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Transformations of elliptic hypergeometric integrals

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Abstract

We prove a pair of transformations relating elliptic hypergeometric integrals of different dimensions, corresponding to the root systems BC_n and A_n ; as a special case, we recover some integral identities conjectured by van Diejen and Spiridonov. For BC_n , we also consider their “Type II” integral. Their proof of that integral, together with our transformation, gives rise to pairs of adjoint integral operators; a different proof gives rise to pairs of adjoint difference operators. These allow us to construct a family of biorthogonal abelian functions generalizing the Koornwinder polynomials, and satisfying the analogues of the Macdonald conjectures. Finally, we discuss some transformations of Type II-style integrals. In particular, we find that adding two parameters to the Type II integral gives an integral invariant under an appropriate action of the Weyl group E_7 .

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1. Introduction

In recent work, van Diejen and Spiridonov [6], [7], [32] have produced a number of conjectural elliptic hypergeometric integration formulae, common generalizations of Spiridonov's elliptic beta integral [31] and q -hypergeometric integration identities due to Gustafson [11]. In particular, for the BC_n root system, they gave two conjectures, "Type I" and "Type II" (proved as Corollary 3.2 and Theorem 6.1 below), and showed that the Type I integral would imply the Type II integral. In an appropriate limit, their Type II integral transforms via residue calculus into a sum originally conjectured by Warnaar [37] (and proved by Rosengren [25]). In fact, Warnaar also conjectured a more general formula, a Bailey-type transformation identity, rather than a Jackson-type summation identity. This suggests that there should be transformation formulae on the integral level as well; this is the topic of the present work.

Their Type I integral can be thought of as the ultimate generalization of an integral identity used by Anderson [1] in his proof of the Selberg integral (which the Type II integral generalizes). While Anderson's proof of this identity (based on a clever change of variables) does not appear to generalize any further, some recent investigations of Forrester and the author [9] of a random matrix interpretation of the Anderson integral suggested a different argument, which as we will see does indeed generalize to the elliptic level. While the argument was not powerful enough to directly prove the Type I integral, it *was* able to prove it for a countably infinite union of submanifolds of parameter space. This suggested that this argument should at least suffice to produce the correct conjecture for a transformation law; in the event, it produced not only a conjecture but a proof. We thus obtain an identity relating an n -dimensional integral with $2n + 2m + 4$ parameters to an m -dimensional integral with transformed parameters; when $m = 0$, this gives the van Diejen-Spiridonov integral, but the proof requires this degree of freedom. A similar identity for the A_n root system follows by a slight modification of the argument; this gives a transformation generalization of a conjecture of Spiridonov [32]. The basic idea for both proofs is that, in an appropriate special case, the transformations can be written as determinants of relatively simple one-dimensional transformations. This "determinantal" case is thus easy to prove; moreover, by taking limits of some of the remaining degrees of freedom, we can transform the n -dimensional determinantal identity into a lower-dimensional, but nondeterminantal instance of the transformation. Indeed, by repeating this process, starting with a sufficiently large instance of the determinantal case, we can obtain a dense set of special cases of the desired transformation, thus proving the theorem.

As mentioned above the Type II integral follows as a corollary of the Type I integral. In many ways, this integral is of greater interest, most notably because it generalizes the inner product density for the Koornwinder polynomials [14].

Since the inner product density generalizes, it would be natural to suppose that the orthogonal polynomials themselves should generalize. It would be too much to expect them to generalize to orthogonal functions, however; indeed, even in the univariate case, the elliptic analogues of the Askey-Wilson polynomials are merely *bi* orthogonal (these analogues are due to Spiridonov and Zhedanov [34], [35] in the discrete case (generalizing work of Wilson [38]), and Spiridonov [32] in the continuous case (generalizing work of Rahman [17])). With this in mind, we will construct in the sequel a family of functions satisfying biorthogonality with respect to the Type II integral.

There are two main ingredients in this construction. The first is a family of difference operators, generalizing some difference operators known to act nicely on the Koornwinder polynomials [20], and satisfying adjointness relations with respect to the elliptic inner product. As a special case, we obtain a difference-operator-based proof of the Type II integral. This suggests that the proof based on the Type I integral should be related to a pair of adjoint *integral* operators, which form the other main ingredient in our construction. It turns out that the $BC_n \leftrightarrow BC_m$ transformation plays an important role in understanding these integral operators; indeed, by taking limits of the transformation so that one side becomes a finite sum, we obtain formulas for the images under the integral operator of a spanning set of its domain. The biorthogonal functions are then constructed as the images of suitable sequences of difference and integral operators. (This construction is new even at the level of Koornwinder polynomials.)

As these functions are biorthogonal with respect to a generalization of the Koornwinder density (and indeed contain the Koornwinder polynomials as a special case, although this turns out to be somewhat subtle to prove), one of course expects that they satisfy analogous properties. While the Hecke-algebraic aspects of the Koornwinder theory (see, for instance, [29]) are still quite mysterious at the elliptic level, the main properties, i.e., the “Macdonald conjectures”, do indeed carry over. Two of these properties, namely the closed forms for the principal specialization and the nonzero values of the inner product, follow quite easily from the construction and adjointness; the third (evaluation symmetry) will be proved in a follow-up paper [21]. The arguments there are along the same lines as those given in [20] for the Koornwinder case, which were based on Okounkov’s BC_n -symmetric interpolation polynomials [15]. Unlike in the Koornwinder case, however, at the elliptic level the required interpolation functions are actually special cases of the biorthogonal functions.

Just as in [20], these interpolation functions satisfy a number of generalized hypergeometric identities, having Warnaar’s multivariate identities and conjectures [37] as special cases. To be precise, they satisfy multivariate analogues of Jackson’s summation and Bailey’s transformation. The former identity has an integral

analogue ([Theorem 9.2](#) below), extending the Type II integral. In fact, the transformation also has an analogue, stated as [Theorem 9.7](#) below; as a special case, this gives our desired transformation analogue of the Type II integral. The simplest version of this transformation states that an eight-parameter version of the Type II integral is invariant with respect to an action of the Weyl group E_7 ; in fact, this action extends formally to an action of E_8 , acting on the parameters in the most natural way.

The plan of the paper is as follows. After defining some notation at the end of this introduction, we proceed in [Section 2](#) to discuss Anderson's integral, as motivation for our proof of the BC_n and A_n integral transformations. These transformations are then stated and proved in [Sections 3](#) and [4](#), respectively; we also briefly discuss in [Section 5](#) some hybrid transformations arising from the fact that the BC_1 and A_1 integrals are the same, but the transformations are not. [Section 6](#) then begins our discussion of the biorthogonal functions by describing the three kinds of difference operators, as well as the spaces of functions on which they act, and filtrations of those spaces with respect to which the operators are triangular. [Section 7](#) discusses the corresponding integral operators (all of which are special cases of a single operator defined from the Type I integral), showing that they are triangular with respect to the same filtrations as the difference operators. Then, in [Section 8](#), we combine these ingredients to construct the biorthogonal functions, and describe their main properties. In [Section 9](#), we discuss our Type II transformations. Finally, in an appendix, we give a general result regarding when an integral of a meromorphic function is meromorphic, and apply it to obtain precise information about the singularities of our integrals.

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Notation. We will need a number of generalized q -symbols in the sequel. First, define the theta function and elliptic Gamma function [[28](#)]:

$$\begin{aligned}\theta(x; p) &:= \prod_{0 \leq k} (1 - p^k x)(1 - p^{k+1}/x), \\ \Gamma(x; p, q) &:= \prod_{0 \leq j, k} (1 - p^{j+1} q^{k+1}/x)(1 - p^j q^k x)^{-1}.\end{aligned}$$

In each case, the presence of multiple arguments before the semicolon indicates a product; thus, for instance,

$$\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q) = \Gamma(z_i z_j; p, q) \Gamma(z_i/z_j; p, q) \Gamma(z_j/z_i; p, q) \Gamma(1/z_i z_j; p, q).$$

These functions satisfy a number of identities, most notably

$$\theta(x; p) = \theta(p/x; p) = (-x)\theta(1/x; p),$$

and

$$\begin{aligned} \Gamma(x; p, q) &= \Gamma(pq/x; p, q)^{-1}, \\ \Gamma(px; p, q) &= \theta(x; q)\Gamma(x; p, q), \\ \Gamma(qx; p, q) &= \theta(x; p)\Gamma(x; p, q). \end{aligned}$$

Using the theta function, one can define an elliptic analogue of the q -symbol; in fact, just as the elliptic Gamma function is symmetric in p and q , we will want our elliptic q -symbol also to be symmetric. Thus, we define

$$\theta(x; p, q)_{l,m} := \prod_{0 \leq k < l} \theta(p^k x; q) \prod_{0 \leq k < m} \theta(q^k x; p),$$

so that

$$\frac{\Gamma((p, q)^{l,m} x; p, q)}{\Gamma(x; p, q)} = (-x)^{-lm} p^{-lm(l-1)/2} q^{-lm(m-1)/2} \theta(x; p, q)_{l,m},$$

where

$$(p, q)^{l,m} := p^l q^m.$$

We also need some multivariate symbols, indexed (as the biorthogonal functions will be) by pairs of partitions. By convention, we will use bold greek letters to refer to such partition pairs, and extend transformations and relations of partitions in the obvious way. We then define, following [20],

$$\begin{aligned} \mathcal{C}_{\lambda}^0(x; t; p, q) &:= \prod_{1 \leq i} \theta(t^{1-i} x; p, q)_{\lambda_i}, \\ \mathcal{C}_{\lambda}^-(x; t; p, q) &:= \prod_{1 \leq i \leq j} \frac{\theta(t^{j-i} x; p, q)_{\lambda_i - \lambda_{j+1}}}{\theta(t^{j-i} x; p, q)_{\lambda_i - \lambda_j}}, \\ \mathcal{C}_{\lambda}^+(x; t; p, q) &:= \prod_{1 \leq i \leq j} \frac{\theta(t^{2-i-j} x; p, q)_{\lambda_i + \lambda_j}}{\theta(t^{2-i-j} x; p, q)_{\lambda_i + \lambda_{j+1}}}. \end{aligned}$$

We note that each of the above \mathcal{C} symbols extends to a holomorphic function on $x \in \mathbb{C}^*$.

Two particular combinations of \mathcal{C} symbols will occur frequently enough to merit their own notation. We define:

$$\begin{aligned} \Delta_{\lambda}^0(a|\dots b_i \dots; t; p, q) &:= \frac{\mathcal{C}_{\lambda}^0(\dots b_i \dots; t; p, q)}{\mathcal{C}_{\lambda}^0(\dots pqa/b_i \dots; t; p, q)}, \\ \Delta_{\lambda}(a|\dots b_i \dots; t; p, q) &:= \Delta_{\lambda}^0(a|\dots b_i \dots; t; p, q) \\ &\quad \times \frac{\mathcal{C}_{2\lambda}^0(pqa; t; p, q)}{\mathcal{C}_{\lambda}^-(pq, t; t; p, q) \mathcal{C}_{\lambda}^+(a, pqa/t; t; p, q)}. \end{aligned}$$

We will also need the following notion, where $0 < |p| < 1$.

Definition 1. A BC_n -symmetric (p -)theta function of degree m is a holomorphic function $f(x_1, \dots, x_n)$ on $(\mathbb{C}^*)^n$ such that

- $f(x_1, \dots, x_n)$ is invariant under permutations of its arguments.
- $f(x_1, \dots, x_n)$ is invariant under $x_i \mapsto 1/x_i$ for each i .
- $f(px_1, x_2, \dots, x_n) = (1/px_1^2)^m f(x_1, x_2, \dots, x_n)$.

A BC_n -symmetric (p)-abelian function is a meromorphic function satisfying the above conditions with $m = 0$.

In particular, a BC_n -symmetric theta function of degree m is a BC_1 -symmetric theta function of degree m in each of its arguments. Now, the space of BC_1 -symmetric theta functions of degree m is $m + 1$ -dimensional, and moreover, any nonzero BC_1 -symmetric theta function vanishes at exactly $2m$ orbits of points (under multiplication by p , and counting multiplicity). Thus we can show that a BC_1 -symmetric theta function vanishes by finding $m + 1$ independent points at which it vanishes.

The canonical example of a BC_n -symmetric theta function of degree 1 is

$$\prod_{1 \leq i \leq n} \theta(ux_i^{\pm 1}; p);$$

indeed, the functions for any $n + 1$ distinct choices of u form a basis of the space of BC_n -symmetric theta functions of degree 1:

$$f(\dots x_i \dots) = \sum_{0 \leq j \leq n} f(u_0, u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n) \frac{\prod_{1 \leq i \leq n} \theta(u_j x_i^{\pm 1}; p)}{\prod_{k \neq j} \theta(u_j u_k^{\pm 1}; p)},$$

for any BC_n -symmetric theta function f of degree 1. More generally, the space of BC_n -symmetric theta functions of degree m is spanned by the set of products of m such functions.

2. The Anderson integral

Many of our arguments in the sequel were inspired by considerations of an extremely special (but quite important) case of [Corollary 3.2](#) below, a multivariate integral identity used in Anderson’s proof of the Selberg integral.

THEOREM 2.1 ([1]). *Let x_1, \dots, x_n and s_1, \dots, s_n be sequences of real numbers such that*

$$x_1 > x_2 > \dots > x_n \quad \text{and} \quad 0 < s_1, s_2, \dots, s_n.$$

Then

$$\int_{x_n \leq y_{n-1} \leq \dots \leq x_2 \leq y_1 \leq x_1} \frac{\prod_{1 \leq i < j \leq n-1} |y_i - y_j| \prod_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq n}} |y_i - x_j|^{s_j - 1}}{\prod_{1 \leq i < j \leq n} |x_i - x_j|^{s_i + s_j - 1}} \times \prod_{1 \leq i \leq n-1} dy_i = \frac{\prod_{1 \leq i \leq n} \Gamma(s_i)}{\Gamma(S)},$$

where $S = \sum_{1 \leq i \leq n} s_i$.

Remark 1. In fact, although Anderson independently discovered the above integral, it turns out that a more general identity (analogous to [Theorem 3.1](#) below) was discovered in 1905 by Dixon [4]; see also the remark above [Theorem 2.3](#) below. However, Anderson was the first to notice the significance of this special case in the theory of the Selberg integral, and so we will refer to it as the Anderson integral in the sequel.

Since the integrand is nonnegative, we can normalize to obtain a probability distribution. It turns out that if the s_i parameters are all positive integers, then this probability distribution has a natural random matrix interpretation.

THEOREM 2.2 ([3], [8, §4]). *Let A be an $S \times S$ complex Hermitian matrix, let $x_1 > x_2 > \dots > x_n$ be the list of distinct eigenvalues of A , and let s_1, s_2, \dots, s_n be the corresponding multiplicities. Let $\Pi : \mathbb{C}^S \rightarrow \mathbb{C}^{S-1}$ be the orthogonal projection onto a hyperplane chosen uniformly at random. Then the $(S-1) \times (S-1)$ Hermitian matrix $B = \Pi A \Pi^\dagger$ has eigenvalues x_i with multiplicity $s_i - 1$, together with $n - 1$ more eigenvalues y_i , distributed according to the Anderson distribution.*

Remark. A similar statement holds over the reals, except that now the multiplicities correspond to $2s_1, 2s_2, \dots, 2s_n$. Similarly, over the quaternions, the multiplicities correspond to $s_1/2, s_2/2, \dots, s_n/2$.

Of particular interest is the generic case, in which the eigenvalues of A are all distinct; that is $s_1 = s_2 = \dots = s_n = 1$ (this is the case considered in [\[3\]](#)). In this

case, the Anderson integral is particularly simple to prove. Indeed, the relevant integral is:

$$\frac{(n-1)!}{\prod_{1 \leq i < j \leq n} (x_i - x_j)} \int_{x_n \leq y_{n-1} \leq \dots \leq x_2 \leq y_1 \leq x_1} \prod_{1 \leq i < j \leq n-1} (y_i - y_j) \prod_{1 \leq i \leq n-1} dy_i.$$

In particular, the integrand is simply a Vandermonde determinant,

$$\prod_{1 \leq i < j \leq n-1} (y_i - y_j) = (-1)^{n(n-1)/2} \det_{1 \leq i, j \leq n-1} (y_i - x_n)^{j-1}.$$

Integrating this row-by-row gives

$$\begin{aligned} & \frac{(-1)^{n(n-1)/2} (n-1)!}{\prod_{1 \leq i < j \leq n} (x_i - x_j)} \det_{1 \leq i, j \leq n-1} \int_{x_{i+1}}^{x_i} (y - x_n)^{j-1} dy \\ &= \frac{(-1)^{n(n-1)/2} (n-1)!}{\prod_{1 \leq i < j \leq n} (x_i - x_j)} \det_{1 \leq i, j \leq n-1} \int_{x_n}^{x_i} (y - x_n)^{j-1} dy \\ &= \frac{(-1)^{n(n-1)/2} (n-1)!}{\prod_{1 \leq i < j \leq n} (x_i - x_j)} \det_{1 \leq i, j \leq n-1} \frac{(x_i - x_n)^j}{j} \\ &= 1. \end{aligned}$$

Now, in general, a Hermitian matrix with multiple eigenvalues can be expressed as a limit of matrices with distinct eigenvalues; this suggests that we should be able to obtain the general integer s Anderson integral as a limit of $s \equiv 1$ Anderson integrals. Indeed, if we integrate over y_i and take a limit $x_{i+1} \rightarrow x_i$, the result is simply the Anderson distribution with parameters

$$x_1 > \dots > x_i > x_{i+2} > \dots > x_n \quad \text{and} \quad s_1, s_2, \dots, s_i + s_{i+1}, \dots, s_{n-1}, s_n.$$

Combining this with the determinantal proof for $s \equiv 1$, we thus obtain by induction a proof of Anderson’s integral for arbitrary positive integer s . We can then obtain the general case via analytic continuation (for which we omit the argument, as it is greatly simplified in the cases of interest below). The resulting proof is less elegant than Anderson’s original proof; however, it has the distinct advantage for our purposes of extending to much more general identities. Indeed, our proofs of Theorems 3.1 and 4.1 below proceed by precisely this sort of induction from large dimensional, but simple, cases.

Remark. Note that the key property of the “determinantal” case is not so much that it is a determinant, but that it is a determinant of univariate instances of the Anderson integral. Indeed, the general Anderson integral can be expressed as a determinant of univariate integrals; in fact, a generalization (different from Dixon’s) of the resulting identity was proved by Varchenko [36] two years before Anderson’s work [1], but without notice that it could be used to prove the Selberg integral. See

also [24] (apparently the first article to observe that Varchenko’s identity could be expressed as a multivariate integral). It would be interesting to know if Varchenko’s generalized formula can be extended to the elliptic level.

The random matrix interpretation also gives the following result. Given a symmetric function f , we define $f(A)$ for a matrix A to be f evaluated at the multiset of eigenvalues of A .

THEOREM 2.3. *Let A be an n -dimensional Hermitian matrix, and let Π be a random orthogonal projection as before. Then for any partition λ , the Schur function s_λ satisfies*

$$\mathbf{E}_\Pi s_\lambda(\Pi A \Pi^\dagger) = \frac{s_\lambda(1_{n-1})}{s_\lambda(1_n)} s_\lambda(A).$$

Proof. Since Π was uniformly distributed, we have

$$\mathbf{E}_\Pi s_\lambda(\Pi A \Pi^\dagger) = \mathbf{E}_\Pi s_\lambda(\Pi U A U^\dagger \Pi^\dagger)$$

for any unitary matrix U . In particular, we can fix Π and take expectations over U , thus obtaining

$$\mathbf{E}_U s_\lambda(\Pi U A U^\dagger \Pi^\dagger) = \mathbf{E}_U s_\lambda(U A U^\dagger \Pi^\dagger \Pi) = \frac{s_\lambda(A) s_\lambda(\Pi^\dagger \Pi)}{s_\lambda(1_n)} = \frac{s_\lambda(A) s_\lambda(1_{n-1})}{s_\lambda(1_n)}.$$

Here we have used the fact

$$\mathbf{E}_U s_\lambda(U A U^\dagger B) = \frac{s_\lambda(A) s_\lambda(B)}{s_\lambda(1_n)}$$

from the theory of zonal polynomials, or equivalently from the fact that Schur functions are irreducible characters of the unitary group. \square

In other words, the Anderson distribution for $s \equiv 1$ acts as a raising integral operator on Schur functions, taking an $n - 1$ -variable Schur function to the corresponding n -variable Schur function. Similarly, the Anderson densities for $s \equiv \frac{1}{2}$ and $s \equiv 2$ act as raising operators on the real and quaternionic zonal polynomials. This suggests that in general, an Anderson distribution with constant s should take polynomials to polynomials (mapping an appropriate Jack polynomial to the corresponding n -variable Jack polynomial).

Indeed, we have the following fact, even for nonconstant s .

THEOREM 2.4. *Let y_1, y_2, \dots, y_{n-1} be distributed according to the Anderson distribution with parameters $s_1, \dots, s_n > 0, x_1 > \dots > x_n > 0$. Then as a*

function of a_1, a_2, \dots, a_m ,

$$\mathbf{E}_y \left(\prod_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq m}} (a_j - y_i) \right) = \frac{\Gamma(S)}{\Gamma(S+m)} \frac{\prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (a_i - x_j)^{1-s_j}}{\prod_{1 \leq i < j \leq m} (a_j - a_i)} \\ \times \prod_{1 \leq i \leq m} \frac{\partial}{\partial a_i} \prod_{1 \leq j \leq n} (a_i - x_j)^{s_j} \prod_{1 \leq i < j \leq m} (a_j - a_i).$$

In particular, the left-hand side is a polynomial in the x_j .

Proof. If $m = 0$, this simply states that $\mathbf{E}_y(1) = 1$; we may thus proceed by induction on m . Suppose the theorem holds for $m = m_0$, and consider what becomes of that instance when $s_n = 1$. In that case, the density is essentially independent of x_n , in that x_n only affects the normalization and the domain of integration. Thus if we multiply both sides by $\prod_{0 \leq i < j \leq n} (x_i - x_j)^{1-s_i-s_j}$, we can differentiate by x_n to obtain an $n - 1$ -dimensional integral. If we then set $x_n = a_{m_0+1}$ and renormalize, the result is the $n - 1$ -dimensional case of the theorem with $m = m_0 + 1$.

That the right-hand side is a polynomial in the x_i is straightforward, and thus the left-hand side is also a polynomial. \square

Remark 1. Compare the proof of [Theorem 7.1](#) given in [Remark 2](#) following the theorem.

Remark 2. The left-hand side of the above identity was studied by Barsky and Carpentier [2] using Anderson’s change of variables; they did not obtain as simple a right-hand side, however.

COROLLARY 2.5. *As an integral operator, the Anderson distribution takes symmetric functions to polynomials; if $s_1 = s_2 = \dots = s_n = s$, it maps symmetric functions to symmetric functions.*

Remark. A q -integral analogue of the corollary was proved by Okounkov [16], who credits a private communication from Olshanski for the corollary itself.

We can also obtain integral operators on symmetric functions by fixing one or two of the x parameters and allowing their multiplicities to vary; the result is then a symmetric function in the remaining x parameters. In particular, Anderson’s proof of the Selberg integral acquires an interpretation in terms of pairs of adjoint integral operators.

3. The $BC_n \leftrightarrow BC_m$ transformation

For all nonnegative integers m, n , and parameters $p, q, t_0, \dots, t_{2m+2n+3}$ satisfying

$$|p|, |q|, |t_0|, \dots, |t_{2m+2n+3}| < 1, \quad \prod_{0 \leq r \leq 2m+2n+3} t_r = (pq)^{m+1},$$

define

$$I_{BC_n}^{(m)}(t_0, t_1, \dots; p, q) := \frac{(p; p)^n (q; q)^n}{2^n n!} \times \int_{T^n} \frac{\prod_{1 \leq i \leq n} \prod_{0 \leq r \leq 2m+2n+3} \Gamma(t_r z_i^{\pm 1}; p, q)}{\prod_{1 \leq i < j \leq n} \Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q) \prod_{1 \leq i \leq n} \Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \leq i \leq n} \frac{dz_i}{2\pi \sqrt{-1} z_i},$$

a contour integral over the unit torus. We can extend this to a meromorphic function on the set

$$\mathcal{P}_{mn} := \{(t_0, t_1, \dots, t_{2m+2n+3}, p, q) \mid \prod_{0 \leq r \leq 2m+2n+3} t_r = (pq)^{m+1}, 0 < |p|, |q| < 1\}$$

by replacing the unit torus with the n -th power of an arbitrary (possibly disconnected) contour that contains the points of the form $p^i q^j t_r, i, j \geq 0$ and excludes their reciprocals. We thus find that the resulting function is analytic away from points where $t_r t_s = p^{-i} q^{-j}$ for some $0 \leq r, s \leq 2m + 2n + 3, 0 \leq i, j$. (In fact, its singularities consist precisely of simple poles along the hypersurfaces $t_r t_s = p^{-i} q^{-j}$ with $0 \leq r < s \leq 2m + 2n + 3, 0 \leq i, j$; see the appendix.)

Note in particular that $I_{BC_0}^{(m)}(t_0, t_1, \dots, t_{2m+3}; p, q) = 1$.

THEOREM 3.1. *The following holds for $m, n \geq 0$ as an identity in meromorphic functions on \mathcal{P}_{mn} :*

$$(3.1) \quad I_{BC_n}^{(m)}(t_0, t_1, \dots, t_{2m+2n+3}; p, q) = \prod_{0 \leq r < s \leq 2m+2n+3} \Gamma(t_r t_s; p, q) I_{BC_m}^{(n)}\left(\frac{\sqrt{pq}}{t_0}, \frac{\sqrt{pq}}{t_1}, \dots, \frac{\sqrt{pq}}{t_{2m+2n+3}}; p, q\right).$$

Remark. If $\sqrt{pq} < |t_r| < 1$ for all r , both contours may be taken to be the unit torus.

Taking $m = 0$ gives the following:

COROLLARY 3.2.

$$I_{BC_n}^{(0)}(t_0, t_1, \dots, t_{2n+3}; p, q) = \prod_{0 \leq r < s \leq 2n+3} \Gamma(t_r t_s; p, q).$$

This is the “Type I” identity conjectured by van Diejen and Spiridonov [7], who showed that it would follow from the fact that the integral vanishes if $t_0 t_1 = pq$, but were unable to prove that fact. The case $n = 1$ of the $m = 0$ integral is, however, known: it is an elliptic beta integral due to Spiridonov [31]; it also happens to agree with the case $n = 1$ of [Theorem 6.1](#) below. A direct proof of the corollary has since been given by Spiridonov [33]; see also the remark following the second proof of [Theorem 6.1](#) below.

[Corollary 3.2](#) has a well-defined limit as $p \rightarrow 0$, in which all but one of the parameters are fixed, and the remaining parameter is determined from the balancing condition; the result is an identity of Gustafson [11]. The corresponding limit of [Theorem 3.1](#) is problematical, as the right-hand side will have parameters behaving as $O(p^{-1/2})$. This can be dealt with at the cost of breaking the symmetry of the integrand; see [23].

Our strategy for proving [Theorem 3.1](#) is as follows. We first observe that in a certain extremely special (“determinantal”) case, each integrand can be expressed as a product of simple determinants, and thus the integrals themselves can be expressed as determinants. The agreement of corresponding entries of the determinants then follows from the $m = n = 1$ instance of the determinantal case ([Lemma 3.3](#) below).

The next crucial observation is that if we take the limit $t_1 \rightarrow pq/t_0$ in an instance of the general identity for given values of m and n , the result is an instance of the general identity with m diminished by 1; similarly, the limit $t_1 \rightarrow 1/t_0$ decreases n by 1. It turns out, however, that the determinantal case is *not* preserved by those operations; thus by starting with ever larger determinantal cases and dropping down to the desired m and n , we obtain an ever increasing collection of proved special cases of the identity. The full set of special cases obtained is in fact dense, and thus the theorem will follow.

Remark. A similar inductive argument based on a determinantal case was applied in [26] to prove the summation analogue of [Corollary 3.2](#) (see also [Remark 3](#) following [Theorem 7.1](#)); it is worth noting, therefore, that the present argument is not in fact a generalization of Rosengren’s. This is not to say that the arguments are unrelated; indeed, in a sense, the two arguments are dual. In fact, Rosengren’s determinantal case turns out to be precisely [Lemma 6.2](#) below, which is thus related to the difference operators we will be considering in the sequel. These, in turn, are related (by [Theorem 7.1](#), among other things) to the integral operators we will define using [Theorem 3.1](#). The duality is most apparent on the series level; if one interprets the sum as a sum over partitions, the two arguments are precisely related by conjugation of partitions. The main distinction for our purposes is that Rosengren’s argument, while superior in the series case (as it does not require analytic continuation), does not appear to extend to the integral case.

