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Proper affine actions and geodesic flows of hyperbolic surfaces

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Abstract

Let $\Gamma_0 \subset \mathsf{O}(2,1)$ be a Schottky group, and let $\Sigma = \mathsf{H}^2/\Gamma_0$ be the corresponding hyperbolic surface. Let $\mathscr{C}(\Sigma)$ denote the space of unit length geodesic currents on Σ. The cohomology group $H^1(\Gamma_0, \mathsf{V})$ parametrizes equivalence classes of affine deformations Γ_{u} of Γ_0 acting on an irreducible representation V of $\mathsf{O}(2,1)$. We define a continuous biaffine map $\Psi : \mathscr{C}(\Sigma) \times H^1(\Gamma_0, \mathsf{V}) \to \mathbb{R}$ which is linear on the vector space $H^1(\Gamma_0, \mathsf{V})$. An affine deformation Γ_{u} acts properly if and only if $\Psi(\mu, [\mathsf{u}]) \neq 0$ for all $\mu \in \mathscr{C}(\Sigma)$. Consequently the set of proper affine actions whose linear part is a Schottky group identifies with a bundle of open convex cones in $H^1(\Gamma_0, \mathsf{V})$ over the Fricke-Teichmüller space of Σ.

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Introduction

As is well known, every discrete group of Euclidean isometries of $\mathbb{R}^n$ contains a free abelian subgroup of finite index. In [36], Milnor asked if every discrete subgroup of *affine transformations* of $\mathbb{R}^n$ must contain a polycyclic subgroup of finite index. Furthermore he showed that this question is equivalent to whether a discrete subgroup of $\mathsf{Aff}(\mathbb{R}^n)$ which acts properly must be *amenable.* By Tits [42], this is equivalent to the existence of a proper affine action of a nonabelian free group. Margulis subsequently showed [33], [34] that proper affine actions of nonabelian free groups do indeed exist. The present paper describes the deformation space of proper affine actions on $\mathbb{R}^3$, for a large class of discrete groups, such as a bundle of convex cones over the Fricke-Teichmüller deformation space of hyperbolic structures on a compact surface with geodesic boundary.

If $\Gamma \subset \mathsf{Aff}(\mathbb{R}^n)$ is a discrete subgroup which acts properly on $\mathbb{R}^n$, then a subgroup of finite index will act freely. In this case the quotient $\mathbb{R}^n/\Gamma$ is a *complete affine manifold* M with fundamental group $\pi_1(M) \cong \Gamma$. When $n = 3$, Fried-Goldman [20] classified all the amenable such Γ (including the case when M is compact). Furthermore the linear holonomy homomorphism

$$\pi_1(M) \xrightarrow{\mathbb{L}} \mathsf{GL}(\mathbb{R}^3)$$

embeds $\pi_1(M)$ onto a discrete subgroup $\Gamma_0 \subset \mathsf{GL}(\mathbb{R}^3)$, which is conjugate to a subgroup of $\mathsf{O}(2,1)^0$ (Fried-Goldman [20]). Mess [35] proved the hyperbolic surface

$$\Sigma = \Gamma_0 \backslash \mathsf{O}(2,1)^0/\mathsf{SO}(2)$$

is noncompact. Goldman-Margulis [23] and Labourie [31] gave alternate proofs of Mess's theorem, using ideas which evolved to the present work.

Thus the classification of proper 3-dimensional affine actions reduces to *affine deformations* of free discrete subgroups $\Gamma_0 \subset \mathsf{O}(2,1)^0$. An *affine deformation* of Γ_0 is a group Γ of affine transformations whose linear part equals Γ_0, that is, a subgroup $\Gamma \subset \mathsf{Isom}^0(\mathbb{E}^{2,1})$ such that the restriction of $\mathbb{L}$ to Γ is an isomorphism $\Gamma \longrightarrow \Gamma_0$.

Equivalence classes of affine deformations of $\Gamma_0 \subset \mathsf{O}(2,1)^0$ are parametrized by the cohomology group $H^1(\Gamma_0, \mathsf{V})$, where V is the linear holonomy representation. Given a cocycle $\mathsf{u} \in Z^1(\Gamma_0, \mathsf{V})$, we denote the corresponding affine deformation by Γ_{u}. Drumm [13], [15], [16], [12] showed that Mess's necessary condition of noncompactness is sufficient: *every noncocompact discrete subgroup of* $\mathsf{O}(2,1)^0$ *admits proper affine deformations.* In particular he found an open subset of $H^1(\Gamma_0, \mathsf{V})$ parametrizing proper affine deformations [18].

This paper gives a criterion for the properness of an affine deformation Γ_{u} in terms of the parameter $[\mathsf{u}] \in H^1(\Gamma_0, \mathsf{V})$.

With no extra work, we take V to be a representation of Γ_0 obtained by composition of the discrete embedding $\Gamma_0 \subset \mathsf{SL}(2,\mathbb{R})$ with any irreducible representation of $\mathsf{SL}(2,\mathbb{R})$. Such an action is called *Fuchsian* in Labourie [31]. Proper actions with irreducible linear parts occur only when V has dimension $4k+3$ (see Abels [1], Abels-Margulis-Soifer [2], [3] and Labourie [31]). In those dimensions, Abels-Margulis-Soifer [2], [3] constructed proper affine deformations of Fuchsian actions of free groups.

THEOREM. *Suppose that Γ_0 contains no parabolic elements. Then the equivalence classes of proper affine deformations of Γ form an open convex cone in $H^1(\Gamma_0, \mathsf{V})$.*

The main tool in this paper is a generalization of the invariant constructed by Margulis [33], [34] and extended to higher dimensions by Labourie [31]. Margulis's invariant α_{u} is a class function $\Gamma_0 \xrightarrow{\alpha_{\mathsf{u}}} \mathbb{R}$ associated to an affine deformation Γ_{u}. It satisfies the following properties:

- $\alpha_{[\mathsf{u}]}(\gamma^n) = |n|\alpha_{[\mathsf{u}]}(\gamma)$;
- $\alpha_{[\mathsf{u}]}(\gamma) = 0 \iff \gamma$ fixes a point in $\mathbb{V}$;
- The function $\alpha_{[\mathsf{u}]}$ depends linearly on $[\mathsf{u}]$;
- The map
$$\begin{aligned} H^1(\Gamma_0, \mathsf{V}) &\longrightarrow \mathbb{R}^{\Gamma_0} \\ [\mathsf{u}] &\longmapsto \alpha_{[\mathsf{u}]} \end{aligned}$$
is injective ([19]).
- Suppose $\dim \mathsf{V} = 3$. If $\Gamma_{[\mathsf{u}]}$ acts properly, then $|\alpha_{[\mathsf{u}]}(\gamma)|$ is the Lorentzian length of the unique closed geodesic in $\mathsf{E}/\Gamma_{[\mathsf{u}]}$ corresponding to γ.

The significance of the *sign* of $\alpha_{[\mathsf{u}]}$ is the following result, due to Margulis [33], [34]:

OPPOSITE SIGN LEMMA. *Suppose Γ acts properly. Either $\alpha_{[\mathsf{u}]}(\gamma) > 0$ for all $\gamma \in \Gamma_0$ or $\alpha_{[\mathsf{u}]}(\gamma) < 0$ for all $\gamma \in \Gamma_0$.*

This paper provides a more conceptual proof of the Opposite Sign Lemma by extending a normalized version of Margulis's invariant α to a continuous function Ψ_{u} on a connected space $\mathscr{C}(\Sigma)$ of probability measures. If

$$\alpha_{[\mathsf{u}]}(\gamma_1) < 0 < \alpha_{[\mathsf{u}]}(\gamma_2),$$

then an element $\mu \in \mathscr{C}(\Sigma)$ "between" γ_1 and γ_2 exists, satisfying $\Psi_{\mathsf{u}}(\mu) = 0$.

Associated to the linear group Γ_0 is a complete hyperbolic surface $\Sigma = \Gamma_0 \backslash \mathsf{H}^2$. Modifying slightly the terminology of Bonahon [6], we call a *geodesic current* a Borel probability measure on the unit tangent bundle invariant under the geodesic flow ϕ. We modify $\alpha_{[\mathsf{u}]}$ by dividing it by the length of the corresponding closed geodesic in the hyperbolic surface. This modification extends to a continuous function $\Psi_{[\mathsf{u}]}$ on the compact space $\mathscr{C}(\Sigma)$ of *geodesic currents* on Σ.

Here is a brief description of the construction, which first appeared in Labourie [31]. For brevity we only describe this for the case $\dim \mathsf{V} = 3$. Corresponding to the affine deformation $\Gamma_{[\mathsf{u}]}$ is a flat affine bundle $\mathbb{E}$ over $U\Sigma$, whose associated flat vector bundle has a parallel Lorentzian structure $\mathbb{B}$. Let φ be the vector field on $U\Sigma$ generating the geodesic flow. For a sufficiently smooth section s of $\mathbb{E}$, the covariant derivative $\nabla_\varphi(s)$ is a section of the flat vector bundle $\mathbb{V}$ associated to $\mathbb{E}$. Let ν denote the section of $\mathbb{V}$ which associates to a point $x \in U\Sigma$ the unit-spacelike vector corresponding to the geodesic associated to x. Let $\mu \in \mathscr{C}(\Sigma)$ be a geodesic current. Then $\Psi_{[\mathsf{u}]}(\mu)$ is defined by:

$$\int_{U\Sigma} \mathbb{B}\big(\nabla_\varphi(s), \nu\big)\, d\mu.$$

Nontrivial elements $\gamma \in \Gamma_0$ correspond to periodic orbits c_γ for φ. The period of c_γ equals the length $\ell(\gamma)$ of the corresponding closed geodesic in Σ. Denote the subset of $\mathscr{C}(\Sigma)$ consisting of measures supported on periodic orbits by $\mathscr{C}_{\mathsf{per}}(\Sigma)$. Denote by μ_γ the geodesic current corresponding to c_γ. Because $\alpha(\gamma^n) = |n|\alpha(\gamma)$ and $\ell(\gamma^n) = |n|\ell(\gamma)$, the ratio $\alpha(\gamma)/\ell(\gamma)$ depends only on μ_γ.

THEOREM. *Let Γ_{u} denote an affine deformation of Γ_0.*

- *The function*

$$\begin{aligned} \mathscr{C}_{\mathsf{per}}(\Sigma) &\longrightarrow \mathbb{R} \\ \mu_\gamma &\longmapsto \frac{\alpha(\gamma)}{\ell(\gamma)} \end{aligned}$$

extends to a continuous function

$$\mathscr{C}(\Sigma) \xrightarrow{\Psi_{[\mathsf{u}]}} \mathbb{R}.$$

- Γ_{u} *acts properly if and only if* $\Psi_{[\mathsf{u}]}(\mu) \neq 0$ *for all* $\mu \in \mathscr{C}(\Sigma)$.

Since $\mathscr{C}(\Sigma)$ is connected, either $\Psi_{[\mathsf{u}]}(\mu) > 0$ for all $\mu \in \mathscr{C}(\Sigma)$ or $\Psi_{[\mathsf{u}]}(\mu) < 0$ for all $\mu \in \mathscr{C}(\Sigma)$. Compactness of $\mathscr{C}(\Sigma)$ implies that Margulis's invariant grows at most linearly with the length of γ:

COROLLARY. *For any* $\mathsf{u} \in Z^1(\Gamma_0, \mathsf{V})$, *there exists* $C > 0$ *such that*

$$|\alpha_{[\mathsf{u}]}(\gamma)| \leq C\ell(\gamma)$$

for all $\gamma \in \Gamma_{\mathsf{u}}$.

