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Abstract

We show how localization and smoothing techniques can be used to establish universality in the bulk of the spectrum for a fixed positive measure μ on $[-1, 1]$. Assume that μ is a regular measure, and is absolutely continuous in an open interval containing some point x. Assume moreover, that μ' is positive and continuous at x. Then universality for μ holds at x. If the hypothesis holds for x in a compact subset of $(-1, 1)$, universality holds uniformly for such x. Indeed, this follows from universality for the classical Legendre weight. We also establish universality in an L_p sense under weaker assumptions on μ.

1. **Introduction and results**

Let μ be a finite positive Borel measure on $(-1, 1)$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$ satisfying the orthonormality conditions

$$\int_{-1}^{1} p_n p_m d\mu = \delta_{mn}.$$

These orthonormal polynomials satisfy a recurrence relation of the form

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \tag{1.1}$$

where

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} > 0 \text{ and } b_n \in \mathbb{R}, \qquad n \geq 1,$$

and we use the convention $p_{-1} = 0$. Throughout we use

$$w = \frac{d\mu}{dx}$$

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to denote the Radon-Nikodym derivative of μ. A classic result of E. A. Rakhmanov [12] asserts that if $w > 0$ a.e. in $[-1, 1]$, then μ belongs to the Nevai-Blumenthal class $\mathcal{M}$, that is,

$$\lim_{n\to\infty} a_n = \tfrac{1}{2} \text{ and } \lim_{n\to\infty} b_n = 0. \tag{1.2}$$

We note that there are pure jump and pure singularly continuous measures in $\mathcal{M}$, despite the fact that one tends to associate it with weights that are a.e. positive. A class of measures that contains $\mathcal{M}$ is the class of *regular measures* on $[-1, 1]$ (see [13]), defined by the condition

$$\lim_{n\to\infty} \gamma_n^{1/n} = 2.$$

Orthogonal polynomials play an important role in random matrix theory [3], [8]. One of the key limits there involves the reproducing kernel

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y). \tag{1.3}$$

Because of the Christoffel-Darboux formula, it may also be expressed as

$$K_n(x, y) = a_n \frac{p_n(x) p_{n-1}(y) - p_{n-1}(x) p_n(y)}{x - y}. \tag{1.4}$$

Define the normalized kernel

$$\tilde{K}_n(x, y) = w(x)^{1/2} w(y)^{1/2} K_n(x, y). \tag{1.5}$$

The simplest case of the universality law is the limit

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right)}{\tilde{K}_n(x,x)} = \frac{\sin \pi(a-b)}{\pi(a-b)}. \tag{1.6}$$

Typically this holds uniformly for x in a compact subinterval of $(-1, 1)$ and a, b in compact subsets of the real line. Of course, when $a = b$, we interpret the quotient $\sin \pi(a-b)/(\pi(a-b))$ as 1. We cannot hope to survey the vast body of results on universality limits here — the reader may consult [1], [2], [3], [8] and the forthcoming proceedings of the conference devoted to the 60th birthday of Percy Deift.

Our goal here is to present what we believe is a new approach, based on localization and smoothing. Our main result is:

THEOREM 1.1. *Let μ be a finite positive Borel measure on $(-1, 1)$ that is regular. Let $J \subset (-1, 1)$ be compact, and such that μ is absolutely continuous in an open set containing J. Assume moreover, that w is positive and continuous at* *each point of J. Then uniformly for $x \in J$ and a, b in compact subsets of the real line, we have*

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(x + \frac{a}{\widetilde{K}_n(x,x)}, x + \frac{b}{\widetilde{K}_n(x,x)}\right)}{\widetilde{K}_n(x,x)} = \frac{\sin \pi(a-b)}{\pi(a-b)}. \tag{1.7}$$

If J consists of just a single point x, then the hypothesis is that μ is absolutely continuous in some neighborhood $(x-\varepsilon, x+\varepsilon)$ of x, while $w(x) > 0$ and w is continuous at x. This alone is sufficient for universality at x.

COROLLARY 1.2. *Let $m \geq 1$ and*

$$R_m(y_1, y_2, \ldots, y_m) = \det\bigl(\widetilde{K}_n(y_i, y_j)\bigr)_{i,j=1}^m$$

denote the m-point correlation function. Uniformly for $x \in J$, and for given $\{\xi_j\}_{j=1}^m$, we have

$$\lim_{n\to\infty} \frac{1}{\widetilde{K}_n(x,x)^m} R_m\left(x + \frac{\xi_1}{\widetilde{K}_n(x,x)}, x + \frac{\xi_2}{\widetilde{K}_n(x,x)}, \ldots, x + \frac{\xi_m}{\widetilde{K}_n(x,x)}\right) = \det\left(\frac{\sin \pi(\xi_i - \xi_j)}{\pi(\xi_i - \xi_j)}\right)_{i,j=1}^m.$$

COROLLARY 1.3. *Let r, s be nonnegative integers and*

$$K_n^{(r,s)}(x,x) = \sum_{k=0}^{n-1} p_k^{(r)}(x) p_k^{(s)}(x). \tag{1.8}$$

Let

$$\tau_{r,s} = \begin{cases} 0, & r+s \text{ odd} \\ \dfrac{(-1)^{(r-s)/2}}{r+s+1}, & r+s \text{ even}. \end{cases} \tag{1.9}$$

Then uniformly for $x \in J$,

$$\lim_{n\to\infty} \frac{1}{n^{r+s+1}} K_n^{(r,s)}(x,x) = \frac{1}{\pi w(x) \left(1-x^2\right)^{(r+s+1)/2}} \tau_{r,s}. \tag{1.10}$$

Remarks. (a) We believe that the hypotheses above are the weakest imposed thus far guaranteeing universality for a fixed weight on $(-1, 1)$. Most hypotheses imposed so far involve analyticity, for example in [5].

(b) The only reason for restricting a, b to be real in (1.7), is that

$$\widetilde{K}_n\left(x + \frac{a}{\widetilde{K}_n(x,x)}, x + \frac{b}{\widetilde{K}_n(x,x)}\right)$$

involves the weight evaluated at arguments involving a and b. If we consider instead

$$K_n\left(x+\frac{a}{\tilde{K}_n(x,x)}, x+\frac{b}{\tilde{K}_n(x,x)}\right),$$

then the limits hold uniformly for a, b in compact subsets of the plane.

We also present L_p results, assuming less about w:

THEOREM 1.4. *Let μ be a finite positive Borel measure on $(-1,1)$ that is regular. Let $p>0$. Let I be a closed subinterval of $(-1,1)$ in which μ is absolutely continuous, and w is bounded above and below by positive constants.*

(a) *If I' is a closed subinterval of I^0,*

$$\lim_{n\to\infty}\int_{I'}\left|\frac{K_n\left(x+\frac{a}{\tilde{K}_n(x,x)}, x+\frac{b}{\tilde{K}_n(x,x)}\right)}{K_n(x,x)}-\frac{\sin\pi(a-b)}{\pi(a-b)}\right|^p dx=0, \tag{1.11}$$

uniformly for a, b in compact subsets of the real line.

(b) *If, in addition, w is Riemann integrable in I, we may replace*

$$\frac{K_n\left(x+\frac{a}{\tilde{K}_n(x,x)}, x+\frac{b}{\tilde{K}_n(x,x)}\right)}{K_n(x,x)} \quad by \quad \frac{\tilde{K}_n\left(x+\frac{a}{\tilde{K}_n(x,x)}, x+\frac{b}{\tilde{K}_n(x,x)}\right)}{\tilde{K}_n(x,x)}$$

in (1.11).

When we assume only that w is bounded below, and do not assume absolute continuity of μ, we can still prove an L_1 form of universality, see Theorem 5.1.

