf{1}_{Z'}$ of subassignments $Z' \subset Z$ of dimension $\leq d$.

We set

$$C(Z)^{\exp} = \oplus_d C^d(Z)^{\exp} \tag{3.3.2}$$

with

$$C^d(Z)^{\exp} := \mathscr{C}^{\leq d}(Z)^{\exp} / \mathscr{C}^{\leq d-1}(Z)^{\exp}. \tag{3.3.3}$$

It is a graded abelian group, and also a $\mathscr{C}(Z)^{\exp}$-module. We call elements of $C(Z)^{\exp}$ constructible exponential Functions.

For S in Def_k and Z in Def_S we define the group $\mathrm{I}_S C(Z)^{\exp}$ of S-integrable constructible exponential Functions by

$$\mathrm{I}_S C(Z)^{\exp} := \mathrm{I}_S C(Z) \otimes_{K_0(\mathrm{RDef}_Z)} K_0(\mathrm{RDef}_Z^{\exp}). \tag{3.3.4}$$

It is a graded subgroup of $C(Z)^{\exp}$.

LEMMA 3.3.1. *For every Z in* Def_k, *the natural morphisms of rings, resp. of graded groups*, $\mathscr{C}(Z) \to \mathscr{C}(Z)^{\exp}$, $\mathscr{C}^{\leq d}(Z) \to \mathscr{C}^{\leq d}(Z)^{\exp}$, *resp.* $C(Z) \to C(Z)^{\exp}$, $\mathrm{I}_S C(Z) \to \mathrm{I}_S C(Z)^{\exp}$ *are injective.*

Proof. This follows from Lemma 3.1.3 by taking tensor products, and by noting that $\mathscr{C}^{\leq d}(Z)^{\exp}$ is isomorphic to $\mathscr{C}^{\leq d}(Z) \otimes_{K_0(\mathrm{RDef}_Z)} K_0(\mathrm{RDef}_Z^{\exp})$. $\square$

PROPOSITION 3.3.2. *Let S be in* Def_k *and let W be a definable subassignment of $h[0,0,m]$. The canonical morphism*

$$K_0(\mathrm{RDef}_S^{\exp}) \otimes_{\mathscr{P}^0(S)} \mathscr{P}^0(S \times W) \longrightarrow K_0(\mathrm{RDef}_{S\times W}^{\exp})$$

is an isomorphism.

Proof. This follows from the Denef-Pas quantifier elimination as stated in [11]. $\square$

3.4. *Inverse image of constructible exponential functions.* Let $f : Z \to Z'$ be a morphism in Def_k. Since f^* as defined on $K_0(\mathrm{RDef}_{Z'}^{\exp})$ and on $\mathscr{C}(Z')$ is compatible with the morphisms $K_0(\mathrm{RDef}_{Z'}) \to \mathscr{C}(Z')$ and $K_0(\mathrm{RDef}_{Z'}) \to K_0(\mathrm{RDef}_{Z'}^{\exp})$, one gets by tensor product a natural pull-back morphism $f^* : \mathscr{C}(Z')^{\exp} \to \mathscr{C}(Z)^{\exp}$.

3.5. *Push-forward for inclusions.* Let $i : Z \hookrightarrow Z'$ be the inclusion between two definable subassignments $Z \subset Z'$. Extension by zero induces a morphism $i_! : K_0(\mathrm{RDef}_Z^{\exp}) \to K_0(\mathrm{RDef}_{Z'}^{\exp})$. Since this is compatible on $K_0(\mathrm{RDef}_Z)$ with $i_! : \mathscr{C}(Z) \to \mathscr{C}(Z')$, we get, by tensor product, a morphism $i_! : \mathscr{C}(Z)^{\exp} \to \mathscr{C}(Z')^{\exp}$.

Because i sends subassignments of Z to subassignments of Z' of the same dimension, there are group morphisms $i_! : \mathscr{C}^{\leq d}(Z)^{\exp} \to \mathscr{C}^{\leq d}(Z')^{\exp}$, and graded group morphisms $i_! : C(Z)^{\exp} \to C(Z')^{\exp}$. If Z' is in Def_S then $f_!$ clearly restricts to a morphism $f_! : \mathrm{I}_S C(Z)^{\exp} \to \mathrm{I}_S C(Z')^{\exp}$.

3.6. *Push-forward for k-projections.* Let Y be in Def_k and let Z be a definable subassignment of $Y[0,r,0]$, for some $r \geq 0$. Denote by $f : Z \to Y$ the morphism induced by projection. It follows from statement (1) in Proposition 5.2.1 of [11] that the map $f_! : K_0(\mathrm{RDef}_Z^{\exp}) \to K_0(\mathrm{RDef}_Y^{\exp})$ induces a ring morphism $f_! : \mathscr{C}(Z)^{\exp} \to \mathscr{C}(Y)^{\exp}$, and because f sends subassignments of Z to subassignments of Y of the same dimension, there are group morphisms $f_! : \mathscr{C}^{\leq d}(Z)^{\exp} \to \mathscr{C}^{\leq d}(Y)^{\exp}$, and graded group morphisms $f_! : C(Z)^{\exp} \to C(Y)^{\exp}$. If Y is in Def_S then $f_!$ clearly restricts to a morphism $f_! : \mathrm{I}_S C(Z)^{\exp} \to \mathrm{I}_S C(Y)^{\exp}$. Note also that the projection formula trivially holds in this setting; that is, for every α in $\mathscr{C}(Y)^{\exp}$ and β in $C(Z)^{\exp}$, $f_!(f^*(\alpha)\beta) = \alpha f_!(\beta)$.

3.7. *Push-forward for $\mathbb{Z}$-projections.* If $f : Z[0,0,m] \to Z$ is the projection and Z is in Def_S, $m \geq 0$, then, by Proposition 3.3.2 and by the fact that f preserves the dimension of definable subassignments of $Z[0,0,m]$, the map $f_! : \mathrm{I}_S C(Z[0,0,m]) \to \mathrm{I}_S C(Z)$ induces a graded group morphism

$$f_! : \mathrm{I}_S C(Z[0,0,m])^{\exp} \to \mathrm{I}_S C(Z)^{\exp}.$$

Lemma-Definition 3.7.1 below is a basic kind of Fubini theorem between the push forwards of Sections 3.6 and 3.7, and Lemma 3.7.2 is a basic form of the change of variables formula.

Lemma-Definition 3.7.1. Let φ be in $\mathrm{I}_S C(Z[0,m,r])^{\exp}$ for some $m, r \geq 0$ and some Z in Def_S and let $f : Z[0,m,r] \to Z$ be the projection. Let $\pi_1, \ldots, \pi_{m+r}$ be any sequence of projections of the form $Z[0,i,j] \to Z[0,i-1,j]$ or $Z[0,i,j] \to Z[0,i,j-1]$ whose composition goes from $Z[0,m,r]$ to Z. Then, $\pi_{m+r!} \circ \ldots \circ \pi_{1!}(\varphi)$ is independent of the sequence $\pi_1, \ldots, \pi_{m+r}$ and we define $f_!(\varphi)$ to be this element.

Proof. This follows from the fact that

$$K_0(\mathrm{RDef}_{Z[0,m,0]}^{\exp}) \otimes_{K_0(\mathrm{RDef}_Z^{\exp})} \mathscr{C}(Z)^{\exp} \otimes_{\mathscr{C}(Z)} \mathscr{C}(Z[0,0,r]) \longrightarrow \mathscr{C}(Z[0,m,r])^{\exp}$$

is an isomorphism. $\square$

Let $\lambda : S[0,n,r] \to S[0,n',r']$ be a morphism in Def_S. Let φ be a function in $\mathscr{C}(S[0,n,r])^{\exp}$. Assume $\varphi = \mathbf{1}_Z \varphi$ with Z a definable subassignment of $S[0,n,r]$ on which λ is injective. Thus λ restricts to an isomorphism λ' between Z and $Z' := \lambda(Z)$. We define $\lambda_+(\varphi)$ in $\mathscr{C}(S[0,n',r'])^{\exp}$ as $[i'_!(\lambda'^{-1})^* i^*](\varphi)$, where i and i' denote respectively the inclusions of Z and Z' in $S[0,n,r]$ and $S[0,n',r']$.

LEMMA 3.7.2. *Let $\lambda : S[0,n,r] \to S[0,n',r']$ be a morphism in* Def_S. *Let φ be a function in $\mathscr{C}(S[0,n,r])^{\exp}$ such that $\varphi = \mathbf{1}_Z\varphi$ with Z a definable subassignment of $S[0,n,r]$ on which λ is injective. Then φ is in $\mathrm{I}_S C(S[0,n,r])^{\exp}$ if and only if $\lambda_+(\varphi)$ is in $\mathrm{I}_S C(S[0,n',r'])^{\exp}$ and if this is the case then*

$$p_!(\varphi) = p'_!(\lambda_+(\varphi)),$$

with $p : S[0,n,r] \to S$ and $p' : S[0,n',r'] \to S$ the projections and $p_!$ and $p'_!$ as in Lemma-Definition 3.7.1.

Proof. Consider the definable isomorphism

$$\lambda \times \mathrm{id} : S[0,n,r] \to S[0,n+n',r+r'] \tag{3.7.1}$$

with inverse g. Since this is an isomorphism, φ is S-integrable if and only if $g^*(\varphi)$ is S-integrable. By construction,

$$\pi'_!\left(g^*(\varphi)\right) = \lambda_+(\varphi), \tag{3.7.2}$$

$$\pi_!\left(g^*(\varphi)\right) = \varphi, \tag{3.7.3}$$

with $\pi : S[0,n+n',r+r'] \to S[0,n,r]$ and $\pi' : S[0,n+n',r+r'] \to S[0,n',r']$ the projections. Now the Lemma follows from Lemma-Definition 3.7.1. □

3.8. *Relative setting.* Let us fix Λ in Def_k that will play the role of a parameter space. For Z in Def_Λ, we consider, similarly, as in [11], the ideal $\mathscr{C}^{\leq d}(Z \to \Lambda)^{\exp}$ of $\mathscr{C}(Z)^{\exp}$ generated by functions $\mathbf{1}_{Z'}$ with Z' a definable subassignment of Z such that all fibers of $Z' \to \Lambda$ have dimension $\leq d$. We set

$$C(Z \to \Lambda)^{\exp} := \oplus_d C^d(Z \to \Lambda)^{\exp} \tag{3.8.1}$$

with

$$C^d(Z \to \Lambda) := \mathscr{C}^{\leq d}(Z \to \Lambda)/\mathscr{C}^{\leq d-1}(Z \to \Lambda). \tag{3.8.2}$$

This graded abelian semigroup may be naturally identified with

$$C(Z \to \Lambda) \otimes_{K_0(\mathrm{RDef}_Z)} K_0(\mathrm{RDef}_Z^{\exp}). \tag{3.8.3}$$

For $Z \to S$ a morphism in Def_Λ, we set

$$\mathrm{I}_S C(Z \to \Lambda)^{\exp} := \mathrm{I}_S C(Z \to \Lambda) \otimes_{K_0(\mathrm{RDef}_Z)} K_0(\mathrm{RDef}_Z^{\exp}). \tag{3.8.4}$$

Lemma 3.3.1 and all results and constructions in Sections 3.5, 3.6, 3.7, including Lemma-Definition 3.7.1 and Lemma 3.7.2, extend immediately with the same proofs to the relative setting.

4. **Integration of constructible exponential functions**

4.1. *The main result.* We can now state the result on extending our construction of motivic integrals from constructible functions to constructible exponential functions.

THEOREM 4.1.1. *Let S be in* Def_k. *There is a unique functor from the category* Def_S *to the category of abelian groups which sends Z to* $\mathrm{I}_S C(Z)^{\exp}$, *assigns to every morphism $f : Z \to Y$ in* Def_S *a morphism* $f_! : \mathrm{I}_S C(Z)^{\exp} \to \mathrm{I}_S C(Y)^{\exp}$ *and which satisfies the following five axioms*:

(A1) Compatibility: *For every morphism $f : Z \to Y$ in* Def_S, *the map* $f_! : \mathrm{I}_S C(Z)^{\exp} \to \mathrm{I}_S C(Y)^{\exp}$ *is compatible with the inclusions of groups* $\mathrm{I}_S C(Z) \to \mathrm{I}_S C(Z)^{\exp}$ *and* $\mathrm{I}_S C(Y) \to \mathrm{I}_S C(Y)^{\exp}$ *and with the map* $f_! : \mathrm{I}_S C(Z) \to \mathrm{I}_S C(Y)$ *as constructed in* [11].

(A2) Disjoint union: *Let Z and Y be definable subassignments in* Def_S. *Assume Z, resp. Y, is the disjoint union of two definable subassignments Z_1 and Z_2, resp. Y_1 and Y_2, of Z, resp. Y. Then, for every morphism $f : Z \to Y$ in* Def_S, *with $f(Z_i) \subset Y_i$ for $i = 1, 2$, under the isomorphisms* $\mathrm{I}_S C(Z)^{\exp} \simeq \mathrm{I}_S C(Z_1)^{\exp} \oplus \mathrm{I}_S C(Z_2)^{\exp}$ *and* $\mathrm{I}_S C(Y)^{\exp} \simeq \mathrm{I}_S C(Y_1)^{\exp} \oplus \mathrm{I}_S C(Y_2)^{\exp}$, *we have $f_! = f_{1!} \oplus f_{2!}$, with $f_i : Z_i \to Y_i$ the restrictions of f.*

(A3) Projection formula: *For every morphism $f : Z \to Y$ in* Def_S, *and every α in $\mathscr{C}(Y)^{\exp}$ and β in* $\mathrm{I}_S C(Z)^{\exp}$, *if $f^*(\alpha)\beta$ is in* $\mathrm{I}_S C(Z)^{\exp}$, *then $f_!(f^*(\alpha)\beta) = \alpha f_!(\beta)$.*

(A4) Projection on k-variables: *Assume that f is the projection $f : Z = Y[0, n, 0] \to Y$ for some Y in* Def_S. *For every φ in* $\mathrm{I}_S C(Z)^{\exp}$, *$f_!(\varphi)$ is as constructed in Section* 3.6.

(A5) Relative balls of large volume: *Let Y be in* Def_S *and consider definable morphisms $\alpha : Y \to \mathbb{Z}$, $\xi : Y \to h_{\mathbb{G}_{m,k}}$, with $\mathbb{G}_{m,k}$ the multiplicative group $\mathbb{A}^1_k \setminus \{0\}$. Suppose that $[\mathbf{1}_Z]$ is in* $\mathrm{I}_S C(Z)^{\exp}$ *and that Z is the definable subassignment of $Y[1, 0, 0]$ defined by* $\mathrm{ord}\, z = \alpha(y)$ *and* $\overline{\mathrm{ac}}\, z = \xi(y)$, *and $f : Z \to Y$ is the morphism induced by the projection $Y[1, 0, 0] \to Y$. If moreover $\alpha(y) < 0$ holds for every y in Y, then*

$$f_!(E(z)[\mathbf{1}_Z]) = 0.$$

Moreover, these group morphisms $f_!$ coincide with the group morphisms constructed in Sections 3.7 *and in* 3.5 *in the corresponding cases.*

When $S = h[0, 0, 0]$, we write $\mathrm{I}C(Y)^{\exp}$ for $\mathrm{I}_S C(Y)^{\exp}$ and μ for the morphism $f_! : \mathrm{I}C(Y)^{\exp} \to \mathrm{I}C(h[0, 0, 0])^{\exp} = K_0(\mathrm{RDef}^{\exp}_{h[0,0,0]}) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A}$ when $f : Y \to h[0, 0, 0]$ is the projection to the final object.

4.2. *Change of variables formula.* We have the following analogue of Theorem 12.1.1 of [11].

THEOREM 4.2.1 (Change of variables formula). *Let $f : X \to Y$ be a definable isomorphism between definable subassignments of dimension d. Let φ be in $\mathscr{C}^{\leq d}(Y)^{\exp}$ with a nonzero class in $C^d(Y)^{\exp}$. Then $[f^*(\varphi)]$ belongs to* $\mathrm{I}_Y C^d(X)^{\exp}$ *and*

$$f_!([f^*(\varphi)]) = \mathbb{L}^{\operatorname{ordjac} f \circ f^{-1}}[\varphi].$$

Proof. Similar to the proof of Theorem 12.1.1 of [11], it is enough to consider the cases where f is an injection or a projection. When f is an injection the statement is true by construction. For projections, one reduces to the case of the projection of a 0-cell as in Proposition 11.4.3 of [11], which follows also by construction. □

4.3. *Relative version.* Fix Λ in Def_k. The proof of Theorem 4.1.1 which we shall give in Sections 5 and 6 readily extends to the following relative version:

THEOREM 4.3.1. *Let Λ belong to* Def_k *and let S belong to* Def_Λ. *There exists a unique functor from* Def_S *to the category of abelian groups assigning to any morphism $f : Z \to Y$ in* Def_S *a morphism*

$$f_{!\Lambda} : \mathrm{I}_S C(Z \to \Lambda)^{\exp} \to \mathrm{I}_S C(Y \longrightarrow \Lambda)^{\exp}$$

satisfying the analogues of axioms (A1)–(A5) *when replacing $C(\_)$ by $C(\_ \to \Lambda)$.*

Let Z be in Def_Λ. For every point λ of Λ, we denote by Z_λ the fiber of Z at λ, as defined in [11, 2.6]. We have a natural restriction morphism $i_\lambda^* : C(Z \to \Lambda)^{\exp} \to C(Z_\lambda)^{\exp}$, which respects the grading. Let $f : Z \to Y$ be a morphism in Def_Λ and let φ be in $C(Z \to \Lambda)^{\exp}$. We denote by $f_\lambda : Z_\lambda \to Y_\lambda$ the restriction of f to the fiber Z_λ. It follows from Proposition 14.2.1 of [11] that if φ is in $\mathrm{I}_Y C(Z \to \Lambda)^{\exp}$, then $i_\lambda^*(\varphi)$ is in $\mathrm{I}_{Y_\lambda} C(Z_\lambda)^{\exp}$. Furthermore, it follows from the constructions that

$$i_\lambda^*(f_{!\Lambda}(\varphi)) = f_{\lambda !}(i_\lambda^*(\varphi)) \tag{4.3.1}$$

for every point λ of Λ.

When $S = \Lambda$ and f is the morphism $Z \to \Lambda$, we write μ_Λ for the morphism $f_{!\Lambda} : \mathrm{I}_\Lambda C(Z \to \Lambda)^{\exp} \to \mathscr{C}(\Lambda)^{\exp} = \mathrm{I}_\Lambda C(\Lambda \to \Lambda)^{\exp}$.

Remark 4.3.2. It follows from the functoriality statement in Theorem 4.1.1, resp. Theorem 4.3.1, that for $f : X \to Y$ and $g : Y \to Z$ in Def_S, $(g \circ f)_! = g_! \circ f_!$, resp. $(g \circ f)_{!\Lambda} = g_{!\Lambda} \circ f_{!\Lambda}$. We shall sometimes refer to that property as the "Fubini Theorem".

4.4. *Global version.* Once we have Theorem 4.2.1 at our disposal it is possible to develop the theory on global subassignments, defined by replacing affine spaces by general algebraic varieties, along the lines of Section 15 of [11]. Since this is essentially straightforward we shall not give more details here.

5. Exponential integrals in dimension 1

We shall start by constructing directly exponential integrals in relative valued field dimension 1.

5.1. *Construction.* Let S be a definable subassignment and consider a definable subassignment $X \subset S[1,0,0]$ and denote by $\pi : X \to S$ the projection. Let M_X be the free group on symbols $[Y \to X, \xi, g, \varphi]$ with $((Y \to X, \xi, g), \varphi)$ in $\mathrm{RDef}_X^{\exp} \times I_S C(X)$.

We construct a map

$$\pi_! : M_X \to C(S)^{\exp} \tag{5.1.1}$$

and show that it factors through the natural surjective group morphism $M_X \to I_S C(X)^{\exp}$, thus obtaining a map

$$\pi_! : I_S C(X)^{\exp} \to C(S)^{\exp}, \tag{5.1.2}$$

which is the integral in relative dimension 1.

Consider $a = [f : Y \to X, \xi, g, \varphi]$ in M_X.

We shall use a suitable isomorphism of the form $\lambda : Y \to Y' \subset Y[0, n, r]$ which is an isomorphism over Y and which is adapted to a in a certain sense. Then we shall define $\pi_!$ by going through the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\lambda"', "\cong"] & X \arrow[r, "\pi"] & S \\
Y' \arrow[r, "\pi'"] \arrow[d, equal] & S' := S[0, n + n_Y, r] \arrow[ur, "p"] & \\
A_1 \cup A_2 \cup B, & &
\end{tikzcd}$$

where $Y \subset X[0, n_Y, 0]$, π' and p are the projections, and where we will write Y' as the disjoint union of A_1, A_2, and B, along which $\pi'_!$ will be easy to define.

