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The Smith-Toda complex $V((p+1)/2)$ does not exist

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Abstract

Using a generalized homotopy fixed point spectral sequence due to Hopkins and Miller, we show that the Smith-Toda complex $V((p+1)/2)$ does not exist for p a prime greater than 5. This extends earlier results of Toda and Ravenel for the primes 2, 3, and 5. It is also shown that if $V((p-1)/2)$ exists, it is not a ring spectrum.

1. Introduction

One of the most significant modern developments in algebraic topology has been the discovery of periodic phenomena in stable homotopy. This began with Adams' work on the J homomorphism [Ada66] and was later continued by L. Smith [Smi70] and H. Toda [Tod71].

Let p be a prime and let $V(0)_k$ denote the cofiber of the degree p map on S^k. Adams and Toda showed that for k sufficiently large there is a nonnilpotent map

$$\Sigma^q V(0)_k \xrightarrow{\alpha} V(0)_k,$$

where $q = 2(p-1)$ if p is odd and $q = 8$ if $p = 2$. Let α^t denote the composite

$$\Sigma^{tq} V(0)_k \longrightarrow \cdots \longrightarrow \Sigma^{2q} V(0)_k \xrightarrow{\Sigma^q \alpha} \Sigma^q V(0)_k \xrightarrow{\alpha} V(0)_k.$$

Then by nonnilpotent, we mean that no α^t is nullhomotopic.

By including the bottom cell of $\Sigma^{tq} V(0)_k$ and projecting to the top cell of $V(0)_k$, we obtain a family of elements, also denoted α_t, in the homotopy of spheres:

$$S^{k+tq} \longrightarrow \Sigma^{tq} V(0)_k \xrightarrow{\alpha^t} V(0)_k \longrightarrow S^{k+1}.$$

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That k must be sufficiently large makes this a stable phenomenon. Thus from now on we work in the stable homotopy category of spectra.

To show that α is nonnilpotent, Adams and Toda showed that it induces an isomorphism in complex K-theory. In modern terms, α is multiplication by a power of v_1 in BP-homology. Explicitly, $\mathrm{BP}_*(\alpha)$ is v_1 when $p > 2$ and v_1^4 when $p = 2$.

Here BP is the Brown-Peterson spectrum. It is a p-local ring spectrum with coefficient ring $\mathrm{BP}_* \cong \mathbf{Z}_{(p)}[v_1, v_2, \dots]$, where v_i is in degree $2(p^i - 1)$. BP is obtained as a wedge summand of the p-localization of MU, the spectrum representing complex bordism. Its coalgebra of cooperations is $\mathrm{BP}_*\mathrm{BP} \cong \mathrm{BP}_*[t_1, t_2, \dots]$, where t_i is also in degree $2(p^i - 1)$.

The natural setting for a discussion of the α family and Smith's attempt to generalize it is that of the Adams-Novikov spectral sequence (ANSS). For X a CW complex or, more generally, a spectrum, the spectral sequence has initial term

$$\mathrm{Ext}(\mathrm{BP}_*X) \equiv \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*X)$$

and converges to the p-localization of the stable homotopy of X when X is connective. (See [Rav86] for a thorough discussion.)

The generators of BP_* give rise to infinite families in the initial term of the ANSS for $X = S^0$ as follows. Let $I_k = (p, v_1, \dots, v_{k-1}) \subset \mathrm{BP}_*$. Then for each $k \geq 0$,

$$0 \longrightarrow \Sigma^{|v_k|}\mathrm{BP}_*/I_k \xrightarrow{v_k} \mathrm{BP}_*/I_k \longrightarrow \mathrm{BP}_*/I_{k+1} \longrightarrow 0 \tag{1.1}$$

is an exact sequence of $\mathrm{BP}_*\mathrm{BP}$-comodules. Let

$$\delta_k : \mathrm{Ext}^*(\mathrm{BP}_*/I_{k+1}) \longrightarrow \Sigma^{|v_k|}\,\mathrm{Ext}^{*+1}(\mathrm{BP}_*/I_k)$$

be the associated connecting homomorphism. Then $\mathrm{Ext}^0(\mathrm{BP}_*/I_k) \cong \mathbf{F}_p[v_k]$ and for $k, t > 0$ we define

$$\alpha_t^{(k)} = \delta_0\delta_1 \cdots \delta_{k-1}(v_k^t) \in \mathrm{Ext}^k(\mathrm{BP}_*),$$

where $\alpha^{(k)}$ denotes the kth letter of the Greek alphabet. This is the "Greek letter construction."

When $p > 2$ and $k = 1$, the elements α_t survive the spectral sequence to detect the "α family" of Adams' construction. The corresponding statement also holds true for $p = 2$. The question of whether or not the more general Greek letter elements survive in the spectral sequence is a difficult one. The most fruitful approach to tackling this is to realize the short exact sequences of (1.1).

Definition 1.1. A *Smith-Toda complex* is a finite spectrum $V(k)$ satisfying

$$\mathrm{BP}_*V(k) \cong \mathrm{BP}_*/(p, v_1, \dots, v_k)$$

as a BP_*-module, hence as a $\mathrm{BP}_*\mathrm{BP}$-comodule. For example, $V(-1) = S^0$ and $V(0) = M(p)$, the mod p Moore spectrum.

Write $\mathscr{A}_*$ for the mod p dual Steenrod algebra. Recall that, for p odd, $\mathscr{A}_* \cong \mathbf{F}_p[\xi_1, \xi_2, \dots] \otimes E(\tau_0, \tau_1, \dots)$ as an algebra, where $E(\dots)$ indicates the exterior $\mathbf{F}_p$-algebra on the indicated generators. The following gives an alternative characterization of $V(k)$ in terms of its mod p homology. The result is well known but we include a proof in the appendix.

THEOREM 1.2. *Let $p > 2$ and X be a finite spectrum. Then*

$$\mathrm{BP}_* X \cong \mathrm{BP}_*/(p, v_1, \dots, v_k)$$

as a BP_*-module if and only if

$$H_*(X; \mathbf{F}_p) \cong E(\tau_0, \tau_1, \dots, \tau_k)$$

as an $\mathscr{A}_*$-comodule.

Such spectra need not be unique. However, it follows from Theorem 1.2 that if $V(k)$ exists, it arises as a cofiber

$$\Sigma^{|v_k|} V(k-1)' \xrightarrow{f} V(k-1) \longrightarrow V(k),$$

where f induces multiplication by v_k in BP-homology. Note that f is not necessarily an honest self-map, i.e., the spectra $V(k-1)'$, and $V(k-1)$ need not be equivalent.

Thus the construction of $V(k)$ is equivalent to the realization of (1.1). Having constructed $V(k)$, it is a simple matter to show that $\alpha_1^{(k)}$ is a permanent cycle in the Adams-Novikov spectral sequence. If, in addition, $V(k-1)$ is a ring spectrum or f is a self-map, then $\alpha_t^{(k)}$ is a permanent cycle for all $t > 0$.

