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Abstract

We prove that Donaldson-Thomas type invariants are equal to weighted Euler characteristics of their moduli spaces. In particular, such invariants depend only on the scheme structure of the moduli space, not the symmetric obstruction theory used to define them. We also introduce new invariants generalizing Donaldson-Thomas type invariants to moduli problems with open moduli space. These are useful for computing Donaldson-Thomas type invariants over stratifications.

Introduction

Donaldson-Thomas type invariants. Donaldson-Thomas invariants ([DT98], [Tho00]) are the virtual counts of stable sheaves (with fixed determinant) on Calabi-Yau threefolds. Heuristically, the Donaldson-Thomas moduli space is the critical set of the holomorphic Chern-Simons functional and the Donaldson-Thomas invariant is a holomorphic analogue of the Casson invariant. Recently [MNOP06], Donaldson-Thomas invariants for sheaves of rank one have been conjectured to have deep connections with Gromov-Witten theory of Calabi-Yau threefolds. They are supposed to encode the integrality properties of such Gromov-Witten invariants, for example.

Mathematically, Donaldson-Thomas invariants are constructed as follows (see [Tho00]). Deformation theory gives rise to a *perfect obstruction theory* [BF97] (or a *tangent-obstruction complex* in the language of [LT98]) on the moduli space of stable sheaves X. As Thomas points out in [Tho00], the *obstruction sheaf* is equal to Ω_X, the sheaf of Kähler differentials, and hence the tangents T_X are *dual* to the obstructions. This expresses a certain symmetry of the obstruction theory and is a mathematical reflection of the heuristic that views X as the critical locus of a holomorphic functional.

Associated to the perfect obstruction theory is the *virtual fundamental class*, an element of the Chow group $A_*(X)$ of algebraic cycles modulo rational equivalence on X. One implication of the symmetry of the obstruction theory is the fact that the virtual fundamental class $[X]^{\mathrm{vir}}$ is of degree zero. It can hence be integrated over the proper space of stable sheaves to a number, the *Donaldson-Thomas invariant* or *virtual count* of X

$$\#^{\mathrm{vir}}(X) = \int_{[X]^{\mathrm{vir}}} 1\,.$$

We take the point of view that the symmetry of the obstruction theory is the distinguishing feature of Donaldson-Thomas invariants, and call any virtual count of a proper scheme with symmetric obstruction theory a *Donaldson-Thomas type* invariant.

Euler characteristics and ν_X. If the moduli space X is smooth, the obstruction sheaf Ω_X is a bundle, so the virtual fundamental class is the top Chern class $e(\Omega_X)$ and so the virtual count is, up to a sign, the Euler characteristic of X:

$$\#^{\mathrm{vir}}(X) = \int_{[X]} e(\Omega_X) = (-1)^{\dim X} \chi(X)\,.$$

We will generalize this formula to arbitrary (embeddable) schemes.

More precisely, we will construct on any scheme X over $\mathbb{C}$ in a canonical way a constructible function $\nu_X : X \to \mathbb{Z}$ (depending only on the scheme structure of X), such that if X is proper and embeddable we have

$$\#^{\mathrm{vir}}(X) = \chi(X, \nu_X) = \sum_{n\in\mathbb{Z}} n\, \chi\{\nu_X = n\}\,, \tag{1}$$

for *any* symmetric obstruction theory on X with associated Donaldson-Thomas type invariant $\#^{\mathrm{vir}}(X)$.

As consequences of this result we obtain:

- Donaldson-Thomas type invariants depend only on the scheme structure of the underlying moduli space, not on the symmetric obstruction theory used to define them.
- Even if X is not proper, and so the virtual count does not make sense as an integral, we can consider the weighted Euler characteristic

  $$\tilde{\chi}(X) = \chi(X, \nu_X)$$

  as a substitute for the virtual count. This generalizes Donaldson-Thomas type invariants to the case of nonproper moduli space X. It also makes Donaldson-Thomas invariants accessible to arguments involving stratifying the moduli space X. For applications, see [BF08] and [BB07].
- The value of ν_X at the point $P \in X$ should be considered as the contribution of the point P to the virtual count of X.

Some of the fundamental properties of ν_X are:

- At smooth points P of X we have $\nu_X(P) = (-1)^{\dim X}$.
- If $f : X \to Y$ is étale, then $f^*\nu_Y = \nu_X$. Thus $\nu_X(P)$ is an invariant of the singularity of X at the point P.
- Multiplicativity: $\nu_{X\times Y}(P, Q) = \nu_X(P)\nu_Y(Q)$.
- If X is the critical scheme of a regular function f on a smooth scheme M, that is, $X = Z(df)$, then

$$\nu_X(P) = (-1)^{\dim M}\big(1 - \chi(F_P)\big),$$

where F_P is the Milnor fibre, that is, the intersection of a nearby fibre of f with a small ball in M centred at P.

Thus, if X is the Donaldson-Thomas moduli space of stable sheaves, one can, heuristically, think of ν_X as the Euler characteristic of the perverse sheaf of vanishing cycles of the holomorphic Chern-Simons functional.

The existence of a symmetric obstruction theory on X puts strong restrictions on the singularities X may have. For example, reduced local complete intersection singularities are excluded. Thus it is not clear how useful or significant ν_X is on general schemes which do not admit symmetric obstruction theories.

Microlocal geometry. Embed X into a smooth scheme M. Then we have a commutative diagram

$$\begin{tikzcd} Z_*(X) \arrow[r, "\mathrm{Eu}", "\sim"'] \arrow[dr, "c_0^M"'] & \mathrm{Con}(X) \arrow[r, "\mathrm{Ch}", "\sim"'] \arrow[d, "c_0^{SM}"] & \mathfrak{L}_X(\Omega_M) \arrow[dl, "0^!"] \\ & A_0(X) & \end{tikzcd} \tag{2}$$

where the two horizontal arrows are isomorphisms. Here $\mathrm{Eu} : Z_*(X) \to \mathrm{Con}(X)$ is MacPherson's local Euler obstruction [Mac74], which maps algebraic cycles to $\mathbb{Z}$-valued constructible functions and $\mathrm{Ch} : \mathrm{Con}(X) \to \mathfrak{L}_X(\Omega_M)$ maps a constructible function to its characteristic cycle, which is a conic Lagrangian cycle on Ω_M supported inside X. The maps to $A_0(X)$ are the degree zero Chern-Mather class, the degree zero Schwartz-MacPherson Chern class, and the intersection with the zero section, respectively. (Of course, the left part of the diagram exists without the embedding into M.)

Now, given a symmetric obstruction theory on X, the cone of curvilinear obstructions $\mathrm{cv} \hookrightarrow \mathrm{ob} = \Omega_X$, pulls back to a cone in $\Omega_M|_X$ via the epimorphism $\Omega_M|X \to \Omega_X$. Via the embedding $\Omega_M|_X \hookrightarrow \Omega_M$ we obtain a conic subscheme $C \hookrightarrow \Omega_M$, the *obstruction cone* for the embedding $X \hookrightarrow M$. The virtual fundamental class is $[X]^{\mathrm{vir}} = 0^![C]$.

The key fact is that C is *Lagrangian*. Because of this, there exists a unique constructible function ν_X on X such that $\mathrm{Ch}(\nu_X) = [C]$ and $c_0^{SM}(\nu_X) = [X]^{\mathrm{vir}}$. Then (1) follows by a simple application of MacPherson's theorem [Mac74] (or equivalently from the microlocal index theorem of Kashiwara [KS90]).

The cycle $\mathfrak{c}_X$ such that $\mathrm{Eu}(\mathfrak{c}_X) = \nu_X$ is easily written down. It can be thought of as the (signed) support of the intrinsic normal cone of X.

The class $\alpha_X = c^M(\mathfrak{c}_X) = c^{SM}(\nu_X)$, whose degree zero component is the virtual fundamental class of any symmetric obstruction theory on X, was introduced by Aluffi [Alu00] (although with a different sign) and we call it therefore the *Aluffi class* of X.

We do not know if every scheme admitting a symmetric obstruction theory can locally be written as the critical locus of a regular function on a smooth scheme. This limits the usefulness of the above formula for $\nu_X(P)$ in terms of the Milnor fibre. Hence we provide an alternative formula (19), similar in spirit, which always applies.

If $\mathcal{M}$ is a regular holonomic $\mathcal{D}$-module on M whose characteristic cycle is $[C]$, then

$$\nu_X(P) = \sum_i (-1)^i \dim_{\mathbb{C}} H^i_{\{P\}}(X, \mathcal{M}_{\mathrm{DR}}) ,$$

for any point $P \in M$. Here $H^i_{\{P\}}$ denotes cohomology with supports in the subscheme $\{P\} \hookrightarrow M$ and $\mathcal{M}_{\mathrm{DR}}$ denotes the perverse sheaf associated to $\mathcal{M}$ via the Riemann-Hilbert correspondence, as incarnated, for example, by the De Rham complex $\mathcal{M}_{\mathrm{DR}}$. It would be interesting to construct $\mathcal{M}$ or $\mathcal{M}_{\mathrm{DR}}$ in special cases, for example the moduli space of sheaves. Maybe, as opposed to $[C]$ and $[X]^{\mathrm{vir}}$, this more subtle data could actually depend on the symmetric obstruction theory.

Conventions. We will always work over the field of complex numbers $\mathbb{C}$. All schemes and algebraic stacks we consider are of finite type (over $\mathbb{C}$). The relevant facts about algebraic cycles and intersection theory on stacks can be found in [Vis89] and [Kre99].

We will often have to assume that our Deligne-Mumford stacks have the resolution property or are embeddable into smooth stacks. We therefore consider quasiprojective Deligne-Mumford stacks (see [Kre09]):

Definition 0.1 (Kresch). A separated Deligne-Mumford stack X, of finite type over $\mathbb{C}$, with quasiprojective coarse moduli space is called *quasiprojective*, if any of the following equivalent conditions are satisfied:

(i) X has the resolution property, that is, every coherent $\mathcal{O}_X$-module is a quotient of a locally free coherent $\mathcal{O}_X$-module.

(ii) X admits a finite flat cover $Y \to X$, where Y is a quasiprojective scheme.

(iii) X is isomorphic to a quotient stack $[Y/\mathrm{GL}_n]$, for some n, where Y is a quasiprojective scheme with a linear GL_n-action.

(iv) X can be embedded as a locally closed substack into a smooth separated Deligne-Mumford stack of finite type with projective coarse moduli space.