Once we have found λ we will write a' for $[Y' \to Y', \xi' := \xi \circ \lambda^{-1}, g' := g \circ \lambda^{-1}, \varphi' := \lambda^{-1*} f^*(\varphi)]$ in $M_{Y'}$. We will first define $\pi'_!(a')$, and then define $\pi_!(a)$ as $p_! \pi'_!(a')$, where $p_!$ is as in Lemma-Definition 3.7.1.

Write φ as $[\varphi_0] + [\varphi_1]$ with φ_i in $\mathscr{C}^{\leq i}(X \to S)$ and φ_1 not in $\mathscr{C}^{\leq 0}(X \to S) \setminus \{0\}$.

Let Y_0 be the graph of ξ, and let $\lambda_0 : Y \to Y_0$ be the natural morphism. Let φ_{iY} be $f^*(\varphi_i)$ and let φ_{iY_0} be $\lambda_0^{-1*}(\varphi_{iY})$. By taking common refinements, one can take a single cell decomposition of Y_0 adapted to the φ_{iY_0} as in Theorem 2.2.1(2) and such that all six conditions of Theorem 2.2.2 are fulfilled for $g_{Y_0} := g \circ \lambda_0^{-1}$. Indeed, take a finite partition P_0 into cells adapted to φ_{0Y_0}, another finite partition P_1 adapted to φ_{1Y_0}, and another one P_2 as given by Theorem 2.2.2 for the function $g_{Y_0} : Y_0 \to h[1,0,0]$. The refinement P_3 is constructed as follows. For each triple (Z_0, Z_1, Z_2) of cells with $Z_i \in P_i$ having presentation λ_i, the intersection $Z_0 \cap Z_1 \cap Z_2$, if nonempty, is a cell with presentation $\otimes_{Y_0}\lambda_i$ and is defined to be in P_3. The cell decomposition P_3 is as desired.

Now we can construct λ. Write Z_{3i} for the cells in P_3 with presentation $\lambda_{3i} : Z_{3i} \to S[1, n_Y + n_i, r_i]$. Let n be the maximum of the n_i and r the maximum of the r_i. We extend the maps λ_{3i} to maps $\lambda'_{3i} : Z_{3i} \to S[1, n_Y + n, r]$, by sending to 0 on additional coordinates. Define λ as $\lambda : Y \to Y' \subset S[1, n_Y + n, r]$ sending y in Y to $\lambda'_{3i}(\lambda_0(y))$ for the unique i with $\lambda_0(y) \in Z_{3i}$.

Let φ'_i be $\lambda^{-1*}(\varphi_{iY})$ for $i = 0, 1$. Since ξ is a coordinate function on Y_0 and hence also on Y', one has *a priori* that $\xi' = \tilde{\xi} \circ \pi'$ for a unique $\tilde{\xi} : S' \to h[0,1,0]$. By the above applications of Theorems 2.2.1 and 2.2.2 and the definition of λ, one has that $\varphi'_i = \pi'^*(\psi_i)$ for some $\psi_i \in \mathscr{C}(S')$, and that properties (1) up to (6) of Theorem 2.2.2 are fulfilled for $g' = g \circ \lambda^{-1}$. There are now uniquely determined parts A, $B \subset Y'$, such that $g'(x, \cdot) : y \mapsto g'(x, y)$ is constant on B_x for each x, and nonconstant and injective on A_x for each x, where $A_x = \{y \in h[1,0,0] \mid (x, y) \in A\}$ and $B_x = \{y \in h[1,0,0] \mid (x, y) \in B\}$ are the fibers, and x runs over S'.

By construction $g'_{|B} = \tilde{g} \circ \pi'_{|B}$ for a unique definable $\tilde{g} : \pi'(B) \to h[1,0,0]$.

On the part A we proceed differently. It follows from the fact that $g'(x, \cdot)$ is nonconstant on the fiber A_x and the construction of λ that A is a 1-cell having the identity morphism as presentation. By the previous use of Theorem 2.2.2, A is the disjoint union of A_1 and A_2, with

(5.1.3)
$$A_1 := \{(x, y) \in A \mid g'(x, \cdot) \text{ maps } A_x \text{ onto a ball of volume } \mathbb{L}^{-j} \text{ with } j \le 0\}.$$

(5.1.4)
$$A_2 := \{(x, y) \in A \mid g'(x, \cdot) \text{ maps } A_x \text{ onto a ball of volume } \mathbb{L}^{-j} \text{ with } j > 0\}$$

(recall the normalization is such that the subassignment of all $x \in h[1,0,0]$ with $\mathrm{ord}(x) \ge j$ is a ball of volume $\mathbb{L}^{-j}$). Note that the A_i are cells which have the identity maps as presentations.

By construction and property (6) of Theorem 2.2.2, there are definable morphisms $r : S' \to h[1,0,0]$ and $\eta : S' \to h[0,1,0]$ such that

$$g'(x, y) - r(x) \equiv \eta(x) \bmod (t) \tag{5.1.5}$$

for $(x, y) \in A_2$, that is, either $r(x) - g'(x, y)$ has order > 0 and $\eta(x) = 0$, or, it has order 0 and angular component equal to $\eta(x)$.

Lemma-Definition 5.1.1. Consider λ, A_1, A_2, B, r, and η as constructed above. Define $\pi'_!(a')$ in $C(S')^{\exp}$ as

$$\pi'_!(a') := \mathbf{e}^{\tilde{\xi}} E(\tilde{g})\, \pi'_! \left(\mathbf{1}_B \varphi'\right) + \mathbf{e}^{\tilde{\xi}+\eta} E(r)\, \pi'_! \left(\mathbf{1}_{A_2} \varphi'\right)$$

where $\pi'_!$ in the right-hand side is as in [11]. Then, $\pi'_!(a')$ lies in $\mathrm{I}_S C(S')^{\exp}$ and is independent of the choice of r and η. Furthermore, $p_!(\pi'_!(a'))$, where $p_!$ is as in Lemma-Definition 3.7.1, is independent of the choice of λ, so that we can define $\pi_!(a)$ in $C(S)^{\exp}$ as

$$\pi_!(a) := p_!(\pi'_!(a')).$$

We extend $\pi_!$ to a group morphism $\pi_! : M_X \to C(S)^{\exp}$.

Proof. That $\pi'_!(\varphi')$ lies in $\mathrm{I}_S C(S')^{\exp}$ follows from the fact that $\pi'_!(\mathbf{1}_B \varphi')$ and $\pi'_! \left(\mathbf{1}_{A_2} \varphi'\right)$ are in $\mathrm{I}_S C(S')$, which is true by the main theorem of [11] and the fact that φ is in $I_S C(X)$. The independence from the choice of r and η is clear by relation (R3) for $K_0(\mathrm{RDef}^{\exp}_{S'})$.

We prove the independence from the choice of $\lambda : Y \to Y'$. Although this is similar to the proof of Lemma-Definition 9.1.2 in [11], using furthermore relation (R4), we give details. If another map $\hat{\lambda} : Y \to \hat{Y}$ with the same properties and with partition $\hat{A}_1, \hat{A}_2, \hat{B}$ is given, there exists a third map $\breve{\lambda} : Y \to \breve{Y}$ with the same properties and with partition $\breve{A}_1, \breve{A}_2, \breve{B}$, such that $\breve{\lambda}^{-1}(\breve{B})$ contains both $\hat{\lambda}^{-1}(\hat{B})$ and $\lambda^{-1}(B)$; for example, the map $\breve{\lambda} := \lambda \otimes_Y \hat{\lambda}$ has this property. Necessarily, $\breve{\lambda}^{-1}(\breve{B})$ is equal to the union of $\lambda^{-1}(B)$ with a 0-cell and is also equal to the union of $\hat{\lambda}^{-1}(\hat{B})$ with a 0-cell, since g' is injective on A_x. Since A is a 1-cell adapted to φ'_i, it follows that

$$p_! \pi'_! \left(\mathbf{1}_B \varphi'\right) = \hat{p}_! \hat{\pi}'_! \left(\mathbf{1}_{\hat{B}} \hat{\varphi}'\right) = \breve{p}_! \breve{\pi}'_! \left(\mathbf{1}_{\breve{B}} \breve{\varphi}'\right), \tag{5.1.6}$$

with obvious notation (this also follows from Lemma-Definition 9.1.2 in [11]). By Lemma-Definition 3.7.1, one finds

$$p_!(\mathbf{e}^{\tilde{\xi}} E(\tilde{g})\pi'_! \left(\mathbf{1}_B \varphi'\right)) = \hat{p}_!(\mathbf{e}^{\tilde{\hat{\xi}}} E(\tilde{\hat{g}})\hat{\pi}'_! \left(\mathbf{1}_{\hat{B}} \hat{\varphi}'\right)) = \breve{p}_!(\mathbf{e}^{\tilde{\breve{\xi}}} E(\tilde{\breve{g}})\breve{\pi}'_! \left(\mathbf{1}_{\breve{B}} \breve{\varphi}'\right)). \tag{5.1.7}$$

We now compare integrals over A, $\hat{A}$ and $\breve{A}$. Note that automatically we have the following inclusions

$$\breve{\lambda}^{-1}(\breve{A}) \subset \lambda^{-1}(A), \qquad \breve{\lambda}^{-1}(\breve{A}) \subset \hat{\lambda}^{-1}(\hat{A}), \tag{5.1.8}$$

$$\breve{\lambda}^{-1}(\breve{A}_1) \subset \lambda^{-1}(A_1), \qquad \breve{\lambda}^{-1}(\breve{A}_1) \subset \hat{\lambda}^{-1}(\hat{A}_1), \tag{5.1.9}$$

but maybe not so for A_2. By Lemma-Definition 9.1.2 in [11], one has

$$p_! \pi'_! \left(\mathbf{1}_A \varphi'\right) = \hat{p}_! \hat{\pi}'_! \left(\mathbf{1}_{\hat{A}} \hat{\varphi}'\right) = \breve{p}_! \breve{\pi}'_! \left(\mathbf{1}_{\breve{A}} \breve{\varphi}'\right). \tag{5.1.10}$$

The subassignment $\breve{\lambda}^{-1}(\breve{A}_2)$ corresponds to $\lambda^{-1}(A_2)$ with a 1-cell C adjoined and with a 0-cell removed, since balls can be partitioned into smaller balls but not into bigger balls. In our construction, since A and A_2 are adapted to φ_i', the pushforward is stable under removing a 0-cell from A (or from A_2). Relation (R4) ensures that the integral over C is equal to zero. Together with (5.1.10), this proves the lemma. (The idea behind the integral over C being equal to zero, is that if a big ball (namely a ball of volume $\mathbb{L}^{-j}$ with $j \leq 0$) gets partitioned into a combination of big balls and small balls (namely of volume $\mathbb{L}^{-j}$ for some $j > 0$) by a refining cell decomposition, the small balls fit together to fill the full line on the residue field level so that Relation (R4) can be applied (see the proof of 9.1.2 of [11] for more details about such refinements and how smaller balls fit together).) □

Example 5.1.2. If $g : \mathbb{Z}_p \to \mathbb{Z}_p$ is an analytic isometry and ψ_p is the additive character on $\mathbb{Q}_p$ sending $x \in \mathbb{Q}_p$ to $\exp(2\pi i \bar{x}/p)$, with $\bar{x}$ a representative of x mod $p\mathbb{Z}_p$ in $\mathbb{Z}[1/p]$ and exp the complex exponential, then

$$\int_{\mathbb{Z}_p} \psi_p(g(x))|dx| = 0$$

with $|dx|$ the Haar measure on $\mathbb{Q}_p$. Indeed, one can perform the change of variables $z = g(x)$ and then compute

$$\int_{\mathbb{Z}_p} \psi_p(g(x))|dx| = \int_{\mathbb{Z}_p} \psi_p(z)|dz| = \frac{1}{p}\sum_{a=0}^{p} \exp(2\pi i a/p) = 0.$$

This example explains intuitively why the integral over A_1 in the Lemma-Definition 5.1.1 gives a zero contribution.

Lemma-Definition 5.1.3. The map $\pi_!$ constructed in Lemma-Definition 5.1.1 factors through the natural group homomorphism $M_X \to I_S C(X)^{\exp}$. We write $\pi_!$ for the induced group homomorphism

$$\pi_! : I_S C(X)^{\exp} \to C(S)^{\exp}.$$

Proof. We have to check that $\pi_! : M_X \to C(S)^{\exp}$ factors through $K_0(\mathrm{RDef}_X^{\exp}) \times I_S C(X)$, and that it factors further through the tensor product $K_0(\mathrm{RDef}_X^{\exp}) \otimes_{K_0(\mathrm{RDef}_X)} I_S C(X) = I_S C(X)^{\exp}$.

That $\pi_!$ factors through relation (R1) is clear since its definition is independent of the choice of λ, cf. Lemma-Definition 5.1.1. Relation (R2) is clear by construction. Relation (R3) also follows since we can choose λ in such a way that $\bar{h}$ factors through the projection π', and then one can compare the original construction and definition with the ones where g is replaced by $g + h$. We prove relation (R4). Assume that a is of the form $[Y[0,1,0] \to X, \xi + p, g, \varphi]$ with $p : Y[0,1,0] \to h[0,1,0]$ the projection and that $Y[0,1,0] \to X$, g, and ξ factorize through the projection $Y[0,1,0] \to Y$. It follows by construction and from relation (R4) for $K_0(\mathrm{RDef}_S^{\exp})$ that $\pi_!(a)$ is zero.

By construction, the obtained map $K_0(\mathrm{RDef}_X^{\exp}) \times I_S C(X) \to C(S)^{\exp}$ is bilinear in the factors $K_0(\mathrm{RDef}_X^{\exp})$ and $I_S C(X)$ over the ring $K_0(\mathrm{RDef}_X)$; hence it factors through $K_0(\mathrm{RDef}_X^{\exp}) \otimes_{K_0(\mathrm{RDef}_X)} I_S C(X)$. □

5.2. *Change of variables in relative dimension* 1. One deduces from Theorem 2.2.2 the following change of variables statement in relative dimension 1:

PROPOSITION 5.2.1 (Change of variables). *Let X and Y be definable subassignments of dimension r of $S[1,0,0]$ for some S in* Def_k *and let $f : X \to Y$ be a definable isomorphism over S. Suppose that X and Y are equidimensional of relative dimension* 1 *relative to the projection to S. Then, φ is in $\mathrm{I}_S C^r(Y)^{\exp}$ if and only if $\mathbb{L}^{-\mathrm{ordjac} f} f^*(\varphi)$ is in $\mathrm{I}_S C^r(X)^{\exp}$ and if this is the case then*

$$\pi_{Y!}(\varphi) = \pi_{X!}(\mathbb{L}^{-\mathrm{ordjac} f} f^*(\varphi))$$

holds in $C(S)^{\exp}$ with $\pi_Y : Y \to S$ and $\pi_X : X \to S$ is the projection, $\pi_{Y!}, \pi_{X!}$ as in Lemma-Definition 5.1.3, *and* ordjac *as in* [11].

Proof. By linearity we may assume that φ is of the form

$$\varphi = \mathbf{e}^{\xi} E(g)[Z \to Y]\varphi_0, \tag{5.2.1}$$

with φ_0 in $\mathrm{I}_S C(Y)$, $Z \subset Y[0,n,0]$, and $\xi : Z \to h[0,1,0]$ and $g : Z \to h[1,0,0]$ definable morphisms. By pulling back along $Z \to Y$, we may assume that $Z = Y$. Choose λ as in the construction of $\pi_{Y!}(\varphi)$ in Lemma-Definition 5.1.3. By changing λ we may suppose that Theorem 2.2.2 is also applied to the function $p_1 \circ f$, with $p_1 : X \to h[1,0,0]$ the projection. But then $\lambda \circ f$ can be used to compute $\pi_{X!}(\mathbb{L}^{-\mathrm{ordjac} f} f^*(\varphi))$ as in Lemma-Definition 5.1.3 and is seen to be equal to $\pi_{Y!}(\varphi)$. □

6. Proof of Theorem 4.1.1

6.1. *Notation.* If $p : X \to Z$ is a morphism in RDef_Z and φ a Function in $C^i(Z)$ which is the class of ψ in $\mathscr{C}^{\leq i}(Z)$, the class of $p^*(\psi)$ in $C^i(X)$ depends only of φ, so we denote it by $p^*(\varphi)$. This construction extends by linearity to a morphism $p^* : C(Z) \to C(X)$.

6.2. *A special case.* Replacing $K_0(\mathrm{RDef}_Z^{\exp})$ by the subring $K_0(\mathrm{RDef}_Z^{\mathrm{e}})$, one defines subobjects $\mathscr{C}(Z)^{\mathrm{e}}$, $C(Z)^{\mathrm{e}}$ and $\mathrm{I}_S C(Z)^{\mathrm{e}}$ of $\mathscr{C}(Z)^{\exp}$, $C(Z)^{\exp}$ and $\mathrm{I}_S C(Z)^{\exp}$ as defined in Section 3.3; cf. Lemma 3.1.3.

Let us first prove that Theorem 4.1.1 restricted to this setting holds:

PROPOSITION 6.2.1. *Let S be in* Def_k. *There is a unique functor from the category* Def_S *to the category of abelian groups which sends Z to $\mathrm{I}_S C(Z)^{\mathrm{e}}$, assigns to every morphism $f : Z \to Y$ in* Def_S *a morphism $f_! : \mathrm{I}_S C(Z)^{\mathrm{e}} \to \mathrm{I}_S C(Y)^{\mathrm{e}}$*

and which satisfies axioms (A1) *to* (A4) *of Theorem* 4.1.1. *Moreover, these group morphisms $f_!$ coincide with the group morphisms constructed in Section* 3.7 *and in Section* 3.5 *in the corresponding cases.*

Proof. Let $f : Z \to Y$ be a morphism in Def_S. Consider φ in $\mathrm{I}_S C(Z)^{\mathrm{e}}$ of the form

$$\mathbf{e}^{\eta}[X \to Z]\varphi_0 \tag{6.2.1}$$

with $p : X \to Z$ in RDef_Z, $\eta : X \to h[0,1,0]$ and φ_0 in $\mathrm{I}_S C(Z)$. We have

$$\varphi = p_!(\mathbf{e}^{\eta} p^* \varphi_0). \tag{6.2.2}$$

Hence, if we denote by $\delta_{f,\eta} : X \to Y[0,1,0]$ the morphism

$$x \longmapsto ((f \circ p)(x), \eta(x)), \tag{6.2.3}$$

the axioms force

$$f_!(\varphi) := \pi_{Y!}(\mathbf{e}^{\xi} \delta_{f,\eta!}(p^* \varphi_0)), \tag{6.2.4}$$

with π_Y the projection $Y[0,1,0] \to Y$, ξ the canonical coordinate on the fibers of π_Y, and $\pi_{Y!}$ uniquely determined by (A4). Since $\mathrm{I}_S C(Z)^{\mathrm{e}}$ is generated by functions φ as above, this proves the uniqueness part of the statement. For existence, one uses (6.2.4) to define $p_!$ by additivity. Note that this definition is clearly compatible with the relations involved in the definition of $C(Z)^{\mathrm{e}}$. Note also that (A2) and (A1) are obvious and that (A4), that is, compatibility with Section 3.6, is easily checked. The projection formula (A3) follows easily from the projection formula in [11].