In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, x, y, s, t. The same symbol does not necessarily denote the same constant in different occurrences. We shall write $C = C(\alpha)$ or $C \neq C(\alpha)$ respectively to denote dependence on, or independence of, the parameter α. Given measures μ^*, $\mu^\#$, we use K_n^*, $K_n^\#$ and p_n^*, $p_n^\#$ to denote respectively their reproducing kernels and orthonormal polynomials. Similarly superscript $*, \#$ are used to distinguish other quantities associated with them. The superscript L denotes quantities associated with the Legendre weight 1 on $[-1,1]$. For $x \in \mathbb{R}$ and $\delta > 0$, we set

$$I(x,\delta) = [x-\delta, x+\delta].$$

The distance from a point x to a set J is denoted $\operatorname{dist}(x, J)$. For such a J, we let

$$I(J,\delta) = \{x : \operatorname{dist}(x, J) \le \delta\}.$$

By $[x]$ we denote the greatest integer $\leq x$. Recall that the nth Christoffel function for a measure μ is

$$\lambda_n(x) = 1/K_n(x,x) = \min_{\deg(P)\leq n-1} \left(\int_{-1}^{1} P^2 d\mu\right) / P^2(x).$$

The most important new idea in this paper is a localization principle for universality. We use it repeatedly in various forms, but the following basic inequality is typical. Suppose that μ, μ^* are measures with $\mu \leq \mu^*$ in $[-1,1]$. Then for $x, y \in [-1,1]$,

$$\begin{aligned} \frac{|K_n(x,y) - K_n^*(x,y)|}{K_n(x,x)} &\leq \left(\frac{K_n(y,y)}{K_n(x,x)}\right)^{1/2} \left[1 - \frac{K_n^*(x,x)}{K_n(x,x)}\right]^{1/2} \\ &= \left(\frac{\lambda_n(x)}{\lambda_n(y)}\right)^{1/2} \left[1 - \frac{\lambda_n(x)}{\lambda_n^*(x)}\right]^{1/2}. \end{aligned}$$

Observe that on the right-hand side, we have only Christoffel functions, and their asymptotics are very well understood.

This paper is organised as follows. In Section 2, we present some asymptotics for Christoffel functions. In Section 3, we prove our localization principle, including the above inequality. In Section 4, we approximate locally the measure μ in Theorem 1.1 by a scaled Jacobi weight and then prove Theorem 1.1. In Section 5, we prove the L_1 result Theorem 5.1, and in Section 6, prove the L_p result Theorem 1.4. In Section 7, we prove Corollaries 1.2 and 1.3.

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2. **Christoffel functions**

We use λ_n^L to denote the nth Christoffel function for the Legendre weight on $[-1,1]$. The methods used to prove the following result are very well known, but I could not find this theorem as stated in the literature. The issue is that known asymptotics for Christoffel functions do not include the increment a/n. We could use existing results in [7], [9], [10], [14] to treat the case where $x + a/n \in J$, and add a proof for the case where this fails, but the amount of effort seems almost the same.

THEOREM 2.1. *Let μ be a regular measure on* $[-1,1]$. *Assume that μ is absolutely continuous in an open set containing a compact set J, and in J, $w = \mu'$*

is positive and continuous. Let $A > 0$. Then uniformly for $a \in [-A, A]$, and $x \in J$,

$$\lim_{n\to\infty} \lambda_n \left(x + \frac{a}{n}\right) \Big/ \lambda_n^L \left(x + \frac{a}{n}\right) = w(x). \tag{2.1}$$

Moreover, uniformly for $n \geq n_0(A)$, $x \in J$, and $a \in [-A, A]$,

$$\lambda_n \left(x + \frac{a}{n}\right) \sim \frac{1}{n}. \tag{2.2}$$

The constants implicit in $\sim$ do not depend on ρ.

Remarks. (a) The notation $\sim$ means that the ratio of the two Christoffel functions is bounded above and below by positive constants independent of n, x and a.

(b) We emphasize that we are assuming that w is continuous in J when regarded as a function defined on $(-1, 1)$.

(c) Using asymptotics for λ_n^L, we can rewrite (2.1) as

$$\lim_{n\to\infty} n\lambda_n \left(x + \frac{a}{n}\right) = \pi\sqrt{1-x^2}w(x).$$

Proof. Let $\varepsilon > 0$ and choose $\delta > 0$ such that μ is absolutely continuous in $I(J, \delta) \subset (-1, 1)$, and such that

$$(1+\varepsilon)^{-1} \leq \frac{w(x)}{w(y)} \leq 1+\varepsilon, \quad x \in I(J, \delta) \text{ with } |x-y| \leq \delta. \tag{2.3}$$

(This is possible because of compactness of J and continuity and positivity of w at every point of J.) Let us fix $x_0 \in J$ and recall that $I(x_0, \delta) = [x_0 - \delta, x_0 + \delta]$. Define a measure μ^* with

$$\mu^* = \mu \text{ in } [-1, 1] \backslash I(x_0, \delta)$$

and in $I(x_0, \delta)$, let μ^* be absolutely continuous, with absolutely continuous component w^* satisfying

$$w^* = w(x_0)(1+\varepsilon) \text{ in } I(x_0, \delta). \tag{2.4}$$

Because of (2.3), $d\mu \leq d\mu^*$ in $[-1, 1]$, so that if λ_n^* is the nth Christoffel function for μ^*, we have for all x,

$$\lambda_n(x) \leq \lambda_n^*(x). \tag{2.5}$$

We now find an upper bound for $\lambda_n^*(x)$ for $x \in I(x_0, \delta/2)$. There exists $r \in (0, 1)$ depending only on δ such that

$$0 \leq 1 - \left(\frac{t-x}{2}\right)^2 \leq r \text{ for } x \in I(x_0, \delta/2) \text{ and } t \in [-1, 1] \backslash I(x_0, \delta). \tag{2.6}$$

(In fact, we may take $r = 1-(\delta/4)^2$.) Let $\eta \in \left(0, \frac{1}{2}\right)$ and choose $\sigma > 1$ so close to 1 that

$$\sigma^{1-\eta} < r^{-\eta/4}. \tag{2.7}$$

Let $m = m(n) = n - 2\,[\eta n/2]$. Fix $x \in I\,(x_0, \delta/2)$ and choose a polynomial P_m of degree $\leq m-1$ such that

$$\lambda_m^L(x) = \int_{-1}^{1} P_m^2 \quad \text{and} \quad P_m^2(x) = 1.$$

Thus P_m is the minimizing polynomial in the Christoffel function for the Legendre weight at x. Let

$$S_n(t) = P_m(t)\left(1 - \left(\frac{t-x}{2}\right)^2\right)^{[\eta n/2]},$$

a polynomial of degree $\leq m - 1 + 2\,[\eta n/2] \leq n-1$ with $S_n(x) = 1$. Then using (2.4) and (2.6),

$$\begin{aligned}
\lambda_n^*(x) &\leq \int_{-1}^{1} S_n^2 d\mu^* \\
&\leq w\,(x_0)\,(1+\varepsilon) \int_{I(x_0,\delta)} P_m^2 \\
&\quad + \|P_m\|^2_{L_\infty([-1,1]\backslash I(x_0,\delta))}\, r^{2[\eta n/2]} \int_{[-1,1]\backslash I(x_0,\delta)} d\mu^* \\
&\leq w\,(x_0)\,(1+\varepsilon)\lambda_m^L(x) + \|P_m\|^2_{L_\infty[-1,1]}\, r^{2[\eta n/2]} \int_{-1}^{1} d\mu^*.
\end{aligned}$$

Now we use the key idea from [7, Lemma 9, p. 450]. For $m \geq m_0\,(\sigma)$, we have

$$\|P_m\|^2_{L_\infty[-1,1]} \leq \sigma^m \int_{-1}^{1} P_m^2 = \sigma^m \lambda_m^L(x).$$

(This holds more generally for any polynomial P of degree $\leq m-1$, and is a consequence of the regularity of the Legendre weight. Alternatively, we could use classic bounds for the Christoffel functions for the Legendre weight.) Then from (2.7), uniformly for $x \in I\,(x_0, \delta/2)$,

$$\begin{aligned}
\lambda_n^*(x) &\leq w\,(x_0)\,(1+\varepsilon)\lambda_m^L(x)\left\{1 + C\left[\sigma^{1-\eta} r^{\eta/2}\right]^n\right\} \\
&\leq w\,(x_0)\,(1+\varepsilon)\lambda_m^L(x)\,\{1+o(1)\},
\end{aligned}$$

so as $\lambda_n \leq \lambda_n^*$,

$$\sup_{x\in I(x_0,\delta/2)} \frac{\lambda_n(x)}{\lambda_n^L(x)} \;\leq\; w(x_0)(1+\varepsilon)\{1+o(1)\} \sup_{x\in I(x_0,\delta)} \frac{\lambda_m^L(x)}{\lambda_n^L(x)}. \tag{2.8}$$