For $k = 1$, 2, and 3, $V(k)$ was constructed for $p > 2k$ by Adams, Smith, and Toda, respectively. In fact, these results are sharp. The first negative results were obtained by Toda, who showed that $V(1)$ cannot be constructed when $p = 2$, and likewise, for $V(2)$ when $p = 3$. Later, Ravenel [Rav86, 7.5.1] showed that $V(3)$ does not exist when $p = 5$.

The purpose of this paper is to prove

THEOREM 1.3. *If $p \geq 7$, $V((p+1)/2)$ does not exist. If $V((p-1)/2)$ exists, it is not a ring spectrum.*

Our main tool is a spectral sequence due to M. J. Hopkins and H. R. Miller. If X is a finite spectrum, the spectral sequence has the form

$$H^*(\Phi; E_{n*}X) \Longrightarrow E_{n*}^{h\Phi} X.$$

Here Φ is a maximal finite subgroup of the Morava stabilizer group S_n, $n \geq 1$, E_n is the Landweber exact spectrum with coefficient ring

$$E_{n*} \cong W\mathbf{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u, u^{-1}],$$

graded via $|u| = -2$, and $E_n^{h\Phi}$ is the homotopy fixed point spectrum of E_n under the action of Φ. As usual, $W\mathbf{F}_{p^n}$ denotes the ring of Witt vectors with coefficients in the field $\mathbf{F}_{p^n}$.

We discuss this spectral sequence in more detail in the next section. For now, we wish merely to indicate what is involved in the proof of Theorem 1.3. We will consider the Hopkins-Miller spectral sequence when $n = p-1$ and refer to it simply as the HMSS *for* X.

Recall that E_{n*} (for any n) is a BP_*-algebra via the assignment $v_i = u^{1-p^i} u_i$, with the convention that $u_n = 1$ and $u_i = 0$ for $i > n$. Because E_n is constructed from the Landweber exact functor theorem, we have

$$E_{n*}V(k-1) \cong E_{n*}/(p, v_1, \ldots, v_{k-1}).$$

Now the action of S_n on E_{n*} is related to the structure of the Hopf algebra $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$ and v_k is invariant mod I_k in that structure. In particular, $v_k^t \in H^0(\Phi; E_{n*}V(k-1))$ for all $t \geq 0$.

In Sections 2 and 3 we prove

THEOREM 1.4. *If $V(k-1)$ exists, then v_k is a permanent cycle in the* HMSS *for $V(k-1)$.*

THEOREM 1.5. *Let $m = (p+1)/2$ and assume $p \geq 7$. If $V(m-1)$ exists, then v_m^2 supports a differential in the* HMSS *for $V(m-1)$.*

Assuming Theorems 1.4 and 1.5, we can give the

Proof of Theorem 1.3. Theorems 1.4 and 1.5 together show that there is no map

$$\Sigma^{|v_m|} V(m-1)' \longrightarrow V(m-1)$$

inducing multiplication by v_m in E_{p-1}-homology. By the remarks following Theorem 1.2, this suffices to prove the theorem. □

While Theorem 1.3 holds for all odd primes, it should be pointed out that Theorem 1.5 is only true for $p \geq 7$. When $p = 3$ and $p = 5$, v_m and v_m^2 are both permanent cycles in the HMSS for $V(m-1)$ while v_m^3 is not. We leave it to the reader to verify these statements.

It is also interesting to note that Theorem 1.3 is the best result possible using our techniques. (See Remark 4.2 for an explanation.) One might then speculate that the result is sharp. However, Ravenel [Rav86, 5.6.13] has shown that for $p \geq 3$, there is a possible target for $d_{2p-1}(v_4)$ in the ANSS. It was the detection of this "universal differential" that motivated Hopkins and Miller to develop the machinery we employ here. In fact, they were able to show, with M. Mahowald, that $V(p-2)$ does not exist (unpublished).

Notations and conventions. The spectrum $E_{p-1}^{h\Phi}$ is usually denoted by EO_{p-1}. We further abbreviate the cohomology groups $H^{s,t}(\Phi; E_{n*}/I_{k+1})$ by $\mathrm{HM}^{s,t}V(k)$, whether $V(k)$ is known to exist or not.

We will need to compute $\mathrm{HM}^{s,t}V(k)$ and $(EO_{p-1})_r V(k)$ for various values of s, t, r, and k. For general k this is difficult to do directly, so we instead reduce to the case $V(-1) = S^0$ using long exact sequences. It is then convenient to adopt the following terminology. If a given class $x \in \mathrm{HM}^{s,t}V(k)$ is in the image of $\mathrm{HM}^{s,t}V(k-1) \to \mathrm{HM}^{s,t}V(k)$, we say that it lives on the *bottom* $V(k-1)$ of $V(k)$. Otherwise, it lives on the *top* $V(k-1)$ and gives rise to a nonzero class in $\mathrm{HM}^{s+1,t-|v_k|}V(k-1)$ via the evident connecting homomorphism.

It will often be the case that x is ultimately determined in this way by a unique class in HM^*S^0, which we call the *name* of x on that cell. Such a cell is said to *support* x. For example, in Proposition 3.2 we show that if $k \geq 3$, then $v_k \in \mathrm{HM}^0 V(k-1)$ lives on the top cell of the bottom $V(0)$ of the top $V(k-2)$ of $V(k-1)$. More succinctly, we simply write

$$V(k-1) \longrightarrow V(k-2) \longleftarrow V(0) \longrightarrow S^0$$

for this cell and say that we have *pinched* at $V(k-1)$ and $V(0)$.

2. **The Hopkins-Miller spectral sequence**

2.1. *Preliminaries.* We will now briefly describe the work of Hopkins and Miller. An account of some of this work is given in [Rez98]. Extensions to the theory can be found in [GH03], [**?**], and [DH04].

For a given $n \geq 1$, the Morava stabilizer group S_n is the automorphism group, over $\mathbf{F}_{p^n}$, of a canonical, height n formal group law Γ_n, defined over $\mathbf{F}_p$. If X is a finite spectrum, Morava's theory ([Mor85], also [Dev95]) provides an action of S_n on $E_{n*}X$. Furthermore, $\mathrm{Gal} \equiv \mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p)$ acts on $E_{n*}X$ via its action on $W\mathbf{F}_{p^n}$, making $E_{n*}X$ into a G_n-module, where $G_n \equiv S_n \rtimes \mathrm{Gal}$. In fact, this action is induced by an action (in the stable homotopy category) of G_n on E_n ([Dev97, see discussion p. 767]).

Morava's change of rings theorem, in conjunction with [MS95], leads to a spectral sequence

$$H_c^*(G_n; E_{n*}X) \Longrightarrow \pi_* L_{K(n)}X,$$

where $L_{K(n)}$ denotes localization with respect to $K(n)$, the nth Morava K-theory spectrum. The initial term of this spectral sequence resembles that of a homotopy fixed point spectral sequence, but with two crucial differences. First, the group

G_n is not discrete. In fact, it is profinite and the initial term involves continuous cohomology. Second, G_n acts on E_n only up to homotopy. To form homotopy fixed point spectra in the usual way requires that the action be "on the nose".