Now let us prove functoriality, namely, that $g_! \circ f_! = (g \circ f)_!$ for morphisms $f : Z \to Y$ and $g : Y \to W$ in Def_S. As above consider φ in $\mathrm{I}_S C(Z)^{\mathrm{e}}$ of the form $\mathbf{e}^{\eta}[X \to Z]\varphi_0 = p_!(\mathbf{e}^{\eta} p^* \varphi_0)$ with $p : X \to Z$ in RDef_Z, $\eta : X \to h[0,1,0]$ and φ_0 in $\mathrm{I}_S C(Z)$. We have

$$(g_! \circ f_!)(\varphi) = g_!(\pi_{Y!}((\mathbf{e}^{\xi} \delta_{f,\eta!}(p^* \varphi_0))) \tag{6.2.5}$$

and

$$\begin{aligned}(g \circ f)_!(\varphi) &= \pi_{W!}(\mathbf{e}^{\xi} \delta_{g \circ f,\eta!}(p^* \varphi_0)) \\ &= \pi_{W!}(\mathbf{e}^{\xi}((g \times \mathrm{id})_! \circ \delta_{f,\eta!})(p^* \varphi_0));\end{aligned} \tag{6.2.6}$$

hence it is enough to check that for every ψ in $\mathrm{I}_S C(Y[0,1,0])$

$$g_!(\pi_{Y!}(\mathbf{e}^{\xi} \psi)) = \pi_{W!}(\mathbf{e}^{\xi}((g \times \mathrm{id})_! \psi)). \tag{6.2.7}$$

We may assume ψ is of the form $[p : X \to Y[0,1,0]]\, \pi_Y^*(\psi_0)$ with ψ_0 in $\mathrm{I}_S C(Y)$ the class of a function in some $\mathscr{C}^i(Y)$, and with the above use of notation. Since

$\pi_{Y!}(\mathbf{e}^{\xi}\psi) = \mathbf{e}^{\xi\circ p}[X \to Y]\,\psi_0$, we have

$$g_!(\pi_{Y!}(\mathbf{e}^{\xi}\psi)) = \pi_{W!}(\mathbf{e}^{\xi}\delta_{g,\xi\circ p!}(p^*\psi)). \tag{6.2.8}$$

We now deduce (6.2.7) since

$$\begin{aligned}\delta_{g,\xi\circ p!}(p^*\psi) &= (g\times \mathrm{id})_!(p_!([X]p^*\pi_Y^*(\psi_0))) \\ &= (g\times \mathrm{id})_!([X \to Y[0,1,0]]\,\pi_Y^*(\psi_0)) = (g\times \mathrm{id})_!(\psi). \quad \square\end{aligned} \tag{6.2.9}$$

Remark 6.2.2. Note that in relative dimension 1, the morphisms $f_!$ in Proposition 6.2.1 coincide with those constructed in Section 5, by the construction in Section 5, the change of variables formula Proposition 5.2.1, and the construction in the proof of Proposition 6.2.1.

6.3. *Uniqueness.* The proof is similar to the one in Proposition 6.2.1. Let $f : Z \to Y$ be a morphism in Def_S. Consider φ in $\mathrm{I}_S C(Z)^{\exp}$ of the form

$$E(g)\mathbf{e}^{\eta}[X \to Z]\varphi_0 \tag{6.3.1}$$

with $p : X \to Z$ in RDef_Z, $g : X \to h[1,0,0]$ and $\eta : X \to h[0,1,0]$ definable morphisms, and φ_0 in $\mathrm{I}_S C(Z)$. We have

$$\varphi = p_!(E(g)\mathbf{e}^{\eta}p^*\varphi_0). \tag{6.3.2}$$

Thus, if we denote by $\delta_{f,g,\eta} : X \to Y[1,1,0]$ the morphism sending x to $((f\circ p)(x), g(x), \eta(x))$, the axioms force us to set

$$f_!(\varphi) := \pi_{Y!}(\pi_{Y[0,1,0]!}(E(x)\mathbf{e}^{\xi}\delta_{f,g,\eta!}(p^*\varphi_0))), \tag{6.3.3}$$

with $\pi_{Y[0,1,0]} : Y[1,1,0] \to Y[0,1,0]$ and $\pi_Y : Y[0,1,0] \to Y$ the projections, and x and ξ respectively the canonical coordinate on the fibers of $\pi_{Y[0,1,0]}$ and π_Y. The map $\pi_{Y!}$ is determined by (A4) and for the map $\pi_{Y[0,1,0]!}$ one is forced to use the construction of Section 5.

6.4. *Preliminaries.* Let Z be in Def_S. In Lemma-Definition 3.7.1 we defined push-forward morphisms

$$\pi_! : \mathrm{I}_S C(Z[0,r,m])^{\exp} \longrightarrow \mathrm{I}_S C(Z)^{\exp} \tag{6.4.1}$$

and in Section 5 we constructed a pushforward

$$\pi_! : \mathrm{I}_S C(Z[1,0,0])^{\exp} \longrightarrow \mathrm{I}_S C(Z)^{\exp}, \tag{6.4.2}$$

with π denoting the projection $Z[0,r,m] \to Z$ and $Z[1,0,0] \to Z$, respectively.

We may mix these two constructions as follows:

Lemma-Definition 6.4.1. Let Y be in Def_S. Let φ in $\mathrm{I}_S C(Y[1,n,r])^{\exp}$. Consider the following commutative diagram of projections

$$\begin{tikzcd}
 & Y[1,n,r] \arrow[dl, "\pi_1"'] \arrow[dd, "\pi"] \arrow[dr, "\pi'_1"] & \\
Y[1,0,0] \arrow[dr, "\pi_2"'] & & Y[0,n,r] \arrow[dl, "\pi'_2"] \\
 & Y &
\end{tikzcd}.$$

Now,

$$\pi_{2!}\pi_{1!}(\varphi) = \pi'_{2!}\pi'_{1!}(\varphi)$$

and we define $\pi_!(\varphi)$ to be the common value of $\pi_{2!}\pi_{1!}(\varphi)$ and $\pi'_{2!}\pi'_{1!}(\varphi)$.

Proof. The proof of Proposition-Definition 11.2.2 in [11] carries over to the present setting. $\square$

6.5. *A Fubini result for projections* $Y[2,0,0] \to Y$.

PROPOSITION 6.5.1. *Let Y be in* Def_S. *Consider an object $p : X \to Y[1,0,0]$ in* $\mathrm{RDef}_{Y[1,0,0]}$ *and let $g : X \to h[1,0,0]$ be a morphism in* Def_k. *Denote by π_Y the projection $Y[1,0,0] \to Y$, by z the canonical coordinate on the fibers of π_Y, and set $\gamma_g := (\pi_Y \circ p, g) : X \to Y[1,0,0]$. For every Function ψ in* $\mathrm{I}_S C(Y[1,0,0])^{\mathrm{e}}$,

$$\pi_{Y!}(E(g)[X \to Y]\psi) = \pi_{Y!}(E(z)\gamma_{g!}(p^*(\psi))).$$

Proof. By using a construction with a cell decomposition adapted to $X \to Y[1,0,0]$, by pulling back, and by Lemma-Definition 6.4.1, we may assume $X = Y[1,0,0]$. Similarly, by a cell decomposition construction using Theorem 2.2.2 and by Lemma-Definition 6.4.1, we can reduce to the case where g is either constant or injective. When g is constant the statement is clear and when g is injective it is a direct consequence of Proposition 5.2.1. $\square$

Let Y be in Def_S. For $i = 1, 2$, we denote by $\pi_i : Y[2,0,0] \to Y[1,0,0]$ the projection $(y, z_1, z_2) \mapsto (y, z_i)$ and by π_Y the projection $Y[1,0,0] \to Y$.

PROPOSITION 6.5.2. *Let ψ be in* $\mathrm{I}_S C(Y[2,0,0])^{\mathrm{e}}$. *Then*

$$\pi_{Y!}(\pi_{1!}(E(z_2)\psi)) = \pi_{Y!}(\pi_{2!}(E(z_2)\psi)).$$

Proof. We shall use bicells as defined in Section 7.4 of [11]. By Proposition 7.4.1 of [11], any definable subassignment Z of $Y[2,0,0]$ admits a bicell decomposition and, furthermore, for any φ in $\mathscr{C}(Z)$ there is a bicell decomposition of Z adapted to φ in the sense of loc. cit. More generally, for any φ in $\mathscr{C}(Z)^{\mathrm{e}}$, there is a bicell decomposition of Z adapted to φ. Indeed, this follows from the proof of loc. cit. and the fact that statement (2) in Theorem 2.2.1 still holds when replacing $\mathscr{C}(X)$ by $\mathscr{C}(X)^{\mathrm{e}}$. Hence, we may assume ψ is the characteristic function of a bicell Z in $Y[2,0,0]$. By the argument given at the beginning of the proof of Proposition 11.2.4 of [11], we may assume that the bicell Z is presented by the identity morphism.

Let us consider first the case when Z is a $(1,1)$-bicell. We start with the following special case:

LEMMA 6.5.3. *Let C be a definable subassignment of Y and consider definable morphisms $c : C \to h[1,0,0]$, α, $\beta : C \to h[0,0,1]$, and ξ, $\eta : C \to h_{\mathbb{G}_{m,k}}$. Consider the subassignment Z of $Y[2,0,0]$ defined by $(y \in C)$,*

$$\begin{aligned} \mathrm{ord}(z_1 - z_2) &= \alpha(y), \\ \overline{\mathrm{ac}}(z_1 - z_2) &= \xi(y), \\ \mathrm{ord}(z_2 - c(y)) &= \beta(y), \\ \overline{\mathrm{ac}}(z_2 - c(y)) &= \eta(y). \end{aligned}$$

Then $\pi_{Y!}(\pi_{1!}(E(z_2)\mathbf{1}_Z)) = \pi_{Y!}(\pi_{2!}(E(z_2)\mathbf{1}_Z))$.

Proof. As in the proof of Lemma 11.2.5 of [11], we may assume, after partitioning C, that one of the following conditions is satisfied everywhere on C:

(1) $\beta > \alpha$,

(2) $\beta < \alpha$,

(3) $\beta = \alpha$ and $\xi + \eta \neq 0$,

(4) $\beta = \alpha$ and $\xi + \eta = 0$.

If condition (1) or (3) holds, Z can be rewritten as a product of two 1-cells, cf. loc. cit., and the result is clear. If (2) is satisfied, then Z is also defined $(y \in C)$ by

$$\begin{aligned} \mathrm{ord}(z_1 - z_2) &= \alpha(y), \\ \overline{\mathrm{ac}}(z_1 - z_2) &= \xi(y), \\ \mathrm{ord}(z_2 - c(y)) &= \beta(y), \\ \overline{\mathrm{ac}}(z_2 - c(y)) &= \eta(y), \end{aligned}$$

and one computes

$$\pi_{Y!}(\pi_{1!}(E(z_2)\mathbf{1}_Z)) = E(c) I_{\alpha,\xi} I_{\beta,\eta} \tag{6.5.1}$$

and

$$\pi_{Y!}(\pi_{2!}(E(z_2)\mathbf{1}_Z)) = E(c)\mathbb{L}^{-\alpha-1} I_{\beta,\eta} \tag{6.5.2}$$

with $I_{\alpha,\xi}$, resp. $I_{\beta,\eta}$, the integral of $E(z)$ over the subassignment of $h[1,0,0]$ defined by $\mathrm{ord}(z) = \alpha$ and $\overline{\mathrm{ac}}(z) = \xi$, resp. $\mathrm{ord}(z) = \beta$ and $\overline{\mathrm{ac}}(z) = \eta$. To deduce the requested equality note that $I_{\beta,\eta}=0$ when $\beta<0$, and that when $\beta\geq 0$, then $\alpha>0$, hence $I_{\alpha,\xi}=\mathbb{L}^{-\alpha-1}$. The case of condition (4) also follows from an easy direct computation. $\square$

When Z is a $(1,1)$-bicell one proceeds similarly, as in the proof of Proposition 11.5.4 of [11]. More precisely assume Z is of the form $(y\in C)$

$$\begin{aligned}\operatorname{ord}(z_1-d(y,z_2))&=\alpha(y),\\ \overline{\mathrm{ac}}(z_1-d(y,z_2))&=\xi(y),\\ \operatorname{ord}(z_2-c(y))&=\beta(y),\\ \overline{\mathrm{ac}}(z_2-c(y))&=\eta(y).\end{aligned}$$

If d depends only on y, Z is a product of 1-cells and the statement is clear. Otherwise $d(y,z_2)$ can be supposed injective, as in [11], as a function of z_2 for fixed y. One deduces, with exactly the same proof as in loc. cit., the statement from Lemma 6.5.3 using the change of variables in relative dimension 1 (Proposition 5.2.1).

When Z is a $(1,0)$-bicell one proceeds exactly as in loc. cit., using change of variables in relative dimension 1. In the remaining cases of a $(0,1)$ or a $(0,0)$-bicell, Z is a product of cells, and the statement is clear. $\square$

We may now prove the following version of the Fubini theorem:

PROPOSITION 6.5.4. *Let φ be in $\mathrm{I}_S\mathcal{C}(Y[2,0,0])^{\exp}$. Then*

$$\pi_{Y!}(\pi_{1!}(\varphi))=\pi_{Y!}(\pi_{2!}(\varphi)).$$

Proof. By using a construction as in the proof of Lemma-Definition 6.4.1, but now with a bicell decomposition, we may assume $\varphi=E(g)\psi$ with $g:Y[2,0,0]\to h[1,0,0]$ in RDef_S and ψ in $\mathrm{I}_S\mathcal{C}(Y[2,0,0])^{\mathrm{e}}$. By Proposition 6.5.1 we have

$$\begin{aligned}\pi_{Y!}(\pi_{1_!}(\varphi))&=\pi_{Y!}(\pi_{1_!}(E(g)\psi))\\ &=\pi_{Y!}(\pi_{1_!}(E(z_2)\gamma_{1!}(\psi))),\end{aligned}\tag{6.5.3}$$

with $\gamma_1:=(\pi_1,g):X\to Y[2,0,0]$. Hence, by Proposition 6.5.2,

$$\begin{aligned}\pi_{Y!}(\pi_{1_!}(\varphi))&=\pi_{Y!}(\pi_{2_!}(E(z_2)\gamma_{1!}(\psi)))\\ &=\pi_{Y!}(E(z)(\pi_{2_!}\circ\gamma_{1!})(\psi))).\end{aligned}\tag{6.5.4}$$

The result follows, since by Proposition 6.2.1,

$$(\pi_{2_!}\circ\gamma_{1!})(\psi)=(\pi_2\circ\gamma_1)_!(\psi)\tag{6.5.5}$$

where $\pi_2\circ\gamma_1:X\to Y[1,0,0]$ is the morphism (π,g), with $\pi:Y[2,0,0]\to Y$ the projection, which is independent of the order of the two variables in $h[2,0,0]$, again, by Proposition 6.2.1. $\square$

6.6. *Projections.* Let us first consider the projection $p : Y[m,0,0] \to Y$. Let φ be in $\mathrm{I}_S C(Y[m,0,0])^{\mathrm{exp}}$. If $m > 1$, we set, by induction on m, $p_!(\varphi) = \pi_{Y!}(\pi_{1!}(\varphi))$, where $\pi_1 : Y[m,0,0] \to Y[m-1,0,0]$ is the projection on the first $m-1$ coordinates. By Proposition 6.5.4, this definition is invariant under permutation of coordinates.

In the general case of a projection

$$p : Y[m,n,r] \longrightarrow Y,$$

for any φ in $\mathrm{I}_S C(Y[m,n,r])^{\mathrm{exp}}$, we set $p_!(\varphi) := (p_{2!} \circ p_{1!})(\varphi)$ for

$$Y[m,n,r] \xrightarrow{p_1} Y[m,0,0] \xrightarrow{p_2} Y.$$

It follows from Lemma-Definition 3.7.1, Lemma-Definition 6.4.1 and Proposition 6.5.4 that, for any decomposition of p into projections

$$Y[m,n,r] \xrightarrow{p_1} Y[m',n',r'] \xrightarrow{p_2} Y,$$

with $m' \leq m$, $n' \leq n$ and $r' \leq r$, we have $p_!(\varphi) = (p_{2!} \circ p_{1!})(\varphi)$ and the definition of $p_!(\varphi)$ is invariant under permutation of coordinates.

Now if Z is a definable subassignment of some $h[m,n,r]$ and φ belongs to $\mathrm{I}_S C(Y \times Z)^{\mathrm{exp}}$ we denote by $\tilde{\varphi}$ the Function in $\mathrm{I}_S C(Y[m,n,r])^{\mathrm{exp}}$ which is obtained from φ by extension by zero outside Z. If $\varphi = \sum_{1 \leq i \leq j} \varphi_i [X_i] E(g_i)$, with φ_i in $\mathrm{I}_S C(Y \times Z)^{\mathrm{e}}$, X_i in $\mathrm{RDef}_{Y \times Z}$ and $g_i : X_i \to h[1,0,0]$, we have, with a slight abuse of notations, $\tilde{\varphi} = \sum_{1 \leq i \leq j} j_!(\varphi_i) j_!([X_i]) E(g_i)$, with $j : Z \to h[m,n,r]$ the inclusion. We write the projection $p : Y \times Z \to Y$ as $\pi \circ j$, with π the projection $Y[m,n,r] \to Y$, and we set

$$p_!(\varphi) := \pi_!(\tilde{\varphi}) \tag{6.6.1}$$

for φ in $\mathrm{I}_S C(Y \times Z)^{\mathrm{exp}}$.

The projection formula (A3) trivially holds for $p_!$ and also the following form of Fubini's theorem.

PROPOSITION 6.6.1. *Consider a diagram of projections*

$$Y \times Z \times W \xrightarrow{p_1} Y \times Z \xrightarrow{p_2} Y,$$

with Z and W in Def_k. *Then for any φ in* $\mathrm{I}_S C(Y \times Z \times W)^{\mathrm{exp}}$,

$$(p_2 \circ p_1)_!(\varphi) = (p_{2!} \circ p_{1!})(\varphi). \qquad \square$$

6.7. *Definable injections.* Let $i : X \to Y$ be a morphism in Def_S which is a definable injection and let $g : X \to h[1,0,0]$ and $\eta : X \to h[0,1,0]$ be morphisms. Now if $\varphi = \varphi_0 E(g) \mathbf{e}^{\eta}[W \to X]$ lies in $\mathrm{I}_S C(X)^{\mathrm{exp}}$ with φ_0 in $\mathrm{I}_S C(X)$, W in RDef_X, and if we write W' for the unique element of RDef_Y such that for each

$x \in X$ the fiber W_x equals $W'_{i(x)}$ and W'_y empty for y outside $i(X)$, then

$$i_!(\varphi) := i_!(\varphi_0) E(g_{W'}) \mathbf{e}^{\eta_{W'}} [W' \to Y], \tag{6.7.1}$$

where $\eta_{W'}$ and $g_{W'}$ are η and g seen on W'.

This definition extends uniquely by linearity to give a morphism

$$i_! : \mathrm{I}_S C(X)^{\exp} \to \mathrm{I}_S C(Y)^{\exp}.$$

Also it is quite clear that if $j : Y \to Z$ is another definable injection in Def_S, $(j \circ i)_! = j_! \circ i_!$.

6.8. *General case.* To define $f_!$ for $f : X \to Y$ a general morphism in Def_S, we proceed as in [11]. We decompose f as a composition $f = \pi_f \circ i_f$ with i_f the definable injection $X \to X \times Y$ given by $x \mapsto (x, f(x))$ and $\pi_f : X \times Y \to Y$ the canonical projection, and we set $f_! = \pi_{f!} \circ i_{f!}$.

It is quite clear that when f is an injection the new definition coincides with the previous one. Also, when f is a projection $Y \times Z \to Y$, the new definition coincides with the previous one. Indeed, the analogue of Lemma 11.5.2 of [11] holds for similar reasons in the present setting, and so the proof in Lemma 11.6.1 of [11] extends directly.

6.9. *End of proof.* We still have to check that if $f : X \to Y$ and $g : Y \to Z$ are morphisms in Def_S, then $g_! \circ f_! = (g \circ f)_!$. This is proved in a formal way exactly as in Proposition 11.7.1 of [11], since the analogue of Lemma 11.7.2 in [11] holds in the present setting. Axioms (A1)–(A5) follow directly by construction.

7. **Fourier transform**

Let $p : X \to \Lambda$ be a morphism in Def_k, with all fibers of relative dimension d. We shall denote by $\mathcal{I}_\Lambda(X)^{\exp}$ or $\mathcal{I}_p(X)^{\exp}$ the $\mathscr{C}(\Lambda)^{\exp}$-module of functions φ in $\mathscr{C}(X)^{\exp}$ whose class $[\varphi]$ in $C^d(X \to \Lambda)^{\exp}$ lies in $\mathrm{I}_\Lambda C(Z \to \Lambda)^{\exp}$. We shall also write $\mu_\Lambda(\varphi)$ or $\mu_p(\varphi)$ to denote the function $\mu_\Lambda([\varphi])$ in $\mathscr{C}(\Lambda)^{\exp}$.