Margulis [33], [34] originally used the invariant α to detect properness: if Γ acts properly, then all the $\alpha(\gamma)$ are all positive, or all of them are negative. (Since conjugation by a homothety scales the invariant:

$$\alpha_{\lambda\mathsf{u}} = \lambda\alpha_{\mathsf{u}}$$

uniform positivity and uniform negativity are equivalent.) Goldman [22], [23] conjectured that this necessary condition is sufficient: that is, properness is equivalent to the positivity (or negativity) of Margulis's invariant. This has been proved when Σ is a three-holed sphere; see Jones [26] and Charette-Drumm-Goldman [11]. In this case the cone corresponding to proper actions is defined by the inequalities corresponding to the three boundary components $\gamma_1, \gamma_2, \gamma_3 \subset \partial\Sigma$:

$$\big\{[\mathsf{u}] \in H^1(\Gamma_0, \mathsf{V}) \mid \alpha_{[\mathsf{u}]}(\gamma_i) > 0 \text{ for } i = 1,2,3\big\}$$
$$\bigcup \big\{[\mathsf{u}] \in H^1(\Gamma_0, \mathsf{V}) \mid \alpha_{[\mathsf{u}]}(\gamma_i) < 0 \text{ for } i = 1,2,3\big\}.$$

However, when Σ is a one-holed torus, Charette [9] showed that the positivity of $\alpha_{[\mathsf{u}]}$ requires infinitely many inequalities, suggesting the original conjecture is generally false. Recently Goldman-Margulis-Minsky [24] have disproved the original conjecture, using ideas inspired by Thurston's unpublished manuscript [41].

This paper is organized as follows. Section 1 collects facts about convex cocompact hyperbolic surfaces and the recurrent part of the geodesic flow. Section 2 reviews constructions in affine geometry such as affine deformations, and defines the Margulis invariant. Section 3 defines the flat affine bundles associated to an affine deformation. Section 4 introduces the horizontal lift of the geodesic flow to the flat affine bundle whose dynamics reflects the dynamics of the affine deformation of the Fuchsian group Γ_0. Section 5 extends the normalized Margulis invariant to the function $\Psi_{[\mathsf{u}]}$ on geodesic currents (first treated by Labourie [31]). Section 6 shows that for every *nonproper* affine deformation $[\mathsf{u}]$ there is a geodesic current $\mu \in \mathscr{C}(\Sigma)$ for which $\Psi_{[\mathsf{u}]}(\mu) = 0$. Section 7 shows that, conversely, for every *proper* affine deformation $[\mathsf{u}]$, $\Psi_{[\mathsf{u}]}(\mu) \neq 0$ for every $\mu \in \mathscr{C}(\Sigma)$. The Opposite Sign Lemma follows as a corollary.

Notation and terminology. Denote the tangent bundle of a smooth manifold M by TM. For a given Riemannian metric on M, *the unit tangent bundle* UM consists of unit vectors in TM. For any subset $S \subset M$, denote the induced bundle over S by US. Denote the (real) hyperbolic plane by H^2 and its boundary by $\partial\mathsf{H}^2 = S^1_\infty$. For distinct $a, b \in S^1_\infty$, denote the geodesic having a, b as ideal endpoints by $\overleftrightarrow{ab}$.

Let $\mathbb{V}$ be a vector bundle over a manifold M. Denote by $\mathscr{R}^0(\mathbb{V})$ the vector space of sections of $\mathbb{V}$. Let ∇ be a connection on $\mathbb{V}$. If $\mathsf{s} \in \mathscr{R}^0(\mathbb{V})$ is a section and ξ is a vector field on M, then

$$\nabla_\xi(\mathsf{s}) \in \mathscr{R}^0(\mathbb{V})$$

denotes the covariant derivative of s with respect to ξ. If f is a function, then ξf denotes the directional derivative of f with respect to ξ.

An affine bundle $\mathbb{E}$ over M is a fiber bundle over M whose fiber is an affine space E with structure group the group $\mathsf{Aff}(\mathsf{E})$ of affine automorphisms of E. The *vector bundle* $\mathbb{V}$ *associated to* $\mathbb{E}$ is the vector bundle whose fiber $\mathbb{V}_x$ over $x \in M$ is the vector space associated to the fiber $\mathbb{E}_x$. That is, $\mathbb{V}_x$ is the vector space consisting of all translations $\mathbb{E}_x \longrightarrow \mathbb{E}_x$.

Denote the space of continuous sections of a bundle $\mathbb{E}$ by $\Gamma(\mathbb{E})$. If $\mathbb{V}$ is a vector bundle, then $\Gamma(\mathbb{V})$ is a vector space. If $\mathbb{E}$ is an affine bundle with underlying vector bundle $\mathbb{V}$, then $\Gamma(\mathbb{E})$ is an affine space with underlying vector space $\Gamma(\mathbb{V})$. If $s_1, s_2 \in \Gamma(\mathbb{E})$ are two sections of $\mathbb{E}$, the *difference* $s_1 - s_2$ is the section of $\mathbb{V}$ whose value at $x \in M$ is the translation $s_1(x) - s_2(x)$ of $\mathbb{E}_x$.

Denote the convex set of Borel probability measures on a topological space X by $\mathcal{P}(X)$. Denote the cohomology class of a cocycle z by $[z]$. A *flow* is an action of the additive group $\mathbb{R}$ of real numbers. The transformations in the flow defined by a vector field ξ are denoted ξ_t, for $t \in \mathbb{R}$.

1. **Hyperbolic geometry**

Let $\mathbb{R}^{2,1}$ be the 3-dimensional real vector space with inner product

$$\mathsf{B}(\mathsf{v}, \mathsf{w}) := \mathsf{v}_1\mathsf{w}_1 + \mathsf{v}_2\mathsf{w}_2 - \mathsf{v}_3\mathsf{w}_3$$

and $\mathsf{G}_0 = \mathsf{O}(2,1)^0$ the identity component of its group of isometries. G_0 consists of linear isometries of $\mathbb{R}^{2,1}$ which preserve both an orientation of $\mathbb{R}^{2,1}$ and a *time-orientation* (or *future*) of $\mathbb{R}^{2,1}$, that is, a connected component of the *open light-cone*

$$\{\mathsf{v} \in \mathbb{R}^{2,1} \mid \mathsf{B}(\mathsf{v}, \mathsf{v}) < 0\}.$$

Then

$$\mathsf{G}_0 = \mathsf{O}(2,1)^0 \cong \mathsf{PSL}(2, \mathbb{R}) \cong \mathsf{Isom}^0(\mathsf{H}^2)$$

where H^2 denotes the real hyperbolic plane.

1.1. *The hyperboloid model.* We define H^2 as follows. We work in the irreducible representation $\mathbb{R}^{2,1}$ (isomorphic to the adjoint representation of G_0). Let B denote the invariant bilinear form (isomorphic to a multiple of the Killing form). The two-sheeted hyperboloid

$$\{\mathsf{v} \in \mathbb{R}^{2,1} \mid \mathsf{B}(\mathsf{v}, \mathsf{v}) = -1\}$$

has two connected components. If p_0 lies in this hyperboloid, then its connected component equals

$$\{\mathsf{v} \in \mathbb{R}^{2,1} \mid \mathsf{B}(\mathsf{v}, \mathsf{v}) = -1, \mathsf{B}(\mathsf{v}, \mathsf{p}_0) < 0\}. \tag{1.1}$$

For clarity, we fix a timelike vector p_0, for example,

$$\mathsf{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{2,1},$$

and define H^2 as the connected component (1.1).

The Lorentzian metric defined by B restricts to a Riemannian metric of constant curvature -1 on H^2. The identity component G_0 of $\mathsf{O}(2,1)^0$ is the group $\mathsf{Isom}^0(\mathbb{E}^{2,1})$ of orientation-preserving isometries of H^2. The stabilizer of p_0,

$$\mathsf{K}_0 := \mathsf{PO}(2),$$

is the *maximal compact subgroup* of G_0. Evaluation at p_0

$$\begin{aligned} \mathsf{G}_0/\mathsf{K}_0 &\longrightarrow \mathbb{R}^{2,1} \\ g\mathsf{K}_0 &\longmapsto g\mathsf{p}_0 \end{aligned}$$

identifies H^2 with the homogeneous space $\mathsf{G}_0/\mathsf{K}_0$.

The action of G_0 canonically lifts to a *simply transitive* (left-) action on the unit tangent bundle $U\mathsf{H}^2$. The unit spacelike vector

$$\dot{\mathsf{p}_0} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^{2,1} \tag{1.2}$$

defines a unit tangent vector to H^2 at p_0, which we denote:

$$(\mathsf{p}_0, \dot{\mathsf{p}_0}) \in U\mathsf{H}^2.$$

Evaluation at $(\mathsf{p}_0, \dot{\mathsf{p}_0})$

$$\begin{aligned} \mathsf{G}_0 &\xrightarrow{\mathscr{E}} U\mathsf{H}^2 \\ g_0 &\longmapsto g_0(\mathsf{p}_0, \dot{\mathsf{p}_0}) \end{aligned} \tag{1.3}$$

G_0-equivariantly identifies G_0 with the unit tangent bundle $U\mathsf{H}^2$. Here G_0 acts on itself by left-multiplication. The corresponding action on $U\mathsf{H}^2$ is the action induced by isometries of H^2. Under this identification, K_0 corresponds to the fiber of $U\mathsf{H}^2$ above p_0.

1.2. *The geodesic flow.* The *Cartan subgroup*

$$\mathsf{A}_0 := \mathsf{PO}(1,1).$$

is the one-parameter subgroup comprising

$$a(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{bmatrix}$$

for $t \in \mathbb{R}$. Under this identification,

$$a(t)\mathsf{p}_0 \in \mathsf{H}^2$$

describes the geodesic through p_0 tangent to $\dot{\mathsf{p}_0}$.

Right-multiplication by $a(-t)$ on G_0 identifies with the geodesic flow $\widetilde{\varphi}_t$ on $U\mathsf{H}^2$. First note that $\widetilde{\varphi}_t$ on $U\mathsf{H}^2$ commutes with the action of G_0 on $U\mathsf{H}^2$, which corresponds to the action on G_0 on itself by left-multiplications. The $\mathbb{R}$-action on G_0 corresponding to $\widetilde{\varphi}_t$ commutes with left-multiplications on G_0. Thus this $\mathbb{R}$-action on G_0 must be right-multiplication by a one-parameter subgroup. At the basepoint $(\mathsf{p}_0, \dot{\mathsf{p}_0})$, the geodesic flow corresponds to right-multiplication by A_0. Hence this one-parameter subgroup must be A_0. Therefore right-multiplication by A_0 on G_0 induces the geodesic flow $\widetilde{\varphi}_t$ on $U\mathsf{H}^2$.

Denote the vector field corresponding to the geodesic flow by $\widetilde{\varphi}$:

$$\widetilde{\varphi}(x) := \left.\frac{d}{dt}\right|_{t=0} \widetilde{\varphi}_t(x).$$

1.3. *The convex core.* Since Γ_0 is a Schottky group, the complete hyperbolic surface is *convex cocompact.* That is, there exists a compact geodesically convex subsurface $\mathsf{core}(\Sigma)$ such that the inclusion

$$\mathsf{core}(\Sigma) \subset \Sigma$$

is a homotopy equivalence. Here $\mathsf{core}(\Sigma)$ is called the *convex core* of Σ, and is bounded by closed geodesics $\partial_i(\Sigma)$ for $i = 1, \ldots, k$. Equivalently the discrete subgroup Γ_0 is finitely generated and contains only hyperbolic elements.

