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Abstract

We prove that for a large class of subvarieties of abelian varieties over global function fields, the Brauer-Manin condition on adelic points cuts out exactly the rational points. This result is obtained from more general results concerning the intersection of the adelic points of a subvariety with the adelic closure of the group of rational points of the abelian variety.

1. Introduction

The notation in this section remains in force throughout the paper, except in Section 3.3, and in Section 4 where we allow also the possibility that K is a number field.

Let k be a field. Let K be a finitely generated extension of k of transcendence degree 1. We assume that k is relatively algebraically closed in K, since the content of our theorems will be unaffected by this assumption. Let $\bar{K}$ be an algebraic closure of K. We will use this notation consistently for an algebraic closure, and we will choose algebraic closures compatibly whenever possible. Thus $\bar{k}$ is the algebraic closure of k in $\bar{K}$. Let K^s be the separable closure of K in $\bar{K}$. Let Ω_{all} be the set of all nontrivial valuations of K that are trivial on k. Let Ω be a cofinite subset of Ω_{all}. If k is finite, we may weaken the cofiniteness hypothesis to assume only that $\Omega \subseteq \Omega_{\text{all}}$ has Dirichlet density 1. For each $v \in \Omega$, let K_v be the completion of K at v, and let $\mathbf{F}_v$ be the residue field. Equip K_v with the v-adic topology. Define the ring of adèles $\mathbf{A}$ as the restricted direct product $\prod_{v\in\Omega}(K_v, \mathbb{O}_v)$ of the K_v with respect to their valuation subrings $\mathbb{O}_v$. Then $\mathbf{A}$ is a topological ring, in which $\prod_{v\in\Omega}\mathbb{O}_v$ is open and has the product topology.

If A is an abelian variety over K, then $A(K)$ embeds diagonally into $A(\mathbf{A}) \simeq \prod_v A(K_v)$. Define the adelic topology on $A(K)$ as the topology induced from

$A(\mathbf{A})$. For any fixed v define the v-adic topology on $A(K)$ as the topology induced from $A(K_v)$. Let $\overline{A(K)}$ be the closure of $A(K)$ in $A(\mathbf{A})$.

For any extension of fields $F' \supset F$ and any F-variety X, let $X_{F'}$ be the base extension of X to F'. Call a K-variety X *constant* if $X \simeq Y_K$ for some k-variety Y, and call X *isotrivial* if $X_{\overline{K}} \simeq Y_{\overline{K}}$ for some variety Y defined over $\overline{k}$.

From now on, X is a closed K-subscheme of A. Call X *coset-free* if $X_{\overline{K}}$ does not contain a translate of a positive-dimensional abelian subvariety of $A_{\overline{K}}$.

When k is finite and $\Omega = \Omega_{\text{all}}$, the intersection $X(\mathbf{A}) \cap \overline{A(K)} \subset A(\mathbf{A})$ is closely related to the Brauer-Manin obstruction to the Hasse principle for X/K; see Section 4. For curves over number fields, V. Scharaschkin and A. Skorobogatov independently raised the question of whether the Brauer-Manin obstruction is the only obstruction to the Hasse principle, and proved that this is so when the Jacobian has finite Mordell-Weil group and finite Shafarevich-Tate group. The connection with the adelic intersection is stated explicitly in [Sch99], and is based on global duality statements originating in the work of Cassels: see Remark 4.4. See also [Sko01], [Fly04], [Poo06], and [Sto07], which contains many conjectures and theorems relating descent information, the method of Chabauty and Coleman, the Brauer-Manin obstruction, and Grothendieck's section conjecture.

In this paper we answer (most cases of) a generalization of the function field analogue of a question raised for curves over number fields in [Sch99], concerning whether the Brauer-Manin condition cuts out exactly the rational points; see Theorem D. This question is still wide open in the number field case. Along the way, we prove results about adelic intersections similar to the "adelic Mordell-Lang conjecture" suggested in [Sto07, Question 3.12]. Again, these are open in the number field case. In particular, we prove the following theorems.

THEOREM A. *If* $\operatorname{char} k = 0$, *then* $X(K) = X(\mathbf{A}) \cap \overline{A(K)}$.

THEOREM B. *Suppose that* $\operatorname{char} k = p > 0$, *that* $A_{\overline{K}}$ *has no nonzero isotrivial quotient, and that* $A(K^s)[p^\infty]$ *is finite. Suppose that* X *is coset-free. Then* $X(K) = X(\mathbf{A}) \cap \overline{A(K)}$.

Remark 1.1. The proposition in [Vol95] states that in the "general case" in which A is ordinary and the Kodaira-Spencer class of A/K has maximal rank, we have $A(K^s)[p^\infty] = 0$.

CONJECTURE C. *For any closed* K*-subscheme* X *of any* A, *we have* $\overline{X(K)} = X(\mathbf{A}) \cap \overline{A(K)}$, *where* $\overline{X(K)}$ *is the closure of* $X(K)$ *in* $X(\mathbf{A})$.

Remark 1.2. If $A_{\overline{K}}$ has no nonzero isotrivial quotient and X is coset-free, then $X(K)$ is finite [Hru96, Th. 1.1]; thus $\overline{X(K)} = X(K)$. Hence Conjecture C predicts in particular that the hypothesis on $A(K^s)[p^\infty]$ in Theorem B is unnecessary.

Remark 1.3. Here is an example to show that the statement $X(K) = X(\mathbf{A}) \cap \overline{A(K)}$ can fail for a constant curve in its Jacobian. Let X be a curve of genus ≥ 2 over a finite field k. Choose a divisor of degree 1 on X to embed X in its Jacobian A. Let $F\colon A \to A$ be the k-Frobenius map. Let K be the function field of X. Let $P \in X(K)$ be the point corresponding to the identity map $X \to X$. Let $P_v \in X(\mathbf{F}_v)$ be the reduction of P at v.

For each v, the Teichmüller map $\mathbf{F}_v \to K_v$ identifies $\mathbf{F}_v$ with a subfield of K_v. Any $Q \in A(K_v)$ can be written as $Q = Q_0 + Q_1$ with $Q_0 \in A(\mathbf{F}_v)$ and Q_1 in the kernel of the reduction map $A(K_v) \to A(\mathbf{F}_v)$; then $\lim_{m\to\infty} F^m(Q_1) = 0$, so $\lim_{n\to\infty} F^{n!}(Q) = \lim_{n\to\infty} F^{n!}(Q_0) = Q_0$. In particular, taking $Q = P$, we find that $\left(F^{n!}(P)\right)_{n\geq 1}$ converges in $A(\mathbf{A})$ to the point $(P_v) \in X(\mathbf{A}) = \prod_v X(K_v)$, where we have identified P_v with its image under the Teichmüller map $X(\mathbf{F}_v) \hookrightarrow X(K_v)$. If (P_v) were in $X(K)$, then in $X(K_v)$ we would have $P_v \in X(\mathbf{F}_v) \cap X(K) = X(k)$, which contradicts the definition of P_v if v is a place of degree greater than 1 over k. Thus (P_v) is in $X(\mathbf{A}) \cap \overline{A(K)}$ but not in $X(K)$.

In the final section of this paper, we restrict to the case of a global function field, and extend Theorem B to prove (under mild hypotheses) that for a subvariety of an abelian variety, the Brauer-Manin condition cuts out exactly the rational points; see Section 4 for the definitions of $X(\mathbf{A})^{\mathrm{Br}}$ and $\widehat{\mathrm{Sel}}$. Our result is as follows.

THEOREM D. *Suppose that K is a global function field of characteristic p, that $A_{\bar{K}}$ has no nonzero isotrivial quotient, and that $A(K^s)[p^\infty]$ is finite. Suppose that X is coset-free. Then $X(K) = X(\mathbf{A})^{\mathrm{Br}} = X(\mathbf{A}) \cap \widehat{\mathrm{Sel}}$.*

To our knowledge, Theorem D is the first result giving a wide class of varieties of general type such that the Brauer-Manin condition cuts out exactly the rational points.

2. Characteristic 0

Throughout this section, we assume $\operatorname{char} k = 0$. In this case, results follow rather easily.