The base case for the determinantal identity is the following:

LEMMA 3.3. *The theorem holds for $m = n = 1$, is we assume the parameters have the form*

$$(t_0, t_1, t_2, \dots, t_7) = (a_0, q/a_0, a_1, q/a_1, b_0, p/b_0, b_1, p/b_1).$$

In other words, if we define

$$F(a_0, a_1; b_0, b_1; p, q) := \frac{(p; p)(q; q)}{2} \int_C \frac{\theta(z^2; q)\theta(z^{-2}; p)}{\theta(a_0z^{\pm 1}, a_1z^{\pm 1}; q)\theta(b_0z^{\pm 1}, b_1z^{\pm 1}; p)} \frac{dz}{2\pi\sqrt{-1}z},$$

with contour as appropriate, then

$$(3.2) \quad F(a_0, a_1; b_0, b_1; p, q) = \frac{\theta(a_0a_1/q, a_0/a_1; p)\theta(b_0b_1/p, b_0/b_1; q)}{\theta(a_0a_1/q, a_0/a_1; q)\theta(b_0b_1/p, b_0/b_1; p)} F\left(\frac{\sqrt{pq}}{b_0}, \frac{\sqrt{pq}}{b_1}; \frac{\sqrt{pq}}{a_0}, \frac{\sqrt{pq}}{a_1}; p, q\right).$$

Proof. It suffices to prove the Laurent series expansion

$$b_1\theta(b_0b_1^{\pm 1}; p) \frac{z^{-1}\theta(z^2; p)}{\theta(b_0z^{\pm 1}, b_1z^{\pm 1}; p)} = (p; p)^{-2} \sum_{k \neq 0} \frac{b_0^k + (p/b_0)^k - b_1^k - (p/b_1)^k}{1 - p^k} z^k,$$

valid for $|p| < |b_0|, |b_1| < 1$, and z in a neighborhood of the unit circle. Indeed, the desired integral is the constant term of the product of two such expressions, and is thus expressed as an infinite sum, each term of which already satisfies the desired transformation!

Consider the sum

$$\sum_{k > 0} \frac{b_0^k + (p/b_0)^k - b_1^k - (p/b_1)^k}{1 - p^k} \frac{z^k + z^{-k}}{k},$$

which clearly differentiates (by $z \frac{d}{dz}$) to the stated sum. If we expand $(1 - p^k)^{-1}$ in a geometric series, each term can then be summed over k as a linear combination of logarithms. We conclude that

$$\sum_{k > 0} \frac{b_0^k + (p/b_0)^k - b_1^k - (p/b_1)^k}{1 - p^k} \frac{z^k + z^{-k}}{k} = \log \left(\frac{\theta(b_1z^{\pm 1}; p)}{\theta(b_0z^{\pm 1}; p)} \right).$$

Now, the derivative

$$z \frac{d}{dz} \log \left(\frac{\theta(b_1 z^{\pm 1}; p)}{\theta(b_0 z^{\pm 1}; p)} \right)$$

is an elliptic function antisymmetric under $z \mapsto z^{-1}$, with only simple poles, and those at points of the form $(p^k b_j)^{\pm 1}$. It follows that

$$z \frac{d}{dz} \log \left(\frac{\theta(b_1 z^{\pm 1}; p)}{\theta(b_0 z^{\pm 1}; p)} \right) = C(b_0, b_1, p) \frac{z^{-1} \theta(z^2; p)}{\theta(b_0 z^{\pm 1}, b_1 z^{\pm 1}; p)},$$

for some factor $C(b_0, b_1, p)$ independent of z . Comparison of asymptotics at $z = b_0$ gives the desired result. \square

Remark. If we take the limit $p \rightarrow 1$ in the above Laurent series expansion, we obtain

$$\begin{aligned} \lim_{p \rightarrow 1} (1-p)(p; p)^2 b_1 \theta(b_0 b_1^{\pm 1}; p) \frac{z^{-1} \theta(z^2; p)}{\theta(b_0 z^{\pm 1}, b_1 z^{\pm 1}; p)} \\ = \sum_{k \neq 0} \frac{b_0^k + b_0^{-k} - b_1^k - b_1^{-k}}{k} z^k. \end{aligned}$$

If $|b_0| = |b_1| = 1$, $\Re(b_0) > \Re(b_1)$, then the limit is (up to a factor of $2\pi \sqrt{-1} \operatorname{sgn}(\Im(z))$) the Fourier series expansion of the indicator function for the arcs such that $\Re(b_0) \geq \Re(z) \geq \Re(b_1)$. In particular, this explains how an integral like the Anderson integral, with its relatively complicated domain of integration, can be a limiting case of [Corollary 3.2](#). The corresponding limit applied to [Theorem 3.1](#) gives an identity of Dixon [\[4\]](#). For a rigorous discussion of this (and other limits), see [\[23\]](#).

LEMMA 3.4. *If $m = n$, then [\(3.1\)](#) holds on the codimension $2n + 2$ subset parametrized by:*

$$t_{2r} = a_r, t_{2r+1} = q/a_r, t_{2n+2+2r} = b_r, t_{2n+2+2r+1} = p/b_r.$$

Proof. By taking a determinant of instances of [\(3.2\)](#), we obtain the identity:

$$\begin{aligned} (3.3) \quad \det_{1 \leq i, j \leq n} \left(\int_{C^n} \frac{\theta(z^2; p) \theta(z^{-2}; q)}{\theta(a_0 z^{\pm 1}, a_i z^{\pm 1}; q) \theta(b_0 z^{\pm 1}, b_j z^{\pm 1}; p)} \frac{dz}{2\pi \sqrt{-1} z} \right) \\ = \det_{1 \leq i, j \leq n} \left(\frac{\theta(a_0 a_i / q, a_0 / a_i; p) \theta(b_0 b_j / p, b_0 / b_j; q)}{\theta(a_0 a_i / q, a_0 / a_i; q) \theta(b_0 b_j / p, b_0 / b_j; p)} \right. \\ \left. \times \int_{C^m} \frac{\theta(z^2; p) \theta(z^{-2}; q)}{\theta(\frac{\sqrt{pq}}{a_0} z^{\pm 1}, \frac{\sqrt{pq}}{a_i} z^{\pm 1}; p) \theta(\frac{\sqrt{pq}}{b_0} z^{\pm 1}, \frac{\sqrt{pq}}{b_j} z^{\pm 1}; q)} \frac{dz}{2\pi \sqrt{-1} z} \right). \end{aligned}$$

Consider the determinant on the left. The p -theta functions in that integral are independent of j , while the q -theta functions are independent of i . We may thus expand that determinant of integrals as an integral of a product of two determinants (using the integral analogue of the Cauchy-Binet identity):

$$\begin{aligned}
 (3.4) \quad & \det_{1 \leq i, j \leq n} \left(\int_{C^n} \frac{\theta(z^2; p)\theta(z^{-2}; q)}{\theta(a_0 z^{\pm 1}, a_i z^{\pm 1}; q)\theta(b_0 z^{\pm 1}, b_j z^{\pm 1}; p)} \frac{dz}{2\pi\sqrt{-1}z} \right) \\
 &= \frac{1}{n!} \int_{C^n} \det_{1 \leq i, j \leq n} \left(\frac{\theta(z_j^{-2}; q)}{\theta(a_0 z_j^{\pm 1}, a_i z_j^{\pm 1}; q)} \right) \\
 &\quad \times \det_{1 \leq j, i \leq n} \left(\frac{\theta(z_j^2; p)}{\theta(b_0 z_j^{\pm 1}, b_i z_j^{\pm 1}; p)} \right) \prod_{1 \leq j \leq n} \frac{dz_j}{2\pi\sqrt{-1}z_j}.
 \end{aligned}$$

These determinants can in turn be explicitly evaluated, using the following identity:

$$\begin{aligned}
 (3.5) \quad & \det_{1 \leq i, j \leq n} \left(\frac{1}{a_i^{-1}\theta(a_i z_j^{\pm 1}; q)} \right) \\
 &= (-1)^{n(n-1)/2} \frac{\prod_{1 \leq i < j \leq n} a_i^{-1}\theta(a_i a_j^{\pm 1}; q) \prod_{1 \leq i < j \leq n} z_i^{-1}\theta(z_i z_j^{\pm 1}; q)}{\prod_{1 \leq i, j \leq n} a_i^{-1}\theta(a_i z_j^{\pm 1}; q)}.
 \end{aligned}$$

(This is, for instance, a special case of a determinant identity of Warnaar [37], and can also be obtained as a special case of the Cauchy determinant.) The resulting identity is precisely the desired result. \square

As mentioned above, the other key element to the proof is an understanding of the limit of (3.1) as $t_1 \rightarrow pq/t_0$. On the left-hand side, the integral is perfectly well-defined when $t_1 = pq/t_0$, but the right-hand side ends up identifying two poles that should be separated. Thus we need to understand how $I_{BC_n}^{(m)}(t_0, t_1, \dots)$ behaves as $t_1 \rightarrow 1/t_0$.

LEMMA 3.5. *We have the limit:*

$$\lim_{t_1 \rightarrow t_0^{-1}} \frac{I_{BC_n}^{(m)}(t_0, t_1, \dots; p, q)}{\Gamma(t_0 t_1; p, q) \prod_{2 \leq r \leq 2m+2n+3} \Gamma(t_0 t_r, t_1 t_r; p, q)} = I_{BC_{n-1}}^{(m)}(t_2, t_3, \dots; p, q).$$

Proof. If we deform the contour on the left through the points t_1 and $1/t_1$, the resulting integral will have a finite limit, and will thus be annihilated by the factor of $\Gamma(t_0 t_1)$ in the denominator. In other words, the desired limit is precisely the limit of the sum of residues corresponding to the change of contour. By symmetry, each variable contributes equally, as do t_1 and $1/t_1$; we thus find (using the identity

$$\lim_{y \rightarrow x} \Gamma(x/y)(1 - x/y) = 1/(p; p)(q; q),$$

which is easily verified):

$$\begin{aligned}
 & \lim_{t_1 \rightarrow t_0^{-1}} \frac{I_{BC_n}^{(m)}(t_0, t_1, \dots; p, q)}{\Gamma(t_0 t_1; p, q) \prod_{2 \leq r \leq 2m+2n+3} \Gamma(tr t_0, tr t_1; p, q)} \\
 = & \lim_{t_1 \rightarrow t_0^{-1}} \frac{(p; p)^{n-1} (q; q)^{n-1} \Gamma(t_0/t_1; p, q) \prod_{2 \leq r \leq 2m+2n+3} \Gamma(tr/t_1; p, q)}{2^{n-1} (n-1)! \Gamma(1/t_1^2; p, q) \prod_{2 \leq r \leq 2m+2n+3} \Gamma(t_0 tr; p, q)} \\
 & \times \int_{C^{n-1}} \frac{\prod_{1 \leq i < n} \Gamma(t_0 z_i^{\pm 1}; p, q)}{\prod_{1 \leq i < n} \Gamma(z_i^{\pm 1}/t_1; p, q)} \\
 & \times \frac{\prod_{1 \leq i < n} \prod_{2 \leq r \leq 2m+2n+3} \Gamma(tr z_i^{\pm 1}; p, q)}{\prod_{1 \leq i < j < n} \Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q) \prod_{1 \leq i < n} \Gamma(z_i^{\pm 2}; p, q)} \prod_{1 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1} z_i} \\
 = & I_{BC_{n-1}}^{(m)}(t_2, t_3, \dots; p, q)
 \end{aligned}$$

as required. □

We can now prove [Theorem 3.1](#).

Proof. For $m, n \geq 0$, let \mathcal{C}_{mn} be the set of parameters $c_0 c_1 \dots c_{m+n+1} = (pq)^{m+1}$ such that the theorem holds on the manifold with

$$t_2 i t_{2i+1} = c_i, 0 \leq i \leq m+n+1.$$

Thus, for instance, [Lemma 3.4](#) states that the point $(q, q, q, \dots, q, p, p, p, \dots, p)$ is in \mathcal{C}_{nn} .

The key idea is that if $(c_0, c_1, c_2, c_3, \dots, c_{m+n+1}) \in \mathcal{C}_{mn}$, then we also have:

$$\begin{aligned}
 & (c_0 c_1, c_2, c_3, \dots, c_{m+n+1}) \in \mathcal{C}_{m(n-1)}, \\
 & (c_0 c_1/pq, c_2, c_3, \dots, c_{m+n+1}) \in \mathcal{C}_{(m-1)n},
 \end{aligned}$$

so long as the generic point on the corresponding manifolds gives well-defined integrals; in other words, so long as none of the c_k are of the form $p^i q^j, i, j > 0$ or $p^{-i} q^{-j}, i, j \leq 0$. Indeed, if we use [Lemma 3.5](#) to take the limit $t_2 \rightarrow pq/t_0$ in the generic identity corresponding to $(c_0, c_1, \dots, c_{m+n+1})$, we find that on the left-hand side, the Gamma factors corresponding to t_2 and t_0 cancel, while on the right-hand side, the residue formula gives an $n-1$ -dimensional integral; the result is the generic identity corresponding to $(c_0 c_1/pq, \dots, c_{m+n+1})$. The other combination follows symmetrically.

Thus, starting with the point $(q, q, \dots, q, p, p, \dots, p) \in C_{NN}$ for N sufficiently large, we can combine the q 's with each other to obtain an arbitrary collection of values of the form $q^{j+1} p^{-k}$ with $j, k \geq 0$, and similarly combine the p 's to values of the form $p^{j+1} q^{-k}$, subject only to the global condition that their

product is $(pq)^{m+1}$. In other words (taking $N \rightarrow \infty$), the theorem holds for a dense set of points, and thus holds in general. \square

4. The $A_n \leftrightarrow A_m$ transformation

Consider the following family of A_n -type integrals:

$$I_{A_n}^{(m)}(Z|t_0, \dots, t_{m+n+1}; u_0, \dots, u_{m+n+1}; p, q) := \frac{(p; p)^n (q; q)^n}{(n+1)!} \times \int_{\prod_{0 \leq i \leq n} z_i = Z} \frac{\prod_{0 \leq i \leq n} \prod_{0 \leq r < m+n+2} \Gamma(t_r z_i, u_r/z_i; p, q)}{\prod_{0 \leq i < j \leq n} \Gamma(z_i/z_j, z_j/z_i; p, q)} \prod_{1 \leq i \leq n} \frac{dz_i}{2\pi \sqrt{-1} z_i}.$$

If $|u_r| < |Z|^{1/(n+1)} < 1/|t_r|$, we may take the contour to be the torus of radius $|Z|^{1/(n+1)}$; outside this range, we must choose the contour to meromorphically continue the integral. Such contour considerations can be greatly simplified by multiplying by a test function $f(Z)$ holomorphic on \mathbb{C}^* and integrating over Z . In the resulting integral, the correct contour has the form C^{n+1} , where C contains all points of the form $p^j q^k u_r$, $j, k \geq 0$, $0 \leq r \leq m+n+1$ and excludes all points of the form $p^{-j} q^{-k}/t_r$.

Note that unlike the BC_n case, the A_n integral is not equal to 1 for $n = 0$; instead, we pick up the value of the integrand at Z :

$$I_{A_0}^{(m)}(Z|t_0, \dots, t_{m+1}; u_0, \dots, u_{m+1}; p, q) = \prod_{0 \leq r < m+2} \Gamma(t_r Z, u_r/Z; p, q).$$

We also observe that the Z parameter is not a true degree of freedom; indeed:

$$I_{A_n}^{(m)}(c^{n+1} Z | \dots t_i \dots; \dots u_i \dots; p, q) = I_{A_n}^{(m)}(Z | \dots c t_i \dots; \dots c^{-1} u_i \dots; p, q).$$

In particular, we could in principle always take $Z = 1$ (in which case it will be omitted), although this is sometimes notationally inconvenient.

THEOREM 4.1. *For otherwise generic parameters in the hypersurface such that $\prod_{0 \leq r < m+n+2} t_r u_r = (pq)^{m+1}$,*

$$I_{A_n}^{(m)}(Z | \dots t_i \dots; \dots u_i \dots; p, q) = \prod_{0 \leq r, s < m+n+2} \Gamma(t_r u_s; p, q) I_{A_m}^{(n)}(Z | \dots T^{\frac{1}{m+1}}/t_i \dots; \dots U^{\frac{1}{m+1}}/u_i \dots; p, q),$$

where $T = \prod_{0 \leq r < m+n+2} t_r$, $U = \prod_{0 \leq r < m+n+2} u_r$.

Remark. It appears that this can be viewed as an integral analogue of a series transformation of Rosengren [27] and Kajihara and Noumi [12], in that the latter should be derivable via residue calculus from the former.

For $m = 0$, we obtain the following integral conjectured by Spiridonov [32]:

COROLLARY 4.2. *For otherwise generic parameters in the hypersurface such that $\prod_{0 \leq r < n+2} t_r u_r = pq$,*

$$I_{A_n}^{(0)}(\dots t_i \dots; \dots u_i \dots; p, q) = \prod_{0 \leq r, s < n+2} \Gamma(t_r u_s; p, q) \prod_{0 \leq r < n+2} \Gamma(U/u_r, T/t_r; p, q).$$

The main difficulty with applying the BC_n approach in this case is the fact that the variables are coupled by the condition $\prod_i z_i = Z$; in general the integral over this domain of the usual sort of product of determinants will not be expressible as a determinant of univariate integrals. Another difficulty is that, in any event, even in the “right” specialization, the integrand is not quite expressible as a product of determinants. As we shall see, it turns out that these problems effectively cancel each other.

In particular, we note the extra factor in the following determinant identity.

LEMMA 4.3 ([10]).

$$\det_{0 \leq i, j < n} \left(\frac{\theta(tx_i y_j; p)}{\theta(t, x_i y_j; p)} \right) = \frac{\theta(t \prod_{0 \leq i < n} x_i y_i; p)}{\theta(t; p)} \prod_{0 \leq i < j < n} x_j y_j \theta(x_i/x_j, y_i/y_j; p) \prod_{0 \leq i, j < n} \theta(x_i y_j; p)^{-1}.$$

Proof. Consider the function

$$F(t; \dots x_i \dots; \dots y_i \dots) := \prod_{0 \leq i < j < n} (x_j y_j)^{-1} \theta(x_i/x_j, y_i/y_j; p)^{-1} \times \prod_{0 \leq i, j < n} \theta(x_i y_j; p) \det_{0 \leq i, j < n} \left(\frac{\theta(tx_i y_j; p)}{\theta(t, x_i y_j; p)} \right).$$

This is clearly holomorphic on $(\mathbb{C}^*)^{2n}$ for t fixed; moreover, since

$$F(t; px_0, x_1, \dots, x_{n-1}; \dots y_i \dots) = - \left(t \prod_{0 \leq j < n} (x_j y_j) \right)^{-1} F(t; x_0, x_1, \dots, x_{n-1}; \dots y_i \dots)$$

we conclude that F vanishes if $t \prod_{0 \leq j < n} (x_j y_j) = 1$; indeed, $F(x_0)$ is a degree one theta function, and thus uniquely determined by its multiplier. Thus the function

$$\theta \left(t \prod_{0 \leq i < n} x_i y_i; p \right)^{-1} F(t; \dots x_i \dots; \dots y_i \dots)$$

is still holomorphic, and indeed we verify that it is an abelian function of all variables except t , so is in fact a function of t alone. The remaining factors can thus be recovered from the limiting case:

$$\lim_{\substack{y_i \rightarrow x_i^{-1} \\ i=0, \dots, n-1}} F(t; \dots x_i \dots; \dots y_i \dots) = 1. \quad \square$$

Remark. A presumably related application of this determinant to hypergeometric series identities can be found in [12].

LEMMA 4.4. *The theorem holds for the special case*

$$I_{A_{n-1}}^{(n-1)} \left(Z \mid \dots x_i \dots, \dots \frac{q}{y_i} \dots; \dots \frac{p}{x_i} \dots, \dots y_i \dots; p, q \right).$$

Proof. We first observe that the integral:

$$\int \frac{\theta(sxz; p)}{\theta(s, xz; p)} \frac{\theta(ty/z; q)}{\theta(t, y/z; q)} \frac{dz}{2\pi\sqrt{-1z}}$$

is symmetric in x and y , as follows from the change of variable $z \mapsto yz/x$. It thus follows that the determinant

$$\det_{0 \leq i, j < n} \left(\int \frac{\theta(sx_i z; p)}{\theta(s, x_i z; p)} \frac{\theta(ty_j/z; q)}{\theta(t, y_j/z; q)} \frac{dz}{2\pi\sqrt{-1z}} \right)$$

is invariant under exchanging the roles of the x and y variables. As before, we can write this as a multiple integral of a product of two determinants:

$$\begin{aligned} & n! \det_{0 \leq i, j < n} \left(\int \frac{\theta(sx_i z; p)}{\theta(s, x_i z; p)} \frac{\theta(ty_j/z; q)}{\theta(t, y_j/z; q)} \frac{dz}{2\pi\sqrt{-1z}} \right) \\ &= \int \det_{0 \leq i, j < n} \left(\frac{\theta(sx_i z_j; p)}{\theta(s, x_i z_j; p)} \right) \det_{0 \leq i, j < n} \left(\frac{\theta(ty_i/z_j; q)}{\theta(t, y_i/z_j; q)} \right) \prod_{0 \leq i < n} \frac{dz_i}{2\pi\sqrt{-1z_i}} \\ &= \prod_{0 \leq i < j < n} x_j y_j \theta(x_i/x_j; p) \theta(y_i/y_j; q) \\ &\times \int \frac{\theta(sXZ; p) \theta(tY/Z; q)}{\theta(s; p) \theta(t; q)} \frac{\prod_{0 \leq i < j < n} \theta(z_i/z_j; p) \theta(z_j/z_i; q)}{\prod_{0 \leq i, j < n} \theta(x_i z_j; p) \theta(y_i/z_j; q)} \prod_{0 \leq i < n} \frac{dz_i}{2\pi\sqrt{-1z_i}}, \end{aligned}$$

where $X = \prod_i x_i$, $Y = \prod_i y_i$, $Z = \prod_i z_i$. We thus conclude:

$$\begin{aligned} & \int \frac{\theta(sXZ; p)\theta(tY/Z; q)}{\theta(s; p)\theta(t; q)} \frac{\prod_{0 \leq i < j < n} \theta(z_i/z_j; p)\theta(z_j/z_i; q)}{\prod_{0 \leq i, j < n} \theta(x_i z_j; p)\theta(y_i/z_j; q)} \prod_{0 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1}z_i} \\ &= \prod_{0 \leq i < j < n} \frac{\theta(x_i/x_j; q)\theta(y_i/y_j; p)}{\theta(x_i/x_j; p)\theta(y_i/y_j; q)} \\ & \times \int \frac{\theta(sYZ; p)\theta(tX/Z; q)}{\theta(s; p)\theta(t; q)} \frac{\prod_{0 \leq i < j < n} \theta(z_i/z_j; p)\theta(z_j/z_i; q)}{\prod_{0 \leq i, j < n} \theta(y_i z_j; p)\theta(x_i/z_j; q)} \prod_{0 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1}z_i}. \end{aligned}$$

Now, if we replace s in this identity by $p^k s$, we find:

$$\begin{aligned} & \int \frac{\theta(sXZ; p)\theta(tY/Z; q)}{(XZ)^k \theta(s; p)\theta(t; q)} \frac{\prod_{0 \leq i < j < n} \theta(z_i/z_j; p)\theta(z_j/z_i; q)}{\prod_{0 \leq i, j < n} \theta(x_i z_j; p)\theta(y_i/z_j; q)} \prod_{0 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1}z_i} \\ &= \prod_{0 \leq i < j < n} \frac{\theta(x_i/x_j; q)\theta(y_i/y_j; p)}{\theta(x_i/x_j; p)\theta(y_i/y_j; q)} \\ & \times \int \frac{\theta(sYZ; p)\theta(tX/Z; q)}{(YZ)^k \theta(s; p)\theta(t; q)} \frac{\prod_{0 \leq i < j < n} \theta(z_i/z_j; p)\theta(z_j/z_i; q)}{\prod_{0 \leq i, j < n} \theta(y_i z_j; p)\theta(x_i/z_j; q)} \prod_{0 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1}z_i}. \end{aligned}$$

As this is true for all integers k , we find that

$$\begin{aligned} & \int f(XZ) \frac{\prod_{0 \leq i < j < n} \theta(z_i/z_j; p)\theta(z_j/z_i; q)}{\prod_{0 \leq i, j < n} \theta(x_i z_j; p)\theta(y_i/z_j; q)} \prod_{0 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1}z_i} \\ &= \prod_{0 \leq i < j < n} \frac{\theta(x_i/x_j; q)\theta(y_i/y_j; p)}{\theta(x_i/x_j; p)\theta(y_i/y_j; q)} \\ & \times \int f(YZ) \frac{\prod_{0 \leq i < j < n} \theta(z_i/z_j; p)\theta(z_j/z_i; q)}{\prod_{0 \leq i, j < n} \theta(y_i z_j; p)\theta(x_i/z_j; q)} \prod_{0 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1}z_i}, \end{aligned}$$

for any function f holomorphic in a neighborhood of the contour (the dependence on s and t having been absorbed in f). But this implies

$$\begin{aligned} & \int_{\prod_{0 \leq i < n} z_i = Z} \frac{\prod_{0 \leq i < j < n} \theta(z_i/z_j; p)\theta(z_j/z_i; q)}{\prod_{0 \leq i, j < n} \theta(x_i z_j; p)\theta(y_i/z_j; q)} \prod_{1 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1}z_i} \\ &= \prod_{0 \leq i < j < n} \frac{\theta(x_i/x_j; q)\theta(y_i/y_j; p)}{\theta(x_i/x_j; p)\theta(y_i/y_j; q)} \\ & \times \int_{\prod_{0 \leq i < n} z_i = ZX/Y} \frac{\prod_{0 \leq i < j < n} \theta(z_i/z_j; p)\theta(z_j/z_i; q)}{\prod_{0 \leq i, j < n} \theta(y_i z_j; p)\theta(x_i/z_j; q)^{-1}} \prod_{1 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1}z_i}. \end{aligned}$$

Applying the change of variables $z_i \rightarrow (X/Y)^{1/n} z_i$ on the right gives the desired result. □

We also have the following analogue of [Lemma 3.5](#), with essentially the same proof.

LEMMA 4.5. *We have the limit:*

$$\begin{aligned} \lim_{u_0 \rightarrow t_0^{-1}} \frac{I_{A_n}^{(m)}(Z|t_0, \dots, t_{m+n+1}; u_0, \dots, u_{m+n+1}; p, q)}{\Gamma(t_0 u_0; p, q) \prod_{0 < r < m+n+2} \Gamma(t_0 u_r, t_r u_0; p, q)} \\ = I_{A_{n-1}}^{(m)}(t_0 Z|t_1, \dots, t_{m+n+1}; u_1, \dots, u_{m+n+1}; p, q). \end{aligned}$$

[Theorem 4.1](#) follows as in the proof of [Theorem 3.1](#), except that in the definition of \mathcal{C}_{mn} , we take $t_i u_i = c_i$; we have

$$(q, q, \dots, q, p, p, \dots, p) \in \mathcal{C}_{nn}.$$

and if $(c_0, \dots, c_{m+n+1}) \in \mathcal{C}_{mn}$, then

$$\begin{aligned} (c_0 c_1, c_2, \dots, c_{m+n+1}) &\in \mathcal{C}_{m(n-1)}, \\ (c_0 c_1 / pq, c_2, \dots, c_{m+n+1}) &\in \mathcal{C}_{(m-1)n} \end{aligned}$$

as long as both sides of the corresponding identities are generically well-defined. As before, this shows that \mathcal{C}_{mn} is dense, and thus [Theorem 4.1](#) holds in general.

5. Mixed transformations

Consider the integral associated to A_1 . Eliminating z_2 from the integral using the relation $z_1 z_2 = 1$, we find that the result is invariant under $z_1 \mapsto z_1^{-1}$, and is thus an instance of the BC_1 integral. Indeed, if

$$\prod_i t_i u_i = (pq)^{m+1},$$

then

$$\begin{aligned} I_{A_1}^{(m)}(t_0 \dots t_{m+2}; u_0 \dots u_{m+2}; p, q) \\ = \frac{(p; p)(q; q)}{2} \int \frac{\Gamma(t_r z^{\pm 1}, u_r z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi \sqrt{-1}z} \\ = I_{BC_1}^{(m)}(t_0 \dots t_{m+2}, u_0 \dots u_{m+2}; p, q). \end{aligned}$$

As a consequence, we obtain an identity between the $m = 1$ integrals of types A_n and BC_n .

COROLLARY 5.1. *If $\prod_{0 \leq i \leq n+2} t_i u_i = (pq)^2$, then*

$$\begin{aligned} I_{A_n}^{(1)}(\dots t_i \dots; \dots u_i \dots; p, q) \\ = \prod_{0 \leq i < j \leq n+2} \Gamma(T/t_i t_j, U/u_i u_j; p, q) I_{BC_n}^{(1)}(\dots (U/T)^{1/4} t_i \dots, \dots (T/U)^{1/4} u_i \dots; p, q), \end{aligned}$$

where $T = \prod_{0 \leq i \leq n+2} t_i$, $U = \prod_{0 \leq i \leq n+2} u_i$.

In particular, since the BC_n integral is symmetric in its $2n + 6$ parameters, we obtain an S_{2n+6} symmetry of $I_{A_n}^{(1)}$. We thus obtain a total of $n + 4$ essentially different transformations of the A_n integral, corresponding to the $n + 4$ double cosets of $S_{n+3} \times S_{n+3}$ in S_{2n+6} .