For $\mathbb{Z}$-valued functions f, g on sets X, Y, respectively, we denote by $f \boxdot g$ the function on $X \times Y$ defined by $(f \boxdot g)(x, y) = f(x)g(y)$, for all $(x, y) \in X \times Y$.

We will often use homological notation for complexes. This means that $E_n = E^{-n}$, for a complex $\ldots \to E^i \to E^{i+1} \to \ldots$ in some abelian category.

For a complex of sheaves E, we denote the cohomology sheaves by $h^i(E)$.

Let us recall a few sign conventions: If $E = [E_1 \xrightarrow{\alpha} E_0]$ is a complex concentrated in the interval $[-1, 0]$, then the dual complex $E^\vee = [E_0^\vee \xrightarrow{-\alpha^\vee} E_1^\vee]$ is a complex concentrated in the interval $[0, 1]$. Thus the shifted dual $E^\vee[1]$ is given by $E^\vee[1] = [E_0^\vee \xrightarrow{\alpha^\vee} E_1^\vee]$ and concentrated, again, in the interval $[-1, 0]$.

If $\theta : E \to F$ is a homomorphism of complexes concentrated in the interval $[-1, 0]$, such that $\theta = (\theta_1, \theta_0)$, then the shifted dual $\theta^\vee[1] : F^\vee[1] \to E^\vee[1]$ is given by $\theta^\vee[1] = (\theta_0^\vee, \theta_1^\vee)$.

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1. A few invariants of schemes and stacks

1.1. *The signed support of the intrinsic normal cone* $\mathfrak{c}_X$. Let X be a scheme. Suppose X is embedded as a closed subscheme of a smooth scheme M. Consider the normal cone $C = C_{X/M}$ and its projection $\pi : C \to X$. Define the cycle $\mathfrak{c}_{X/M}$ on X by

$$\mathfrak{c}_{X/M} = \sum_{C'} (-1)^{\dim \pi(C')} \operatorname{mult}(C') \pi(C') .$$

The sum is over all irreducible components C' of C. By $\pi(C')$ we denote the irreducible closed subset (prime cycle) of X obtained as the image of C' under π. Alternatively, we can define $\pi(C')$ as the (set-theoretic) intersection of C' with the zero section of $C \to X$. The multiplicity of the component C' in the fundamental cycle $[C]$ of C is denoted by $\operatorname{mult}(C')$. Hence $\operatorname{mult}(C')$ is the length of C at the generic point of C'. Note that even though $[C] = \sum_{C'} \operatorname{mult}(C')C'$ is an effective cycle of homogeneous degree $\dim M$, the cycle $\mathfrak{c}_{X/M}$ is neither effective nor homogeneous.

PROPOSITION 1.1. *Let X be a Deligne-Mumford stack. There is a unique* (*integral*) *cycle $\mathfrak{c}_X$ on X with the property that for any étale map $U \to X$, and any closed embedding $U \to M$ of U into a smooth scheme M we have*

$$\mathfrak{c}_X|_U = \mathfrak{c}_{U/M} .$$

Proof. Suppose given a commutative diagram of schemes

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & N \arrow[d] \\ X \arrow[r] & M \end{tikzcd}$$

where $Y \to N$ and $X \to M$ are closed embeddings, $Y \to X$ is étale and $M \to N$ is smooth, there is a short exact sequence of cones on Y

$$0 \longrightarrow T_{N/M}|_Y \longrightarrow C_{Y/N} \longrightarrow C_{X/M}|_Y \longrightarrow 0 .$$

This shows that $\mathfrak{c}_{X/M}|_Y = \mathfrak{c}_{Y/N}$.

Comparing any two embeddings of X with the diagonal, we get from this the uniqueness of $\mathfrak{c}_X$ for embeddable X. Then we deduce that $\mathfrak{c}_X$ commutes with étale maps and thus glues with respect to the étale topology. □

If X is smooth, $\mathfrak{c}_X = (-1)^{\dim X}[X]$.

PROPOSITION 1.2. *The basic properties of $\mathfrak{c}_X$ are as follows*:

(i) *If $f : X \to Y$ is a smooth morphism of Deligne-Mumford stacks, then $f^*\mathfrak{c}_Y = (-1)^{\dim X/Y}\mathfrak{c}_X$.*

(ii) *If X and Y are Deligne-Mumford stacks, then $\mathfrak{c}_{X\times Y} = \mathfrak{c}_X \times \mathfrak{c}_Y$.*

Proof. Both of these follow from the product property of normal cones: $C_{X/M} \times C_{Y/N} = C_{X\times Y/M\times N}$. □

Remark 1.3. Maybe it would be appropriate to call $\mathfrak{c}_X$ the *distinguished cycle* of X, in view of its relation to distinguished varieties in intersection theory [Ful84, Def. 6.1.2].

1.2. *The Euler obstruction ν_X of $\mathfrak{c}_X$.* Consider MacPherson's local Euler obstruction $\mathrm{Eu} : Z_*(X) \to \mathrm{Con}(X)$, which maps integral algebraic cycles on X to constructible integer-valued functions on X. Because Eu commutes with étale maps and both Z_* and Con are sheaves with respect to the étale topology, Eu is well-defined for Deligne-Mumford stacks X and defines an isomorphism $Z_*(X) \to \mathrm{Con}(X)$.

If V is a prime cycle of dimension p on the Deligne-Mumford stack X, the constructible function $\mathrm{Eu}(V)$ takes the value

$$\int_{\mu^{-1}(P)} c(\widetilde{T}) \cap s\big(\mu^{-1}(P), \widetilde{V}\big) \tag{3}$$

at the point $P \in X$. Here $\mu : \widetilde{V} \to V$ is the Nash blowup of V (the unique integral closed substack dominating V of the Grassmannian of rank p quotients of Ω_V, or the closure in the Grassmannian of Ω_X of the canonical section over the smooth locus of V). The vector bundle $\widetilde{T}$ is the dual of the universal quotient bundle. Moreover, c is the total Chern class and s the Segre class of the normal cone to a closed immersion. A proof that $\mathrm{Eu}(V)$ is constructible can be found in [Ken90].

Definition 1.4. Let X be a Deligne-Mumford stack. We introduce the canonical integer valued constructible function

$$\nu_X = \mathrm{Eu}(\mathfrak{c}_X)$$

on X.

On a smooth stack X, the function ν_X is locally constant and equals $(-1)^{\dim X}$.

PROPOSITION 1.5. *Let X and Y be Deligne-Mumford stacks.*

(i) *If $f : X \to Y$ is a smooth morphism, then $f^*\nu_Y = (-1)^{\dim X/Y}\nu_X$,*

(ii) $\nu_{X\times Y} = \nu_X \boxdot \nu_Y$.

Proof. Both facts follow from the compatibility of Eu with products. □

Relation with Milnor numbers and vanishing cycles. Suppose that M is a smooth scheme and $f : M \to \mathbb{A}^1$ is a regular function. Let $X = Z(df)$ be the critical locus of f. Then for any $\mathbb{C}$-valued point P of X, we have

$$\nu_X(P) = (-1)^{\dim M}\big(1 - \chi(F_P)\big)\,, \tag{4}$$

where $\chi(F_P)$ is the Euler characteristic of the Milnor fibre of f at P. For the proof, see [PP01, Cor. 2.4 (iii)]. Hence our constructible function ν_X is equal to the function denoted μ in the literature (see, for example, [PP01]).

Let Φ_f be the perverse sheaf of vanishing cycles on X. It is a constructible complex on X and therefore has a fibrewise Euler characteristic

$$\chi(\Phi_f)(P) = \sum_i (-1)^i \dim H^i_{\{P\}}(X, \Phi_f)\,,$$

which is a constructible function on X. As a consequence of (4), we have

$$\nu_X = (-1)^{\dim M - 1} \chi(\Phi_f)\,. \tag{5}$$

1.3. *Weighted Euler characteristics.* The Euler characteristic with compact supports χ is a $\mathbb{Z}$-valued function on isomorphism classes of pairs (X, f), where X is a scheme and f a constructible function on X. It satisfies the properties:

(i) If X is separated and smooth, $\chi(X, 1) = \chi(X)$ is the usual topological Euler characteristic of X.
(ii) $\chi(X, f + g) = \chi(X, f) + \chi(X, g)$.
(iii) If X is the disjoint union of a closed subscheme Z and its open complement U, then $\chi(X, f) = \chi(U, f|_U) + \chi(Z, f|_Z)$.
(iv) $\chi(X \times Y, f \boxdot g) = \chi(X, f)\, \chi(Y, g)$.
(v) If $X \to Y$ is a finite étale morphism of degree d, then $\chi(X, f|_X) = d\, \chi(Y, f)$, for any constructible function f on Y.

These properties suffice to prove that χ extends uniquely to a $\mathbb{Q}$-valued function on pairs (X, f), where X is a Deligne-Mumford stack and f a $\mathbb{Z}$-valued constructible function on X. (Use the fact [LMB00, Cor. 6.1.1] that every Deligne-Mumford stack is generically the quotient of a scheme by a finite group.) Properties (i)–(v) continue to hold. We write $\chi(X)$ for $\chi(X, 1)$. The rational number $\chi(X)$ is often called the *orbifold Euler characteristic* of X.

PROPOSITION 1.6 (Gauß-Bonnet). *If the Deligne-Mumford stack X is smooth and proper*, *we have that*

$$\chi(X) = \int_{[X]} e(T_X)\,,$$

the integral of the Euler class (*top Chern class*) *of the tangent bundle.*

Proof. First one proves that $\chi(I_X) = \chi(\bar{X})$, where I_X is the inertia stack and $\bar{X}$ the coarse moduli space of X. This can be done by passing to stratifications and is therefore not difficult. Then, invoking the Lefschetz trace formula for the identify on a smooth and proper X we get (see [Beh04] for details)

$$\int_{[I_X]} e(T_{I_X}) = \sum_i (-1)^i \dim H^i(X, \mathbb{C}) = \chi(\bar{X})\,.$$

Putting these two remarks together, we get the Gauß-Bonnet theorem for I_X. To prove the theorem for X, note that the part of I_X whose dimension is equal to

$\dim X$ is a closed and open substack Y of I_X, which comes with a finite étale representable morphism $Y \to X$. By induction on the dimension, the theorem holds for Y. Then it also holds for X by Property (v) of the Euler characteristic. □

Note that for stacks, $\chi(X)$ differs from the alternating sum of the dimensions of the compactly supported cohomology groups.