Another criterion for convex cocompactness involves the *ends* $e_i(\Sigma)$ of Σ. Each component $e_i(\Sigma)$ of the closure of $\Sigma \setminus \mathsf{core}(\Sigma)$ is diffeomorphic to a product

$$e_i(\Sigma) \xrightarrow{\approx} \partial_i(\Sigma) \times [0, \infty).$$

The corresponding end of $U\Sigma$ is diffeomorphic to the product

$$e_i(U\Sigma) \xrightarrow{\approx} \partial_i(\Sigma) \times S^1 \times [0, \infty).$$

The corresponding Cartesian projections

$$e_i(\Sigma) \longrightarrow \partial_i(\Sigma)$$

and the identity map on $\mathsf{core}(\Sigma)$ extend to a deformation retraction

$$\Sigma \xrightarrow{\Pi_{\mathsf{core}(\Sigma)}} \mathsf{core}(\Sigma). \tag{1.4}$$

We use the following standard compactness results (see Kapovich [27, §4.17, pp. 98–102], Canary-Epstein-Green [8], or Marden [32] for discussion and proof):

LEMMA 1.1. *Let $\Sigma = \mathsf{H}^2/\Gamma_0$ be the hyperbolic surface where Γ_0 is convex cocompact and let $R \geq 0$. Then*

$$\mathcal{K}_R := \{y \in \Sigma \mid d(y, \mathsf{core}(\Sigma)) \leq R\} \tag{1.5}$$

is a compact geodesically convex subsurface, upon which the restriction of $\Pi_{\mathsf{core}(\Sigma)}$ is a homotopy equivalence.

1.4. *The recurrent subset.* Let $U_{\mathsf{rec}}\Sigma \subset U\Sigma$ denote the union of all recurrent orbits of φ.

LEMMA 1.2. *Let Σ be as above, $U\Sigma$ its unit tangent bundle and φ_t the geodesic flow on $U\Sigma$. Then:*

- *$U_{\mathsf{rec}}\Sigma \subset U\Sigma$ is a compact φ-invariant subset.*
- *Every φ-invariant Borel probability measure on $U\Sigma$ is supported in $U_{\mathsf{rec}}\Sigma \subset U\Sigma$.*
- *The space of φ-invariant Borel probability measures on $U\Sigma$ is a convex compact space $\mathscr{C}(\Sigma)$ with respect to the weak $\star$-topology.*

Proof. Clearly $U_{\mathsf{rec}}\Sigma$ is invariant. The subbundle $U\mathsf{core}(\Sigma)$ comprising unit vectors over points in $\mathsf{core}(\Sigma)$ is compact. Any geodesic which exits $\mathsf{core}(\Sigma)$ into an end $e_i(\Sigma)$ remains in the same end $e_i(\Sigma)$. Therefore every recurrent geodesic ray eventually lies in $\mathsf{core}(\Sigma)$.

On the unit tangent bundle $U\mathsf{H}^2$, every φ-orbit tends to a unique ideal point on $S^1_\infty = \partial\mathsf{H}^2$; let

$$U\mathsf{H}^2 \xrightarrow{\eta} S^1_\infty$$

denote the corresponding map. Then the φ-orbit of $\tilde{x} \in U\mathsf{H}^2$ defines a geodesic which is recurrent in the forward direction if and only if $\eta(\tilde{x}) \in \Lambda$, where $\Lambda \subset S^1_\infty$ is the *limit set* of Γ_0. (See, for example, §1.4 of [4].) The geodesic is recurrent in the backward direction if $\eta(-\tilde{x}) \in \Lambda$. Then every recurrent orbit of the geodesic flow lies in the compact set consisting of vectors $x \in U\mathsf{core}(\Sigma)$ with

$$\eta(\tilde{x}), \eta(-\tilde{x}) \in \Lambda.$$

Thus $U_{\mathsf{rec}}\Sigma$ is compact.

By the Poincaré recurrence theorem [25], every φ-invariant probability measure on $U\Sigma$ is supported in $U_{\mathsf{rec}}\Sigma$. Since $U_{\mathsf{rec}}\Sigma$ is compact, the space of Borel probability measures $\mathcal{P}(U_{\mathsf{rec}}\Sigma)$ on $U_{\mathsf{rec}}\Sigma$ is a compact convex set with respect to the weak topology. The closed subset of φ-invariant probability measures on $U_{\mathsf{rec}}\Sigma$ is also a compact convex set. Since every φ-invariant probability measure on $U\Sigma$ is supported on $U_{\mathsf{rec}}\Sigma$, the assertion follows. □

The following well known fact will be needed later.

LEMMA 1.3. *$U_{\mathsf{rec}}\Sigma$ is connected.*

Proof. Let $\Lambda \subset S^1_\infty$ be the limit set as above. For every $\lambda \in \Lambda$, the orbit $\Gamma_0\lambda$ is dense in Λ. (See for example Marden [32, Lemma 2.4.1].) Denote

$$\Lambda^* := \{(\lambda_1, \lambda_2) \in \Lambda \times \Lambda \mid \lambda_1 \neq \lambda_2\}.$$

The preimage $\widetilde{U_{\mathsf{rec}}\Sigma}$ of $U_{\mathsf{rec}}\Sigma \subset U\Sigma$ under the covering space $U\mathsf{H}^2 \xrightarrow{\Pi} U\Sigma$ is the union

$$\widetilde{U_{\mathsf{rec}}\Sigma} = \bigcup_{(\lambda_1,\lambda_2)\in\Lambda^*} \overleftrightarrow{\lambda_1\lambda_2}.$$

Choose a hyperbolic element $\gamma \in \Gamma_0$ and let $\lambda_\pm \in \Lambda$ denote its fixed points. The image

$$c_\gamma := \Pi\big(\overleftrightarrow{\lambda_-\lambda+}\big)$$

is the unique closed geodesic in $U_{\mathsf{rec}}\Sigma$ corresponding to γ.

We claim that the image U_+ under Π of the subset

$$\bigcup_{\lambda\in\Lambda} \overleftrightarrow{\lambda_-\lambda} \subset U_{\mathsf{rec}}\Sigma$$

is connected. Suppose that $W_1, W_2 \subset U\Sigma$ are disjoint open subsets such that $U_+ \subset W_1 \cup W_2$. Since c_γ is connected, either W_1 or W_2 contains c_γ; suppose that $W_1 \supset c_\gamma$. We claim that $U_+ \subset W_1$. Otherwise $\overleftrightarrow{\lambda_-\lambda}$ meets W_2 for some $\lambda \in \Lambda$. Connectedness of $\overleftrightarrow{\lambda_-\lambda}$ implies $\overleftrightarrow{\lambda_-\lambda} \subset W_2$. Now

$$\lim_{n\to\infty} \gamma^n\lambda = \lambda_+,$$

which implies

$$\lim_{n\to\infty} \gamma^n\big(\overleftrightarrow{\lambda_-\lambda}\big) = \lim_{n\to\infty}\big(\overleftrightarrow{\lambda_-(\gamma^n\lambda)}\big) = \overleftrightarrow{\lambda_-\lambda_+}.$$

Thus for $n >> 1$, the geodesic $\overleftrightarrow{\lambda_\lambda}$ lies in $\Pi^{-1}(W_1)$, a contradiction.

Thus $U_+ \subset W_1$, implying that U_+ is connected. Density of $\Gamma_0\lambda$ in Λ implies U_+ is dense in $U_{\mathsf{rec}}\Sigma$. Therefore $U_{\mathsf{rec}}\Sigma$ is connected. □

2. Affine geometry

This section collects general properties on affine spaces, affine transformations, and affine deformations of linear group actions. We are primarily interested in affine deformations of linear actions factoring through the irreducible $2r+1$-dimensional real representation V_r of

$$\mathsf{G}_0 \cong \mathsf{PSL}(2,\mathbb{R}),$$

where r is a positive integer.

2.1. *Affine spaces and their automorphisms.* Let V be a real vector space. Denote its group of linear automorphisms by $\mathsf{GL}(\mathsf{V})$.

An *affine space* E (modeled on V) is a space equipped with a simply transitive action of V. Call V the *vector space underlying* E, and refer to its elements as *translations.* Translation τ_{v} by a vector $\mathsf{v} \in \mathsf{V}$ is denoted by *addition:*

$$\begin{aligned} \mathsf{E} &\xrightarrow{\tau_{\mathsf{v}}} \mathsf{E} \\ x &\longmapsto x + \mathsf{v}. \end{aligned}$$

Let $x, y \in \mathsf{E}$. Denote the unique vector $\mathsf{v} \in \mathsf{V}$ such that $\tau_{\mathsf{v}}(x) = y$ by *subtraction:*

$$\mathsf{v} = y - x.$$

Let E be an affine space with associated vector space V. Choice of an arbitrary point $O \in \mathsf{E}$ (the *origin*) identifies E with V via the map

$$\begin{aligned} \mathsf{V} &\longrightarrow \mathsf{E} \\ \mathsf{v} &\longmapsto O + \mathsf{v}. \end{aligned}$$

An *affine automorphism* of E is the composition of a linear mapping (using the above identification of E with V) and a translation; that is,

$$\begin{aligned} \mathsf{E} &\xrightarrow{g} \mathsf{E} \\ O + \mathsf{v} &\longmapsto O + \mathbb{L}(g)(\mathsf{v}) + \mathsf{u}(g) \end{aligned}$$

which we write simply as

$$\mathsf{v} \longmapsto \mathbb{L}(g)(\mathsf{v}) + \mathsf{u}(g).$$

The affine automorphisms of E form a group $\mathsf{Aff}(\mathsf{E})$, and $(\mathbb{L}, \mathsf{u})$ defines an isomorphism of $\mathsf{Aff}(\mathsf{E})$ with the semidirect product $\mathsf{GL}(\mathsf{V}) \ltimes \mathsf{V}$. The linear mapping $\mathbb{L}(g) \in \mathsf{GL}(\mathsf{V})$ is the *linear part* of the affine transformation g, and

$$\mathsf{Aff}(\mathsf{E}) \xrightarrow{\mathbb{L}} \mathsf{GL}(\mathsf{V})$$

is a homomorphism. The vector $\mathsf{u}(g) \in \mathsf{V}$ is the *translational part* of g. The mapping

$$\mathsf{Aff}(\mathsf{E}) \xrightarrow{\mathsf{u}} \mathsf{V}$$

satisfies a *cocycle identity:*

$$\mathsf{u}(\gamma_1\gamma_2) = \mathsf{u}(\gamma_1) + \mathbb{L}(\gamma_1)\mathsf{u}(\gamma_2) \tag{2.1}$$

for $\gamma_1, \gamma_2 \in \mathsf{Aff}(\mathsf{E})$.

2.2. *Affine deformations of linear actions.* Let $\Gamma_0 \subset \mathsf{GL}(\mathsf{V})$ be a group of linear automorphisms of a vector space V. Denote the corresponding Γ_0-module by V as well.

