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Abstract

We prove that every finite split embedding problem is solvable over the field $K((X_1, \dots, X_n))$ of formal power series in $n \geq 2$ variables over an arbitrary field K, as well as over the field $\mathrm{Quot}(A[[X_1, \dots, X_n]])$ of formal power series in $n \geq 1$ variables over a Noetherian integrally closed domain A. This generalizes a theorem of Harbater and Stevenson, who settled the case $K((X_1, X_2))$.

Introduction

A central problem in the study of absolute Galois groups is the solvability of finite embedding problems. A sufficient condition for all finite embedding problems for a given profinite group G to be solvable is that each Frattini embedding problem and each finite split embedding problem for G are solvable [FJ05, Prop. 22.5.8], [Mat91, p. 430]. The first condition is equivalent to G being projective. Thus, a natural question for an arbitrary profinite group G is whether each split embedding problem for G is solvable. Specifically, Dèbes and Deschamps made the following conjecture for absolute Galois groups of Hilbertian fields.

CONJECTURE A [DD97, §2.1.2]. *If F is a Hilbertian field, then every finite split embedding problem over F is solvable.*

Note that Conjecture A implies a positive answer to the Inverse Galois Problem: is every finite group realizable over $\mathbb{Q}$?

In this work we prove Conjecture A for a large class of Hilbertian fields. Indeed we prove the conjecture when F is the quotient field of a Noetherian integrally closed domain A of dimension at least 2 which is complete in some broad sense.

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Following [HS05] we abbreviate "finite split embedding problem" by *FSEP*. An FSEP for a field F is an epimorphism $\pi\colon G \rtimes \Gamma \to \Gamma$ of finite groups, where $\Gamma = \mathrm{Gal}(F'/F)$ is the Galois group of a Galois extension F'/F, G is a finite group on which Γ acts, $G \rtimes \Gamma$ is the corresponding semidirect product, and π is the projection on Γ. A *solution* of the problem is a Galois extension L of F which contains F' and an isomorphism $\gamma\colon \mathrm{Gal}(L/F) \to G \rtimes \Gamma$ such that $\pi \circ \gamma = \mathrm{res}_E$.

Our result follows a route paved for the first time more than twenty years ago by Harbater in [Har87]. In that paper Harbater introduced the concept of "formal patching" and used it to prove that if F is the quotient field of a complete local ring, then every Realization Problem over the field $F(x)$ of rational functions over F is solvable. One can view Realization Problems as FSEPs where the group Γ is trivial. Later Harbater [Har94] used formal patching to reduce Abhyankar's generalized conjecture to the special case settled by Raynaud [Ray94].

Harbater's proof [Har87] is phrased in the language of formal geometry. It was later translated to the language of Rigid Analytic Geometry by Liu [Liu95] and Serre [Ser92]. Haran and Völklein [HV96] gave a self contained algebraic proof of this result, introducing the concept of "algebraic patching".

The next step is due to Pop [Pop96], who used methods of rigid analytic geometry to prove that if F is a complete valued field (and more generally, if F is an ample field), then every *constant* finite split embedding problem over F is regularly solvable. That is, given a Galois extension F' of F with group Γ which acts on a finite group G, there exists a Galois extension L of $F'(x)$ (where x is transcendental over F') and an isomorphism $\psi\colon \mathrm{Gal}(L/F(x)) \to G \rtimes \Gamma$ such that L/F is regular and $\pi \circ \psi = \mathrm{res}_E$ (where π is the canonical projection).

Haran and Jarden [HJ98a] extended algebraic patching to give a self contained (along with [HV96]) algebraic proof of that result.

Using formal patching, Lefcourt [Lef99] showed that if F is the quotient field of a complete domain with respect to a prime ideal, then every realization problem is solvable over $F(x)$. Extending the method of algebraic patching from complete based fields to complete domains, [Par08] gives a self-contained algebraic proof to Lefcourt's result.

If F is a Hilbertian field, then one can specialize each solution over $F(x)$ and solve FSEPs over F. Thus, by [Pop96] and [HJ98a] every FSEP is solvable over a Hilbertian ample field, and by [Lef99] every realization problem is solvable over a Hilbertian field which is the quotient field of a complete domain.