COROLLARY 5.2. *Let k be an integer $0 \leq k \leq n + 3$. Then*

$$I_{A_n}^{(1)}(t_0, \dots, t_{n+2}; u_0, \dots, u_{n+2}; p, q) = \prod_{\substack{0 \leq r < k \\ k \leq s \leq n+2}} \Gamma(t_r u_s, t_s u_r, T/t_r t_s, U/u_r u_s; p, q) I_{A_n}^{(1)}(t'_0, \dots, t'_{n+2}; u'_0, \dots, u'_{n+2}; p, q),$$

where

$$t'_r = \begin{cases} (T/U)^{(n+1-k)/2(n+1)} (T_k/U_k)^{1/(n+1)} u_r, & 0 \leq r < k \\ (U/T)^{k/2(n+1)} (T_k/U_k)^{1/(n+1)} t_r, & k \leq r \leq n+2, \end{cases}$$

$$u'_r = \begin{cases} (U/T)^{(n+1-k)/2(n+1)} (U_k/T_k)^{1/(n+1)} t_r, & 0 \leq r < k \\ (T/U)^{k/2(n+1)} (U_k/T_k)^{1/(n+1)} u_r, & k \leq r \leq n+2, \end{cases}$$

$$T = \prod_{0 \leq r \leq n+2} t_r, \quad U = \prod_{0 \leq r \leq n+2} u_r,$$

$$T_k = \prod_{0 \leq r < k} t_r, \quad U_k = \prod_{0 \leq r < k} u_r.$$

For $k = 0$, we obtain the identity transformation, while for $k = n + 3$, we simply switch the t_i and u_i parameters (corresponding to taking $z \mapsto 1/z$ in the integral). The case $k = 1$ was stated as equation (6.11) of [32] (conditional on Corollary 4.2). Again, apparently related series identities are known; see [27] and [12].

6. Difference operators

The following identity was originally conjectured by van Diejen and Spiridonov [7] (their ‘‘Type II’’ integral):

THEOREM 6.1. *For otherwise generic parameters satisfying $|p|, |q|, |t| < 1$ and $t^{2n-2} \prod_{0 \leq r \leq 5} t_r = pq$,*

$$(6.1) \quad \frac{(p; p)^n (q; q)^n \Gamma(t; p, q)^n}{2^n n!} \int_{C^n} \prod_{1 \leq i < j \leq n} \frac{\Gamma(t z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\ \times \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 5} \Gamma(t_r z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \frac{dz_i}{2\pi \sqrt{-1} z_i} \\ = \prod_{0 \leq j < n} \Gamma(t^{j+1}; p, q) \prod_{0 \leq r < s \leq 5} \Gamma(t^j t_r t_s; p, q),$$

where the contour $C = C^{-1}$ contains all points of the form $p^i q^j t_r$ for $i, j \geq 0$, excludes their reciprocals, and contains the contours $p^i q^j tC$ for $i, j \geq 0$. (In particular, if $|t_r| < 1$ for $0 \leq r \leq 5$, C may be taken to be the unit circle.)

Proof ([7]). Suppose $t^{2n} \prod_{0 \leq r \leq 5} t_r = pq$, and consider the double integral

$$\begin{aligned} & \int_{C^{n+1}} \int_{C^n} \frac{\prod_{\substack{0 \leq i \leq n \\ 1 \leq j \leq n}} \Gamma(\sqrt{t} x_i^{\pm 1} y_j^{\pm 1}; p, q)}{\prod_{0 \leq i < j \leq n} \Gamma(x_i^{\pm 1} x_j^{\pm 1}; p, q) \prod_{1 \leq i < j \leq n} \Gamma(y_i^{\pm 1} y_j^{\pm 1}; p, q)} \\ & \times \prod_{0 \leq i \leq n} \frac{\Gamma(t^n t_0 x_i^{\pm 1}; p, q) \prod_{1 \leq r \leq 5} \Gamma(t_r x_i^{\pm 1}; p, q)}{\Gamma(x_i^{\pm 2}; p, q)} \\ & \times \prod_{1 \leq i \leq n} \frac{\Gamma(pqt^{-n-1/2} y_i^{\pm 1} / t_0, t^{-1/2} t_0 y_i^{\pm 1}; p, q)}{\Gamma(y_i^{\pm 2}; p, q)} \\ & \times \prod_{1 \leq i \leq n} \frac{dy_i}{2\pi \sqrt{-1} y_i} \prod_{0 \leq i \leq n} \frac{dx_i}{2\pi \sqrt{-1} x_i}. \end{aligned}$$

Both the x and y integrals can be evaluated via [Corollary 3.2](#); comparing both sides gives a recurrence for the left-hand side of [\(6.1\)](#), the unique solution of which is the right-hand side, as required. \square

We will discuss this proof (of which Anderson’s proof of the Selberg integral is a limiting case) in greater detail in the sequel; for the moment, however, it will be instructive to consider a different proof. The main ingredient in the alternate proof is the following identity:

LEMMA 6.2. *Let n be a nonnegative integer, and let u_0, u_1, u_2, u_3, t satisfy $t^{n-1} u_0 u_1 u_2 u_3 = p$. Then*

$$\begin{aligned} (6.2) \quad & \sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 3} \theta(u_r z_i^{\sigma_i}; p)}{\theta(z_i^{2\sigma_i}; p)} \prod_{1 \leq i < j \leq n} \frac{\theta(t z_i^{\sigma_i} z_j^{\sigma_j}; p)}{\theta(z_i^{\sigma_i} z_j^{\sigma_j}; p)} \\ & = \prod_{0 \leq i < n} \theta(t^i u_0 u_1, t^i u_0 u_2, t^i u_0 u_3; p) \\ & = \prod_{0 \leq i < n} \theta(t^i u_0 u_1, t^i u_0 u_2, t^i u_1 u_2; p). \end{aligned}$$

Proof. We first observe that the condition on the u_r ensures that every term in the above sum is invariant under all translations $z_i \rightarrow pz_i$, and thus the same is true of their sum. Moreover, the sum is manifestly invariant under permutations of

the z_i as well as reflections $z_i \rightarrow 1/z_i$. Thus if we multiply the sum by

$$\prod_{1 \leq i \leq n} z_i^{-1} \theta(z_i^2; p) \prod_{1 \leq i < j \leq n} z_i^{-1} \theta(z_i z_j, z_i z_j^{-1}; p),$$

the result is a (holomorphic) theta function, anti-invariant under the same group. But any such theta function is a multiple of the above product; it thus follows that the desired sum has no singularities in z_i , and must therefore be independent of z_i .

To evaluate the sum, we may therefore specialize $z_i = u_0 t^{n-i}$, in which case all but one of the terms in the sum vanish, so that the sum is given by the remaining term (with $\sigma_i = 1$ for all i):

$$\frac{\prod_{0 \leq r \leq 3} \theta(u_0 u_r t^{n-i}; p)}{\theta(u_0^2 t^{2n-2i}; p)} \prod_{1 \leq i < j \leq n} \frac{\theta(u_0^2 t^{2n+1-i-j}; p)}{\theta(u_0^2 t^{2n-i-j}; p)}.$$

The factors involving u_0^2 cancel, and we are thus left with the evaluation claimed above. □

Proof of Theorem 6.1. Divide the integral by the claimed evaluation, and consider the result as a meromorphic function on the set

$$t^{2n-2} t_0 t_1 t_2 t_3 t_4 t_5 = pq.$$

We claim that this function is invariant under the translations

$$(t_0, t_1, t_2, t_3, t_4, t_5) \rightarrow (p^{1/2} t_0, p^{1/2} t_1, p^{1/2} t_2, p^{-1/2} t_3, p^{-1/2} t_4, p^{-1/2} t_5),$$

$$(t_0, t_1, t_2, t_3, t_4, t_5) \rightarrow (q^{1/2} t_0, q^{1/2} t_1, q^{1/2} t_2, q^{-1/2} t_3, q^{-1/2} t_4, q^{-1/2} t_5),$$

and all permutations thereof. It will then follow that the ratio is a constant; to evaluate the constant, we may then consider the limit $t_1 \rightarrow t^{1-n} t_0^{-1}$ as in Lemma 3.5 above. (In other words, we apply the special case of the residue formula of van Diejen and Spiridonov in which the resulting sum consists of precisely one term.)

Since both sides are symmetric in p and q , it suffices to consider the q translation. If we factor the integrand as

$$\Delta^{(n)}(z_1, z_2, \dots, z_n) \Delta^{(n)}(z_1^{-1}, z_2^{-1}, \dots, z_n^{-1}) \prod_{1 \leq i \leq n} \frac{dz_i}{2\pi \sqrt{-1} z_i},$$

where

$$\begin{aligned} &\Delta^{(n)}(z_1, z_2, \dots, z_n) \\ &= \prod_{1 \leq i \leq n} \frac{\Gamma(t_0 z_i, t_1 z_i, t_2 z_i, t_3 z_i, t_4 z_i, t_5 z_i, p z_i / (t^{n-1} t_0 t_1 t_2); p, q)}{\Gamma(z_i^2, p / (z_i t^{n-1} t_0 t_1 t_2); p, q)} \\ &\quad \times \prod_{1 \leq i < j \leq n} \frac{\Gamma(t z_i z_j^{\pm 1}; p, q)}{\Gamma(z_i z_j^{\pm 1}; p, q)}, \end{aligned}$$

and similarly let $\tilde{\Delta}^{(n)}$ be the corresponding product with parameters

$$(q^{-1/2}t_3, q^{-1/2}t_4, q^{-1/2}t_5, q^{1/2}t_0, q^{1/2}t_1, q^{1/2}t_2)$$

(permuting the parameters to make the transformation an involution), then we find that

$$\frac{\tilde{\Delta}^{(n)}(\dots q^{1/2}z_i \dots)}{\Delta^{(n)}(\dots z_i \dots)} = \prod_{1 \leq i \leq n} \frac{\theta(t_0z_i, t_1z_i, t_2z_i, pz_i/t^{n-1}t_0t_1t_2; p)}{\theta(z_i^2; p)} \prod_{1 \leq i < j \leq n} \frac{\theta(tz_iz_j; p)}{\theta(z_iz_j; p)},$$

and thus

$$(6.3) \quad \sum_{\sigma_i \in \{\pm 1\}^n} \frac{\tilde{\Delta}^{(n)}(\dots q^{1/2}z_i^{\sigma_i} \dots)}{\Delta^{(n)}(\dots z_i^{\sigma_i} \dots)} = \prod_{0 \leq i < n} \theta(t^i t_0 t_1, t^i t_0 t_2, t^i t_1 t_2; p)$$

by Lemma 6.2. Similarly,

$$(6.4) \quad \sum_{\sigma_i \in \{\pm 1\}^n} \frac{\Delta^{(n)}(\dots q^{1/2}z_i^{\sigma_i} \dots)}{\tilde{\Delta}^{(n)}(\dots z_i^{\sigma_i} \dots)} = \prod_{0 \leq i < n} \theta(t^i t_3 t_4/q, t^i t_3 t_5/q, t^i t_4 t_5/q; p).$$

Now, consider the integral:

$$\int_C \tilde{\Delta}^{(n)}(\dots q^{1/2}z_i \dots) \Delta^{(n)}(\dots z_i^{-1} \dots) \prod_{1 \leq i \leq n} \frac{dz_i}{2\pi \sqrt{-1}z_i},$$

where the contour is chosen to contain the points $p^i q^j t_r$ for $i, j \geq 0$, exclude their reciprocals, and contain the contours tC and tC^{-1} ; here we note that the poles of $\tilde{\Delta}^{(n)}(\dots q^{1/2}z_i \dots)$ are a subset of the poles of $\Delta^{(n)}(\dots z_i \dots)$, and so this constraint on the contour is still reasonable. If we then perform the change of variable $z_i \mapsto q^{-1/2}/z_i$, we find that the new contour is legal for the transformed parameters. In other words, we have

$$\begin{aligned} & \int_C \tilde{\Delta}^{(n)}(\dots q^{1/2}z_i \dots) \Delta^{(n)}(\dots z_i^{-1} \dots) \prod_{1 \leq i \leq n} \frac{dz_i}{2\pi \sqrt{-1}z_i} \\ &= \int_{C'} \Delta^{(n)}(\dots q^{1/2}z_i \dots) \tilde{\Delta}^{(n)}(\dots z_i^{-1} \dots) \prod_{1 \leq i \leq n} \frac{dz_i}{2\pi \sqrt{-1}z_i}. \end{aligned}$$

Since the constraints on the contours are symmetrical under $z_i \mapsto 1/z_i$, we may symmetrize the integrands, losing the same factor of 2^n on both sides. The theorem follows upon applying equations (6.3) and (6.4) to simplify the symmetrized integrands. \square

Remark. One can also prove [Corollary 3.2](#) by a similar argument, based on the straightforward identity

$$\sum_{\sigma \in \{\pm 1\}^n} \frac{\theta(\prod_{0 \leq r < n+2} t_r / \prod_{1 \leq i \leq n} z_i^{\sigma_i}; p) \prod_{1 \leq i \leq n} \prod_{0 \leq r < n+2} \theta(t_r z_i^{\sigma_i}; p)}{\prod_{1 \leq i \leq j \leq n} \theta(z_i^{\sigma_i} z_j^{\sigma_j}; p)} = \prod_{0 \leq r < s < n+2} \theta(t_r t_s; p).$$

Define a q -difference operator $D_q^{(n)}(u_0, u_1, u_2, u_3; t, p)$ by setting

$$(D_q^{(n)}(u_0, u_1, u_2, u_3; t, p) f)(\dots z_i \dots) := \sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \leq i \leq n} \frac{\theta(u_r z_i^{\sigma_i}; p)}{\theta(z_i^{2\sigma_i}; p)} \prod_{1 \leq i < j \leq n} \frac{\theta(t z_i^{\sigma_i} z_j^{\sigma_j}; p)}{\theta(z_i^{\sigma_i} z_j^{\sigma_j}; p)} f(\dots q^{\sigma_i/2} z_i \dots).$$

Thus [Lemma 6.2](#) gives a formula for the image of 1 under

$$D_q^{(n)}(u_0, u_1, u_2, u_3; t, p)$$

when $t^{n-1} u_0 u_1 u_2 u_3 = p$. Moreover, the resulting proof of [Theorem 6.1](#) would appear to be based on an adjointness relation between two such difference operators, as we will confirm below.

To make this precise, we need some suitable spaces of functions on which to act. Let $A^{(n)}(u_0; p, q)$ be the space of BC_n -symmetric p -abelian functions f such that

$$\prod_{1 \leq i \leq n} \theta(pqz_i^{\pm 1}/u_0; p, q)_{0,m} f(\dots z_i \dots)$$

is holomorphic for sufficiently large m ; that is, f is smooth except at the points $p^k u_0/q^l$, $p^k q^l/u_0$ for $k \in \mathbb{Z}$, $1 \leq l \leq m$, where it has at most simple poles. The canonical (multiplication) map from the tensor product of $A^{(n)}(u_0; p, q)$ and $A^{(n)}(u_0; q, p)$ to the space of meromorphic functions on $(\mathbb{C}^*)^n$ is generically injective; denote the image by $\mathcal{A}^{(n)}(u_0; p, q)$. In particular, we observe that if $f \in \mathcal{A}^{(n)}(u_0; p, q)$, then

$$\prod_{1 \leq i \leq n} \theta(pqz_i^{\pm 1}/u_0; p, q)_{l,m} f(\dots z_i \dots) \propto \prod_{1 \leq i \leq n} \frac{\Gamma(u_0 z_i^{\pm 1}; p, q)}{\Gamma(p^{-l} q^{-m} u_0 z_i^{\pm 1}; p, q)} f(\dots z_i \dots)$$

is holomorphic for sufficiently large l, m .

Remark. Our main motivation for considering the large space $\mathcal{A}^{(n)}(u_0; p, q)$, rather than the smaller spaces in which the functions are actually abelian, is that

such product functions already appear in the family of univariate biorthogonal functions considered by Spiridonov [32, App. A].

We now define

$$\mathfrak{D}_q^{(n)}(u_0, u_1, u_2; t, p) f := \frac{D_q^{(n)}(u_0, u_1, u_2, t^{1-n} p / u_0 u_1 u_2; t, p) f}{\prod_{1 \leq i \leq n} \theta(t^{n-i} u_0 u_1, t^{n-i} u_0 u_2, t^{n-i} u_1 u_2; p)}.$$

We will also need a shift operator $T_{\omega, q}^{(n)}$:

$$(T_{\omega, q}^{(n)} f)(\dots z_i \dots) = f(\dots q^{1/2} z_i \dots).$$

Note that this maps BC_n -symmetric q -abelian functions to BC_n -symmetric q -abelian functions.

LEMMA 6.3. *The operator $\mathfrak{D}_q^{(n)}(u_0, u_1, u_2; t, p)$ induces a linear transformation*

$$\mathfrak{D}_q^{(n)}(u_0, u_1, u_2; t, p) : \mathcal{A}^{(n)}(\sqrt{q}u_0; p, q) \rightarrow \mathcal{A}^{(n)}(u_0; p, q).$$

Moreover, the corresponding map

$$\begin{aligned} \mathfrak{D}_q^{(n)}(u_0, u_1, u_2; t, p) : A^{(n)}(\sqrt{q}u_0; p, q) \otimes A^{(n)}(\sqrt{q}u_0; q, p) \\ \rightarrow A^{(n)}(u_0; p, q) \otimes A^{(n)}(u_0; q, p) \end{aligned}$$

can be decomposed as

$$\mathfrak{D}_q^{(n)}(u_0, u_1, u_2; t, p) = \mathfrak{D}_q^{(n)}(u_0, u_1, u_2; t, p) \otimes T_{\omega, q}^{(n)}.$$

Proof. Let

$$g \in A^{(n)}(\sqrt{q}u_0; p, q), \quad h \in A^{(n)}(\sqrt{q}u_0; q, p).$$

A straightforward computation, using the fact that h is q -abelian, gives:

$$\mathfrak{D}_q^{(n)}(u_0, u_1, u_2; t, p)(gh) = (\mathfrak{D}_q^{(n)}(u_0, u_1, u_2; t, p)g)(T_{\omega, q}^{(n)}h)$$

as required. That $T_{\omega, q}^{(n)}h \in A^{(n)}(u_0; q, p)$ is straightforward; that

$$\mathfrak{D}_q^{(n)}(u_0, u_1, u_2; t, p)g$$

is p -abelian follows as in the proof of Lemma 6.2. Finally, we observe that this function is holomorphic at $z_i = u_0$, as required. \square

The desired adjointness relation can then be stated as follows. For parameters satisfying $t^{2n-2}u_0u_1t_0t_1t_2t_3 = pq$, define a scalar product between $\mathcal{A}^{(n)}(u_0; p, q)$

and $\mathcal{A}^{(n)}(u_1; p, q)$ as follows:

$$\begin{aligned} &\langle f, g \rangle_{t_0, t_1, t_2, t_3; u_0, u_1; t, p, q} \\ &:= \frac{1}{Z} \int_{\mathbb{C}^n} f(\dots z_i \dots) g(\dots z_i \dots) \prod_{1 \leq i < j \leq n} \frac{\Gamma(t z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\ &\times \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 5} \Gamma(t_r z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \frac{dz_i}{2\pi \sqrt{-1} z_i}, \end{aligned}$$

where

$$Z = \frac{2^n n!}{(p; p)^n (q; q)^n \Gamma(t; p, q)^n} \prod_{1 \leq i \leq n} \Gamma(t^i; p, q) \prod_{0 \leq r < s \leq 5} \Gamma(t^{n-i} t_r t_s; p, q),$$

$$t_4 = u_0,$$

$$t_5 = u_1,$$

and the contour is chosen as in [Theorem 6.1](#), except that we first absorb the singularities of f and g into the factors $\Gamma(u_r z_i^{\pm 1}; p, q)$ of the integrand. In particular, we have

$$\langle 1, 1 \rangle_{t_0, t_1, t_2, t_3; u_0, u_1; t, p, q} = 1.$$

THEOREM 6.4. *If $f \in \mathcal{A}^{(n)}(q^{1/2} u_0; p, q)$, $g \in \mathcal{A}^{(n)}(u_1; p, q)$ and the parameters (which are otherwise generic) satisfy $t^{2n-2} u_0 u_1 t_0 t_1 t_2 t_3 = pq$, then*

$$\begin{aligned} &\langle \mathcal{D}_q^{(n)}(u_0, t_0, t_1; t, p) f, g \rangle_{t_0, t_1, t_2, t_3; u_0, u_1; t, p, q} \\ &= \langle f, \mathcal{D}_q^{(n)}(u'_1, t'_2, t'_3; t, p) g \rangle_{t'_0, t'_1, t'_2, t'_3; u'_0, u'_1; t, p, q}, \end{aligned}$$

where

$$(t'_0, t'_1, t'_2, t'_3, u'_0, u'_1) = (q^{1/2} t_0, q^{1/2} t_1, q^{-1/2} t_2, q^{-1/2} t_3, q^{1/2} u_0, q^{-1/2} u_1).$$

Proof. The second proof of [Theorem 6.1](#) applies, essentially without change. □

To understand the significance of this result, we need to introduce a filtration of the space $\mathcal{A}^{(n)}(u_0; p, q)$. Let Λ_n be the set of partitions of at most n parts, and let \subset denote the inclusion partial order; we also let \subset denote the product partial order on $\Lambda_n \times \Lambda_n$. Then for any pair of partitions $\lambda, \mu \in \Lambda_n$, we define

$$\mathcal{A}_{\lambda\mu}^{(n)}(u_0; t; p, q)$$

to be the subspace of $\mathcal{A}^{(n)}(u_0; p, q)$ consisting of functions f such that whenever $(\kappa, \nu) \not\subset (\lambda, \mu)$, we have the limit

$$\lim_{z_i \rightarrow p^{-\kappa_i} q^{-\nu_i} t^{i-1} u_0} \prod_{1 \leq i \leq n} \theta(pqz_i^{\pm 1}/u_0; p, q)_{l,m} f(\dots z_i \dots) = 0$$

whenever

$$\prod_{1 \leq i \leq n} \theta(pqz_i^{\pm 1}/u_0; p, q)_{l,m} f(\dots z_i \dots)$$

is holomorphic and $\kappa, \nu \subset (l, m)^n$. Note that enlarging l or m multiplies the equation by a (possibly zero) scalar, so we really have only one equation for each pair (κ, ν) .

Remark. In the univariate case ([38], [17], [18], [34], [35], [32]), this filtration simply corresponds to a sequence of allowed poles. Given the role played by vanishing conditions in the theory of Koornwinder polynomials [15], [20], it would seem to be natural to generalize the forbiddance of a pole to the vanishing (after clearing the denominator) at an appropriate point, thus obtaining our filtration.

LEMMA 6.5. *For generic u_0, p, q, t , the filtration $\mathcal{A}_{\lambda}^{(n)}(u_0; t; p, q)$ is tight in the sense that*

$$\dim \mathcal{A}_{\lambda}^{(n)}(u_0; t; p, q) = 1 + \dim \sum_{\kappa \subsetneq \lambda} \mathcal{A}_{\kappa}^{(n)}(u_0; t; p, q),$$

for any partition pair $\lambda \in \Lambda_n^2$. In particular, each space in the filtration is finite-dimensional.

Proof. Let $\lambda = (\lambda, \mu)$, $\kappa = (\kappa, \nu)$. Since the spaces

$$\mathcal{A}_{\lambda\mu}^{(n)}(u_0; t; p, q)$$

and

$$\sum_{(\kappa, \nu) \subsetneq (\lambda, \mu)} \mathcal{A}_{\kappa\nu}^{(n)}(u_0; t; p, q)$$

differ by a single equation, their dimensions differ by at most 1; it thus suffices to construct a function in the former but not in the latter.

Define a function $F_{\lambda\mu}^{(n)}(u_0 : \dots z_i \dots)$ by the following product:

$$\begin{aligned} & F_{\lambda\mu}^{(n)}(u_0 : \dots z_i \dots; t; p, q) \\ &= \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \lambda_1}} \frac{\theta(p^j q t^{-\lambda'_j} z_i^{\pm 1}/u_0; q)}{\theta(p^j q z_i^{\pm 1}/u_0; q)} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \mu_1}} \frac{\theta(pq^j t^{-\mu'_j} z_i^{\pm 1}/u_0; p)}{\theta(pq^j z_i^{\pm 1}/u_0; p)}. \end{aligned}$$

It follows as in the proof of Lemma 6.3 of [15] that

$$F_{\lambda\mu}^{(n)}(u_0 : \dots z_i \dots) \in \mathcal{A}_{\lambda\mu}^{(n)}(u_0; t; p, q);$$

on the other hand, we find that

$$\lim_{\substack{z_i \rightarrow p^{-\lambda_i} q^{-\mu_i} t^{i-1} u_0 \\ i=1\dots n}} \prod_{1 \leq i \leq n} \theta(pqz_i^{\pm 1}/u_0; p, q)_{l,m} F_{\lambda\mu}^{(n)}(u_0 : \dots z_i \dots; t; p, q)$$

is generically nonzero. □

Remark. The function $F_{\lambda\mu}^{(n)}$ is a special case of the interpolation functions introduced below (Definition 5). Indeed, one can show that

$$F_{\lambda}^{(n)}(u_0; t, p, q) = \mathcal{R}_{\lambda}^{*(n)}(; pqt^{-n}/u_0, u_0; t; p, q).$$

The existence of such a factorizable special case of the interpolation functions will turn out to be crucial to the arguments of [21].

The reason we have introduced this filtration is the following fact:

LEMMA 6.6. *The difference operator $\mathcal{D}_q^{(n)}(u_0, t_0, t_1; t, p)$ is triangular with respect to the above filtration; that is, for all $\lambda \in \Lambda_n^2$,*

$$\mathcal{D}_q^{(n)}(u_0, t_0, t_1; t, p) \mathcal{A}_{\lambda}^{(n)}(\sqrt{q}u_0; t; p, q) \subset \mathcal{A}_{\lambda}^{(n)}(u_0; t; p, q),$$

with equality for generic values of the parameters.

Proof. Let $\lambda = (\lambda, \mu)$. Choose $l \geq \lambda_1, m \geq \mu_1$, and consider a function

$$f \in \mathcal{A}_{\lambda\mu}^{(n)}(\sqrt{q}u_0; t; p, q).$$

For $\kappa \subset l^n, \nu \subset m^n$, define

$$C_{\kappa\nu}(f) = \lim_{\substack{z_i \rightarrow p^{-\kappa_i} q^{1/2-\nu_i} t^{i-1} u_0 \\ i=1\dots n}} \prod_{1 \leq i \leq n} \theta(pq^{1/2}z_i^{\pm 1}/u_0; p, q)_{l,m} f(\dots z_i \dots),$$

$$\begin{aligned} C'_{\kappa\nu}(f) &= \lim_{\substack{z_i \rightarrow p^{-\kappa_i} q^{-\nu_i} t^{i-1} u_0 \\ i=1\dots n}} \\ &\times \prod_{1 \leq i \leq n} \theta(pqz_i^{\pm 1}/u_0; p, q)_{l,m} (\mathcal{D}_q^{(n)}(u_0, t_0, t_1; t, p) f)(\dots z_i \dots). \end{aligned}$$

We claim that we can write

$$C'_{\kappa\nu} = \sum_{\nu \subset \rho} c_{\kappa\nu\rho} C_{\kappa\rho},$$

where the coefficients $c_{\kappa\nu\rho}$ are meromorphic and independent of the choice of f . Indeed, this follows readily from the definition of \mathcal{D} ; compare the proof of Theorem 3.2 of [20]. More precisely, we see that a given term of the corresponding

sum involves the specialization

$$\lim_{z_i \rightarrow p^{-\kappa_i} q^{\sigma_i/2 - \nu_i} t^{i-1} u_0} \prod_{1 \leq i \leq n} (pq^{1/2} z_i^{\pm 1} / u_0; p, q)_{l, m} f(\dots z_i \dots);$$

if the sequence $\frac{1-\sigma_i}{2} + \nu_i$ does not induce a partition, then the remaining factors vanish, while if it does give a partition, that partition necessarily contains ν . We also find that the diagonal coefficient $c_{\kappa \nu \nu}$ is generically nonzero; the result follows. □

Now, given a pair of spaces with corresponding tight filtrations, equipped with a (sufficiently general) scalar product, there is a unique (up to scalar multiples) orthogonal pair of bases compatible with the filtration. In the case of the above scalar product, this suggests the following definition.

“Definition”. For all partition pairs $\lambda \in \Lambda_n^2$, the function

$$R_\lambda^{(n)}(\dots z_i \dots; t_0, t_1, t_2, t_3; u_0, u_1; t; p, q)$$

is defined to be the unique (up to scalar multiples) element of $\mathcal{A}_\lambda^{(n)}(u_0; t; p, q)$ such that

$$\langle R_\lambda^{(n)}(\dots z_i \dots; t_0, t_1, t_2, t_3; u_0, u_1; t; p, q), g \rangle_{t_0, t_1, t_2, t_3; u_0; u_1; t; p, q} = 0$$

whenever $g \in \mathcal{A}_\kappa^{(n)}(u_1; t; p, q)$ for some $\kappa \subsetneq \lambda$.