PROPOSITION 2.1. *For any v, the v-adic topology on $A(K)$ equals the discrete topology.*

Proof. The question is isogeny-invariant, so we reduce to the case where A is simple. Let $A(\mathbf{F}_v)$ denote the group of $\mathbf{F}_v$-points on the Néron model of A over $\mathcal{O}_v$. Let $A^1(K_v)$ be the kernel of the reduction map $A(K_v) \to A(\mathbf{F}_v)$. The Lang-Néron theorem [LN59, Th. 1] implies that either A is constant and $A(K)/A(k)$ is finitely generated, or A is nonconstant and $A(K)$ itself is finitely generated. In either case, the subgroup $A^1(K) := A(K) \cap A^1(K_v)$ is finitely generated. By the theory of formal groups (cf. [Ser92, p. 118, Th. 2]), $A^1(K_v)$ has a descending filtration by open subgroups in which the quotients of consecutive terms are torsion-free (this is where we use $\operatorname{char} k = 0$), so the induced filtration on the finitely generated group $A^1(K)$ has only finitely many nonzero quotients. Thus $A^1(K)$ is discrete. Since $A^1(K_v)$ is open in $A(K_v)$, the group $A(K)$ is discrete in $A(K_v)$. □

Remark 2.2. The literature contains results close to Proposition 2.1. It is mentioned in the third subsection of the introduction to [Man63a] for elliptic curves with nonconstant j-invariant, and it appears for abelian varieties with K/k-trace zero in [BV93].

COROLLARY 2.3. *The adelic topology on $A(K)$ equals the discrete topology.*

Proof. The adelic topology is at least as strong as (i.e., has at least as many open sets as) the v-adic topology for any v. □

We can improve the result by imposing conditions in only the residue fields $\mathbf{F}_v$ instead of the completions K_v, that is, "flat" instead of "deep" information in the sense of [Fly04]. In fact, we have:

PROPOSITION 2.4. *There exist $v, v' \in \Omega$ of good reduction for A such that $A(K) \to A(\mathbf{F}_v) \times A(\mathbf{F}_{v'})$ is injective.*

Proof. Let B be the K/k-trace of A. Pick any $v \in \Omega$ of good reduction. The kernel H of $A(K) \to A(\mathbf{F}_v)$ meets $B(k)$ trivially. By Silverman's specialization theorem [Lan83, Ch. 12, Th. 2.3], there exists $v' \in \Omega$ such that H injects under reduction modulo v'. □

Proof of Theorem A. By Corollary 2.3, $X(\mathbf{A}) \cap \overline{A(K)} = X(\mathbf{A}) \cap A(K) = X(K)$. □

3. Characteristic p

Throughout this section, $\operatorname{char} k = p$.

3.1. *Field-theoretic lemmas.*

LEMMA 3.1. *For any v, if $\alpha \in K_v$ is algebraic over K, then α is separable over K.*

Proof. Replacing K by its relative separable closure in $L := K(\alpha)$, we may assume that L is purely inseparable over K. Then the valuation v on K admits a unique extension w to L, and the inclusion of completions $K_v \to L_w$ is an isomorphism. By [Ser79, I.§4, Prop. 10] (loc. cit. Hypothesis (F) holds for localizations of finitely generated algebras over a field), we have an "$n = \sum e_i f_i$" result, which in our case says $[L : K] = [L_w : K_v] = 1$. So $\alpha \in K$. □

If L is a finite extension of K, let $\mathbf{A}_L$ be the corresponding ring of adèles, defined as a restricted direct product over places of L. There is a natural inclusion $\mathbf{A} \hookrightarrow \mathbf{A}_L$.

LEMMA 3.2. *Let L be a finite extension of K. Then in $\mathbf{A}_L$ we have $\mathbf{A} \cap L = K$.*

Proof. Fix $v \in \Omega$. By [Bou98, VI.§8.5, Cor. 3] and the fact that [Ser79, Hypothesis (F)] holds, the natural map $K_v \otimes_K L \to \prod_{w|v} L_w$ is an isomorphism. Hence in $\prod_{w|v} L_w$ we have $K_v \cap L = K$. The result follows. □

3.2. *Abelian varieties.*

LEMMA 3.3. *For any $n \in \mathbf{Z}_{\geq 1}$, the quotient $A(K_v)/nA(K_v)$ is Hausdorff.*

Proof. Equivalently, we must show that $nA(K_v)$ is closed in $A(K_v)$. Suppose (P_i) is a sequence in $nA(K_v)$ that converges to $P \in A(K_v)$. Write $P_i = nQ_i$ with $Q_i \in A(K_v)$. Then $n(Q_i - Q_{i+1}) \to 0$ as $i \to \infty$.

Let $\mathcal{O}_v$ be the valuation ring of K_v, and let $\mathcal{A}$ be the Néron model of A over $\mathcal{O}_v$. Applying [Gre66, Cor. 1] to $\mathcal{A}[n]$ shows that for any sequence (R_i) in $A(K_v)$ with $nR_i \to 0$, the distance of R_i to the nearest point of $A(K_v)[n]$ tends to 0.

Thus by induction on i we may adjust each Q_i by a point in $A(K_v)[n]$ so that $Q_i - Q_{i+1} \to 0$ as $i \to \infty$. Since $A(K_v)$ is complete, (Q_i) converges to some $Q \in A(K_v)$, and $nQ = P$. Thus $nA(K_v)$ is closed. □

Remark 3.4. In the case where k is finite, Lemma 3.3 is immediate since $A(K_v)$ is compact and its image under multiplication-by-n is closed.

The following is a slight generalization of [Vol95, Lemma 2], with a more elementary proof.

PROPOSITION 3.5. *If $A(K^s)[p^\infty]$ is finite, then for any v, the v-adic topology on $A(K)$ is at least as strong as the topology induced by all subgroups of finite p-power index.*

Proof. For convenience choose algebraic closures $\bar{K}, \bar{K}_v$ of K, K_v such that $K^s \subseteq \bar{K} \subseteq \bar{K}_v$. As in the proof of Proposition 2.1, there is an open subgroup U of $A(K_v)$ such that $B := A(K) \cap U$ is finitely generated. It suffices to show that for every $e \in \mathbf{Z}_{\geq 0}$, there exists an open subgroup V of $A(K_v)$ such that $B \cap V \subseteq p^e A(K)$.

Choose m such that $p^m A(K^s)[p^\infty] = 0$. Let $M = e + m$. Then $B/p^M B$ is finite. By Lemma 3.3, $A(K_v)/p^M A(K_v)$ is Hausdorff, so the image of $B/p^M B$ in $A(K_v)/p^M A(K_v)$ is discrete. Hence there is an open subgroup V of $A(K_v)$ such that $B \cap V = \ker(B \to A(K_v)/p^M A(K_v))$.

Suppose $b \in B \cap V$. Then $b = p^M c$ for some $c \in A(K_v) \cap A(\bar{K})$. By Lemma 3.1, we obtain $c \in A(K^s)$. If $\sigma \in \operatorname{Gal}(K^s/K)$, then ${}^\sigma c - c \in A(K^s)[p^M]$

is killed by p^m. Thus $p^m c \in A(K)$. So $b = p^e p^m c \in p^e A(K)$. Hence $B \cap V \subseteq p^e A(K)$. ☐

PROPOSITION 3.6. *The adelic topology on $A(K)$ is at least as strong as the topology induced by all subgroups of finite index.*

Proof. As in the proof of Proposition 2.1, the Lang-Néron theorem implies that $A(\mathbf{A})$ has an open subgroup whose intersection with $A(K)$ is finitely generated. It suffices to study the topology induced on that finitely generated subgroup, so we may reduce to the case in which k is finitely generated over a finite field $\mathbf{F}_q$. This case is proved in [Mil72], which adapts and extends [Ser64] and [Ser71]. (The paper [Mil72] uses not the adelic topology as we have defined it, but the topology coming from the closed points of a finite-type $\mathbf{Z}$-scheme with function field K. Since the adelic topology is stronger, [Mil72] contains what we want.) ☐