Harbater and Stevenson [HS05] took the next step and proved that every FSEP is solvable over the field $K((X_1, X_2))$ of formal power series in two variables over an arbitrary field K. They showed that every FSEP over this field arises from an FSEP over $K((X_1))(X_2)$, and the latter problem has a solution that can be lifted to the solution of the original FSEP. Note that the field $K((X_1, X_2))$ is the quotient field of the complete local ring $K[\![X_1, X_2]\!]$, and it is Hilbertian by a theorem of Weissauer.

Our goal is to generalize the results of Pop and Haran-Jarden on the one hand, and those of Lefcourt and Harbater-Stevenson on the other hand.

MAIN THEOREM. *Let F be the quotient field of an integral domain A satisfying the following conditions*:

(a) *A is a Noetherian integrally closed domain.*

(b) *A has a proper nonzero ideal $\mathfrak{p}$ such that $v_{\mathfrak{p}}(x) = \max(n \mid x \in \mathfrak{p}^n)$ extends to a discrete valuation of $\mathrm{Quot}(A)$, and A is complete with respect to $v_{\mathfrak{p}}$.*

Then every constant FSEP *over $F(x)$ is regularly solvable.*

As mentioned above, if F is Hilbertian, then one can specialize the regular solution of constant FSEPs, thus solving FSEPs over F.

THEOREM B. *Let $F = \mathrm{Quot}(A)$ be a Hilbertian field, where A satisfies conditions (a) and (b) above, then every* FSEP *over F is solvable.*

The conditions of Theorem B hold for a large spectrum of fields:

COROLLARY C. *Every* FSEP *over F is solvable in the following cases*:

(a) *$F = K((X_1, \dots, X_n)) = \mathrm{Quot}(K[\![X_1, \dots, X_n]\!])$ is the field of formal power series over an arbitrary field K in $n \geq 2$ variables.*

(b) *$F = \mathrm{Quot}(A[\![X_1, \dots, X_n]\!])$ is the field of formal power series in $n \geq 1$ variables series over a Noetherian integrally closed domain which is not a field (for example $A = \mathbb{Z}$ or more generally, A is a Dedekind domain).*

Note that [HS05] proves that there are "many" such solutions for every FSEP over $K((X_1, X_2))$ — that is, the cardinality of the set of solutions is equal to the cardinality of F. We do not prove this much in the general case given by Corollary C.

Corollary C would follow from the result of Haran-Jarden-Pop if the fields F involved in the theorems were ample. However this is unknown and one suspects that they are not.

To prove the Main Theorem, we generalize the fundamental ideas of algebraic patching introduced in [HV96] and [HJ98a]. The basic framework for algebraic patching of groups over complete domains was introduced in [Par08]. In this paper we recall these generalized ideas, and adapt them to the solution of FSEPs.

Our general strategy is similar to that of [HJ98a]. The basic step is to realize all cyclic subgroups of G as Galois extensions of $F'(x)$, and embed these realizations in suitable "analytic" rings. In order to construct these rings, we first construct a complete ring whose quotient field is the given extension F' of F. The naive approach is to choose that ring as the integral closure B of A in F'. However B may be ramified over A, which creates a severe technical difficulty in the following construction of the analytic rings. We therefore choose $D = B[\frac{1}{f}]$, where f is the discriminant of a suitable primitive element of F'/F. The ring D need not be complete with respect to the absolute value that corresponds to $v_{\mathfrak{p}}$ (which extends to an absolute value of F'). So, we settle for less and instead construct a complete norm for D which need not be an absolute value.

This norm is good enough for the construction of our 'analytic' rings (which were fields in [HJ98a]). They are quotient rings of rings of convergent power series in several dependent variables over D. The properties of power series over a normed ring are not as nice as the properties of power series over a valued field (which were used in [HV96] and [HJ98a]). To overcome this difficulty we embed our rings of convergent power series over D in rings of convergent power series over the completion $\hat{F}'$ of F' with respect to $v_{\mathfrak{p}}$. The strong properties of the latter rings are transferred, in a weaker form, to our rings. This is where the assumption that $v_{\mathfrak{p}}$ is a valuation is used — this seemingly technical assumption allows this embedding (otherwise, we could not extend v_p from A to F and to F', so we could not define the completion $\hat{F}'$), which allows us to indirectly exploit the nice properties of rings of convergent power series over a complete field.