Nevertheless, Hopkins and Miller have shown that if $G \subset G_n$ is a finite subgroup one may form the homotopy fixed point spectrum E_n^{hG} and if X is a finite spectrum there is a spectral sequence

$$H^*(G; E_{n*}X) \Longrightarrow \pi_*(E_n^{hG} \wedge X).$$

In [DH04] this is accomplished for any closed subgroup $G \subset G_n$, provided one works with continuous cohomology. More precisely, the spectral sequence is the $K(n)$-local E_n^{Gal}-based Adams spectral sequence for $E_n^{hG} \wedge X$, where E_n^{Gal} is the Landweber exact spectrum with coefficient ring $\mathbf{Z}_p[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$.

This construction is natural in G and agrees with the usual spectral sequence when $G = G_n$. Also, the map $X \wedge S^0 \xrightarrow{\sim} X$ makes the spectral sequence for X into a module over the spectral sequence for S^0. Similarly, if X is a ring spectrum, the spectral sequence for X is one of algebras over the spectral sequence for S^0.

A suitable geometric boundary theorem holds as well: Suppose

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a cofiber sequence of finite spectra with $E_{n*}(h) = 0$. The resulting short exact sequence

$$0 \longrightarrow E_{n*}X \longrightarrow E_{n*}Y \longrightarrow E_{n*}Z \longrightarrow 0$$

gives rise to a connecting homomorphism

$$H_c^*(G; E_{n*}Z) \xrightarrow{\delta} H_c^{*+1}(G; E_{n*}X).$$

Suppose $z \in \pi_*(E_n^{hG} \wedge Z)$ is detected by $\bar{z} \in H_c^*(G; E_{n*}Z)$. Then $\delta(\bar{z})$ is a permanent cycle and $h(z) \in \pi_{*-1}(E_n^{hG} \wedge X)$ is detected by $\delta(z)$, or else $h(z)$ is in higher filtration.

2.2. *The case $n = p-1$.* For the rest of the paper, we take p odd and $n = p-1$.

In this case, S_{p-1} has, up to conjugation, exactly one maximal finite subgroup, $\Phi \cong \mathbf{Z}/p \rtimes \mathbf{Z}/(p-1)^2$, where the action of the right factor on the left is given by

$$\mathbf{Z}/(p-1)^2 \to \mathbf{Z}/(p-1) \cong \mathrm{Aut}(\mathbf{Z}/p).$$

(See [Hew95] and [Hew99].) Following Hopkins and Miller, we write EO_{p-1} for the homotopy fixed point spectrum $E_{p-1}^{h\Phi}$. If X is a finite spectrum, we refer to the spectral sequence

$$H^*(\Phi; (E_{p-1})_*X) \Longrightarrow (EO_{p-1})_*X$$

as the *Hopkins-Miller spectral sequence*, or HMSS, for X.

The initial term. Our interest in this work began when Hopkins showed us how to compute the initial term of this spectral sequence. To wit,

THEOREM 2.1 (Hopkins-Mahowald).

$$H^*(\Phi; (E_{p-1})_*) \cong \mathbf{F}_{p^{p-1}}[\Delta^{\pm}]\langle\alpha\rangle[\beta],$$

where $|\Delta| = (0, 2p(p-1)^2)$, $|\alpha| = (1, 2(p-1))$, *and* $|\beta| = (2, 2p(p-1))$. (*The cohomological degree is given first.*)

Actually, this presentation ignores much of $H^0(\Phi; (E_{p-1})_*)$. The theorem is true in positive cohomological degrees, whereas $H^0(\Phi; (E_{p-1})_*)/p$ has a splitting with $\mathbf{F}_{p^{p-1}}[\Delta^{\pm}]$ as a direct summand.

As this theorem has not been published, we indicate how the proof goes. The calculation of $H^*(\Phi; (E_{p-1})_*)$ involves choosing coordinates for $(E_{p-1})_*$ on which the action of Φ is easily described. Let $\mathfrak{m} = (p, u_1, \ldots, u_{p-2})$ be the maximal ideal of $(E_{p-1})_*$. There are elements w, $w_1, \ldots, w_{p-2} \in (E_{p-1})_*$ with

$$\begin{aligned} w &\equiv u \mod(p, \mathfrak{m}^2), \\ w_i &\equiv u_i \mod(p, u_1, \ldots, u_{i-1}, \mathfrak{m}^2), \end{aligned}$$

and Φ action is given by

$$\begin{aligned} \sigma(ww_i) &= ww_i + ww_{i-1}, \quad \text{for } 2 \le i \le p-1, \\ \tau(w) &= \eta w, \end{aligned}$$

(following the convention that $w_{p-1} = 1$). The relation $(1 + \sigma + \cdots + \sigma^{p-1})w = 0$, where σ generates $\mathbf{Z}/p$, τ generates $\mathbf{Z}/(p-1)^2$, and η is a primitive $(p-1)^2$ root of unity.

To obtain this representation of $(E_{p-1})_*$, the following formulae are required for the action of Φ on the generators u, u_i, which we also record for later use. Recall that an element $g \in S_n$ (n any positive integer) has a unique expression of the form $g = \sum_{j=0}^{n-1} a_j S^j$, where $a_j \in W\mathbf{F}_{p^n}$ and a_0 is a unit. We have [DH95, Prop. 3.3 and Th. 4.4]

$$g(u) \equiv a_0 u + a_{n-1}^{\chi} u u_1 + \cdots + a_1^{\chi^{n-1}} u u_{n-1} \mod(p, \mathfrak{m}^2)$$

and

$$g(uu_i) \equiv a_0^{\chi^i} u u_i + \cdots + a_{i-1}^{\chi} u u_1 \mod(p, \mathfrak{m}^2),$$

where χ denotes the Frobenius automorphism of $W\mathbf{F}_{p^n}$.

LEMMA 2.2. *Suppose* $g = \sum_{j=0}^{n-1} a_j S^j \in S_n$ *has order* p. *Then* $a_0 \equiv 1 \bmod p$ *and* a_1 *is a unit.*

Proof. Recall that S_n is the group of units in the ring of integers of a certain division algebra D. (See [Rav86] for details.) Write ν for the valuation and note that $S^n = p$ implies $\nu(S) = 1/n$. It then suffices to show that $\nu(g-1) = 1/n$, for then $\nu((g-1)/S) = 0$ and so $(g-1)/S \in S_n$.