The $o(1)$ term is independent of x_0. Now for large enough n, and some C independent of η, m, n, x_0,

$$\sup_{x\in[-1,1]} \lambda_m^L(x)/\lambda_n^L(x) \le 1 + C\eta. \tag{2.9}$$

Indeed if $\{p_k^L\}$ denote the orthonormal Legendre polynomials, they admit the bound [9, p. 170]

$$\left|p_k^L(x)\right| \le C\left(1 - x^2 + \frac{1}{k^2}\right)^{-1/4}, \qquad x \in [-1,1].$$

Then uniformly for $x \in [-1,1]$,

$$\begin{aligned} 0 \le 1 - \frac{\lambda_n^L(x)}{\lambda_m^L(x)} &= \lambda_n^L(x) \sum_{k=m}^{n-1} \left(p_k^L(x)\right)^2 \\ &\le C\lambda_n^L(x)(n-m) \max_{\frac{n}{2}\le k\le n} \left(1 - x^2 + \frac{1}{k^2}\right)^{-1/2} \\ &\le C\eta n \lambda_n^L(x)\left(1 - x^2 + \frac{1}{n^2}\right)^{-1/2} \\ &\le C\eta, \end{aligned}$$

by classical bounds for Christoffel functions [9, p. 108, Lemma 5]. Thus we have (2.9), and then (2.8) and (2.3) give for $n \ge n_0 = n_0(x_0, \delta)$,

$$\sup_{x\in I(x_0,\delta/2)} \frac{\lambda_n(x)}{\lambda_n^L(x)w(x)} \le (1+\varepsilon)^2(1+C\eta).$$

By covering J with finitely many such intervals $I\ (x_0, \delta/2)$, we obtain for some maximal threshold $n_1 = n_1\ (\varepsilon, \delta, J)$, that for $n \ge n_1$,

$$\sup_{x\in I(J,\delta/2)} \frac{\lambda_n(x)}{\left(\lambda_n^L(x)w(x)\right)} \le (1+\varepsilon)^2(1+C\eta).$$

It is essential here that C is independent of ε, η. Now let $A > 0$ and $|a| \le A$. There exists $n_2 = n_2(A)$ such that for $n \ge n_2$ and all $|a| \le A$ and all $x \in J$, we have $x + \frac{a}{n} \in I\ (J, \delta/2)$. We deduce that

$$\limsup_{n\to\infty} \sup_{a\in[-A,A],x\in J} \frac{\lambda_n\left(x+\frac{a}{n}\right)}{\lambda_n^L\left(x+\frac{a}{n}\right)w(x)} \le (1+\varepsilon)^2\,(1+C\eta)\,.$$

As the left-hand side is independent of the parameters ε, η, we deduce that

$$\limsup_{n\to\infty}\left(\sup_{a\in[-A,A],x\in J} \frac{\lambda_n\left(x+\frac{a}{n}\right)}{\lambda_n^L\left(x+\frac{a}{n}\right)w(x)}\right) \le 1. \tag{2.10}$$

In a similar way, we can establish the converse bound

$$\limsup_{n\to\infty}\left(\sup_{a\in[-A,A],x\in J}\frac{\lambda_n^L\left(x+\frac{a}{n}\right)w(x)}{\lambda_n\left(x+\frac{a}{n}\right)}\right)\le 1. \tag{2.11}$$

Indeed with m, x and η as above, let us choose a polynomial P of degree $\le m-1$ such that

$$\lambda_m(x)=\int_{-1}^{1}P_m^2(t)\,d\mu(t)\quad\text{and}\quad P_m^2(x)=1.$$

Then with S_n as above, and proceeding as above,

$$\begin{aligned}\lambda_n^L(x)&\le\int_{-1}^{1}S_n^2\\&\le[w(x_0)^{-1}(1+\varepsilon)]\int_{I(x_0,\delta)}P_m^2d\mu+\|P_m\|^2_{L_\infty([-1,1]\setminus I(x_0,\delta))}r^{2[\eta n/2]}\int_{[-1,1]\setminus I(x_0,\delta)}1\\&\le[w(x_0)^{-1}(1+\varepsilon)]\,\lambda_m(x)\left\{1+C[\sigma^{1-\eta}r^{\eta/2}]^n\right\},\end{aligned}$$

and so as above,

$$\begin{aligned}\sup_{x\in I(x_0,\delta/2)}\frac{\lambda_m^L(x)}{\lambda_m(x)}&\le\left[w(x_0)^{-1}(1+\varepsilon)(1+o(1))\right]\sup_{x\in I(x_0,\delta/2)}\frac{\lambda_m^L(x)}{\lambda_n^L(x)}\\&\le\left[w(x_0)^{-1}(1+\varepsilon)\right]\{1+o(1)\}\,(1+C\eta)\,.\end{aligned}$$

As n runs through all the positive integers, so does $m=n-2[\eta/2]$. (Indeed, the difference between successive such m is at most 1.) Then (2.11) follows and using monotonicity of λ_n in n, much as above. Together (2.10) and (2.11) give (2.1). Finally, (2.2) follows from standard bounds for the Christoffel function for the Legendre weight. □

3. **Localization**

THEOREM 3.1. *Assume that μ,μ^* are regular measures on $[-1,1]$ which are absolutely continuous in an open interval containing a compact set J. Assume that $w=\mu'$ is positive and continuous in J and*

$$d\mu=d\mu^*\ \textit{in}\ J.$$

Let $A>0$. Then as $n\to\infty$,

$$\sup_{a,b\in[-A,A],x\in J}\left|\left(K_n-K_n^*\right)\left(x+\frac{a}{n},x+\frac{b}{n}\right)\right|/n=o(1)\,. \tag{3.1}$$

Proof. We initially assume that

$$d\mu\le d\mu^*\ \text{in}\ (-1,1). \tag{3.2}$$

The idea is to estimate the L_2 norm of $K_n(x,t) - K_n^*(x,t)$ over $[-1,1]$, and then to use Christoffel function estimates. Now

$$\begin{aligned}\int_{-1}^{1} &\left(K_n(x,t) - K_n^*(x,t)\right)^2 d\mu(t)\\ &= \int_{-1}^{1} K_n^2(x,t)\, d\mu(t) - 2\int_{-1}^{1} K_n(x,t) K_n^*(x,t)\, d\mu(t) + \int_{-1}^{1} K_n^{*2}(x,t)\, d\mu\,(t)\\ &= K_n(x,x) - 2K_n^*(x,x) + \int_{-1}^{1} K_n^{*2}(x,t)\, d\mu(t),\end{aligned}$$

by the reproducing kernel property. As $d\mu \le d\mu^*$, we also have

$$\int_{-1}^{1} K_n^{*2}(x,t)\, d\mu(t) \le \int_{-1}^{1} K_n^{*2}(x,t)\, d\mu^*(t) = K_n^*(x,x).$$

Thus

$$\int_{-1}^{1} \left(K_n(x,t) - K_n^*(x,t)\right)^2 d\mu(t) \le K_n(x,x) - K_n^*(x,x). \tag{3.3}$$

Next for any polynomial P of degree $\le n-1$, we have the Christoffel function estimate

$$|P(y)| \le K_n(y,y)^{1/2} \left(\int_{-1}^{1} P^2 d\mu\right)^{1/2}. \tag{3.4}$$

Applying this to $P(t) = K_n(x,t) - K_n^*(x,t)$ and using (3.3) gives, for all $x, y \in [-1,1]$,

$$\left|K_n(x,y) - K_n^*(x,y)\right| \le K_n(y,y)^{1/2} \left[K_n(x,x) - K_n^*(x,x)\right]^{1/2}$$

so

$$\frac{|K_n(x,y) - K_n^*(x,y)|}{K_n(x,x)} \le \left(\frac{K_n(y,y)}{K_n(x,x)}\right)^{1/2} \left[1 - \frac{K_n^*(x,x)}{K_n(x,x)}\right]^{1/2}. \tag{3.5}$$

Now we set $x = x_0 + a/n$ and $y = x_0 + b/n$, where $a, b \in [-A, A]$ and $x_0 \in J$. By Theorem 2.1, uniformly for such x, we have $K_n^*(x,x)/K_n(x,x) = 1 + o(1)$, because they both have the same asymptotics as for the Legendre weight on $[-1,1]$. Moreover, uniformly for $a, b \in [-A, A]$,

$$K_n\left(x_0 + \frac{b}{n}, x_0 + \frac{b}{n}\right) \sim K_n\left(x_0 + \frac{a}{n}, x_0 + \frac{a}{n}\right) \sim n,$$

and so

$$\sup_{a,b\in[-A,A],x_0\in J} \left|\left(K_n - K_n^*\right)\left(x_0 + \frac{a}{n}, x_0 + \frac{b}{n}\right)\right| \Big/ n = o(1).$$