An *affine deformation* of Γ_0 is a representation

$$\Gamma_0 \xrightarrow{\rho} \mathsf{Aff}(\mathsf{E})$$

such that $\mathbb{L}\circ\rho$ is the inclusion $\Gamma_0 \hookrightarrow \mathsf{GL}(\mathsf{V})$. We confuse ρ with its image $\Gamma := \rho(\Gamma_0)$, to which we also refer as an *affine deformation* of Γ_0. Note that ρ embeds Γ_0 as the subgroup Γ of $\mathsf{GL}(\mathsf{V})$. In terms of the semidirect product decomposition

$$\mathsf{Aff}(\mathsf{E}) \cong \mathsf{V} \rtimes \mathsf{GL}(\mathsf{V})$$

an affine deformation is the graph $\rho = \rho_{\mathsf{u}}$ (with image denoted $\Gamma = \Gamma_{\mathsf{u}}$) of a *cocycle*

$$\Gamma_0 \xrightarrow{\mathsf{u}} V;$$

that is, a map satisfying the cocycle identity (2.1). Write

$$\gamma = \rho(\gamma_0) = (\mathsf{u}(\gamma_0), \gamma_0) \in V \rtimes \Gamma_0$$

for the corresponding affine transformation:

$$x \overset{\gamma}{\longmapsto} \gamma_0 x + \mathsf{u}(\gamma_0).$$

Cocycles form a vector space $Z^1(\Gamma_0, \mathsf{V})$. Cocycles $\mathsf{u}_1, \mathsf{u}_2 \in Z^1(\Gamma_0, \mathsf{V})$ are *cohomologous* if their difference $\mathsf{u}_1 - \mathsf{u}_2$ is a *coboundary,* a cocycle of the form

$$\Gamma_0 \xrightarrow{\delta\mathsf{v}_0} \mathsf{V}$$
$$\gamma \longmapsto \mathsf{v}_0 - \gamma\mathsf{v}_0$$

where $\mathsf{v}_0 \in \mathsf{V}$. Cohomology classes of cocycles form a vector space $H^1(\Gamma_0, \mathsf{V})$. Affine deformations $\rho_{\mathsf{u}_1}, \rho_{\mathsf{u}_2}$ are conjugate by translation by v_0 if and only if

$$\mathsf{u}_1 - \mathsf{u}_2 = \delta\mathsf{v}_0.$$

Thus $H^1(\Gamma_0, \mathsf{V})$ parametrizes translational conjugacy classes of affine deformations of $\Gamma_0 \subset \mathsf{GL}(\mathsf{V})$.

An important affine deformation of Γ_0 is the *trivial affine deformation:* When $\mathsf{u} = 0$, the affine deformation Γ_{u} equals Γ_0 itself.

2.3. *Margulis's invariant of affine deformations.* Consider the case that $\mathsf{G}_0 = \mathsf{PSL}(2,\mathbb{R})$ and $\mathbb{L}$ is an irreducible representation of G_0. For every positive integer r, let $\mathbb{L}_r$ denote the irreducible representation of G_0 on the $2r$-symmetric power V_r of the standard representation of $\mathsf{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$. The dimension of V_r equals $2r+1$. The central element $-\mathbb{I} \in \mathsf{SL}(2,\mathbb{R})$ acts by $(-1)^{2r} = 1$, so this representation of $\mathsf{SL}(2,\mathbb{R})$) defines a representation of

$$\mathsf{PSL}(2,\mathbb{R}) = \mathsf{SL}(2,\mathbb{R})/\{\pm\mathbb{I}\}.$$

The representation $\mathbb{R}^{2,1}$ introduced in Section 1 is V_1, the case when $r = 1$. Furthermore the G_0-invariant nondegenerate skew-symmetric bilinear form on $\mathbb{R}^2$ induces a nondegenerate *symmetric* bilinear form B on V_r, which we normalize in the following paragraph.

An element $\gamma \in \mathsf{G}_0$ is *hyperbolic* if it corresponds to an element $\tilde{\gamma}$ of $\mathsf{SL}(2,\mathbb{R})$ with distinct real eigenvalues. Necessarily these eigenvalues are reciprocals λ, λ^{-1}, which we can uniquely specify by requiring $|\lambda| < 1$. Furthermore we choose eigenvectors $\mathsf{v}_+, \mathsf{v}_- \in \mathbb{R}^2$ such that:

- $\tilde{\gamma}(\mathsf{v}_+) = \lambda \mathsf{v}_+$;
- $\tilde{\gamma}(\mathsf{v}_-) = \lambda^{-1} \mathsf{v}_-$;
- The ordered basis $\{\mathsf{v}_-, \mathsf{v}_+\}$ is positively oriented.

Then the action $\mathbb{L}_r$ has eigenvalues λ^{2j}, for

$$j = -r, 1-r, \cdots -1, 0, 1, \ldots, r-1, r,$$

where the symmetric product

$$\mathsf{v}_-^{r-j} \mathsf{v}_+^{r+j} \in \mathsf{V}_r$$

is an eigenvector with eigenvalue λ^{2j}. In particular γ fixes the vector

$$\mathsf{x}^0(\gamma) := c\mathsf{v}_-^r \mathsf{v}_+^r,$$

where the scalar c is chosen so that

$$\mathsf{B}(\mathsf{x}^0(\gamma), \mathsf{x}^0(\gamma)) = 1.$$

Call $\mathsf{x}^0(\gamma)$ the *neutral vector* of γ.

The subspaces

$$\mathsf{V}^-(\gamma) := \sum_{j=1}^{r} \mathbb{R}\big(\mathsf{v}_-^{r+j}\mathsf{v}_+^{r-j}\big),$$
$$\mathsf{V}^+(\gamma) := \sum_{j=1}^{r} \mathbb{R}\big(\mathsf{v}_-^{r-j}\mathsf{v}_+^{r+j}\big)$$

are γ-invariant and V enjoys a γ-invariant B-orthogonal direct sum decomposition

$$\mathsf{V} \;=\; \mathsf{V}^-(\gamma) \,\oplus\, \mathbb{R}\big(\mathsf{x}^0(\gamma)\big) \,\oplus\, \mathsf{V}^+(\gamma).$$

For any norm $\| \;\|$ on V, there exists $C, k > 0$ such that

$$\|\gamma^n(\mathsf{v})\| \leq Ce^{-kn}\|\mathsf{v}\| \text{for } \mathsf{v} \in \mathsf{V}^+(\gamma) \tag{2.2}$$
$$\|\gamma^{-n}(\mathsf{v})\| \leq Ce^{-kn}\|\mathsf{v}\| \text{for } \mathsf{v} \in \mathsf{V}^-(\gamma).$$

Furthermore,

$$\mathsf{x}^0(\gamma^n) = |n|\mathsf{x}^0(\gamma)$$

if $n \in \mathbb{Z}, n \neq 0$, and

$$\mathsf{V}^\pm(\gamma^n) = \begin{cases} \mathsf{V}^\pm(\gamma) & \text{if } n > 0 \\ \mathsf{V}^\mp(\gamma) & \text{if } n < 0. \end{cases}$$

For example, consider the hyperbolic one-parameter subgroup A_0 comprising $a(t)$ where

$$a(t) : \begin{cases} \mathsf{v}_+ & \longmapsto e^{t/2}\mathsf{v}_+ \\ \mathsf{v}_- & \longmapsto e^{-t/2}\mathsf{v}_- \,. \end{cases}$$

In that case the action on V_1 corresponds to the one-parameter group of isometries of $\mathbb{R}^{2,1}$ defined by

$$a(t) := \begin{bmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & 1 & 0 \\ \sinh(t) & 0 & \cosh(t) \end{bmatrix} \tag{2.3}$$

with neutral vector

$$\mathsf{x}^0\big(a(t)\big) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.4}$$

when $t \neq 0$.

Next suppose that $g \in \mathsf{Aff}(\mathsf{E})$ is an affine transformation whose linear part $\gamma = \mathbb{L}(\gamma)$ is hyperbolic. Then there exists a unique affine line $l_g \subset E$ which is g-invariant. The line l_g is parallel to the neutral vector $\mathsf{x}^0(\gamma)$. The restriction of g

to l_g is a translation by the vector

$$\mathsf{B}\big(gx - x\,,\, \mathsf{x}^0(\gamma)\big)\ \mathsf{x}^0(\gamma)$$

where $\mathsf{x}^0(\gamma)$ is chosen so that $\mathsf{B}(\mathsf{x}^0(\gamma), \mathsf{x}^0(\gamma)) = 1$.

Suppose that $\Gamma_0 \subset \mathsf{G}_0$ is a *Schottky group,* that is, a nonabelian discrete subgroup containing only hyperbolic elements. Such a discrete subgroup is a free group of rank at least two. The adjoint representation of G_0 defines an isomorphism of G_0 with the identity component $\mathsf{O}(2,1)^0$ of the orthogonal group of the 3-dimensional Lorentzian vector space $\mathbb{R}^{2,1}$.

Let $\mathsf{u} \in Z^1(\Gamma_0, \mathsf{V})$ be a cocycle defining an affine deformation ρ_{u} of Γ_0. In [33], [34], Margulis constructed the invariant

$$\begin{aligned} \Gamma_0 &\xrightarrow{\alpha_{\mathsf{u}}} \mathbb{R} \\ \gamma &\longmapsto \mathsf{B}(\mathsf{u}(\gamma), \mathsf{x}^0(\gamma)). \end{aligned}$$

This well-defined class function on Γ_0 satisfies

$$\alpha_{\mathsf{u}}(\gamma^n) = |n|\alpha_{\mathsf{u}}(\gamma)$$

and depends only the cohomology class $[\mathsf{u}] \in H^1(\Gamma_0, \mathsf{V})$. Furthermore $\alpha_{\mathsf{u}}(\gamma) = 0$ if and only if $\rho_{\mathsf{u}}(\gamma)$ fixes a point in E. Two affine deformations of a given Γ_0 are conjugate if and only if they have the same Margulis invariant (Drumm-Goldman [19]). An affine deformation γ_{u} of Γ_0 is *radiant* if it satisfies any of the following equivalent conditions:

- Γ_{u} fixes a point;
- Γ_{u} is conjugate to Γ_0;
- The cohomology class $[\mathsf{u}] \in H^1(\Gamma_0, \mathsf{V})$ is zero;
- The Margulis invariant α_{u} is identically zero.

(For further discussion see [9], [10], [18], [22], [21], [23], [28], [31].)

The centralizer of Γ_0 in the general linear group $\mathsf{GL}(\mathbb{R}^3)$ consists of *homotheties*

$$\mathsf{v} \xmapsto{h_\lambda} \lambda\mathsf{v}$$

where $\lambda \neq 0$. The homothety h_λ conjugates an affine deformation (ρ_0, u) to $(\rho_0, \lambda\mathsf{u})$. Thus conjugacy classes of nonradiant affine deformations are parametrized by the projective space $\mathbb{P}\big(H^1(\Gamma_0, \mathsf{V})\big)$. Our main result is that conjugacy classes of proper actions comprise a convex domain in $\mathbb{P}\big(H^1(\Gamma_0, \mathsf{V})\big)$.

3. **Flat bundles associated to affine deformations**

We define two fiber bundles over $U\Sigma$. The first bundle $\mathbb{V}$ is a *vector bundle* associated to the original group Γ_0. The second bundle $\mathbb{E}$ is an *affine bundle* associated to the affine deformation Γ. The vector bundle underlying $\mathbb{E}$ is $\mathbb{V}$. The vector bundle $\mathbb{V}$ has a flat linear connection and the affine bundle $\mathbb{E}$ has a flat affine connection, each denoted ∇. (For the theory of connections on affine bundles, see Kobayashi-Nomizu [29].) We define a vector field Φ on $\mathbb{E}$ which is uniquely determined by the following properties:

- Φ is ∇-horizontal;
- Φ covers the vector field φ defining the geodesic flow on $U\Sigma$.