Definition 1.7. Let X be a Deligne-Mumford stack. Introduce the weighted Euler characteristic

$$\tilde{\chi}(X) = \chi(X, \nu_X) = \chi\big(X, \mathrm{Eu}(\mathfrak{c}_X)\big) \in \mathbb{Q}\,.$$

More generally, given a morphism $Z \to X$, define

$$\tilde{\chi}(Z, X) = \chi(Z, \nu_X|_Z)\,.$$

This definition is particularly useful for locally closed substacks $Z \subset X$.

PROPOSITION 1.8. *The weighted Euler characteristic $\tilde{\chi}(Z, X)$ has the basic properties*:

(i) *If X is smooth*, $\tilde{\chi}(Z, X) = (-1)^{\dim X} \chi(Z)$.

(ii) *If $Z \to X$ is smooth*, $\tilde{\chi}(Z, X) = (-1)^{\dim Z/X} \tilde{\chi}(Z)$.

(iii) *If $Z = Z_1 \cup Z_2$ is the disjoint union of two locally closed substacks*, $\tilde{\chi}(Z_1, X) + \tilde{\chi}(Z_2, X) = \tilde{\chi}(Z, X)$.

(iv) $\tilde{\chi}(Z_1 \times Z_2, X_1 \times X_2) = \tilde{\chi}(Z_1, X_1)\, \tilde{\chi}(Z_2, X_2)$.

(v) *Given a commutative diagram*

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array}$$

with $X \to Y$ smooth and $Z \to W$ finite étale, we have that $\tilde{\chi}(Z, X) = (-1)^{\dim X/Y} \deg(Z/W)\, \tilde{\chi}(W, Y)$.

Proof. All these properties follow by combining the properties of ν_X with those of the orbifold Euler characteristic. □

Remark 1.9. Suppose X is the disjoint union of an open substack U and a closed substack Z. We have $\tilde{\chi}(X) = \tilde{\chi}(U) + \tilde{\chi}(Z, X)$. But because, in general, $\tilde{\chi}(Z, X) \neq \tilde{\chi}(Z)$, we also have $\tilde{\chi}(X) \neq \tilde{\chi}(U) + \tilde{\chi}(Z)$.

Remark 1.10. If X is smooth and proper,

$$\tilde{\chi}(X) = \int_{[X]} e(\Omega_X)\,.$$

Both Proposition 1.12 and Theorem 4.18 can be viewed as generalizations of this formula.

1.4. *The Aluffi class.* The Mather class is a homomorphism $c^M : Z_*(X) \to A_*(X)$. It exists for Deligne-Mumford stacks as well as for schemes. The definition is a globalization of the construction of the local Euler obstruction. For a prime cycle V of degree p on X, we have

$$c^M(V) = \mu_*\big(c(\widetilde{T}) \cap [\widetilde{V}]\big)\,,$$

with the same notation as in (3). We will only need to use the degree zero part $c_0^M : Z_*(X) \to A_0(X)$.

Definition 1.11. Applying c^M to our cycle $\mathfrak{c}_X$, we obtain the *Aluffi class*

$$\alpha_X = c^M(\mathfrak{c}_X) \in A_*(X)\,,$$

The class α_X was introduced by Aluffi [Alu00], although one should note that the sign conventions there differ from ours.

If X is smooth, its Aluffi class equals

$$\alpha_X = (-1)^{\dim X} c(T_X) \cap [X]\,.$$

PROPOSITION 1.12. *Let X be a proper Deligne-Mumford stack. The formula (note that only the degree zero component of α_X enters into it)*

$$\tilde{\chi}(X) = \int_X \alpha_X \tag{6}$$

is true in the following cases:

(i) *if X is a global finite group quotient,*

(ii) *if X is a gerbe over a scheme,*

(iii) *if X is smooth.*

Proof. In the smooth case, Formula (6) is the Gauß-Bonnet theorem.

For schemes, the proposition is true by a direct application of MacPherson's theorem, which says that

$$\chi\big(X, \mathrm{Eu}(\mathfrak{c})\big) = \int_X c^M(\mathfrak{c})\,,$$

for any cycle $\mathfrak{c} \in Z_*(X)$.

Let $f : X \to Y$ be a finite étale morphism of Deligne-Mumford stacks, representable or not. Then both the local Euler obstruction and the Chern-Mather class commute with pulling back via f. It follows that $\tilde{\chi}(X) = d\,\tilde{\chi}(Y)$ and $\int_X \alpha_X = d \int_Y \alpha_Y$, where $d \in \mathbb{Q}$ is the degree of f. Thus Formula (6) holds for X if and only if it holds for Y.

If $X = [Y/G]$ is a global quotient of a scheme Y by a finite group G, there is the finite étale morphism $Y \to X$, proving Case (i). If X is a gerbe, the morphism $X \to \overline{X}$ from X to its coarse moduli space is finite étale, proving Case (ii). $\square$

Remark 1.13. Of course, it is very tempting to conjecture Formula (6) to hold true in general.

2. **Remarks on virtual cycle classes**

Let X denote a scheme or a Deligne-Mumford stack. Let L_X be the cotangent complex of X. Recall from [BF97] that a *perfect obstruction theory* for X is a derived category morphism $\phi : E \to L_X$, such that

(i) $E \in D(\mathcal{O}_X)$ is perfect, of perfect amplitude contained in the interval $[-1, 0]$,
(ii) ϕ induces an isomorphism on h^0 and an epimorphism on h^{-1}.

Let us fix a perfect obstruction theory $E \to L_X$ for X.

Recall that E defines a vector bundle stack $\mathfrak{E}$ over X: whenever we write E locally as a complex of vector bundles $E = [E_1 \to E_0]$, the stack $\mathfrak{E}$ becomes the stack quotient $\mathfrak{E} = [E_1^\vee / E_0^\vee]$.

Recall also the *intrinsic normal cone* $\mathfrak{C}_X$. Whenever $U \to X$ is étale and $U \to M$ a closed immersion into a smooth scheme M, the pullback $\mathfrak{C}_X|_U$ is canonically isomorphic to the stack quotient $[C_{U/M}/(T_M|_U)]$, where $C_{U/M}$ is the normal cone. The morphism $E \to L_X$ defines a closed immersion of cone stacks $\mathfrak{C}_X \hookrightarrow \mathfrak{E}$.

Recall, finally, that the obstruction theory $E \to L_X$ defines a *virtual fundamental class* $[X]^{\mathrm{vir}} \in A_{\operatorname{rk} E}(X)$, as the intersection of the fundamental class $[\mathfrak{C}_X]$ with the zero section of $\mathfrak{E}$:

$$[X]^{\mathrm{vir}} = 0^!_{\mathfrak{E}}[\mathfrak{C}_X]\,.$$

(For the last statement in the absence of global resolutions, see [Kre99].) Here $A_r(X)$ denotes the Chow group of r-cycles modulo rational equivalence on X with values in $\mathbb{Z}$.

2.1. *Obstruction cones.*

Definition 2.1. We call $\mathrm{ob} = h^1(E^\vee)$ the *obstruction sheaf* of the obstruction theory $E \to L_X$.

Our goal is to prove that if X is a quasiprojective Deligne-Mumford stack and $\Omega \to \mathrm{ob}$ an epimorphism, where Ω is an arbitrary vector bundle over X, then the obstruction theory gives rise to a cone C inside Ω such that $[X]^{\mathrm{vir}} = 0^!_\Omega[C]$. For the case of schemes, this was already observed by Li and Tian in [LT98].

A *local resolution* of E is a derived category homomorphism $F \to E^\vee[1]|_U$, over some étale open subset U of X, where F is a vector bundle over U and the homomorphism $F \to E^\vee[1]|_U$ is such that its cone is a locally free sheaf over U concentrated in degree -1. Alternatively, a local resolution may be defined as a local presentation $F \to \mathfrak{E}|_U$ (over an étale open U of X) of the vector bundle stack $\mathfrak{E}$ associated to E.

Recall that for every local resolution $F \to E^\vee[1]|_U$ there is an associated cone $C \hookrightarrow F$, the *obstruction cone*, defined via the cartesian diagram of cone stacks over U

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d] \\ \mathfrak{C}|_U \arrow[r] & \mathfrak{E}|_U \end{tikzcd}$$

where $\mathfrak{C}$ is the intrinsic normal cone of X.

Note that every local resolution $F \to E^\vee[1]|_U$ comes with a canonical epimorphism of coherent sheaves $F \to \mathrm{ob}\,|_U$.

PROPOSITION 2.2. *let Ω be a vector bundle on X and $\Omega \to \mathrm{ob}$ an epimorphism of coherent sheaves. Then there exists a unique closed subcone $C \subset \Omega$ such that for every local resolution $F \to E^\vee[1]|_U$, with obstruction cone $C' \subset F$, and every lift ϕ*

$$\begin{tikzcd} & F \arrow[d] \\ \Omega|_U \arrow[ur, "\phi"] \arrow[r] & \mathrm{ob}\,|_U \end{tikzcd} \tag{7}$$

we have that $C|_U = \phi^{-1}(C')$, in the scheme-theoretic sense.

Proof. Étale locally on X, presentations F and lifts ϕ always exist. The uniqueness of C follows.

So far, we have only considered ob as a coherent sheaf on X. We can extend it to a sheaf on the big étale site of X in the canonical way. We may then think of ob as the coarse moduli sheaf of $\mathfrak{E}$. Let cv be the coarse moduli sheaf of the intrinsic normal cone $\mathfrak{C}$. The key facts are

(i) $\mathrm{cv} \hookrightarrow \mathrm{ob}$ is a subsheaf,

(ii) the diagram

$$\begin{tikzcd} \mathfrak{C} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{E} \arrow[d] \\ \mathrm{cv} \arrow[r, hook] & \mathrm{ob} \end{tikzcd} \tag{8}$$

is a cartesian diagram of stacks over X.

Both of these facts are local in the étale topology of X, so we may assume that E has a global resolution $E^\vee = [H \to F]$. Let $C' \subset F$ be the obstruction cone. Then C' is invariant under the action of H on F. Note that ob is the sheaf-theoretic quotient of F by H and cv the sheaf-theoretic quotient of C' by H. Simple sheaf theory on the big étale site of X (exactness of the associated sheaf functor) implies

(i) $\mathrm{cv} \hookrightarrow \mathrm{ob}$ is a subsheaf,

(ii) the diagram

(9)
$$\begin{tikzcd} C' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d] \\ \mathrm{cv} \arrow[r, hook] & \mathrm{ob} \end{tikzcd}$$

is a cartesian diagram of sheaves on the big étale site of X. This implies that Diagram (8) is cartesian, proving the key facts.