Now we drop the extra hypothesis (3.2). Define a measure ν by $\nu = \mu = \mu^*$ in J. In $[-1,1]\backslash J$, let

$$d\nu(x) = \max\left\{\operatorname{dist}(x,J), w(x), w^*(x)\right\} dx + d\mu_s(x) + d\mu_s^*(x),$$

where w, w^* and μ_s, μ_s^* are respectively the absolutely continuous and singular components of μ, μ^*. Then $d\mu \leq d\nu$ and $d\mu^* \leq d\nu$, and ν is regular as its absolutely continuous component is positive in $(-1, 1)$, and hence lies in the even smaller class $\mathcal{M}$. Moreover, ν is absolutely continuous in an open interval containing J, and $\nu' = w$ in J. The case above shows that the reproducing kernels for μ and μ^* have the same asymptotics as that for ν, in the sense of (3.1), and hence the same asymptotics as each other. □

4. **Smoothing**

In this section, we approximate μ of Theorem 1.1 by a scaled Legendre Jacobi measure $\mu^\#$ and then prove Theorem 1.1. Recall that $\tilde{K}_n$ is the normalized kernel, given by (1.5). Our smoothing result (which may also be viewed as localization) is:

THEOREM 4.1. *Let μ be as in Theorem 1.1. Let $A > 0$, $\varepsilon \in \left(0, \frac{1}{2}\right)$ and choose $\delta > 0$ such that (2.3) holds. Let $x_0 \in J$. Then there exists C and n_0 such that for $n \geq n_0$,*

$$\sup_{a,b\in[-A,A],x\in I\left(x_0,\frac{\delta}{2}\right)\cap J} \left|(\tilde{K}_n - K_n^L)\left(x + \frac{a}{n}, x + \frac{b}{n}\right)\right| \Big/ n \leq C\varepsilon^{1/2}, \tag{4.1}$$

where C is independent of ε, δ, n, x_0.

Proof. Fix $x_0 \in J$ and let $w^\#$ be the scaled Legendre weight

$$w^\# = w(x_0) \text{ in } (-1, 1).$$

Note that

$$K_n^\#(x, y) = \frac{1}{w(x_0)} K_n^L(x, y). \tag{4.2}$$

(Recall that the superscript L indicates the Legendre weight on $[-1, 1]$.) Because of our localization result Theorem 3.1, we may replace $d\mu$ by $w^*(x)\,dx$, where

$$w^* = w \text{ in } I(x_0, \delta)$$

and

$$w^* = w(x_0) \text{ in } [-1, 1]\backslash I(x_0, \delta),$$

without affecting the asymptotics for $K_n\left(x+\frac{a}{n}, x+\frac{b}{n}\right)$ in the interval $I\left(x_0, \frac{\delta}{2}\right)$. (Note that ε and δ play no role in Theorem 3.1.) Thus, in the sequel, we assume that $w = w(x_0) = w^\#$ in $[-1, 1]\backslash I(x_0, \delta)$, while not changing w in $I(x_0, \delta)$. Observe that (2.3) implies that

$$(1 + \varepsilon)^{-1} \leq \frac{w}{w^\#} \leq 1 + \varepsilon, \text{ in } [-1, 1]. \tag{4.3}$$

Then, much as in the previous section,

$$\begin{aligned}
&\int_{-1}^{1}\left(K_n(x,t)-K_n^{\#}(x,t)\right)^2 w^{\#}(t)\,dt \\
&= \int_{-1}^{1} K_n^2(x,t)w^{\#}(t)dt - 2\int_{-1}^{1} K_n(x,t)K_n^{\#}(x,t)w^{\#}(t)\,dt + \int_{-1}^{1} K_n^{\#2}(x,t)w^{\#}(t)\,dt \\
&= \int_{-1}^{1} K_n^2(x,t)w(t)\,dt + \int_{I(x_0,\delta)} K_n^2(x,t)(w^{\#}-w)(t)\,dt - 2K_n(x,x) + K_n^{\#}(x,x) \\
&= K_n^{\#}(x,x) - K_n(x,x) + \int_{I(x_0,\delta)} K_n^2(x,t)(w^{\#}-w)(t)\,dt.
\end{aligned}$$

Recall that $w = w^{\#}$ in $[-1,1]\backslash I\,(x_0,\delta)$. By (4.3),

$$\int_{I(x_0,\delta)} K_n^2(x,t)\left(w^{\#}-w\right)(t)\,dt \le \varepsilon \int_{I(x_0,\delta)} K_n^2(x,t)w(t)\,dt \le \varepsilon K_n(x,x).$$

Thus

$$\int_{-1}^{1}\left(K_n(x,t)-K_n^{\#}(x,t)\right)^2 w^{\#}(t)\,dt \le K_n^{\#}(x,x) - (1-\varepsilon)\,K_n(x,x). \tag{4.4}$$

Applying an obvious analogue of (3.4) to $P(t) = K_n\,(x,t) - K_n^{\#}(x,t)$ and using (4.4) gives for $x,\ y \in [-1,1]$,

$$\left|K_n(x,y) - K_n^{\#}(x,y)\right| \le K_n^{\#}(y,y)^{1/2}\left[K_n^{\#}(x,x) - (1-\varepsilon)\,K_n(x,x)\right]^{1/2}$$

so

$$\frac{\left|K_n(x,y)-K_n^{\#}(x,y)\right|}{K_n^{\#}(x,x)} \le \left(\frac{K_n^{\#}(y,y)}{K_n^{\#}\,(x,x)}\right)^{1/2}\left[1-(1-\varepsilon)\,\frac{K_n(x,x)}{K_n^{\#}(x,x)}\right]^{1/2}.$$

In view of (4.3), we also have

$$\frac{K_n(x,x)}{K_n^{\#}(x,x)} = \frac{\lambda_n^{\#}(x)}{\lambda_n(x)} \ge \frac{1}{1+\varepsilon},$$

so for all $x,\ y \in [-1,1]$,

$$\begin{aligned}
\frac{\left|K_n(x,y)-K_n^{\#}(x,y)\right|}{K_n^{\#}(x,x)} &\le \left(\frac{K_n^{\#}(y,y)}{K_n^{\#}\,(x,x)}\right)^{1/2}\left[1-\frac{1-\varepsilon}{1+\varepsilon}\right]^{1/2} \le \sqrt{2\varepsilon}\left(\frac{K_n^{\#}(y,y)}{K_n^{\#}(x,x)}\right)^{1/2} \\
&= \sqrt{2\varepsilon}\left(\frac{K_n^{L}(y,y)}{K_n^{L}(x,x)}\right)^{1/2} = \sqrt{2\varepsilon}\left(\frac{\lambda_n^{L}(x)}{\lambda_n^{L}(y)}\right)^{1/2}.
\end{aligned}$$

Here we have used (4.2). Now we set $x = x_1 + \frac{a}{n}$ and $y = x_1 + \frac{b}{n}$, where $x_1 \in I\left(x_0, \frac{\delta}{2}\right)$ and $a, b \in [-A, A]$. By classical estimates for Christoffel functions for the Legendre weight (or even Theorem 2.1), uniformly for $a, b \in [-A, A]$, and

$x_1 \in J$,

$$\lambda_n^L\Big(x_1 + \frac{b}{n}\Big) \sim \lambda_n^L\Big(x_1 + \frac{a}{n}\Big) \sim n^{-1},$$

and also the constants implicit in $\sim$ are independent of ε, δ and x_1 (this is crucial!). Thus for some C and n_0 depending only on A and J, we have for $n \geq n_0$,

$$\sup_{a,b\in[-A,A],x_1\in I(x_0,\frac{\delta}{2})\cap J} \left|(K_n - K_n^{\#})\Big(x_1 + \frac{a}{n}, x_1 + \frac{b}{n}\Big)\right| \Big/ n \leq C\sqrt{\varepsilon}.$$