We derive a direct alternate description of the corresponding flow Φ_t in terms of the Lie group G_0 and its semidirect product $\mathsf{V} \rtimes \mathsf{G}_0$.

3.1. *Semidirect products and homogeneous affine bundles.* Consider a Lie group G_0, a vector space V, and a linear representation

$$\mathsf{G}_0 \xrightarrow{\mathbb{L}} \mathsf{GL}(\mathsf{V}).$$

Let G be the corresponding semidirect product $\mathsf{V} \rtimes \mathsf{G}_0$. Multiplication in G is defined by

$$(\mathsf{v}_1, g_1) \cdot (\mathsf{v}_2, g_2) := (\mathsf{v}_1 + g_1 \mathsf{v}_2, g_1 g_2). \tag{3.1}$$

Projection

$$\mathsf{G} \xrightarrow{\Pi} \mathsf{G}_0$$
$$(\mathsf{v}, g) \longmapsto g$$

defines a trivial bundle with fiber V over G_0. It is equivariant with respect to the action of G on the total space by left-multiplication and the action of G on the base obtained from left-multiplication on G_0 and the homomorphism $\mathbb{L}$. Since $\mathbb{L}$ is a homomorphism, (3.1) implies equivariance of Π.

When $r = 1$, that is, $\mathsf{V} = \mathbb{R}^{2,1}$, then G is the *tangent bundle* $T\mathsf{G}_0$, with its natural Lie group structure. Compare Goldman-Margulis [23] and [22].

Since the fiber of Π equals the vector space V, the fibration $\mathsf{G} \xrightarrow{\Pi} \mathsf{G}_0$ has the structure of a (trivial) *affine bundle* over G_0. Furthermore this structure is G-invariant: (3.1) implies that the action of $\gamma_1 = (\mathsf{v}_1, g_1)$ on the total space covers the action of g_1 on the base. On the fibers the action is affine with linear part $g_1 = \mathbb{L}(\gamma_1)$ and translational part $\mathsf{v}_1 = \mathsf{u}(\gamma_1)$.

Denote the total space of this G-homogeneous affine bundle over G_0 by $\widetilde{\mathbb{E}}$.

Consider Π also as a (trivial) *vector bundle.* By (3.1), this structure is G_0-invariant. Via $\mathbb{L}$, this G_0-homogeneous vector bundle becomes a G-homogeneous vector bundle $\widetilde{\mathbb{V}} \longrightarrow \mathsf{G}_0$.

This G*-homogeneous vector bundle underlies* $\widetilde{\mathbb{E}}$*:* Let $(\mathsf{v}_2' - \mathsf{v}_2, 1)$ be the translation taking (v_2, g_2) to (v_2', g_2). Then (3.1) implies that $\gamma_1 = (\mathsf{v}_1, g_1)$ acts on

$(\mathsf{v}_2' - \mathsf{v}_2, g_2)$ by $\mathbb{L}$:

$$\big((\mathsf{v}_1 + g_1(\mathsf{v}_2')) - (\mathsf{v}_1 + g_1(\mathsf{v}_2)), g_1 g_2\big) \;=\; \big(\mathbb{L}(g_1)(\mathsf{v}_2' - \mathsf{v}_2), g_1 g_2\big).$$

Of course both $\widetilde{\mathbb{E}}$ and $\widetilde{\mathbb{V}}$ identify with G, but each has different actions of the discrete group $\Gamma \cong \Gamma_0$, imparting the different structures of a *flat affine bundle* and a *flat vector bundle* to the respective quotients.

In our examples, $\mathbb{L}$ preserves a a bilinear form B on V. The G_0-invariant bilinear form $V \times V \xrightarrow{\mathsf{B}} \mathbb{R}$ defines a bilinear pairing $\widetilde{\mathbb{V}} \times \widetilde{\mathbb{V}} \xrightarrow{\mathbb{B}} \mathbb{R}$ of vector bundles.

3.2. *Homogeneous connections.* The G-homogeneous affine bundle $\widetilde{\mathbb{E}}$ and the G-homogeneous vector bundle $\widetilde{\mathbb{V}}$ admit flat connections (each denoted $\tilde{\nabla}$) as follows. To specify $\tilde{\nabla}$, it suffices to define the covariant derivative of a section $\tilde{s}$ over a smooth path $g(t)$ in the base. For either $\widetilde{\mathbb{E}} = \mathsf{G}$ or $\widetilde{\mathbb{V}} = \mathsf{G}$, a section is determined by a smooth path $\mathsf{v}(t) \in \mathsf{V}$:

$$\begin{aligned} \mathbb{R} &\xrightarrow{\tilde{s}} \mathsf{V} \rtimes \mathsf{G}_0 = \mathsf{G} \\ t &\longmapsto (\mathsf{v}(t), g(t)). \end{aligned}$$

Define

$$\frac{D}{dt}\tilde{s}(t) := \frac{d}{dt}\mathsf{v}(t) \in \mathsf{V}.$$

Now if $\tilde{s}$ is a section of $\widetilde{\mathbb{E}}$ or $\widetilde{\mathbb{V}}$, and X is a tangent vector field, define

$$\tilde{\nabla}_X(\tilde{s}) = \frac{D}{dt}\tilde{s}(g(t))$$

where $g(t)$ is any smooth path with $g'(t) = X\big(g(t)\big)$. The resulting covariant differentiation operators define connections on $\mathbb{V}$ and $\mathbb{E}$ which are invariant under the respective G-actions.

3.3. *Flatness.* For each $\mathsf{v} \in \mathsf{V}$,

$$\begin{aligned} \mathsf{G}_0 &\xrightarrow{\tilde{s}_{\mathsf{v}}} \mathsf{V} \rtimes \mathsf{G}_0 = \mathsf{G} \\ g &\longmapsto (\mathsf{v}, g) \end{aligned}$$

defines a section whose image is the coset $\{\mathsf{v}\} \times \mathsf{G}_0 \subset \mathsf{G}$. Clearly these sections are parallel with respect to $\tilde{\nabla}$. Since the sections $\tilde{s}_{\mathsf{v}}$ foliate G, the connections are flat.

If $\mathbb{L}(\Gamma)$ preserves a bilinear form B on V, the bilinear pairing $\mathbb{B}$ on $\widetilde{\mathbb{V}}$ is parallel with respect to $\tilde{\nabla}$.

4. Sections and subbundles

Now we describe the sections and subbundles of the homogeneous bundles over $\mathsf{G}_0 \cong \mathsf{PSL}(2,\mathbb{R})$ associated to the irreducible $(2r+1)$-dimensional representation V_r.

4.1. *The flow on the affine bundle.* Right-multiplication by $a(-t)$ on G defines a flow $\widetilde{\Phi}_t$ on $\widetilde{\mathbb{E}}$. Since $\mathsf{A}_0 \subset \mathsf{G}_0$, this flow covers the flow φ_t on G_0 defined by right-multiplication by $a(-t)$ on G_0, where $a(-t)$ is defined by (2.3). That is, the diagram

$$\begin{tikzcd} \widetilde{\mathbb{E}} \arrow[r, "\widetilde{\Phi}_t"] \arrow[d, "\tilde{\Pi}"'] & \widetilde{\mathbb{E}} \arrow[d, "\tilde{\Pi}"] \\ U\mathsf{H}^2 \arrow[r, "\widetilde{\varphi}_t"'] & U\mathsf{H}^2 \end{tikzcd}$$

commutes. The vector field $\widetilde{\Phi}$ on $\widetilde{\mathbb{E}}$ generating $\widetilde{\Phi}_t$ covers the vector field $\widetilde{\varphi}$ generating $\widetilde{\varphi}_t$.

Furthermore,

$$\tilde{s}_{\mathsf{v}}\big(ga(-t)\big) = \big(\mathsf{v}, ga(-t)\big) = (\mathsf{v}, g)\big(0, a(-t)\big) = \tilde{s}_{\mathsf{v}}(g)a(-t)$$

implies

$$\tilde{s}_{\mathsf{v}} \circ \widetilde{\varphi}_t = \widetilde{\Phi}_t \circ \tilde{s}_{\mathsf{v}},$$

whence $\widetilde{\Phi}$ is the $\widetilde{\nabla}$-*horizontal lift* of $\widetilde{\varphi}$.

The flow $\widetilde{\Phi}_t$ commutes with the action of G. Thus $\widetilde{\Phi}_t$ *is a flow on the flat* G-*homogeneous affine bundle* $\widetilde{\mathbb{E}}$ *covering* φ_t.

Right-multiplication by $a(t)$ on G also defines a flow on the flat G-homogeneous vector bundle $\widetilde{\mathbb{V}}$ covering $\widetilde{\varphi}_t$. Identifying $\widetilde{\mathbb{V}}$ as the vector bundle underlying $\widetilde{\mathbb{E}}$, we see that the $\mathbb{R}$-action is just the linearization $D\widetilde{\Phi}_t$ of the action $\widetilde{\Phi}_t$:

$$\begin{tikzcd} \widetilde{\mathbb{V}} \arrow[r, "D\widetilde{\Phi}_t"] \arrow[d] & \widetilde{\mathbb{V}} \arrow[d] \\ U\mathsf{H}^2 \arrow[r, "\widetilde{\varphi}_t"'] & U\mathsf{H}^2 . \end{tikzcd}$$

4.2. *The neutral section.* The G-action and the flow $D\widetilde{\Phi}_t$ on $\widetilde{\mathbb{V}}$ preserve a section $\tilde{\nu} \in \widetilde{\mathbb{V}}$ defined as follows. The one-parameter subgroup A_0 fixes the neutral vector $\mathsf{v}_0 \in \mathsf{V}$ defined in (2.4). Let $\tilde{\nu}$ denote the section of $\widetilde{\mathbb{V}}$ defined by:

$$\begin{aligned} U\mathsf{H}^2 \approx \mathsf{G}_0 &\xrightarrow{\tilde{\nu}} \mathsf{V} \rtimes \mathsf{G}_0 \approx \widetilde{\mathbb{V}} \\ g &\longmapsto (g\mathsf{v}_0, g). \end{aligned}$$

In terms of the group operation on G, this section is given by *right-multiplication* by $\mathsf{v}_0 \in \mathsf{V} \subset \mathsf{G}$ acting on $g \in \mathsf{G}_0 \subset \mathsf{G}$. Since $\mathbb{L}$ is a homomorphism,

$$\tilde{\nu}(hg) = h\tilde{\nu}(g),$$

so that $\tilde{\nu}$ defines a G-invariant section of $\widetilde{\mathbb{V}}$.

Although $\tilde{\nu}$ is not parallel in every direction, it is parallel along the flow $\widetilde{\varphi}_t$:

LEMMA 4.1. $\widetilde{\nabla}_{\widetilde{\varphi}}(\tilde{\nu}) = 0$.

Proof. Let $g \in \mathsf{G}_0$. Then

$$\widetilde{\nabla}_{\widetilde{\varphi}}(\tilde{\nu})(g) = \left.\frac{D}{dt}\right|_{t=0} \tilde{\nu}(\widetilde{\varphi}_t(g)) = \left.\frac{D}{dt}\right|_{t=0} g a(-t)\mathsf{v}_0 = 0$$

since $a(-t)\mathsf{v}_0 = \mathsf{v}_0$ is constant. □

The section $\tilde{\nu}$ is the diffuse analogue of the neutral eigenvector $\mathsf{x}^0(\gamma)$ of a hyperbolic element $\gamma \in \mathsf{O}(2,1)^0$ discussed in subsection 2.3. Another approach to the flat connection ∇ and the neutral section ν is given in Labourie [31].