7.1. *Fourier transform over the residue field.* Fix Λ in Def_k and an integer $d \geq 0$. We consider the subassignment $\Lambda[0, 2d, 0]$ with first d residue field coordinates $x = (x_1, \dots, x_d)$ and last d residue coordinates $y = (y_1, \dots, y_d)$ and denote by $p_1 : \Lambda[0, 2d, 0] \to \Lambda[0, d, 0]$ and $p_2 : \Lambda[0, 2d, 0] \to \Lambda[0, d, 0]$ the projection onto the x-variables and y-variables, respectively. In this section we shall write $\mathbf{e}(\xi)$ for $\mathbf{e}^\xi$. We view $\sum_{1 \leq i \leq d} x_i y_i$ as a morphism $\Lambda[0, 2d, 0] \to h[0, 1, 0]$ and we consider the function

$$\mathbf{e}(xy) := \mathbf{e}\Big(\sum_{1 \leq i \leq d} x_i y_i \Big)$$

in $\mathscr{C}(\Lambda[0, 2d, 0])^{\exp}$.

We define the Fourier transform

$$\mathfrak{f} : \mathscr{C}(\Lambda[0,d,0])^{\exp} \longrightarrow \mathscr{C}(\Lambda[0,d,0])^{\exp} \tag{7.1.1}$$

by

$$\mathfrak{f}(\varphi) := p_{1!\Lambda[0,d,0]}([\mathbf{e}(xy)\, p_2^*(\varphi)]), \tag{7.1.2}$$

for φ in $\mathscr{C}(\Lambda[0,d,0])^{\exp}$. The morphism $\mathfrak{f}$ is $\mathscr{C}(\Lambda)^{\exp}$-linear.

For φ in $\mathscr{C}(\Lambda[0,d,0])^{\exp}$ we write $\check{\varphi}$ for $\iota^*(\varphi)$, with $\iota : \Lambda[0,d,0] \to \Lambda[0,d,0]$ the Λ-morphism sending x to $-x$.

THEOREM 7.1.1. *Let φ be in $\mathscr{C}(\Lambda[0,d,0])^{\exp}$. Then*

$$(\mathfrak{f} \circ \mathfrak{f})(\varphi) = \mathbb{L}^d\, \check{\varphi}.$$

Proof. The proof is essentially the same as the standard one for finite fields. More precisely, we work on $\Lambda[0,3d,0]$ with coordinates

$$(x_1, \dots, x_d, y_1, \dots, y_d, z_1, \dots, z_d).$$

We shall denote by $x : \Lambda[0,3d,0] \to \Lambda[0,d,0]$, $(x,y) : \Lambda[0,3d,0] \to \Lambda[0,2d,0]$ the projections onto the corresponding components, etc. If f is a function on $\Lambda[0,d,0]$ we shall write $f(x)$ instead of $x^* f$, etc. By induction and Fubini's theorem (cf. Remark 4.3.2) we may assume $d = 1$.

Let φ be in $\mathscr{C}(\Lambda[0,d,0])^{\exp}$. Then

$$\begin{aligned}(\mathfrak{f} \circ \mathfrak{f})(\varphi) &= \mu_x[\mathbf{e}(y(x+z))\varphi(z)] \\ &= \mu_x[\mathbf{e}(yu)\varphi(u-x)],\end{aligned} \tag{7.1.3}$$

after performing the change of variables $(x,y,z) \mapsto (x,y,u = x+z)$. Since

$$\mathbb{L}\, \check{\varphi}(x) = \mu_x[\mathbf{1}_{u=0}\mathbf{1}_{y=y}\varphi(-x)], \tag{7.1.4}$$

we have

$$((\mathfrak{f} \circ \mathfrak{f})(\varphi) - \mathbb{L}\, \check{\varphi})(x) = \mu_x[\mathbf{1}_{u\neq 0}\mathbf{e}(yu)\varphi(u-x)], \tag{7.1.5}$$

which, after performing the change of variables $(x,y,u) \mapsto (x, w = yu, u)$, may be rewritten as

$$((\mathfrak{f} \circ \mathfrak{f})(\varphi) - \mathbb{L}\, \check{\varphi})(x) = \mu_x[\mathbf{1}_{u\neq 0}\mathbf{e}(w)\varphi(u-x)], \tag{7.1.6}$$

whose right-hand side is zero by the Fubini theorem and relation (R4). □

7.2. *Fourier transform over the valued field.* Fix Λ in Def_k and an integer $d \geq 0$. We consider the subassignment $\Lambda[2d,0,0]$ with first d valued field coordinates $x = (x_1, \dots, x_d)$ and last d valued field coordinates $y = (y_1, \dots, y_d)$ and denote by $p_1 : \Lambda[2d,0,0] \to \Lambda[d,0,0]$ and $p_2 : \Lambda[2d,0,0] \to \Lambda[d,0,0]$ the projection onto the x-variables and y-variables, respectively. We view $\sum_{1 \leq i \leq d} x_i y_i$

as a morphism $\Lambda[0,2d,0] \to h[1,0,0]$ and we consider the function $E(xy) := E(\sum_{1\leq i\leq d} x_i y_i)$ in $\mathscr{C}(\Lambda[2d,0,0])^{\exp}$.

LEMMA 7.2.1. *Let φ be in $\mathscr{I}_\Lambda(\Lambda[d,0,0])^{\exp}$. The class $[E(xy)p_2^*(\varphi)]$ of $E(xy)p_2^*(\varphi)$ in $C^d(p_1 : \Lambda[2d,0,0] \to \Lambda[d,0,0])^{\exp}$ is integrable rel. p_1.*

Proof. Indeed, it follows by construction that, if ψ is a function in $\mathscr{C}(\Lambda[d,0,0])$ whose class $[\psi]$ in $C^d(\Lambda[d,0,0] \to \Lambda)$ is Λ-integrable, the class of the pull-back $p_2^*(\psi)$ in $C^d(p_1 : \Lambda[2d,0,0] \to \Lambda[d,0,0])$ is integrable rel. p_1. The statement follows. □

Thanks to Lemma 7.2.1, one may define the Fourier transform

$$\mathfrak{F} : \mathscr{I}_\Lambda(\Lambda[d,0,0])^{\exp} \longrightarrow \mathscr{C}(\Lambda[d,0,0])^{\exp} \tag{7.2.1}$$

by

$$\mathfrak{F}(\varphi) := \mu_{p_1}(E(xy)p_2^*(\varphi)), \tag{7.2.2}$$

for φ in $\mathscr{I}_\Lambda(\Lambda[d,0,0])^{\exp}$. The morphism $\mathfrak{F}$ is $\mathscr{C}(\Lambda)^{\exp}$-linear.

7.3. *Some examples.* Let us compute some simple examples. Consider definable functions $\alpha : \Lambda \to \mathbb{Z}$ and $\xi = (\xi_1, \dots, \xi_d) : \Lambda \to h[0,d,0]$ with ξ_i nowhere zero and set

$$Z_\alpha := \{(\lambda, x = (x_1, \dots, x_d)) \in \Lambda[d,0,0] \mid \operatorname{ord}(x_i) \geq \alpha(\lambda)\},$$

$$W_\alpha := \{(\lambda, x = (x_1, \dots, x_d)) \in \Lambda[d,0,0] \mid \operatorname{ord}(x_i) = \alpha(\lambda)\},$$

$$W_{\alpha,\xi} := \{(\lambda, x = (x_1, \dots, x_d)) \in \Lambda[d,0,0] \mid \operatorname{ord}(x_i) = \alpha(\lambda), \overline{\operatorname{ac}}(x_i) = \xi_i(\lambda)\},$$

$$\varphi_\alpha := \mathbf{1}_{Z_\alpha}, \quad \psi_\alpha := \mathbf{1}_{W_\alpha}, \quad \text{and} \quad \psi_{\alpha,\xi} := \mathbf{1}_{W_{\alpha,\xi}}.$$

PROPOSITION 7.3.1. *The following formulas hold*:

(1) $\mathfrak{F}(\varphi_\alpha) = \mathbb{L}^{-d\alpha}\varphi_{-\alpha+1}$.

(2) $\mathfrak{F}(\psi_\alpha) = \mathbb{L}^{-d\alpha}\varphi_{-\alpha+1} - \mathbb{L}^{-d\alpha-d}\varphi_{-\alpha}$.

(3) $\mathfrak{F}(\psi_{\alpha,\xi}) = \mathbb{L}^{-d\alpha-d}(\varphi_{-\alpha+1} + \mathbf{e}(i)\psi_{-\alpha})$, *with i the morphism $\Lambda[d,0,0] \to \Lambda[d,1,0]$ given by $(\lambda, x) \mapsto (\lambda, x, \sum_i \xi_i(\lambda)\overline{\operatorname{ac}}\, x_i)$.*

Proof. By induction on d, we may assume $d = 1$. Let us start by proving (3). It is enough to check that the restriction of $\mathfrak{F}(\psi_{\alpha,\xi})$ to the subassignment defined by $\operatorname{ord} x = \beta$ and $\overline{\operatorname{ac}}\, x = \eta$ is equal to 0 if $\alpha + \beta < 0$, to $\mathbb{L}^{-\alpha-1}$ if $\alpha + \beta > 0$, and is equal to $\mathbf{e}(\xi\eta)\mathbb{L}^{-\alpha-1}$ if $\alpha + \beta = 0$. The case $\alpha + \beta < 0$ follows from (A5) of Theorem 4.1.1. The cases $\alpha + \beta > 0$ and $\alpha + \beta = 0$ follow from relation (R3) and the construction of the direct image formalism in [11]. Cases (1) and (2) are easier. The reader may also choose to prove first the case of $\alpha = 0$ and deduce the case of general α from it. □

We compute some more examples.

LEMMA 7.3.2. *Assume* $d = 1$ *and let* $\gamma : \Lambda \to \mathbb{Z}$ *and* $\xi : \Lambda \to h[0,1,0]$ *be definable functions. Then*

(1) *If* $\gamma > 0$ *on* Λ, $\mu_\Lambda(\psi_{\gamma,\xi} E(x)) = \mathbb{L}^{-\gamma-1}$, $\mu_\Lambda(\psi_\gamma E(x)) = (\mathbb{L}-1)\mathbb{L}^{-\gamma-1}$ *and* $\mu_\Lambda(\varphi_\gamma E(x)) = \mathbb{L}^{-\gamma}$.

(2) *If* $\gamma < 0$ *on* Λ, $\mu_\Lambda(\psi_{\gamma,\xi} E(x)) = \mu_\Lambda(\psi_\gamma E(x)) = 0$.

(3) *If* $\gamma = 0$ *on* Λ, $\mu_\Lambda(\psi_{\gamma,\xi} E(x)) = \mathbf{e}(\xi)\mathbb{L}^{-1}$ *and* $\mu_\Lambda(\psi_\gamma E(x)) = -\mathbb{L}^{-1}$.

Proof. Statement (1) and the first part of (3) are obvious from relation (R3) and the construction in [11], and (2) follows from (A5) of Theorem 4.1.1.

The last part of (3) follows from the first part using cell decomposition, since, by relation (R4), $\mu_\Lambda(i_! i^* \mathbf{e}(\xi)) = -1$, with ξ the residue field variable on $\Lambda[0,1,0]$ and i the inclusion of the subassignment defined by $\xi \neq 0$ in $\Lambda[0,1,0]$. □

If follows readily from Proposition 7.3.1 that

$$\mathfrak{F} \circ \mathfrak{F}(\varphi_\alpha) = \mathbb{L}^{-d}\, \varphi_\alpha \tag{7.3.1}$$

and

$$\mathfrak{F} \circ \mathfrak{F}(\psi_\alpha) = \mathbb{L}^{-d}\, \psi_\alpha. \tag{7.3.2}$$

The corresponding statement for $\psi_{\alpha,\xi}$ will follow from the general Fourier inversion for Schwartz-Bruhat functions to be proved in Theorem 7.5.1. Though not at all necessary, let us provide a direct proof of that fact:

PROPOSITION 7.3.3. *The following holds*

$$\mathfrak{F} \circ \mathfrak{F}(\psi_{\alpha,\xi}) = \mathbb{L}^{-d}\, \psi_{\alpha,-\xi}.$$

Proof. We may assume $d = 1$. By (1) and (3) of Proposition 7.3.1, it is enough to compute $\mathfrak{F}(\mathbf{e}(i)\psi_{-\alpha})$, with i the morphism $\Lambda[1,0,0] \to \Lambda[1,1,0]$ given by $(\lambda, x) \mapsto (\lambda, x, \xi(\lambda)\overline{\mathrm{ac}}\, x)$. But this can be done similarly as in (3) of Proposition 7.3.1 and the last part of (3) in Lemma 7.3.2. □

7.4. *Convolution.* We denote by $x + y$ the morphism $\Lambda[2d,0,0] \to \Lambda[d,0,0]$ given by $(x_1, \ldots, x_d, y_1, \ldots, y_d) \mapsto (x_1 + y_1, \ldots, x_d + y_d)$. We shall also work on $\Lambda[3d,0,0]$ with coordinates $(x_1, \ldots, x_d, y_1, \ldots, y_d, z_1, \ldots, z_d)$. We shall denote by $x : \Lambda[3d,0,0] \to \Lambda[d,0,0]$, $(x,y) : \Lambda[3d,0,0] \to \Lambda[2d,0,0]$ the projections onto the corresponding components, etc. If f is a function on $\Lambda[d,0,0]$ we shall write $f(x)$ instead of $x^* f$, etc.

Proposition-Definition 7.4.1. Let f and g be two functions in

$$\mathscr{I}_\Lambda(\Lambda[d,0,0])^{\exp}.$$

The function $p_1^*(f)p_2^*(g)$ lies in $\mathcal{I}_{x+y}(\Lambda[2d,0,0])^{\exp}$ and the function

$$f * g := \mu_{x+y}(p_1^*(f)p_2^*(g))$$

lies in $\mathcal{I}_\Lambda(\Lambda[d,0,0])^{\exp}$, where p_1 and p_2 are as in Section 7.2.

Proof. It follows directly from [11, Th. 14.1.1] that, if φ and ψ are functions in $\mathscr{C}(\Lambda[d,0,0])$ whose classes in $C^d(\Lambda[d,0,0] \to \Lambda)$ are in $\mathrm{I}_\Lambda C(\Lambda[d,0,0] \to \Lambda)$, then the class of $p_1^*(\varphi)p_2^*(\psi)$ in $C^d(\Lambda[2d,0,0] \to \Lambda)$ lies in $\mathrm{I}_\Lambda C(\Lambda[2d,0,0] \to \Lambda)$. One deduces that the function $p_1^*(f)p_2^*(g)$ lies in $\mathcal{I}_\Lambda(\Lambda[2d,0,0])^{\exp}$, hence also in $\mathcal{I}_{x+y}(\Lambda[2d,0,0])^{\exp}$. Since, by the Fubini theorem,

$$\mu_\Lambda(p_1^*(f)p_2^*(g)) = \mu_\Lambda(\mu_{x+y}(p_1^*(f)p_2^*(g))), \tag{7.4.1}$$

it follows that $f * g$ lies in $\mathcal{I}_\Lambda(\Lambda[d,0,0])^{\exp}$. □

PROPOSITION 7.4.2. *The convolution product $(f,g) \mapsto f * g$ is $\mathscr{C}(\Lambda)^{\exp}$-linear and it endows $\mathcal{I}_\Lambda(\Lambda[d,0,0])^{\exp}$ with an associative and commutative law.*

Proof. $\mathscr{C}(\Lambda)^{\exp}$-linearity and commutativity being clear, we check associativity. This follows from the fact that, if f, g and h are functions in $\mathcal{I}_\Lambda(\Lambda[d,0,0])^{\exp}$,

$$(f * g) * h = \mu_{x+y+z}(p_1^*(f)p_2^*(g)p_3^*(h)). \tag{7.4.2}$$

by the Fubini theorem. □

PROPOSITION 7.4.3. *Let f and g be two functions in $\mathcal{I}_\Lambda(\Lambda[d,0,0])^{\exp}$. Then*

$$\mathfrak{F}(f * g) = \mathfrak{F}(f)\,\mathfrak{F}(g).$$

Proof. The proof is just the same as the usual one. Let us consider the function $E(x(y+z))f(y)g(z)$ on $\Lambda[3d,0,0]$. It is integrable rel. x, and by the Fubini theorem we have

$$\begin{aligned} \mu_x(E(x(y+z))f(y)g(z)) &= \mu_x((E(xy)f(y))(E(xz)g(z))) \\ &= \mathfrak{F}(f)\mathfrak{F}(g). \end{aligned} \tag{7.4.3}$$

On the other hand, by the change of variables formula, $(x,y,z) \mapsto (x, u = y+z, z)$, $\mu_x(E(x(y+z))f(y)g(z))$ may be expressed as

$$\mu_x(E(xu)\mu_{x,u}(f(u-z)g(z))) = \mathfrak{F}(f * g), \tag{7.4.4}$$

which ends the proof. □

For φ in $\mathscr{C}(\Lambda[d,0,0])^{\exp}$ we write $\check{\varphi}$ for $\iota^*(\varphi)$, with $\iota : \Lambda[d,0,0] \to \Lambda[d,0,0]$ the Λ-morphism sending x to $-x$.

We have the following partial Fourier inversion:

PROPOSITION 7.4.4. *Let φ be a function in $\mathscr{I}_\Lambda(\Lambda[d,0,0])^{\exp}$. For every α in $\mathbb{Z}$, $\varphi_\alpha\mathfrak{F}(\varphi)$ lies in $\mathscr{I}_\Lambda(\Lambda[d,0,0])^{\exp}$ and*

$$\mathfrak{F}(\varphi_\alpha\mathfrak{F}(\varphi)) = \mathbb{L}^{-\alpha d}\,\check{\varphi} * \varphi_{-\alpha+1}.$$

Proof. We shall work on $\Lambda[3d,0,0]$, keeping notation and conventions from the proof of Proposition 7.4.3. The integrability of $\varphi_\alpha\mathfrak{F}(\varphi)$ follows from the fact that the function $E(yz)\varphi_\alpha(y)\varphi(z)$ on $\Lambda[2d,0,0]$ lies in $\mathscr{I}_\Lambda(\Lambda[2d,0,0])^{\exp}$. We consider the function $E(y(x+z))\varphi_\alpha(y)\varphi(z)$ on $\Lambda[3d,0,0]$. It is integrable rel. x, and by the Fubini theorem we have

$$\begin{aligned}(7.4.5)\quad \mu_x(E(y(x+z))\varphi_\alpha(y)\varphi(z)) &= \mu_x(\mu_{(x,y)}(E(y(x+z))\varphi_\alpha(y)\varphi(z)))\\ &= \mu_x(\varphi_\alpha(y)E(xy)\mu_{(x,y)}(E(yz)\varphi(z)))\\ &= \mu_x(E(xy)\varphi_\alpha(y)\mathfrak{F}(\varphi)(y))\\ &= \mathfrak{F}(\varphi_\alpha\mathfrak{F}(\varphi)).\end{aligned}$$

On the other hand, performing the change of variables $(x,y,z)\mapsto(u=x+z,y,z)$, we have, accordingly,

$$\begin{aligned}(7.4.6)\quad \mu_x(E(y(x+z))\varphi_\alpha(y)\varphi(z)) &= \mu_x(\mu_{(x,z)}(E(y(x+z))\varphi_\alpha(y)\varphi(z)))\\ &= \mu_{u-z}(\mu_{(u,z)}(E(uy)\varphi_\alpha(y)\varphi(z)))\\ &= \mu_{u-z}(\varphi(z)\mu_{(u,z)}(E(uy)\varphi_\alpha(y)))\\ &= \mu_{u-z}(\varphi(z)\mathfrak{F}(\varphi_\alpha)(u))\\ &= \mu_{u+z}(\check{\varphi}(z)\mathfrak{F}(\varphi_\alpha)(u))\\ &= \check{\varphi} * \mathfrak{F}(\varphi_\alpha),\end{aligned}$$

which concludes the proof. □

7.5. *Schwartz-Bruhat functions.* We define the space $\mathscr{S}_\Lambda(\Lambda[d,0,0])^{\exp}$ of Schwartz-Bruhat functions over Λ as the $\mathscr{C}(\Lambda)^{\exp}$-submodule of $\mathscr{I}_\Lambda(\Lambda[d,0,0])^{\exp}$ consisting of functions f such that

$$(7.5.1)\qquad f\cdot\varphi_\alpha = f \quad \text{for } \alpha \ll 0$$

and

$$(7.5.2)\qquad f * \varphi_\alpha = \mathbb{L}^{-\alpha d} f \quad \text{for } \alpha \gg 0.$$

Condition (7.5.1) stands for "compactly supported" and condition (7.5.2) for "locally constant". Here the quantifier $\alpha \ll 0$ in (7.5.1), resp. $\alpha \gg 0$ in (7.5.2), means there exists a definable function $\alpha_0 : \Lambda \to \mathbb{Z}$ such that (7.5.1), resp. (7.5.2), holds for every definable function $\alpha : \Lambda \to \mathbb{Z}$ such that $\alpha \le \alpha_0$, resp. $\alpha \ge \alpha_0$.