Now the extension of the p-adic valuation to D is given by

$$\nu(a) = \frac{\nu(N_{\mathbf{Q}_p(a)/\mathbf{Q}_p}(a))}{[\mathbf{Q}_p(a) : \mathbf{Q}_p]}.$$

Since $g-1$ is a root of the Eisenstein polynomial

$$X^{p-1} + pX^{p-2} + \binom{p}{2} X^{p-3} + \cdots + p,$$

we have $\nu(g-1) = \nu(p)/n = 1/n$, as desired. $\square$

Therefore we can write $\sigma = \sum_{j=0}^{\infty} a_j S^j$ with $a_0 \equiv 1 \bmod p$ and $a_1 \in (W\mathbf{F}_q)^\times$. Furthermore, the construction of σ and τ is such that $\tau = \eta$ for some primitive $(p-1)^2$ root of unity $\eta \in W\mathbf{F}_q$.

Then for $1 \leq i \leq n$,

$$\begin{aligned} \sigma(uu_i) &\equiv uu_i + a_i uu_{i-1} && \operatorname{mod}(p, u_1, \ldots, u_{i-2}, \mathfrak{m}^2), \\ \tau(uu_i) &\equiv \eta^{p^i} uu_i && \operatorname{mod}(p, \mathfrak{m}^2), \end{aligned} \tag{2.1}$$

where $a_i \in (W\mathbf{F}_q)^\times$ and we make the conventions $u_0 = p$ and $u_n = 1$.

To get the desired representation of $(E_{p-1})_*$, Hopkins and Miller start with $s = (1-\sigma)(v_1)/p \in (E_{p-1})_*$. From the definition of the action of S_n on $(E_{p-1})_*$ [Dev95, (5.2)] and the formula $\eta_R(v_1) = v_1 + pt_1$ in $\mathrm{BP}_*\mathrm{BP}$, we have $s = t_1(\sigma^{-1})$, where we write t_i for the image of t_i under the map of Hopf algebroids

$$(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP}) \longrightarrow ((E_{p-1})_*^{\mathrm{Gal}}, \mathrm{Map}_c(S_n, (E_{p-1})_*)^{\mathrm{Gal}}).$$

This map is a composition of several Hopf algebroid maps. By unraveling the definitions of these maps in [Dev95], it is clear that if $g \in S_n$ is given by $g = \sum_{j=0}^{\infty} a_j S^j$, and then $t_i(g) \equiv a_0^{-1} a_i u^{1-p^i} \bmod \mathfrak{m}$.

In particular, $s \equiv cu^{1-p} \bmod \mathfrak{m}$, where $c \in (W\mathbf{F}_q)^\times$. Now let

$$t = s \prod_{j=0}^{p-1} \sigma^j(u) \qquad \text{and} \qquad w = \frac{1}{n^2} \sum_{j=1}^{n^2} \eta^{-j} \tau^j(t).$$

Thus, $w \equiv cu \bmod (p, \mathfrak{m}^2)$, $(1 + \sigma + \cdots + \sigma^{p-1})w = 0$, and $\tau(w) = \eta w$. Finally, let $ww_i = (\sigma - 1)^{n-i}(w)$.

The differentials. The differentials are determined by the Toda differential in the Adams-Novikov spectral sequence (ANSS) and the nilpotence of $\beta_1 \in \pi_* S^0$ as follows. There is a map from the ANSS to the HMSS for S^0,

$$\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \longrightarrow H^*(\Phi; (E_{p-1})_*).$$

By [Rav78], this map sends α_1 to α, β_1 to β, and $\beta_{p/p}$ to $\Delta\beta$. We will omit the phrase "up to multiplication by a unit" from this discussion. (See [Rav86, §1.3] for notation.) Because α_1 and β_1 are permanent cycles, so are α and β.

Let K denote the kernel of the projection $H^0(G;(E_{p-1})_*) \to \mathbf{F}_q[\Delta^{\pm}]$. Because $\beta \cdot K = 0$, d_{2p-1} vanishes on K. Also, from the Toda differential $d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p$ ([Tod67], [Tod68]) and the evident sparseness in our spectral sequence, we conclude that

$$d_{2p-1}(\Delta) = \alpha\beta^{p-1}.$$

These facts and the multiplicative structure of the spectral sequence completely determine d_{2p-1}.

For reasons of degree, the next possible differential is d_{2n^2+1}. As $\beta_1^{pn+1} = 0$ in $\pi_* S^0$ ([Tod67]), [Tod68]), β^{pn+1} is hit by some differential in our spectral sequence. The last differential which could do this is d_{2n^2+1} and therefore

$$d_{2n^2+1}(\Delta^{p-1}\alpha) = \beta^{n^2+1}.$$

Again, d_{2n^2+1} vanishes on K. This determines d_{2n^2+1} completely and the spectral sequence collapses after this point for reasons of degree; hence

PROPOSITION 2.3. *If s is odd, then $E_\infty^{s,t} = 0$ unless $1 \le s \le 2n-1$ and*

$$t \equiv 2n + (s-1)pn + 2pn^2 x \mod 2p^2n^2, \quad \textit{where } x \not\equiv -1 \bmod p.$$

If $s > 0$ is even, then $E_\infty^{s,t} = 0$ unless $2 \le s \le 2n^2$ and

$$t \equiv spn \mod 2p^2n^2.$$

We also need a handle on the zero line. From (2.1), it is easy to show that

PROPOSITION 2.4. $\mathrm{HM}^0 S^0$ *is concentrated in degrees $t \equiv 0 \bmod 2n$.*

Finally, at the other extreme,

PROPOSITION 2.5. *We have*

$$\mathrm{HM}^* V(p-2) \cong \mathbf{F}_{p^n}[u^{\pm n^2}]\langle\alpha\rangle[\beta]$$

with $|\alpha| = (1, 2n)$ and $|\beta| = (2, 2pn)$.

Proof. Basic group cohomology techniques show that

$$H^*(G;N) \cong H^*(\mathbf{Z}/p;N)^{\mathbf{Z}/n^2}$$

if N is a $W\mathbf{F}_{p^n}$-module, because n^2 is a unit in $W\mathbf{F}_{p^n}$. In this case, $N \cong \mathbf{F}_{p^n}[u^{\pm}]$, a trivial $\mathbf{Z}/p$-module. Writing a and b for exterior and polynomial generators in degrees 1 and 2, respectively, one computes $\tau(a) = ea$ and $\tau(b) = eb$, where $e \in (\mathbf{Z}/p)^{\times}$ satisfies $\tau^{-1}\sigma\tau = \sigma^e$.

To complete the picture, we have to relate η^{p-1} and e. Applying $\tau^{-1}(\cdot)\tau$ to the presentation of σ guaranteed by Lemma 2.2, it follows that $\eta^{p-1} \equiv e \bmod p$, hence the proposition. $\square$

3. Proof of Theorem 1.4

In this section we present the proof of Theorem 1.4.

LEMMA 3.1. *Let $-1 \le i \le p-2$. Then any of the following conditions on d imply that* $\mathrm{HM}^{1,d} V(i) = 0$:

i) $d \equiv 2pnk \bmod 2pn^2$ *with* $k \not\equiv 1 \bmod n$;

ii) $d \equiv 2n + 2pnk \bmod 2pn^2$ *with* $k \not\equiv 0 \bmod n$;

iii) $d \equiv 4n + 2pnk \bmod 2pn^2$ *with* $0 \le k \le i-1$.