LEMMA 3.7. *Suppose that $A(K)$ is finitely generated. Then $\left(\overline{A(K)}\right)_{\text{tors}} = A(K)_{\text{tors}}$.*

Proof. Let

$$T := \ker\left(A(K) \to \prod_{v \in \Omega} \frac{A(\mathbf{F}_v)}{A(\mathbf{F}_v)_{\text{tors}}}\right),$$

where $A(\mathbf{F}_v)$ is the group of $\mathbf{F}_v$-points on the Néron model of A. Since $A(K)$ is finitely generated and the groups $A(\mathbf{F}_v)/A(\mathbf{F}_v)_{\text{tors}}$ are torsion-free, there is a finite subset $S \subset \Omega$ such that $T = A(K) \cap U$ for the open subgroup

$$U := \ker\left(A(\mathbf{A}) \to \prod_{v \in S} \frac{A(\mathbf{F}_v)}{A(\mathbf{F}_v)_{\text{tors}}}\right)$$

of $A(\mathbf{A})$. The finitely generated group $A(K)/T$ is contained in the torsion-free group $\prod_{v \in S} \frac{A(\mathbf{F}_v)}{A(\mathbf{F}_v)_{\text{tors}}}$, so $A(K)/T$ is free, and we have $A(K) \simeq T \oplus F$ as topological groups, where F is a discrete free abelian group of finite rank.

We claim that the topology of T is that induced by the subgroups nT for $n \geq 1$. For $n \geq 1$, the subgroup nT is open in T by Proposition 3.6. If $t \in T$, then some positive multiple of t is in the kernel of $A(K_v) \to A(\mathbf{F}_v)$, and then p-power multiples of this multiple tend to 0. Applying this to a finite set of generators of T, we see that any open neighborhood of 0 in T contains nT for some $n \in \mathbf{Z}_{>0}$.

It follows that $\overline{T} \simeq T \otimes \hat{\mathbf{Z}}$. Now

$$\left(\overline{A(K)}\right)_{\text{tors}} = (\overline{T} \oplus F)_{\text{tors}} = \overline{T}_{\text{tors}} \simeq (T \otimes \hat{\mathbf{Z}})_{\text{tors}} \simeq T_{\text{tors}}. \qquad ☐$$

Remark 3.8. When k is finite, an easier proof of Lemma 3.7 is possible: Combined with the fact that $A(\mathbf{A})$ is profinite, Proposition 3.6 implies that $\overline{A(K)} \simeq A(K) \otimes \hat{\mathbf{Z}}$; the torsion subgroup of the latter equals $A(K)_{\text{tors}}$.

The following proposition is a function field analogue of [Sto07, Prop. 3.6]. Our proof must be somewhat different, however, since [Sto07] made use of strong "image of Galois" theorems whose function field analogues have recently been disproved [Zar07].

PROPOSITION 3.9. *Suppose that $A(K^s)[p^\infty]$ is finite. Let Z be a finite K-subscheme of A. Then $Z(\mathbf{A}) \cap \overline{A(K)} = Z(K)$.*

Proof. In this first paragraph we show that replacing K by a finite extension L does not destroy the hypothesis that $A(K^s)[p^\infty]$ is finite. This is obvious if L is separable over K, so assume that L is purely inseparable. Choose $n \in \mathbf{Z}_{\geq 0}$ with $L^{p^n} \subseteq K$. Then $(L^s)^{p^n} \subseteq K^s$, so $p^n A(L^s)[p^\infty] \subseteq A(K^s)[p^\infty]$. Thus $p^n A(L^s)[p^\infty]$ is finite. But multiplication-by-p^n has finite fibers, so $A(L^s)[p^\infty]$ itself is finite.

Next we claim that if we prove the conclusion after base extension to a finite extension L, then the desired conclusion over K holds. Namely, suppose that we prove $Z(\mathbf{A}_L) \cap \overline{A(L)} = Z(L)$. Then

$$Z(\mathbf{A}) \cap \overline{A(K)} \subseteq Z(\mathbf{A}_L) \cap \overline{A(L)} = Z(L),$$

so

$$Z(\mathbf{A}) \cap \overline{A(K)} \subseteq Z(\mathbf{A}) \cap Z(L) = Z(K),$$

where the last equality uses Lemma 3.2.

Thus we may replace K by a finite extension to assume that Z consists of a finite set of K-points of A. (The same idea was used in [Sto07].) A point $P \in \overline{A(K)}$ is represented by a sequence $(P_n)_{n\geq 1}$ in $A(K)$ such that for every v, the limit $\lim_{n\to\infty} P_n$ exists in $A(K_v)$. If in addition $P \in Z(\mathbf{A})$, then there is a point $Q_v \in Z(K)$ whose image in $Z(K_v)$ equals $\lim_{n\to\infty} P_n \in A(K_v)$. The $P_n - Q_v$ are eventually contained in the kernel of $A(K) \to A(\mathbf{F}_v)$, which is finitely generated, so there are finitely generated subfields $k_0 \subseteq k$, $K_0 \subseteq K$ with K_0/k_0 a function field such that all the P_n and the points of $Z(K)$ are in $A(K_0)$. By Proposition 3.5, the sequence $(P_n - Q_v)_{n\geq 1}$ is eventually divisible in $A(K_0)$ by an arbitrarily high power of p. For any other $v' \in \Omega$, the same is true of $(P_n - Q_{v'})_{n\geq 1}$. Then $Q_{v'} - Q_v \in A(K_0)$ is divisible by every power of p. Since $A(K_0)$ is finitely generated, $Q_{v'} - Q_v$ is a torsion point in $A(K_0)$. This holds for every $v' \in \Omega$, and $A(K_0)_{\text{tors}}$ is finite. Thus $R := P - Q_v \in \overline{A(K_0)}$ is a torsion point in $\overline{A(K_0)}$. Lemma 3.7 applied to K_0 yields $R \in A(K_0)_{\text{tors}}$. Hence $P = R + Q_v \in A(K)$, and so $P \in Z(\mathbf{A}) \cap A(K) = Z(K)$. □

LEMMA 3.10. *Fix $v \in \Omega$. Let Γ_v be the closure of $A(K)$ in $A(K_v)$. Then for every $e \in \mathbf{Z}_{\geq 0}$, the map $A(K)/p^e A(K) \to \Gamma_v / p^e \Gamma_v$ is surjective.*

Proof. Let $\mathcal{O}_v$ be the valuation ring of K_v, and let $\mathfrak{m}_v$ be its maximal ideal. Let $\mathscr{A}$ over $\mathcal{O}_v$ be the Néron model. For $r \in \mathbf{Z}_{\geq 1}$, let G_r be the kernel of $A(K_v) =$

$\mathscr{A}(\mathbb{O}_v) \to \mathscr{A}(\mathbb{O}_v/\mathfrak{m}_v^r)$. It follows from [Ser92, p. 118, Th. 2] that G_r/G_{r+1} is isomorphic to $(\mathbb{O}_v/\mathfrak{m}_v)^{\dim A}$, which is killed by p, so that each G_r is an abelian pro-p-group, and hence a topological $\mathbf{Z}_p$-module. There are only finitely many points of order p in $A(K_v)$, and $\bigcap_{r\geq 1} G_r = \{0\}$, so some G_r contains no nontrivial p-torsion points, and hence is torsion-free. In particular, $A(K_v)$ has an open subgroup $A^\circ(K_v)$ that is a torsion-free topological $\mathbf{Z}_p$-module, and we may choose $A^\circ(K_v)$ so that $A^\circ(K) := A(K) \cap A^\circ(K_v)$ is finitely generated.

The group $\Gamma_v^\circ := \Gamma_v \cap A^\circ(K_v)$ is the closure of $A^\circ(K)$, so there is an isomorphism of topological groups $\Gamma_v^\circ \cong \mathbf{Z}_p^{\oplus m}$ for some $m \in \mathbf{Z}_{\geq 0}$. In particular, for any $e \in \mathbf{Z}_{\geq 0}$, the group $p^e\Gamma_v^\circ$ is open in Γ_v°, which is open in Γ_v. So the larger group $p^e\Gamma_v$ also is open in Γ_v. But the image of $A(K)$ in the discrete group $\Gamma_v/p^e\Gamma_v$ is dense, so the map $A(K)/p^eA(K) \to \Gamma_v/p^e\Gamma_v$ is surjective. □

3.3. *A uniform Mordell-Lang conjecture.* We thank Zoé Chatzidakis, Françoise Delon, and Thomas Scanlon for many of the ideas used in this section. See [Del98] for the definitions of *separable*, *p-basis*, *p-free*, *p-components*, etc. By *iterated p-components* we mean p-components of p-components of ... of p-components (all with respect to a given p-basis).