THEOREM 7.5.1. *Fourier transform induces an isomorphism*

$$\mathfrak{F} : \mathcal{S}_\Lambda(\Lambda[d,0,0])^{\mathrm{exp}} \simeq \mathcal{S}_\Lambda(\Lambda[d,0,0])^{\mathrm{exp}} \tag{7.5.3}$$

and, for every φ in $\mathcal{S}_\Lambda(\Lambda[d,0,0])^{\mathrm{exp}}$,

$$(\mathfrak{F}\circ\mathfrak{F})(\varphi) = \mathbb{L}^{-d}\,\check{\varphi}. \tag{7.5.4}$$

Proof. Let φ be in $\mathcal{S}_\Lambda(\Lambda[d,0,0])^{\mathrm{exp}}$. Note that, for $\alpha \ll 0$,

$$\mathfrak{F}(\varphi)\varphi_\alpha = \mathfrak{F}(\varphi). \tag{7.5.5}$$

Indeed, by Proposition 7.4.3 and Proposition 7.3.1 (1), for $\alpha \gg 0$,

$$\mathfrak{F}(\varphi) = \mathfrak{F}(\varphi * \varphi_\alpha)\,\mathbb{L}^{\alpha d} = \mathfrak{F}(\varphi)\varphi_{-\alpha+1}. \tag{7.5.6}$$

It follows from Proposition 7.4.4 that $\mathfrak{F}(\varphi)$ lies in $\mathcal{I}_\Lambda(\Lambda[d,0,0])^{\mathrm{exp}}$ and that

$$\mathfrak{F}(\mathfrak{F}(\varphi)) = \mathbb{L}^{-\alpha d}\,\check{\varphi} * \varphi_{-\alpha+1}, \tag{7.5.7}$$

for $\alpha \ll 0$. Since φ lies in $\mathcal{S}_\Lambda(\Lambda[d,0,0])^{\mathrm{exp}}$, $\check{\varphi}$ also,

$$\check{\varphi} * \varphi_{-\alpha+1} = \mathbb{L}^{(\alpha-1)d}\,\check{\varphi} \tag{7.5.8}$$

for $\alpha \ll 0$, and we deduce (7.5.4). So we are left to prove that $\mathfrak{F}(\varphi)$ lies in $\mathcal{S}_\Lambda(\Lambda[d,0,0])^{\mathrm{exp}}$. It is enough to check that, for $\alpha \ll 0$,

$$\mathfrak{F}(\varphi)\,\mathbb{L}^{-(-\alpha+1)d} = \mathfrak{F}(\varphi) * \varphi_{-\alpha+1}, \tag{7.5.9}$$

which follows from the relations

$$\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\varphi))) = \mathfrak{F}(\varphi_\alpha\mathfrak{F}(\mathfrak{F}(\varphi))) = \mathbb{L}^{-\alpha d}\,\check{\mathfrak{F}(\varphi)} * \varphi_{-\alpha+1}, \tag{7.5.10}$$

for $\alpha \ll 0$ by Proposition 7.4.4, and, by (7.5.4),

$$\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\varphi))) = \mathbb{L}^{-d}\,\check{\mathfrak{F}(\varphi)}, \tag{7.5.11}$$

which concludes the proof. □

Now we can prove Fourier inversion for integrable functions with integrable Fourier transform.

THEOREM 7.5.2. *Let φ be in $\mathcal{I}_\Lambda(\Lambda[d,0,0])^{\mathrm{exp}}$. Assume $\mathfrak{F}(\varphi)$ belongs also to $\mathcal{I}_\Lambda(\Lambda[d,0,0])^{\mathrm{exp}}$. Then the functions $(\mathfrak{F}\circ\mathfrak{F})(\varphi)$ and $\mathbb{L}^{-d}\,\check{\varphi}$ have the same class in $C^d(\Lambda[d,0,0]\to\Lambda)^{\mathrm{exp}}$.*

Proof. By induction on d we may assume $d=1$. Take φ in $\mathcal{I}_\Lambda(\Lambda[1,0,0])^{\mathrm{exp}}$. By Lemma 7.5.3 and additivity, we may assume a 1-cell $\lambda : Z \to Z_C \subset \Lambda[1,s,r]$ exists, such that, denoting by i the inclusion $Z \to \Lambda[1,0,0]$ and by j the inclusion $Z_C \to \Lambda[1,s,r]$, $\varphi = i_!(i^*(\varphi))$ and $\psi := j_!\lambda_!(i^*(\varphi))$ lies in $\mathcal{S}_{\Lambda[0,r,s]}(\Lambda[1,r,s])^{\mathrm{exp}}$. Denoting by π the projection $\Lambda[1,r,s]\to\Lambda$, $\varphi = \pi_!(\psi)$, we see that the result follows formally from Lemma 7.5.4 and Theorem 7.5.1. □

LEMMA 7.5.3. *For every φ in $\mathscr{I}_{\Lambda}(\Lambda[1,0,0])^{\exp}$ there exists a cell decomposition of $\Lambda[1,0,0]$ such that, for every 1-cell $\lambda : Z \to Z_C \subset \Lambda[1,s,r]$, the function $j_!\lambda_!(i^*\varphi)$ lies in $\mathscr{I}_{\Lambda[0,r,s]}(\Lambda[1,r,s])^{\exp}$, where i denotes the inclusion $Z \to \Lambda[1,0,0]$ and j the inclusion $Z_C \to \Lambda[1,s,r]$.*

Proof. This follows easily from Section 5.1, or even from Theorem 2.2.1. □

LEMMA 7.5.4. *For r and s in $\mathbb{N}$, denote by π the projection $\Lambda[d,r,s] \to \Lambda[d,0,0]$ and recall notation from Section 3.6. For any φ in $\mathscr{I}_{\Lambda[0,r,s]}(\Lambda[d,r,s])^{\exp}$, if the function $\pi_!(\varphi)$ lies in $\mathscr{I}_{\Lambda}(\Lambda[d,0,0])^{\exp}$, then*

$$\mathfrak{F}(\pi_!(\varphi)) = \pi_!(\mathfrak{F}(\varphi)).$$

Proof. This follows from the fact that μ_{p_1} commutes with $\pi_!$. □

8. Exponential integrals over the p-adics

8.1. *Definable sets over the p-adics.* Let K be a finite field extension of $\mathbb{Q}_p$ with valuation ring R. We recall the notion of (globally) subanalytic subsets of K^n and of semialgebraic subsets of K^n. Let $\mathscr{L}_{\mathrm{Mac}} = \{0, +, -, \cdot, \{P_n\}_{n>0}\}$ be the language of Macintyre and $\mathscr{L}_{\mathrm{an}} = \mathscr{L}_{\mathrm{Mac}} \cup \{^{-1}, \cup_{m>0} K\{x_1, \dots, x_m\}\}$, where P_n stands for the set of nth powers in $K^\times$, where $K\{x_1, \dots, x_m\}$ is the ring of restricted power series over K (that is, formal power series converging on R^m), and each element f of $K\{x_1, \dots, x_m\}$ is interpreted as the restricted analytic function $K^m \to K$ given by

$$x \mapsto \begin{cases} f(x) & \text{if } x \in R^m \\ 0 & \text{otherwise.} \end{cases} \tag{8.1.1}$$

By subanalytic we mean $\mathscr{L}_{\mathrm{an}}$-definable with coefficients from K and by semialgebraic we mean $\mathscr{L}_{\mathrm{Mac}}$-definable with coefficients from K. Note that subanalytic, resp. semialgebraic, sets can be given by a quantifier free formula with coefficients from K in the language $\mathscr{L}_{\mathrm{Mac}}$, resp. $\mathscr{L}_{\mathrm{an}}$.

In this section we let $\mathscr{L}$ be either the language $\mathscr{L}_{\mathrm{Mac}}$ or $\mathscr{L}_{\mathrm{an}}$ and by $\mathscr{L}$-definable we will mean semialgebraic, resp. subanalytic when $\mathscr{L}$ is $\mathscr{L}_{\mathrm{Mac}}$, resp. $\mathscr{L}_{\mathrm{an}}$. Everything in this section will hold for both languages and we will give the appropriate references for both languages where needed.

For each definable set $X \subset K^n$, let $\mathscr{C}(X)$ be the $\mathbb{Q}$-algebra of functions on X generated by functions $|f|$ and $\mathrm{ord}(f)$ for all definable functions $f : X \to K^\times$.[1]

For an $\mathscr{L}$-definable set X, let $\mathscr{C}^{\leq d}(X)$ be the ideal of $\mathscr{C}(X)$ generated by the characteristic functions $\mathbf{1}_Z$ of $\mathscr{L}$-definable subsets $Z \subset X$ of dimension $\leq d$. (For the definition of the dimension of $\mathscr{L}$-definable sets, see [30] and [19].) Note that

[1] Instead of taking the $\mathbb{Q}$-algebra we could as well take the $\mathbb{Z}[1/q, \{1/(1-q^i)\}_{i<0}]$-algebra, with q the residue cardinality of K.

the support of a function in $\mathscr{C}(X)$ is in general not $\mathcal{L}$-definable; cf. the function $(x, y) \mapsto |x| - \mathrm{ord}(y)$ on $K \times K^\times$.

By $C^d(X)$ we denote the quotient

$$C^d(X) := \mathscr{C}^{\leq d}(X)/\mathscr{C}^{\leq d-1}(X). \tag{8.1.2}$$

Finally we set

$$C(X) := \bigoplus_{d \geq 0} C^d(X). \tag{8.1.3}$$

It is a module over $\mathscr{C}(X)$. If φ is in $\mathscr{C}(X)$ with support contained in a $\mathcal{L}$-definable set of dimension d, we denote by $[\varphi]_d$ its image in $C^d(X)$.

8.2. *The p-adic measure.* Suppose that $X \subset K^n$ is an $\mathcal{L}$-definable set of dimension $d \geq 0$. The set X contains a definable nonempty open submanifold $X' \subset K^n$ such that $X \setminus X'$ has dimension $< d$; cf. [19]. There is a canonical d-dimensional measure on X' coming from the embedding in K^n, which is constructed as follows; cf. [31]. For each d-element subset J of $\{1, \ldots, n\}$, with $j_i < j_{i+1}$, j_i in J, let dx_J be the d-form $dx_{j_1} \wedge \ldots \wedge dx_{j_d}$ on K^n, with $x = (x_1, \ldots, x_n)$ standard global coordinates on K^n. Let x_0 be a point on X' such that x_I are local coordinates around x_0 for some $I \subset \{1, \ldots, n\}$. For each d-element subset J of $\{1, \ldots, n\}$ let g_J be the $\mathcal{L}$-definable function determined at a neighborhood of x_0 in X' by $g_J dx_I = dx_J$. There is a unique volume form $|\omega_0|_{X'}$ on X' which is locally equal to $(\max_J |g_J|)|dx_I|$ around every point x_0 in X'. Indeed, $|\omega_0|_{X'}$ is equal to $\sup_J |dx_J|$. The canonical d-dimensional measure on X', cf. [31], [28], is the one induced by the volume form $|\omega_0|_{X'}$. We extend this measure to X by zero and denote it by μ^d.

This measure allows us to define the subgroup $\mathrm{I}C^d(X)$ of $C^d(X)$ for an $\mathcal{L}$-definable set X of dimension d, as the group generated by elements $[\varphi]_d$ with φ in $\mathscr{C}(X)$ integrable for μ^d. We define $\mathrm{I}C^e(X)$ for general e as the subgroup of $C^e(X)$ consisting of elements $[\varphi]_e$ with φ with support contained in an $\mathcal{L}$-definable subset $Z \subset X$ of dimension e and with $[\varphi_{|Z}]_e$ in $\mathrm{I}C^e(Z)$. Finally, we define the graded group $\mathrm{I}C(X)$ as $\oplus_r \mathrm{I}C^r(X)$.

8.3. *Jacobian.* Using the pullback of differential forms under analytic maps, it is possible to define the norm of the Jacobian $|\mathrm{Jac} f|$ of an $\mathcal{L}$-definable bijection $f : X \subset K^n \to Y \subset K^m$ as follows. There exist definable K-analytic manifolds $X' \subset X$ and $Y' \subset Y$ such that $X \setminus X'$ and $Y \setminus Y'$ have dimension $< d$ with $d = \dim X$ and such that $f_{|X'}$ is a K-bi-analytic bijection onto Y'. For subsets I and J of $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$ respectively, we denote by $U_{I,J}$ the definable subset of X' consisting of points x_0 such that $|dx_I|$ coincides with $|\omega_0|_{X'}$ on a neighborhood of x_0 and $|dy_J|$ coincides with $|\omega_0|_{Y'}$ on a neighborhood of $f(x_0)$. On $U_{I,J}$ we may write $f^*(dy_J) = g_{I,J}\,dx_I$. The functions $|g_{I,J}|$ are constructible on $U_{I,J}$ and there exists a unique constructible function h on X' restricting to $|g_{I,J}|$ on each $U_{I,J}$. We define $|\mathrm{Jac}\, f|$ as the class of h in $C^d(X)$ which does not depend on the choices made.

The p-adic change of variables formula (cf. [22]) may be restated as follows:

PROPOSITION 8.3.1. *Let $f : X \subset K^n \to Y \subset K^m$ be an $\mathcal{L}$-definable bijection, with $d = \dim X$. For every measurable subset A of Y one has*

$$\mu^d(A) = \int_{f^{-1}(A)} |\mathrm{Jac}\, f|\, \mu^d. \tag{8.3.1}$$

8.3.2. *A variant.* For the proof of Theorem 8.5.3 below we shall need the following variant of $|\mathrm{Jac}\, f|$. Let X be a definable subset of $Y \times K^n$ for Y a definable subset of K^m and consider the morphism $f : X \to Y$ induced by projection. Assume first X is of dimension r, Y of dimension s, f is surjective and all fibers of f have dimension $r - s$. In this setting we define a constructible function $\delta(f)$ defined almost everywhere on X as follows. We can choose (cf. [19]) definable manifolds $X' \subset X$ and $Y' \subset Y$ such that $X \setminus X'$ has dimension $< r$, $Y \setminus Y'$ has dimension $< s$, f restricts to a locally analytic morphism $f' : X' \to Y'$ and for every point y in Y', $f^{-1}(y) \setminus f'^{-1}(y)$ is of dimension $< r - s$, and such that f' is regular (that is, the Jacobian matrix has everywhere maximal rank). Denote by y_i the coordinates on K^m and by z_i the coordinates on K^n. Consider subsets I, I' of $\{1, \dots, m\}$ and J, J' of $\{1, \dots, n\}$, and denote by $U_{I,J,I',J'}$ the set of points x of X' such that on a neighborhood of x, $|\omega_0|_{X'}$ coincides with $|dy_I \wedge dz_J|$, $|\omega_0|_{f'^{-1}(f(x))}$ coincides with $|dz_{J'}|$, and on a neighborhood of $f(x)$, $|\omega_0|_{Y'}$ coincides with $|dy_{I'}|$. On $U_{I,J,I',J'}$ we may write

$$dy_{I'} \wedge dz_{J'} = g_{I,J,I',J'}\, dy_I \wedge dz_J, \tag{8.3.2}$$

with $g_{I,J,I',J'}$ definable. There is a unique constructible function g on X' restricting to $|g_{I,J,I',J'}|$ on X'. We denote its class in $C^r(X)$, which is independent of the choices made, by $\delta(f)$. Note that when f is an isomorphism $\delta(f) = |\mathrm{Jac}\, f|$.

The proof of the following chain rule is clear:

LEMMA 8.3.3. *Let Z be a definable subset of dimension t of K^m, Y be a definable subset of dimension s of $Z \times K^n$ and X be a definable subset of dimension r of $Y \times K^q$. Assume the morphisms $f : X \to Y$ and $g : Y \to Z$ are induced by projections and are surjective, and that all their fibers have dimension $r - s$ and $s - t$, respectively. Then the equality*

$$\delta(g \circ f) = \delta(f)(\delta(g) \circ f)$$

holds. □

8.4. *p-adic cell decomposition.* Recall that P_n is the set of nth powers in $K^\times$, and for $\lambda \in K$ let λP_n be $\{\lambda x \mid x \in P_n\}$. Cells are defined by induction on the number of variables:

Definition 8.4.1. An $\mathcal{L}$-cell $A \subset K$ is a (nonempty) set of the form

$$\{t \in K \mid |\alpha| \underset{1}{\square} |t-c| \underset{2}{\square} |\beta|,\ t-c \in \lambda P_n\}, \tag{8.4.1}$$

with constants $n > 0$, λ, c in K, α, β in $K^\times$, and $\square_i$ either $<$ or no condition. An $\mathcal{L}$-cell $A \subset K^{m+1}$, $m \geq 0$, is a set of the form

$$\begin{aligned}\{(x,t) \in K^{m+1} \mid\ & x \in D,\ |\alpha(x)| \square_1 |t-c(x)| \square_2 |\beta(x)|, \\ & t - c(x) \in \lambda P_n\},\end{aligned} \tag{8.4.2}$$

with $(x,t) = (x_1, \dots, x_m, t)$, $n > 0$, λ in K, $D = \pi_m(A)$ a cell where π_m is the projection $K^{m+1} \to K^m$, $\mathcal{L}$-definable functions $\alpha, \beta : K^m \to K^\times$ and $c : K^m \to K$, and $\square_i$ either $<$ or no condition, such that the functions α, β, and c are analytic on D. We call c the center of the cell A and λP_n the coset of A. In either case, if $\lambda = 0$ we call A a 0-cell and if $\lambda \neq 0$ we call A a 1-cell. (Recall that P_n denotes the set of n-th powers in $K^\times$.)

In the p-adic semialgebraic case, cell decomposition theorems are due to Cohen [12] and Denef [14], [16] and they were extended in [4] to the subanalytic setting where one can find the following version:

THEOREM 8.4.2 (p-adic cell decomposition). *Let $X \subset K^{m+1}$ and $f_j : X \to K$ be $\mathcal{L}$-definable for $j = 1, \dots, r$. Then there exists a finite partition of X into $\mathcal{L}$-cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that*

$$|f_j(x,t)| = |h_{ij}(x)| \cdot |(t - c_i(x))^{a_{ij}} \lambda_i^{-a_{ij}}|^{\frac{1}{n_i}}, \quad \textit{for each } (x,t) \in A_i,$$

with $(x,t) = (x_1, \dots, x_m, t)$, integers a_{ij}, and $h_{ij} : K^m \to K$ $\mathcal{L}$-definable functions which are analytic on $\pi_m(A_i)$, $j = 1, \dots, r$. If $\lambda_i = 0$, we use the convention that $a_{ij} = 0$.

We shall also use the following lemma from [5]:

LEMMA 8.4.3. *Let $X \subset K^{m+1}$ be $\mathcal{L}$-definable and let G_j be functions in $\mathcal{C}(X)$ in the variables $(x_1, \dots, x_m, t)$ for $j = 1, \dots, r$. Then there exists a finite partition of X into $\mathcal{L}$-cells A_i with center c_i and coset $\lambda_i P_{n_i}$ such that each restriction $G_j|_{A_i}$ is a finite sum of functions of the form*

$$|(t - c_i(x))^a \lambda^{-a}|^{\frac{1}{n_i}} v(t - c_i(x))^s h(x),$$

where h is in $\mathcal{C}(K^m)$, and $s \geq 0$ and a are integers.

8.5. *Integration.* By $\mathrm{Def}(\mathcal{L})$ we denote the category of $\mathcal{L}$-definable subsets $X \subset K^n$ for $n > 0$, with $\mathcal{L}$-definable maps as morphisms. We can now state a general integration result which states uniqueness and existence of a certain integral operator. This integral operator is introduced as a push-forward operator of functions under $\mathcal{L}$-definable maps, inspired by integration in the fibers with a measure on the fibers coming from Leray-differential forms.

In [17], see also [15], Denef proved stability of p-adic constructible functions under integration with respect to parameters in the semialgebraic case. Denef's result had a major influence on our work [11] and the present one. It was later generalized to the subanalytic case by the first author in [4] and [5].

PROPOSITION 8.5.1 ([17], [4], [5]). *Let W be a definable subset of K^n of dimension r.*

(1) *Let φ be in $\mathcal{C}(W \times K^m)$. Assume for every x in W the function $t \mapsto \varphi(x,t)$ is integrable on K^m. Then the function*
$$g(x) := \int_{K^m} \varphi(x,t)|dt|$$
lies in $\mathcal{C}(W)$.