Since our adjoint difference operators preserve the filtrations, they would necessarily be diagonal in the corresponding bases, if they were well-defined. Unfortunately, we have as yet no reason to believe that the scalar product is nondegenerate relative to the filtration; that is, that its restriction to $\mathcal{A}_\lambda^{(n)}(u_0; t; p, q)$ and $\mathcal{A}_\lambda^{(n)}(u_1; t; p, q)$ is nondegenerate for all partition pairs. If this condition were to fail for a given pair κ , then the function $R_\lambda^{(n)}$ would not be uniquely determined for $\lambda \supsetneq \kappa$, and the argument breaks down.

There is one special case in which we can prove the scalar product is generically nondegenerate.

PROPOSITION 6.7. *For generic parameters satisfying $t^{2n-2} t_0 t_1 t_2 t_3 u_0 u_1 = pq$, and any partition $\lambda \in \Lambda_n$, the scalar product $\langle \rangle_{t_0, t_1, t_2, t_3; u_0, u_1; t; p, q}$ is nondegenerate between $\mathcal{A}_{0\lambda}^{(n)}(u_0; t; p, q)$ and $\mathcal{A}_{0\lambda}^{(n)}(u_1; t; p, q)$.*

Proof. To show a scalar product generically nondegenerate, it suffices to exhibit a nondegenerate specialization. Choose l such that the spaces

$$\theta(pq/u_0; p, q)_{0, l} \mathcal{A}_{0\lambda}^{(n)}(u_0; t; p, q) \quad \text{and} \quad \theta(pq/u_1; p, q)_{0, l} \mathcal{A}_{0\lambda}^{(n)}(u_1; t; p, q)$$

consist of holomorphic functions, and specialize the parameters so that every parameter except u_0, u_1 is real, between 0 and 1, while u_0 and u_1 are complex conjugates satisfying $0 < |u_0| = |u_1| < q^l$. (This is possible as long as $p < q^{2l-1}t^{2n-2}$.) Then the contour in the scalar product can be taken to be the unit torus, on which the weight function is clearly strictly positive. Moreover, the filtrations with respect to u_0 and u_1 are conjugate to each other. The scalar product thus becomes a positive definite Hermitian inner product, and is therefore nondegenerate. \square

This in particular proves the existence and uniqueness of the above biorthogonal functions, as long as one of the partitions is trivial. In general, however, it is unclear how to construct a manifestly nondegenerate instance of the scalar product. We will therefore give a more direct construction of these functions, and by computing their scalar products show that this problem generically does not arise. (In addition, the above construction gives functions that are only guaranteed to be orthogonal when the corresponding pairs of partitions are distinct but comparable; it will follow below (as one would expect) that comparability is not necessary.)

To do this, we need a different adjoint pair of difference operators.

First, define

$$\mathcal{D}_q^{-(n)}(u_0; t, p) := D_q^{(n)}(u_0, qu_0, p/u_0, \frac{1}{t^{n-1}u_0q}; t, p).$$

Next, define

$$\begin{aligned} & (\mathcal{D}_q^{+(n)}(u_0:u_1:u_2, u_3, u_4; t, p)f)(\dots z_i \dots) \\ & := \prod_{1 \leq i \leq n} \frac{\theta(pqt^{n-i}u_1/u_0; p)}{\prod_{2 \leq r \leq 5} \theta(u_r t^{n-i}u_1; p)} \\ & \times \sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \leq i \leq n} \frac{\prod_{1 \leq r \leq 5} \theta(u_r z_i^{\sigma_i}; p)}{\theta(pqz_i^{\sigma_i}/u_0; p)\theta(z_i^{2\sigma_i}; p)} \prod_{1 \leq i < j \leq n} \frac{\theta(tz_i^{\sigma_i}z_j^{\sigma_j}; p)}{\theta(z_i^{\sigma_i}z_j^{\sigma_j}; p)} f(\dots q^{\sigma_i/2}z_i \dots), \end{aligned}$$

where $u_5 = p^2q/t^{n-1}u_0u_1u_2u_3u_4$. Note that aside from the normalization factor, $\mathcal{D}_q^{+(n)}$ is symmetric in u_1 through u_5 .

These act as lowering and raising operators with respect to the filtration:

LEMMA 6.8. *For all $\lambda \in \Lambda_n^2$ with $(0, 1)^n \subset \lambda$,*

$$\mathcal{D}_q^{-(n)}(u_0; t, p)\mathcal{A}_\lambda^{(n)}(q^{3/2}u_0; t; p, q) \subset \mathcal{A}_{\lambda-(0,1)^n}^{(n)}(u_0; t; p, q).$$

Similarly, for all $\lambda \in \Lambda_n^2$,

$$\mathcal{D}_q^{+(n)}(u_0:u_1:u_2, u_3, u_4; t, p)\mathcal{A}_\lambda^{(n)}(q^{-1/2}u_0; t; p, q) \subset \mathcal{A}_{\lambda+(0,1)^n}^{(n)}(u_0; t; p, q).$$

Moreover, the restriction of $\mathcal{D}^{-(n)}$ is generically surjective, while the restriction of $\mathcal{D}^{+(n)}$ is generically injective.

Proof. As above. □

THEOREM 6.9. *If $f \in \mathcal{A}^{(n)}(q^{-1/2}u_0; p, q)$, $g \in \mathcal{A}^{(n)}(u_1; p, q)$ and*

$$t^{2n-2}u_0u_1t_0t_1t_2t_3 = pq,$$

then

$$\begin{aligned} \langle \mathcal{D}_q^{+(n)}(u_0:t_0:t_1, t_2, t_3; t, p) f, g \rangle_{t_0, t_1, t_2, t_3; u_0, u_1; t, p, q} \\ = C \langle f, \mathcal{D}_q^{-(n)}(u'_1; t, p) g \rangle_{t'_0, t'_1, t'_2, t'_3; u'_0, u'_1; t, p, q}, \end{aligned}$$

where

$$(t'_0, t'_1, t'_2, t'_3, u'_0, u'_1) = (q^{1/2}t_0, q^{1/2}t_1, q^{1/2}t_2, q^{1/2}t_3, q^{-1/2}u_0, q^{-3/2}u_1)$$

and

$$C = \prod_{1 \leq i \leq n} \frac{\theta(t^{n-i}t_1t_2, t^{n-i}t_1t_3, t^{n-i}t_2t_3, pq t^{n-i}t_0/u_0; p) \theta(pt_0u_1 t^{2n-1-i}; p)^{-1}}{\theta(t^{n-i}t_0u_1/q, t^{n-i}t_1u_1/q, t^{n-i}t_2u_1/q, t^{n-i}t_3u_1/q, t^{n-i}u_0u_1/q, t^{n-i}u_0u_1/q^2; p)}.$$

7. Integral operators

Just as our second proof of [Theorem 6.1](#) is related to an adjoint pair of difference operators, the argument of van Diejen and Spiridonov is related to an adjoint pair of *integral* operators. To understand these operators, we first need to understand what happens to the $I_{BC_n}^{(0)}$ integral when the integrand is multiplied by an element of $\mathcal{A}(t_0; p, q)$. We define a corresponding integral operator as follows.

Definition 2. If $f \in \mathcal{A}(u_0; p, q)$, then $\mathcal{J}^{*(n)}(u_0; p, q) f$ is the function on the set $\prod_{0 \leq r \leq 2n+3} u_r = pq$ defined by

$$\begin{aligned} (\mathcal{J}^{*(n)}(u_0; p, q) f)(u_1, \dots, u_{2n+3}) &= \frac{(p; p)^n (q; q)^n}{2^n n! \prod_{0 \leq r < s \leq 2n+3} \Gamma(u_r u_s; p, q)} \\ &\times \int_{\mathcal{C}^n} f(\dots z_i \dots) \prod_{1 \leq i < j \leq n} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\ &\times \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 2n+3} \Gamma(u_r z_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)} \frac{dz_i}{2\pi \sqrt{-1} z_i}, \end{aligned}$$

with the usual conventions about the choice of contour.

In particular, by [Corollary 3.2](#), it follows that

$$\mathcal{J}^{*(n)}(u_0; p, q) 1 = 1.$$

To determine the action of this integral operator in general, it suffices to consider f in a spanning set. We may thus restrict our attention to functions of the

form

$$f(\dots z_i \dots) = \prod_{1 \leq i \leq n} \frac{\prod_{1 \leq j \leq l} x_j^{-1} \theta(x_j z_i^{\pm 1}; q) \prod_{1 \leq j \leq m} y_j^{-1} \theta(y_j z_i^{\pm 1}; p)}{\theta(pqz_i^{\pm 1}/u_0; p, q)_{l,m}}.$$

If we write the theta functions in the numerator as a ratio of elliptic Γ functions, and similarly absorb the denominator factors into a ratio of elliptic Γ functions, we find that the resulting integral is proportional to an integral of type $I_{BC_n}^{(l+m)}$ in which the extra $2l + 2m$ parameters have pairwise products p^2q and pq^2 . If we then apply [Theorem 3.1](#), we find that the right-hand side becomes a sum via residue calculus. We thus obtain the following result.

THEOREM 7.1. *If*

$$f(\dots z_i \dots) = \prod_{1 \leq i \leq n} \frac{\prod_{1 \leq j \leq l} x_j^{-1} \theta(x_j z_i^{\pm 1}; q) \prod_{1 \leq j \leq m} y_j^{-1} \theta(y_j z_i^{\pm 1}; p)}{\theta(pqz_i^{\pm 1}/u_0; p, q)_{l,m}},$$

and $\prod_{0 \leq r \leq 2n+3} u_r = pq$, then

(7.1)

$$\begin{aligned} (\mathcal{F}^{*(n)}(u_0; p, q)f)(u_1, \dots, u_{2n+3}) &= \prod_{1 \leq r \leq 2n+3} \frac{1}{\theta(pq/u_0 u_r; p, q)_{l,m}} \\ &\times \left(\prod_{1 \leq i \leq l} (1 + R(x_i)) \frac{\theta(p^{-l} u_0 x_i; q) \prod_{1 \leq r \leq 2n+3} \theta(u_r x_i; q)}{x_i^n \theta(x_i^2; q)} \prod_{1 \leq i < j \leq l} \frac{\theta(p x_i x_j; q)}{\theta(x_i x_j; q)} \right) \\ &\times \left(\prod_{1 \leq i \leq m} (1 + R(y_i)) \frac{\theta(q^{-m} u_0 y_i; p) \prod_{1 \leq r \leq 2n+3} \theta(u_r y_i; p)}{y_i^m \theta(y_i^2; p)} \prod_{1 \leq i < j \leq m} \frac{\theta(q y_i y_j; p)}{\theta(y_i y_j; p)} \right), \end{aligned}$$

where $R(x_k)$ is an operator acting on $g(\dots x_i \dots)$ via the substitution $x_k \mapsto x_k^{-1}$; thus the factors in parentheses are sums of 2^l and 2^m terms respectively.

Since the factors in parentheses are clearly holomorphic in u_1, \dots, u_{2n+3} , and the given functions span $\mathcal{A}(u_0; p, q)$, we obtain the following as an immediate consequence:

COROLLARY 7.2. *If $f \in \mathcal{A}(u_0; p, q)$ is such that*

$$\theta(pqz_i^{\pm 1}/u_0; p, q)_{l,m} f(\dots z_i \dots)$$

is holomorphic, then

$$\prod_{1 \leq r \leq 2n+3} \theta(pq/u_0 u_r; p, q)_{l,m} (\mathcal{F}^{*(n)}(u_0; p, q)f)(u_1, \dots, u_{2n+3})$$

is holomorphic on the set $\prod_{0 \leq r \leq 2n+3} u_r = pq$.

Remark 1. Similarly, the left-hand side of (7.1) is manifestly a holomorphic q -theta function in the x 's, and a holomorphic p -theta function in the y 's; that this is true of the right-hand side follows from a symmetrization argument analogous to those we have just encountered in studying difference operators. And, indeed, the two sums are really just minor variants of the difference operators we have already seen.

Remark 2. As the above argument is based on Theorem 3.1, it cannot be (easily) directly applied in the limit $p \rightarrow 0$. In fact, one can also derive this result from Corollary 3.2, for which direct, nonelliptic, proofs are known in the $p \rightarrow 0$ limit [11]. The basic observation is that if two of the parameters have product q , i.e., if two of the Γ factors combine to produce a factor of the form

$$\prod_{1 \leq i \leq n} \frac{1}{\theta(az_i^{\pm 1}; q)},$$

then the integrand is essentially invariant under $a \mapsto 1/a$ (aside from an overall constant). However, the *integral* does not share this invariance, because inverting a changes the constraint on the contour. The two contours differ only in whether they contain the points $z = a^{\pm 1}$; as a result, the difference in the two integrals is (proportional to) the $n - 1$ -dimensional integral of the residue at that point. This $n - 1$ -dimensional integral simplifies to the above form, with $l = 1, m = 0$; the difference of the original n -dimensional integrals simplifies to the desired right-hand side. This argument can then be repeated as necessary to prove the theorem for arbitrary values of $l, m \geq 0$.

Remark 3. The fact that we obtain an $l + m$ -tuple sum is, of course, directly related to the fact that we needed $2l + 2m$ Γ factors to represent the numerator of f . In general, if we took

$$f(\dots z_i \dots) = \prod_{1 \leq i \leq n} \frac{\prod_{1 \leq j \leq m} \theta(x_j z_i^{\pm 1}; p, q)_{a_j, b_j}}{\theta(pqz_i^{\pm 1}/u_0; p, q)_{\sum a_j, \sum b_j}},$$

residue calculus would again give a sum, this time a $2m$ -tuple sum (i.e., the product of an m -tuple sum for p and an m -tuple sum for q). On the other hand, we could also compute $\mathcal{J}^{*(n)}(u_0; p, q) f$ by specialization of Theorem 7.1, which would give a sum with $2^{\sum a_j + b_j}$ terms. The fact that this sum simplifies underlies Rosengren's arguments in Section 7 of [26].

Remark 4. It is particularly striking that the right-hand side factors as a product of two sums, one involving only q -theta functions, and one involving only p -theta functions. This factorization phenomenon appears to hold quite generally in the theory of elliptic hypergeometric integrals, but only when the relevant balancing condition holds.

Since $\mathcal{F}^{*(n)}(u_0; p, q)$ takes BC_n -symmetric functions to A_{2n+2} -symmetric functions, it is not quite suitable for our purposes. However, we can readily obtain BC -symmetric functions by suitable specialization.

Definition 3. Define operators $\mathcal{F}_t^{+(n)}(u_0; p, q)$, $\mathcal{F}_t^{(n)}(u_0; u_1, u_2; p, q)$, and $\mathcal{F}_t^{- (n)}(u_0; u_1, u_2, u_3, u_4; p, q)$ by:

$$\begin{aligned} &(\mathcal{F}_t^{+(n)}(u_0; p, q)f)(z_1, \dots, z_{n+1}) \\ &= (\mathcal{F}^{*(n)}(u_0; p, q)f)\left(\frac{t^{-n-1}pq}{u_0}, \dots, \sqrt{t}z_i^{\pm 1}, \dots\right), \\ &(\mathcal{F}_t^{(n)}(u_0; u_1, u_2; p, q)f)(z_1, \dots, z_n) \\ &= (\mathcal{F}^{*(n)}(u_0; p, q)f)\left(u_1, u_2, \frac{t^{-n}pq}{u_0u_1u_2}, \dots, \sqrt{t}z_i^{\pm 1}, \dots\right), \\ &(\mathcal{F}_t^{- (n)}(u_0; u_1, u_2, u_3, u_4; p, q)f)(z_1, \dots, z_{n-1}) \\ &= (\mathcal{F}^{*(n)}(u_0; p, q)f)\left(u_1, u_2, u_3, u_4, \frac{t^{1-n}pq}{u_0u_1u_2u_3u_4}, \dots, \sqrt{t}z_i^{\pm 1}, \dots\right). \end{aligned}$$

THEOREM 7.3. *The above operators are triangular with respect to the filtration of $\mathcal{A}^{(n)}(u_0; p, q)$; to be precise,*

$$\begin{aligned} &\mathcal{F}_t^{+(n)}(u_0; p, q)\mathcal{A}_\lambda^{(n)}(u_0; t; p, q) \subseteq \mathcal{A}_\lambda^{(n+1)}(t^{1/2}u_0; t; p, q), \\ &\mathcal{F}_t^{(n)}(u_0; u_1, u_2; p, q)\mathcal{A}_\lambda^{(n)}(u_0; t; p, q) \subseteq \mathcal{A}_\lambda^{(n)}(t^{1/2}u_0; t; p, q), \end{aligned}$$

and, if $\lambda_n = (0, 0)$,

$$\mathcal{F}_t^{- (n)}(u_0; u_1, u_2, u_3, u_4; p, q)\mathcal{A}_\lambda^{(n)}(u_0; t; p, q) \subseteq \mathcal{A}_\lambda^{(n-1)}(t^{1/2}u_0; t; p, q).$$

Also, $\mathcal{F}_t^{+(n)}(u_0; p, q)$ is generically injective, and $\mathcal{F}_t^{- (n)}(u_0; u_1, u_2, u_3, u_4; p, q)$ is generically surjective.

Proof. It suffices to consider the action of the operators on the functions

$$\begin{aligned} &F_{\lambda\mu}^{(n)}(u_0 : \dots z_i \dots; t; p, q) \\ &= \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \lambda_1}} \frac{\theta(p^jqt^{-\lambda'_j}z_i^{\pm 1}/u_0; q)}{\theta(p^jqz_i^{\pm 1}/u_0; q)} \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \mu_1}} \frac{\theta(pq^jt^{-\mu'_j}z_i^{\pm 1}/u_0; p)}{\theta(pq^jz_i^{\pm 1}/u_0; p)} \end{aligned}$$

considered above. Applying [Theorem 7.1](#), we find that each term of the resulting sum is also of this form, with appropriately constrained partitions. The one exception is $\mathcal{F}_t^{- (n)}$ in the case when λ_n or $\mu_n > 0$, which we will consider below. \square

As promised, the integral operators indeed satisfy appropriate adjointness relations.

THEOREM 7.4. *If $f \in \mathcal{A}^{(n)}(u_0; p, q)$, $g \in \mathcal{A}^{(n)}(t^{-1/2}u_1; p, q)$ and*

$$t^{2n-2}u_0u_1t_0t_1t_2t_3 = pq,$$

then

$$\begin{aligned} \langle \mathcal{F}_t^{(n)}(u_0:t_0, t_1; p, q)f, g \rangle_{t'_0, t'_1, t'_2, t'_3; u'_0, u'_1; t; p, q} \\ = \langle f, \mathcal{F}_t^{(n)}(u'_1:t'_2, t'_3; p, q)g \rangle_{t_0, t_1, t_2, t_3; u_0, u_1; t; p, q}, \end{aligned}$$

where

$$(t'_0, t'_1, t'_2, t'_3, u'_0, u'_1) = (t^{1/2}t_0, t^{1/2}t_1, t^{-1/2}t_2, t^{-1/2}t_3, t^{1/2}u_0, t^{-1/2}u_1).$$

Similarly, if $f \in \mathcal{A}^{(n)}(u_0; p, q)$, $g \in \mathcal{A}^{(n-1)}(u'_1; p, q)$ and $t^{2n-2}u_0u_1t_0t_1t_2t_3 = pq$, then

$$\begin{aligned} \langle \mathcal{F}_t^{-(n)}(u_0:t_0, t_1, t_2, t_3; p, q)f, g \rangle_{t'_0, t'_1, t'_2, t'_3; u'_0, u'_1; t; p, q} \\ = \langle f, \mathcal{F}_t^{+(n-1)}(u'_1; p, q)g \rangle_{t_0, t_1, t_2, t_3, u_0, u_1; t; p, q}, \end{aligned}$$

where

$$(t'_0, t'_1, t'_2, t'_3, u'_0, u'_1) = (t^{1/2}t_0, t^{1/2}t_1, t^{1/2}t_2, t^{1/2}t_3, t^{1/2}u_0, t^{-1/2}u_1).$$

Proof. In each case, the definition of the integral operators allows us to express the inner products as double integrals; the stated identities correspond to changing the order of integration.

Note added December, 2009: There is a difficulty performing this exchange of integrals in the second case, as it does not appear possible to choose consistent contours in general. However, if g is either p - or q -elliptic, there is no difficulty—if one fixes the “degrees” and numerators of f and g , there is an open subset of parameter space for which one can use a common contour for all variables. This holds even if f lies in the larger space

$$\mathcal{A}^{(n)}(u_0; p, q) \otimes \mathcal{A}^{(n)}(t_0; p, q),$$

and thus (setting $g = 1$ and swapping t_0 and u_1) gives rise to a formula for the inner product by iterating the lowering operator. But this implies that the inner product factors as a tensor product, so the p - and q -elliptic cases suffice to prove adjointness in general. \square

Recall that for the operators \mathcal{D}^- and \mathcal{F}^- , we were only able to show triangularity with respect to a portion of the filtration; for some functions, the methods we used were insufficient to understand the images. The key observation for dealing with those cases is that the difficult case for \mathcal{D}^- is precisely the (generic) image of \mathcal{F}^+ , and similarly the difficult case for \mathcal{F}^- is the image of \mathcal{D}^+ . Thus to complete our understanding of the action of these operators on the filtration, it will suffice to prove the following result.

THEOREM 7.5. *For any function $f \in \mathcal{A}(u_0; p, q)$,*

$$\mathcal{D}_q^{-(n)}(q^{-3/2}t^{1/2}u_0; t, p)\mathcal{F}_t^{+(n-1)}(u_0; p, q)f = 0.$$

Similarly, for any function $f \in \mathcal{A}^{(n)}(q^{-1/2}u_0; p, q)$,

$$(7.2) \quad \mathcal{F}_t^{-(n)}(u_0:t_0, t_1, t_2, t_3; p, q) \mathcal{D}_q^{+(n)}(u_0:t_0, t_1, t_2, t_3; t, p) f = 0.$$

Proof. For the first identity, take

$$f(\dots z_i \dots) = \prod_{1 \leq i \leq n} \frac{\prod_{1 \leq j \leq l} x_j^{-1} \theta(x_j z_i^{\pm 1}; q) \prod_{1 \leq j \leq m} y_j^{-1} \theta(y_j z_i^{\pm 1}; p)}{\theta(pqz_i^{\pm 1}/u_0; p, q)_{l,m}};$$

we can thus compute its image via [Theorem 7.1](#) and the definition of $\mathcal{D}^{-(n)}$. The vanishing of the resulting sum follows as a special case of [Lemma 7.6](#) below.

For the second identity, we can argue as in the proof of adjointness of the difference operators to express the image as the integral of $f(\dots z_i \dots)$ with respect to an appropriate BC_n -symmetric density. That this density vanishes identically follows from [Lemma 7.8](#) below. \square

LEMMA 7.6. *For arbitrary parameters satisfying $vw \prod_{1 \leq i \leq n} q_i^2 = 1$, and generic z_1, \dots, z_n ,*

$$\prod_{1 \leq i \leq n} (1 + R(z_i)) \frac{\theta(vq_i z_i, wq_i z_i; p)}{\theta(z_i^2; p)} \prod_{1 \leq i < j \leq n} \frac{q_j^{-1} \theta(q_i q_j z_i z_j, q_j z_i / q_i z_j; p)}{\theta(z_i z_j, z_i / z_j; p)} = 0.$$

Proof. For $n = 1$, the summand is manifestly antisymmetric under $R(z_i)$, and thus the lemma follows in that case. Thus assume $n > 1$, set $v = u/Q$, $w = (uQ)^{-1}$ with $Q := \prod_{1 \leq i \leq n} q_i$, and consider the sum as a function of u . We readily verify that it is a BC_1 -symmetric theta function in u of degree n ; we thus need only show that it vanishes at more than n independent points. If $u = Qz_n/q_n$, the terms involving $R(z_n)$ vanish; moreover, if we pull out BC_{n-1} -symmetric factors, we obtain a special case of the $n - 1$ -dimensional sum. By symmetry, the identity holds for any point of the form $u = Qz_i^{\pm 1}/q_i$; since $n > 1$, these $2n$ points are generically independent, and the result follows. \square

We note the following related result in passing:

COROLLARY 7.7. *For arbitrary parameters satisfying $tuvv \prod_{1 \leq i \leq n} q_i^2 = 1$, and generic z_1, \dots, z_n ,*

$$t\theta(uv, uv, vw; p) \prod_{1 \leq i \leq n} (1 + R(z_i)) \frac{\theta(tz_i/q_i, uq_i z_i, vq_i z_i, wq_i z_i; p)}{z_i \theta(z_i^2; p)} \\ \times \prod_{1 \leq i < j \leq n} \frac{q_j^{-1} \theta(q_i q_j z_i z_j, q_j z_i / q_i z_j; p)}{\theta(z_i z_j, z_i / z_j; p)}$$

is symmetric under permutations of t, u, v, w .

Proof. The sum is manifestly symmetric in u, v, w , so it suffices to show that it is invariant under the exchange of t and u . Thus take the difference of the given sum and its image upon exchanging t and u . If we then set $t = q_{n+1}y_{n+1}$, $u = q_{n+1}/y_{n+1}$, we obtain the $n + 1$ -dimensional instance of the lemma. \square

LEMMA 7.8. *For generic values of y_1, \dots, y_n ,*

$$\prod_{1 \leq i \leq n} (1 + R(y_i)) \prod_{1 \leq i < j \leq n} \frac{\theta(uy_i y_j; p)}{\theta(y_i y_j; p)} \prod_{1 \leq i \leq n} \frac{\prod_{1 \leq r \leq n-1} \theta(u^{-1/2} z_r^{\pm 1} y_i; p)}{y_i^{n-2} \theta(y_i^2; p)} = 0.$$

Proof. When $n = 1$, the summand is antisymmetric under $R(y_1)$, and the sum therefore vanishes. Now, consider the sum for general n as a function of z_{n-1} . This is manifestly a BC_1 -symmetric theta function of degree n ; it thus suffices to show that it vanishes at more than n independent points. If $z_{n-1} = u^{1/2}y_n$, the terms coming from $R(y_n)$ vanish; we thus obtain an instance of the $n - 1$ -dimensional sum, which vanishes by induction. By symmetry, the sum vanishes at any point of the form $z_{n-1} = u^{1/2}y_i^{\pm 1}$; this gives $2n$ independent values at which the sum vanishes, proving the lemma. \square

A similar argument applies to the following result, which can also be obtained from Theorem 3.1 via residue calculus.

THEOREM 7.9. *Choose integers $m \geq l \geq 0$, and suppose $q^{m-1}t_0 t_1 t_2 t_3 = q$. Then we have the following identity.*

$$\begin{aligned} & \prod_{1 \leq i \leq m} (1 + R(x_i)) \frac{\theta(t_0 x_i, t_1 x_i, t_2 x_i, t_3 x_i; p) \prod_{1 \leq r \leq l} \theta(q^{-1/2} x_i x_r^{\pm 1}; p)}{x_i^{l+1} \theta(x_i^2; p)} \\ & \times \prod_{1 \leq i < j \leq m} \frac{\theta(q x_i x_j; p)}{\theta(x_i x_j; p)} \\ = & \prod_{0 \leq i < m-l} \frac{\theta(q^i t_0 t_1, q^i t_0 t_2, q^i t_0 t_3; p)}{(-q^{1/2})^{m-1} t_0} \\ & \times \prod_{1 \leq i \leq l} (1 + R(y_i)) \frac{\theta(q^{1/2} y_i, q^{1/2} y_i, q^{1/2} y_i, q^{1/2} y_i; p) \prod_{1 \leq r \leq m} \theta(q^{-1/2} y_i x_r^{\pm 1}; p)}{y_i^{m+1} \theta(y_i^2; p)} \\ & \times \prod_{1 \leq i < j \leq l} \frac{\theta(q y_i y_j; p)}{\theta(y_i y_j; p)}. \end{aligned}$$

Proof. By the usual symmetry argument, we find that both sides are BC_m -symmetric theta functions of degree l in x . By induction, both sides agree if x_m is of the form t_r or $q^{-1/2}y_i^{\pm 1}$; this gives $2l + 4$ independent points at which the functions agree, which shows that they agree everywhere. \square

This gives rise to some commutation relations between our difference and integral operators.