We now construct the subsheaf $C \subset \Omega$ as the fibred product of sheaves on the big étale site of X

(10)
$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Omega \arrow[d] \\ \mathrm{cv} \arrow[r] & \mathrm{ob}\,. \end{tikzcd}$$

Then any diagram such as (7) gives rise to a cartesian diagram of big étale sheaves

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Omega \arrow[d, "\phi"] \\ C' \arrow[r] & F. \end{tikzcd}$$

This latter diagram is cartesian, because Diagrams (9) and (10) are. This proves the claimed property of C, as well as the fact that C is a closed subcone of Ω, in the scheme-theoretic sense. $\square$

Definition 2.3. We call $C \subset \Omega$ the *obstruction cone* associated to the epimorphism $\Omega \to \mathrm{ob}$.

Remark 2.4. In [BF97], it was shown that the subsheaf $\mathrm{cv} \hookrightarrow \mathrm{ob}$ classifies small curvilinear obstructions. Note that ob is in general bigger than the actual sheaf of obstructions, which is the abelian subsheaf of ob generated by cv. Thus cv is intrinsic to X, whereas ob depends on $E \to L_X$.

If X is smooth, then ob is a vector bundle and $\mathrm{cv} = X$, so the obstruction cone is the kernel of $\Omega \to \mathrm{ob}$.

2.2. *The virtual fundamental class.*

LEMMA 2.5. *If X is a quasiprojective Deligne-Mumford stack, every perfect obstruction theory $E \to L_X$ has a global resolution.*

Proof. Let $D(\mathscr{O}_X)$ be the derived category of sheaves of $\mathscr{O}_X$-modules on the (small) étale site of X and let $D(\text{Qcoh-}\mathscr{O}_X)$ be the derived category of the category of quasicoherent $\mathscr{O}_X$-modules. First prove that the natural functor $D^+(\text{Qcoh-}\mathscr{O}_X)\to D^+_{\mathrm{qcoh}}(\mathscr{O}_X)$ is an equivalence of categories. For this, show that the inclusion of categories $(\text{Qcoh-}\mathscr{O}_X) \to (\mathscr{O}_X\text{-mods})$ has a right adjoint Q, which commutes with pushforward along morphisms of quasiprojective stacks. Prove that quasicoherent sheaves are acyclic for Q and satisfy that $QF \to F$ is an isomorphism. Thus the right derivation of Q provides a quasi-inverse to the inclusion. To reduce all these claims to the affine case use a groupoid $U_1 \rightrightarrows U_0$ presenting X, which is étale and has affine U_1, U_0. For the details of the proof, see Section 3 of Exposé II in SGA6.

Next, prove that every quasicoherent sheaf on X is a direct limit of coherent sheaves. For this, it is convenient to choose a finite flat cover $f : Y \to X$, where Y is a quasiprojective scheme. Construct a right adjoint $f^!$ to f_*, from (Qcoh-$\mathcal{O}_X$) to (Qcoh-$\mathcal{O}_Y$). (This can be done étale locally over X.) Now, let F be a quasicoherent $\mathcal{O}_X$-module. There exist coherent sheaves G_i on Y, such that $f^!F = \lim_{\to} G_i$, because Y is quasiprojective. Since it admits a right adjoint, f_* commutes with direct limits and so we have $f_*f^!F = \lim_{\to} f_*G_i$. The trace map $f_*f^!F \to F$ is onto, and so we get a surjection $\lim_{\to} f_*G_i \to F$. Since the f_*G_i are coherent, F is, indeed, an inductive limit of coherent modules.

With these preparations, we can now construct a global resolution of the perfect complex $E \in D^b_{\mathrm{coh}}(\mathcal{O}_X)$. First, we may assume that E is given by a 2-term complex $E = [E_1 \to E_0]$, where E_0 and E_1 are quasicoherent. (Our above argument gives an infinite complex of quasicoherents, which we may cut off, because the kernel of a morphism between quasicoherent sheaves is quasicoherent.) Then we choose coherent sheaves G_i on X such that $E_0 = \lim_{\to} G_i$. The images of the G_i in $h^0(E) = \mathrm{cok}(E_1 \to E_0)$ stabilize, because X is noetherian and hence the coherent $\mathcal{O}_X$-module $h^0(E)$ satisfies the ascending chain condition. So there exists a coherent sheaf $G_i \to E_0$, which maps surjectively onto $h^0(E)$. Now find a locally free coherent F_0 mapping onto G_i (and hence onto $h^0(E)$), and define $F_1 = E_1 \times_{E_0} F_0$. Then F_1 is automatically locally free coherent and $F = [F_1 \to F_0]$ maps quasi-isomorphically to $[E_1 \to E_0]$. Thus F provides us with the required global resolution of E. $\square$

PROPOSITION 2.6. *Consider a quasiprojective Deligne-Mumford stack X and a perfect obstruction theory $E \to L_X$ with obstruction sheaf* ob. *Let Ω be a vector bundle over X and $\Omega \to$* ob *an epimorphism of coherent sheaves. Let $C \subset \Omega$ be the associated obstruction cone. Then C is of pure dimension* $\mathrm{rk}\, E + \mathrm{rk}\, \Omega$ *and we have*

$$[X]^{\mathrm{vir}} = 0^!_{\Omega}[C]\,.$$

Proof. Let $F \to E^{\vee}[1]$ be a global resolution of E with obstruction cone $C' \subset F$. Start by constructing the fibred product of coherent sheaves

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d] \\ \Omega \arrow[r] & \mathrm{ob} \end{tikzcd} \tag{11}$$

and choosing an epimorphism of coherent sheaves $F' \to \mathcal{P}$, where F' is locally free. Wet get a commutative diagram of sheaf epimorphisms

$$\begin{tikzcd} F' \arrow[r, two heads] \arrow[d, two heads] & F \arrow[d, two heads] \\ \Omega \arrow[r, two heads] & \mathrm{ob} \end{tikzcd}$$

which we can now consider as a diagram of sheaves on the big étale site of X.

Remark. Diagram (11) is a cartesian diagram of sheaves on the small étale site of X. This is because fibred products of (small) coherent sheaves do not commute with base change, and so if we had taken the fibred product of big sheaves, $\mathcal{P}$ would not have ended up coherent. After having chosen F', we do not any longer have use for the cartesian property of the diagram, and so we pass back to big sheaves, as commutativity of diagrams and the property of being an epimorphism are stable under base change.

Now, of course, $F' \to \Omega$ and $F' \to F$ are epimorphisms of vector bundles. The preimage of $\mathrm{cv} \hookrightarrow \mathrm{ob}$ in Ω is C, and in F is C'. It follows that C' and C have the same preimage in F'. This implies by standard arguments the claim about the dimension of C and the fact that $[C'] \cap [O_F] = [C] \cap [O_\Omega]$. □

3. **Symmetric obstruction theories**

We will summarize the main features of symmetric obstruction theories. For proofs, see [BF08]. Throughout this section, X will denote a Deligne-Mumford stack.

3.1. *Nondegenerate symmetric bilinear forms.*

Definition 3.1. Let $E \in D^b_{\mathrm{coh}}(\mathcal{O}_X)$ be a perfect complex. A *nondegenerate symmetric bilinear form of degree 1* on E is an isomorphism $\theta : E \to E^\vee[1]$, satisfying $\theta^\vee[1] = \theta$.

Of course, it has to be understood that θ is a morphism in the derived category, and invertible as such. The duals appearing in the definition are derived.

Example 3.2. A simple example of a perfect complex with nondegenerate symmetric bilinear form of degree 1 is given as follows. Let F be a vector bundle on X, endowed with a symmetric bilinear form, inducing a homomorphism $\alpha : F \to F^\vee$. To define the complex $E = [F \to F^\vee]$, put $F^\vee$ in degree 0 and F in degree -1. Since the components of E are locally free, we can compute the derived dual as componentwise dual. We find $E^\vee[1] = E$. So we may and will define θ to be the identity, that is, $\theta_1 = \mathrm{id}_F$ and $\theta_0 = \mathrm{id}_{F^\vee}$:

$$\begin{tikzcd} E \arrow[d, "\theta"] & = & {[F} \arrow[r, "\alpha"] \arrow[d, "1"'] & {F^\vee]} \arrow[d, "1"] \\ E^\vee[1] & = & {[F} \arrow[r, "\alpha"] & {F^\vee].} \end{tikzcd}$$

Note that θ is an isomorphism, whether or not α is nondegenerate.

Example 3.3. As a special case of Example 3.2, consider a regular function f on a smooth variety M. The Hessian of f defines a symmetric bilinear form on $T_M|_X$, where $X = Z(df)$. Hence we get a nondegenerate symmetric bilinear form on the complex $[T_M|_X \to \Omega_M|_X]$, which is, by the way, a perfect obstruction theory for X.

Definition 3.4. Let A and B be perfect complexes endowed with nondegenerate symmetric forms $\theta : A \to A^\vee[1]$ and $\eta : B \to B^\vee[1]$. An *isometry* $\Phi : (B, \eta) \to (A, \theta)$ is an isomorphism $\Phi : B \to A$, such that the diagram

$$\begin{tikzcd} B \arrow[r, "\Phi"] \arrow[d, "\eta"'] & A \arrow[d, "\theta"] \\ B^\vee[1] & A^\vee[1] \arrow[l, "\Phi^\vee[1]"'] \end{tikzcd}$$

commutes in $D(\mathcal{O}_X)$. Since η and θ are isomorphisms, this amounts to saying that $\Phi^{-1} = \Phi^\vee[1]$ (if we use η and θ to identify).

3.2. *Symmetric obstruction theories.*

Definition 3.5. A perfect obstruction theory $E \to L_X$ for X is called *symmetric*, if E is endowed with a nondegenerate symmetric bilinear form $\theta : E \to E^\vee[1]$.