Then also, from (4.2),

$$\sup_{a,b\in[-A,A],x_1\in I\left(x_0,\frac{\delta}{2}\right)\cap J} \left|(w(x_0)K_n - K_n^L)\Big(x_1 + \frac{a}{n}, x_1 + \frac{b}{n}\Big)\right| \Big/ n \leq C\sqrt{\varepsilon}.$$

Finally, note that for $n \geq n_0$, $x_1 \in I\left(x_0, \frac{\delta}{2}\right) \cap J$ and $a, b \in [-A, A]$,

$$(1+\varepsilon)^{-1} \leq \frac{w\big(x_1 + \frac{a}{n}\big)^{1/2} w\big(x_1 + \frac{b}{n}\big)^{1/2}}{w(x_0)} \leq 1 + \varepsilon.$$

Changing x_1 to x gives (4.1). □

Proof of Theorem 1.1. Let $A, \varepsilon_1 > 0$. Choose $\varepsilon > 0$ so small that the right-hand side $C\varepsilon^{1/2}$ of (4.1) is less than ε_1. Choose $\delta > 0$ such that (2.3) holds. Now cover J by, say M intervals $I\big(x_j, \frac{\delta}{2}\big)$, $1 \leq j \leq M$, each of length δ. For each j, there exists a threshold $n_0 = n_0(j)$ for which (4.1) holds for $n \geq n_0(j)$ with $I\big(x_0, \frac{\delta}{2}\big)$ replaced by $I\big(x_j, \frac{\delta}{2}\big)$. Let n_1 denote the largest of these. Then we obtain, for $n \geq n_1$,

$$\sup_{a,b\in[-A,A],x\in J} \left|(\tilde{K}_n - K_n^L)\Big(x + \frac{a}{n}, x + \frac{b}{n}\Big)\right| \Big/ n \leq \varepsilon_1.$$

It follows that

$$\text{(4.5)} \qquad \lim_{n\to\infty} \left(\sup_{a,b\in[-A,A],x\in J} \left|(\tilde{K}_n - K_n^L)\Big(x + \frac{a}{n}, x + \frac{b}{n}\Big)\right|\right) = 0.$$

Finally the universality limit for the Legendre weight (see for example [5]) gives as $n \to \infty$,

$$\text{(4.6)} \qquad \frac{\pi\sqrt{1-x^2}}{n} K_n^L\left(x + \frac{u\pi\sqrt{1-x^2}}{n}, x + \frac{v\pi\sqrt{1-x^2}}{n}\right) \to \frac{\sin \pi(u-v)}{\pi(u-v)},$$

uniformly for u, v in compact subsets of the real line, and x in compact subsets of $(-1, 1)$. Setting

$$a = u\pi\sqrt{1-x^2} \quad \text{and} \quad b = v\pi\sqrt{1-x^2}$$

in (4.5), we obtain as $n \to \infty$, uniformly for $x \in J$ and u, v in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{\pi\sqrt{1-x^2}}{n} \tilde{K}_n\left(x + \frac{u\pi\sqrt{1-x^2}}{n}, x + \frac{v\pi\sqrt{1-x^2}}{n}\right) = \frac{\sin\pi(u-v)}{\pi(u-v)}. \tag{4.7}$$

Since uniformly for $x \in J$, by Theorem 2.1,

$$\tilde{K}_n(x,x)^{-1} = K_n^L(x,x)^{-1}(1+o(1)) = \pi\sqrt{1-x^2}/n\,(1+o(1)),$$

we then also obtain the conclusion of Theorem 1.1. □

For future use, we also record that

$$\lim_{n\to\infty} \frac{1}{n}\tilde{K}_n\left(x + \frac{a}{n}, x + \frac{b}{n}\right) = \frac{\sin\left((a-b)/\sqrt{1-x^2}\right)}{\pi(a-b)} \tag{4.8}$$

uniformly for $x \in J$ and $a, b \in [-A, A]$.

5. Universality in L_1

In this section, we prove:

THEOREM 5.1. *Let μ be a finite positive Borel measure on $(-1,1)$ that is regular. Let I be a closed subinterval of $(-1,1)$ such that*

$$w \geq C_0 \text{ in } I. \tag{5.1}$$

Then if I' is a closed subinterval of I^0, uniformly for a, b in compact subsets of the plane,

$$\lim_{n\to\infty} \int_{I'} \left| \frac{1}{n} K_n\left(x + \frac{\pi a\sqrt{1-x^2}}{n}, x + \frac{\pi b\sqrt{1-x^2}}{n}\right) - \frac{1}{\pi w(x)\sqrt{1-x^2}} \frac{\sin\pi(a-b)}{\pi(a-b)} \right| dx = 0. \tag{5.2}$$

Let $\Delta > 0$, also with Δ less than half the length of I. Define a measure $\mu^{\#}$ by

$$\mu^{\#} = \mu \text{ in } [-1,1] \backslash I$$

and in I, we define $d\mu^{\#}(x) = w^{\#}(x)\,dx$, where

$$w^{\#}(x) = \frac{1}{2\Delta}\int_{x-\Delta}^{x+\Delta} w = \frac{1}{2}\int_{-1}^{1} w\,(x + s\Delta)\,ds. \tag{5.3}$$

LEMMA 5.2. *Let I' be a closed subinterval of I^0.*

(a) *$\mu^{\#}$ is absolutely continuous in I^0 and $w^{\#} \geq \frac{1}{2}C_0$ in I^0.*

(b) *$\mu^{\#}$ is regular on $[-1,1]$.*

(c) *There exists* $C_1 > 0$, *independent of* Δ, *such that for* $n \geq 1$,

$$\sup_{t \in I'} \frac{1}{n} K_n(t,t) \leq C_1 \quad \text{and} \quad \sup_{t \in I'} \frac{1}{n} K_n^{\#}(t,t) \leq C_1. \tag{5.4}$$

(d)

$$\lim_{n \to \infty} \frac{1}{n} \int_{I'} \left| K_n - K_n^{\#} \right| (t,t)\, dt = \frac{1}{\pi} \int_{I'} \left| \frac{1}{w(t)} - \frac{1}{w^{\#}(t)} \right| \frac{dt}{\sqrt{1-t^2}}. \tag{5.5}$$

(e) *For some* $C_2 > 0$ *independent of* Δ,

$$\int_{I'} \frac{1}{\sqrt{1-t^2}} \left| \frac{1}{w(t)} - \frac{1}{w^{\#}(t)} \right| dt \leq C_2 \sup_{|u| \leq \Delta} \int_I |w(t+u) - w(t)|\, dt. \tag{5.6}$$

Proof. (a) is immediate.

(b) This follows from Theorem 5.3.3 in [13, p. 148]. As μ is regular, that theorem shows that the restriction of μ to $[-1,1] \backslash I$ is regular. Hence the restriction of $\mu^{\#}$ is trivially regular in $[-1,1] \backslash I$. The restriction of $\mu^{\#}$ to I is regular as its absolutely continuous component $w^{\#} > 0$ there. Then the theorem just cited shows that $\mu^{\#}$ is regular as a measure on all of $[-1,1]$.

(c) In view of (5.1), we have for $x \in I'$,

$$\lambda_n(x) \geq C_0 \inf_{\deg(P) \leq n-1} \int_I P^2 / P^2(x) \geq C_0 C_1 / n.$$

Here we are using classical bounds for the Legendre weight translated to the interval I, and the constant C_1 depends only on the intervals I' and I. Then the first bound in (5.4) follows, and that for $\lambda_n^{\#}$ is similar. Since the lower bound on $\mu^{\#}$ in I is independent of Δ, it follows that the constants we obtain in (5.4) will also be independent of Δ.

(d) Since μ is regular, and $\mu' = w$ is bounded below by a positive constant in I, we have a.e. in I,

$$\lim_{n \to \infty} \frac{K_n(x,x)}{n} = \frac{1}{\pi w(x) \sqrt{1-x^2}}.$$

See for example [7, p. 449, Thm. 8] or [14, Thm. 1]. A similar limit holds for $K_n^{\#}/n$. We also have the uniform bound in (c). Then Lebesgue's Dominated Convergence Theorem gives the result.