COROLLARY 7.10. For any function $f \in \mathcal{A}^{(n)}(q^{1/2}u_0; p, q)$,

$$\begin{aligned} & \mathcal{F}_t^{(n)}(u_0:t_0, t_1; p, q)\mathcal{D}_q^{(n)}(u_0, t_0, t_1; t, p) f \\ &= \mathcal{D}_q^{(n)}(t^{1/2}u_0, t^{1/2}t_0, t^{1/2}t_1; t, p)\mathcal{F}_t^{(n)}(q^{1/2}u_0:q^{1/2}t_0, q^{1/2}t_1; p, q) f, \\ & \mathcal{F}_t^{(n)}(u_0:t_0, t_1; p, q)\mathcal{D}_q^{(n)}(u_0, t_0, t_2; t, p) f \\ &= \mathcal{D}_q^{(n)}(t^{1/2}u_0, t^{1/2}t_0, t^{-1/2}t_2; t, p)\mathcal{F}_t^{(n)}(q^{1/2}u_0:q^{1/2}t_0, q^{-1/2}t_1; p, q) f, \\ & \mathcal{F}_t^{+(n)}(u_0; p, q)\mathcal{D}_q^{(n)}(u_0, t_0, t_1; t, p) f \\ &= \mathcal{D}_q^{(n+1)}(t^{1/2}u_0, t^{-1/2}t_0, t^{-1/2}t_1; t, p)\mathcal{F}_t^{+(n)}(q^{1/2}u_0; p, q) f, \end{aligned}$$

while for any function $f \in \mathcal{A}^{(n)}(q^{-1/2}u_0; p, q)$,

$$\begin{aligned} & \mathcal{F}_t^{(n)}(u_0:t_0, t_1; p, q)\mathcal{D}_q^{+(n)}(u_0:t_0:t_1, t_2, t_3; t, p) f \\ &= \mathcal{D}_q^{+(n)}(t^{1/2}u_0:t^{1/2}t_0:t^{1/2}t_1, t^{-1/2}t_2, t^{-1/2}t_3; t, p) \\ & \quad \times \mathcal{F}_t^{(n)}(q^{-1/2}u_0:q^{1/2}t_0, q^{1/2}t_1; p, q) f. \end{aligned}$$

Proof. In each case, arguing as in the proof of adjointness of difference operators transforms the left-hand side into an integral of f against a BC_n -symmetric density which itself can be transformed via the theorem to give the right-hand side. □

8. Biorthogonal functions

Now that we have suitable difference and integral operators, we are in a position to construct the desired biorthogonal functions.

Definition 4. For each integer $n \geq 0$, we define a family of functions

$$\tilde{\mathcal{R}}_{\lambda}^{(n)}(z_1, \dots, z_n; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \in \mathcal{A}_{\lambda}^{(n)}(u_0; t; p, q)$$

indexed by a partition pair λ of length at most n and with parameters satisfying $t^{2n-2}t_0t_1t_2t_3u_0u_1 = pq$, as follows. For $n = 0$, we take

$$\tilde{\mathcal{R}}^{(0)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) := 1.$$

Otherwise, if $\lambda_n = (0, 0)$, we set

$$\begin{aligned} & \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \\ &:= \mathcal{F}_t^{+(n-1)}(t^{-1/2}u_0; p, q) \\ & \quad \cdot \tilde{\mathcal{R}}_{\lambda}^{(n-1)}(; t^{1/2}t_0:t^{1/2}t_1, t^{1/2}t_2, t^{1/2}t_3; t^{-1/2}u_0, t^{1/2}u_1; t; p, q). \end{aligned}$$

If $(0, 1)^n \subset \lambda$, set

$$\begin{aligned} & \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \\ & := \mathcal{D}_q^{+(n)}(u_0:t_0:t_1, t_2, t_3; t, p) \\ & \quad \cdot \tilde{\mathcal{R}}_{\lambda-(0,1)^n}^{(n)}(; q^{1/2}t_0:q^{1/2}t_1, q^{1/2}t_2, q^{1/2}t_3; q^{-1/2}u_0, q^{-3/2}u_1; t; p, q). \end{aligned}$$

Finally, if $(1, 0)^n \subset \lambda$, but $(0, 1)^n \not\subset \lambda$, set

$$\begin{aligned} & \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \\ & := \mathcal{D}_p^{+(n)}(u_0:t_0:t_1, t_2, t_3; t, q) \\ & \quad \cdot \tilde{\mathcal{R}}_{\lambda-(1,0)^n}^{(n)}(; p^{1/2}t_0:p^{1/2}t_1, p^{1/2}t_2, p^{1/2}t_3; p^{-1/2}u_0, p^{-3/2}u_1; t; p, q). \end{aligned}$$

Remark. The above definition closely resembles, and indeed was inspired by, Okounkov’s integral representation for interpolation polynomials [15]; in fact, in an appropriate limit, our $\mathcal{G}^{+(n)}$ becomes Okounkov’s integral operator (which can thus be expressed as a contour integral, rather than a q -integral).

It is clear that this inductively defines a family of functions as described; note also that the last relation still holds if $(1, 1)^n \subset \lambda$, since the corresponding p - and q -difference operators “commute”. In addition, it is clear that these functions should agree with the functions R we attempted to define above, aside from the fact that the scalar multiplication freedom has been eliminated:

PROPOSITION 8.1. *The functions $\tilde{\mathcal{R}}$ satisfy the normalization condition*

$$\tilde{\mathcal{R}}_{\lambda}^{(n)}(\dots t^{n-i} t_0 \dots; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) = 1.$$

Since the “diagonal” coefficients of the $+$ operators with respect to the filtration are generically nonzero, we find that they form a section of the filtration; that is:

PROPOSITION 8.2. *For any partition pair λ , and for generic values of the parameters, the functions*

$$\tilde{\mathcal{R}}_{\kappa}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q)$$

for $\kappa \subset \lambda$ form a basis of $\mathcal{A}_{\lambda}^{(n)}(t; p, q)$.

Also, since each of the $+$ operators used above factors as a tensor product, we find that the same holds for our family of functions.

LEMMA 8.3. *Each function $\tilde{\mathcal{R}}_{\lambda}^{(n)}$ is a product of a q -abelian and a p -abelian function; more precisely,*

$$\begin{aligned} \tilde{\mathcal{R}}_{\lambda\mu}^{(n)}(\dots z_i \dots; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \\ = \tilde{\mathcal{R}}_{\lambda 0}^{(n)}(\dots z_i \dots; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \\ \quad \times \tilde{\mathcal{R}}_{0\mu}^{(n)}(\dots z_i \dots; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q). \end{aligned}$$

Similarly, from adjointness and [Theorem 7.5](#), we can conclude:

THEOREM 8.4. *The functions $\tilde{\mathcal{R}}_{\lambda}^{(n)}$ satisfy the biorthogonality relation*

$$\begin{aligned} \langle \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q), \\ \tilde{\mathcal{R}}_{\kappa}^{(n)}(; t_0:t_1, t_2, t_3; u_1, u_0; t; p, q) \rangle_{t_0, t_1, t_2, t_3, u_0, u_1; t, p, q} = 0 \end{aligned}$$

whenever $\kappa \neq \lambda$. In particular, $\tilde{\mathcal{R}}_{\lambda}^{(n)}$ is orthogonal to the space $\mathcal{A}_{\kappa}^{(n)}(t; p, q)$ whenever $\lambda \not\subset \kappa$.

Remark. In particular, it follows that our functions agree with the univariate biorthogonal functions considered in [\[32, App. A\]](#). Note that in the univariate case, the definition involves only the raising difference operators; the integral operators are unnecessary. This gives rise to a generalized Rodriguez-type formula; compare [\[18\]](#).

THEOREM 8.5. *The functions $\tilde{\mathcal{R}}_{\lambda}^{(n)}$ satisfy the difference equations:*

$$\begin{aligned} \mathcal{D}_p^{(n)}(u_0, t_0, t_1; t, q) \tilde{\mathcal{R}}_{\lambda}^{(n)}(; p^{1/2}t_0: p^{1/2}t_1, p^{-1/2}t_2, p^{-1/2}t_3; p^{1/2}u_0, p^{-1/2}u_1; t; p, q) \\ = \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q), \\ \mathcal{D}_q^{(n)}(u_0, t_0, t_1; t, p) \tilde{\mathcal{R}}_{\lambda}^{(n)}(; q^{1/2}t_0: q^{1/2}t_1, q^{-1/2}t_2, q^{-1/2}t_3; q^{1/2}u_0, q^{-1/2}u_1; t; p, q) \\ = \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q), \end{aligned}$$

and the integral equation

$$\begin{aligned} \mathcal{I}_t^{(n)}(u_0:t_0, t_1; p, q) \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \\ = \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t^{1/2}t_0:t^{1/2}t_1, t^{-1/2}t_2, t^{-1/2}t_3; t^{1/2}u_0, t^{-1/2}u_1; t; p, q). \end{aligned}$$

Proof. Since each of the operators respects the factorization of $\tilde{\mathcal{R}}_{\lambda\mu}^{(n)}$, it suffices to consider the cases $\lambda = 0$ or $\mu = 0$, which are clearly equivalent. In particular, the inner product is now generically nondegenerate, and thus $\tilde{\mathcal{R}}_{\lambda 0}^{(n)}$ and $\tilde{\mathcal{R}}_{0\mu}^{(n)}$ are uniquely determined by biorthogonality and the normalization condition. Since each of the three operators we are considering has triangular adjoint, the left-hand sides satisfy biorthogonality; on the other hand, we readily compute that each operator preserves the normalization condition. \square

Remark. This gives rise to an alternate proof of the commutation relations of [Corollary 7.10](#), by comparing the actions of the two sides on the appropriate basis of biorthogonal functions. Similarly, one obtains the commutation relations:

$$\begin{aligned}
 & \mathcal{F}_t^{(n+1)}(u_0:t_0, t_1; p, q) \mathcal{F}_t^{+(n)}(t^{-1/2}u_0; p, q) \\
 &= \mathcal{F}_t^{+(n)}(u_0; p, q) \mathcal{F}_t^{(n)}(t^{-1/2}u_0:t^{1/2}t_0, t^{1/2}t_1; p, q), \\
 & \mathcal{F}_t^{(n)}(u_0:t_0, t_1; p, q) \mathcal{F}_t^{(n)}(t^{-1/2}u_0:t^{-1/2}t_0, t^{1/2}t_2; p, q) \\
 &= \mathcal{F}_t^{(n)}(u_0:t_0, t_2; p, q) \mathcal{F}_t^{(n)}(t^{-1/2}u_0:t^{-1/2}t_0, t^{1/2}t_1; p, q), \\
 & \mathcal{D}_q^{(n)}(u_0, t_0, t_1; t, p) \mathcal{D}_q^{(n)}(q^{1/2}u_0, q^{1/2}t_0, q^{-1/2}t_2; t, p) \\
 &= \mathcal{D}_q^{(n)}(u_0, t_0, t_2; t, p) \mathcal{D}_q^{(n)}(q^{1/2}u_0, q^{1/2}t_0, q^{-1/2}t_1; t, p), \\
 & \mathcal{D}_q^{+(n)}(u_0:t_0:t_1, t_2, t_3; t, p) \mathcal{D}_q^{(n)}(q^{-1/2}u_0, q^{1/2}t_0, q^{1/2}t_1; t, p) \\
 &= \mathcal{D}_q^{(n)}(u_0, t_0, t_1; t, p) \mathcal{D}_q^{+(n)}(q^{1/2}u_0:q^{1/2}t_0:q^{1/2}t_1, q^{-1/2}t_2, q^{-1/2}t_3; t, p).
 \end{aligned}$$

In contrast to [Corollary 7.10](#), it is unclear how to prove these commutation relations directly.

COROLLARY 8.6. *For any partition λ ,*

$$\begin{aligned}
 T_{\omega, p}^{(n)} \tilde{\mathcal{R}}_{0\lambda}^{(n)}(; p^{1/2}t_0:p^{1/2}t_1, p^{-1/2}t_2, p^{-1/2}t_3; p^{1/2}u_0, p^{-1/2}u_1; t; p, q) \\
 = \tilde{\mathcal{R}}_{0\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q).
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 \tilde{\mathcal{R}}_{0\lambda}^{(n)}(; p^{k_0}t_0:p^{k_1}t_1, p^{k_2}t_2, p^{k_3}t_3; p^{l_0}u_0, p^{l_1}u_1; t; p, q) \\
 = \tilde{\mathcal{R}}_{0\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q)
 \end{aligned}$$

for all choices of integers k_γ, l_γ such that $k_0 + k_1 + k_2 + k_3 + l_0 + l_1 = 0$.

Proof. The first claim follows from the fact that

$$\mathcal{D}_p^{(n)}(u_0, t_0, t_1; t, q) f = T_{\omega, p}^{(n)} f$$

for any p -abelian function f . Now, when $l_0 = 0$, the second claim follows from the definition of $\mathcal{R}_{0\lambda}^{(n)}$ and the fact that $\mathcal{D}_q^{+(n)}(u_0:t_0:t_1, t_2, t_3; t, p)$ is a p -abelian function of the u_γ and t_γ parameters. Iterating the first claim and using the fact that $\tilde{\mathcal{R}}_{0\lambda}^{(n)}$ is p -abelian we have an instance of the second claim with $l_0 = 1$, and thus the claim holds in general. \square

To see how the operators act when t_0 is not among the parameters of the operator, we need to determine how $\tilde{\mathcal{R}}^{(n)}$ changes when we switch t_0 and t_1 . This leaves the biorthogonality relation unchanged, and so multiplies the function by a constant; to determine that constant, it suffices to compute the following evaluation.

PROPOSITION 8.7. *For generic values of the parameters,*

$$\begin{aligned} \tilde{\mathcal{R}}_{\lambda}^{(n)}(\dots t^{n-i} t_1 \dots; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \\ = \frac{\mathcal{C}_{\lambda}^0(t^{n-1} t_1 t_2, t^{n-1} t_1 t_3, pqt^{n-1} t_0/u_0, t^{1-n}/t_1 u_1; t; p, q)}{\mathcal{C}_{\lambda}^0(t^{n-1} t_0 t_2, t^{n-1} t_0 t_3, pqt^{n-1} t_1/u_0, t^{1-n}/t_0 u_1; t; p, q)}. \end{aligned}$$

Proof. This follows by comparing the actions of $\mathcal{D}_q^{+(n)}(u_0:t_0:t_1, t_2, t_3; t, p)$ and $\mathcal{D}_q^{+(n)}(u_0:t_1:t_0, t_2, t_3; t, p)$. This gives a recurrence for the desired specialization, having the right-hand side as unique solution. \square

It will be convenient at this point to introduce ‘‘hatted’’ parameters. These are defined as follows. First, we have:

$$\hat{t}_0 = \sqrt{t_0 t_1 t_2 t_3 / pq} = \frac{t^{1-n}}{\sqrt{u_0 u_1}}.$$

The remaining parameters are then defined by giving invariants of the transformation. To be precise, we define $\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{u}_0,$ and \hat{u}_1 by insisting that

$$\hat{t}_0 \hat{t}_1 = t_0 t_1, \quad \hat{t}_0 \hat{t}_2 = t_0 t_2, \quad \hat{t}_0 \hat{t}_3 = t_0 t_3, \quad \frac{\hat{u}_0}{\hat{t}_0} = \frac{u_0}{t_0}, \quad \frac{\hat{u}_1}{\hat{t}_0} = \frac{u_1}{t_0}.$$

Note in particular that

$$t^{2n-2} \hat{t}_0 \hat{t}_1 \hat{t}_2 \hat{t}_3 \hat{u}_0 \hat{u}_1 = pq.$$

The action of the hat transformation on the t parameters is, of course, quite familiar from the theory of Koornwinder polynomials [14], [29] (aside from the factor of p required to preserve symmetry); the action on the u parameters is then essentially forced by the balancing condition. We furthermore define $z_i(\lambda; \hat{t}_0) := (p, q)^{\lambda_i} t^{n-i} \hat{t}_0$.

In the following formulas, the ratios of Γ functions that appear are sometimes ill-defined, in that some of the factors vanish. These should be interpreted by multiplying the argument of each Γ function by the same scale factor, then taking the limit as that scale factor approaches 1. Alternatively, it turns out in each case that the ratio can be formally expressed in terms of theta functions alone, and that upon doing so, the resulting formula is well-defined. Similar comments apply to ratios of θ functions. In particular, we note that

$$\begin{aligned} \prod_{1 \leq i \leq n} \frac{\theta(vz_i(\lambda; w)^{\pm 1}; p)}{\theta(vz_i(0, 0; w)^{\pm 1}; p)} &= \prod_{1 \leq i \leq n} \frac{\Gamma(qvz_i(\lambda; w)^{\pm 1}, vz_i(0, 0; w)^{\pm 1}; p, q)}{\Gamma(vz_i(\lambda; w)^{\pm 1}, qvz_i(0, 0; w)^{\pm 1}; p, q)} \\ &\propto \frac{\mathcal{C}_{\lambda}^0(t^{n-1} qvw, t^{n-1} pqw/v; p, q)}{\mathcal{C}_{\lambda}^0(t^{n-1} vw, t^{n-1} pw/v; p, q)}, \end{aligned}$$

where the constant of proportionality is independent of v .

COROLLARY 8.8. *We have the difference equations*

$$\begin{aligned} & \mathfrak{D}_q^{(n)}(u_0, t_0, t_1, t; p) \mathfrak{D}_q^{(n)}(q^{1/2}u_0, q^{-1/2}t_2, q^{-1/2}t_3; t, p) \\ & \quad \cdot \tilde{\mathfrak{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; qu_0, q^{-1}u_1; t; p, q) \\ & = \frac{\mathfrak{E}_\lambda^{\mathfrak{D}}(\hat{t}_1:\hat{t}_0; t; p; q)}{\mathfrak{E}_\lambda^{\mathfrak{D}}(\hat{u}_0:\hat{t}_0; t; p; q)} \tilde{\mathfrak{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q), \\ & \mathfrak{D}_q^{(n)}(u_0, t_2, t_3; t, p) \mathfrak{D}_q^{(n)}(q^{1/2}u_0, q^{-1/2}t_0, q^{-1/2}t_1; t, p) \\ & \quad \cdot \tilde{\mathfrak{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; qu_0, q^{-1}u_1; t; p, q) \\ & = \frac{\mathfrak{E}_\lambda^{\mathfrak{D}}(\hat{t}_1/q:\hat{t}_0; t; p; q)}{\mathfrak{E}_\lambda^{\mathfrak{D}}(\hat{u}_0:\hat{t}_0; t; p; q)} \tilde{\mathfrak{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q), \end{aligned}$$

where

$$\mathfrak{E}_\lambda^{\mathfrak{D}}(v; w; t; p; q) := \prod_{1 \leq i \leq n} \frac{\theta(vz_i(\lambda; w)^{\pm 1}; p)}{\theta(vz_i(0, 0; w)^{\pm 1}; p)}.$$

Similarly,

$$\begin{aligned} & \mathfrak{F}_t^{(n)}(t^{1/2}u_0:t^{-1/2}t_2, t^{-1/2}t_3; p, q) \mathfrak{F}_t^{(n)}(u_0:t_0, t_1; p, q) \\ & \quad \cdot \tilde{\mathfrak{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; p, q) \\ & = \frac{\mathfrak{E}_\lambda^{\mathfrak{F}}(\hat{t}_1:\hat{t}_0; t; p, q)}{\mathfrak{E}_\lambda^{\mathfrak{F}}(\hat{u}_0:\hat{t}_0; t; p, q)} \tilde{\mathfrak{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; tu_0, u_1/t; p, q), \\ & \mathfrak{F}_t^{(n)}(t^{1/2}u_0:t^{-1/2}t_0, t^{-1/2}t_1; p, q) \mathfrak{F}_t^{(n)}(u_0:t_2, t_3; p, q) \\ & \quad \cdot \tilde{\mathfrak{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; p, q) \\ & = \frac{\mathfrak{E}_\lambda^{\mathfrak{F}}(\hat{t}_1/t:\hat{t}_0; t; p, q)}{\mathfrak{E}_\lambda^{\mathfrak{F}}(\hat{u}_0:\hat{t}_0; t; p, q)} \tilde{\mathfrak{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; tu_0, u_1/t; p, q), \end{aligned}$$

where

$$\mathfrak{E}_\lambda^{\mathfrak{F}}(v; w; t; p, q) := \prod_{1 \leq i \leq n} \frac{\Gamma(vz_i(\lambda; w)^{\pm 1}, tvz_i(0, 0; w)^{\pm 1}; p, q)}{\Gamma(tvz_i(\lambda; w)^{\pm 1}, vz_i(0, 0; w)^{\pm 1}; p, q)}.$$

The $-$ and $+$ operators give similar equations:

THEOREM 8.9.

$$\begin{aligned} & \mathfrak{D}_q^{-(n)}(u_0; t, p) \mathfrak{D}_q^{+(n)}(q^{3/2}u_0: \frac{t_0}{q^{1/2}}: \frac{t_1}{q^{1/2}}, \frac{t_2}{q^{1/2}}, \frac{t_3}{q^{1/2}}; t, p) \\ & \quad \cdot \tilde{\mathfrak{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; qu_0, q^{-1}u_1; t; p, q) \\ & = C \frac{\mathfrak{E}_\lambda^{\mathfrak{D}}(\hat{t}_0/q:\hat{t}_0; p)}{\mathfrak{E}_\lambda^{\mathfrak{D}}(\hat{u}_0:\hat{t}_0; p)} \tilde{\mathfrak{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q), \end{aligned}$$

where

$$C = \prod_{1 \leq i \leq n} \frac{\theta(t^{n-i}u_0t_0, t^{n-i}u_0t_1, t^{n-i}u_0t_2, t^{n-i}u_0t_3, pt^{n-i}t_0/qu_0, pq t^{n-i}, t^{n-i}/qu_0u_1; p)}{\theta(pt^{2n-i-1}t_0u_1, t^{n-i}t_0t_1/q, t^{n-i}t_0t_2/q, t^{n-i}t_0t_3/q; p)}$$

Similarly,

$$\begin{aligned} & \mathcal{F}_t^{-(n+1)}(t^{1/2}u_0; \frac{t_0}{t^{1/2}}, \frac{t_1}{t^{1/2}}, \frac{t_2}{t^{1/2}}, \frac{t_3}{t^{1/2}}; p, q) \mathcal{F}_t^{+(n)}(u_0; p, q) \\ & \cdot \tilde{\mathcal{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; p, q) \\ & = \frac{\mathcal{E}_\lambda^{\mathcal{F}}(t/\hat{t}_0:\hat{t}_0; t; p, q)}{\mathcal{E}_\lambda^{\mathcal{F}}(\hat{u}_0:\hat{t}_0; t; p, q)} \tilde{\mathcal{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; tu_0, u_1/t; p, q). \end{aligned}$$

Proof. In each case, by adjointness, both sides satisfy biorthogonality, and must therefore be proportional. To determine the constant of proportionality, we can compare one of the corresponding equations from [Corollary 8.8](#). Indeed, the fact of proportionality shows that the relevant products of difference (or integral) operators differ in their action only by a diagonal transformation; as a result, we can compute the ratio of their constants of proportionality using *any* section of the filtration. In particular, it is straightforward to compute diagonal coefficients using the sections with which we proved triangularity in the first place, thus giving the desired result. \square

Remark. We thus find that for $v \in \{\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_0/q, \hat{t}_1/q, \hat{t}_2/q, \hat{t}_3/q\}$, we have a difference operator $\mathcal{D}(v)$ (of “order” 2) such that

$$\begin{aligned} \mathcal{D}(v)\tilde{\mathcal{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; qu_0, q^{-1}u_1; t; p, q) \\ = \frac{\mathcal{E}_\lambda^{\mathcal{D}}(v:\hat{t}_0; t; p; q)}{\mathcal{E}_\lambda^{\mathcal{D}}(\hat{u}_0:\hat{t}_0; t; p; q)} \tilde{\mathcal{R}}_\lambda^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q); \end{aligned}$$

moreover

$$\mathcal{D}_q^{+(n)}(u_0:t_0:t_1, t_2, t_3; t, p)\mathcal{D}^{-(n)}(q^{-1/2}u_0; t, p)$$

essentially gives us such an operator for $v = \hat{t}_0$. We conjecture that such an operator exists for all v ; since the “eigenvalue” is effectively just a BC_1 -symmetric theta function of degree n in v , this conjecture certainly holds for $n \leq 7$. Such a collection of difference operators, together with the various spaces of higher-degree difference operators obtained by composing them, would seem to give the analogue of the center of the affine Hecke algebra applicable to our biorthogonal functions. Indeed, in the Koornwinder limit, the conjecture certainly holds, and the resulting space of operators is precisely the subspace of the center of the affine Hecke algebra having degree at most 1.

In particular, this gives us a recurrence for the nonzero values of the inner product. Define

$$\Delta^{(n)}(z; t_0, t_1, t_2, t_3, t_4, t_5; t; p, q) = \prod_{1 \leq i < j \leq n} \frac{\Gamma(tz_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 5} \Gamma(trz_i^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 2}; p, q)};$$

in other words, this is simply the density with respect to which our functions are biorthogonal.

THEOREM 8.10. *For any partition pair λ of length at most n , and for generic values of the parameters,*

$$\begin{aligned} & \left\langle \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q), \right. \\ & \quad \left. \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_1, u_0; t; p, q) \right\rangle_{t_0, t_1, t_2, t_3, u_0, u_1; t, p, q} \\ &= \frac{\Delta^{(n)}(\dots z_i(0, 0; \hat{t}_0) \dots; \hat{t}_0, \hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{u}_0, \hat{u}_1; t; p, q)}{\Delta^{(n)}(\dots z_i(\lambda; \hat{t}_0) \dots; \hat{t}_0, \hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{u}_0, \hat{u}_1; t; p, q)} \\ &= \Delta_{\lambda} (t^{2n-2} \hat{t}_0^2 |t^n, t^{n-1} \hat{t}_0 \hat{t}_1, t^{n-1} \hat{t}_0 \hat{t}_2, t^{n-1} \hat{t}_0 \hat{t}_3, t^{n-1} \hat{t}_0 \hat{u}_0, t^{n-1} \hat{t}_0 \hat{u}_1; t; p, q)^{-1} \\ &= \Delta_{\lambda} \left(\frac{1}{u_0 u_1} |t^n, t^{n-1} t_0 t_1, t^{n-1} t_0 t_2, t^{n-1} t_0 t_3, \frac{t^{1-n}}{t_0 u_0}, \frac{t^{1-n}}{t_0 u_1}; t; p, q \right)^{-1}. \end{aligned}$$

This of course, is the direct analogue of the formula for the inner products of Koornwinder polynomials.

If $t_0 t_1 = p^{-l} q^{-m} t^{1-n}$, then the integral converts via residue calculus to a sum, and we thus obtain the following discrete biorthogonality property.

THEOREM 8.11. *For any partition pairs $\lambda, \kappa \subset (l, m)^n$, and for otherwise generic parameters satisfying $t_0 t_1 = p^{-l} q^{-m} t^{1-n}$,*

$$\begin{aligned} & \sum_{\mu \subset (l, m)^n} \tilde{\mathcal{R}}_{\lambda}^{(n)}(\dots z_i(\mu; t_0) \dots; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \\ & \quad \times \tilde{\mathcal{R}}_{\kappa}^{(n)}(\dots z_i(\mu; t_0) \dots; t_0:t_1, t_2, t_3; u_1, u_0; t; p, q) \\ & \quad \times \frac{\Delta^{(n)}(\dots z_i(\mu; t_0) \dots; t_0, t_1, t_2, t_3, u_0, u_1; t; p, q)}{\Delta^{(n)}(\dots z_i(0, 0; t_0) \dots; t_0, t_1, t_2, t_3, u_0, u_1; t; p, q)} = 0 \end{aligned}$$

unless $\lambda = \kappa$.

Remark. Note that when $t_0 t_1 = p^{-l} q^{-m} t^{1-n}$,

$$z_i(\mu; t_0) = z_{n+1-i}((l, m)^n - \mu; t_1)^{-1},$$

and thus summing over $z_i(\mu; t_1)$ gives the same result.

This result leads to a very important special case of the $\tilde{\mathcal{R}}$ functions.

COROLLARY 8.12. *If $t_1 u_1 = t^{1-n}$, then*

$$\tilde{\mathcal{R}}_{\lambda}^{(n)}(\dots z_i(\kappa; t_1) \dots; t_0; t_1, t_2, t_3; u_0, u_1; t; p, q) = 0$$

unless $\lambda \subseteq \kappa$. Moreover, in this case $\tilde{\mathcal{R}}_{\lambda}^{(n)}$ is independent of t_2 and t_3 , and up to scalar multiplication, is independent of t_0 .

Proof. First suppose that $t_0 t_1 = p^{-l} q^{-m} t^{1-n}$ for l, m such that $\lambda, \kappa \subset (l, m)^n$, and consider the discrete biorthogonality relation. We observe that for $f \in \mathcal{A}^{(n)}(u_1; p, q)$ such that

$$\prod_{1 \leq i \leq n} \theta(pqz_i^{\pm 1}/u_1; p, q)_{l,m} f(\dots z_i \dots)$$

is holomorphic, and for partitions $\lambda \subset (l, m)^n$,

$$\begin{aligned} \lim_{v \rightarrow t^{1-n}/u_1} \prod_{1 \leq i \leq n} \theta(pqz_i(\lambda; v)^{\pm 1}/u_1; p, q)_{l,m} f(\dots z_i(\lambda; v) \dots) \\ = \lim_{\substack{z_i \rightarrow (p,q)^{\kappa_i} t^{i-1} u_1 \\ i=1, \dots, n}} \prod_{1 \leq i \leq n} \theta(pqz_i^{\pm 1}/u_1; p, q)_{l,m} f(\dots z_i \dots). \end{aligned}$$

In other words, if $f \in \mathcal{A}_{\kappa}^{(n)}(u_1; t; p, q)$, then the inner product of our function with f can be expressed as a sum over partition pairs contained in κ , by the very definition of the filtration. The desired vanishing property follows immediately. Moreover, this orthogonality is independent of the specific values for t_2, t_3 , and thus changing t_2 or t_3 can at most multiply our function by a scalar; this scalar must then be 1 by the normalization formula.