The goal of this section is to deduce a uniform version (Proposition 3.16) of the function field Mordell-Lang conjecture from a version in [Hru96]. Under some hypotheses, the uniform version asserts the finiteness of the intersection of a subvariety X of an abelian variety A with any coset of a subgroup $p^eA(F)$ of $A(F)$, where F is allowed to range over p-basis-preserving extensions of an initial ground field K.

Remark 3.11. The p-basis condition on F, or something like it, is necessary for the truth of Proposition 3.16; with no condition, F might be algebraically closed, and then $p^eA(F) = A(F)$, so the desired finiteness would fail assuming $\dim X > 0$. The p-basis condition is used in the proof of Proposition 3.16 to imply separability of F over K, which guarantees that a nonisotriviality hypothesis on A over K is preserved by base extension to F; see Lemma 3.13 and its proof.

LEMMA 3.12. *Let $\mathscr{B}$ be a p-basis for a field K of characteristic p. Let L be an extension of K such that $\mathscr{B}$ is also a p-basis for L. Suppose that c is an element of L that is not algebraic over K. Then there exists a separably closed extension F of L such that $\mathscr{B}$ is a p-basis of F and the* $\operatorname{Aut}(F/K)$*-orbit of c is infinite.*

Proof. Fix a transcendence basis T for L/K. Let Ω be an algebraically closed extension of K such that the transcendence basis of Ω/K is identified with the set $\mathbf{Z} \times T$. Identify L with a subfield of Ω in such a way that each $t \in T$ maps to the transcendence basis element for Ω/K labelled by $(0,t) \in \mathbf{Z} \times T$. The map of sets $\mathbf{Z} \times T \to \mathbf{Z} \times T$ mapping (i,t) to $(i+1,t)$ extends to an automorphism

$\sigma \in \mathrm{Aut}(\Omega/K)$. For $i \in \mathbf{Z}$, let $L_i = \sigma^i(L)$. Let L_∞ be the compositum of the L_i in Ω. Then $\sigma(L_\infty) = L_\infty$. Let F be the separable closure of L_∞ in Ω. Thus $\sigma(F) = F$. The σ-orbit of c is infinite, since $L_i \cap L_j$ is algebraic over K whenever $i \neq j$.

The p-basis hypothesis implies that L is separable over K. Applying σ^i shows that L_i is separable over K. Moreover, the L_i are algebraically disjoint over K, so their compositum L_∞ is separable over K, by the last corollary in [Bou03, V.§16.7] and Proposition 3(b) in [Bou03, V.§15.2]. Thus $\mathscr{B}$ is p-free in L_∞.

The p-basis hypothesis also implies that $L = L^p(\mathscr{B})$. Thus $L_i = L_i^p(\mathscr{B})$ for all $i \in \mathbf{Z}$, and $L_\infty = L_\infty^p(\mathscr{B})$.

Combining the previous two paragraphs shows that $\mathscr{B}$ is a p-basis for L_∞, and hence also for F. □

LEMMA 3.13. *Let $K \subseteq F$ be a separable extension such that the field $K^{p^\infty} := \bigcap_{n\geq 1} K^{p^n}$ is algebraically closed. Let A be an abelian variety over K such that no nonzero quotient of $A_{\overline{K}}$ is the base extension of an abelian variety over K^{p^∞}. Then no nonzero quotient of $A_{\overline{F}}$ is the base extension of an abelian variety over F^{p^∞}.*

Proof. Suppose not. Thus there exists a nonzero abelian variety B over F^{p^∞} and a surjective homomorphism $\phi\colon A_{\overline{F}} \to B_{\overline{F}}$. Choose a finitely generated extension F_0 of K^{p^∞} over which B is defined. Whenever A and B are abelian varieties over a field L, any homomorphism $A \to B$ is definable over $L(A[\ell^\infty], B[\ell^\infty])$ for any prime $\ell \neq \operatorname{char} L$; thus, in our situation, ϕ is definable over a finite separable extension F_1 of KF_0. Since K^{p^∞} is algebraically closed, we may choose a place $F_0 \dashrightarrow K^{p^\infty}$ extending the identity on K^{p^∞}, such that B has good reduction at this place. By [Del98, Fact 1.4], K and F^{p^∞} are linearly disjoint over K^{p^∞}, so the place $F_0 \dashrightarrow K^{p^\infty}$ extends to a place $KF_0 \dashrightarrow K$ that is the identity on K, and then to a place $F_1 \dashrightarrow K_1$ for some finite extension K_1 of K. Reduction of $\phi\colon A_{F_1} \to B_{F_1}$ yields a homomorphism $A_{K_1} \to B_{K_1}$, and since the place restricted to F_0 has values in K^{p^∞}, the abelian variety B_{K_1} is the base extension of an abelian variety over K^{p^∞}. This contradicts the hypothesis on A. □

LEMMA 3.14 (a version of the Mordell-Lang conjecture). *Let F be a separably closed field of characteristic p. Suppose that A is an abelian variety over F such that no nonzero quotient of $A_{\overline{F}}$ is the base extension of an abelian variety over F^{p^∞}. Suppose that X is a coset-free closed F-subvariety of A. Then there exists $e \in \mathbf{Z}_{\geq 1}$ such that for every $a \in A(F)$, the intersection $X(F) \cap (a + p^e A(F))$ is finite.*

Proof. This is a special case of [Hru96, Lemma 6.2]. □

Using Lemma 3.14 and the compactness theorem in model theory, we can deduce a version of the Mordell-Lang conjecture that is more uniform as we vary the ground field, Proposition 3.16.

But first we introduce some notation. Suppose that K is a field of characteristic p such that $[K : K^p] < \infty$. Let F be any field extension with the same p-basis. Suppose that A is an abelian variety over K. Let R_n be the restriction of scalars of A from K to K^{p^n}, so that R_n is an abelian variety over K^{p^n}. Let A_n be the base extension of R_n by the isomorphism $K^{p^n} \to K$ that takes an element to its p^n-th root, so that $A_n(F) = R_n(F^{p^n}) = A(F)$. For $n \in \mathbf{Z}_{\geq 0}$, we have a natural morphism $A_{n+1} \to A_n$ such that $A_{n+1}(F) \to A_n(F)$ is compatible with the identifications $A_{n+1}(F) = A(F)$ and $A_n(F) = A(F)$.

Suppose moreover that X is a K-variety. Let $\mathcal{Y}_n$ be the set of (not necessarily closed) K-subvarieties $Y \subseteq A_n \times X$ such that the projection from Y to A_n has finite fibers. For such Y, use the identification $A_n(F) = A(F)$ to view $Y(F)$ as a subset of $A(F) \times X(F)$. Taking inverse images under $A_{n+1} \times X \to A_n \times X$ defines a map of sets $\mathcal{Y}_n \to \mathcal{Y}_{n+1}$. Let $\mathcal{Y} = \varinjlim \mathcal{Y}_n$. For $Y \in \mathcal{Y}$, the set $Y(F) \subseteq A(F)$ is independent of which $\mathcal{Y}_n$ we consider Y as coming from. Each $\mathcal{Y}_n$ is closed under taking finite unions of elements, so the same is true of $\mathcal{Y}$.

Remark 3.15. Alternatively, one can think of Y as a "variety" defined not only by polynomial equations and inequations in the coordinates on A and X, but also by polynomial equations and inequations in the iterated p-components of coordinates on A and usual coordinates on X.