4.3. *Stable and unstable subbundles.* Let $\mathsf{V}^0 \subset \mathsf{V}$ denote the line of vectors fixed by A_0, that is, the line spanned by v_0. The eigenvalues of $a(t)$ acting on V are the $2r+1$ distinct positive real numbers

$$e^{rt}, e^{(r-1)t}, \ldots, 1, \ldots, e^{(1-r)t}, e^{-rt}$$

and the eigenspace decomposition of V is invariant under $a(t)$. Let V^+ denote the sum of eigenspaces for eigenvalues > 1 and V^- denote the sum of eigenspaces for eigenvalues < 1. The corresponding decomposition

$$\mathsf{V} = \mathsf{V}^- \oplus \mathsf{V}^0 \oplus \mathsf{V}^+ \tag{4.1}$$

is $a(t)$-invariant and defines a (left)G-invariant decomposition of the vector bundle $\widetilde{\mathbb{V}}$ into subspaces which are invariant under $D\widetilde{\Phi}_t$.

4.4. *Bundles over $U\Sigma$.* Let $\Gamma \subset \mathsf{G}$ be an affine deformation of a discrete subgroup $\Gamma_0 \subset \mathsf{G}_0$. Since Γ is a discrete subgroup of G, the quotient $\mathbb{E} := \widetilde{\mathbb{E}}/\Gamma$ is an affine bundle over $U\Sigma = U\mathsf{H}^2/\Gamma_0$ and inherits a flat connection ∇ from the flat connection $\widetilde{\nabla}$ on $\widetilde{\mathbb{E}}$. Furthermore the flow $\widetilde{\Phi}_t$ on $\widetilde{\mathbb{E}}$ descends to a flow Φ_t on $\mathbb{E}$ which is the horizontal lift of the flow φ_t on $U\Sigma$.

The vector bundle $\mathbb{V}$ underlying $\mathbb{E}$ is the quotient

$$\mathbb{V} := \widetilde{\mathbb{V}}/\Gamma = \widetilde{\mathbb{V}}/\Gamma_0$$

and inherits a flat linear connection ∇ from the flat linear connection $\widetilde{\nabla}$ on $\widetilde{\mathbb{V}}$. The flow $D\widetilde{\Phi}_t$ on $\widetilde{\mathbb{V}}$ covering $\widetilde{\varphi}_t$, the neutral section $\tilde{\nu}$, and the stable-unstable splitting

(4.1) all descend to a flow $D\Phi_t$, a section ν and a splitting

$$\mathbb{V} = \mathbb{V}^- \oplus \mathbb{V}^0 \oplus \mathbb{V}^+ \tag{4.2}$$

of the flat vector bundle $\mathbb{V}$ over $U\Sigma$.

There exists a Euclidean metric g on $\mathbb{V}$ with the following properties:

- The neutral section ν is bounded with respect to g;
- The bilinear form $\mathbb{B}$ is bounded with respect to g;
- The flat linear connection ∇ is bounded with respect to g;
- *Hyperbolicity:* The flow $D\Phi_t$ exponentially expands the subbundle V^+ and exponentially contracts V^-. Explicitly, there exist constants $C, k > 0$, so that

$$\|D(\Phi_t)\mathsf{v}\| \leq Ce^{kt}\|\mathsf{v}\| \quad \text{as } t \longrightarrow -\infty, \tag{4.3}$$

for $\mathsf{v} \in \mathbb{V}^+$, and

$$\|D(\Phi_t)\mathsf{v}\| \leq Ce^{-kt}\|\mathsf{v}\| \quad \text{as } t \longrightarrow +\infty \tag{4.4}$$

for $\mathsf{v} \in \mathbb{V}^-$.

To construct such a metric, first choose a Euclidean inner product on the vector space V. Then extend it to a left G_0-invariant Euclidean metric on the vector bundle

$$\widetilde{\mathbb{V}} \approx \mathsf{G} = \mathsf{V} \rtimes \mathsf{G}_0.$$

Hyperbolicity follows from the description of the adjoint action of A_0 as in (2.2). With this Euclidean metric, the space $\Gamma(\mathbb{V})$ of continuous sections of $\mathbb{V}$ is a Banach space.

5. Proper Γ-actions and proper $\mathbb{R}$-actions

Let X be a locally compact Hausdorff space with homeomorphism group $\mathsf{Homeo}(X)$. Suppose that H is a closed subgroup of $\mathsf{Homeo}(X)$ with respect to the compact-open topology. Recall that H acts *properly* if the mapping

$$\begin{aligned} H \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

is a proper mapping. That is, for every pair of compact subsets $K_1, K_2 \subset X$, the set

$$\{g \in H \mid gK_1 \cap K_2 \neq \varnothing\}$$

is compact. The usual notion of a *properly discontinuous* action is a proper action where H is given the discrete topology. In this case, the quotient X/H is a Hausdorff space. If H acts freely, then the quotient mapping $X \longrightarrow X/H$ is a covering space. For background on proper actions, see Bourbaki [7], Koszul [30] and Palais [37].

The question of properness of the Γ-action is equivalent to that of properness of an action of $\mathbb{R}$.

PROPOSITION 5.1. *An affine deformation* Γ *acts properly on* E *if and only if* A_0 *acts properly by right-multiplication on* G/Γ.

The following lemma is a key tool in our argument. (A related statement is proved in Benoist [5, Lemma 3.1.1]. For a proof in a different context see Rieffel [38], [39].)

LEMMA 5.2. *Let* X *be a locally compact Hausdorff space. Let* A *and* B *be commuting groups of homeomorphisms of* X, *each of which acts properly on* X. *Then the following conditions are equivalent*:

(1) A *acts properly on* X/B;

(2) B *acts properly on* X/A;

(3) $A \times B$ *acts properly on* X.

Proof. We prove (3) $\Longrightarrow$ (1). Suppose $A \times B$ acts properly on X but A does not act properly on X/B. Then a B-invariant subset $H \subset X$ exists such that:

- $H/B \subset X/B$ is compact;
- The subset
$$A_{H/B} := \{a \in A \mid a(H/B) \cap (H/B) \neq \varnothing\}$$
is not compact.

We claim a compact subset $K \subset X$ exists satisfying $B \cdot K = H$.

Denote the quotient mapping by
$$X \xrightarrow{\Pi_B} X/B.$$
For each $x \in \Pi_B^{-1}(H/B)$, choose a precompact open neighborhood $U(x) \subset X$. The images $\Pi_B(U(x))$ define an open covering of H/B. Choose a finite subset $x_1, \dots, x_l$ such that the open sets $\Pi_B(U(x_i))$ cover H/B. Then the union
$$K' := \bigcup_{i=1}^{l} \overline{U(x_i)}$$
is a compact subset of X such that $\Pi_B(K') \supset H/B$. Taking $K = K' \cap \Pi_B^{-1}(H/B)$ proves the claim.

Since A acts properly on X, the subset
$$(A \times B)_K := \{(a,b) \in A \times B \mid aK \cap bK \neq \varnothing\}$$
is compact. However $A_{H/B}$ is the image of the compact set $(A \times B)_K$ under Cartesian projection $A \times B \longrightarrow A$ and is compact, a contradiction.

We prove that (1) $\Longrightarrow$ (3). Suppose that A acts properly on X/B and $K \subset X$ is compact. Cartesian projection $A \times B \longrightarrow A$ maps

$$(A \times B)_K \longrightarrow A_{(B \cdot K)/B}. \tag{5.1}$$

A acts properly on X/B implies $A_{(B \cdot K)/B}$ is a compact subset of A. Because (5.1) is a proper map, $(A \times B)_K$ is compact, as desired.

Thus (3) $\Longleftrightarrow$ (1) . The proof that (3) $\Longleftrightarrow$ (2) is similar. □

Proof of Proposition 5.1. Apply Lemma 5.2 with $X = \mathsf{G}$ and A the action of $\mathsf{A}_0 \cong \mathbb{R}$ by right-multiplication and B the action of Γ by left-multiplication. The lemma implies that Γ acts properly on $\mathsf{E} = \mathsf{G}/\mathsf{G}_0$ if and only if G_0 acts properly on $\Gamma\backslash\mathsf{G}$.

Apply the Cartan decomposition $\mathsf{G}_0 = \mathsf{K}_0\mathsf{A}_0\mathsf{K}_0$. Since K_0 is compact, the action of G_0 on E is proper if and only if its restriction to A_0 is proper. □

6. **Labourie's diffusion of Margulis's invariant**

In [31], Labourie defined a function

$$U\Sigma \xrightarrow{F_{\mathsf{u}}} \mathbb{R},$$

corresponding to the invariant α_{u} defined by Margulis [33], [34]. Margulis's invariant $\alpha = \alpha_{\mathsf{u}}$ is an $\mathbb{R}$-valued class function on Γ_0 whose value on $\gamma \in \Gamma_0$ equals

$$B\big(\rho_{\mathsf{u}}(\gamma)O - O, \mathsf{x}^0(\gamma)\big)$$

where O is the origin and $\mathsf{x}^0(\gamma) \in \mathsf{V}$ is the neutral vector of γ (see §2.3 and the references listed there).

Now the origin $O \in E$ will be replaced by a section s of $\mathbb{E}$, the vector $\mathsf{x}^0(\gamma) \in \mathsf{V}$ will be replaced by the neutral section ν of $\mathbb{V}$, and the linear action of Γ_0 on V will be replaced by the geodesic flow φ_t on $U\Sigma$.

Let s be a section of $\mathbb{E}$. Suppose s is C^1 along φ. That is, the function

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{E} \\ t &\longmapsto s\big(\varphi_t(p)\big) \end{aligned}$$

is C^1 for all $p \in U\Sigma$. Its covariant derivative with respect to φ is a continuous section $\nabla_\varphi(s) \in \mathcal{R}^0(\mathbb{V})$. Pairing with $\nu \in \mathcal{R}^0(\mathbb{V})$ via

$$\mathbb{V} \times \mathbb{V} \xrightarrow{\mathbb{B}} \mathbb{R}$$

produces a continuous function $U\Sigma \xrightarrow{F_{\mathsf{u},s}} \mathbb{R}$ defined by:

$$F_{\mathsf{u},s} := \mathbb{B}(\nabla_\varphi(s), \nu).$$

6.1. *The invariant is continuous.* Let $\mathcal{S}(\mathbb{E})$ denote the space of continuous sections s of $\mathbb{E}$ over $U_{\mathsf{rec}}\Sigma$ which are differentiable along φ and the covariant derivative $\nabla_\varphi(s)$ is continuous. If $s \in \mathcal{S}(\mathbb{E})$, then $F_{\mathsf{u},s}$ is continuous.

For each probability measure $\mu \in \mathcal{P}(U\Sigma)$,

$$\int_{U\Sigma} F_{\mathsf{u},s}\, d\mu$$

is a well-defined real number and the function

$$\begin{aligned} \mathcal{P}(U\Sigma) &\longrightarrow \mathbb{R} \\ \mu &\longmapsto \int_{U\Sigma} F_{\mathsf{u},s}\, d\mu \end{aligned}$$

is continuous in the weak $\star$-topology on $\mathcal{P}(U\Sigma)$. Furthermore its restriction to the subspace $\mathcal{C}(\Sigma) \subset \mathcal{P}(U\Sigma)$ of Φ-invariant measures is also continuous in the weak $\star$-topology.