(2) *Let φ be in $\mathrm{IC}^{r+m}(W \times K^m)$. Then, there exists a function g in $\mathcal{C}(K^n)$ such that for all x in $W \setminus Z$, with Z an $\mathcal{L}$-definable set of dimension $< r$ in K^n, one has*
$$g(x) = \int_{K^m} \varphi(x,t)|dt|.$$

Proof. When $W = K^n$, statement (1) is proved in [17] in the semialgebraic case and in [4] in the subanalytic case and statement (2) is proved in [5]. The proofs carry over literally to general W. □

Remark 8.5.2. The point in (2) of Proposition 8.5.1 is that it is possible that the subset of W consisting of those points x in W such that $t \in K^m \mapsto \varphi(x,t)$ is integrable may not be definable.

We shall now prove the following analogue of Theorem 10.1.1 of [11]. Note that the proof will be much easier, since integrable functions are already defined and Proposition 8.5.1 is available.

THEOREM 8.5.3. *There exists a unique functor from* $\mathrm{Def}(\mathcal{L})$ *to the category of groups sending an $\mathcal{L}$-definable set X to the group* $\mathrm{IC}(X)$ *such that a morphism $f : X \to Y$ in* $\mathrm{Def}(\mathcal{L})$ *is sent to a group morphism $f_! : \mathrm{IC}(X) \to \mathrm{IC}(Y)$ satisfying the following axioms*

(A1) Disjoint union: *Assume that X, resp. Y, is the disjoint union of two $\mathcal{L}$-definable sets X_1 and X_2, resp. Y_1 and Y_2, such that $f(X_i) \subset Y_i$. Write $f_i : X_i \to Y_i$ for the restrictions. Then we have $f_! = f_{1!} \oplus f_{2!}$ under the isomorphisms $\mathrm{IC}(X) \simeq \mathrm{IC}(X_1) \oplus \mathrm{IC}(X_2)$ and $\mathrm{IC}(Y) \simeq \mathrm{IC}(Y_1) \oplus \mathrm{IC}(Y_2)$.*

(A2) Projection formula: *For every* α *in* $\mathscr{C}(Y)$ *and* β *in* $\mathrm{I}C(X)$, *if* $(\alpha \circ f)\beta$ *is in* $\mathrm{I}C(X)$, *then* $f_!((\alpha \circ f)\beta) = \alpha f_!(\beta)$.

(A3) Projection for 1-cells: *Let* $X \subset K^{n+1}$ *be a* 1*-cell of dimension* r *and* Y *its image under the projection on* K^n, $f : X \to Y$ *the projection. Let* φ *be a* μ^r*- integrable function in* $\mathscr{C}(X)$. *By Proposition* 8.5.1 *there exists an* $\mathscr{L}$*-definable set* $Z \subset Y$ *such that* $Y \setminus Z$ *has dimension* $< r - 1$ *and such that the function* $g : Y \to \mathbb{Q} : y \mapsto \int_{f^{-1}(y)} \mathbf{1}_{Y \setminus Z}(y)\varphi(y,t)|dt|$ *lies in* $\mathscr{C}^{\leq r-1}(Y)$. *Then* $f_!([\varphi]_r)$ *is equal to the class of* g *in* $\mathrm{I}C^{r-1}(Y)$.

(A4) Projection for 0-cells: *Let* $X \subset K^{n+1}$ *be a* 0*-cell of dimension* r *and* Y *its image under the projection on* K^n, $f : X \to Y$ *the projection. Then* $f_!(\mathbf{1}_X)$ *is equal to the class of* $(|(\mathrm{Jac} f)| \circ f^{-1})^{-1}\mathbf{1}_Y$ *in* $\mathrm{I}C^r(Y)$, *where* $|\mathrm{Jac} f|$ *is as in* 8.3.

Proof. We will freely use classical forms of the change of variables formula, without mentioning it. Let us first check uniqueness. Since, by the graph construction, any morphism $f : X \to Y$ is the composition of the graph injection $i_f : X \to X \times Y$ and the projection $p : X \times Y \to Y$, it is enough, by functoriality, to prove uniqueness for $i_{f!}$ and $p_!$. For projections $X \times Y \to Y$, one can assume $X = K^m$ and $Y = K^n$ by (A1). By induction on m, it is enough to define $p_!$ when $m = 1$. Consider φ in $\mathscr{C}^r(K^{n+1})$ and assume it is integrable. By cell decomposition and linearity we may assume the support of φ is contained in a cell Z of dimension r. If Z is a 1-cell, $f_!([\varphi]_r)$ is given by (A3). In case Z is a 0-cell, we may assume by (A2) that $\varphi = \mathbf{1}_Z$, and then $f_!([\varphi]_r)$ is given by (A4). Finally, since $q \circ i_f = \mathrm{id}_X$, with q the projection $X \times Y \to X$, and since q induces a bijection between the graph of f and X, uniqueness for $i_{f!}$ reduces to that of $q_!$ (an essentially similar argument is given with full details in the uniqueness section of the proof of Theorem 10.1.1 of [11]).

Let us now define $f_!$ for projections. Let X be a definable subset of $Y \times K^n$ for Y a definable subset of K^m and consider the morphism $f : X \to Y$ induced by projection. Assume first that X is of dimension r, Y of dimension s, f is surjective and all fibers of f have dimension $r - s$.

Let φ be a μ^r-integrable function in $\mathscr{C}(X)$. There exists a definable subset Z of Y, with dimension $< s$, such that the function

$$g : y \mapsto \int_{f^{-1}(y)} \mathbf{1}_{Y \setminus Z}(y)\delta(f)\varphi\mu^{r-s} \tag{8.5.1}$$

lies in $\mathscr{C}(Y)$, with $\delta(f)$ as in Section 8.3.2, and μ^{r-s} is the measure as in Section 8.2. Indeed, by Fubini and induction, and possibly after considering a partition of X and Y, we may assume $n = 1$. Then, we may by cell decomposition assume X is a cell. If X is a 0-cell the statement is clear, if X is a 1-cell the statement follows from Proposition 8.5.1. We may then define $f_!([\varphi]_r)$ to be the class of g in

$C^{r-s}(Y)$. It follows from Fubini that $f_!([\varphi]_r)$ lies in $\mathrm{I}C^{r-s}(Y)$. Note that certainly (A3) and (A4) hold. Also, there is a unique way to extend that construction to a morphism $f_! : \mathrm{I}C(X) \to \mathrm{I}C(Y)$ satisfying (A1) and (A2) for every morphism $f : X \to Y$ induced by a projection $Y \times K^m \to Y$. Indeed, it is enough to construct $f_!$ on $\mathrm{I}C^r(X)$ and after cutting X and Y into finitely many definable pieces, one may assume the above condition is verified. Furthermore, by Lemma 8.3.3, $(g \circ f)_! = g_! \circ f_!$ for composable morphisms induced by projections.

Let us now define $i_!$ when $i : X \to Y$ is a definable injection. Let φ be in $C^e(X)$. Consider a definable subset X' of dimension e of X such that the support of a representative $\tilde{\varphi}$ of φ is contained in X'. Denote by $\lambda : i(X') \to X'$ the inverse of the isomorphism induced by i. We define $i_!(\varphi)$ as the image of $[(\tilde{\varphi} \circ \lambda)]_e |\mathrm{Jac}(\lambda)|$ in $C^e(Y)$ under the inclusion $C^e(i(X')) \to C^e(Y)$. Certainly $i_!(\varphi)$ is integrable if φ is, hence we deduce a morphism $\lambda_! : \mathrm{I}C(X) \to \mathrm{I}C(Y)$.

For a general morphism $f : X \to Y$ one considers the factorization $f = \pi_f \circ i_f$, with $i_f : X \to X \times Y$ the inclusion of the graph and $\pi_f : X \times Y \to Y$ the projection, and one sets $f_! := \pi_{f!} \circ i_{f!}$. One is then left with checking that the construction coincides with the previous one for injections and projections and that $(g \circ f)_! = g_! \circ f_!$ for composable morphisms. This is purely formal and performed exactly as in the proof of the corresponding statements in the proof of Theorem 10.1.1 and Proposition 12.1.2 of [11]. □

8.6. *Exponential constructible functions.* Fix an additive character $\psi : K \to \mathbb{C}^\times$ which is trivial on the maximal ideal M of R and such that $\psi(x) \neq 1$ for some x in K with $\mathrm{ord}(x) = 0$.

For X an $\mathscr{L}$-definable set, we let $\mathscr{C}(X)^{\mathrm{exp}}$ be the $\mathbb{Q}$-algebra generated by $\mathscr{C}(X)$ and all functions $\psi(f)$, where $f : X \to K$ is $\mathscr{L}$-definable (cf. footnote on page 1043).

Similarly, for each $d \geq 0$ we define $\mathscr{C}^{\leq d}(X)^{\mathrm{exp}}$ as the $\mathbb{Q}$-algebra generated by $\mathscr{C}^{\leq d}(X)$ and all functions $\psi(f)$ with $f : X \to K$ $\mathscr{L}$-definable.

We set

$$C(X)^{\mathrm{exp}} = \oplus_d C^d(X)^{\mathrm{exp}} \tag{8.6.1}$$

with

$$C^d(X)^{\mathrm{exp}} := \mathscr{C}^{\leq d}(X)^{\mathrm{exp}} / \mathscr{C}^{\leq d-1}(X)^{\mathrm{exp}}. \tag{8.6.2}$$

We call elements of $C(Z)^{\mathrm{exp}}$ constructible exponential Functions.

For $d \geq 0$ we define the group $\mathrm{I}C^d(X)^{\mathrm{exp}}$ of integrable constructible exponential Functions as the subgroup of $C^d(X)^{\mathrm{exp}}$ generated by elements $\psi(f)\varphi_0$ with $f : X \to K$ $\mathscr{L}$-definable and φ_0 in $\mathrm{I}C^d(X)$. Thus $\mathrm{I}C(X)^{\mathrm{exp}} = \oplus_{d \geq 0} \mathrm{I}C^d(X)^{\mathrm{exp}}$ is a graded subgroup of $C(X)^{\mathrm{exp}}$.

The following exponential analogue of Proposition 8.5.1 is our main p-adic result:

PROPOSITION 8.6.1. *Let W be a definable subset of K^n of dimension r.*

(1) *Let φ be in $\mathscr{C}(W \times K^m)^{\exp}$ of the form $\varphi_0 \psi(f)$ with φ_0 in $\mathscr{C}(W \times K^m)$ and f an $\mathscr{L}$-definable function from $W \times K^m$ to K. Assume for every x in W the function $t \mapsto \varphi(x,t)$ is integrable on K^m. Then the function*
$$g(x) := \int_{K^m} \varphi(x,t)|dt|$$
lies in $\mathscr{C}(W)^{\exp}$.

(2) *Let φ be in $\mathrm{I}C^{r+m}(W \times K^m)^{\exp}$. Then, there exists a function g in $\mathscr{C}(K^n)^{\exp}$ such that for all x in $W \setminus Z$, with Z an $\mathscr{L}$-definable set of dimension $< r$ in K^n, one has*
$$g(x) = \int_{K^m} \varphi(x,t)|dt|.$$

Proof. (1) follows easily from (2), so let us prove (2). By the Fubini theorem it is enough to consider the case $m = 1$. By linearity of the integral operator it is enough to prove the proposition when $\varphi = \varphi_0 \psi(f)$ with φ_0 in $\mathrm{I}C^{r+1}(W \times K)$ and $f : W \times K \to K$ an $\mathscr{L}$-definable morphism.

We partition $W \times K$ into $\mathscr{L}$-definable parts B_1, B_2, and B_3:
$$B_1 := \{(x,t) \in W \times K \mid f(x,\cdot) \text{ is } C^1 \text{ at } t \text{ and } \frac{\partial f}{\partial t}(x,t) \neq 0\},$$
$$B_2 := \{(x,t) \in W \times K \mid f(x,\cdot) \text{ is not } C^1 \text{ at } t\},$$
$$B_3 := \{(x,t) \in W \times K \mid f(x,\cdot) \text{ is } C^1 \text{ at } t \text{ and } \frac{\partial f}{\partial t}(x,t) = 0\},$$
where C^1 at a point means continuously differentiable in an open neighborhood and $f(x,\cdot)$ denotes the function $K \to K : t \mapsto f(x,t)$ for each x in W.

Note that it follows directly from cell decomposition Theorem 8.4.2, cf. also [19] and [30], that for every x in W the set $B_{2x} := \{t \in K \mid (x,t) \in B_2\}$ is finite and of uniformly bounded cardinality when x varies and also that surjective $\mathscr{L}$-definable maps admit $\mathscr{L}$-definable sections. Hence there exists a partition of B_2 into finitely many $\mathscr{L}$-definable sets B_{2i} such that $f(x,t) = g_i(x)$ for each i and for each (x,t) in B_{2i} for some $\mathscr{L}$-definable functions $g_i : W \to K$.

Similarly, there exists a partition B_3 into finitely many $\mathscr{L}$-definable sets B_{3i} such that $f(x,t) = r_i(x)$ for each i and for each (x,t) in B_{3i} for some $\mathscr{L}$-definable functions $r_i : W \to K$. Indeed, this follows from the fact that for every x the image of the function $t \mapsto f(x,t)$ is discrete, hence finite and uniformly bounded when x varies (again a consequence of cell decomposition Theorem 8.4.2, cf. also [19] and

[30]) and from the already mentioned fact that surjective $\mathcal{L}$-definable maps admit $\mathcal{L}$-definable sections.

Hence, for the functions $\mathbf{1}_{B_{\ell i}}\varphi$ with $\ell = 2, 3$ the proposition follows. By linearity of the integral operator we only have to prove the proposition for the function $\mathbf{1}_{B_1}\varphi$.

By the implicit function theorem, the set $\{t \mid f(x,t) = z,\ (x,t) \in B_1\}$ is discrete for each x in W and z in K, hence finite and uniformly bounded when x and z vary, by the cell decomposition Theorem 8.4.2 (or by [19], [30]). By the existence of $\mathcal{L}$-definable sections for surjective $\mathcal{L}$-definable maps, there exists a partition of B_1 into finitely many $\mathcal{L}$-definable parts B_{1i} such that $f(x,\cdot)$ is injective on B_{1ix} for each i and each x, with $B_{1ix} := \{t \in K \mid (x,t) \in B_{1i}\}$. Hence, we may as well suppose that $f(x,\cdot)$ is injective on $B_{1x} := \{t \in K \mid (x,t) \in B_1\}$ itself. Then, we let T be the transformation

$$T : \begin{cases} B_1 \mapsto T(B_1) \\ (x,t) \mapsto (x,y) := (x, f(x,t)), \end{cases} \tag{8.6.3}$$

and let $|\mathrm{Jac}T|$ be the Jacobian of T as in Section 8.3. Writing $T(B_1)_x$ for $\{t \in K \mid (x,t) \in T(B_1)\}$, one has by the change of variables rule for each x in $\pi_{B_1}(B_1)$

$$\begin{aligned} &\int_{B_{1x}} \varphi_0(x,t)\psi(f(x,t))|dt| \\ &\qquad = \int_{T(B_1)_x} (|\mathrm{Jac}T| \circ T^{-1}(x,y))^{-1}\varphi_0(T^{-1}(x,y))\psi(y)|dy|. \end{aligned} \tag{8.6.4}$$

Now apply Lemma 8.4.3 to the function

$$\varphi_1 : \begin{cases} T(B_1) \to \mathbb{Q} \\ (x,y) \mapsto (|\mathrm{Jac}T| \circ T^{-1}(x,y))^{-1}\varphi_0(T^{-1}(x,y)) \end{cases} \tag{8.6.5}$$

with respect to the variable y to obtain a partition of $T(B_1)$ into $\mathcal{L}$-cells A with center c and coset λP_m such that each $\varphi_{1|A}$ is a finite sum of functions of the form

$$H(x,y) = |(y-c(x))^a\lambda^{-a}|^{\frac{1}{m}} v(y-c(x))^s h(x), \tag{8.6.6}$$

where $h : W \to \mathbb{Q}$ is in $\mathcal{C}(W)$, and $s \geq 0$ and a are integers.

CLAIM 8.6.2. *Possibly after refining the partition, we can assure that for each A either the projection $A' := \pi_W(A) \subset W$ has zero μ^r-measure, or we can write $\varphi_{1|A}$ as a sum of terms H of the form (8.6.6) such that H is μ^r-integrable over A and $H(x,\cdot)$ is integrable over $A_x := \{y \mid (x,y) \in A\}$ for all x in A'.*

As this claim is very similar to Claim 2 of [5] we will only give an indication of its proof. Partitioning further, we may suppose that $v(y - c(x))$ either takes only one value on A or infinitely many values. In the case that $v(y-c(x))$ only takes one value on A, we may suppose that the exponents a and s as in (8.6.6) are zero. In the other case, we just keep a and s. Now, in both cases, apply Lemma 8.4.3 to each h and to the norms of all the $\mathscr{L}$-definable functions appearing in the description of the cells A in a similar way and do this inductively for each variable. This way, the claim is reduced to a summation problem over a Presburger set of integers, which is easily solved. This proves the claim.

Fix a cell A and a term H as in the claim. The cell A has by definition the following form

$$A=\{(x,y)\mid x\in A',\ v(\alpha(x))\underset{1}{\square}v(y-c(x))\underset{2}{\square}v(\beta(x)),\, y-c(x)\in\lambda P_m\},$$

where $A'=\pi_W(A)$ is a cell, $\square_i$ is $<$ or no condition, and $\alpha,\beta:W\to K^\times$ and $c:W\to K$ are $\mathscr{L}$-definable functions. We focus on a cell A of dimension $r+1$, in particular, $\lambda\neq 0$ and A' is of dimension r.

For x in A', we denote by $I(x)$ the value

$$I(x)=\int_{y\in A_x}H(x,y)\,\psi(y)\,|dy|,$$

where $A_x=\{y\in K\mid (x,y)\in A\}$. Write

$$G(j):=\int_{v(u)=j,\ u\in\lambda P_m}\psi(u)\,|du|. \tag{8.6.7}$$

We easily find

$$I(x)=\psi(c(x))\,h(x)|\lambda|^{-a/m}\sum_{(8.6.9)}q^{-ja/m}\,j^sG(j), \tag{8.6.8}$$

where the summation is over

$$J:=\{j\mid v(\alpha(x))\underset{1}{\square}j\underset{2}{\square}v(\beta(x)),\ \ j\equiv v(\lambda)\bmod m\}. \tag{8.6.9}$$

By Hensel's Lemma, there exists an integer $e\geq 0$ such that all units u with $u\equiv 1\bmod \pi_0^e$ are m-th powers (here, π_0 is such that $v(\pi_0)=1$). Hence, $G(j)$ is zero whenever $j\leq -e$ (since in this case one essentially sums a nontrivial character over a finite group). Also, when $j>0$ then $G(j)=\int_{v(u)=j,\ u\in\lambda P_m}|du|$, which is independent of ψ. We find that $I(y)$ is equal to

(8.6.10)

$$\psi(c(y))\,h(y)|\lambda|^{-a/m}\cdot\Bigg(\sum_{\substack{-e\leq j\leq 0\\ j\in J}}q^{-ja/m}\,j^sG(j)+\sum_{\substack{0<j\\ j\in J}}q^{-ja/m}\,j^sG(j)\Bigg).$$

The factors of (8.6.10) before the brackets clearly are in $\mathscr{C}(W)^{\exp}$. The (parametrized) finite sum inside the brackets of (8.6.10) can be written as a finite sum of generators of $\mathscr{C}(W)^{\exp}$ since each $G(j)=p^{-j}\alpha_{j\bmod e+n}$ with each $\alpha_{j\bmod e+n}$

some $\mathbb{Q}$-linear combination of values of ψ which only depends on $j \bmod e+n$, and hence, it is also in $\mathscr{C}(W)^{\exp}$. The infinite sum inside the brackets of (8.6.10) is in $\mathscr{C}(W)$ by Proposition 8.5.1 and the above discussion. This finishes the proof of the proposition. □

One may extend Theorem 8.5.3 to the exponential setting as follows:

THEOREM 8.6.3. *There exists a unique functor from* $\mathrm{Def}(\mathscr{L})$ *to the category of groups sending an* $\mathscr{L}$*-definable set* X *to the group* $\mathrm{I}C(X)^{\exp}$ *such that a morphism* $f : X \to Y$ *in* $\mathrm{Def}(\mathscr{L})$ *is sent to a group morphism* $f_! : \mathrm{I}C(X)^{\exp} \to \mathrm{I}C(Y)^{\exp}$ *satisfying the following axioms*:

(A1) Compatibility: *For every morphism* $f : X \to Y$ *in* $\mathrm{Def}(\mathscr{L})$, *the map* $f_! : \mathrm{I}C(X)^{\exp} \to \mathrm{I}C(Y)^{\exp}$ *is compatible with the inclusions of groups* $\mathrm{I}C(X) \to \mathrm{I}C(X)^{\exp}$ *and* $\mathrm{I}C(Y) \to \mathrm{I}C(Y)^{\exp}$ *and with the map* $f_! : \mathrm{I}C(X) \to \mathrm{I}C(Y)$ *as constructed in Theorem* 8.5.3.