We thus find that the result holds whenever t_0 is of the above form. Since the given quantity is a product of abelian functions of t_0 for any choice of λ, κ , the fact that it holds for t_0 of the form $p^{-l} q^{-m} t^{1-n}/t_1$ implies that it holds in general. Symmetry in t_0, t_2, t_3 then shows that the dependence on t_0 is only via the normalization. \square

With this in mind, we consider the following alternate normalization in this case.

Definition 5. The interpolation functions $\mathcal{R}_{\lambda}^{*(n)}(; t_0, u_0; t; p, q)$ are defined by

$$\begin{aligned} \mathcal{R}_{\lambda}^{*(n)}(; t_0, u_0; t; p, q) &= \Delta_{\lambda}^0(t^{n-1} t_0 / u_0 | t^{n-1} t_0 t_1, t_0 / t_1; t; p, q) \\ &\quad \times \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_1; t_0, t_2, t_3; u_0, t^{1-n} / t_0; t; p, q), \end{aligned}$$

where the right-hand side is independent of the choice of t_1, t_2, t_3 , as long as $t^{n-1}t_1t_2t_3u_0 = pq$. The *multivariate elliptic binomial coefficient* $\left[\begin{smallmatrix} \lambda \\ \kappa \end{smallmatrix} \right]_{[a,b];t;p,q}$ is defined by

$$\left[\begin{smallmatrix} \lambda \\ \kappa \end{smallmatrix} \right]_{[a,b];t;p,q} := \Delta_{\kappa} \left(\frac{a}{b} | t^n, 1/b; t; p, q \right) \times \mathcal{R}_{\kappa}^{*(n)} (\dots z_i (\lambda; t^{1-n} a^{1/2}) \dots; t^{1-n} a^{1/2}, b/a^{1/2}; t; p, q),$$

for $n \geq \ell(\lambda), \ell(\kappa)$.

Remark. An alternate definition uses the fact that

$$\begin{aligned} \mathcal{R}_{l^n + \lambda, m^n + \mu}^{*(n)} (; t_0, u_0; t; p, q) &= \frac{(pq/t_0u_0)^{-2l|\mu| - 2m|\lambda|} \prod_{1 \leq i \leq n} \theta(t_0x_i^{\pm 1}; p, q)_{l,m}}{\prod_{1 \leq i \leq n} \theta((pq/u_0)x_i^{\pm 1}; p, q)_{l,m}} \\ &\quad \times \mathcal{R}_{\lambda, \mu}^{*(n)} (; p^l q^m t_0, u_0/p^l q^m; t; p, q) \end{aligned}$$

(which follows by two applications of [equation \(9.1\)](#) below) together with the action of $\mathcal{G}^{+(n)}$ to obtain an integral representation generalizing that of [\[15\]](#).

We note in particular that

$$\begin{aligned} \mathcal{R}_{\kappa}^{*(n)} (\dots z_i \dots, a, ta, \dots, t^{m-1}a; a, b; t; p, q), \\ = \frac{\Delta_{\kappa} \left(\frac{t^{n-1}a}{b} | t^{n-m}, 1/b; t; p, q \right)}{\Delta_{\kappa} \left(\frac{t^{n-1}a}{b} | t^n, 1/b; t; p, q \right)} \mathcal{R}_{\kappa}^{*(n-m)} (\dots z_i \dots; t^m a, b; t; p, q), \end{aligned}$$

and thus the multivariate elliptic binomial coefficient is independent of n (as long as $\ell(\lambda), \ell(\kappa) \leq n$, that is).

The significance of these interpolation functions is that one can express connection coefficients for the biorthogonal functions in terms of multivariate elliptic binomial coefficients. (The proof requires a more thorough study of these binomial coefficients, and will thus be deferred to [\[21\]](#).)

THEOREM 8.13 ([\[21\]](#)). *If we define connection coefficients $c_{\lambda\mu}$ by*

$$\tilde{\mathcal{R}}_{\lambda}^{(n)} (; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) = \sum_{\mu \subset \lambda} c_{\lambda\mu} \tilde{\mathcal{R}}_{\mu}^{(n)} (; t_0:t_1v, t_2, t_3; u_0, u_1/v; t; p, q),$$

then

$$c_{\lambda\mu} = \frac{\Delta_{\lambda}^0 (1/u_0u_1|1/v, t^{n-1}t_2t_3, pqt^{n-1}t_0/u_0, t_1v/u_1; t; p, q)}{\Delta_{\mu}^0 (v/u_0u_1|v, t^{n-1}t_2t_3, pqt^{n-1}t_0/u_0, t_1v/u_1; t; p, q)} \left[\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix} \right]_{[1/u_0u_1, 1/v];t;p,q}.$$

If the biorthogonal function on the right is specialized to an interpolation function, we obtain the following generalization of Okounkov’s binomial formula for Koornwinder polynomials [15]:

COROLLARY 8.14.

$$\begin{aligned} \tilde{\mathcal{R}}_{\lambda}^{(n)}(; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \\ = \sum_{\kappa \subseteq \lambda} c_{\kappa} \mathcal{R}_{\kappa}^{*(n)}(\dots z_i(\lambda; \hat{t}_0) \dots; \hat{t}_0, \hat{u}_0; t; p, q) \mathcal{R}_{\kappa}^{*(n)}(; t_0, u_0; t; p, q), \end{aligned}$$

where

$$c_{\kappa} = \Delta_{\kappa}(t^{n-1}t_0/u_0|t^n, pq/u_0t_1, pq/u_0t_2, pq/u_0t_3; t; p, q).$$

Since c_{κ} above remains the same when the parameters are replaced by their hatted analogues, we obtain the following corollary, the analogue of the “evaluation symmetry” property of Koornwinder polynomials.

COROLLARY 8.15. *For any partition pairs λ, κ of length at most n , and for generic values of the parameters,*

$$\begin{aligned} \tilde{\mathcal{R}}_{\lambda}^{(n)}(\dots z_i(\kappa; t_0) \dots; t_0:t_1, t_2, t_3; u_0, u_1; t; p, q) \\ = \tilde{\mathcal{R}}_{\kappa}^{(n)}(\dots z_i(\lambda; \hat{t}_0) \dots; \hat{t}_0:\hat{t}_1, \hat{t}_2, \hat{t}_3; \hat{u}_0, \hat{u}_1; t; p, q). \end{aligned}$$

Before leaving the topic of biorthogonal functions, it remains to justify our assertions that these are a generalization of Koornwinder polynomials. The inner product clearly can be degenerated into the Koornwinder inner product; the difficulty is the filtration. Indeed, in order to degenerate the inner product, we must take $p \rightarrow 0, u_0 \rightarrow \{0, \infty\}$, at which point the definition of the filtration breaks. It turns out that the filtration actually does have a well-defined limit; however, we have been unable to find an argument for this other than as a corollary of the following result.

THEOREM 8.16. *Fix otherwise generic parameters t_0, t_1, t_2, t_3 . Then the limits*

$$\begin{aligned} \lim_{u_0 \rightarrow 0} \lim_{p \rightarrow 0} \tilde{\mathcal{R}}_{0\lambda}^{(n)}(\dots z_i \dots; t_0:t_1, t_2, t_3; u_0, \frac{pq}{t^{2n-2}u_0t_0t_1t_2t_3}; t; p, q), \\ \lim_{u_0 \rightarrow \infty} \lim_{p \rightarrow 0} \tilde{\mathcal{R}}_{0\lambda}^{(n)}(\dots z_i \dots; t_0:t_1, t_2, t_3; u_0, \frac{pq}{t^{2n-2}u_0t_0t_1t_2t_3}; t; p, q) \end{aligned}$$

agree, and give a family of BC_n -symmetric Laurent polynomials orthogonal with respect to the Koornwinder inner product. Moreover, these polynomials are diagonal with respect to dominance of monomials, and thus are precisely the Koornwinder polynomials (normalized to have principal specialization 1).

Proof. The key observation is that, although the definition of the *filtration* blows up in the limit, the raising difference and integral operators have perfectly fine limits. Consequently, the above limits are indeed well-defined; as the choice $u_0 \rightarrow 0$ or $u_0 \rightarrow \infty$ has no effect on the limiting operators, it can have no effect on the limiting functions. Since the space of BC_n -symmetric p -theta functions of degree m tends in the limit $p \rightarrow 0$ to the space of BC_n -symmetric Laurent polynomials of degree at most m in each variable, our functions become rational functions in that limit. Taking the limit $u_0 \rightarrow 0, \infty$, causes the poles of the rational functions to move to 0 and ∞ , thus giving Laurent polynomials. Finally, we observe that because the above limits agree, biorthogonality becomes orthogonality in the limit. (Recall that $\mathcal{R}_{0\lambda}^{(m)}$ is p -abelian in its parameters; so the factor of p in u_1 can be moved around arbitrarily before taking the limit.) We have thus proved the first claim.

To see that these agree with Koornwinder polynomials, we observe that the operator $\mathcal{D}_q^{(n)}(u_0, t_0, t_1; t)$ also has a well-defined limit; standard arguments ([20, Th. 3.2]) show that the limit is triangular with respect to dominance of monomials. It thus follows from Theorem 8.5 that the limiting polynomials are eigenfunctions of a pair of triangular difference operators, and thus must themselves be triangular. The normalization then follows from Proposition 8.1. \square

Remark. In order to determine the constant of proportionality, i.e., determine the leading coefficient of the limiting polynomial, we simply determine how the raising operators affect the leading coefficient. For the difference operator, this is straightforward; for the integral operator, we can appeal to Theorem 7.1 and, by using the fact that

$$\sum_{\lambda \subset m^n} (-1)^{mn-|\lambda|} m_\lambda(y_1, y_2, \dots, y_n) e_{m^n-\lambda}(z_1, z_2, \dots, z_m, 1/z_1, 1/z_2, \dots, 1/z_m) = \prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (y_i + 1/y_i - z_j - 1/z_j)$$

(where m_λ is a BC_n -symmetric monomial), reduce to the difference operator case. The result, of course, is simply Macdonald’s “evaluation” conjecture; Theorem 8.10 then gives the nonzero values of the inner product. (For more details, see [22].) The remaining “symmetry” conjecture does not follow from the methods given above, however (although there are at least two different arguments for deducing it from evaluation: [5], [15]). The argument we will give in [21] does descend to the Koornwinder case; indeed, the result is precisely the proof given in [20].

It follows from [20, Th. 7.25] that the filtration has the following limit.

COROLLARY 8.17. *Choose an integer $n \geq 0$, and a partition λ of at most n parts. Then the limits $u_0 \rightarrow 0$ or $u_0 \rightarrow \infty$ of the space $\mathcal{A}_{0\lambda}^{(n)}(u_0; t; 0, q)$ agree, and are given by the span*

$$\langle P_\mu(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}; q, t) \rangle_{\mu \subset \lambda},$$

where P_μ is an ordinary Macdonald polynomial.

Remark. It would be nice to have a direct proof of this corollary, or the corresponding result for a refined partial order; in particular, for the dominance partial order, the limiting filtration should correspond to dominance of monomials.

9. Type II transformations

The connection coefficient formula for our biorthogonal functions, [Theorem 8.13](#), has a number of nice consequences for the multivariate elliptic binomial coefficients. For instance, by taking $v = 1$, we obtain the limiting case

$$\lim_{b \rightarrow 1} \frac{\Delta_\lambda^0(a|b; t; p, q)}{\Delta_\kappa^0(a/b|1/b; t; p, q)} \left[\begin{matrix} \lambda \\ \kappa \end{matrix} \right]_{[a,b];t;p,q} = \delta_{\lambda\kappa}.$$

Also, if we perform the change of basis corresponding to $t_1 \rightarrow t_1 v$, then the change of basis corresponding to $t_1 v \rightarrow t_1 v w$, the result should be the same as if we directly changed $t_1 \rightarrow t_1 v w$. We thus obtain the following sum:

THEOREM 9.1 ([\[21\]](#)). *For otherwise generic parameters satisfying $bcde = pqa$,*

$$\begin{aligned} \left[\begin{matrix} \lambda \\ \kappa \end{matrix} \right]_{[a,c];t;p,q} &= \frac{\Delta_\kappa^0(a/c|1/c, bd, be, pqa/b; t; p, q)}{\Delta_\lambda^0(a|c, bd, be, pqa/b; t; p, q)} \\ &\times \sum_{\kappa \subset \mu \subset \lambda} \Delta_\mu^0(a/b|c/b, pqa, d, e; t; p, q) \left[\begin{matrix} \lambda \\ \mu \end{matrix} \right]_{[a,b];t;p,q} \left[\begin{matrix} \mu \\ \kappa \end{matrix} \right]_{[a/b,c/b];t;p,q}. \end{aligned}$$

In particular,

$$\sum_{\kappa \subset \mu \subset \lambda} \left[\begin{matrix} \lambda \\ \mu \end{matrix} \right]_{[a,b];t;p,q} \left[\begin{matrix} \mu \\ \kappa \end{matrix} \right]_{[a/b,1/b];t;p,q} = \delta_{\lambda\kappa}.$$

Remark. Although this identity, along with the other sums mentioned in this section, does indeed follow from [Theorem 8.13](#), we should mention that the argument in [\[21\]](#) proceeds in the opposite direction, using these identities (and others) to prove the binomial formula, and from this, [Theorem 8.13](#). On the other hand, the above argument provides a more straightforward *interpretation* of the identity than that given in [\[21\]](#).

If we take $\lambda = (l, m)^n, \kappa = 0$ above, the above identity turns out to be a product of two general instances of Warnaar’s Jackson-type summation (conjectured in [37], and proved by Rosengren [25]). Warnaar’s Schlosser-type summation is also a special case; see [21].

Our reason for discussing this here is that there is an integral analogue of the above sum, generalizing [Theorem 6.1](#).

THEOREM 9.2. *For otherwise generic parameters satisfying $|p|, |q| < 1$ and $t^{2n-2}t_0t_1t_2t_3u_0u_1 = pq$,*

$$\begin{aligned} & \langle \mathcal{R}_\lambda^{*(n)}(; t_0, u_0; t; p, q), \mathcal{R}_\kappa^{*(n)}(; t_1, u_1; t; p, q) \rangle_{t_0, t_1, t_2, t_3, u_0, u_1; t; p, q} \\ &= \Delta_\lambda^0(t^{n-1}t_0/u_0 | t^{n-1}t_0t_2, t^{n-1}t_0t_3; t; p, q) \\ & \times \Delta_\kappa^0(t^{n-1}t_1/u_1 | t^{n-1}t_1t_2, t^{n-1}t_1t_3, t^{n-1}t_1t_0, t^{n-1}t_1u_0; t; p, q) \\ & \times \mathcal{R}_\lambda^{*(n)}\left(\dots z_i(\kappa; t_1/\sqrt{t^{n-1}t_1u_1}) \dots; t_0\sqrt{t^{n-1}t_1u_1}, u_0\sqrt{t^{n-1}t_1u_1}; t; p, q\right). \end{aligned}$$

Proof. Using the connection coefficient identity, we may express both interpolation functions as linear combinations of biorthogonal functions. Substituting in the known values for the inner products of the biorthogonal functions, we thus obtain a sum over partition pairs $\mu \subset \lambda, \kappa$. That this sum gives the desired right-hand side is itself a special case of the connection coefficient identity.

Alternatively, we can mimic the proof of [Theorem 6.1](#), using the fact that $\mathcal{D}_q^{(n)}(u_0, t_0, t_3; t, p)$ acts nicely on $\mathcal{R}_\lambda^{*(n)}(; t_0, u_0; t; p, q)$. If we define

$$\begin{aligned} & F_{\lambda\kappa}^{(n)}(t_0, t_1, t_2, t_3, u_0, u_1; t; p, q) \\ & := \frac{\langle \mathcal{R}_\lambda^{*(n)}(; t_0, u_0; t; p, q), \mathcal{R}_\kappa^{*(n)}(; t_1, u_1; t; p, q) \rangle_{t_0, t_1, t_2, t_3, u_0, u_1; t; p, q}}{\Delta_\lambda^0(t^{n-1}t_0/u_0 | t^{n-1}t_0t_2, t^{n-1}t_0t_3; t; p, q) \Delta_\kappa^0(t^{n-1}t_1/u_1 | t^{n-1}t_1t_2, t^{n-1}t_1t_3; t; p, q)} \end{aligned}$$

then adjointness gives

$$\begin{aligned} & F_{\lambda\kappa}^{(n)}(t_0, t_1, t_2, t_3, u_0, u_1; t; p, q) \\ &= F_{\lambda\kappa}^{(n)}(q^{1/2}t_0, q^{-1/2}t_1, q^{1/2}t_2, q^{-1/2}t_3, q^{1/2}u_0, q^{-1/2}u_1; t; p, q) \\ &= F_{\lambda\kappa}^{(n)}(p^{1/2}t_0, p^{-1/2}t_1, p^{1/2}t_2, p^{-1/2}t_3, p^{1/2}u_0, p^{-1/2}u_1; t; p, q). \end{aligned}$$

Thus, the usual density argument shows that

$$F_{\lambda\kappa}^{(n)}(t_0, t_1, t_2, t_3, u_0, u_1; t; p, q) = F_{\lambda\kappa}^{(n)}(wt_0, t_1/w, t_2w, t_3/w, u_0w, u_1/w; t; p, q)$$

for any $w \in \mathbb{C}^*$. Taking the limit $w \rightarrow \sqrt{t^{n-1}t_1u_1}$ and expanding via residue calculus, we obtain a sum over partition pairs contained in κ_1^n , in which only the

term associated to κ survives. (Recall that the contour must be deformed around the poles of $\mathcal{R}_\kappa^{*(n)}$.) We thus find

$$F_{\lambda\kappa}^{(n)}(t_0, t_1, t_2, t_3, u_0, u_1; t; p, q) \propto \mathcal{R}_\lambda^{*(n)}(\dots z_i(\kappa; t_1/\sqrt{t^{n-1}t_1u_1}) \dots; t_0\sqrt{t^{n-1}t_1u_1}, u_0\sqrt{t^{n-1}t_1u_1}; t; p, q),$$

where the constant of proportionality is independent of λ . This constant can be resolved by taking the limit $w \rightarrow \sqrt{t^{n-1}t_0u_0}$ in the case $\lambda = 0$. \square

Remark 1. The left-hand side above is invariant under exchanging (λ, t_0, u_0) and (κ, t_1, u_1) . That the right-hand side is invariant is a special case of evaluation symmetry (Corollary 8.15). We can also use that same special case of evaluation symmetry to see that this generalizes Theorem 9.1. Indeed, if we specialize so that $t_0t_1 = p^{-1}q^{-m}t^{1-n}$ with $\lambda, \kappa \subset (l, m)^n$, then the above left-hand side becomes a sum over $\mu \subset (l, m)^n$. Using evaluation symmetry, the factor

$$\mathcal{R}_\kappa^{*(n)}(\dots z_i(\mu; t_0) \dots; t_1, u_1) = \mathcal{R}_\kappa^{*(n)}(\dots z_i((l, m)^n - \mu; t_1) \dots; t_1, u_1)$$

can be rewritten in terms of

$$\mathcal{R}_\mu^{*(n)}(\dots z_i((l, m)^n - \kappa; x) \dots; x, y)$$

for suitable x and y . Replacing κ by $(l, m)^n - \kappa$ gives Theorem 9.1.

Similarly, replacing κ by $(l, m)^n - \kappa$ in the general version and comparing the results, we find

$$(9.1) \quad \mathcal{R}_{l^n - \lambda, m^n - \mu}^{*(n)}(; t_1, u_1; t; p, q) = \frac{(pq/t_1u_1)^{2l|\mu|+2m|\lambda|} \prod_{1 \leq i \leq n} \theta(t_1x_i^{\pm 1}; p, q)_{l,m} \mathcal{R}_{\lambda, \mu}^{*(n)}(; u_1/p^l q^m, p^l q^m t_1; t; p, q)}{\prod_{1 \leq i \leq n} \theta((pq/u_1)x_i^{\pm 1}; p, q)_{l,m}}$$

as both sides have the same inner product with $\mathcal{R}_\lambda^{*(n)}(; t_0, u_0; t; p, q)$.

Remark 2. Note that the second proof of the theorem did not use the connection coefficient identity, and is thus independent of [21].

If we take $\kappa = 0$ above, we obtain the following identity, generalizing Kadell’s lemma (see, for instance Corollary 5.14 of [20]).

COROLLARY 9.3. *For otherwise generic parameters satisfying $|p|, |q| < 1$ and $t^{2n-2}t_0t_1t_2t_3t_4t_5 = pq$,*

$$\langle \mathcal{R}_\lambda^{*(n)}(; t_0, t_1; t; p, q) \rangle_{t_0, t_1, t_2, t_3, t_4, t_5; t; p, q} = \Delta_\lambda^0(t^{n-1}t_0/t_1 | t^{n-1}t_0t_2, t^{n-1}t_0t_3, t^{n-1}t_0t_4, t^{n-1}t_0t_5; t; p, q).$$

The connection coefficient argument gives the following identity as well.

THEOREM 9.4. *For otherwise generic parameters satisfying $|p|, |q| < 1$ and $t^{2n-2}t_0t_1t_2t_3u_0u_1 = pq$,*

$$\begin{aligned} & \langle \mathcal{R}_\lambda^{*(n)}(; t_0, u_0; t; p, q), \mathcal{R}_\kappa^{*(n)}(; t_0, u_1; t; p, q) \rangle_{t_0, t_1, t_2, t_3, u_0, u_1; t; p, q} \\ &= \Delta_\lambda^0 \left(\frac{t^{n-1}t_0}{u_0} \middle| t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, t^{n-1}t_0u_1; t; p, q \right) \\ & \times \Delta_\kappa^0 \left(\frac{t^{n-1}t_0}{u_1} \middle| t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, t^{n-1}t_0u_0; t; p, q \right) \\ & \times \tilde{\mathcal{R}}_\lambda^{(n)}(\dots z_i(\kappa, t'_0) \dots; t'_0:t'_1, t'_2, t'_3; u'_0, u'_1; t; p, q), \end{aligned}$$

where the primed parameters are determined by

$$\begin{aligned} t^{n-1}t'_0t'_1 &= t^{n-1}t_0t_1, & t^{n-1}t'_0t'_2 &= t^{n-1}t_0t_2, & t^{n-1}t'_0t'_3 &= t^{n-1}t_0t_3, \\ t^{n-1}t'_0u'_0 &= t^{n-1}t_0u_0, & t^{n-1}t'_0u'_1 &= \frac{1}{t^{n-1}t_0u_1}, & t_0'^2 &= \frac{t_0}{t^{n-1}u_1}. \end{aligned}$$

Remark. The above transformation of the parameters is involutive, and conjugates the exchange $u_0 \leftrightarrow u_1$ to the “hat” transformation.

A further application of connection coefficients gives the following result, containing both Theorems 9.2 and 9.4 as special cases.

THEOREM 9.5. *For otherwise generic parameters satisfying $|p|, |q| < 1$ and $t^{2n-2}t_0t_1t_2t_3u_0u_1 = pq$,*

$$\begin{aligned} & \langle \mathcal{R}_\lambda^{*(n)}(; t_0v, u_0; t; p, q), \mathcal{R}_\kappa^{*(n)}(; t_0, u_1; t; p, q) \rangle_{t_0, t_1, t_2, t_3, u_0, u_1; t; p, q} \\ &= \Delta_\lambda^0 (t^{n-1}t_0v/u_0 \middle| t^{n-1}vt_0t_1, t^{n-1}vt_0t_2, t^{n-1}vt_0t_3, t^{n-1}t_0u_1; t; p, q) \\ & \times \Delta_\kappa^0 (t^{n-1}t_0/u_1 \middle| t^{n-1}t_0t_1, t^{n-1}t_0t_2, t^{n-1}t_0t_3, t^{n-1}t_0u_0; t; p, q) \\ & \times \tilde{\mathcal{R}}_\lambda^{(n)}(\dots z_i(\kappa, t'_0) \dots; t'_0v:t'_1, t'_2, t'_3; u'_0, u'_1/v; t; p, q), \end{aligned}$$

with primed parameters as above.

Now, Theorem 9.1 is sufficiently general that the univariate argument for deriving Bailey-type transformations from Jackson-type summations applies, giving the following identity.

THEOREM 9.6 ([21]). *The sum*

$$\begin{aligned} & \frac{\Delta_\lambda^0(a|b, apq/bf; t; p, q)}{\Delta_\kappa^0(a/c|b/c, apq/bd; t; p, q)} \\ & \times \sum_{\kappa \subset \mu \subset \lambda} \frac{\Delta_\mu^0(a/b|c/b, f, g; t; p, q)}{\Delta_\mu^0(a/b|1/b, d, e; t; p, q)} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{[a, b]; t; p, q} \begin{bmatrix} \mu \\ \kappa \end{bmatrix}_{[a/b, c/b]; t; p, q} \end{aligned}$$

is symmetric in b and b' , where $bb'de = capq$, $bb'fg = apq$.

Remark 1. Again, taking $\lambda = (l, m)^n$, $\kappa = 0$, gives an identity conjectured by Warnaar, in this case his conjectured multivariate Frenkel-Turaev transformation [37].

Remark 2. This identity can also be obtained by comparing various ways of computing connection coefficients for biorthogonal functions in which t_0, t_3, u_0 are left fixed, but t_1, t_2, u_1 change.

It turns out that this identity also has an integral analogue. For each integer $n \geq 0$, and partition pairs λ, μ of length at most n , we define a (meromorphic) function

$$\begin{aligned}
 II_{\lambda, \mu}^{(n)}(t_0, t_1:t_2, t_3:t_4, t_5, t_6, t_7; t; p, q) &:= \frac{(p; p)^n (q; q)^n \Gamma(t; p, q)^n}{2^n n!} \\
 &\times \int_{C^n} \mathcal{R}_\lambda^{*(n)}(\dots x_i \dots; t_0, t_1; t; p, q) \mathcal{R}_\mu^{*(n)}(\dots x_i \dots; t_2, t_3; t; p, q) \\
 &\times \prod_{1 \leq i < j \leq n} \frac{\Gamma(tx_i^{\pm 1} x_j^{\pm 1}; p, q)}{\Gamma(x_i^{\pm 1} x_j^{\pm 1}; p, q)} \prod_{1 \leq i \leq n} \frac{\prod_{0 \leq r \leq 7} \Gamma(t_r x_i^{\pm 1}; p, q)}{\Gamma(x_i^{\pm 2}; p, q)} \frac{dx_i}{2\pi \sqrt{-1} x_i},
 \end{aligned}$$

on the domain $t^{2n-2} t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7 = p^2 q^2$, where the contour C^n is constrained in the usual way by the poles of the integrand.

THEOREM 9.7. *If $t^{2n-2} t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7 = p^2 q^2$ for some nonnegative integer n , then*

$$\begin{aligned}
 &II_{\lambda, \mu}^{(n)}(t_0, t_1:t_2, t_3:t_4, t_5, t_6, t_7; t; p, q) \\
 &= \Delta_\lambda^0(t^{n-1} t_0/t_1 | t^{n-1} t_0 t_4, t^{n-1} t_0 t_5; t; p, q) \prod_{1 \leq j \leq n} \prod_{\substack{r, s \in \{0, 1, 4, 5\} \\ r < s}} \Gamma(t^{n-j} t_r t_s; p, q) \\
 &\times \Delta_\mu^0(t^{n-1} t_2/t_3 | t^{n-1} t_2 t_6, t^{n-1} t_2 t_7; t; p, q) \prod_{1 \leq j \leq n} \prod_{\substack{r, s \in \{2, 3, 6, 7\} \\ r < s}} \Gamma(t^{n-j} t_r t_s; p, q) \\
 &\times II_{\lambda, \mu}^{(n)}(t_0/u, t_1/u:ut_2, ut_3:t_4/u, t_5/u, ut_6, ut_7; t; p, q),
 \end{aligned}$$

where u is chosen so that

$$u^2 = \sqrt{\frac{t_0 t_1 t_4 t_5}{t_2 t_3 t_6 t_7}} = \frac{pqt^{1-n}}{t_2 t_3 t_6 t_7} = \frac{t_0 t_1 t_4 t_5}{pqt^{1-n}}.$$

Proof. If, following the second proof of [Theorem 9.2](#), we attempt to mimic the difference operator proof of [Theorem 6.1](#), we immediately encounter the difficulty that we no longer have adjointness between two instances of $\mathcal{D}_q^{(n)}$, but rather between an instance of $\mathcal{D}_q^{(n)}$ and an instance of $\mathcal{D}_q^{+(n)}$. The one exception is when

$t^{n-1}t_0t_1t_4t_5 = p$, in which case

$$\mathfrak{D}_q^{(n)}(t_0, t_1, t_4; t, p) \quad \text{and} \quad \mathfrak{D}_q^{(n)}(\sqrt{q}t_2, \sqrt{q}t_3, \sqrt{q}t_6; t, p)$$

are adjoint; the resulting transformation is precisely the case $u^2 = q$ of the theorem.