6.2. *The invariant is independent of the section.*

LEMMA 6.1. *Let $s_1, s_2 \in \mathcal{S}(\mathbb{E})$ and μ be a φ-invariant Borel probability measure on $U\Sigma$. Then*

$$\int_{U\Sigma} F_{\mathsf{u},s_1}\, d\mu = \int_{U\Sigma} F_{\mathsf{u},s_2}\, d\mu.$$

Proof. The difference $\mathsf{s} = s_1 - s_2$ of the sections s_1, s_2 of the affine bundle $\mathbb{E}$ is a section of the vector bundle $\mathbb{V}$ and

$$\nabla_\varphi(s_1) - \nabla_\varphi(s_2) = \nabla_\varphi(\mathsf{s}).$$

Therefore

$$\begin{aligned} \int_{U\Sigma} F_{\mathsf{u},s_1}\, d\mu - \int_{U\Sigma} F_{\mathsf{u},s_2}\, d\mu &= \int_{U\Sigma} \mathbb{B}(\nabla_\varphi(\mathsf{s}), \nu)\, d\mu \\ &= \int_{U\Sigma} \varphi\mathbb{B}(\mathsf{s}, \nu)\, d\mu - \int_{U\Sigma} \mathbb{B}(\mathsf{s}, \nabla_\varphi(\nu))\, d\mu. \end{aligned}$$

The first term vanishes since μ is φ-invariant:

$$\begin{aligned} \int_{U\Sigma} \varphi\mathbb{B}(\mathsf{s}, \nu)\, d\mu &= \int_{U\Sigma} \frac{d}{dt}(\varphi_t)^*\big(\mathbb{B}(\mathsf{s}, \nu)\big)\, d\mu \\ &= \frac{d}{dt}\int_{U\Sigma} (\varphi_t)^*\big(\mathbb{B}(\mathsf{s}, \nu)\big)\, d\mu \\ &= \frac{d}{dt}\int_{U\Sigma} \mathbb{B}(\mathsf{s}, \nu) d\big((\varphi_t)_*\mu\big) = 0. \end{aligned}$$

The second term vanishes since $\nabla_\varphi(\nu) = 0$ (Lemma 4.1). $\square$

Thus
$$\Psi_{[\mathsf{u}]}(\mu) := \int_{U\Sigma} F_{\mathsf{u},s}\, d\mu$$
is a well-defined continuous function $\mathscr{C}(\Sigma) \xrightarrow{\Psi_{[\mathsf{u}]}} \mathbb{R}$ which is independent of the section s used to define it.

6.3. *Periodic orbits.* Suppose
$$t \longmapsto \varphi_t(x_0)$$
defines a periodic orbit of the geodesic flow φ with *period* $T > 0$. That is, T is the least positive number such that $\varphi_T(x_0) = x_0$. Suppose that γ is the corresponding element of $\Gamma_0 \cong \pi_1(\Sigma)$. Then T equals the length $\ell(\gamma)$ of the corresponding closed geodesic on Σ.

If $x_0 \in U\Sigma$ and $T > 0$, let $\varphi^{x_0}_{[0,T]}$ denote the map

$$\begin{aligned} [0,T] &\xrightarrow{\varphi^{x_0}_{[0,T]}} U\Sigma \\ t &\longmapsto \varphi_t(x_0). \end{aligned} \tag{6.1}$$

Then
$$\mu_\gamma := \frac{1}{T}(\varphi^{x_0}_{[0,T]})_*\big(\mu_{[0,T]}\big) \tag{6.2}$$
defines the *geodesic current associated to the periodic orbit* γ*,* where $\mu_{[0,T]}$ denotes Lebesgue measure on $[0,T]$.

PROPOSITION 6.2 (Labourie [31, Prop. 4.2]). *Let* $\gamma \in \Gamma_0$ *be hyperbolic and let* $\mu_\gamma \in \mathscr{C}_{\mathsf{per}}(\Sigma)$ *be the corresponding geodesic current. Then*
$$\alpha(\gamma) = \ell(\gamma) \int_{U\Sigma} F_{\mathsf{u},s}\, d\mu_\gamma. \tag{6.3}$$

6.3.1. *Proof.* Let $x_0 \in U\Sigma$ be a point on the periodic orbit, and consider the map (6.1) and the geodesic current (6.2).

Choose a section $s \in \mathscr{S}(\mathbb{E})$. Pull s back by the covering space
$$U\mathsf{H}^2 \approx \mathsf{G}_0 \xrightarrow{\Pi} U\Sigma$$
to a section of $\widetilde{\mathbb{E}} = \Pi^*\mathbb{E}$. Since $\widetilde{\mathbb{E}} \longrightarrow U\mathsf{H}^2$ is a trivial $\mathscr{E}$-bundle, this section is the graph of a map
$$\mathsf{G}_0 \xrightarrow{\tilde{\mathsf{v}}} \mathsf{E}$$
satisfying
$$\tilde{\mathsf{v}} \circ \gamma = \rho(\gamma)\tilde{\mathsf{v}}.$$
Then
$$\int_{U\Sigma} F_{\mathsf{u},s}\, d\mu_\gamma = \frac{1}{T}\int_0^T F_{\mathsf{u},s}(\varphi_t(x_0))dt, \tag{6.4}$$
which we evaluate by passing to the covering space $U\mathsf{H}^2 \approx \mathsf{G}_0$.

Lift the basepoint x_0 to $\tilde{x}_0 \in U\mathsf{H}^2$. Then $\tilde{x}_0$ corresponds via $\mathscr{E}$ to some $g_0 \in \mathsf{G}_0$, where $\mathscr{E}$ is as defined in (1.3). The path $\varphi^{x_0}_{[0,T]}$ lifts to the map

$$\begin{aligned} [0,T] &\xrightarrow{\widetilde{\varphi}^{\tilde{x}_0}_{[0,T]}} U\mathsf{H}^2 \\ t &\longmapsto \widetilde{\varphi}_t(\tilde{x}_0) \longleftrightarrow g_0 a(-t). \end{aligned}$$

The periodic orbit lifts to the trajectory

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathsf{G}_0 \\ t &\longmapsto g_0 a(-t). \end{aligned}$$

Since the periodic orbit corresponds to the deck transformation γ (which acts by left-multiplication on G_0),

$$\gamma g_0 = g_0 a(-T)$$

which implies

$$\gamma = g_0 a(-T) g_0^{-1}. \tag{6.5}$$

Evaluate $\nabla_\varphi s$ and ν along the trajectory $\widetilde{\varphi}^{\tilde{x}_0}_{[0,T]}$ by lifting to the covering space and computing in G:

$$\big(\nabla_\varphi \tilde{s}\big)(\varphi_t \tilde{x}_0) = \frac{D}{dt}\tilde{\mathsf{v}}(g_0 a(-t)). \tag{6.6}$$

In semidirect product coordinates, the lift $\tilde{\nu}$ is defined by the map

$$\begin{aligned} \mathsf{G}_0 &\xrightarrow{\tilde{\nu}} \mathsf{V} \\ g &\longmapsto \mathbb{L}(g)\mathsf{v}_0. \end{aligned} \tag{6.7}$$

LEMMA 6.3. *For any $g \in \mathsf{G}_0$ and $t \in \mathbb{R}$,*

$$\tilde{\nu}\big(ga(-t)\big) = \mathsf{x}^0(\gamma).$$

Proof.

$$\begin{aligned} \tilde{\nu}\big(ga(-t)\big) &= \mathbb{L}\big(ga(-t)\big)\,\mathsf{v}_0 && \big(\text{by (6.7)}\big) \\ &= \mathbb{L}\big(ga(-t)\big)\,\mathsf{x}^0\big(a(-T)\big) \\ &= \mathsf{x}^0\big((ga(-t))a(-T)(ga(-t))^{-1}\big) \\ &= \mathsf{x}^0\big((ga(-T)g^{-1}\big) = \mathsf{x}^0(\gamma) && (\text{by (6.5)}). \end{aligned}$$

This concludes the proof. $\square$

Conclusion of proof of Proposition 6.2. Evaluate the integrand in (6.4):

$$\begin{aligned} F_{\mathsf{u},s}(\varphi_t x_0) &= \mathbb{B}(\nabla_\varphi s, \nu)(\varphi_t x_0) \\ &= \mathbb{B}\big(\nabla_\varphi \tilde{s}(g_0 a(-t)), \tilde{\nu}(g_0 a(-t)\big) \\ &= \mathsf{B}\left(\frac{D}{dt}\tilde{\mathsf{v}}(g_0 a(-t)), \mathsf{x}^0(\gamma)\right) \ \big(\text{by Lemma 6.3 and (6.6)}\big) \\ &= \frac{d}{dt}\mathsf{B}\big(\tilde{\mathsf{v}}(g_0 a(-t)), \mathsf{x}^0(\gamma)\big) \end{aligned}$$

and

$$\begin{aligned} \int_0^T F_{\mathsf{u},s}(\varphi_t x_0)dt &= \mathsf{B}\big(\tilde{\mathsf{v}}(g_0 a(-T)), \mathsf{x}^0(\gamma)\big) - \mathsf{B}\big(\tilde{\mathsf{v}}(g_0), \mathsf{x}^0(\gamma)\big) \\ &= \mathsf{B}\big(\tilde{\mathsf{v}}(\gamma g_0) - \tilde{\mathsf{v}}(g_0), \mathsf{x}^0(\gamma)\big) \\ &= \mathsf{B}\big(\rho(\gamma)\tilde{\mathsf{v}}(g_0) - \tilde{\mathsf{v}}(g_0), \mathsf{x}^0(\gamma)\big) = \alpha(\gamma) \end{aligned}$$

as claimed. □

7. Nonproper deformations

Now we prove that $0 \notin \Psi_{[\mathsf{u}]}(\mathscr{C}(\Sigma))$ implies Γ_{u} acts properly.

PROPOSITION 7.1. *Suppose that Γ_{u} is a nonproper affine deformation. Then there exists a geodesic current $\mu \in \mathscr{C}(\Sigma)$ (that is, a φ-invariant Borel probability measure) such that $\Psi_{[\mathsf{u}]}(\mu) = 0$.*

Proof. By Proposition 5.1, we may assume that the flow Φ_t defines a nonproper action on $\mathbb{E}$. Choose compact subsets $K_1, K_2 \subset \mathbb{E}$ for which the set of $t \in \mathbb{R}$ such that

$$\Phi_t(K_1) \cap K_2 \neq \varnothing$$

is noncompact. Choose an unbounded sequence $t_n \in \mathbb{R}$, and sequences $P_n \in K_1$ and $Q_n \in K_2$ such that

$$\Phi_{t_n} P_n = Q_n.$$

Passing to subsequences, assume that $t_n \nearrow +\infty$ and

$$\lim_{n\to\infty} P_n = P_\infty, \qquad \lim_{n\to\infty} Q_n = Q_\infty \tag{7.1}$$

for some $P_\infty, Q_\infty \in \mathbb{E}$. For $n = 1, \ldots, \infty$, the images

$$p_n = \Pi(P_n),\ q_n = \Pi(Q_n)$$

are points in $U\Sigma$ such that

$$\lim_{n\to\infty} p_n = p_\infty, \qquad \lim_{n\to\infty} q_n = q_\infty \tag{7.2}$$

and $\varphi_{t_n}(p_n) = q_n$. Choose $R > 0$ such that

$$\begin{aligned} d(p_\infty, U\,\mathsf{core}(\Sigma)) &< R, \\ d(q_\infty, U\,\mathsf{core}(\Sigma)) &< R. \end{aligned}$$

Passing to a subsequence, assume that all p_n, q_n lie in the compact set $U\mathcal{H}_R$ where $\mathcal{H}_R$ is the R-neighborhood of the core, defined in (1.5). Since $\mathcal{H}_R$ is geodesically convex, the curves

$$\{\varphi_t(p_n) \mid 0 \le t \le t_n\}$$

also lie in $U\mathcal{H}_R$(Lemma 1.1).