Proof. Suppose $d \equiv 2pnk \bmod 2pn^2$. We examine which cells of $V(i)$ could support $\mathrm{HM}^{1,d} V(i)$. Now, if $j > 0$, the pinch map $V(j) \to V(j-1)$ lowers internal degree by $|v_j| \equiv 2n \bmod 2pn$. For $s > 0$, $\mathrm{HM}^{s,t} S^0 = 0$ unless $t \equiv 2n \bmod 2pn$ (if s is odd) or $t \equiv 0 \bmod 2pn$ (if s is even). Thus the only possibility is

$$V(i) \longleftarrow V(0) \longrightarrow S^0.$$

But $\mathrm{HM}^{2,d} S^0 = 0$ if $d \not\equiv 2pn \bmod 2pn^2$; hence i) is proved.

We prove ii) by downward induction on i. The case $i = p-2$ is obtained by noting that $d \not\equiv 2n \bmod 2n^2$ and appealing to Proposition 2.5. Now assume ii) holds for some $V(i+1)$ with $i < p-2$. There is an exact sequence

$$\mathrm{HM}^{1,d-|v_{i+1}|} V(i) \longrightarrow \mathrm{HM}^{1,d} V(i) \longrightarrow \mathrm{HM}^{1,d} V(i+1). \tag{3.1}$$

By induction, the right group is zero. Arguing as above, the middle group is supported by cells of the form

$$V(i) \longleftarrow V(j) \longrightarrow V(j-1) \longleftarrow S^0$$

with $j > 0$. In fact, by Proposition 2.3 we must have $j = k$. On the other hand, by i) the left group is zero unless $k = i+1$. But then $j = i+1$, which is not possible.

Finally, iii) follows from ii) by another downward induction on i. The initial case is again secured by Proposition 2.5. In the exact sequence (3.1), the right group is zero by induction and the left group is zero by ii). $\square$

PROPOSITION 3.2. *Let $3 \le k \le p-1$. The class $v_k \in \mathrm{HM}^0 V(k-1)$ lives on the cell*

$$V(k-1) \longrightarrow V(k-2) \longleftarrow V(0) \longrightarrow S^0$$

where its name is $\beta\Delta^r$ with $r \equiv 1 \bmod p$.

Proof. We first show that $\delta(v_k) \neq 0$. Suppose not, and that $\delta(v_k) = 0$. Then v_k is in the image of the map

$$\mathrm{HM}^0 V(k-2) \longrightarrow \mathrm{HM}^0 V(k-1);$$

i.e., v_k lifts to a fixed point $\gamma \in u^s \mathbf{F}_q[[u_{k-1}, \ldots, u_{n-1}]]$, where $s = 1 - p^k$. If $\mu \in u^s \mathbf{F}_q[[u_{k-1}, \ldots, u_{n-1}]]$ is a monomial, we write $\mu \in \gamma$ if μ appears as a term (up to multiplication by a nonzero scalar) when γ is expressed as a sum of monomials.

We claim that $u^s u_k \notin \gamma$, which provides the desired contradiction. Recall from (2.1) the formula

$$(\sigma - 1)(u^s u_i) \equiv c_i u^s u_{i-1} \bmod (p, u_1, \ldots, u_{i-2}, \mathfrak{m}^2).$$

Suppose $u^s \in \gamma$. Since $(\sigma - 1)\gamma = 0$ and $u^s u_{n-1} \in (\sigma - 1)u^s$, there must be another monomial $\mu \in \gamma$ with $u^s u_{n-1} \in (\sigma - 1)\mu$. But by the formula above, this is not possible; hence $u^s \notin \gamma$. Iterating this procedure yields $u^s u_i \notin \gamma$ for $k \leq i \leq n$, hence the claim.

Next we show that the map

$$\mathrm{HM}^{1,|\delta(v_k)|} V(0) \longrightarrow \mathrm{HM}^{1,|\delta(v_k)|} V(k-2)$$

is surjective. It suffices to show that

$$\mathrm{HM}^{2,|\delta(v_k)|-|v_i|} V(i-1) = 0$$

for $i = 1, 2, \ldots, k-2$. Since $|\delta(v_k)| - |v_i| \equiv -2n \bmod 2pn$, this follows by arguing as in the proof of Lemma 3.1 i).

Let $y \in \mathrm{HM}^{1,|\delta(v_k)|} V(0)$ be the class which hits $\delta(v_k)$. If $k \geq 3$, then

$$|y| = |v_k| - |v_{k-1}| \equiv 2pn + 2pn^2 \bmod 2p^2 n^2.$$

By Theorem 2.1, $\mathrm{HM}^1 S^0$ is zero in this degree, and so y lives on the top cell of $V(0)$. The class in that degree is $\beta \Delta^r$ with $r \equiv 1 \bmod p$. □

Remark 3.3. This proof shows that v_1 lives on the top cell of $V(0)$ and v_2 lives on the top cell of $V(1)$. For reasons of degree, their names are α and β, respectively. In particular, α and β "are" α_1 and β_1.

LEMMA 3.4. *Let $k < p - 1$. If $d \in \mathbf{Z}$ satisfies*

$$d \equiv 2n + 2pn + 2p^2 ny \bmod 2p^2 n^2,$$

where $y \not\equiv 0 \bmod n$, and $V(k)$ exists, then

$$\pi_{d-1}(EO_{p-1} \wedge V(k)) = 0.$$

Proof. We show that the relevant groups are zero on each cell of $V(k)$. Specifically, in the HMSS for S^0, $E_\infty^{s,t} = 0$ whenever

$$t - s = d - 1 - j - \sum_{h=1}^{j} |v_{i_h}| \tag{3.2}$$

with $0 \le j \le k+1$ and $0 \le i_1 < i_2 < \cdots < i_j \le k$. Here the i_h simply record where pinching occurs.

First, suppose $s = 0$. Reducing (3.2) mod $2n$, we have $t \equiv -(j+1)$. By Proposition 2.4, $E_\infty^{0,t} = 0$.

Next, suppose s is odd. By Proposition 2.3, we may take $1 \le s \le 2n-1$ and $t \equiv 2n + (s-1)pn \bmod 2pn^2$. Substituting this into (3.2) and reducing mod $2n$, we are led to $s = j+1$. Substituting this into (3.2) and reducing mod p yields $j = 0$ or $j = 1$ (if $i_1 = 0$). The second case cannot occur because s is odd, therefore we may take $j = 0$ and $s = 1$. But $E_\infty^{1,d} = 0$ by Proposition 2.3.