To extend this argument, we will thus need to extend the difference operators. With this in mind, we define a difference operator $D_{l,m}^{(n)}(u_0, u_1, u_2; t; p, q)$ for $l, m \geq 0$ as follows.

$$D_{0,1}^{(n)}(u_0, u_1, u_2; t; p, q) = D_q^{(n)}(u_0, u_1, u_2, p/t^{n-1}u_0u_1u_2; t, p),$$

$$D_{1,0}^{(n)}(u_0, u_1, u_2; t; p, q) = D_p^{(n)}(u_0, u_1, u_2, q/t^{n-1}u_0u_1u_2; t, q),$$

$$\begin{aligned} D_{l+l',m+m'}^{(n)}(u_0, u_1, u_2; t; p, q) \\ = D_{l,m}^{(n)}(u_0, u_1, u_2; t; p, q)D_{l',m'}^{(n)}(S_{l,m}^{1/2}u_0, S_{l,m}^{1/2}u_1, S_{l,m}^{1/2}u_2; t; p, q), \end{aligned}$$

where $S_{l,m} = (p, q)^{l,m}$. Since $D_{l,m}^{(n)}$ is a composition of p - and q -difference operators, it itself is a difference operator; that it is well-defined follows from the fact that the two ways of computing $D_{1,1}^{(n)}$ agree.

LEMMA 9.8. *Let u_0, u_1, u_2, u_3 be such that $t^{n-1}u_0u_1u_2u_3 = pq/S_{l,m}$. Then*

$$\begin{aligned} D_{l,m}^{(n)}(u_0, u_1, u_2; t; p, q)\mathfrak{R}_\lambda^{*(n)}(; S_{l,m}^{1/2}u_0, S_{l,m}^{1/2}u_1; t; p, q) \\ = \frac{\Delta_\lambda^0(t^{n-1}u_0/u_1|pq/u_1u_2, pq/u_1u_3; t; p, q)}{\prod_{1 \leq i \leq n} \prod_{0 \leq r < s \leq 3} \Gamma(t^{n-i}u_r u_s; p, q)} \mathfrak{R}_\lambda^{*(n)}(; u_0, u_1; t; p, q). \end{aligned}$$

In particular,

$$D_{l,m}^{(n)}(u_0, u_1, u_2; t; p, q) = D_{l,m}^{(n)}(u_0, u_1, u_3; t; p, q).$$

Proof. The first claim holds when $(l, m) \in \{(0, 1), (1, 0)\}$; an easy induction gives it in general.

Since $D_{l,m}^{(n)}(u_0, u_1, u_2; t; p, q)$ is a difference operator, it is uniquely determined by this action; since the given formula is symmetric between u_2 and u_3 , the operator itself is symmetric. \square

LEMMA 9.9. *The different instances of $D_{l,m}^{(n)}$ are related by*

$$\begin{aligned} D_{l,m}^{(n)}(u_0, u_1, u_2; t; p, q) \\ = \prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \frac{\Gamma(u'_r x_i^{\pm 1}; p, q)}{\Gamma(u_r x_i^{\pm 1}; p, q)} D_{l,m}^{(n)}(u'_0, u'_1, u'_2; t; p, q) \prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \frac{\Gamma(S_{l,m}^{1/2}u_r x_i^{\pm 1}; p, q)}{\Gamma(S_{l,m}^{1/2}u'_r x_i^{\pm 1}; p, q)}. \end{aligned}$$

Proof. If $u'_3 = u_3$, the result follows by a simple induction; the general case then follows by combination of that case with the symmetry between u_0, u_1, u_2 and u_3 . □

In particular, we can define the operator

$$D_{l,m}^{(n)}(t; p, q) := \prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \Gamma(u_r x_i^{\pm 1}; p, q) D_{l,m}^{(n)}(u_0, u_1, u_2; t; p, q) \prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \frac{1}{\Gamma(S_{l,m}^{1/2} u_r x_i^{\pm 1}; p, q)},$$

which is independent of u_0, u_1, u_2 .

This operator is self-adjoint with respect to the cross-terms in the II density; that is, with respect to

$$\prod_{1 \leq i < j \leq n} \frac{\Gamma(tx_i^{\pm 1} x_j^{\pm 1}; p, q)}{\Gamma(x_i^{\pm 1} x_j^{\pm 1}; p, q)} \prod_{1 \leq i \leq n} \frac{dx_i}{2\pi \sqrt{-1} x_i}.$$

This follows from the fact that

$$\begin{aligned} \int f D_{l,m}^{(n)}(t_0, t_1, t_2; t; p, q) g \Delta^{(n)}(t_0, \dots, t_5; t; p, q) \\ = \int g D_{l,m}^{(n)}(t'_3, t'_4, t'_5; t; p, q) f \Delta^{(n)}(t'_0, \dots, t'_5; t; p, q), \end{aligned}$$

where

$$(t'_0, t'_1, t'_2, t'_3, t'_4, t'_5) = (S_{l,m}^{1/2} t_0, S_{l,m}^{1/2} t_1, S_{l,m}^{1/2} t_2, S_{l,m}^{-1/2} t_3, S_{l,m}^{-1/2} t_4, S_{l,m}^{-1/2} t_5),$$

which in turn follows by induction from the cases $(l, m) \in \{(0, 1), (1, 0)\}$.

We also have a sort of commutation relation satisfied by $D_{l,m}^{(n)}(t; p, q)$.

LEMMA 9.10. *If l, m, l', m' are nonnegative integers and*

$$u_0 u_1 u_2 u_3 = S_{l,m} S_{l',m'} p^2 q^2,$$

then we have the following identity of difference operators.

$$\begin{aligned} D_{l,m}^{(n)}(t; p, q) \prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \Gamma(u_r x_i^{\pm 1}; p, q) D_{l',m'}^{(n)}(t; p, q) \prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \frac{1}{\Gamma(S_{l',m'}^{-1/2} u_r x_i^{\pm 1}; p, q)} \\ = \prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \Gamma(S_{l,m}^{-1/2} u_r x_i^{\pm 1}; p, q) D_{l',m'}^{(n)}(t; p, q) \\ \times \prod_{\substack{1 \leq i \leq n \\ 0 \leq r \leq 3}} \frac{1}{\Gamma((S_{l,m} S_{l',m'})^{-1/2} u_r x_i^{\pm 1}; p, q)} D_{l,m}^{(n)}(t; p, q). \end{aligned}$$

Proof. By comparing actions on

$$\mathfrak{R}_\lambda^{*(n)}((S_{l,m}S_{l',m'})^{1/2}u_0, (S_{l,m}S_{l',m'})^{1/2}t_0; t; p, q),$$

we find that

$$\begin{aligned} D_{l,m}^{(n)}(u_0, t_0, t_1; t; p, q)D_{l',m'}^{(n)}(S_{l,m}^{1/2}u_0, S_{l,m}^{1/2}t_0, S_{l,m}^{-1/2}t_2; t; p, q) \\ = D_{l',m'}^{(n)}(u_0, t_0, t_2; t; p, q)D_{l,m}^{(n)}(S_{l',m'}^{1/2}u_0, S_{l',m'}^{1/2}t_0, S_{l',m'}^{-1/2}t_1; t; p, q). \end{aligned}$$

Expressing this in terms of $D_{l,m}^{(n)}(t; p, q)$ and simplifying give the desired result. □

Now, consider an integral of the form

$$\begin{aligned} \int [D_{l,m}^{(n)}(t_0, t_1, t_4; t; p, q)f] \\ \times [D_{l',m'}^{(n)}(t_2, t_3, t_6; t; p, q)g]\Delta^{(n)}(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7; t; p, q), \end{aligned}$$

where $f \in \mathcal{A}^{(n)}(S_{l,m}^{1/2}t_1; p, q)$, $g \in \mathcal{A}^{(n)}(S_{l',m'}^{1/2}t_3; p, q)$, and the parameters satisfy the relations

$$t^{n-1}t_0t_1t_4t_5 = pq \frac{S_{l',m'}}{S_{l,m}}, \quad t^{n-1}t_2t_3t_6t_7 = pq \frac{S_{l,m}}{S_{l',m'}}.$$

If we rewrite this integral in terms of $D^{(n)}(t; p, q)$ and $\Delta^{(n)}(t; p, q)$ and apply self-adjointness of $D_{l,m}^{(n)}(t; p, q)$, the resulting composition of difference operators can be transformed by the commutation relation. The result is of the same form, and we thus obtain the following identity.

$$\begin{aligned} \int (D_{l,m}^{(n)}(t_0, t_1, t_4; t; p, q)f) \\ \times (D_{l',m'}^{(n)}(t_2, t_3, t_6; t; p, q)g)\Delta^{(n)}(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7; t; p, q) \\ = \int (D_{l',m'}^{(n)}(t'_0, t'_1, t'_4; t; p, q)f) \\ \times (D_{l,m}^{(n)}(t'_2, t'_3, t'_6; t; p, q)g)\Delta^{(n)}(t'_0, t'_1, t'_2, t'_3, t'_4, t'_5, t'_6, t'_7; t; p, q), \end{aligned}$$

where

$$t'_r = \begin{cases} (S_{l,m}/S_{l',m'})^{1/2}t_r & r \in \{0, 1, 4, 5\} \\ (S_{l',m'}/S_{l,m})^{1/2}t_r & r \in \{2, 3, 6, 7\}. \end{cases}$$

If we set

$$\begin{aligned} f &= \mathfrak{R}_\lambda^{*(n)}(; S_{l,m}^{1/2}t_0, S_{l,m}^{1/2}t_1; t; p, q), \\ g &= \mathfrak{R}_\mu^{*(n)}(; S_{l',m'}^{1/2}t_2, S_{l',m'}^{1/2}t_3; t; p, q), \end{aligned}$$

we obtain the special case $u^2 = S_{l',m'}/S_{l,m}$ of the theorem. Since this set is dense, the theorem holds in general. \square

Remark 1. Note that the above proof did not use [Theorem 8.13](#), and is thus independent of the results of [\[21\]](#). In fact, one can use this result to prove [Theorem 8.13](#), as follows. Connection coefficients for interpolation functions can be obtained from the special case $t_2t_7 = pq$ (essentially [Theorem 9.5](#)), by comparing the result to that of [Theorem 9.2](#). One can then reverse the first proof of [Theorem 9.2](#) to show that the functions given by the binomial formula are indeed biorthogonal; [Theorem 8.13](#) then follows via [Theorem 9.1](#).

Remark 2. In the special case $t^{n-1}t_0t_2 = 1/S_{l,m}$, we recover [Theorem 9.6](#). Also, the univariate case $n = 1$ is precisely the case $n = m = 1$ of the A_n transformation.

Remark 3. Similarly, using our integral operators, one can give a direct proof for the case $u^2 = t$, which presumably only extends to an argument valid for $u^2 \in t^{\mathbb{Z}}$. This is, however, probably the simplest proof in the univariate case (since then the integral is independent of t).

We can simplify this transformation somewhat by adding an appropriate normalization factor. Define a meromorphic function

$$\begin{aligned} \tilde{I}_{\lambda,\mu}^{(n)}(t_0, t_1:t_2, t_3:t_4, t_5, t_6, t_7; t; p, q) \\ := Z_{\lambda\mu} II_{\lambda,\mu}^{(n)}(t^{1/2}t_0, t^{1/2}t_1:t^{1/2}t_2, t^{1/2}t_3:t^{1/2}t_4, t^{1/2}t_5, t^{1/2}t_6, t^{1/2}t_7; t; p, q), \end{aligned}$$

where

$$\begin{aligned} Z_{\lambda\mu} &= \prod_{0 \leq r < s \leq 7} \Gamma^+(t t_r t_s; t, p, q) Z_{\lambda} Z_{\mu}, \\ Z_{\lambda} &= \mathcal{C}_{\lambda}^0(t^n, pq/tt_1t_2, pq/tt_1t_3; t; p, q) \prod_{4 \leq r \leq 7} \mathcal{C}_{\lambda}^0(pq/tt_1t_r; t; p, q) \\ Z_{\mu} &= \mathcal{C}_{\mu}^0(t^n, pq/tt_0t_3, pq/tt_1t_3; t; p, q) \prod_{4 \leq r \leq 7} \mathcal{C}_{\mu}^0(pq/tt_3t_r; t; p, q) \end{aligned}$$

and the condition on the parameters is now $t^{2n+2}t_0t_1t_2t_3t_4t_5t_6t_7 = p^2q^2$. Here $\Gamma^+(x; t, p, q)$ is defined by

$$\Gamma^+(x; t, p, q) := \prod_{i,j,k \geq 0} (1 - t^i p^j q^k x)(1 - t^{i+1} p^{j+1} q^{k+1} / x),$$

so that, for instance that,

$$\Gamma^+(tx; t, p, q) = \Gamma^+(x; t, p, q)\Gamma(x; p, q).$$

Note that for generic p, q, t , the integer n can be deduced from the balancing condition on the parameters, and thus could in principle be omitted from the notation

for \tilde{II} . Note also that it follows from the residue calculus of the appendix that \tilde{II}_{00} is holomorphic for each n ; it may very well be holomorphic for $\lambda, \mu \neq 0$, but this would require a deeper understanding of the singularities of \mathcal{R}^* as a function of the parameters.

COROLLARY 9.11. *We have*

$$\begin{aligned} \tilde{II}_{\lambda, \mu}^{(n)}(t_0, t_1:t_2, t_3:t_4, t_5, t_6, t_7; t; p, q) \\ = \tilde{II}_{\lambda, \mu}^{(n)}(t_0/u, t_1/u:ut_2, ut_3:t_4/u, t_5/u, ut_6, ut_7; t; p, q), \end{aligned}$$

where u is chosen so that

$$u^2 = \sqrt{\frac{t_0 t_1 t_4 t_5}{t_2 t_3 t_6 t_7}} = \frac{pqt^{-n-1}}{t_2 t_3 t_6 t_7} = \frac{t_0 t_1 t_4 t_5}{pqt^{-n-1}}.$$

Since \tilde{II} is also invariant under permutations of t_4, t_5, t_6, t_7 , it is in fact invariant under an action of the Weyl group D_4 . Since there are three double cosets $S_4 \backslash D_4 / S_4$, there is one other type of nontrivial transformation, namely:

$$\begin{aligned} \tilde{II}_{\lambda, \mu}^{(n)}(t_0, t_1:t_2, t_3:t_4, t_5, t_6, t_7; t; p, q) \\ = \tilde{II}_{\lambda, \mu}^{(n)}(u/t_1, u/t_0:u/t_3, u/t_2:v/t_4, v/t_5, v/t_6, v/t_7; t; p, q), \end{aligned}$$

where $u^2 = t_0 t_1 t_2 t_3$, $v^2 = t_4 t_5 t_6 t_7$, and $t^{n+1}uv = pq$. In terms of the unnormalized integral, this reads

$$\begin{aligned} II_{\mu, \nu}^{(n)}(t_0, t_1:t_2, t_3:t_4, t_5, t_6, t_7; t; p, q) \\ = \prod_{1 \leq j \leq n} \prod_{0 \leq r \leq 3, 4 \leq s \leq 7} \Gamma(t^{n-j} t_r t_s; p, q) \\ \times \Delta_{\mu}^0(t^{n-1} t_0/t_1 | t^{n-1} t_0 t_4, t^{n-1} t_0 t_5, t^{n-1} t_0 t_6, t^{n-1} t_0 t_7; t; p, q) \\ \times \Delta_{\nu}^0(t^{n-1} t_2/t_3 | t^{n-1} t_2 t_4, t^{n-1} t_2 t_5, t^{n-1} t_2 t_6, t^{n-1} t_2 t_7; t; p, q) \\ \times II_{\mu, \nu}^{(n)}(u/t_1, u/t_0:u/t_3, u/t_2:v/t_4, v/t_5, v/t_6, v/t_7; t; p, q). \end{aligned}$$

The reason for the factors of $t^{1/2}$ in the definition of $\tilde{II}^{(n)}$ is that the integral satisfies a further identity.

THEOREM 9.12. *Let $n \geq m \geq 0$ be nonnegative integers such that $\ell(\lambda), \ell(\mu) \leq m$, and suppose the parameters satisfy $t^{n-m} t_0 t_2 = 1$. Then*

$$\begin{aligned} \tilde{II}_{\lambda, \mu}^{(n)}(t_0, t_1:t_2, t_3:t_4, t_5, t_6, t_7; t; p, q) \\ = \tilde{II}_{\lambda, \mu}^{(m)}(1/t_2, t_1:1/t_0, t_3:t_4, t_5, t_6, t_7; t; p, q). \end{aligned}$$

Proof. To compute the left-hand side, we must take a limit (as the condition on the contour cannot be satisfied); what we find is that we must take residues in $n - m$ of the variables, effectively setting those variables to $t^{1/2}t_0, \dots, t^{n-m-1/2}t_0$, or equivalently (taking reciprocals) to $t^{n-m-1/2}t_2, \dots, t^{1/2}t_2$. The result is the desired m -dimensional instance of \tilde{II} . \square

Remark. Note that the requirement that $\ell(\lambda), \ell(\mu) \leq m$ and $n \geq m \geq 0$ with $n, m \in \mathbb{Z}$ is equivalent to a requirement that both sides be well-defined.

We thus find that we have a *formal* symmetry under a larger group, isomorphic to the Weyl group A_1D_4 .

If one of the partition pairs is trivial, the effective symmetry group becomes larger. To be precise, define

$II_{\lambda}^{(n)}(t_0, t_1:t_2, t_3, t_4, t_5, t_6, t_7; t; p, q) := II_{\lambda,0}^{(n)}(t_0, t_1:t_2, t_3:t_4, t_5, t_6, t_7; t; p, q)$, and similarly for $\tilde{II}_{\lambda}^{(n)}$. This function is now manifestly symmetric under permutations of t_2 through t_7 ; together with the symmetry of [Theorem 9.7](#), this gives rise to the Weyl group D_6 . Since $S_6 \setminus D_6 / S_6$ has four double cosets, we thus obtain a further transformation.

COROLLARY 9.13. *We have*

$$\begin{aligned}
 &II_{\lambda}^{(n)}(t_0, t_1:t_2, t_3, t_4, t_5, t_6, t_7; t; p, q) \\
 &= \Delta_{\lambda}^0(t^{n-1}t_0/t_1 | t^{n-1}t_0t_2, \dots, t^{n-1}t_0t_7; t; p, q) \prod_{1 \leq i \leq n} \prod_{0 \leq r < s \leq 7} \Gamma(t^{n-i}t_r t_s; p, q) \\
 &\quad \times II_{\lambda}^{(n)}(u/t_1, u/t_0:u/t_2, u/t_3, u/t_4, u/t_5, u/t_6, u/t_7; t; p, q),
 \end{aligned}$$

where $u^2 = \sqrt{t_0t_1t_2t_3t_4t_5t_6t_7} = pq/t^{n-1}$.

Remark. In the limit $t_6t_7 = p^{l+1}q^{m+1}$, the right-hand side becomes a sum; taking $\lambda = 0$ and reparametrizing, we obtain the following integral representation for a Warnaar-type sum:

$$\begin{aligned}
 &II^{(n)}(pq/u_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7; t; p, q) \\
 &= \prod_{1 \leq i \leq n} \frac{\Gamma(t^{n-i}t, t^{n-i}u_0^2; p, q) \prod_{1 \leq r < s \leq 7} \Gamma(t^{n-i}t_r t_s; p, q)}{\prod_{1 \leq r \leq 7} \Gamma(t^{n-i}u_0 t_r; p, q)} \\
 &\quad \times \sum_{\mu \subset (l,m)^n} \Delta_{\mu}(t^{n-1}u_0^2/pq | t^n, u_0/t_1, u_0/t_2, \dots, u_0/t_7; t; p, q),
 \end{aligned}$$

assuming $t_1 = p^l q^m u_0$ and

$$t^{2n-2}t_2t_3t_4t_5t_6t_7 = p^{1-l}q^{1-m}.$$

Of course, other, less symmetric, integral representations can be obtained from transformations of the left-hand side.

The formal group (adding in the dimension-changing transformation) now becomes the Weyl group D_7 ; it is, however, unclear what significance this has, since we cannot in general compose such transformations. Thus rather than obtaining the full Weyl group, we only obtain a union of two $D_6 \setminus D_7 / D_6$ double cosets (out of three). This gives rise to several new dimension-altering transformations, some of which correspond to well-defined integrals. Thus for instance, we find that

$$\begin{aligned} \tilde{I}_\lambda^{(n)}(t_0, t_1 : t_2, t_3, t_4, t_5, t_6, t_7; t; p, q) \\ = \tilde{I}_\lambda^{(n+m)}(t_0/u, t_1u : t_2/u, t_3/u, t_4/u, t_5/u, t_6/u, t_7u; t; p, q), \end{aligned}$$

where

$$u^2 = \sqrt{t_0 t_2 t_3 t_4 t_5 t_6 / t_1 t_7} = \frac{pqt^{-n-1}}{t_1 t_7} = \frac{t_0 t_2 t_3 t_4 t_5 t_6}{pqt^{-n-1}},$$

such that $u^2 = t^m$ with $m \in \mathbb{Z}$, $n, n + m \geq \ell(\lambda)$. (In all, there are essentially nine distinct dimension altering transformations, coming from the twelve legal $S_6 \setminus D_7 / S_6$ double cosets not in D_6 (modulo inverses).)

If $\lambda = 0$, the group enlarges even further; in that case, the main group is the Weyl group E_7 , while the ‘‘formal’’ group is the Weyl group E_8 . Moreover, the action of E_8 comes from the usual root system, with roots of the form

$$\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}\right)$$

(with an even number of $-$ signs) and permutations of

$$(1, 1, 0, 0, 0, 0, 0), (1, -1, 0, 0, 0, 0, 0).$$

(Thus, for instance, [Corollary 9.11](#) corresponds to the reflection in the root $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$.) The subgroup E_7 is then the stabilizer of the root $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, corresponding to $\sqrt{t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7} = pq/t^{n+1}$. Since there are again four double cosets $S_8 \setminus E_7 / S_8$, we do not obtain any new forms of the main transformation (and similarly for the dimension-altering transformation). The various subgroups considered above are related to E_8 as follows:

$$\begin{aligned} E_7 &= \text{Stab}_{E_8}(\sqrt{t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7}) \\ D_7 &= \text{Stab}_{E_8}(\sqrt{t_1^3 t_2 t_3 t_4 t_5 t_6 t_7 / t_0}) \\ D_6 &= \text{Stab}_{E_8}(t_1 / t_0, \sqrt{t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7}) \\ A_1 D_4 &= \text{Stab}_{E_8}(\sqrt{t_1^3 t_2 t_3 t_4 t_5 t_6 t_7 / t_0}, \sqrt{t_0 t_1 t_3^3 t_4 t_5 t_6 t_7 / t_2, t_1 t_3}) \\ D_4 &= \text{Stab}_{E_8}(t_1 / t_0, t_3 / t_2, t_1 t_3, \sqrt{t_0 t_1 t_2 t_3 t_4 t_5 t_6 t_7}). \end{aligned}$$

If $t = q$, the integrand can be expressed as a product of two determinants, and is itself expressible as a determinant. This gives rise to a system of three-term quadratic recurrences, which turn out to be a form of Sakai's elliptic Painlevé equation [30]; this generalizes the result of [13] for the univariate case (the authors of which also observed the existence of an E_7 symmetry in that case). This also generalizes results of [8] at the Selberg level (showing that certain Selberg-type integrals give solutions of the ordinary Painlevé equations). Also of interest are the cases $t = q^2$, $t = \sqrt{q}$, when the integral can be expressed as a pfaffian, and thus satisfies a system of four-term quadratic recurrences. See [19] for more details. Finally, to obtain in the obvious way (but see [23]) a reasonable degeneration of the integral in the limit $p \rightarrow 0$, we would need two "upper" parameters, of order $O(p)$, while the remaining parameters would have order $O(1)$; we would then use the fact that $\Gamma(pq/x; p, q) = \Gamma(x; p, q)^{-1}$ to move the upper parameters to the denominator. This property is in fact not invariant under E_8 , or even under the above E_7 ; instead we obtain a *different* instance of E_7 (as the stabilizer of the root $(0, 0, 0, 0, 0, 1, 1)$, assuming the upper parameters are t_6 and t_7) from the E_8 action, while the E_7 action reduces to E_6 . (The one-dimensional instance of the resulting integral identity is a trivial consequence of the hypergeometric series representation of Rahman [17].) If we further degenerate the integral to the multivariate Askey-Wilson case (a.k.a. the Koornwinder density), the symmetry group reduces to D_5 , and the corresponding identity was proved in [20].

10. Appendix: Meromorphy of integrals

In the above work, we have made heavy use of the fact that the various contour integrals we consider are meromorphic functions of the parameters. This does not quite follow from the meromorphy of the integrands, as can be seen from the following two examples:

$$\int_{|z|=1} \frac{1}{1+t(z+1/z)} \frac{dz}{2\pi\sqrt{-1}z} = (1-4t^2)^{-1/2}, \quad |t| < 1/2,$$

$$\int_{|z|=1} e^{1/(z-2)} \frac{dz}{2\pi\sqrt{-1}(z-t)} = e^{1/(t-2)}, \quad |t| < 1.$$

In both cases, the integrand is meromorphic in a neighborhood of the contour, but there are obstructions to meromorphically continuing the integral. (The second integrand, of course, has an essential singularity, but so do the integrands of interest to us.) It turns out, however, that these are typical of the only two such obstructions: an initial contour that separates branches of a component of the polar divisor of the integrand, or such a component that leaves the domain of meromorphy.

We will prove this fact in [Theorem 10.2](#) below, but first a lemma about meromorphy of residues is needed. Note that with g as described in the hypotheses of

the lemma, the polar divisor of g is a codimension 1 analytic subvariety of $D \times P$, and thus each component is either of the form $D \times \chi_0$, or can be viewed as a family of point sets in D parametrized by P .

LEMMA 10.1. *Let D be a nonempty open subset of $\mathbb{C}P^1$, and let P be an irreducible normal holomorphic variety. Let g be a meromorphic function on $D \times P$, and let χ be a component of the polar divisor of g which is closed in $\overline{D} \times P$. Let P' be the subset of P on which the fibers of χ are disjoint from the other polar divisors of g (the complement of a codimension 1 subvariety), and define a function $f(p)$ on P' by*

$$f(p) = \sum_{d:(d,p) \in \chi} \operatorname{Res}_{z=d} g(z, p).$$

Then f extends uniquely to a meromorphic function on all of P .

Proof. Note that on any compact subset of P , χ is bounded away from the complement of D , and thus its fibers lie in a compact subset of D , and so are finite in number. In particular, the above sum is thus well-defined, and gives a holomorphic function on P' .

Now, by Levi's theorem, we can freely remove any codimension-2 subvariety W of P without affecting the extension of f ; in particular, we may assume that P is regular (since its singular locus is codimension-2 by normality). We can then further restrict to a neighborhood of a general regular point, to reduce to the case $P \subset \mathbb{C}^n$ an open polydisc; we can also then write $g(d, p) = g_1(d, p)/g_2(d, p)$ for g_1, g_2 holomorphic. Let $Z \subset P$ be the locus for which $g_2(d, p)$ is identically 0 as a function of d ; then by multiplying g by a suitable function of p alone, we can remove all codimension-1 components of Z , leaving a codimension-2 locus which can be removed by another application of Levi's theorem.

Now, consider a point $p_0 \in P$. Reducing P as necessary, we can assume that the fiber of χ over p_0 consists of a single point d_0 ; we can then reduce D to assure that $g_2(d, p_0)$ also vanishes only at d_0 . But then by the Weierstrass preparation theorem, $g_2(d, p)$ is the product of a monic polynomial in d with holomorphic coefficients and a holomorphic function nowhere vanishing on D , which can be absorbed into g_1 . Moreover, $g_2(d, p)$ factors as $h_1(d, p)h_2(d, p)$ where the monic polynomial $h_1(d, p)$ vanishes precisely along χ , and the monic polynomial $h_2(d, p)$ is relatively prime to $h_1(d, p)$. We can thus write

$$g(d, p) = \frac{i_1(d, p)}{h_1(d, p)} + \frac{i_2(d, p)}{h_2(d, p)} + i_0(d, p),$$

where i_0 is holomorphic in d , and i_1, i_2 are polynomials with meromorphic coefficients of degree less than $\deg(h_1), \deg(h_2)$ respectively. But the above sum of residues is then precisely the leading coefficient of $i_1(d, p)$, and is thus meromorphic in a neighborhood of p_0 . □

Remark. In fact, $i_1(d, p)\Delta(p)$ is holomorphic, where $\Delta(p)$ is the resultant of the polynomials $h_1(d, p)$ and $h_2(d, p)$.

Given a closed contour C in $\mathbb{C}P^1$, every point not in C of course has an associated winding number; we extend this by linearity to formal linear combinations of contours.