Choose a section $s \in \mathcal{S}(\mathbb{E})$. We use the splitting (4.2) of $\mathbb{V}$ and the section s of $\mathbb{E}$ to decompose the points $P_1, \dots, P_\infty$ and $Q_1, \dots, Q_\infty$. For $n = 1, \dots, \infty$, write

$$\begin{aligned} P_n &= s(p_n) + \big(\mathsf{p}_n^- + a_n\nu + \mathsf{p}_n^+\big), \\ Q_n &= s(q_n) + \big(q_n^- + b_n\nu + q_n^+\big) \end{aligned} \tag{7.3}$$

where $a_n, b_n \in \mathbb{R}$, $\mathsf{p}_n^-, \mathsf{p}_n^- \in \mathbb{V}^-$, and $q_n^+, q_n^+ \in \mathbb{V}^+$. Since

$$D\Phi_t(\nu) = \nu$$

and $\Phi_{t_n}(P_n) = Q_n$, taking $\mathbb{V}^0$-components in (7.3) yields:

$$\int_0^{t_n} (F_{\mathsf{u},s} \circ \varphi_t)(p_n)\, dt \;=\; b_n - a_n. \tag{7.4}$$

Since s is continuous, (7.2) implies $s(p_n) \to s(p_\infty)$ and $s(q_n) \to s(q_\infty)$. By (7.1),

$$\lim_{n \longrightarrow \infty} (b_n - a_n) \;=\; b_\infty - a_\infty.$$

Thus (7.4) implies

$$\lim_{n\to\infty} \int_0^{t_n} (F_{\mathsf{u},s} \circ \varphi_t)(p_n)\, dt \;=\; b_\infty - a_\infty$$

so that

$$\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} (F_{\mathsf{u},s} \circ \varphi_t)(p_n)\, dt \;=\; 0 \tag{7.5}$$

(because $t_n \nearrow +\infty$).

Now we construct the measure $\mu \in \mathcal{C}(\Sigma)$ such that

$$\Psi_{[\mathsf{u}]}(\mu) \;=\; \int_{U\Sigma} F_{\mathsf{u},s}\, d\mu \;=\; 0.$$

Define approximating probability measures

$$\mu_n \in \mathcal{P}(U\mathcal{H}_R)$$

by pushing forward Lebesgue measure on the orbit through p_n and dividing by t_n:

$$\mu_n := \frac{1}{t_n}\,(\varphi^{p_n}_{[0,t_n]})_*\mu_{[0,t_n]}$$

where $\varphi^{p_n}_{[0,t_n]}$ is as defined in (6.1) and $\mu_{[0,t_n]}$ is Lebesgue measure on $[0,t_n]$. Compactness of $U\mathcal{K}_R$ guarantees a weakly convergent subsequence μ_n in $\mathcal{P}(U\mathcal{K}_R)$. Let

$$\mu_\infty := \lim_{n\longrightarrow\infty} \mu_n.$$

Thus by (7.5),

$$\Psi_{[\mathsf{u}]}(\mu_\infty) = \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}(F_{\mathsf{u},s}\circ\varphi_t)(p_n)dt = 0.$$

Finally we show that μ_∞ is φ-invariant. For any $f \in L^\infty(U\mathcal{K}_R)$ and $\lambda \in \mathbb{R}$,

$$\left|\int f d\mu_n - \int (f\circ\varphi_\lambda)d\mu_n\right| < \frac{2\lambda}{t_n}\|f\|_\infty$$

for n sufficiently large. Passing to the limit, we have

$$\left|\int f\, d\mu_\infty - \int (f\circ\varphi_\lambda)\, d\mu_\infty\right| = 0$$

as desired. □

8. **Proper deformations**

Now we prove that $\Psi_{[\mathsf{u}]}(\mu) \neq 0$ for a proper deformation Γ_{u}. The proof uses a lemma ensuring that the section s can be chosen only to vary in the neutral direction ν. The proof of this fact uses the hyperbolicity of the geodesic flow. The uniform positivity or negativity of $\Psi_{[\mathsf{u}]}$ implies Margulis's "Opposite Sign Lemma" (Margulis' [33], [34], Abels [1], Drumm [14], [17]) as a corollary.

PROPOSITION 8.1. *Suppose that Φ defines a proper action. Then*

$$\Psi_{[\mathsf{u}]}(\mu) \neq 0$$

for all $\mu \in \mathcal{C}(\Sigma)$.

COROLLARY 8.2 (Margulis [33], [34]). *Suppose that $\gamma_1, \gamma_2 \in \Gamma_0$ satisfy*

$$\alpha(\gamma_1) < 0 < \alpha(\gamma_2).$$

Then Γ does not act properly.

Proof of Corollary 8.2. Proposition 6.2 guarantees

$$\Psi_{[\mathsf{u}]}(\mu_{\gamma_1}) < 0 < \Psi_{[\mathsf{u}]}(\mu_{\gamma_2}).$$

Convexity of $\mathscr{C}(\Sigma)$ implies a continuous path $\mu_t \in \mathscr{C}(\Sigma)$ exists, with $t \in [1,2]$, for which

$$\mu_1 = \mu_{\gamma_1},$$
$$\mu_2 = \mu_{\gamma_2}.$$

The function

$$\mathscr{C}(\Sigma) \xrightarrow{\Psi_{[\mathsf{u}]}} \mathbb{R}$$

is continuous. The intermediate value theorem implies $\Psi_{[\mathsf{u}]}(\mu_t) = 0$ for some $1 < t < 2$. Proposition 8.1 implies that Γ does not act properly. $\square$

8.1. *Neutralizing sections.* A section $s \in \mathscr{S}(\mathbb{E})$ is *neutralized* if and only if

$$\nabla_\varphi(s) \in \mathbb{V}^0.$$

This will enable us to relate the properness of the flow $\{\Phi_t\}_{t\in\mathbb{R}}$ to the invariant $\Psi_{[\mathsf{u}]}(\mu)$. To construct neutralized sections, we need the following technical facts.

The flow $\{D\Phi_t\}_{t\in\mathbb{R}}$ on $\mathbb{V}$ and the flow $\{\Phi_t\}_{t\in\mathbb{R}}$ on $\mathbb{E}$ induce (by push-forward) one-parameter groups of continuous bounded operators $(D\Phi_t)_*$ and $(\Phi_t)_*$ on $\Gamma(\mathbb{V})$ and $\Gamma(\mathbb{E})$ respectively over the recurrent part $U_{\mathsf{rec}}\Sigma$. We begin with an elementary observation, whose proof is immediate. (Compare §4.4.)

LEMMA 8.3. *Let $\xi \in \Gamma(\mathbb{V})$. Suppose that*

$$t \longmapsto (D\Phi_t)_*(\xi)$$

is a path in the Banach space $\Gamma(\mathbb{V})$ which is differentiable at $t = 0$. Then $\xi \in \mathscr{S}(\mathbb{E})$ and

$$\left.\frac{d}{dt}\right|_{t=0} (D\Phi_t)_*(\xi) = \nabla_\varphi(\xi).$$

Here is our main lemma:

LEMMA 8.4. *A neutralized section exists over the recurrent part $U_{\mathsf{rec}}\Sigma$.*

Proof. By the hyperbolicity condition (4.3),

$$\|(D\Phi_t)_*(\xi^+)\|_+ \le Ce^{-kt}\|\xi^+\|$$

for any section $\xi^+ \in \Gamma(\mathbb{V}^+)$. Consequently the improper integral

$$\zeta^+ := \int_0^\infty (D\Phi_t)_*(\xi^+)dt$$

converges. Moreover, Lemma 8.3 implies that $\zeta^+ \in \mathscr{S}(\mathbb{E})$ with

$$\nabla_\varphi(\zeta^+) = -\xi^+.$$

Similarly (4.4) implies that, for any section $\xi^- \in \Gamma(\mathbb{V}^-)$, the improper integral

$$\zeta^- := \int_{-\infty}^{0} (D\Phi_t)_*(\xi^-)dt$$

converges, and $\zeta^- \in \mathcal{S}(\mathbb{E})$ with

$$\nabla_\varphi(\zeta^-) = \xi^-.$$

Now let $s \in \mathcal{S}(\mathbb{E})$. Decompose $\nabla_\varphi(s)$ by the splitting (4.2):

$$\nabla_\varphi(s) = \nabla_\varphi^-(s) + \nabla_\varphi^0(s) + \nabla_\varphi^+(s),$$

where $\nabla_\varphi^\pm(s) \in \Gamma(\mathbb{V}^\pm)$. Apply the above discussion to $\xi^- = \nabla_\varphi^-(s)$ and $\xi^+ = \nabla_\varphi^+(s)$. Then

$$s_0 = s + \int_0^\infty (D\Phi_t)_*\big(\nabla_\varphi^-(s)\big)\,dt - \int_0^\infty (D\Phi_{-t})_*\big(\nabla_\varphi^+(s)\big)\,dt$$

lies in $\mathcal{S}(\mathbb{E})$ with $\nabla_\varphi(s_0) = \nabla_\varphi^0(s)$. Hence s_0 is neutralized, as claimed. □

8.2. *Conclusion of the proof.* Let $\mu \in \mathscr{C}(\Sigma)$ so that

$$\Psi_{[\mathsf{u}]}(\mu) = \int F_{\mathsf{u},s}d\mu = 0.$$

Define, for $T > 0$ and $p \in U_{\mathsf{rec}}\Sigma$,

$$g_T(p) := \int_0^T F_{\mathsf{u},s}(\varphi_t p)dt.$$

Since μ is φ-invariant, by Fubini's Theorem,

$$\int g_T d\mu = 0.$$

Therefore, for every $T > 0$, since $U_{\mathsf{rec}}\Sigma$ is connected by Lemma 1.3, there exists $p_T \in U_{\mathsf{rec}}\Sigma$ such that

$$g_T(p_T) = 0.$$

We may assume that $s \in \mathcal{S}(\mathbb{E})$ is neutralized. Then

$$\frac{d}{dt}(D\Phi_t)_*(s) = \nabla_\varphi(s) = F_{\mathsf{u},s}\nu,$$

and

$$(\Phi_T s)(p) = s(\varphi_T p) + \left(\int_0^T F_{\mathsf{u},s}(\varphi_t p)dt\right)\nu \tag{8.1}$$

for any $T > 0$. Thus

$$(\Phi_T s)(p_T) = s(\varphi_T p_T).$$

Let K be the compact set $s(U_{\mathrm{rec}}\Sigma)$. Then, for all $T > 0$,

$$\Phi_T(K) \cap K \neq \varnothing,$$

and $\{\Phi_t\}_{t\in\mathbb{R}}$ is not proper, as claimed. This completes the proof.

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