Finally, suppose $s > 0$ is even. In this case, we may take $2 \le s \le 2n^2$ and $t \equiv spn \bmod 2p^2n^2$. Working mod $2n$ as above, we get $s = 2nl + j + 1$, where $0 \le l < n$. Note that j must be odd. Substituting into (3.2) and reducing mod p leads to $l = j-1$ (if $i_1 > 0$) or $l = j-2$ (if $i_1 = 0$). To finish the proof, we need to reduce (3.2) mod $2p^2n$, which is a bit unwieldy. There are four cases:

$$t - s \equiv 2n + 2pn - 1 - j - \begin{cases} (2pn+2n)j, & i_1 > 1, \\ (2pn+2n)(j-1) + 2n, & i_1 = 1, \\ (2pn+2n)(j-1), & i_1 = 0, i_2 > 1, \\ (2pn+2n)(j-2) + 2n, & i_1 = 0, i_2 = 1. \end{cases}$$

For each case, we substitute $t = spn$, $s = 2nl + j + 1$, and either $l = j-1$ (first two cases) or $l = j-2$ (last two cases), reduce mod $2p$, and solve for j. The first and third cases yield $j \equiv -1 \bmod 2p$, which is not possible. The second and fourth cases yield $j \equiv 1 \bmod 2p$, i.e., $j = 1$. In the fourth case, this is not possible, because there we are assuming that $i_2 = 1$, and so $j > 1$. This leaves the second case, with $j = 1$. But $E_\infty^{2,d-2n} = 0$ by Proposition 2.3. $\square$

Proof of Theorem 1.4. The cases $k = 1$, 2 and 3 are immediate, because v_1, v_2, and v_3 are known to be permanent cycles in the corresponding Adams-Novikov spectral sequences. So let $k \ge 4$. Let $i : V(1) \to V(k-2)$ be the inclusion of the bottom $V(1)$. By Proposition 3.2, $\delta(v_k)$ equals $i_*\delta(v_3)$ times a pth power of Δ. Therefore $\delta(v_k)$ is a permanent cycle.

By Lemma 3.4, $\pi_{|v_k|-1}(EO_{p-1} \wedge V(k-2)) = 0$, so

$$\pi_{|v_k|}(EO_{p-1} \wedge V(k-1)) \xrightarrow{h} \pi_{|v_k|-|v_{k-1}|-1}(EO_{p-1} \wedge V(k-2))$$

is surjective. Let $y \in \pi_{|v_k|}(EO_{p-1} \wedge V(k-1))$ be a class which hits $\delta(v_k)$. Note that y necessarily has filtration zero; i.e., $y \in \mathrm{HM}^{0,|v_k|}V(k-1)$ and $\delta(y) = \delta(v_k)$.

Of course, we may not have $y = v_k$ on the nose. Arguing as in the proof of Lemma 3.1 i), $\mathrm{HM}^{2,|\beta v_k|}V(k-2) = 0$, and so $\delta(\beta y) = \delta(\beta v_k)$ implies $\beta y = \beta v_k$. Therefore βv_k is a permanent cycle, which suffices. □

4. **Proof of Theorem 1.5**

In this section we prove Theorem 1.5. Throughout, $m = (p+1)/2$.

LEMMA 4.1. *The class $\beta v_m^2 \in \mathrm{HM}^2 V(m-1)$ is nonzero.*

Proof. By Proposition 3.2, $\delta(v_m) \neq 0$. Therefore $\delta(\beta v_m) = \beta\delta(v_m) \neq 0$, so $\beta v_m \neq 0$. To finish the proof, it suffices to show that $\mathrm{HM}^{1,|\beta v_m^2|}V(m) = 0$. But this follows from part iii) of Lemma 3.1. □

Remark 4.2. In fact, $\beta v_k^2 \neq 0$ if and only if $m \leq k \leq p-1$.

PROPOSITION 4.3. *The class $v_m^2 \in \mathrm{HM}^0 V(m-1)$ lives on the cell*

$$V(m-1) \longleftarrow V(1) \longrightarrow V(0) \longleftarrow S^0$$

where its name is $\alpha\Delta^r$ with $r \equiv -2 \bmod p$.

Proof. By Lemma 4.1, it suffices to determine the name of βv_m^2. Let $d = |\beta v_m^2|$. Since $d \equiv 4n \bmod 2pn$, there are two ways such a class can arise. One is by pinching at some $V(i)$ with $1 \leq i \leq m-1$:

$$V(m-1) \longleftarrow V(i) \longrightarrow V(i-1) \longleftarrow S^0.$$

Since $d - |v_i| \equiv 2n + 2pn(2-i) \bmod 2pn^2$, this only works if $i = 1$. In that case, $d - |v_1| \equiv 2n + 2pn - 4pn^2 \bmod 2p^2n^2$ and the Proposition is proved.

The other possibility is to pinch at $V(i)$ and $V(j)$ with $1 \leq i < j \leq m-1$:

$$V(m-1) \longleftarrow V(i) \longrightarrow V(i-1) \longleftarrow V(j) \longrightarrow V(j-1) \longleftarrow S^0.$$

The resulting class is in degree $d - |v_i| - |v_j| \equiv 2pn(3-i-j) \bmod 2pn^2$. But the relevant cohomology is zero unless $i + j \equiv 1 \bmod n$, or $i + j = p$, which is not possible. □

Notation 1. We denote by $[v_m^2]$ an element in $\mathrm{HM}^{0,|v_m^2|}V(2)$ which reduces to $v_m^2 \bmod I_m$, the existence of such being secured by Proposition 4.3.

LEMMA 4.4. *The class $\beta v_3[v_m^2] \in \mathrm{HM}^2 V(2)$ is nonzero.*

Proof. We will show that $\beta^{(p-3)/2}v_3[v_m^2] \neq 0$. Let $d = |\beta^{(p-3)/2}v_3 v_m^2|$. By Lemma 4.1, $\beta^{(p-3)/2}[v_m^2] \neq 0$, so it suffices to show that $\mathrm{HM}^{p-4,d}V(3) = 0$. To see this, we claim that the map

$$\mathrm{HM}^{1,d+|v_4|+\cdots+|v_{p-2}|}V(p-2) \longrightarrow \mathrm{HM}^{p-4,d}V(3),$$

obtained by pinching to the top $V(3)$ of $V(p-2)$, is surjective. Assuming this, Lemma 4.4 follows by noting that $d+|v_4|+\cdots+|v_{p-2}| \equiv -4n \bmod 2n^2$ and appealing to Proposition 2.5.