(e) Recall that I is a positive distance from ± 1, while $w, w^{\#}$ are bounded below in I by $C_0/2$. Then

$$\int_{I'} \frac{1}{\sqrt{1-t^2}} \left| \frac{1}{w(t)} - \frac{1}{w^\#(t)} \right| dt \le C \int_{I'} \left| w^\#(t) - w(t) \right| dt$$
$$\le C \int_{I'} \int_{-1}^{1} |w(t+s\Delta) - w(t)| \, ds \, dt$$
$$= C \int_{-1}^{1} \int_{I'} |w(t+s\Delta) - w(t)| \, dt \, ds$$
$$\le C \sup_{|u| \le \Delta} \int_{I'} |w(t+u) - w(t)| \, dt. \qquad \square$$

Proof of Theorem 5.1. As per usual,

$$\int_{-1}^{1} (K_n^\# - K_n)^2(x,t) \, d\mu^\#(t) = \int_{-1}^{1} K_n^{\#2}(x,t) \, d\mu^\#(t) - 2\int_{-1}^{1} K_n^\#(x,t) K_n(x,t) \, d\mu^\#(t)$$
$$+ \int_{-1}^{1} K_n^2(x,t) \, d\mu(t) + \int_I K_n^2(x,t) \, d(\mu^\# - \mu)(t)$$
$$= K_n^\#(x,x) - K_n(x,x) + \int_I K_n^2(x,t) \, d(\mu^\# - \mu)(t)$$
$$\le K_n^\#(x,x) - K_n(x,x) + \int_I K_n^2(x,t)(w^\# - w)(t) \, dt.$$

Recall that $\mu = \mu^\#$ outside I and that $\mu^\#$ is absolutely continuous in I. Then the Christoffel function estimate (3.4) gives for $x, y \in [-1, 1]$,

$$\text{(5.7)} \quad \left| K_n - K_n^\# \right| (x, y)$$
$$\le K_n^\#(y,y)^{1/2} \left(K_n^\#(x,x) - K_n(x,x) + \int_I K_n^2(x,t) \left(w^\# - w \right)(t) \, dt \right)^{1/2}.$$

We now replace x by $x + a\pi\sqrt{1-x^2}/n$, y by $x + b\pi\sqrt{1-x^2}/n$, integrate over I', and then use the Cauchy-Schwarz inequality. We obtain

$$\text{(5.8)} \quad \int_{I'} \left| K_n - K_n^\# \right| \left(x + \frac{a\pi\sqrt{1-x^2}}{n}, x + \frac{b\pi\sqrt{1-x^2}}{n} \right) dx \le T_1^{1/2} T_2^{1/2},$$

where

$$T_1 = \int_{I'} K_n^\# \left(x + \frac{b\pi\sqrt{1-x^2}}{n}, x + \frac{b\pi\sqrt{1-x^2}}{n} \right) dx,$$

$$\text{(5.9)} \quad T_2 = \int_{I'} (K_n^\# - K_n) \left(x + \frac{a\pi\sqrt{1-x^2}}{n}, x + \frac{a\pi\sqrt{1-x^2}}{n} \right) dx$$
$$+ \int_{I'} \left[\int_I K_n^2 \left(x + \frac{a\pi\sqrt{1-x^2}}{n}, t \right) (w^\# - w)(t) \, dt \right] dx$$
$$=: T_{21} + T_{22}.$$

Now let $A > 0$ and $a, b \in [-A, A]$. Choose a subinterval I'' of I^0 such that $I' \subset (I'')^0$. Observe that for some n_0 depending only on A and I', I'',

$$x + \frac{b\pi\sqrt{1-x^2}}{n} \in I'' \text{ for } x \in I', \quad b \in [-A, A], \quad n \geq n_0. \tag{5.10}$$

Then (c) of Lemma 5.2 shows that for $n \geq n_0$,

$$T_1 \leq C_2 n, \tag{5.11}$$

where C_2 is independent of n and $b \in [-A, A]$. Next, we make the substitution $s = x + \frac{a\pi\sqrt{1-x^2}}{n}$ in T_{21}. Observe that

$$\frac{ds}{dx} = 1 - \frac{a\pi x}{n\sqrt{1-x^2}} \in \left[\tfrac{1}{2}, 2\right],$$

for $n \geq n_1$, where n_1 depends only on A and I. We can also assume that (5.10) holds, with a replacing b, for $n \geq n_1$. Hence for $n \geq \max\{n_0, n_1\}$ and all a in $[-A, A]$,

$$\begin{aligned} |T_{21}| &\leq \int_{I'} \left|K_n^\# - K_n\right| \left(x + \frac{a\pi\sqrt{1-x^2}}{n}, x + \frac{a\pi\sqrt{1-x^2}}{n}\right) dx \\ &\leq 2\int_{I''} \left|K_n^\# - K_n\right| (s,s)\, ds. \end{aligned}$$

Thus, using (d) and (e) of the above lemma,

$$\limsup_{n\to\infty} \frac{1}{n} T_{21} \leq C \sup_{|u|\leq\Delta} \int_{I''} |w(t+u) - w(t)|\, dt,$$

where C does not depend on Δ and a. Next,

$$|T_{22}| \leq \int_I \left|w - w^\#\right|(t) \left[\int_{I'} K_n^2\left(x + \frac{a\pi\sqrt{1-x^2}}{n}, t\right) dx\right] dt.$$

Here for $n \geq \max\{n_0, n_1\}$,

$$\begin{aligned} &\int_{I'} K_n^2\left(x + \frac{a\pi\sqrt{1-x^2}}{n}, t\right) dx \\ &\qquad \leq \frac{1}{C_0}\int_{I'} K_n^2\left(x + \frac{a\pi\sqrt{1-x^2}}{n}, t\right) w\left(x + \frac{a\pi\sqrt{1-x^2}}{n}\right) dx \\ &\qquad \leq \frac{2}{C_0}\int_{I''} K_n^2(s,t)w(s)\, ds \leq \frac{2}{C_0} K_n(t,t). \end{aligned}$$

Then using (c) of the previous lemma, we obtain

$$|T_{22}| \leq Cn \int_I \left|w - w^\#\right|(t)\, dt \leq Cn \sup_{|u|\leq\Delta} \int_{I''} |w(t+u) - w(t)|\, dt;$$

compare (5.6). Substituting all the above estimates in (5.8), we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \int_{I'} \left|K_n - K_n^{\#}\right| \left(x + \frac{a\pi\sqrt{1-x^2}}{n}, x + \frac{b\pi\sqrt{1-x^2}}{n}\right) dx \\ \leq C\left(\sup_{|u|\leq\Delta} \int_{I''} |w(t+u) - w(t)|\, dt\right)^{1/2},$$

uniformly for $a, b \in [-A, A]$, where C is independent of Δ. Now as $\mu^{\#}$ is regular and absolutely continuous in I, and $w^{\#}$ is continuous in I^0, Theorem 2.1 shows that

$$\lim_{n\to\infty} \frac{1}{n} K_n^{\#}\left(x + \frac{a\pi\sqrt{1-x^2}}{n}, x + \frac{b\pi\sqrt{1-x^2}}{n}\right) \\ = \frac{\sin \pi(a-b)}{\pi(a-b)} \frac{1}{\pi\sqrt{1-x^2}w^{\#}(x)},$$

uniformly for $x \in I'$ and $a, b \in [-A, A]$. It follows that

$$\limsup_{n\to\infty} \int_{I'} \left| \frac{1}{n} K_n\left(x + \frac{a\pi\sqrt{1-x^2}}{n}, x + \frac{b\pi\sqrt{1-x^2}}{n}\right) \right. \\ \left. - \frac{\sin \pi(a-b)}{\pi(a-b)} \frac{1}{\pi\sqrt{1-x^2}w(x)} \right| dx \\ \leq \left|\frac{\sin \pi(a-b)}{\pi(a-b)}\right| \int_{I'} \frac{1}{\pi\sqrt{1-x^2}} \left|\frac{1}{w^{\#}(x)} - \frac{1}{w(x)}\right| dx \\ + C\left(\sup_{|u|\leq\Delta} \int_{I''} |w(t+u) - w(t)|\, dt\right)^{1/2},$$

uniformly for $a, b \in [-A, A]$, where C is independent of Δ. Since the left-hand side is independent of Δ, we may apply (e) of the previous lemma, and then let $\Delta \to 0+$ to get the result. Of course, as w is integrable, we have as $\Delta \to 0+$,

$$\sup_{|u|\leq\Delta} \int_{I''} |w(t+u) - w(t)|\, dt \to 0. \qquad \square$$

6. Universality in L_p

The case $p = 1$ of Theorem 1.4(a) is an easy consequence of Theorem 5.1 and the following lemma:

LEMMA 6.1. *Assume the hypotheses of Theorem 1.4(a). Let $A > 0$ and I' be a closed subinterval of I^0. As $n \to \infty$, uniformly for $a, b \in [-A, A]$,*

$$\frac{1}{n}\int_{I'}\left|K_n\left(x+\frac{a\pi\sqrt{1-x^2}}{n}, x+\frac{b\pi\sqrt{1-x^2}}{n}\right) - K_n\left(x+\frac{a}{\tilde{K}_n(x,x)}, x+\frac{b}{\tilde{K}_n(x,x)}\right)\right| dx \to 0. \tag{6.1}$$

Proof. Choose a subinterval I'' of I^0 such that $I' \subset (I'')^0$. Define $r_n(x)$ by

$$\frac{1}{\tilde{K}_n(x,x)} = \frac{\pi\sqrt{1-x^2}}{n} r_n(x).$$

Then the integrand in (6.1) may be written as

$$\left| K_n\left(x+\frac{a\pi\sqrt{1-x^2}}{n}, x+\frac{b\pi\sqrt{1-x^2}}{n}\right) - K_n\left(x+\frac{a\pi\sqrt{1-x^2}}{n}r_n(x), x+\frac{b\pi\sqrt{1-x^2}}{n}r_n(x)\right) \right|$$

$$\leq \left|\frac{\partial}{\partial s}K_n\left(s, x+\frac{b\pi\sqrt{1-x^2}}{n}\right)\right|_{|s=\xi} \frac{|a|\,\pi\sqrt{1-x^2}}{n}\,|1-r_n(x)| + \left|\frac{\partial}{\partial t}K_n\left(x+\frac{a\pi\sqrt{1-x^2}}{n}r_n(x), t\right)\right|_{|t=\zeta} \frac{|b|\pi\sqrt{1-x^2}}{n}\,|1-r_n(x)|$$

where ξ lies between $x + a\pi\sqrt{1-x^2}/n$ and $x + (a\pi\sqrt{1-x^2}/n)r_n(x)$, with a similar restriction on ζ. Now by Lemma 5.2(c) and Cauchy-Schwarz,

$$\sup_{s,t\in I} |K_n(s,t)| \leq Cn.$$

By Bernstein's inequality [4, p. 98, Cor. 1.2],

$$\sup_{s\in I'', t\in I}\left|\frac{\partial}{\partial s}K_n(s,t)\right| \leq C_1 C n^2$$

with a similar bound for $\frac{\partial}{\partial t}K_n$. Here C_1 depends only on I and I''. Then for some C_2 independent of a, b, n, x,

$$\frac{1}{n}\left| K_n\left(x+\frac{a\pi\sqrt{1-x^2}}{n}, x+\frac{b\pi\sqrt{1-x^2}}{n}\right) - K_n\left(x+\frac{a\pi\sqrt{1-x^2}}{n}r_n(x), x+\frac{b\pi\sqrt{1-x^2}}{n}r_n(x)\right) \right| \leq C\,|1-r_n(x)|.$$

Hence the integral in the left-hand side of (6.1) is bounded above by

$$C\int_{I'} |1-r_n(x)|\,dx.$$

Of course C is independent of n. Next [7, p. 449, Thm. 8],

$$r_n(x) = \frac{n}{K_n(x,x)w(x)\pi\sqrt{1-x^2}} \to 1 \text{ a.e. in } I. \tag{6.2}$$

We shall shortly show that

$$r_n(x) \le C \text{ for } x \in I' \text{ and } n \ge n_0. \tag{6.3}$$

Then Lebesgue's Dominated Convergence Theorems shows that

$$\lim_{n\to\infty} \int_{I'} |1 - r_n(x)|\, dx = 0.$$

To prove (6.3), choose $M > 0$ such that $w \le M$ in I. Define a measure μ^* by

$$d\mu = d\mu^* \text{ in } [-1,1]\backslash I; \quad d\mu^*(x) = M\,dx \text{ in } I.$$

Then $d\mu \le d\mu^*$ in $[-1,1]$ and so $\lambda_n \le \lambda_n^*$ in $[-1,1]$. As the absolutely continuous component of μ^* is positive and continuous in I, Theorem 2.1 shows that for some $C > 0$,

$$\lambda_n^*(x) \le \frac{C}{n} \text{ for } x \in I' \text{ and } n \ge 1.$$

Then

$$\frac{n}{K_n(x,x)} = n\lambda_n(x) \le C \text{ for } x \in I' \text{ and } n \ge 1. \tag{6.4}$$

The definition (6.2) of r_n, the fact that w is bounded below in I, and this last inequality, give (6.3). □

Proof of Theorem 1.4(a). As w is bounded above and below in I, the lemma and Theorem 5.1 show that

$$\lim_{n\to\infty}\int_{I'}\left|K_n\left(x+\frac{a}{\tilde{K}_n(x,x)},\, x+\frac{b}{\tilde{K}_n(x,x)}\right)\frac{w(x)\pi\sqrt{1-x^2}}{n}-\frac{\sin\pi(a-b)}{\pi(a-b)}\right|dx=0$$

uniformly for $a,b \in [-A,A]$. Now as in (6.2), a.e. in I,

$$\frac{1}{K_n(x,x)} = \frac{w(x)\pi\sqrt{1-x^2}}{n}(1+o(1)).$$

Moreover, by (6.4), Lemma 5.2(c), and Cauchy-Schwarz, both

$$\frac{1}{n}K_n\left(x+\frac{a}{\tilde{K}_n(x,x)}, x+\frac{b}{\tilde{K}_n(x,x)}\right)$$

and

$$K_n\left(x+\frac{a}{\tilde{K}_n(x,x)}, x+\frac{b}{\tilde{K}_n(x,x)}\right)\Big/K_n(x,x)$$

are bounded above uniformly for $a, b \in [-A, A]$, $x \in I'$, and $n \geq n_0$. We deduce that

$$\lim_{n\to\infty} \int_{I'} \left| K_n \left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)} \right) \Big/ K_n(x,x) - \frac{\sin \pi(a-b)}{\pi(a-b)} \right| dx = 0.$$

Finally, as we have just noted, the integrand in the last integral is bounded above uniformly for $a, b \in [-A, A]$, $x \in I'$, and $n \geq n_0$, so we may replace the first power by the pth power, for any $p > 1$. For $p < 1$, we can use Hölder's inequality. □

In proving Theorem 1.4(b), our last step is to replace

$$\frac{K_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)} \right)}{K_n(x,x)} \quad \text{by} \quad \frac{\tilde{K}_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)} \right)}{\tilde{K}_n(x,x)}.$$

This is more difficult than one might expect — it is only here that we need Riemann integrability of w in I. For general Lebesgue measurable w, it seems difficult to deal with the factor $\tilde{K}_n(x,x) = w(x) K_n(x,x)$ below.

LEMMA 6.2. *Assume that w is Riemann integrable and bounded below by a positive constant in I. Let I' be a compact subinterval of I. Let $p, A > 0$. Then uniformly for $a, b \in [-A, A]$,*

$$\lim_{n\to\infty} \int_{I'} \left| \sqrt{ w\left(x + \frac{a}{\tilde{K}_n(x,x)} \right) w\left(x + \frac{b}{\tilde{K}_n(x,x)} \right) } \Big/ w(x) - 1 \right|^p dx = 0.$$

Proof. Let $a, b \in [-A, A]$. From (6.4), for a suitable integer n_0 and some $L > 0$,

$$\left| \frac{a}{\tilde{K}_n(x,x)} \right| \leq \frac{L}{n} \quad \text{and} \quad \left| \frac{b}{\tilde{K}_n(x,x)} \right| \leq \frac{L}{n},$$

uniformly for $x \in I'$, $a, b \in [-A, A]$, and $n \geq n_0$. Next, as w is Riemann integrable in I, it is continuous a.e. in I [11, p. 23]. For $x \in I$ and $n \geq 1$, let

$$\Omega_n(x) = \sup \left\{ |w(x+s) - w(x)| : |s| \leq \frac{L}{n} \right\}.$$

Note that for $x \in I', n \geq n_0$ and $a, b \in [-A, A]$,

$$\left| w\left(x + \frac{a}{\tilde{K}_n(x,x)} \right) - w(x) \right| \leq \Omega_n(x).$$