THEOREM 10.2. *Let D be a nonempty connected open subset of $\mathbb{C}P^1$, let C be a finite complex linear combination of contours in D , and let P be an irreducible normal holomorphic variety. Let g be a function meromorphic on $D \times P$, and suppose the function f is defined on an open subset U of P by*

$$f(p) = \int_C g(z, p) dz.$$

(Thus, in particular, we assume that the polar divisor of g in $D \times U$ is disjoint from $C \times U$.)

Suppose that each irreducible component χ of the polar divisor of g is either of the form $D \times \chi_0$ or satisfies the assumptions:

1. For every point $u \in U$, every point $d \in D$ such that $(d, u) \in \chi$ has the same winding number with respect to C ; call this the winding number of χ .
2. If (d, p) is a limit point of χ in $\overline{D} \times P$ outside $D \times P$, then the winding number of d with respect to C is the same as that of χ itself.

Then $f(p)$ extends uniquely to a meromorphic function on all of P .

Proof. Again, we may as well assume that P is an open polydisc in \mathbb{C}^n for some n . We may then assume that the polar divisor of g contains no components of the form $D \times \chi_0$, since we can in that case simply multiply g by a holomorphic function to remove that pole.

Now, let U' be an open subset of P , and consider a component χ of the polar divisor of g on $D \times U'$. This is contained in a unique component of the full polar divisor, with winding number w_0 , say; on the other hand, if U' is not contained in U , χ can easily intersect C or have well-defined winding number different from the “true” winding number w_0 , in which case we call it “problematical”. We claim that every point $p \in P$ has a neighborhood with only finitely many problematical polar components. Indeed, by condition (2) above, we can choose a bounded neighborhood U' of p such that the problematical components of g on $D \times U'$ are bounded away from the complement of D , and are thus contained in $D' \times U'$ for some compact subset of D' , in which g can support only finitely many poles.

If p is such that we can choose U' so that the problematical components are disjoint from all components with a different “true” winding number (which will hold for p away from a codimension-1 subvariety), then we can obtain a new

contour C' by deforming C and adding small circles in such a way that the integral

$$\int_{C'} g(z, p) dz$$

is well-defined on U' , and such that on the intersection of two such components, the functions agree. Indeed, we can clearly deform C in such a way that the problematical components have well-defined winding numbers with respect to the new contour; by adding small circles around the problematical components (shrinking U' as necessary to allow these circles to be fixed) we can make these winding numbers equal to the “true” winding numbers. Any two such contours will give the same integral, by Cauchy’s theorem, and thus these functions agree on intersections.

At a general point, we can still deform C to give well-defined winding numbers to the problematical components, but now we have the difficulty that they might intersect components with different winding numbers. Here, we can observe that the above analytic continuation can be written as

$$\int_{C'} g(z, p) dz + \text{finite sum of residues}$$

where instead of adding a small circle around the problematical components, we simply add the corresponding residue. The first term is certainly meromorphic (in fact, holomorphic near p); that the residue terms are meromorphic (and thus that the theorem holds) results from the lemma. \square

We need only the following special case. Here

$$(x; p, q)_\infty := \prod_{0 \leq i, j} (1 - p^i q^j x).$$

COROLLARY 10.3. *Let $F(z; t_0, \dots, t_{m-1}; u_0, \dots, u_{n-1}; p, q)$ be a function holomorphic on the domain*

$$z, t_0, \dots, t_{m-1}, u_0, \dots, u_{n-1} \in \mathbb{C}^*, \quad 0 < |p|, |q| < 1.$$

Then the function defined for $|t_r|, |u_r| < 1$ by

$$\begin{aligned} &G(t_0, \dots, t_{m-1}; u_0, \dots, u_{n-1}; p, q) \\ &= \prod_{0 \leq r < m, 0 \leq s < n} (t_r u_s; p, q)_\infty \int_{|z|=1} \Delta(z; t_0, \dots, t_{m-1}; p, q) \frac{dz}{2\pi \sqrt{-1}z} \end{aligned}$$

where

$$\Delta(z; t_0, \dots, t_{m-1}; p, q) = \frac{F(z; t_0, \dots, t_{m-1}; p, q)}{\prod_{0 \leq r < m} (t_r z; p, q)_\infty \prod_{0 \leq r < n} (u_r/z; p, q)_\infty},$$

extends uniquely to a holomorphic function on the domain

$$t_0, \dots, t_{m-1}, u_0, \dots, u_{n-1} \in \mathbb{C}^*, 0 < |p|, |q| < 1.$$

Away from the divisor of $\prod_{0 \leq r < m, 0 \leq s < n} (t_r u_s; p, q)_\infty$, this extension can be obtained by replacing the unit circle by any (possibly disconnected) contour that contains the points $p^i q^j u_r$ and excludes the points $1/p^i q^j t_r$, for $0 \leq i, j$.

In particular, our multidimensional integrals can all be expressed as iterated contour integrals of this form (in general restricted to a subvariety of parameter space), so are meromorphic by straightforward induction. This does, however, tend to grossly overestimate the polar divisor. This overestimation can easily occur even in the one-dimensional case, in the presence of symmetry.

In the case of the BC_1 integral, one role of the balancing condition, as we have seen, is to make the summation limits factor into p -abelian and q -abelian factors, which occurs because the density satisfies the relation

$$\Delta(p^i q^j z) \Delta(z) = \Delta(p^i z) \Delta(q^j z)$$

for $i, j \in \mathbb{Z}$. As observed by Spiridonov (personal communication), this only determines the balancing condition up to a sign. However, one special case of this relation is the identity

$$\Delta(\pm p^{i/2} q^{j/2}) \Delta(\pm p^{-i/2} q^{-j/2}) = \Delta(\pm p^{i/2} q^{-j/2}) \Delta(\pm p^{-i/2} q^{j/2}),$$

when both sides are defined; using the fact that $\Delta(z) = \Delta(1/z)$, we conclude that

$$\Delta(\pm p^{i/2} q^{j/2})^2 = \Delta(\pm p^{i/2} q^{-j/2})^2$$

so that

$$\Delta(p^{i/2} q^{j/2}) = \pm \Delta(p^{i/2} q^{-j/2}), \text{ and } \Delta(-p^{i/2} q^{j/2}) = \pm \Delta(-p^{i/2} q^{-j/2}).$$

The balancing condition for the BC_1 integral then has the effect of choosing the sign in this identity:

$$\Delta(\pm p^{i/2} q^{j/2}) = -\Delta(\pm p^{i/2} q^{-j/2}).$$

This motivates the hypotheses for the following result.

LEMMA 10.4. *Let $\Delta(z; \mathbf{p})$ be a BC_1 -symmetric function on $\mathbb{C}^* \times P$, with P an irreducible normal subvariety of $\{t_0, t_1, \dots, t_{d-1}, p, q \in \mathbb{C}^* : |p|, |q| < 1\}$. Suppose furthermore that the following conditions are satisfied.*

1. *The function*

$$\prod_{0 \leq r < d} (t_r z^{\pm 1}; p, q) \Delta(z; \mathbf{p})$$

is holomorphic.

- 2. At a generic point of P , the denominator has only simple zeros; it has triple zeros only in codimension 2.
- 3. For any integers i, j ,

$$\Delta(\pm p^{i/2}q^{j/2}; \mathbf{p}) = -\Delta(\pm p^{i/2}q^{-j/2}; \mathbf{p}),$$

as an identity of meromorphic functions on P .

Then the function on P defined for $|t_r| < 1$ by

$$\prod_{0 \leq r < s < d} (t_r t_s; p, q)_\infty \int_{|z|=1} \Delta(z) \frac{dz}{2\pi \sqrt{-1}z}$$

extends to a holomorphic function on P .

Proof. The integral extends meromorphically to this domain by [Corollary 10.3](#); that it has at most simple poles along the subvarieties $t_r t_s = p^{-i}q^{-j}$, $i, j \geq 0$ follows immediately from the fact that at a generic point of such a subvariety, there are no higher-order collisions of poles. However, these considerations still leave open the possibility that the given function might have poles along the subvarieties $t_r^2 = p^{-i}q^{-j}$, $i, j \geq 0$.

We thus need, without loss of generality, to show that the above function is holomorphic at a generic point \mathbf{p}_0 such that $t_0 = \pm p^{-l/2}q^{-m/2}$, $i, j \geq 0$. Now, in a neighborhood of such a point, the analytic continuation is given by

$$\int_C \Delta(z) \frac{dz}{2\pi \sqrt{-1}z},$$

where $C = C^{-1}$ is a contour containing $p^i q^j t_r$ for $i, j \geq 0$, $0 \leq r < d$. Now, let C' be a modified symmetric contour that still contains $p^i q^j t_r$ for $r > 0$ and $p^i q^j t_0$ for $i \geq l$ or $j \geq m$, but excludes $p^i q^j t_0$ for $0 \leq i \leq l$, $0 \leq j \leq m$. Then we claim that

$$\int_C \Delta(z) \frac{dz}{2\pi \sqrt{-1}z} + \int_{C'} \Delta(z) \frac{dz}{2\pi \sqrt{-1}z}$$

is holomorphic on a neighborhood of \mathbf{p}_0 . Indeed, anywhere that two poles coalesce, the poles have the same overall winding number with respect to the two contours. Thus to show the first term is holomorphic, it suffices to prove that the difference of the two terms is holomorphic. But this is just a sum of residues; it is therefore sufficient to prove that

$$\sum_{0 \leq i \leq l} \sum_{0 \leq j \leq m} \text{Res}_{z=t_0 p^i q^j} \Delta(z; \mathbf{p}),$$

is holomorphic near \mathbf{p}_0 , or in other words that

$$\lim_{\mathbf{p} \rightarrow \mathbf{p}_0} \sum_{0 \leq i \leq l} \sum_{0 \leq j \leq m} \text{Res}_{z=t_0 p^i q^j} \Delta(z; \mathbf{p})$$

is well-defined. We claim in fact that

$$\lim_{\mathbf{p} \rightarrow \mathbf{p}_0} (1 - \pm p^{l/2} q^{m/2} t_0) [\text{Res}_{z=t_0 p^i q^j} \Delta(z; \mathbf{p}) + \text{Res}_{z=t_0 p^i q^{m-j}} \Delta(z; \mathbf{p})] = 0,$$

which then makes the poles of the summands cancel pairwise, giving the desired result. Now,

$$\begin{aligned} &\lim_{\mathbf{p} \rightarrow \mathbf{p}_0} (1 - \pm p^{l/2} q^{m/2} t_0) \text{Res}_{z=t_0 p^i q^j} \Delta(z; \mathbf{p}) \\ &\quad \propto \lim_{\mathbf{p} \rightarrow \mathbf{p}_0} \lim_{z \rightarrow t_0 p^i q^j} (1 - \pm p^{l/2} q^{m/2} t_0) (1 - t_0 p^i q^j / z) \Delta(z; \mathbf{p}), \end{aligned}$$

and this limit is well-defined, again because at most two poles coalesce at any given point. Now, if we pull out the denominator factors $(t_0 z^{\pm 1}; p, q)_\infty$ of $\Delta(z; \mathbf{p})$, we can explicitly compute their contributions to the limit, and use the fact that limits of holomorphic functions can be exchanged to conclude that

$$\begin{aligned} &\lim_{\mathbf{p} \rightarrow \mathbf{p}_0} \lim_{z \rightarrow t_0 p^i q^j} (1 - \pm p^{l/2} q^{m/2} t_0) (1 - t_0 p^i q^j / z) \Delta(z; \mathbf{p}) \\ &\quad = \frac{1}{2} \lim_{\mathbf{p} \rightarrow \mathbf{p}_0} (1 - \pm p^{l/2} q^{m/2} t_0)^2 \Delta(\pm p^{i-l/2} q^{j-m/2}; \mathbf{p}). \end{aligned}$$

The claim follows. □

Similarly, for higher dimensional integrals, the simple inductive argument leads to predictions of extremely high order poles along the divisors $t_r t_s = p^{-l} q^{-m}$, $l, m \geq 0$. That this does not occur for our integrals follows via a similar argument from the fact that

$$\Delta(p^a q^b z_0, p^c q^d z_0, z_3, \dots, z_n; \mathbf{p}) = -\Delta(p^a q^d z_0, p^c q^b z_0, z_3, \dots, z_n; \mathbf{p})$$

for our integrands; the consequence is that when moving the contour over a given collection of poles of the form $p^i q^j t_0$, $0 \leq i \leq l$, $0 \leq j \leq m$, the residues of residues that arise all cancel pairwise. Somewhat more generally, we have the following.

LEMMA 10.5. *Let $\Delta(z_1, \dots, z_n; \mathbf{p})$ be a symmetric meromorphic function on $(\mathbb{C}^*)^n \times P$, where P is an irreducible normal subvariety of the domain*

$$\{t_0, t_1, \dots, t_{d-1}, u_0, \dots, u_{d-1}, p, q \in \mathbb{C}^* : |p|, |q| < 1\}.$$

Suppose furthermore that the following conditions are satisfied.

1. *The function*

$$\prod_{1 \leq i \leq n} \prod_{0 \leq r < d} (t_r z_i, u_r / z_i; p, q) \Delta(z_1, \dots, z_n; \mathbf{p})$$

is holomorphic.

2. *At a generic point of P , the denominator has only simple zeros.*

3. For any integers $a, b, c, d \geq 0$,

$$\Delta(p^a q^b z, p^c q^d z, z_3, \dots, z_n; \mathbf{p}) = -\Delta(p^a q^d z, p^c q^b z, z_3, \dots, z_n; \mathbf{p}),$$

as an identity of meromorphic functions on $(\mathbb{C}^*)^{n-1} \times P$.

For generic $\mathbf{p} \in P$, choose a contour $C_{\mathbf{p}}$ containing all points of the form $u_r p^i q^j$ with $i, j \geq 0, 0 \leq r < d$, and excluding all points of the form $(t_r p^i q^j)^{-1}$ with $i, j \geq 0, 0 \leq r < d$. Let $C'_{\mathbf{p}}$ be a similar contour that differs from $C_{\mathbf{p}}$ by excluding the points $u_0 p^i q^j$ with $0 \leq i \leq l, 0 \leq j \leq m$. Then

$$\begin{aligned} & \int_{C_{\mathbf{p}}^n} \Delta(z; \mathbf{p}) \prod_{1 \leq i \leq n} \frac{dz_i}{2\pi \sqrt{-1} z_i} - \int_{C'_{\mathbf{p}}^n} \Delta(z; \mathbf{p}) \prod_{1 \leq i \leq n} \frac{dz_i}{2\pi \sqrt{-1} z_i} \\ &= n \int_{C_{\mathbf{p}}^{m-1}} \sum_{0 \leq (a,b) \leq (l,m)} \lim_{z_n \rightarrow p^a q^b t_0} (1 - p^a q^b t_0 / z_n) \Delta(z; \mathbf{p}) \prod_{1 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1} z_i}. \end{aligned}$$

Proof. A straightforward induction using the symmetry of the integrand tells us that

$$\begin{aligned} & \int_{C_{\mathbf{p}}^n} \Delta(z; \mathbf{p}) \prod_{1 \leq i \leq n} \frac{dz_i}{2\pi \sqrt{-1} z_i} - \int_{C'_{\mathbf{p}}^n} \Delta(z; \mathbf{p}) \prod_{1 \leq i \leq n} \frac{dz_i}{2\pi \sqrt{-1} z_i} \\ &= \sum_{1 \leq j \leq n} \int_{C_{\mathbf{p}}^{j-1} \times C_{\mathbf{p}}^{n-j}} \\ & \quad \times \sum_{0 \leq (a,b) \leq (l,m)} \lim_{z_n \rightarrow p^a q^b t_0} (1 - p^a q^b t_0 / z_n) \Delta(z; \mathbf{p}) \prod_{1 \leq i < n} \frac{dz_i}{2\pi \sqrt{-1} z_i}; \end{aligned}$$

indeed, the sum is simply the contributions from residues as we deform the contours in z_n, z_{n-1}, \dots, z_1 . It thus suffices to show that these n terms all agree. But the difference between the j th term and the $j + 1$ st term is an $n - 2$ -dimensional integral of a sum of double residues:

$$\begin{aligned} & \sum_{0 \leq (a,b),(c,d) \leq (l,m)} \lim_{z_n \rightarrow p^a q^b t_0} \\ & \quad \times \lim_{z_{n-1} \rightarrow p^c q^d t_0} (1 - p^a q^b t_0 / z_n)(1 - p^a q^d t_0 / z_{n-1}) \Delta(z_1, \dots, z_n; \mathbf{p}). \end{aligned}$$

But again we can pull out the known pole factors and interchange limits of the resulting holomorphic function; we conclude that the $(a, b), (c, d)$ and $(a, d), (c, b)$ terms cancel. \square

Applying this to the type I integral gives the following result; similar results apply to the integral operators.

THEOREM 10.6. *The function*

$$\prod_{0 \leq r < s \leq 2m+2n+3} (t_r t_s; p, q)_{\infty} I_{BC_n}^{(m)}(t_0, \dots, t_{2m+2n+3}; p, q)$$

extends to a holomorphic function on the domain $\prod_r t_r = (pq)^m, |p|, |q| < 1$.

Proof. Indeed, the integrand satisfies the hypotheses of the two lemmas; the second lemma readily shows that the integral has a simple pole along each subvariety $t_r t_s p^a q^b = 1$ (with residue equal to a sum of $n - 1$ -dimensional integrals), while an induction using the first lemma shows that the potential singularities for $t_r^2 p^a q^b = 1$ are not present. \square

Remark. The situation for the A_n integral is much more complicated, as we must integrate against a test function in Z to allow the use of a product contour (which is legal since an inductive argument shows the A_n integral to be meromorphic); but this extra integration, while preserving meromorphy, can easily remove singularities.

For the type II integral, a similar argument applies to contour deformations; the additional poles coming from the cross terms are sufficiently generic that the multidimensional lemma still holds. There is an important difference in that we have the additional constraint that the contour $C = C^{-1}$ should contain the contours $t p^i q^j C, i, j \geq 0$. Thus when deforming through the collection of points $t^i q^j p^k t_0, 0 \leq (i, j, k) \leq (a, b, c)$, it is necessary to first deform through the points with $i = 0$, then those with $i = 1$, and so forth; otherwise the contour constraint will be broken. With that caveat, however, the results still apply, and we obtain the following result.

THEOREM 10.7. *Let $II_n^{(m)}(t_0, t_1, \dots, t_{2m+3}; t; p, q)$ be the $2m+4$ -parameter analogue of the type II integral, $m > 0$. The function*

$$\prod_{0 \leq i < n} \prod_{0 \leq r < s < 2m} (t^i t_r t_s; p, q)_{\infty} II_n^{(m)}(t_0, \dots, t_{2m+3}; t; p, q)$$

extends to a holomorphic function on the domain $t^{2n-2} \prod_r t_r = (pq)^m, |t|, |p|, |q| < 1$.

Proof. At a generic point with $t^i p^j q^k t_r t_s = 1, r < s$, we can simply deform the contour through the points $t^a p^b q^c t_r, 0 \leq a \leq i, 0 \leq b \leq j, 0 \leq c \leq k$ to obtain a holomorphic integral. We thus find that the desired integral is a sum of integrals over the new contour, with integrands given by multiple residues at a sequence of points with $a = 0, a = 1, \dots$. The only such integrals that are singular at $t^i p^j q^k t_r t_s = 1$ are those involving $i + 1$ -tuple residues, and those have simple poles. Since we can obtain at most n -tuple residues from an n -tuple integral, we conclude that we have at most simple poles, and those only when $i < n$.

For $r = s$, if we first deform through the points $t^a p^b q^c t_r$ with $0 \leq b \leq j$, $0 \leq c \leq k$, $0 \leq a < i/2$, we find that the only integrals with possible singularities are those of $i/2$ -tuple residues; these are then generically holomorphic when $t^i p^j q^k t_r^2 = 1$ by induction. \square

Remark. A similar result applies to $II_{\lambda\mu}^{(n)}$, with the caveat that the interpolation functions may have poles independent of z_1, \dots, z_n ; these poles would then in general survive as poles of the integral.

References

- [1] G. W. ANDERSON, A short proof of Selberg's generalized beta formula, *Forum Math.* **3** (1991), 415–417. [MR 92j:33006](#) [Zbl 0723.33002](#)
- [2] D. BARSKY and M. CARPENTIER, [Polynômes de Jacobi généralisés et intégrales de Selberg](#), *Electron. J. Combin.* **3** (1996), Research Paper 1, approx. 33 pp. [MR 97k:33005](#) [Zbl 0883.33010](#)
- [3] Y. BARYSHNIKOV, [GUEs and queues](#), *Probab. Theory Related Fields* **119** (2001), 256–274. [MR 2002a:60165](#) [Zbl 0980.60042](#)
- [4] A. L. DIXON, On a generalisation of Legendre's formula $KE' - (K - E)K' = \frac{1}{2}\pi$, *Proc. London Math. Soc.* **3** (1905), 206–224. [JFM 36.0506.01](#)
- [5] J. F. VAN DIEJEN, [Self-dual Koornwinder-Macdonald polynomials](#), *Invent. Math.* **126** (1996), 319–339. [MR 98k:33025](#) [Zbl 0869.33012](#)
- [6] J. F. VAN DIEJEN and V. P. SPIRIDONOV, An elliptic Macdonald-Morris conjecture and multiple modular hypergeometric sums, *Math. Res. Lett.* **7** (2000), 729–746. [MR 2002m:33022](#) [Zbl 0981.33013](#)
- [7] ———, Elliptic Selberg integrals, *Internat. Math. Res. Notices* **20** (2001), 1083–1110. [MR 2002j:33016](#) [Zbl 1010.33010](#)
- [8] P. J. FORRESTER and N. S. WITTE, [Application of the \$\tau\$ -function theory of Painlevé equations to random matrices: \$P_{VI}\$, the JUE, CyUE, cJUE and scaled limits](#), *Nagoya Math. J.* **174** (2004), 29–114. [MR 2005h:33047](#) [Zbl 1056.15023](#)
- [9] P. J. FORRESTER and E. M. RAINS, [Interpretations of some parameter dependent generalizations of classical matrix ensembles](#), *Probab. Theory Related Fields* **131** (2005), 1–61. [MR 2006g:05222](#) [Zbl 1056.05142](#)
- [10] G. FROBENIUS, Über die elliptischen Functionen zweiter Art, *J. für die reine und angew. Math.* **93** (1882), 53–68. [JFM 14.0389.01](#)
- [11] R. A. GUSTAFSON, [Some \$q\$ -beta and Mellin-Barnes integrals with many parameters associated to the classical groups](#), *SIAM J. Math. Anal.* **23** (1992), 525–551. [MR 92m:33006](#) [Zbl 0764.33008](#)
- [12] Y. KAJIHARA and M. NOUMI, [Multiple elliptic hypergeometric series. An approach from the Cauchy determinant](#), *Indag. Math.* **14** (2003), 395–421. [MR 2005g:33032](#) [Zbl 1051.33009](#)
- [13] K. KAJIWARA, T. MASUDA, M. NOUMI, Y. OHTA, and Y. YAMADA, [\$_{10}E_9\$ solution to the elliptic Painlevé equation](#), *J. Phys. A* **36** (2003), L263–L272. [MR 2004g:39033](#) [Zbl 1044.39014](#)
- [14] T. H. KOORNWINDER, Askey-Wilson polynomials for root systems of type BC , in *Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications*, (Tampa, FL, 1991), *Contemp. Math.* **138**, Amer. Math. Soc., Providence, RI, 1992, pp. 189–204. [MR 94e:33039](#) [Zbl 0797.33014](#)

- [15] A. OKOUNKOV, [BC-type interpolation Macdonald polynomials and binomial formula for Koornwinder polynomials](#), *Transform. Groups* **3** (1998), 181–207. [MR 99h:33061](#) [Zbl 0941.17005](#)
- [16] ———, [\(Shifted\) Macdonald polynomials: \$q\$ -integral representation and combinatorial formula](#), *Compositio Math.* **112** (1998), 147–182. [MR 99h:05120](#) [Zbl 0897.05085](#)
- [17] M. RAHMAN, An integral representation of a ${}_{10}\phi_9$ and continuous bi-orthogonal ${}_{10}\phi_9$ rational functions, *Canad. J. Math.* **38** (1986), 605–618. [MR 87i:33011](#) [Zbl 0599.33015](#)
- [18] M. RAHMAN and S. K. SUSLOV, Classical biorthogonal rational functions, in *Methods of Approximation Theory in Complex Analysis and Mathematical Physics (Leningrad, 1991)*, *Lecture Notes in Math.* **1550**, Springer-Verlag, New York, 1993, pp. 131–146. [MR 95m:42034](#) [Zbl 0789.33005](#)
- [19] E. M. RAINS, Recurrences of elliptic hypergeometric integrals, in *Elliptic Integrable Systems* (M. Noumi and K. Takasaki, eds.), *Rokko Lectures in Math.* **18** (2005), 183–199.
- [20] ———, [BC_n-symmetric polynomials](#), *Transform. Groups* **10** (2005), 63–132. [MR 2007g:33017](#) [Zbl 1080.33018](#)
- [21] ———, [BC_n-symmetric Abelian functions](#), *Duke Math. J.* **135** (2006), 99–180. [MR 2008e:33043](#) [Zbl 1101.33012](#)
- [22] ———, A difference-integral representation of Koornwinder polynomials, in *Jack, Hall-Littlewood and Macdonald Polynomials*, *Contemp. Math.* **417**, Amer. Math. Soc., Providence, RI, 2006, pp. 319–333. [MR 2008e:33042](#) [Zbl 1132.33332](#)
- [23] ———, [Limits of elliptic hypergeometric integrals](#), *Ramanujan J.* **18** (2009), 257–306. [MR 2495549](#) [Zbl pre05620218](#)
- [24] D. RICHARDS and Q. ZHENG, [Determinants of period matrices and an application to Selberg's multidimensional beta integral](#), *Adv. in Appl. Math.* **28** (2002), 602–633. [MR 2003f:33018](#) [Zbl 1011.33014](#)
- [25] H. ROSENGREN, A proof of a multivariable elliptic summation formula conjectured by Warnaar, in *q -Series with Applications to Combinatorics, Number Theory, and Physics (Urbana, IL, 2000)*, *Contemp. Math.* **291**, Amer. Math. Soc., Providence, RI, 2001, pp. 193–202. [MR 2002m:33027](#) [Zbl 1040.33014](#)
- [26] ———, [Elliptic hypergeometric series on root systems](#), *Adv. Math.* **181** (2004), 417–447. [MR 2005b:33022](#) [Zbl 1066.33017](#)
- [27] ———, [New transformations for elliptic hypergeometric series on the root system \$A_n\$](#) , *Ramanujan J.* **12** (2006), 155–166. [MR 2008e:33044](#) [Zbl 1103.33012](#)
- [28] S. N. M. RUIJSENAARS, [First order analytic difference equations and integrable quantum systems](#), *J. Math. Phys.* **38** (1997), 1069–1146. [MR 98m:58065](#) [Zbl 0877.39002](#)
- [29] S. SAHI, [Nonsymmetric Koornwinder polynomials and duality](#), *Ann. of Math.* **150** (1999), 267–282. [MR 2002b:33018](#) [Zbl 0941.33013](#)
- [30] H. SAKAI, [Rational surfaces associated with affine root systems and geometry of the Painlevé equations](#), *Comm. Math. Phys.* **220** (2001), 165–229. [MR 2003c:14030](#) [Zbl 1010.34083](#)
- [31] V. P. SPIRIDONOV, [On the elliptic beta function](#), *Uspekhi Mat. Nauk* **56** (2001), 181–182. [MR 2003b:33032](#) [Zbl 0997.33009](#)
- [32] ———, [Theta hypergeometric integrals](#), *Algebra i Analiz* **15** (2003), 161–215. [MR 2005m:33030](#) [Zbl 1071.33011](#)
- [33] ———, [Short proofs of the elliptic beta integrals](#), *Ramanujan J.* **13** (2007), 265–283. [MR 2007i:33042](#) [Zbl 1118.33008](#)

- [34] V. SPIRIDONOV and A. ZHEDANOV, Classical biorthogonal rational functions on elliptic grids, *C. R. Math. Acad. Sci. Soc. R. Can.* **22** (2000), 70–76. [MR 2001c:33035](#) [Zbl 0974.33016](#)
- [35] ———, [Spectral transformation chains and some new biorthogonal rational functions](#), *Comm. Math. Phys.* **210** (2000), 49–83. [MR 2001e:33008](#) [Zbl 0989.33008](#)
- [36] A. N. VARCHENKO, The determinant of a matrix of multidimensional hypergeometric integrals, *Dokl. Akad. Nauk SSSR* **308** (1989), 777–780. [MR 91c:32032](#) [Zbl 0737.05033](#)
- [37] S. O. WARNAAR, [Summation and transformation formulas for elliptic hypergeometric series](#), *Constr. Approx.* **18** (2002), 479–502. [MR 2003h:33018](#) [Zbl 1040.33013](#)
- [38] J. A. WILSON, [Orthogonal functions from Gram determinants](#), *SIAM J. Math. Anal.* **22** (1991), 1147–1155. [MR 92c:33018](#) [Zbl 0742.33008](#)

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