As for the claim, it suffices to show that

$$\mathrm{HM}^{p-3-k,d+|v_4|+\cdots+|v_{k+3}|}V(k+2)=0$$

for $k=1,2,\ldots,p-5$. Now $d+|v_4|+\cdots+|v_{k+3}| \equiv 2n(k+3) \bmod 2pn$, so we have to pinch all the way to the top cell to support this group:

$$\mathrm{HM}^{p,d-|v_1|-|v_2|-|v_3|+|v_{k+3}|}S^0.$$

Since $\mathrm{HM}^{p,t}S^0=0$ unless $t \equiv 2n+pn^2 \bmod 2pn^2$, this group is zero. □

PROPOSITION 4.5. *The class $v_3[v_m^2] \in \mathrm{HM}^0 V(2)$ lives on the top cell of $V(2)$, where its name is $\alpha\beta\Delta^k$, with $k \equiv -1 \bmod p$.*

Proof. By Lemma 4.4, we may consider $\beta v_3[v_m^2]$ instead. Because $d \equiv 6n \bmod 2pn$, this class lives on the top cell of $V(2)$. Since $d-|v_2|-|v_1| \equiv 2n+4pn-2pn^2 \bmod 2p^2n^2$, the name is as claimed. □

Proof of Theorem 1.5. It follows from Proposition 4.5 and the differentials in the HMSS for S^0 that $v_3[v_m^2]$ supports a differential. Now the differentials in the HMSS for $V(2)$ are v_3-linear because $V(2)$ admits a v_3 self-map, by [Tod71]. Therefore $[v_m^2]$ supports a differential.

Next, we claim that the map

$$\pi_{|v_m^2|}(EO_{p-1}\wedge V(2)) \longrightarrow \pi_{|v_m^2|}(EO_{p-1}\wedge V(m-1)) \tag{4.1}$$

is surjective. By Lemma 3.4, $\pi_{|v_m^2|-|v_k|-1}(EO_{p-1}\wedge V(k-1))=0$ for $k \not\equiv 0 \bmod n$, which suffices. In filtration zero, (4.1) is just reduction mod I_m, so it follows that v_m^2 cannot be a permanent cycle. □

Remark 4.6. We have produced a differential on v_m^2 without saying exactly what it is. Now let $[v_m^2]$ denote a lift of v_m^2 to $V(1)$, which exists by Proposition 4.3. Then

$$d_{2n(p-2)+1}([v_m^2])=\beta^{n(p-2)}\widehat{\beta}\Delta^r,$$

where $r \equiv 0 \bmod p$ and $\widehat{\beta}$ is a class which lives on the cell

$$V(1) \longleftarrow V(0) \longrightarrow S^0$$

with name β. (We leave the verification of this to the reader.) Unfortunately we do not know what happens on $V(m-1)$. The class $\widehat{\beta}$ persists until that point, but it is not clear that the differential does.

Appendix A. Proof of Theorem 1.2

For the reader's convenience, we give the following:

Proof of Theorem 1.2. Write $H = H\mathbf{F}_p$ and suppose $H_*X \cong E(\tau_0, \dots, \tau_k)$. Let $X \xrightarrow{f} H$ correspond to $1 \in H^0X$. Then $H_*(f)$ is a map of $\mathscr{A}_*$-comodules and as such is the inclusion $H_*X \hookrightarrow \mathscr{A}_*$. Since f is surjective in dimension zero, $\pi_0(f)$ is onto by the Whitehead theorem. Then let $S^0 \xrightarrow{g} X$ map to the unit $S^0 \to H$.

Now BP_* can be computed from $H_*\mathrm{BP}$ in precisely the same way that MU_* is computed from $H_*\mathrm{MU}$, namely via the Adams spectral sequence

$$\mathrm{Ext}_{\mathscr{A}_*}(\mathbf{F}_p, H_*\mathrm{BP}) \Longrightarrow \mathrm{BP}_*.$$

Although this is a standard computation, we summarize the argument for the sake of completeness. (See [Rav86, §3.1] or [Swi75, Chap. 20] for more details.) One computes $H_*\mathrm{BP} \cong \mathbf{F}_p[\xi_1, \xi_2, \dots]$ as a comodule algebra over $\mathscr{A}_*$. Then

$$\begin{aligned}\mathrm{Ext}_{\mathscr{A}_*}(\mathbf{F}_p, H_*\mathrm{BP}) &\cong \mathrm{Ext}_{E(\tau_0,\dots)}(\mathbf{F}_p, \mathbf{F}_p)\\ &\cong \mathbf{F}_p\big[[\tau_0], [\tau_1], \dots\big],\end{aligned}$$

where $[\tau_i] \in \mathrm{Ext}^{1,2p^i-1}$. Here the first isomorphism is a change of rings argument and the second is by explicit calculation. The spectral sequence collapses because it is concentrated in even degrees. One solves the extension problems to obtain $\pi_*\mathrm{BP} \cong \mathbf{Z}_{(p)}[v_1, v_2, \dots]$, where $v_i \equiv [\tau_i]$ mod decomposables.

The calculation of BP_*X proceeds in the same way. The map $S^0 \xrightarrow{g} X$ furnishes a map of Adams spectral sequences

$$\begin{array}{ccc}\mathrm{Ext}_{\mathscr{A}_*}(\mathbf{F}_p, H_*(\mathrm{BP}\wedge S^0)) & \Longrightarrow & \mathrm{BP}_*S^0\\ & & \downarrow\\ \mathrm{Ext}_{\mathscr{A}_*}(\mathbf{F}_p, H_*(\mathrm{BP}\wedge X)) & \Longrightarrow & \mathrm{BP}_*X.\end{array} \tag{A.2}$$

We have $H_*(\mathrm{BP}\wedge X) \cong H_*\mathrm{BP} \otimes_{\mathbf{F}_p} E(\tau_0, \dots, \tau_k)$,

$$\begin{aligned}\mathrm{Ext}_{\mathscr{A}_*}(\mathbf{F}_p, H_*(\mathrm{BP}\wedge X)) &\cong \mathrm{Ext}_{E(\tau_{k+1},\dots)}(\mathbf{F}_p, \mathbf{F}_p)\\ &\cong \mathbf{F}_p\big[[\tau_{k+1}], \dots\big],\end{aligned}$$

and the map of (A.2) is the obvious one. It follows that $\mathrm{BP}_*X \cong \mathrm{BP}_*/I_{k+1}$ and $\mathrm{BP}_*(g)$ is the natural projection.

Now suppose $\mathrm{BP}_*X \cong \mathrm{BP}_*/I_{k+1}$. To begin, we review how $\mathscr{A}_*$ can be recovered from BP_*H. Let $h : \mathrm{BP} \to H$ be the map of ring spectra which classifies the additive formal group law over $\mathbf{F}_p$. Zahler [Zah72] has shown that $t_i \mapsto c(\xi_i)$ under $(h\wedge h)_* : \mathrm{BP}_*\mathrm{BP} \to \mathscr{A}_*$, where as usual, $c : \mathscr{A}_* \to \mathscr{A}_*$ is the conjugation.

Since $(1 \wedge h)_* : H_*\mathrm{BP} \to \mathscr{A}_*$ is the inclusion of $\mathbf{F}_p[\xi_1, \ldots]$, it follows that $\mathrm{BP}_* H \cong \mathbf{F}_p[t_1, \ldots]$, where we write t_i for $(1 \wedge h)_*(t_i)$.