(A2) Disjoint union: *Assume that* X, *resp.* Y, *is the disjoint union of two* $\mathscr{L}$*-definable sets* X_1 *and* X_2, *resp.* Y_1 *and* Y_2, *such that* $f(X_i) \subset Y_i$. *Write* $f_i : X_i \to Y_i$ *for the restrictions. Then* $f_! = f_{1!} \oplus f_{2!}$ *under the isomorphisms* $\mathrm{I}C(X)^{\exp} \simeq \mathrm{I}C(X_1)^{\exp} \oplus \mathrm{I}C(X_2)^{\exp}$ *and* $\mathrm{I}C(Y)^{\exp} \simeq \mathrm{I}C(Y_1)^{\exp} \oplus \mathrm{I}C(Y_2)^{\exp}$.

(A3) Projection formula: *For every* α *in* $\mathscr{C}(Y)^{\exp}$ *and* β *in* $\mathrm{I}C(X)^{\exp}$, *if* $(\alpha \circ f)\beta$ *is in* $\mathrm{I}C(X)^{\exp}$, *then* $f_!((\alpha \circ f)\beta) = \alpha f_!(\beta)$.

(A4) Projection for 1-cells: *Let* $X \subset K^{n+1}$ *be a* 1*-cell of dimension* r *and* Y *its image under the projection to* K^n, $f : X \to Y$ *the projection. Let* φ *be a* μ^r*- integrable function in* $\mathscr{C}(X)^{\exp}$. *By Proposition* 8.6.1 *there exists an* $\mathscr{L}$*-definable set* $Z \subset Y$ *such that* $Y \setminus Z$ *has dimension* $< r-1$ *and such that the the function* $g : Y \to \mathbb{Q} : y \mapsto \int_{f^{-1}(y)} \mathbf{1}_{Y\setminus Z}(y)\varphi(y,t)|dt|$ *lies in* $\mathscr{C}^{\leq r-1}(Y)^{\exp}$. *We let* $f_!([\varphi]_r)$ *be the class of* g *in* $\mathrm{I}C^{r-1}(Y)^{\exp}$.

Proof. The proof is quite formal and similar to proofs we have already given. Indeed, uniqueness is proved along similar lines to those given in Section 6.3; for existence one can proceed similarly, as in the proof of Theorem 8.5.3, using Proposition 8.6.1 instead of Proposition 8.5.1. □

8.7. *Variants*: *adding sorts and relative versions.* By analogy with the motivic framework, we now expand the language $\mathscr{L}$ to a three sorted language $\mathscr{L}'$ having $\mathscr{L}$ as language for the valued field sort, the ring language $\mathbf{L}_{\mathrm{Rings}}$ for the residue field, and the Presburger language $\mathbf{L}_{\mathrm{PR}}$ for the value group together with maps ord and $\overline{\mathrm{ac}}$ as in Section 2.1. By taking the product of the measure μ^m with the counting measure on $k_K^n \times \mathbb{Z}^r$ one defines a measure still denoted by μ^m on $K^m \times k_K^n \times \mathbb{Z}^r$.

One defines the dimension of an $\mathcal{L}'$-definable subset X of $K^m \times k_K^n \times \mathbb{Z}^r$ as the dimension of its projection $\pi(X) \subset K^m$. If X is of dimension d, one defines a measure μ^d on X extending the previous construction on X by setting

$$\mu^d(W) := \int_{\pi(X)} \pi_!(\mathbf{1}_W)\mu^d \tag{8.7.1}$$

with $\pi_!(\mathbf{1}_W)$ the function $y \mapsto \operatorname{card}(\pi^{-1}(y) \cap W)$.

For such an X, one defines $\mathscr{C}(X)$ as the $\mathbb{Q}$-algebra of functions on X generated by functions α and $p^{-\alpha}$ with $\alpha : X \to \mathbb{Z}$ definable in $\mathcal{L}'$. Note that this definition coincides with the previous one when $n = r = 0$. Since $\mathcal{L}'$ is interpretable in $\mathcal{L}$, the formalism developed in this section extends to $\mathcal{L}'$-definable objects in a natural way. In particular the definitions of $\mathscr{C}^{\leq d}$, C^d, $\mathrm{I}C$, $|\mathrm{Jac}|$, δ, $\mathscr{C}^{\exp}$, etc., extend readily to $\mathcal{L}'$-definable objects and we have:

THEOREM 8.7.1. (1) *The statement of Theorem* 8.5.3 *extends to* $\mathrm{Def}(\mathcal{L}')$ *after adding the additional axiom*:

(A5) *Let* $\pi : X \times k_K^n \times \mathbb{Z}^r \to X$ *be the projection with* X *in* $\mathrm{Def}(\mathcal{L}')$. *For any* φ *in* $\mathrm{I}C(X \times k_K^n \times \mathbb{Z}^r)$ *and every* x *in* X,

$$\pi_!(\varphi)(x) = \sum_{\pi(y)=x} \varphi(y).$$

(2) *The statement of Theorem* 8.6.3 *extends to* $\mathrm{Def}(\mathcal{L}')$.

8.7.2. Fix Λ in $\mathrm{Def}(\mathcal{L}')$. We consider the category $\mathrm{Def}_\Lambda(\mathcal{L}')$ whose objects are $\mathcal{L}'$-definable morphisms $f : S \to \Lambda$, a morphism $g : (f : S \to \Lambda) \to (f' : S' \to \Lambda)$ being a morphism $g : S \to S'$ in $\mathrm{Def}(\mathcal{L}')$ with $f' \circ g = f$. For $f : S \to \Lambda$ in $\mathrm{Def}_\Lambda(\mathcal{L}')$ we define the relative dimension of S over Λ as the maximum of the dimension of the fibers of f. For d in $\mathbb{Z}$, define $\mathscr{C}^{\leq d}(S \to \Lambda)$ as the ideal of $\mathscr{C}(S)$ generated by the characteristic functions of $\mathcal{L}'$-definable subsets of S of relative dimension $\leq d$ over Λ. Set

$$C^d(S \to \Lambda) := \mathscr{C}^{\leq d}(S \to \Lambda)/\mathscr{C}^{\leq d-1}(S \to \Lambda), \tag{8.7.2}$$

and define the graded group

$$C(S \to \Lambda) := \oplus_d C^d(S \to \Lambda). \tag{8.7.3}$$

For every λ in Λ there exists a graded group homomorphism called restriction to λ,

$$|_{f^{-1}(\lambda)} : C(S \to \Lambda) \to C(f^{-1}(\lambda)), \tag{8.7.4}$$

sending φ in $C(S \to \Lambda)$ to its restriction to the fiber $f^{-1}(\lambda)$.

We define $\mathrm{I}C(S \to \Lambda)$ as the graded subgroup of $C(S \to \Lambda)$ consisting of $\varphi \in C(S \to \Lambda)$ such that, for every λ in Λ, the restriction $\varphi|_{f^{-1}(\lambda)}$ lies in $\mathrm{I}C(f^{-1}(\lambda))$.

One defines similarly $\mathscr{C}^{\leq d}(S \to \Lambda)^{\exp}$, $C(S \to \Lambda)^{\exp}$, $\mathrm{I}C(S \to \Lambda)^{\exp}$ and

(8.7.5) $$|_{f^{-1}(\lambda)} : C(S \to \Lambda)^{\exp} \to C(f^{-1}(\lambda))^{\exp}.$$

If $g : S \to S'$ is an isomorphism in $\mathrm{Def}_\Lambda(\mathscr{L}')$ between subsets of relative dimension d, one denotes by $|\mathrm{Jac}_\Lambda g|$ the function in $C^d(S \to \Lambda)$ such that

(8.7.6) $$|\mathrm{Jac}_\Lambda g||_{f^{-1}(\lambda)} = |\mathrm{Jac}(g_{|f^{-1}(\lambda)})|$$

for every λ in Λ.

PROPOSITION 8.7.3. *For $g : S \to S'$ a morphism in $\mathrm{Def}_\Lambda(\mathscr{L}')$, there exists a unique morphism*

(8.7.7) $$g_{!\Lambda} : \mathrm{I}C(S \to \Lambda) \to \mathrm{I}C(S' \to \Lambda)$$

which sends $\varphi \in \mathrm{I}C(S \to \Lambda)$ to the unique $\psi \in \mathrm{I}C(S' \to \Lambda)$ such that for each $\lambda \in \Lambda$

(8.7.8) $$(g|_{S_\lambda})_!(\varphi|_{f^{-1}(\lambda)}) = \psi|_{S'_\lambda},$$

with S_λ and S'_λ the fibers and $(g|_{S_\lambda})_!$ the direct image as constructed above, and similarly a morphism

(8.7.9) $$g_{!\Lambda} : \mathrm{I}C(S \to \Lambda)^{\exp} \to \mathrm{I}C(S' \to \Lambda)^{\exp}.$$

Furthermore, these morphisms $g_{!\Lambda}$ satisfy the relative analogues of properties (A1)–(A4) *of Theorem* 8.5.3, (A1)–(A4) *of Theorem* 8.6.3, *and* (A5) *of Theorem* 8.7.1 *respectively, where in* (A4) *of Theorem* 8.5.3, $|\mathrm{Jac}|$ *is replaced by its relative analogue* $|\mathrm{Jac}_\Lambda|$.

Proof. One can either note that the proofs of Theorems 8.5.3, 8.6.3 and 8.7.1 carry over literally to the relative case, or deduce it from the absolute case using Propositions 8.5.1 and 8.6.1. □

Since $\mathrm{I}C(\Lambda \to \Lambda) = \mathscr{C}(\Lambda)$, when g is the morphism $S \to \Lambda$, one gets from (8.7.7) a morphism

(8.7.10) $$\mu_\Lambda : \mathrm{I}C(S \to \Lambda) \to \mathscr{C}(\Lambda).$$

9. Specialization and transfer

In this section we obtain new results on specialization to p-adic and $\mathbb{F}_q((t))$-integration and a transfer principle for exponential integrals with parameters from $\mathbb{Q}_p$ and from $\mathbb{F}_q((t))$. Some of the results which are announced in [9] are generalized here to exponential constructible functions. The specialization principle given here generalizes the one of [18].

9.1. *Specialization to valued local fields.*

9.1.1. *Notation.* Let k be a number field with ring of integers $\mathcal{O}$. We denote by $\mathcal{A}_{\mathcal{O}}$ the collection of the p-adic completions of all the finite field extensions of k. We denote by $\mathcal{B}_{\mathcal{O}}$ the set of all local fields of positive characteristic over $\mathcal{O}$, that is, endowed with an $\mathcal{O}$-algebra structure. For $N > 0$, denote by $\mathcal{C}_{\mathcal{O},N}$ the collection of all K in $\mathcal{A}_{\mathcal{O}} \cup \mathcal{B}_{\mathcal{O}}$ with residue field of characteristic $> N$ and write $\mathcal{C}_{\mathcal{O}}$ for $\mathcal{C}_{\mathcal{O},1}$. By $\mathcal{A}_{\mathcal{O},N}$, resp. $\mathcal{B}_{\mathcal{O},N}$, denote $\mathcal{C}_{\mathcal{O},N} \cap \mathcal{A}_{\mathcal{O}}$, resp. $\mathcal{C}_{\mathcal{O},N} \cap \mathcal{B}_{\mathcal{O}}$.

For K in $\mathcal{C}_{\mathcal{O}}$, we write R_K for its valuation ring, M_K for the maximal ideal, k_K for its residue field, and $q(K)$ for the number of elements of k_K. For each choice of a uniformizing parameter ϖ_K of R_K, there is a unique multiplicative map $\overline{\mathrm{ac}} : K^\times \to k_K^\times$ which extends the projection $R_K^\times \to k_K^\times$ and sends ϖ_K to 1, and we extend this by setting $\overline{\mathrm{ac}}(0) = 0$. We denote by $\mathcal{D}_K$ the collection of additive characters $\psi : K \to \mathbb{C}^\times$ such that

$$\psi(x) = \exp((2\pi i/p)\mathrm{Tr}_{k_K}(\bar{x}))$$

for $x \in R_K$, with p the characteristic of k_K, Tr_{k_K} the trace of k_K over its prime subfield and $\bar{x}$ the natural projection modulo M_K of x into k_K. Here we identify $\mathbb{F}_p$ with $\mathbb{Z}/p\mathbb{Z}$ and observe that exp denotes the complex exponential.

9.1.2. *Interpretation of functions.* As a language that can be interpreted in all the fields of $\mathcal{C}_{\mathcal{O}}$, we shall use $\mathcal{L}_{\mathcal{O}} := \mathcal{L}_{\mathrm{DP,P}}(\mathcal{O}[\![t]\!])$, that is, the language $\mathcal{L}_{\mathrm{DP,P}}$ with coefficients in k for the residue field sort and coefficients in $\mathcal{O}[\![t]\!]$ for the valued field sort. (Instead of $\mathcal{O}[\![t]\!]$, any subring of $\mathcal{O}[\![t]\!]$ containing $\mathcal{O}[t]$ can be used as a coefficient ring.) To say that a definable subassignment is defined in the language $\mathcal{L}_{\mathcal{O}}$, we say that it belongs to $\mathrm{Def}(\mathcal{L}_{\mathcal{O}})$, and for a constructible function we say likewise that it belongs to $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})$, $\mathcal{C}(S, \mathcal{L}_{\mathcal{O}})^{\exp}$, and so on, when it is defined in $\mathcal{L}_{\mathcal{O}}$.

For every uniformizing parameter ϖ_K of R_K, one may consider K as an $\mathcal{O}[\![t]\!]$-algebra via the morphism

$$\lambda_{\mathcal{O},K} : \mathcal{O}[\![t]\!] \to K : \sum_{i \in \mathbb{N}} a_i t^i \mapsto \sum_{i \in \mathbb{N}} a_i \varpi_K^i. \tag{9.1.1}$$

Hence, if one interprets elements a of $\mathcal{O}[\![t]\!]$ as $\lambda_{\mathcal{O},K}(a)$, an $\mathcal{O}[\![t]\!]$-formula ϑ defines for all K in $\mathcal{C}_{\mathcal{O}}$ a definable subset φ_K of $K^m \times k_K^n \times \mathbb{Z}^r$ for some m, n, r, for every choice of uniformizing parameter ϖ_K of R_K. On the other hand, the formula ϑ gives rise to a definable subassignment X of $h[m, n, r]$ and if ϑ' gives rise to the same subassignment X then $\vartheta_K = \vartheta'_K$ for all K in $\mathcal{C}_{\mathcal{O},N}$ for some large enough N, independently of the choice of uniformizing parameter.[2]

[2]This follows either from Ax and Kochen [1], [2], [3], Eršov [20], Cohen [12], Pas [29], or others, or from a small variant of Proposition 5.2.1 of [18] (a result of Ax-Kochen-Eršov type that uses ultraproducts and follows from the theorem of Denef-Pas).

With a slight abuse of notation, for X a definable subassignment of $h[m,n,r]$ in $\mathrm{Def}(\mathscr{L}_{\mathbb{O}})$, we write X_K for the subassignment defined by ϑ_K where ϑ is a $\mathrm{Def}(\mathscr{L}_{\mathbb{O}})$-formula defining X, which is well determined for K in $\mathscr{C}_{\mathbb{O},N}$ for some large enough N, as explained above. Similarly, if $f : X \to Y$ is an $\mathscr{L}_{\mathbb{O}}$-definable morphism, we obtain a function $f_K : X_K \to Y_K$ for all K in $\mathscr{C}_{\mathbb{O},N}$ for some N.

With a similar abuse of notation, we can interpret a function φ in $\mathscr{C}(X, \mathscr{L}_{\mathbb{O}})$ as a function $X_K \to \mathbb{Q}$, for N large enough and K in $\mathscr{C}_{\mathbb{O},N}$, as follows.

First suppose that φ is in $K_0(\mathrm{RDef}_X(\mathscr{L}_{\mathbb{O}}))$ and of the form $[\pi : W \to X]$ for some $\mathscr{L}_{\mathbb{O}}$-definable subassignment W in $\mathrm{RDef}_X(\mathscr{L}_{\mathbb{O}})$. For K in $\mathscr{C}_{\mathbb{O}}$, consider W_K, which is a subset of $X_K \times (k_K)^\ell$ for some ℓ, and consider the natural projection $\pi_K : W_K \to X_K$. Then one sets

$$\varphi_K : \begin{cases} X_K \to \mathbb{Q} \\ x \mapsto \operatorname{card}\left(\pi_K^{-1}(x)\right). \end{cases} \tag{9.1.2}$$

Similarly as before, this is well determined for N large enough and K in $\mathscr{C}_{\mathbb{O},N}$. By linearity that construction extends to $K_0(\mathrm{RDef}_X(\mathscr{L}_{\mathbb{O}}))$.

Let us now define φ_K when φ lies in $\mathscr{P}(X)$. If one expresses φ in terms of $\mathbb{L}$ and of definable morphisms $\alpha : X \to \mathbb{Z}$, replacing $\mathbb{L}$ by q_K and each α by $\alpha_K : X_K \to \mathbb{Z}$, one gets a function $\varphi_K : X_K \to \mathbb{Q}$ again well determined for K in $\mathscr{C}_{\mathbb{O},N}$ when N is large enough. By tensor product, this defines φ_K for general φ in $\mathscr{C}(X, \mathscr{L}_{\mathbb{O}})$.

Next we interpret φ in $\mathscr{C}(X, \mathscr{L}_{\mathbb{O}})^{\exp}$ as a function $\varphi_{K,\psi_K} : X_K \to \mathbb{C}$, for K in $\mathscr{C}_{\mathbb{O},N}$ when N is large enough and for every ψ_K in $\mathscr{D}_K$, as follows.

First suppose that φ in $K_0(\mathrm{RDef}_X(\mathscr{L}_{\mathbb{O}}))^{\exp}$ is of the form $[W, g, \xi]$ with W an $\mathscr{L}_{\mathbb{O}}$-definable subassignment, where $g : W \to h[1,0,0]$ and $\xi : W \to h[0,1,0]$ are $\mathscr{L}_{\mathbb{O}}$-definable morphisms. For K in $\mathscr{C}_{\mathbb{O}}$, consider W_K, $g_K : W_K \to K$, and $\xi_K : W_K \to k_K$, and consider the projection $\pi : W_K \to X_K$. Then, for ψ_K in $\mathscr{D}_K$,

$$\varphi_{K,\psi_K} : \begin{cases} X_K \to \mathbb{Q} \\ x \mapsto \sum_{y \in \pi_K^{-1}(x)} \psi_K(g_K(y)) \exp((2\pi i/p)\mathrm{Tr}_{k_K}(\xi_K(y))). \end{cases} \tag{9.1.3}$$

Similarly, as before, this is well determined for N large enough and for all K in $\mathscr{C}_{\mathbb{O},N}$ and all ψ_K in $\mathscr{D}_K$. The construction being compatible with the previous one, this defines φ_{K,ψ_K} for general $\varphi \in \mathscr{C}(X, \mathscr{L}_{\mathbb{O}})^{\exp}$, by tensor product.

9.1.3. *Integration.* Let K be in $\mathscr{C}_{\mathbb{O}}$ and consider a (not necessarily definable) subset A of $K^m \times k_K^n \times \mathbb{Z}^r$. Let A' be the image of A under the projection $K^m \times k_K^n \times \mathbb{Z}^r \to K^m$ and define the dimension of A as the dimension of the Zariski closure of A' in $\mathbb{A}_K^m$ with $\dim \varnothing := -1$. Let $f : A \to \Lambda$ be any function, with Λ a subset of $K^{m'} \times k_K^{n'} \times \mathbb{Z}^{r'}$. The relative dimension of f is defined to be the maximum of the dimensions of the fibers; it is also called the relative dimension of A over Λ.

For A and A' as above, we denote by $\bar{A}'$ the Zariski closure of A' in $\mathbb{A}^m_K$. If $\bar{A}'$ is of dimension d, we consider the canonical d-dimensional measure μ^d on $\bar{A}'(K)$, cf. [31], [28], and put the counting measure on $k_K^n \times \mathbb{Z}^r$. We shall still write μ^d for the product measure on $\bar{A}'(K) \times k_K^n \times \mathbb{Z}^r$, formed by taking the product of the above μ^d with the counting measure, and we still denote by μ^d its restriction to A, similarly as in Section 8.