If E is symmetric, we have

$$\operatorname{rk} E = \operatorname{rk}(E^\vee[1]) = -\operatorname{rk} E^\vee = -\operatorname{rk} E$$

and hence $\operatorname{rk} E = 0$. So the expected dimension is zero. Therefore, we can make the following definition:

Definition 3.6. Let X be endowed with a symmetric obstruction theory and assume that X is proper. The *virtual count* (or *Donaldson-Thomas type invariant*) of X is the number

$$\#^{\mathrm{vir}}(X) = \deg[X]^{\mathrm{vir}} = \int_{[X]^{\mathrm{vir}}} 1\,.$$

If X is a scheme (or an algebraic space), $\#(X)$ is an integer, otherwise a rational number.

Remark 3.7. For a symmetric obstruction theory $E \to L_X$, we have ob $= h^1(E^\vee) = h^0(E^\vee[1]) = h^0(E) = \Omega_X$. So the obstruction sheaf is canonically isomorphic to the sheaf of differentials.

Remark 3.8. Let X be endowed with a symmetric obstruction theory. Then for any closed embedding $X \to M$ into a smooth Deligne-Mumford stack M, we get a canonical epimorphism of coherent sheaves $\Omega_M|_X \to \Omega_X =$ ob, and hence a canonical closed subcone $C \hookrightarrow \Omega_M|_X$, the obstruction cone of Definition 2.3. (If X is smooth, C is the conormal bundle of X in M.) Via the inclusion $\Omega_M|_X \hookrightarrow \Omega_M$ we think of C as a closed conic substack of Ω_M. If X is quasiprojective, Proposition 2.6 applies and so C is pure dimensional and

$$\dim C = \dim M = \tfrac{1}{2} \dim \Omega_M.$$

We will show below that C is *Lagrangian*.

Remark 3.9. Any symmetric obstruction theory on X induces (by restriction) in a canonical way a symmetric obstruction theory on $U \to X$, for every étale morphism $U \to X$.

Remark 3.10. If E is a symmetric obstruction theory for X and F a symmetric obstruction theory for Y, then $E \boxplus F$ (see [BF97]) is naturally a symmetric obstruction theory for $X \times Y$.

3.3. *Examples.* For proofs of the following statements, see [BF08].

Lagrangian intersections. Let M be a complex symplectic manifold and V, W two Lagrangian submanifolds. Let X be their scheme-theoretic intersection. Then X carries a canonical symmetric obstruction theory. This generalizes to Deligne-Mumford stacks.

Sheaves on Calabi-Yau threefolds. Let Y be a smooth projective Calabi-Yau threefold and L a line bundle on Y. Let X be any open substack of the stack of stable sheaves of positive rank r with determinant L. (For example, X could be the stack of sheaves of a fixed Hilbert polynomial admitting no strictly semistable sheaves. Then X would be proper.)

Let $\mathscr{E}$ be the universal sheaf and $\mathscr{F}$ the shifted cone of the trace map:

$$\begin{tikzcd} & \mathscr{O} \arrow[dl, "+1"'] & \\ \mathscr{F} \arrow[rr] & & R\mathrm{Hom}(\mathscr{E}, \mathscr{E}) \arrow[ul, "tr"'] \end{tikzcd}$$

Then $R\pi_* \mathscr{F}[2]$ is a symmetric obstruction theory for X. Here $\pi : Y \times X \to X$ is the projection. For the proof, see [Tho00] or [BF08].

Note that X is a μ_r-gerbe over a quasiprojective scheme. Moreover, X is a quasiprojective stack.

Hilbert schemes of local Calabi-Yau threefolds. If we restrict to rank one sheaves, we can consider the following more general situation. Let Y be a smooth projective threefold with a section of the anticanonical bundle whose zero locus we denote by D. Recall that stable sheaves with trivial determinant can be considered as ideal sheaves on Y.

Let X be an open subscheme of the Hilbert scheme of ideal sheaves on Y. We require that X consists entirely of ideal sheaves whose associated subschemes of Y are disjoint from D. Then, with the same notation as above, $R\pi_*\mathscr{F}[2]$ is a symmetric obstruction theory for X. Note that X is a quasiprojective scheme.

Stable maps. Let Y be a Calabi-Yau threefold. Let X be the open substack of the stack of stable maps $\overline{M}_{g,n}(Y,\beta)$, corresponding to stable maps which are immersions from a smooth curve to Y. Then the Gromov-Witten obstruction theory for $\overline{M}_{g,n}(Y,\beta)$ is symmetric over X.

3.4. *Local structure*: *almost closed* 1*-forms.*

Definition 3.11. A differential form ω on a smooth Deligne-Mumford stack M is called *almost closed*, if $d\omega \in I\Omega^2_M$. Here I is the ideal sheaf of the zero locus of ω (in other words the image of $\omega^\vee : T_M \to \mathscr{O}_M$). Equivalently, we may say that $d\omega|_X = 0$ as a section of $\Omega^2_M|_X$, where X is the zero locus of ω, that is, $\mathscr{O}_X = \mathscr{O}_M/I$.

Of course, in local coordinates $x_1, \dots, x_n$, where $\omega = \sum_i f_i\, dx_i$, being almost closed means that

$$\frac{\partial f_i}{\partial x_j} \equiv \frac{\partial f_j}{\partial x_i} \mod (f_1, \dots, f_n)\,,$$

for all $i, j = 1, \dots, n$.

Remark 3.12. Let M be a smooth Deligne-Mumford stack and ω an almost closed 1-form on M with zero locus $X = Z(\omega)$. It is a general principle, that a section of a vector bundle defines a perfect obstruction theory for the zero locus of the section. In our case, this obstruction theory is given by

$$\begin{tikzcd} E \arrow[d] & = & {[T_M|_X} \arrow[r, "\nabla\omega"] \arrow[d, "\omega^\vee"'] & {\Omega_M|_X]} \arrow[d, "1"] \\ \tau_{\geq -1} L_X & = & {[I/I^2} \arrow[r, "d"] & {\Omega_M|_X].} \end{tikzcd}$$

Here $\nabla\omega$ is the composition $d \circ \omega^\vee$ of the other maps in the diagram and we have only displayed the cutoff at -1 of $E \to L_X$, as that is the only part of the obstruction theory that intervenes in our discussion.

This obstruction theory is symmetric, in a canonical way, because under our assumption that ω is almost closed we have that $\nabla\omega$ is self-dual, as a homomorphism of vector bundles over X. (See Example 3.2.) We denote this symmetric obstruction theory by $H(\omega) \to L_X$, where

$$H(\omega) = [T_M|_X \xrightarrow{\nabla\omega} \Omega_M|_X]\,.$$

We will show that, at least locally, every symmetric obstruction theory is given in this way by an almost closed 1-form.

PROPOSITION 3.13. *Suppose $E \to L_X$ is a symmetric obstruction theory for the Deligne-Mumford stack X. Then étale locally in X (Zariski locally if X is a scheme) there exists a closed immersion $X \hookrightarrow M$ of X into a smooth scheme M and an almost closed 1-form ω on M and an isometry $E \to H(\omega)$ such that the diagram*

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & H(\omega) \arrow[dl] \\ & L_X & \end{tikzcd}$$

commutes in the derived category of X.

Proof. Let P be a $\mathbb{C}$-valued point of X. By passing to an étale neighborhood of P, we may assume given a closed immersion $X \hookrightarrow M$ into a smooth scheme M of dimension $\dim M = \dim \Omega_X|_P$. Moreover, we may assume that $E = [E_1 \to E_0]$ is given by a homomorphism of vector bundles such that $\operatorname{rk} E_0 = \operatorname{rk} E_1 = \dim M$ and $E \to L_X$ is given by a homomorphism of complexes

$$\begin{tikzcd} E \arrow[d] & = & {[E_1} \arrow[r, "\alpha"] \arrow[d, "\phi_1"'] & {E_0]} \arrow[d, "\phi_0"] \\ \tau_{\geq -1} L_X & = & {[I/I^2} \arrow[r] & {\Omega_M|_X].} \end{tikzcd}$$

Since ϕ_0 is an isomorphism at P, by passing to a smaller neighborhood of P, we may assume that ϕ_0 is an isomorphism and use it to identify E_0 with $\Omega_M|_X$.

For the symmetric form $\theta : E \to E^\vee[1]$ let us use notation $\theta = (\theta_1, \theta_0)$. Then the equality of derived category morphisms $\theta^\vee[1] = \theta$ implies that, locally, $\theta^\vee[1] = (\theta_0^\vee, \theta_1^\vee)$ and $\theta = (\theta_1, \theta_0)$ are homotopic. So let $h : E_0 \to E_0^\vee$ be a homotopy:

$$h\alpha = \theta_1 - \theta_0^\vee$$
$$\alpha^\vee h = \theta_0 - \theta_1^\vee\,.$$

Now define

$$\lambda_0 = \tfrac{1}{2}(\theta_0 + \theta_1^\vee)$$
$$\lambda_1 = \tfrac{1}{2}(\theta_1 + \theta_0^\vee)\,.$$

One checks that (λ_1, λ_0) is a homomorphism of complexes, and as such, homotopic to (θ_1, θ_0). Thus (λ_1, λ_0) represents the derived category morphism θ, and has the property that $\lambda_1 = \lambda_0^\vee$:

$$\begin{tikzcd}
E \arrow[d, "\theta"'] & = & {[E_1} \arrow[r, "\alpha"] \arrow[d, "\lambda^\vee"'] & {\Omega_M|_X]} \arrow[d, "\lambda"] \\
E^\vee[1] & = & {[T_M|_X} \arrow[r, "\alpha^\vee"] & {E_1^\vee].}
\end{tikzcd}$$

Since θ is a quasi-isomorphism, λ is necessarily an isomorphism at P, hence, without loss of generality, an isomorphism. Use λ to identify. Then we have written our obstruction theory as

$$\begin{tikzcd}
E \arrow[d] & = & {[T_M|_X} \arrow[r, "\alpha"] \arrow[d, "\phi_1"'] & {\Omega_M|_X]} \arrow[d, "1"] \\
\tau_{\geq -1} L_X & = & {[I/I^2} \arrow[r] & {\Omega_M|_X]}
\end{tikzcd}$$

with $\alpha = \alpha^\vee$. Lift $\phi_1 : T_M|_X \to I/I^2$ in an arbitrary fashion to a homomorphism $\omega^\vee : T_M \to I$, defining an almost closed 1-form ω such that $E = H(\omega)$, completing the proof. $\square$

We need a slight amplification of this proposition:

COROLLARY 3.14. *Let E be a symmetric obstruction theory for X and let $X \hookrightarrow M'$ be an embedding into a smooth Deligne-Mumford stack M'. Then étale locally in M', there exists an almost closed* 1*-form ω on M', such that $X = Z(\omega)$ and $E \to L_X$ is isometric to $H(\omega) \to L_X$.*

Proof. Let $P \in X$. The proof of Proposition 3.13 actually gives M is an étale slice though P in M'. Then write M' locally as a product of the slice with a complement to the slice. $\square$

4. **Microlocal geometry**

4.1. *Conic Lagrangians inside Ω_M.* Let M be a smooth scheme. The cotangent bundle Ω_M carries the *tautological* 1-form $\alpha \in \Omega^1(\Omega_M)$. It is the image of the identity under $\pi^*\Omega^1_M \to \Omega^1_{\Omega_M}$, the pullback map for 1-forms under the projection $\pi : \Omega_M \to M$. Its differential $d\alpha$ defines the tautological symplectic structure on Ω_M.