We have at every point of continuity of w and in particular for a.e. $x \in I$,

$$\lim_{n\to\infty} \Omega_n(x) = 0.$$

Moreover, as w is Riemann integrable, Ω_n is bounded above in I, uniformly in n. Then Lebesgue's Dominated Convergence Theorem gives uniformly for $a \in [-A, A]$,

$$\int_{I'} \left| w\left(x + \frac{a}{\widetilde{K}_n(x,x)}\right) - w(x)\right|^p dx \leq \int_{I'} \Omega_n(x)^p dx \to 0, \quad n \to \infty.$$

This, the fact that w is bounded above and below, and some elementary manipulations, give the result. □

Proof of Theorem 1.4(b). Since

$$\frac{K_n\left(x + \frac{a}{\widetilde{K}_n(x,x)}, x + \frac{b}{\widetilde{K}_n(x,x)}\right)}{K_n(x,x)}$$

is bounded uniformly in n, x, a, b (over the relevant ranges) and

$$\frac{\widetilde{K}_n\left(x + \frac{a}{\widetilde{K}_n(x,x)}, x + \frac{b}{\widetilde{K}_n(x,x)}\right)}{\widetilde{K}_n(x,x)} \Big/ \frac{K_n\left(x + \frac{a}{\widetilde{K}_n(x,x)}, x + \frac{b}{\widetilde{K}_n(x,x)}\right)}{K_n(x,x)}$$

$$= \sqrt{w\left(x + \frac{a}{\widetilde{K}_n(x,x)}\right) w\left(x + \frac{b}{\widetilde{K}_n(x,x)}\right)} \Big/ w(x),$$

this follows directly from the lemma above and Theorem 1.4(a). □

7. Proof of Corollaries 1.2 and 1.3

Proof of Corollary 1.2. This follows directly by substituting (1.6) into the determinant defining R_m. □

In proving Corollary 1.3, we need

LEMMA 7.1. *Let $w \geq C$ in I and I', I'' be closed subintervals of I^0 such that I' is contained in the interior of I''. Let $A > 0$. There exists C_2 such that for $n \geq 1$, $x \in I'$, and all $\alpha, \beta \in \mathbb{C}$ with $|\alpha|, |\beta| \leq A$,*

$$\left| \frac{1}{n} K_n\left(x + \frac{\alpha}{n}, x + \frac{\beta}{n}\right)\right| \leq C_2. \tag{7.1}$$

Proof. Recall that $\frac{1}{n} K_n(x,x)$ is uniformly bounded above for $x \in I'$ by Lemma 5.2(c). Applying Cauchy-Schwarz, we obtain for $x, y \in I''$,

$$\frac{1}{n}|K_n(x,y)| \leq \sqrt{\frac{1}{n}K_n(x,x)}\sqrt{\frac{1}{n}K_n(y,y)} \leq C_1. \tag{7.2}$$

Next we note Bernstein's growth lemma for polynomials in the plane [4, Thm. 2.2, p. 101]: if P is a polynomial of degree $\leq n$, we have for $z \notin [-1, 1]$,

$$|P(z)| \leq \left|z + \sqrt{z^2 - 1}\right|^n \|P\|_{L_\infty[-1,1]}.$$

From this we deduce that given $L > 0$, and $0 < \delta < 1$, there exists $C_2 \neq C_2(n, P, z)$ such that for $|\mathrm{Re}(z)| \leq \delta$, and $|\mathrm{Im}\, z| \leq \frac{L}{n}$

$$|P(z)| \leq C_2 \, \|P\|_{L_\infty[-1,1]} \, .$$

Mapping this to I by a linear transformation, we deduce that for $\mathrm{Re}\, z \in I'$ and $|\mathrm{Im}\, z| \leq \frac{L}{n}$,

$$|P(z)| \leq C_3 \, \|P\|_{L_\infty(I'')}$$

where $C_3 \neq C_3(n, P, z)$. We now apply this to $\frac{1}{n} K_n(x, y)$, separately in each variable, obtaining the stated result. □

Proof of Corollary 1.3. Since w is positive and continuous at each point of the compact set J, we may find $C > 0$ and finitely many closed intervals $\{I\}$ such that $w \geq C$ in each I, and such that J is contained in the union of their interiors I^0. From each such interval I, we can choose a subinterval I' as in Lemma 7.1, in such a way that J is contained in the union of the finitely many intervals $\{I'\}$. It suffices to prove (1.11) for just one of the intervals I'. We proceed to do this.

By the lemma, $\left\{ \frac{1}{n} K_n \left(x + \frac{\alpha}{n}, x + \frac{\beta}{n} \right) \right\}_{n=1}^{\infty}$ is analytic in α, β and uniformly bounded for α, β in compact subsets of the plane, and $x \in I'$. Moreover, from (4.8), and continuity of w,

$$\lim_{n \to \infty} \frac{1}{n} w(x) K_n \left(x + \frac{\alpha}{n}, x + \frac{\beta}{n} \right) = \frac{\sin \left((\alpha - \beta)/\sqrt{1 - x^2} \right)}{\pi \, (\alpha - \beta)}$$

uniformly for $x \in I'$ and α, β in compact subsets of I'. By convergence continuation theorems, this last limit then holds uniformly for α, β in compact subsets of the plane. Next, expanding $p_k\left(x + \frac{\alpha}{n}\right)$ and $p_k\left(x + \frac{\beta}{n}\right)$ in Taylor series about x,

$$\begin{aligned} \frac{1}{n} K_n \left(x + \frac{\alpha}{n}, x + \frac{\beta}{n} \right) &= \frac{1}{n} \sum_{k=0}^{n-1} p_k \left(x + \frac{\alpha}{n} \right) p_k \left(x + \frac{\beta}{n} \right) \\ &= \frac{1}{n} \sum_{r,s=0}^{\infty} \frac{\left(\frac{\alpha}{n} \right)^r}{r!} \frac{\left(\frac{\beta}{n} \right)^s}{s!} \sum_{k=0}^{n-1} p_k^{(r)}(x) p_k^{(s)}(x) \\ &= \sum_{r,s=0}^{\infty} \frac{\alpha^r}{r!} \frac{\beta^s}{s!} \frac{1}{n^{r+s+1}} K_n^{(r,s)}(x, x), \end{aligned}$$

with the notation (1.8). Since the series terminates, the interchanges are valid. By using the Maclaurin series of sin and the binomial theorem, we see that

$$\frac{\sin (\alpha - \beta)}{\alpha - \beta} = \sum_{r,s=0}^{\infty} \frac{\alpha^r}{r!} \frac{\beta^s}{s!} \tau_{r,s},$$

where $\tau_{r,s}$ is given by (1.9). Since uniformly convergent sequences of analytic functions have Taylor series coefficients that also converge, we see that for $x \in I$, and each $r, s \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n^{r+s+1}} w(x) K_n^{(r,s)}(x,x) = \frac{\tau_{r,s}}{\pi} \left(1-x^2\right)^{-(r+s+1)/2}.$$

This establishes the limit (1.11), but we must still prove uniformity in x. Let $A, \varepsilon > 0$. By the uniform convergence in Theorem 1.1, there exists n_0 such that for $n \geq n_0$,

$$(7.3)\quad \left| \frac{w(x)\sqrt{1-x^2}}{n} K_n\left(x + \frac{a\pi\sqrt{1-x^2}}{n}, x + \frac{b\pi\sqrt{1-x^2}}{n}\right) - \frac{w(y)\sqrt{1-y^2}}{n} K_n\left(y + \frac{a\pi\sqrt{1-y^2}}{n}, y + \frac{b\pi\sqrt{1-y^2}}{n}\right) \right| \leq \varepsilon,$$

uniformly for $x, y \in J$, $a, b \in [-A, A]$ and $n \geq n_0$. Using Bernstein's growth inequality as in the lemma above, applied to the polynomial in a, b in the left-hand side of (7.3), we obtain that this inequality persists for complex a, b with $|a|, |b| \leq A$, except that we must replace ε by $C\varepsilon$, where C depends only on A, not on n, x, a, b, ε. We can now use Cauchy's inequalities to bound the Taylor series coefficients of the double series in a, b implicit in the left-hand side in (7.3). This leads to bounds on

$$\left| \frac{1}{n^{r+s+1}} w(x) K_n^{(r,s)}(x,x) - \frac{1}{n^{r+s+1}} w(y) K_n^{(r,s)}(y,y) \right|$$

that are uniform in $x, y \in I'$. □

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