We denote by $\mathcal{F}(A)$ the algebra of all functions $A \to \mathbb{C}$. Also, we say a function φ in $\mathcal{F}(A)$ is integrable in dimension d if A and φ are measurable and φ is integrable with respect to the measure μ^d. More generally, we say that a function $\varphi \in \mathcal{F}(A)$ is integrable in dimension e if the support B of φ is of dimension e and the restriction $\varphi|_B$ is integrable in dimension e as defined above. For $e \geq 0$ an integer, we denote by $\mathcal{F}^{\leq e}(A)$ the ideal of $\mathcal{F}(A)$ of functions with support of dimension $\leq e$ and we set

$$F^e(A) := \mathcal{F}^{\leq e}(A)/\mathcal{F}^{\leq e-1}(A) \quad \text{and} \quad F(A) := \oplus_e F^e(A). \tag{9.1.4}$$

We define $\mathrm{I}F^e(A)$ as the subgroup of $F^e(A)$ consisting of functions in $F^e(A)$ which are integrable in dimension e and denote by $\mu : \mathrm{I}F^e(A) \to \mathbb{C}$ as the integration operator. We set $\mathrm{I}F(A) := \oplus_e \mathrm{I}F^e(A)$ and extend μ to $\mu : \mathrm{I}F(A) \to \mathbb{C}$ by linearity.

Let $f : A \to \Lambda$ be a mapping as before. Let $\mathcal{F}^{\leq e}(A \to \Lambda)$ be the ideal of $\mathcal{F}(A)$ of functions with support of relative dimension $\leq e$ over Λ. The groups $F^e(A \to \Lambda)$ and $F(A \to \Lambda)$ are defined correspondingly. For every λ in Λ, there is a natural restriction map, which is a graded group homomorphism,

$$|_{f^{-1}(\lambda)} : F_K(A \to \Lambda) \to F_K(f^{-1}(\lambda)), \tag{9.1.5}$$

defined by sending φ in $F_K(A \to \Lambda)$ to the restriction of φ to the fiber $f^{-1}(\lambda)$. We define $\mathrm{I}F_K(A \to \Lambda)$ as the graded subgroup of $F_K(A \to \Lambda)$ of Functions whose restrictions to all fibers lie in $\mathrm{I}F$, where restriction is as just defined, and we denote by μ_Λ the unique mapping

$$\mu_\Lambda : \mathrm{I}F(A \to \Lambda) \to \mathcal{F}(\Lambda) \tag{9.1.6}$$

such that $\mu_\Lambda(\varphi)(\lambda) = \mu(\varphi|_{f^{-1}(\lambda)})$ for every φ in $\mathrm{I}F(A \to \Lambda)$ and every point λ in Λ.

We still have to go one step further in the interpretation of Functions. Let $f : S \to \Lambda$ be a morphism in $\mathrm{Def}(\mathcal{L}_\mathcal{O})$. Let φ be in $C(S \to \Lambda, \mathcal{L}_\mathcal{O})^{\exp}$. The function φ is the class of a tuple $(\varphi_d)_d$ with φ_d in $\mathcal{C}^{\leq d}(S \to \Lambda, \mathcal{L}_\mathcal{O})^{\exp}$, where only finitely many components are nonzero. Then, for $N > 0$ large enough, $K \in \mathcal{C}_{\mathcal{O},N}$, and

$\psi_K \in \mathscr{D}_K$, each function φ_{d,K,ψ_K} lies in $\mathscr{F}_K^{\leq d}(S_K)$, and by taking the class of $(\varphi_{d,K,\psi_K})_d$, we get a function φ_{K,ψ_K} in $F_K(S_K \to \Lambda_K)$.

The following result says that the motivic exponential integral specializes to the corresponding integrals over the local fields of high enough residue field characteristic.

THEOREM 9.1.4 (Specialization Principle). *Let $f : S \to \Lambda$ be a morphism in* $\mathrm{Def}(\mathscr{L}_{\mathbb{O}})$. *Take φ in $\mathrm{I}C(S \to \Lambda, \mathscr{L}_{\mathbb{O}})^{\exp}$. Then there exists $N > 0$ such that for all K in $\mathscr{C}_{\mathbb{O},N}$, every choice of a uniformizing parameter ϖ_K of R_K, and all ψ_K in $\mathscr{D}_K$, the Function φ_{K,ψ_K} lies in $\mathrm{I}F_K(S_K \to \Lambda_K)$ and*

$$(\mu_\Lambda(\varphi))_{K,\psi_K} = \mu_{\Lambda_K}(\varphi_{K,\psi_K}).$$

Proof. Let us first consider the case where φ lies in $\mathrm{I}C(S \to \Lambda, \mathscr{L}_{\mathbb{O}})$. We can assume φ lies in $\mathrm{I}C_+(S \to \Lambda, \mathscr{L}_{\mathbb{O}})$, using notations from [11]. In [11], the definition of relative integrability of φ and the value of the relative integral were defined simultaneously along the following lines. One may assume S is a definable subassignment of $\Lambda[m,n,r] := \Lambda \times h[m,n,r]$ and, using cell decomposition and induction, it is enough, by Theorem 14.1.1 of [11] to consider the behavior of the integrability condition and the computation of the integral for: 1) projection along $\mathbb{Z}$-variables, 2) projection along residue field variables, 3) projections $\Lambda[m,n,r] \to \Lambda[m-1,n,r]$ when S is a 0-cell, 4) projections $\Lambda[m,n,r] \to \Lambda[m-1,n,r]$ when S is a 1-cell adapted to φ. Note that given φ the cell decompositions involved here will certainly specialize to cell decomposition defined by the specialized conditions when N is large enough. This is a special instance of the compactness argument in model theory. In 1), one can assume φ is a Presburger function, that is lies in $\mathscr{P}_+(S)$ with the notation of loc. cit. In that case, the integrability condition was built from the start to be compatible with specialization, since it was expressed by "summability with $\mathbb{L}$ replaced by $q > 1$". Also the relative integral was defined by summing up series in powers of $\mathbb{L}$ and specializes to summing over $\mathbb{Z}^r$ with respect to the counting measure. Step 2) is tautologically compatible with specialization. In step 3) a function $\mathbb{L}^{-\mathrm{ordjac} f}$, defined almost everywhere occurs, and for N large enough it specializes to $|\mathrm{Jac} f_K|$. By the change of variables formula for integrals over fields in $\mathscr{C}_{\mathbb{O}}$, it follows that 3) is compatible with specialization. Finally step 4) is compatible with specialization since the relative motivic volume of a 1-cell Z specializes to the volume of the corresponding Z_K, for N large enough, by definition.

When φ lies in $\mathscr{C}(S, \mathscr{L}_{\mathbb{O}})^{\exp}$, the statement about compatibility of relative integrability with specialization holds by the previous construction. The construction of the relative integral of φ can be performed along similar lines as before. Specialization for steps 1), 2) and 3) holds for the same reasons as before and only step 4) needs to be considered. It follows from our constructions that it is enough to show that the relative integral of the function $E(z)$ on a 1-cell with special coordinate z specializes to the corresponding one over K, for N large enough, which is clear by construction; see also Lemma 7.3.2. □

For p-adic fields, we can say more, using the formalism of Section 8.7.

THEOREM 9.1.5 (Specialization Principle). *Let* Λ *be in* $\mathrm{Def}(\mathcal{L}_{\mathbb{O}})$ *and let* $f : S \to S'$ *be a morphism in* $\mathrm{Def}_\Lambda(\mathcal{L}_{\mathbb{O}})$. *Let* φ *be in* $\mathrm{IC}(S \to \Lambda, \mathcal{L}_{\mathbb{O}})^{\mathrm{exp}}$. *Then there exists* $N > 0$ *such that for all* K *in* $\mathcal{A}_{\mathbb{O},N}$, *each choice of a uniformizing parameter* ϖ_K *of* R_K, *and all* ψ_K *in* $\mathcal{D}_K$, *the function* φ_{K,ψ_K} *lies in* $\mathrm{IC}(S_K \to \Lambda_K)^{\mathrm{exp}}$ *and is such that*

$$(f_{!\Lambda}(\varphi))_{K,\psi_K} = f_{!\Lambda_K}(\varphi_{K,\psi_K}).$$

Proof. This is similar to the proof of Theorem 9.1.4. □

9.2. *Transfer principle for integrals with parameters.* We start by proving the following abstract form of the transfer principle:

PROPOSITION 9.2.1. *Let* φ *be in* $\mathcal{C}(\Lambda, \mathcal{L}_{\mathbb{O}})^{\mathrm{exp}}$. *Then, there exists an integer* N *such that for all* K_1, K_2 *in* $\mathcal{C}_{\mathbb{O},N}$ *with* $k_{K_1} \simeq k_{K_2}$ *the following holds*:

$$\varphi_{K_1,\psi_{K_1}} = 0 \textit{ for all } \psi_{K_1} \in \mathcal{D}_{K_1}$$

if and only if

$$\varphi_{K_2,\psi_{K_2}} = 0 \textit{ for all } \psi_{K_2} \in \mathcal{D}_{K_2}.$$

Proof. We first consider the case when φ lies in $\mathcal{C}(\Lambda, \mathcal{L}_{\mathbb{O}})^{\mathrm{e}}$. Suppose that Λ is an $\mathcal{L}_{\mathbb{O}}$-definable subassignment of $h[m, n, r]$. We give a proof by induction on m. For $m = 0$, the proof goes as follows. By quantifier elimination, any finite set of formulas needed to describe φ can be taken to be valued field quantifier free. It follows that

$$\varphi_{K_1} = \varphi_{K_2} \tag{9.2.1}$$

for K_1 and K_2 in $\mathcal{C}_{\mathbb{O},N}$ with $k_{K_1} \simeq k_{K_2}$ and N large enough, since two ultraproducts $K = \prod_{\mathcal{U}} K_i$ and $K' = \prod_{\mathcal{U}} K'_i$ of fields K_i and K'_i in $\mathcal{C}_{\mathbb{O}}$ with $k_{K_i} \simeq k_{K'_i}$ over a nonprincipal ultrafilter $\mathcal{U}$ on a set I are elementarily equivalent, as soon as K and K' have characteristic zero.

Now assume $m > 0$. By applying inductively the Cell Decomposition Theorem 2.2.1, we can construct an $\mathcal{L}_{\mathbb{O}}$-definable morphism

$$f : \Lambda \to h[0, n', r'] \tag{9.2.2}$$

for some n', r', and $\tilde{\varphi} \in \mathcal{C}(h[0, n', r'], \mathcal{L}_{\mathbb{O}})^{\mathrm{e}}$, such that $\varphi = f^*(\tilde{\varphi})$. Necessarily, $\tilde{\varphi}$ is unique. By the induction hypothesis,

$$\tilde{\varphi}_{K_1} = 0 \text{ if and only if } \tilde{\varphi}_{K_2} = 0 \tag{9.2.3}$$

for K_1 and K_2 in $\mathscr{C}_{\mathbb{O},N}$ with $k_{K_1} \simeq k_{K_2}$ and N large enough. Since $\varphi_K = f_K^*(\tilde{\varphi}_K)$ for K in $\mathscr{C}_{\mathbb{O},N}$ when N is large enough, the result follows for general m and for φ in $\mathscr{C}(\Lambda, \mathscr{L}_{\mathbb{O}})^{\mathrm{e}}$.

In general, when φ lies in $\mathscr{C}(\Lambda, \mathscr{L}_{\mathbb{O}})^{\exp}$, we write φ as a finite sum of the form

$$\sum_{i=1}^{\ell} E(g_i)e(\xi_i)[X_i \to \Lambda]\varphi_i, \tag{9.2.4}$$

with $\varphi_i \in \mathscr{C}(\Lambda, \mathscr{L}_{\mathbb{O}})$.

After finitely partitioning Λ, we may suppose that there is a partition of $\{1, \dots, \ell\}$ into parts B_r such that

$$\operatorname{ord}(g_i(x_i) - g_j(x_j)) < 0 \tag{9.2.5}$$

for all $i \in B_{r_1}$, all $j \in B_{r_2}$, all $r_1 \neq r_2$, all $\lambda \in \Lambda$ and all $x_i \in X_i$, $x_j \in X_j$ lying above λ, and such that

$$\operatorname{ord}(g_i(x_i) - g_j(x_j)) \geq 0 \tag{9.2.6}$$

for all $i, j \in B_r$, all r, all $\lambda \in \Lambda$ and all $x_i \in X_i$, $x_j \in X_j$ lying above λ.

CLAIM 9.2.2. *There exists $N > 0$ such that for all K in $\mathscr{C}_{\mathbb{O},N}$ the statement*

$$\varphi_{K,\psi_K} = 0 \text{ for every } \psi_K \in \mathscr{D}_K \tag{9.2.7}$$

is equivalent to

$$\sum_{i \in B_r} (e(\xi_i)[X_i \to \Lambda]\varphi_i)_{K,\psi_K} = 0 \text{ for every } r \text{ and for every } \psi_K \in \mathscr{D}_K. \tag{9.2.8}$$

Since the left-hand side of (9.2.8) is in fact independent of the choice of character ψ_K, the proposition directly follows from the claim and the treatment of the case φ in $\mathscr{C}(\Lambda, \mathscr{L}_{\mathbb{O}})^{\mathrm{e}}$.

Let us now prove the claim. By compactness there exists N_0 such that for all $K \in \mathscr{C}_{\mathbb{O},N_0}$ the partition $\{B_r\}_r$ satisfies the following property: for every $i \in B_{r_1}$, every $j \in B_{r_2}$, every $r_1 \neq r_2$, every $\lambda \in \Lambda_K$ and every $x_i \in X_{iK}$, $x_j \in X_{jK}$ lying above λ,

$$\operatorname{ord}(g_{iK}(x_i) - g_{jK}(x_j)) < 0, \tag{9.2.9}$$

and, for every $i, j \in B_r$, every r, every $\lambda \in \Lambda_K$ and every $x_i \in X_{iK}$, $x_j \in X_{jK}$ lying above λ,

$$\operatorname{ord}(g_{iK}(x_i) - g_{jK}(x_j)) \geq 0. \tag{9.2.10}$$

Now the claim follows from Lemma 9.2.3. □

LEMMA 9.2.3. *Let K be in $\mathscr{C}_{\mathcal{O}}$. Let c_i be in $\mathbb{C}$ and $x_i \in K$ with* $\operatorname{ord}(x_i - x_j) < 0$ *for $i \neq j$, $i, j = 1, \dots, n$. For every ψ in $\mathscr{D}_K$ consider the exponential sum*

$$S_\psi := \sum_{i=1}^{n} c_i \psi(x_i).$$

Suppose that $S_\psi = 0$ for all ψ in $\mathscr{D}_K$. Then $c_i = 0$ for all i.

Proof. We shall perform an induction on $m := -\min_i(\operatorname{ord}(x_i))$. If $m = 0$ there is nothing to prove. So let us assume $m \geq 1$.

For every $n \geq 0$, we denote by $\mathscr{D}_K(n)$ the set of restrictions of the characters in $\mathscr{D}_K$ to the ball $\varpi_K^{-n} R$. We denote by p the characteristic of k_K and we set $\pi := \varpi_K$ if K is of characteristic p and $\pi := p$ if K is of characteristic 0. We fix elements $y_1, \dots, y_r$ of $\varpi_K^{-m} R$ whose images in $\varpi_K^{-m} R / \pi \varpi_K^{-m} R$ form an $\mathbb{F}_p$-basis. For $a = (a_1, \dots, a_r)$ in $\{0, \dots, p-1\}^r$, we denote by B_a the ball $\operatorname{ord}(x - \sum_j a_j y_j) \geq \operatorname{ord}(\pi \varpi_K^{-m})$. Let us fix ψ_0 in $\mathscr{D}_K(m-1)$. There are exactly p^r characters in $\mathscr{D}_K(m)$ extending ψ_0. Indeed, such characters are determined by their value on $y_1, \dots, y_r$, hence if we denote by $\zeta_{j,i}$, for $1 \leq i \leq p$, the p distinct complex numbers such that $\zeta_{j,i}^p = \psi_0(p y_j)$, they are in one to one correspondence with the set of tuples $(\zeta_{j,i})_j$, via $\psi \mapsto (\psi(y_j))$.

We may rewrite S_ψ as

$$S_\psi = \sum_{a \in \{0, \dots, p-1\}^r} \prod_{1 \leq j \leq r} \psi(y_j)^{a_j} S_{a, \psi_0} \tag{9.2.11}$$

with

$$S_{a,\psi_0} = \sum_{x_i \in B_a} c_i \psi_0\Big(x_i - \sum_{1 \leq j \leq r} a_j y_j\Big). \tag{9.2.12}$$

For fixed j, the $p \times p$-matrix $A_j := (\zeta_{j,i}^{\ell})_{i,\ell}$, $0 \leq \ell \leq p-1$, is an invertible Vandermonde matrix. It follows that the Kronecker (tensor) product matrix $A_1 \otimes \cdots \otimes A_r$ with coefficients $\prod_{1 \leq j \leq r} \zeta_{j,i}^{\ell_j}$, $0 \leq \ell_j \leq p-1$, is an invertible $p^r \times p^r$-matrix. Thus, the vanishing of S_ψ for every ψ in $\mathscr{D}_K(m)$ implies the vanishing of all the sums S_{a,ψ_0} for every ψ_0 in $\mathscr{D}_K(m-1)$, and the induction hypothesis allows us to conclude. □

Now we can prove the following fundamental transfer principle for exponential integrals:

THEOREM 9.2.4 (Transfer principle for exponential integrals). *Let $S \to \Lambda$ and $S' \to \Lambda$ be morphisms in* $\operatorname{Def}(\mathscr{L}_{\mathcal{O}})$. *Let φ be in* $\mathrm{I}C(S \to \Lambda, \mathscr{L}_{\mathcal{O}})^{\exp}$ *and φ' in* $\mathrm{I}C(S' \to \Lambda, \mathscr{L}_{\mathcal{O}})^{\exp}$. *Then, there exists an integer N such that for all K_1, K_2 in*

$\mathscr{C}_{\mathcal{O},N}$ *with* $k_{K_1} \simeq k_{K_2}$ *the following holds*:

$$\mu_{\Lambda_{K_1}}(\varphi_{K_1,\psi_{K_1}}) = \mu_{\Lambda_{K_1}}(\varphi'_{K_1,\psi_{K_1}}) \textit{ for all } \psi_{K_1} \in \mathscr{D}_{K_1}$$

if and only if

$$\mu_{\Lambda_{K_2}}(\varphi_{K_2,\psi_{K_2}}) = \mu_{\Lambda_{K_2}}(\varphi'_{K_2,\psi_{K_2}}) \textit{ for all } \psi_{K_2} \in \mathscr{D}_{K_2}.$$

Proof. By taking the disjoint union of S and S' over Λ and linearity, it is enough to prove the following particular case of the result: if $S \to \Lambda$ is a morphism in $\mathrm{Def}(\mathscr{L}_{\mathcal{O}})$ and φ is in $\mathrm{I}C(S \to \Lambda, \mathscr{L}_{\mathcal{O}})^{\exp}$, there exists an integer N such that for all K_1, K_2 in $\mathscr{C}_{\mathcal{O},N}$ with $k_{K_1} \simeq k_{K_2}$ the following holds:

$$\mu_{\Lambda_{K_1}}(\varphi_{K_1,\psi_{K_1}}) = 0 \text{ for all } \psi_{K_1} \in \mathscr{D}_{K_1}$$

if and only if

$$\mu_{\Lambda_{K_2}}(\varphi_{K_2,\psi_{K_2}}) = 0 \text{ for all } \psi_{K_2} \in \mathscr{D}_{K_2},$$

which follows directly from Theorem 9.1.4 and Proposition 9.2.1. □

Remark 9.2.5. Without exponentials, a form of Theorem 9.2.4 can be found in [9]. As mentioned in the introduction, it should have a wide range of applications to p-adic representation theory and the Langlands program. It applies in particular to many forms of the Fundamental Lemma. In this direction, we mention work by Cunningham and Hales [13], and recall that the Fundamental Lemma over functions fields has been proved by Laumon and Ngô for unitary groups [25] and more recently by Ngô for Lie algebras [27], and that Waldspurger deduced the case of p-adic fields [33] by representation theoretic techniques. In the paper [6], we explain in detail how our transfer Theorem 9.2.4 applies to the Fundamental Lemma. Theorem 9.2.4 applies also, for instance, to the Jacquet-Ye conjecture [24], a relative version of the Fundamental Lemma involving integrals of additive characters, which has been proved by Ngô [26] over functions fields and by Jacquet [23] in general.

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