Let θ be the vector field on Ω_M which generates the $\mathbb{C}^*$-action on the fibres of Ω_M. It is the image of the identity under $\pi^*\Omega_M \to T_{\Omega_M}$, the map which identifies elements of the vector bundle Ω_M with vertical tangent vectors for the projection π.

The basic relation between these tensors is

$$\alpha = d\alpha(\theta, \cdot)\,.$$

Any local étale coordinate system $x_1, \ldots, x_n$ on M induces the *canonical* coordinate system $x_1, \ldots, x_n, p_1, \ldots, p_n$ on Ω_M. In such canonical coordinates we have $\alpha = \sum_i p_i\, dx_i$ and $\theta = \sum_i p_i(\partial/\partial p_i)$.

Consider an irreducible closed subset $C \subset \Omega_M$. We call C *conic*, if θ is tangent to C at the generic point of C. We call C *Lagrangian*, if $\dim C = \dim M$ and $d\alpha$ vanishes when restricted to the generic point of C.

LEMMA 4.1. *The irreducible closed subset $C \subset \Omega_M$ is conic and Lagrangian if and only if* $\dim C = \dim M$ *and α vanishes when restricted to the generic point of C.*

Proof. Suppose C is Lagrangian. The basic relation shows that $\alpha|_C$ vanishes at smooth points of C if and only if $\theta \in T_C^{\perp} = T_C$ at such points. □

If $V \subset M$ is an irreducible closed subset, the closure in Ω_M of the conormal bundle to any smooth dense open subset of V is conic Lagrangian. This already describes all conic Lagrangians:

LEMMA 4.2. *Let $C \subset \Omega_M$ be a closed irreducible subset. Let $V = \pi(C)$ be its image in M and let $N \subset \Omega_M$ be the closure of the conormal bundle of any smooth dense open subset of V.*

If C is conic and Lagrangian then it is equal to N.

Proof. (See also [Ken90], for a coordinate free proof.) Choose local coordinates $x_1, \ldots, x_n$ for M around a smooth point of V, in such a way that V is cut out by the equations $x_1 = \ldots, x_k = 0$. Then $dx_{k+1} \ldots, dx_n$ are linearly independent at the generic point of V. By generic smoothness of the projection $C \to V$, these forms stay linearly independent at the generic point of C. Since α restricts to $\sum_{i=k+1}^{n} p_i\, dx_i$ at the generic point of C, and α vanishes there, we see that $p_{k+1}, \ldots, p_n$ vanish at the generic point of C. Thus $x_1, \ldots, x_k, p_{k+1}, \ldots p_k$ vanish at the generic point of C.

On the other hand, N is cut out generically by $x_1, \ldots, x_k, p_{k+1}, \ldots p_k$. Thus we have proved that the generic point of C is contained in N. Then $C = N$ for dimension reasons. □

Definition 4.3. A closed subset of Ω_M is called *conic Lagrangian*, if every one of its irreducible components is conic and Lagrangian.

An algebraic cycle on Ω_M is *conic Lagrangian* if its support is conic Lagrangian.

A conic closed subscheme of Ω_M, that is, a closed subscheme of Ω_M which is a cone over a closed subscheme of M, is *conic Lagrangian* if its underlying closed subset is conic Lagrangian.

Remark 4.4. The property of being a conic Lagrangian is local in the étale topology of M, so it makes sense also in the case when M is a smooth Deligne-Mumford stack.

Cycles. Consider a smooth Deligne-Mumford stack M of dimension n. Let $\mathfrak{L}(\Omega_M)\subset Z_n(\Omega_M)$ be the subgroup generated by the conic Lagrangian prime cycles.

If V is a prime cycle (integral closed substack) of M, we consider the closure in Ω_M of the conormal bundle of any smooth dense open subset of V and denote it by $\ell(V)$. Note that $\ell(V)$ is a conic Lagrangian prime cycle on Ω_M. This defines the homomorphism

$$\begin{aligned} L : Z_*(M) &\longrightarrow \mathfrak{L}(\Omega_M) \\ V &\longmapsto (-1)^{\dim V}\ell(V)\,. \end{aligned} \tag{12}$$

Conversely, if W is a conic prime cycle on Ω_M, intersecting (set-theoretically) with the zero section of $\pi : \Omega_M \to M$ or taking the (set-theoretic) image $\pi(W)$, we obtain the same prime divisor in M. Restricting to conic Lagrangian cycles, we obtain a homomorphism

$$\begin{aligned} \pi : \mathfrak{L}(\Omega_M) &\longrightarrow Z_*(M) \\ W &\longmapsto (-1)^{\dim \pi(W)}\pi(W)\,. \end{aligned} \tag{13}$$

By Lemma 4.2 the homomorphisms L and π between $Z_*(M)$ and $\mathfrak{L}(\Omega_M)$ are inverses of each other.

Remark 4.5. The characteristic cycle map $\mathrm{Ch} : \mathrm{Con}(X) \to \mathfrak{L}(\Omega_M)$ is the unique homomorphism making the diagram

$$\begin{tikzcd} Z_*(M) \arrow[rr, "L"] \arrow[dr, "\mathrm{Eu}"'] & & \mathfrak{L}(\Omega_M) \\ & \mathrm{Con}(M) \arrow[ur, "\mathrm{Ch}"'] & \end{tikzcd}$$

commute.

Microlocal view of the Mather class.

PROPOSITION 4.6. *Let M be a smooth Deligne-Mumford stack. The diagram*

$$\begin{tikzcd} Z_*(M) \arrow[r, "L"] \arrow[dr, "c_0^M"'] & \mathfrak{L}(\Omega_M) \arrow[d, "0^!"] \\ & A_0(M) \end{tikzcd}$$

commutes.

Proof. (A proof in the case of schemes can also be deduced by combining (1.2.1) in [Sab85] with Example 4.1.8 of [Ful84].) Assume $V \subset M$ is a prime cycle of dimension p. Let $\mu : \widetilde{M} \to M$ be the Grassmannian of rank-p quotients of Ω_M and $\nu : \widetilde{V} \to V$ the closure inside $\widetilde{M}$ of the canonical rational section $V \dashrightarrow \widetilde{M}$. Then $c_0^M(V) = (-1)^p \nu_*\big(c_p(Q) \cap [\widetilde{V}]\big)$, where Q is the universal quotient bundle on $\widetilde{M}$.

Let us denote the kernel of the universal quotient map by N. Then on $\widetilde{V}$ we have the exact sequence of vector bundles

$$0 \longrightarrow N|_{\widetilde{V}} \longrightarrow \mu^*\Omega_M|_{\widetilde{V}} \longrightarrow Q|_{\widetilde{V}} \longrightarrow 0\,.$$

It implies that $c_p(Q) \cap [\widetilde{V}] = 0^!_{\Omega_M}[N|_{\widetilde{V}}] \in A_0(\widetilde{V})$.

$$\begin{tikzcd}
N|_{\widetilde{V}} \arrow[r] \arrow[d] & \mu^*\Omega_M \arrow[r] \arrow[d] & \Omega_M \arrow[d] & C \arrow[l] \arrow[d] \\
\widetilde{V} \arrow[r] & \widetilde{M} \arrow[r, "\mu"] & M & V \arrow[l]
\end{tikzcd}$$

Let $C = \ell(V) \subset \Omega_M$ be the conic Lagrangian prime cycle defined by V. There is a canonical rational section $C \dashrightarrow \mu^*\Omega_M$ and the closure of the image is equal to $N|_{\widetilde{V}}$. Hence we have a projection map $\eta : N|_{\widetilde{V}} \to C$ which is a proper birational map of integral stacks. It fits into the cartesian diagram:

$$\begin{tikzcd}
\widetilde{V} \arrow[r, "\nu"] \arrow[d, "0"'] & V \arrow[r] \arrow[d, "0"'] & M \arrow[d, "0"] \\
N|_{\widetilde{V}} \arrow[r, "\eta"] & C \arrow[r] & \Omega_M .
\end{tikzcd}$$

Since refined Gysin homomorphisms (see Section 6.2 in [Ful84]) commute with proper pushforward, we have

$$\nu_*(0^!_{\Omega_M}[N|_{\widetilde{V}}]) = 0^!_{\Omega_M}\eta_*[N|_{\widetilde{V}}] = 0^!_{\Omega_M}[C]$$

and

$$c_0^M(V) = (-1)^p 0^!_{\Omega_M}[C] = (-1)^p 0^!_{\Omega_M}\ell(V) = 0^!_{\Omega_M}L(V). \qquad \square$$

COROLLARY 4.7. *If X is a closed substack of M, the diagram*

$$\begin{tikzcd}
Z_*(X) \arrow[r, "L"] \arrow[dr, "c_0^M"'] & \mathfrak{L}_X(\Omega_M) \arrow[d, "0^!_{\Omega_M}"] \\
 & A_0(X)
\end{tikzcd}$$

commutes as well. Here $\mathfrak{L}_X(\Omega_M)$ denotes the subgroup of conic Lagrangian cycles lying over cycles contained in X.

Proof. We just have to remark that the Mather class computed inside M agrees with the Mather class computed inside X. □

Remark 4.8. This proves the existence of Diagram (2).