PROPOSITION 3.16 (a more uniform version of the Mordell-Lang conjecture). *Let k be an algebraically closed field of characteristic p. Let K be a finitely generated extension of k. Fix a (finite) p-basis $\mathcal{B}$ of K. Suppose that A is an abelian variety over K such that no nonzero quotient of $A_{\overline{K}}$ is the base extension of an abelian variety over k. Suppose that X is a coset-free closed K-subscheme of A. Define $\mathcal{Y}$ as in the preceding paragraph. Then there exists $e \in \mathbf{Z}_{\geq 1}$ and $Y \in \mathcal{Y}$ such that for every field extension $F \supseteq K$ having $\mathcal{B}$ as p-basis, if $a \in A(F)$ and $c \in X(F) \cap (a + p^e A(F))$, then $(a, c) \in Y(F)$.*

Proof. Consider the language of fields augmented by a constant symbol α_κ for each element $\kappa \in K$ and by additional finite tuples of constant symbols a and c (to represent coordinates of points on A and X, respectively). We construct a theory $\mathcal{T}$ in this language. Start with the field axioms, and the arithmetic sentences involving the α_κ that hold in K. Include a sentence saying that $\mathcal{B}$ is a p-basis for the universe field F. Include sentences saying that $a, c \in A(F)$ (for some fixed representation of A as a definable set). For each $e \in \mathbf{Z}_{\geq 1}$, include a sentence saying that

$$c \in X(F) \cap (a + p^e A(F)).$$

For each $Y \in \mathcal{Y}$, include a sentence saying that $(a, c) \notin Y(F)$; such a sentence may be constructed since iterated p-components admit a first-order definition.

Suppose that the conclusion of Proposition 3.16 fails. Then we claim that $\mathcal{T}$ is consistent. To prove this, we show that every finite subset $\mathcal{T}_0$ of $\mathcal{T}$ has a model. Given $\mathcal{T}_0$, let $e_{\max}$ be the maximum e that occurs in the sentences, and let $Y_{\max} \in \mathcal{Y}$ be the union of the Y's that occur. The negation of Proposition 3.16 implies that there exists a field extension $F \supseteq K$ having $\mathcal{B}$ as a p-basis, together with $a \in A(F)$ and $c \in X(F) \cap (a + p^{e_{\max}} A(F))$, such that $(a, c) \notin Y_{\max}(F)$. This (F, a, c) is a model for $\mathcal{T}_0$.

By the compactness theorem, there is a model (F, a, c) for $\mathcal{T}$. So F is a field extension of K having $\mathcal{B}$ as a p-basis, and

$$c \in X(F) \cap (a + p^e A(F)).$$

For all $e \in \mathbf{Z}_{\geq 1}$ and for every $Y \in \mathcal{Y}$, we have $(a, c) \notin Y(F)$. Let $\widetilde{K(a)}$ be the smallest extension of $K(a)$ in F that is closed under taking p-components with respect to $\mathcal{B}$, so $\widetilde{K(a)}$ too has $\mathcal{B}$ as p-basis. Then $K(c)$ is not algebraic over $\widetilde{K(a)}$, because an algebraic relation could be used to define a $Y \in \mathcal{Y}$ containing (a, c). By Lemma 3.12 applied to $F/\widetilde{K(a)}$, it is possible to enlarge F to assume that F is separably closed and the $\operatorname{Aut}(F/\widetilde{K(a)})$-orbit of c is infinite while $\mathcal{B}$ is still a p-basis for F. This infinite orbit is contained in the set $X(F) \cap (a + p^e A(F))$, so the latter is infinite too, for every $e \in \mathbf{Z}_{\geq 1}$. By Lemma 3.13 applied to $K \subseteq F$, the abelian variety $A_{\overline{F}}$ has no nonzero quotient defined over F^{p^∞}. Also, the fact that X is coset-free implies that X_F is coset-free. (If $X_{\overline{K}}$ acquires a coset over some extension of $\overline{K}$, it acquires one over some finitely generated extension of $\overline{K}$, and then a standard specialization argument shows that it contains a coset already over $\overline{K}$.) Thus $X_F \subseteq A_F$ is a counterexample to Lemma 3.14. □

Remark 3.17. The proof of Lemma 3.14 involves more model theory than we used elsewhere in this section. On the other hand, the proof of [AV92, Th. A(3)] probably could be adapted to give a purely algebraic proof of Lemma 3.14 in the case where A is ordinary, and this would suffice to prove Proposition 3.16 in the case where A is ordinary.

3.4. *Subvarieties of abelian varieties.* We now return to our situation in which K is the function field of a curve over k.

LEMMA 3.18. *Suppose that $A_{\overline{K}}$ has no nonzero isotrivial quotient, and that X is coset-free. Then there exists a finite K-subscheme Z of X such that $X(\mathbf{A}) \cap \overline{A(K)} \subseteq Z(\mathbf{A})$.*

Proof. By replacing K with its compositum with $\overline{k}$ in $\overline{K}$, we may reduce to the case in which k is algebraically closed.

We have $[K : K^p] = p$. Choose $\alpha \in K - K^p$. Then $\{\alpha\}$ is a p-basis for K. For any v, the field K_v is generated by K and K_v^p, so $[K_v : K_v^p] \le p$. Moreover, Lemma 3.1 implies $\alpha \notin K_v^p$, so $\{\alpha\}$ is a p-basis for K_v.

Let e and Y be as in Proposition 3.16. We think of Y as a subvariety of $A_n \times X$ for some n. For each $a \in A(K)$, let Y_a be the fiber of $Y \to A_n$ above the point of $A_n(K)$ corresponding to a. View Y_a as a finite subscheme of X.

Because $A_{\overline{K}}$ has no nonzero isotrivial quotient, $A(K)$ is finitely generated. Choose a (finite) set of representatives $\mathscr{A} \subseteq A(K)$ for $A(K)/p^e A(K)$. Let $Z = \bigcup_{a \in \mathscr{A}} Y_a$, so that Z is a finite subscheme of X. The conclusion of Proposition 3.16 applied to $F = K_v$ says that

$$X(K_v) \cap (a + p^e A(K_v)) \subseteq Y_a(K_v)$$

for each $a \in \mathscr{A}$. Let Γ_v be the closure of $A(K)$ in $A(K_v)$. Then

$$X(K_v) \cap (a + p^e \Gamma_v) \subseteq Y_a(K_v).$$

Taking the union over $a \in \mathscr{A}$ yields

$$X(K_v) \cap \Gamma_v \subseteq Z(K_v),$$

since by Lemma 3.10, the $a \in \mathscr{A}$ represent all classes in $\Gamma_v / p^e \Gamma_v$. This holds for all v, so $X(\mathbf{A}) \cap \overline{A(K)} \subseteq Z(\mathbf{A})$. ☐

Remark 3.19. In the number field case, one cannot expect finiteness of $X(K_v) \cap \Gamma_v$ in general. Consider the case in which $K_v = \mathbf{Q}_p$. If the "Chabauty condition" $\operatorname{rank} A(K) < \dim A$ fails, it may happen that Γ_v is an open subgroup of $A(K_v)$, in which case $X(K_v) \cap \Gamma_v$ will be infinite if nonempty.

3.5. *End of proof of Theorem* B. We now prove Theorem B. Let Z be as in Lemma 3.18. By Lemma 3.18 and then Proposition 3.9,

$$X(\mathbf{A}) \cap \overline{A(K)} \subseteq Z(\mathbf{A}) \cap \overline{A(K)} = Z(K) \subseteq X(K).$$

The opposite inclusion $X(K) \subseteq X(\mathbf{A}) \cap \overline{A(K)}$ is trivial. This completes the proof. ☐

Remark 3.20. Our proof of Theorem B required both the flat information (from residue fields) implicit in Proposition 3.6 and the deep information (from each K_v) in Proposition 3.5. But it seems possible that the flat information suffices, even for Conjecture C. Specifically, if Ω is a cofinite subset of the set of places of good reduction for X/K, then it is possible that $\overline{X(K)} = X\left(\prod_{v \in \Omega} \mathbf{F}_v\right) \cap \overline{A(K)}$ always holds, where the closures are now taken in $A\left(\prod_{v \in \Omega} \mathbf{F}_v\right)$.