The map h makes H into a BP-module, so we get a universal coefficient spectral sequence

$$E^2 \equiv \mathrm{Tor}^{\mathrm{BP}_*}(\mathrm{BP}_* H, \mathbf{F}_p) \Longrightarrow \mathscr{A}_*. \tag{A.3}$$

Now the following diagram commutes

$$\begin{tikzcd}
\mathrm{BP}_* H \otimes \mathrm{BP}_* \arrow[r] \arrow[d, "(h\wedge 1)\otimes h"'] & \mathrm{BP}_* H \arrow[d, "h\wedge 1"] \\
\mathscr{A}_* \otimes H_* \arrow[r] & \mathscr{A}_*.
\end{tikzcd}$$

Thus the elements of BP_* in positive degrees act trivially on $\mathrm{BP}_* H$, and so

$$E^2 \cong \mathbf{F}_p[t_1, \ldots] \otimes_{\mathbf{F}_p} \mathrm{Tor}^{\mathrm{BP}_*}(\mathbf{F}_p, \mathbf{F}_p).$$

Now $\mathrm{Tor}^{\mathrm{BP}_*}(\mathbf{F}_p, \mathbf{F}_p)$ can be computed using a Koszul resolution, as in [Lan84, §16.10]. Explicitly, let $C_{*,*} = E_{\mathrm{BP}_*}(e_0, e_1, \ldots)$, where $e_i \in C_{1,2p^i-2}$. Define $d : C_{*,*} \to C_{*-1,*}$ by

$$d(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{j=1}^{r} (-1)^{j-1} v_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_r}.$$

Then $C \to \mathbf{F}_p$ is a resolution of $\mathbf{F}_p$ by free, graded BP_*-modules. After applying $(-) \otimes_{\mathrm{BP}_*} \mathbf{F}_p$, the differential becomes trivial, and

$$\begin{aligned}
\mathrm{Tor}^{\mathrm{BP}_*}(\mathbf{F}_p, \mathbf{F}_p) &\cong H(C \otimes_{\mathrm{BP}_*} \mathbf{F}_p) \\
&\cong E(e_0, e_1, \ldots).
\end{aligned}$$

Because the map h makes H into a BP-algebra, (A.3) is actually a spectral sequence of BP_*-algebras. Since the E^2 term is generated by elements in homological degrees 0 and 1, the spectral sequence collapses and $\mathscr{A}_* \cong \mathbf{F}_p[\xi_1, \ldots] \otimes_{\mathbf{F}_p} E(\tau_0, \ldots)$, where $c(\xi_i)$ is detected by t_i and τ_i is detected by e_i up to decomposables.

Now because $\mathrm{BP}_s X = 0$ for $s < 0$, it follows that X is 0-connected and $\pi_0 X \cong \mathbf{F}_p$. Let $X \xrightarrow{f} H$ correspond to the generator in $H^0 X \cong \mathbf{F}_p$. In order to show that $H_* X$ is isomorphic to $E(\tau_0, \ldots, \tau_k)$ as an $\mathscr{A}_*$-comodule, we must show that the image of $H_* X$ under $f_* \equiv H_*(f)$ is precisely $E(\tau_0, \ldots, \tau_k)$.

Now f provides a map of universal coefficient spectral sequences

$$\begin{array}{c} \mathrm{Tor}^{\mathrm{BP}_*}(\mathrm{BP}_*X, \mathbf{F}_p) \Longrightarrow H_*X \\ \downarrow \\ \mathrm{Tor}^{\mathrm{BP}_*}(\mathrm{BP}_*H, \mathbf{F}_p) \Longrightarrow \mathscr{A}_*. \end{array} \tag{A.4}$$

Using the Koszul resolution $D_{*,*} = E_{\mathrm{BP}_*}(e_0, \ldots, e_k)$, it follows easily that

$$\mathrm{Tor}^{\mathrm{BP}_*}(\mathrm{BP}_*X, \mathbf{F}_p) \cong E(e_0, \ldots, e_k).$$

Note that we are not assuming that X is a ring spectrum and therefore we cannot expect H_*X, or the spectral sequence converging to it, to have a multiplicative structure. Nevertheless, if we think of $E(e_0, \ldots, e_k)$ as a ring in the usual way, it is clear that the map of (A.4) is the obvious ring homomorphism at the E^2 term.

Therefore the spectral sequence collapses and $H_*X \cong E(d_0, \ldots, d_k)$ additively, where $|d_i| = 2p^i - 1$. Furthermore, f_* is injective, because the corresponding statement holds true for the associated graded modules. Thus it remains to show that $f_*(H_*X) \subset E(\tau_0, \ldots, \tau_k)$.

Order the monomials of $f_*(H_*X)$ by degree: $1, x_1, x_2, \ldots, x_N$. Then clearly $x_1 = \tau_0$. Now fix $m > 1$ and suppose that $x_i \in E(\tau_0, \ldots, \tau_k)$ for each $i < m$. Write Δ for the coproduct on $\mathscr{A}_*$ and Δ_X for the comodule structure map of H_*X. Then $\Delta \circ f_* = (1 \otimes f_*) \circ \Delta_X$, and so

$$\Delta x_m - 1 \otimes x_m \in \mathscr{A}_* \otimes_{\mathbf{F}_p} E(\tau_0, \ldots, \tau_k).$$

Recall that Δ is given by

$$\Delta(\xi_i) = \sum_{s+t=i} \xi_s^{p^t} \otimes \xi_t \qquad \text{and} \qquad \Delta(\tau_i) = \tau_i \otimes 1 + \sum_{s+t=i} \xi_s^{p^t} \otimes \tau_t,$$

where $\xi_0 = 1$. Now suppose $x_m \notin E(\tau_0, \ldots, \tau_k)$, say $x_m = y\xi_s^t + \cdots$, where s is maximal and t is the highest power of ξ_s occurring in x_m. If $|y| > 0$, then

$$\Delta x_m - 1 \otimes x_m = y \otimes \xi_s^t + \cdots,$$

a contradiction. Thus $|y| = 0$, say $y = 1$.

Now write $t = p^a b$ with $(p, b) = 1$. Then

$$\Delta x_m - 1 \otimes x_m = b\xi_s^{p^a} \otimes \xi_s^{p^a(b-1)} + \cdots$$

and therefore $b = 1$. We conclude by showing that $H_*X = 0$ in degree $|\xi_s^{p^a}|$. Suppose that

$$2p^a(p^s - 1) = (2p^{i_1} - 1) + \cdots + (2p^{i_r} - 1),$$

where $0 \le i_1 < i_2 < \cdots < i_r$. Then clearly $s + a > i_r$, and

$$\begin{aligned} 2(p^{s+a} - 1) &\ge 4(1 + p + \cdots + p^{s+a-1}) \\ &\ge 2(1 + p + \cdots + p^{s+a-1}) + 2p^a \\ &\ge 2(1 + p + \cdots + p^{i_r}) + 2p^a. \end{aligned}$$

In particular, $2p^a(p^s - 1) > (2p^{i_1} - 1) + \cdots + (2p^{i_r} - 1)$, which completes the proof. □

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