4.2. *The fundamental lemma on almost closed* 1*-forms.* Let M denote a smooth scheme. Let ω be a 1-form on M and $X = Z(\omega)$ its scheme-theoretic zero-locus. Considering ω as a linear homomorphism $T_M \to \mathcal{O}_M$, its image $I \subset \mathcal{O}_M$ is the ideal sheaf of X. The epimorphism $\omega^\vee : T_M \to I$ restricts to an epimorphism $\omega^\vee : T_M|_X \to I/I^2$, which gives rise to a closed immersion of cones $C_{X/M} \hookrightarrow \Omega_M|_X$. Via $\Omega_M|_X \hookrightarrow \Omega_M$ we consider $C = C_{X/M}$ as a subscheme of Ω_M.

THEOREM 4.9. *If the* 1*-form* ω *is almost closed, the closed subscheme* $C \subset \Omega_M$ *it defines is conic Lagrangian.*

The proof will follow after an example.

Example 4.10. The case where the zero locus X of ω is smooth is easy: if ω is almost closed with smooth zero locus, $C \subset \Omega_M$ is equal to the conormal bundle $N^\vee_{X/M} \subset \Omega_M$ and is hence conic Lagrangian.

For the general case, this implies that all components of C which lie over smooth points of X are conic Lagrangian.

The proof of Theorem 4.9. We start with two lemmas.

LEMMA 4.11. *Let* B *be an integral noetherian* $\mathbb{C}$*-algebra,* $f \in B$ *nonzero and* $Q : B \to K$ *a morphism to a field, such that* $Q(f) = 0$. *Then there exists a field extension* L/K, *a morphism* $\gamma : B \to L[[t]]$ *and an integer* $m > 0$, *such that*

$$\begin{tikzcd} B \arrow[r, "Q"] \arrow[d, "\gamma"'] & K \arrow[d] \\ L[[t]] \arrow[r, "t=0"] & L \end{tikzcd}$$

commutes and $\gamma(f) = t^m$.

Proof. Without loss of generality, B is local with maximal ideal $\ker Q$. Then we can find a discrete valuation ring A inside the quotient field of B which dominates B. Pass to its completion $\hat{A}$. The image of f in $\hat{A}$ is of the form ut^m, for a unique $m > 0$ and unit u, parameter t for $\hat{A}$. In a suitable extension $\tilde{A}$ of $\hat{A}$, we can find an m-th root of u and change the parameter such that we have that f maps to t^m in $\tilde{A}$. Choosing a field of representatives L' for $\tilde{A}$ we get an isomorphism $\tilde{A} \cong L'[[t]]$ and hence a morphism $\gamma' : B \to L'[[t]]$ satisfying the requirements of the lemma with the residue field of B in place of K. Passing to a common extension L of K and L' over this residue field, we obtain γ. □

LEMMA 4.12. *Let A be an integral noetherian $\mathbb{C}$-algebra and $I \leq A$ an ideal. Let $Q : \bigoplus_{i\geq 0} I^i/I^{i+1} \to K$ be a morphism to a field, which does not vanish identically on the augmentation ideal. Then there exists a field extension L/K, a morphism $\gamma : A \to L[[t]]$ and an integer $m > 0$ such that the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\gamma"'] & A/I \arrow[r, "Q"] & K \arrow[d] \\ L[[t]] \arrow[rr] & & L \end{tikzcd}$$

commutes and

$$Q(f^{(1)}) = \frac{\gamma(f)}{t^m}\Big|_{t=0},$$

for every $f \in I$. Here $f^{(1)}$ denotes the element $f \in I$ considered as an element of the first graded piece of $\bigoplus_{i\geq 0} I^i$.

Proof. Choose $g \in I$ such that Q does not vanish on $g^{(1)}$. Apply Lemma 4.11 to the localization of $\bigoplus_{i\geq 0} I^i$ at the element $g^{(1)}$, the nonzero element $g^{(0)}/g^{(1)}$ and the induced ring morphism to K. We obtain a commutative diagram

$$\begin{tikzcd} \bigoplus_{i\geq 0} I^i \arrow[r, "Q"] \arrow[d, "\tilde{\gamma}"'] & K \arrow[d] \\ L[[t]] \arrow[r] & L \end{tikzcd}$$

and an integer $m > 0$ with the property that $\tilde{\gamma}(g^{(0)}) = t^m \tilde{\gamma}(g^{(1)})$. By restricting $\tilde{\gamma}$ to the degree zero part of $\bigoplus_{i\geq 0} I^i$, we obtain $\gamma : A \to L[[t]]$. Now, for any element $f \in I$ we have the equation $f^{(0)} g^{(1)} = g^{(0)} f^{(1)}$ inside $\bigoplus_{i\geq 0} I^i$. Apply $\tilde{\gamma}$ and cancel out the unit $\tilde{\gamma}(g^{(1)})$ to obtain $\gamma(f) = t^m \tilde{\gamma}(f^{(1)})$. □

To prove the theorem, we may assume that $M = \operatorname{Spec} A$ is affine and admits global coordinates $x_1, \ldots, x_n$ giving rise to an étale morphism $M \to \mathbb{A}^n$. Then we write $\omega = \sum_{i=1}^n f_i \, dx_i$, for regular functions f_i on M.

LEMMA 4.13. *The conic subscheme $C \subset \Omega_M$ defined by the 1-form $\omega = \sum_i f_i \, dx_i$ on M is Lagrangian if for every field $K/\mathbb{C}$, for every path $\gamma : \operatorname{Spec} K[[t]] \to M$ and for every $m > 0$ such that $t^m \mid f_i(\gamma(t))$, for all i, we have*

$$\sum_{i=1}^n d\gamma_i(0) \wedge d\left(\frac{f_i(\gamma(t))}{t^m}\Big|_{t=0}\right) = 0, \tag{14}$$

in $\Omega^2_{K/\mathbb{C}}$. Here $\gamma_i = x_i \circ \gamma$.

Proof. First note that as a normal cone, C is pure-dimensional, of dimension equal to $\dim M$. So To prove that C is Lagrangian, we may show that the 2-form $d\alpha$ defining the symplectic structure on Ω_M vanishes when pulled back via $Q : \operatorname{Spec} K \to C$, for every morphism Q from the spectrum of a field to C. Moreover, let us note that $d\alpha$ will vanish on $\operatorname{Spec} K$ if it vanishes on $\operatorname{Spec} L$ for some extension L/K.

Note that we have a cartesian diagram

$$\begin{tikzcd} \Omega_M \arrow[r] \arrow[d] & \Omega_{\mathbb{A}^n} \arrow[d] \\ M \arrow[r] & \mathbb{A}^n. \end{tikzcd}$$

The coordinates $p_1, \ldots, p_n$ on $\Omega_{\mathbb{A}^n} = \mathbb{A}^{2n}$ pull back to functions on Ω_M, which we denote by the same symbols. Thus $x_1, \ldots, x_n, p_1, \ldots, p_n$ are étale coordinates on Ω_M. In fact $\Omega_M = \operatorname{Spec} A[p_1, \ldots, p_n]$. The 2-form $d\alpha$ is equal to $\sum_i dp_i \wedge dx_i$ in these coordinates.

The ideal defining X is $I = (f_1, \ldots, f_n) \leq A$. The normal cone C is the spectrum of the graded ring $\bigoplus_{i \geq 0} I^i/I^{i+1}$ and the embedding $C \to \Omega_M$ is given by the ring epimorphism $A[p_1 \ldots, p_n] \to \sum_{i \geq 0} I^i/I^{i+1}$ sending p_i to $f_i^{(1)}$.

Thus we have

$$Q^*(d\alpha) = Q^* \sum dp_i \wedge dx_i = \sum dQ^*(f_i^{(1)}) \wedge dQ^*(x_i^{(0)}) .$$

If Q^* vanishes on the entire augmentation ideal, this expression is obviously zero. So assume that Q does not vanish on the entire augmentation ideal, and choose γ, m as in Lemma 4.12. Then we get

$$Q^*(d\alpha) = \sum_{i=1}^{n} d\left(\frac{f_i\big(\gamma(t)\big)}{t^m} \bigg|_{t=0} \right) \wedge d\gamma_i(0) ,$$

which vanishes by hypothesis. □

We will now prove the theorem by verifying the condition given in Lemma 4.13. Thus we choose a field extension $K/\mathbb{C}$, a path $\gamma : \operatorname{Spec} K[[t]] \to M$ and an integer $m > 0$ such that $t^m \mid f_i\big(\gamma(t)\big)$, for all i. We claim that Formula (14) is satisfied in the K-vector space $\Omega^2_{K/\mathbb{C}}$.

We will introduce some notation. Define the field elements $c_i^{(p)}$, $F_i^{(p)} \in K$ by the formulas

$$\gamma_i(t) = \sum_{p=0}^{\infty} \frac{1}{p!} c_i^{(p)} t^p , \qquad (f_i \circ \gamma)(t) = \sum_{p=0}^{\infty} \frac{1}{p!} F_i^{(p)} t^p .$$

We claim that

$$\sum_i F_i^{(m)} dc_i^{(0)} = 0 . \tag{15}$$

This will finish the proof, because

$$\sum_i d\gamma_i(0)\wedge d\left(\frac{f_i\big(\gamma(t)\big)}{t^m}\Big|_{t=0}\right) = -\frac{1}{m!}\,d\sum_i F_i^{(m)}dc_i^{(0)}\,.$$

For future reference, let us remark that the assumption $t^m \mid f_i\big(\gamma(t)\big)$, for all i, is equivalent to

$$\forall p<m:\quad F_i^{(p)}=0\,, \tag{16}$$

for all i.