4. The Brauer-Manin obstruction

In this section we assume that K is either a global function field (i.e., as before, but with k finite) or a number field. We also assume that Ω is the set of all nontrivial places of K, and define the ring of adèles $\mathbf{A}$ in the usual way.

The purpose of this section is to relate the adelic intersections we have been considering to the Brauer-Manin obstruction. This relationship was discovered by Scharaschkin [Sch99] in the number field case. We will give a proof similar in spirit to his, and verify that it works in the function field case.

For convenience, if X is any topological space, let $X_\bullet$ denote the set of connected components, and equip $X_\bullet$ with the quotient topology. This will be needed to avoid annoyances with the archimedean places in the number field case.

Let A be an abelian variety over K. By the adelic topology on $A(K)$, we mean the topology induced by $A(\mathbf{A})_\bullet$; in the function field case this agrees with our terminology in Section 1.

PROPOSITION 4.1. *The adelic topology on $A(K)$ equals the topology induced by subgroups of finite index.*

Proof. Proposition 3.6 gives one inclusion. The other inclusion follows from the fact that $A(\mathbf{A})_\bullet = \prod_{v \in \Omega} A(K_v)_\bullet$ is a profinite group. □

4.1. *Global duality for abelian varieties.* All cohomology below is fppf cohomology, i.e. faithfully flat and of finite presentation (also called "flat cohomology"; see [Mil80], [Mil86]). As usual, for any $n \in \mathbf{Z}_{\geq 1}$, define

$$\operatorname{Sel}^n := \ker\left(H^1(K, A[n]) \to \prod_{v \in \Omega} H^1(K_v, A)\right),$$

$$Ш := \ker\left(H^1(K, A) \to \prod_{v \in \Omega} H^1(K_v, A)\right).$$

Following the notation of [Sto07, §2], define

$$\widehat{A(K)} := \varprojlim \frac{A(K)}{nA(K)},$$

$$\widehat{\operatorname{Sel}} := \varprojlim \operatorname{Sel}^n,$$

$$TШ := \varprojlim Ш[n],$$

where each inverse limit is over positive integers n ordered by divisibility. Taking inverse limits of the usual descent sequence yields

$$0 \to \widehat{A(K)} \to \widehat{\operatorname{Sel}} \to TШ \to 0. \tag{1}$$

For any abelian profinite group G and any prime ℓ, let $G^{(\ell)}$ be the maximal pro-ℓ quotient of G, so that $G \cong \prod_\ell G^{(\ell)} \cong \varprojlim G/nG$, where ℓ ranges over primes, and n ranges over positive integers ordered by divisibility.

LEMMA 4.2. *For each prime ℓ, each of $\widehat{A(K)}^{(\ell)}$, $\widehat{\mathrm{Sel}}^{(\ell)}$, $T\text{Ш}^{(\ell)}$ is a finitely generated $\mathbf{Z}_\ell$-module.*

Proof. For $\widehat{A(K)}^{(\ell)}$ it holds simply because $A(K)$ is finitely generated as a $\mathbf{Z}$-module. For $T\text{Ш}^{(\ell)}$, it follows from the finiteness of $\text{Ш}[\ell]$, which is proved in [Mil70]. Now the result for $\widehat{\mathrm{Sel}}^{(\ell)}$ follows from (1). □

By definition, Sel^n maps to

$$\ker\left(\prod_{v\in\Omega} H^1(K_v, A[n]) \to \prod_{v\in\Omega} H^1(K_v, A)\right) = \prod_{v\in\Omega} \frac{A(K_v)}{nA(K_v)},$$

so as in [Sto07, §2] we obtain a map

$$\widehat{\mathrm{Sel}} \to A(\mathbf{A})_\bullet.$$

For each v we have a pairing

$$A(K_v)_\bullet \times H^1(K_v, A^\vee) \to H^2(K_v, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z},$$

and in fact Tate local duality holds: Each of the first two groups is identified with the Pontryagin dual of the other [Mil86, Th. III.7.8]. An element of $H^1(K, A^\vee)$ maps to 0 in $H^1(K_v, A^\vee)$ for all but finitely many v, so summing the Tate local duality pairing over v defines a pairing

$$A(\mathbf{A})_\bullet \times H^1(K, A^\vee) \to \mathbf{Q}/\mathbf{Z},$$

or equivalently a homomorphism

$$A(\mathbf{A})_\bullet \xrightarrow{\text{Tate}} H^1(K, A^\vee)^D,$$

where the superscript ^{D} denotes Pontryagin dual.

The following global duality statement is a version of the "Cassels dual exact sequence" in which finiteness of Ш is not assumed:

PROPOSITION 4.3 (Cassels dual exact sequence). *The sequence*

$$0 \longrightarrow \widehat{\mathrm{Sel}} \longrightarrow A(\mathbf{A})_\bullet \xrightarrow{\text{Tate}} H^1(K, A^\vee)^D$$

is exact.

Proof. This is part of the main theorem of [GAT07]. □

Remark 4.4. Proposition 4.3 has a long history. For an elliptic curve A over a number field, Proposition 4.3 was proved by Cassels [Cas62], [Cas64] under the assumption that Ш(A) is finite, using a result from [Ser64] to obtain exactness on the left. This, except for the exactness on the left, was generalized by Tate to abelian varieties over number fields with finite Shafarevich-Tate group, at around the same time, though the first published proof appeared much later [Mil86, Th. I.6.13]. The latter reference also proved the global function field analogue except for the p-part in characteristic p. The $\widehat{\mathrm{Sel}}$ version (not requiring finiteness of Ш(A)), except for exactness on the left and the p-part, is implicit in the right three terms of the exact sequence in the middle of page 104 of [Mil86] if there we make the substitution

$$T\text{Ш} \cong \widehat{\mathrm{Sel}}/\widehat{A(K)}$$

from (1). Exactness on the left for the prime-to-p part is implicit in [Ser64], [Ser71] and is mentioned explicitly in [Mil86, Cor. 6.23]. The full statement was proved in [GAT07].

4.2. *Review of the Brauer-Manin obstruction.* Let $X_s := X \times_K K^s$. Let X be a projective K-scheme. Define $\operatorname{Br} X := H^2(X, \mathbf{G}_m)$, where here we may use étale cohomology since it gives the same result as fppf cohomology for smooth quasi-projective commutative group schemes [Mil80, Th. III.3.9]. For simplicity, let $\frac{\operatorname{Br} X}{\operatorname{Br} K}$ denote the cokernel of $\operatorname{Br} K \to \operatorname{Br} X$ even if this map is not injective. The Hochschild-Serre spectral sequence in étale cohomology gives an exact sequence

$$\operatorname{Br} K \to \ker(\operatorname{Br} X \to \operatorname{Br} X_s) \to H^1(K, \operatorname{Pic} X_s) \to H^3(K, \mathbf{G}_m),$$

and the last term is 0 for any global field; see e.g. [Mil86, Cor. 4.21]. From this we extract an injection

$$H^1(K, \operatorname{Pic} X_s) \hookrightarrow \frac{\operatorname{Br} X}{\operatorname{Br} K};$$

if $\operatorname{Br} X_s = 0$, then this is an isomorphism. Composing with the map induced by $\operatorname{Pic}^0 X_s \hookrightarrow \operatorname{Pic} X_s$, we obtain

$$H^1(K, \operatorname{Pic}^0 X_s) \to \frac{\operatorname{Br} X}{\operatorname{Br} K}. \tag{2}$$

Each $A \in \operatorname{Br} X$ induces evaluation maps ev_A fitting in a commutative diagram

$$\begin{array}{ccccccccc}
 & & X(K) & \longrightarrow & X(\mathbf{A})_\bullet & & & & \\
 & & \downarrow{\scriptstyle \mathrm{ev}_A} & & \downarrow{\scriptstyle \mathrm{ev}_A} & \searrow & & & \\
0 & \longrightarrow & \operatorname{Br} K & \longrightarrow & \bigoplus_{v \in \Omega} \operatorname{Br}(K_v) & \longrightarrow & \mathbf{Q}/\mathbf{Z} & \longrightarrow & 0.
\end{array}$$

The diagonal arrows give rise to a pairing

$$X(\mathbf{A})_\bullet \times \frac{\operatorname{Br} X}{\operatorname{Br} K} \to \mathbf{Q}/\mathbf{Z}$$

which induces a map

$$X(\mathbf{A})_\bullet \xrightarrow{\text{BM}} \left(\frac{\operatorname{Br} X}{\operatorname{Br} K}\right)^D .$$

The kernel (inverse image of 0) is called the *Brauer set*, and is denoted $X(\mathbf{A})_\bullet^{\text{Br}}$. It contains $X(K)$ and hence also its closure $\overline{X(K)}$ in $X(\mathbf{A})_\bullet$. One says that the Brauer-Manin obstruction to the Hasse principle is the only one for X if the implication

$$X(\mathbf{A})_\bullet^{\text{Br}} \neq \varnothing \implies X(K) \neq \varnothing$$

holds. (This is equivalent to the usual notion without the $\bullet$.)