Let us now use the fact that ω is almost closed. This means that

$$(d\omega)|_X = 0 \in \Gamma(X,\Omega^2_M|_X)\,.$$

By considering the commutative diagram of schemes

$$\begin{tikzcd} \operatorname{Spec} K[t]/t^m \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} K[[t]] \arrow[r, "\gamma"] & M \end{tikzcd}$$

we see that this implies that

$$\begin{aligned} \gamma^*(d\omega)|_{\operatorname{Spec} K[t]/t^m} &= 0 \in \Gamma(\operatorname{Spec} K[t]/t^m, \Omega^2_{K[[t]]}|_{\operatorname{Spec} K[t]/t^m}) \\ &= \Omega^2_{K[[t]]}\otimes_{K[t]} K[t]/t^m\,. \end{aligned} \tag{17}$$

Let us calculate $\gamma^*(d\omega)$. This calculation takes place inside $\Omega^2_{K[[t]]}$:

$$\begin{aligned} \gamma^*(d\omega) &= \sum_i d(f_i\circ\gamma)\wedge d\gamma_i \\ &= \sum_i \Bigg(\sum_{p=0}^{\infty}\frac{1}{p!}\Big((dF_i^{(p)})t^p + F_i^{(p)}pt^{p-1}dt\Big) \\ &\qquad\qquad \times \sum_{p=0}^{\infty}\frac{1}{p!}\Big((dc_i^{(p)})t^p + c_i^{(p)}pt^{p-1}dt\Big)\Bigg) \\ &= \sum_{p=0}^{\infty}\frac{1}{p!}\Bigg(\sum_{k=0}^{p}\binom{p}{k}\sum_i dF_i^{(k)}\wedge dc_i^{(p-k)}\Bigg)t^p \\ &\quad + \sum_{p=0}^{\infty}\frac{1}{p!}\Bigg(\sum_{k=0}^{p}\binom{p}{k}\sum_i c_i^{(p+1-k)}dF_i^{(k)}\Bigg)\wedge t^p dt \\ &\quad - \sum_{p=0}^{\infty}\frac{1}{p!}\Bigg(\sum_{k=0}^{p}\binom{p}{k}\sum_i F_i^{(k+1)}dc_i^{(p-k)}\Bigg)\wedge t^p dt. \end{aligned}$$

By Property (17), the coefficient of $t^{m-1}dt$ vanishes. Thus, the equation

$$\sum_{k=0}^{m-1}\binom{m-1}{k}\sum_i c_i^{(m-k)}dF_i^{(k)}=\sum_{k=0}^{m-1}\binom{m-1}{k}\sum_i F_i^{(k+1)}dc_i^{(m-k-1)} \tag{18}$$

holds inside $\Omega_{K/\mathbb{C}}$. Now, by Assumption (16), all terms on the left-hand side of (18) vanish, as well as the terms labeled $k=0,\dots,m-2$ of the right-hand side. Hence, the remaining term on the right-hand side of (18) also vanishes. This is the term labeled $k=m-1$ and is equal to the term claimed to vanish in (15). This concludes the proof of Theorem 4.9.

4.3. *Conclusions.*

Obstruction cones are Lagrangian. Let X be a Deligne-Mumford stack with a symmetric obstruction theory. Suppose $X\hookrightarrow M$ is a closed immersion into a smooth Deligne-Mumford stack M and let $C\subset\Omega_M$ be the associated obstruction cone (see Remark 3.8).

The following fact was suggested to hold by R. Thomas at the workshop on Donaldson-Thomas invariants at the University of Illinois at Urbana-Champaign:

THEOREM 4.14. *The obstruction cone C is Lagrangian.*

Proof. This follows by combining Theorem 4.9 with Corollary 3.14. □

COROLLARY 4.15. *For the fundamental cycle of the obstruction cone we have*

$$[C]=L(\mathfrak{c}_X)=\mathrm{Ch}(\nu_X)\,.$$

Proof. Because $[C]$ is Lagrangian, we have $[C]=L(\pi[C])$, with notation as in (12) and (13). It remains to show that $\pi[C]=\mathfrak{c}_X$. But this is a local problem, and so we may assume that our symmetric obstruction theory comes from an almost closed 1-form on M. Then $C=C_{X/M}$. □

Application to Donaldson-Thomas type invariants. Let X be a quasiprojective Deligne-Mumford stack with a symmetric obstruction theory. Let $[X]^{\mathrm{vir}}$ be the associated virtual fundamental class.

PROPOSITION 4.16. *We have*

$$[X]^{\mathrm{vir}}=(\alpha_X)_0=c_0^{SM}(\nu_X)\,,$$

where $(\alpha_X)_0$ is the degree zero part of the Aluffi class.

Proof. Embed X into a smooth Deligne-Mumford stack M. Then combine Proposition 2.6 with Corollaries 4.7 and 4.15 to get $[X]^{\mathrm{vir}}=c_0^M(\mathfrak{c}_X)$. □

Remark 4.17. In the case that $X=Z(df)$, for a regular function f on a smooth scheme M, the virtual fundamental class is the top Chern class of Ω_M, localized to X. Proposition 4.16 in this case is implicit in [Alu00]. Aluffi proves that $\alpha_X = c(\Omega_M) \cap s(X, M) \in A_*(X)$. Thus, $(\alpha_X)_0 = c_n(\Omega_M) \cap [M] \in A_*(X)$, by Proposition 6.1.(a) of [Ful84].

THEOREM 4.18. *If X is proper, the virtual count is equal to the weighted Euler characteristic*

$$\#^{\mathrm{vir}}(X) = \tilde{\chi}(X) = \chi(X, \nu_X),$$

at least if X is smooth, a global finite group quotient or a gerbe over a scheme.

Proof. Combine Propositions 4.16 and 1.12 with one another. □

Remark 4.19. It should be interesting to prove Theorem 4.18 for arbitrary proper Deligne-Mumford stacks with a symmetric obstruction theory.

Remark 4.20. Theorem 4.18 applies to all Examples discussed in Section 3.3 which give rise to proper X. In the case of nonproper X, *define* $\tilde{\chi}(X)$ to be the virtual count.

Remark 4.21. Let us point out that for a Calabi-Yau threefold the Donaldson-Thomas and the Gromov-Witten moduli spaces share a large open part, namely the locus of smooth embedded curves. Both obstruction theories are symmetric on this locus, and the associated virtual count of this open locus is the same, for both theories.

This observation may or may not be significant for the conjectures of Maulik et al [MNOP06].

Another formula for $\nu_X(P)$. Let ω be an almost closed 1-form on a smooth scheme M. Let $X = Z(\omega)$ be the scheme-theoretic zero locus of ω and $P \in X$ a closed point. Let $x_1, \dots, x_n$ be étale coordinates for M around P and $x_1, \dots, x_n$, $p_1 \dots, p_n$ the associated canonical étale coordinates for Ω_M around P. Write $\omega = \sum_{i=1}^n f_i \, dx_i$ in these coordinates.

Let $\eta \in \mathbb{C}$ be a nonzero complex number and consider the image of the morphism $M \to \Omega_M$ given by the section $\frac{1}{\eta}\omega \in \Gamma(M, \Omega)$. We call this image Γ_η. It is a smooth submanifold of Ω_M of real dimension $2n$. It is defined by the equations $\eta p_i = f_i(x)$.

Let Δ be the image of the morphism $M \to \Omega_M$ given by the section $d\rho$ of Ω_M, where $\rho = \sum_i x_i \bar{x}_i$ is the square of the distance function defined on M by the choice of coordinates. Thus Δ is another smooth submanifold of Ω_M of real dimension $2n$. It is defined by the equations $p_i = \bar{x}_i$.

Orient Γ_η and Δ such that the maps $M \to \Gamma_\eta$ and $M \to \Delta$ are orientation preserving.

PROPOSITION 4.22. *For $\varepsilon > 0$ sufficiently small, and $|\eta|$ sufficiently small with respect to ε, we have*

$$\nu_X(P) = L_{S_\varepsilon}(\Gamma_\eta \cap S_\varepsilon, \Delta \cap S_\varepsilon), \tag{19}$$

where $S_\varepsilon = \{\rho = \varepsilon^2\}$ is the sphere of radius ε in Ω_M centred at P and $\Gamma_\eta \cap S_\varepsilon$, $\Delta \cap S_\varepsilon$ are smooth compact oriented submanifolds of S_ε with linking number L.

Proof. Let $C \hookrightarrow \Omega_M$ be the embedding of the normal cone $C_{X/M}$ into Ω_M given by ω. Then $\mathrm{Ch}(\nu_X) = [C]$. The inverse of Ch is calculated in Theorem 9.7.11 of [KS90] (see also [Gin86]). We get

$$\nu_X(P) = I_{\{P\}}([C], [\Delta]),$$

the intersection number at P of the cycles $[C]$ and $[\Delta]$. Note that P is an isolated point of the intersection $C \cap \Delta$, by Lemma 11.2.1 of [Gin86]. We should remark that [KS90] deals with the real case. This introduces various sign changes, which all cancel out.

Now use Example 19.2.4 in [Ful84], which relates intersection numbers to linking numbers. We get

$$\nu_X(P) = L_{S_\varepsilon}([C] \cap S_\varepsilon, \Delta \cap S_\varepsilon),$$

for sufficiently small ε. Next, use Example 18.1.6(d) in [Ful84], which shows that $\lim_{\eta \to 0}[\Gamma_\eta] = [C]$, that is, that there exists an algebraic cycle in $\Omega_M \times \mathbb{A}^1$ which specializes to $[\Gamma_\eta]$ for $\eta \neq 0$ and to $[C]$ for $\eta = 0$. It follows that for sufficiently small η, we can replace $[C]$ in our formula by Γ_η. □

Remark 4.23. Note how Formula (19) is similar in spirit to Formula (4). Combining these two formulas for $\nu_X(P)$, using $\omega = df$, gives an expression for the Euler characteristic of the Milnor fibre in terms of a linking number.

Motivic invariants. Let A be a commutative ring and μ an A-valued motivic measure on the category of finite type schemes over $\mathbb{C}$. For a scheme X, it is tempting to define

$$\tilde{\mu}(X) = \mu(X, \nu_X) = \int_X \nu_X \, d\mu$$

and call it the *virtual motive* of X. Note that $\tilde{\mu}(X)$ encodes the scheme structure of X in a much more subtle way than the usual motive $\mu(X)$, which neglects all nilpotents in the structure sheaf of X.

If X is endowed with a symmetric obstruction theory, $\tilde{\mu}(X)$ may be thought of as a motivic generalization of the virtual count, or a motivic Donaldson-Thomas type invariant.

Here are two caveats:

Remark 4.24. The proper motivic Donaldson-Thomas type invariant should probably motivate not only X but also ν_X. For example, in the case of the singular locus of a hypersurface, motivic vanishing cycles (and not just their Euler characteristics) should play a role.

Remark 4.25. Note that μ will not satisfy Property (v) of Section 1.3, unless $\mu = \chi$. So one encounters difficulties when extending the virtual motive to Deligne-Mumford stacks. To extend μ to stacks one formally inverts $\mu(\mathrm{GL}_n)$, for all n, but then one looses the specialization to χ, as $\chi(\mathrm{GL}_n) = 0$. So one cannot think of the virtual motive of a stack as a generalization of the virtual count, even if the stack admits a symmetric obstruction theory. For example, $\tilde{\mu}(B\mathbb{Z}/2) = 1$.

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