4.3. *The Brauer set for abelian varieties.*

THEOREM E. *We have*

$$\widehat{A(K)} = \overline{A(K)} \subseteq A(\mathbf{A})_\bullet^{\text{Br}} \subseteq \widehat{\text{Sel}}.$$

in $A(\mathbf{A})$.

Proof. Proposition 4.1 shows that $A(K) \to A(\mathbf{A})_\bullet$ induces an isomorphism $\widehat{A(K)} \to \overline{A(K)}$, and the first inclusion is automatic from the previous subsection. It remains to prove the second inclusion. Taking the dual of (2) for A yields the vertical arrow in

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{\text{Sel}} & \longrightarrow & A(\mathbf{A})_\bullet & \xrightarrow{\text{Tate}} & H^1(K, \operatorname{Pic}^0 A_s)^D \\ & & & & & \searrow^{\text{BM}} & \uparrow \\ & & & & & & \left(\dfrac{\operatorname{Br} A}{\operatorname{Br} K}\right)^D . \end{array}$$

Applying [Man71, Prop. 8] to each K_v shows that the triangle commutes (the hypothesis there that the ground field be perfect is not used). The diagram now gives $\ker(\text{BM}) \subseteq \ker(\text{Tate})$, or equivalently $A(\mathbf{A})_\bullet^{\text{Br}} \subseteq \widehat{\text{Sel}}$. □

Remark 4.5. If Ш is finite (or more generally, if its maximal divisible part $Ш_{\text{div}}$ is 0), then $TШ = 0$ and the injection $\widehat{A(K)} \to \widehat{\text{Sel}}$ is an isomorphism; in this case, the conclusion of Theorem E simplifies to

$$\widehat{A(K)} = \overline{A(K)} = A(\mathbf{A})_\bullet^{\text{Br}} = \widehat{\text{Sel}}.$$

Closely related results in the number field case can be found in [Wan96].

4.4. *The Brauer set for subvarieties of abelian varieties.*

PROPOSITION 4.6. *Let* $i : X \to A$ *be a morphism from a projective* K*-scheme to an abelian variety over* K. *Thus* i *induces a map* $X(\mathbf{A})_\bullet \to A(\mathbf{A})_\bullet$.

(a) *We have* $i(X(\mathbf{A})_\bullet^{\text{Br}}) \subseteq \widehat{\text{Sel}}$.

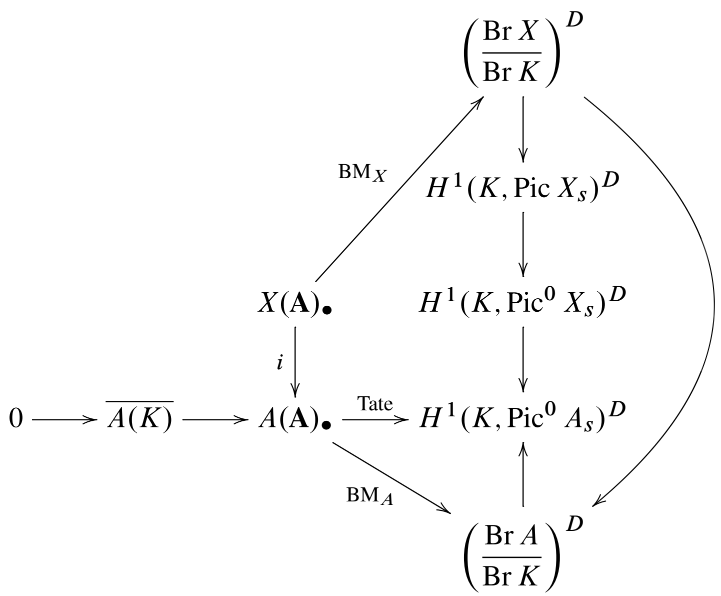
(b) *Suppose that*

- *X is a smooth projective geometrically integral curve*;
- *A is the Jacobian of X*;
- *i is an Albanese map (that is, i sends a point P to the class of $P - D$, where $D \in \operatorname{Div} A_s$ is a fixed divisor of degree 1 whose class is $\operatorname{Gal}(K^s/K)$-invariant); and*
- *Ш is finite (or at least $Ш_{\text{div}} = 0$).*

Then $X(\mathbf{A})_\bullet^{\text{Br}} = i^{-1}(\overline{A(K)})$.

Proof. (a) This follows immediately from Theorem E, since i maps $X(\mathbf{A})_\bullet^{\text{Br}}$ to $A(\mathbf{A})_\bullet^{\text{Br}}$.

(b) We use the diagram



in which the horizontal sequence is the exact sequence of Proposition 4.3 in which we have used the finiteness of Ш to replace $\widehat{\text{Sel}}$ with $\overline{A(K)}$. The lower triangle commutes, as explained in the proof of Theorem E. The "pentagon" at the far right also commutes, since the homomorphism (2) is functorial in X. Thus the whole diagram commutes.

We next claim that the three downward vertical arrows at the right are isomorphisms. The first is an isomorphism because $\operatorname{Br} X_s = 0$ [Gro68, Cor. 5.8]. The second is an isomorphism because $\operatorname{Pic} X_s \simeq \operatorname{Pic}^0 X_s \oplus \mathbf{Z}$ (where the $\mathbf{Z}$ is generated by the class of D) and $H^1(K, \mathbf{Z}) = 0$. The third is an isomorphism because an Albanese map i induces an isomorphism $i^*\colon \operatorname{Pic}^0 A_s \to \operatorname{Pic}^0 X_s$.

The commutativity of the upper left "hexagon" now implies

$$X(\mathbf{A})_\bullet^{\mathrm{Br}} := \ker(\mathrm{BM}_X) = \ker(\mathrm{Tate} \circ i) = i^{-1}(\ker(\mathrm{Tate})) = i^{-1}(\overline{A(K)}). \qquad \square$$

Remark 4.7. Part (b) was originally proved in the number field case in [Sch99] using a slightly different proof. For yet another proof of this case, see [Sto07, Cor. 7.4].

Remark 4.8. If K is a global function field, the conclusion of (b) can be written as $X(\mathbf{A})^{\mathrm{Br}} = X(\mathbf{A}) \cap \overline{A(K)}$ in $A(\mathbf{A})$.

5. Intersections with $\widehat{\mathrm{Sel}}$

From now on, we assume that K is a global function field. The proof of Theorem D will follow that of Theorem B, with $\widehat{\mathrm{Sel}}$ playing the role of $\overline{A(K)}$. We begin by proving $\widehat{\mathrm{Sel}}$-versions of several of the lemmas and propositions of Section 3.2. The following is an analogue of Lemma 3.7.

LEMMA 5.1. *The maps* $A(K)_{\mathrm{tors}} \to \widehat{A(K)}_{\mathrm{tors}} \to \widehat{\mathrm{Sel}}_{\mathrm{tors}}$ *are isomorphisms.*

Proof. Proposition 4.1 yields $\widehat{A(K)} \simeq \overline{A(K)}$, so the first map is an isomorphism by Lemma 3.7. The second map is an isomorphism by (1), since $T\text{Ш}$ is torsion-free by definition. $\square$

The following is an analogue of Proposition 3.5.

PROPOSITION 5.2. *If* $A(K^s)[p^\infty]$ *is finite, then for any* v, *the map* $\widehat{\mathrm{Sel}}^{(p)} \to A(K_v)^{(p)}$ *is injective.*

Proof. Let $K_v' \subseteq K^s$ be the Henselization of K at v. Define $\mathrm{Sel}^{n\prime}$ and $\widehat{\mathrm{Sel}}'$ in the same way as Sel^n and $\widehat{\mathrm{Sel}}$, but using K_v' in place of its completion K_v. By [Mil86, I.3.10(a)(ii)], the natural maps $\mathrm{Sel}^{n\prime} \to \mathrm{Sel}^n$ and $\widehat{\mathrm{Sel}}' \to \widehat{\mathrm{Sel}}$ are isomorphisms. Similarly, by [Mil86, I.3.10(a)(i)] we may replace $A(K_v)^{(p)}$ by $A(K_v')^{(p)} := \varprojlim A(K_v')/p^n A(K_v')$. Hence it suffices to prove injectivity of $(\widehat{\mathrm{Sel}}')^{(p)} \to A(K_v')^{(p)}$.

Choose m such that $p^m A(K^s)[p^\infty] = 0$. Suppose

$$b \in \ker\left((\widehat{\mathrm{Sel}}')^{(p)} \to A(K_v')^{(p)}\right).$$

For each $M \in \mathbf{Z}_{\geq 0}$, let b_M be the image of b in $\mathrm{Sel}'^{p^M} \subseteq H^1(K, A[p^M])$. Then the image of b_M under

$$\mathrm{Sel}'^{p^M} \to \frac{A(K_v')}{p^M A(K_v')} \subseteq H^1(K_v', A[p^M]) \to H^1(K^s, A[p^M])$$

is 0. The Hochschild-Serre spectral sequence (see [Mil80, III.2.21]) gives an exact sequence

$$0 \to H^1(\mathrm{Gal}(K^s/K), A(K^s)[p^M]) \to H^1(K, A[p^M]) \to H^1(K^s, A[p^M]),$$

so b_M comes from an element of $H^1(\mathrm{Gal}(K^s/K), A(K^s)[p^M])$, which is killed by p^m. Thus $p^m b_M = 0$ for all M, so $p^m b = 0$.

By Lemma 5.1, b comes from a point in $A(K)[p^\infty]$, which, in turn, injects into $A(K'_v)^{(p)}$. □

The next result is an analogue of Proposition 3.9.

PROPOSITION 5.3. *Suppose that $A(K^s)[p^\infty]$ is finite. Let Z be a finite K-subscheme of A. Then $Z(\mathbf{A}) \cap \widehat{\mathrm{Sel}} = Z(K)$.*

Proof. One inclusion is easy: $Z(K) \subseteq Z(\mathbf{A})$ and $Z(K) \subseteq A(K) \subseteq \widehat{A(K)} \subseteq \widehat{\mathrm{Sel}}$. Therefore we focus on the other inclusion.

As in the proof of Proposition 3.9, we may replace K by a finite extension to assume that Z consists of a finite set of K-points of A. Suppose $P \in Z(\mathbf{A}) \cap \widehat{\mathrm{Sel}}$. The v-adic component of P in $Z(K_v)$ equals the image of a point $Q_v \in Z(K)$. Then $P - Q_v$ maps to 0 in $A(K_v)$, and in particular in $A(K_v)^{(p)}$, so by Proposition 5.2 the image of $P - Q_v$ in $\widehat{\mathrm{Sel}}^{(p)}$ is 0. This holds for every v, so if v' is another place, then $Q_{v'} - Q_v$ maps to 0 in $\widehat{\mathrm{Sel}}^{(p)}$. The kernel of $A(K) \to \widehat{A(K)}^{(p)} \hookrightarrow \widehat{\mathrm{Sel}}^{(p)}$ is the prime-to-p torsion of $A(K)$, so $Q_{v'} - Q_v \in A(K)_{\mathrm{tors}}$. This holds for all v', so the point $R := P - Q_v$ belongs to $\widehat{\mathrm{Sel}}_{\mathrm{tors}}$. By Lemma 5.1, $R \in A(K)_{\mathrm{tors}}$. Thus $P = R + Q_v \in A(K)$. So $P \in Z(\mathbf{A}) \cap A(K) = Z(K)$. □

Each $\tau \in \mathrm{Sel}^{p^e}$ may be represented by a "covering space": an fppf torsor T under A equipped with a morphism $\phi_T \colon T \to A$ that after base extension to K^s becomes isomorphic to the base extension of $[p^e] \colon A \to A$. (Passing to K^s is enough to trivialize T since A is smooth.)

LEMMA 5.4. *We have $\widehat{\mathrm{Sel}} \subseteq \bigcup_{\tau \in \mathrm{Sel}^{p^e}} \phi_T(T(\mathbf{A}))$.*

Proof. In the commutative diagram

$$\begin{array}{ccc} & & T(\mathbf{A}) \\ & & \downarrow{\scriptstyle \phi_T} \\ \widehat{\mathrm{Sel}} & \hookrightarrow & A(\mathbf{A}) \\ \downarrow & & \downarrow \\ \mathrm{Sel}^{p^e} & \hookrightarrow & \dfrac{A(\mathbf{A})}{p^e A(\mathbf{A})} \end{array}$$

the set $\phi_T(T(\mathbf{A}))$ is a coset of $p^e A(\mathbf{A})$ in $A(\mathbf{A})$, and its image in $\frac{A(\mathbf{A})}{p^e A(\mathbf{A})}$ equals the image of τ under the bottom horizontal map, by the definition of this bottom map. Thus any element of $\widehat{\mathrm{Sel}}$ mapping to τ in Sel^{p^e} belongs to the corresponding $\phi_T(T(\mathbf{A}))$. □

Proof of Theorem D. We have $X(K) \subseteq X(\mathbf{A})^{\mathrm{Br}} \subseteq X(\mathbf{A}) \cap \widehat{\mathrm{Sel}}$, by Proposition 4.6(a), so it will suffice to show that $X(\mathbf{A}) \cap \widehat{\mathrm{Sel}}$ consists of K-rational points.

Let e and Y be as in Proposition 3.16. For each $\tau \in \mathrm{Sel}^{p^e}$, choose $\phi_T \colon T \to A$ as above, choose $t \in T(K^s)$, let $a = \phi_T(t) \in A(K^s)$, and let Y_a be the fiber of $Y \to A_n$ above the point corresponding to a. View Y_a as a finite subscheme of X_{K^s}. It follows from [Mil70] that Sel^{p^e} is finite, so we can choose a finite K-subscheme Z of X such that Z_{K^s} contains all the Y_a as τ ranges over Sel^{p^e}.

For each v, choose a separable closure K_v^s of K_v containing K^s. By the proof of Lemma 3.18, any p-basis of K is also a p-basis for K_v, and hence for K_v^s. Therefore the conclusion of Proposition 3.16 may be applied with $F = K_v^s$ to yield

$$X(K_v^s) \cap (a + p^e A(K_v^s)) \subseteq Y_a(K_v^s)$$

for each a. By definition of ϕ_T, we have $\phi_T(T(K_v^s)) = a + p^e A(K_v^s)$. Thus

$$X(K_v) \cap \phi_T(T(K_v)) \subseteq X(K_v^s) \cap (a + p^e A(K_v^s)) \subseteq Y_a(K_v^s) \subseteq Z(K_v^s).$$

Hence

$$X(K_v) \cap \phi_T(T(K_v)) \subseteq X(K_v) \cap Z(K_v^s) = Z(K_v).$$

This holds for all v, so $X(\mathbf{A}) \cap \phi_T(T(\mathbf{A})) \subseteq Z(\mathbf{A})$. Taking the union over $\tau \in \mathrm{Sel}^{p^e}$, and applying Lemma 5.4, we obtain $X(\mathbf{A}) \cap \widehat{\mathrm{Sel}} \subseteq Z(\mathbf{A})$. Thus $X(\mathbf{A}) \cap \widehat{\mathrm{Sel}} \subseteq Z(\mathbf{A}) \cap \widehat{\mathrm{Sel}}$, and the latter equals $Z(K)$ by Proposition 5.3, so we are done. □

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