Having done that, we are able to patch the realizations of the cyclic subgroups of G to a realization L of G as a Galois group over $F'(x)$. We then define a suitable action of Γ on these rings, and use it to prove that L is a solution of the given FSEP.

The author wishes to acknowledge the work of D. Haran, H. Völklein and M. Jarden, who developed algebraic patching in [HV96], [HJ98a], [HJ98b], [HJ00]. The results of this paper were obtained by generalizing their wonderful ideas.

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1. **Algebraic patching**

In [Par08, §1] a general setup for the patching of Galois groups is presented. We recall the definitions and basic properties, and extend this setup for the solution of FSEPs.

Definition 1.1. Let I be a finite set with $|I| \geq 2$. A *generalized patching data*

$$\mathcal{E} = (E, F_i, Q_i, \Omega; G_i, G)_{i \in I}$$

consists of fields $E \subseteq F_i \subseteq \Omega$, integral domains Q_i contained in the field Ω, and finite groups $G_i \leq G$, $i \in I$, satisfying the following conditions:

(1a) F_i/E is a Galois extension with Galois group G_i, $i \in I$.

(1b) $F_i \subseteq Q_i'$, where $Q_i' = \bigcup_{j \neq i} Q_j$, $i \in I$.

(1c) $F_i \cap \mathrm{Quot}(Q_i) = E$, $i \in I$.

(1d) $G = \langle G_i \mid i \in I \rangle$.

(1e) $\bigcap_{i\in I} Q_i = E$.

Definition 1.1 generalizes [HJ98a, Def. 1.1]. Note that our condition (1c) has no parallel in [HJ98a, Def. 1.1]. This condition is important in order to construct Galois extensions of the rings Q_i, in the following sense:

Definition 1.2. Let $Q \subseteq P$ be integral domains and $\mathrm{Aut}(P)$ the group of automorphisms of P. Define $\mathrm{Aut}(P/Q) := \{\sigma \in \mathrm{Aut}(P) \mid \sigma x = x \text{ for all } x \in Q\}$. We say that P/Q is a *finite Galois domain extension*, if $P = Q[a]$ and $f = \mathrm{irr}(a, \mathrm{Quot}(Q))$ satisfies:

(a) $f \in Q[X]$, so that $P \cong Q[X]/\langle f\rangle$.

(b) f factors in $P[X]$ into a product of distinct linear monic polynomials.

We call $\mathrm{Gal}(P/Q) = \mathrm{Aut}(P/Q)$ the *Galois group* of P/Q.

Fix a generalized patching data $\mathcal{E} = (E, F_i, Q_i, \Omega; G_i, G)_{i\in I}$. We extend $\mathcal{E}$ by more rings and algebras. For each $i \in I$ let $P_i = Q_i F_i$ be the compositum of Q_i and F_i in Ω. By condition (1c) $\mathrm{Quot}(Q_i) \cap F_i = E$. By [Par08, Lem. 1.3, Lem. 1.4] P_i/Q_i is a Galois domain extension, the Galois group of P_i/Q_i is isomorphic (via the restriction of automorphisms) to $G_i = \mathrm{Gal}(F_i/E)$, P_i is a free Q_i-module of rank $|G_i|$, and $P_i^{\mathrm{Gal}(P_i/Q_i)} = Q_i$. Identify $\mathrm{Gal}(P_i/Q_i)$ with G_i via this isomorphism.

Consider the algebra

$$N = \mathrm{Ind}_1^G \Omega = \Big\{ \sum_{\theta\in G} a_\theta \theta \mid a_\theta \in \Omega \Big\}$$

of dimension $|G|$ over Ω. Addition and multiplication are defined in N componentwise — thus $1 = \sum_{\theta\in G}\theta$, Ω is embedded diagonally in N, and G acts on N by

$$\Big(\sum_{\theta\in G} a_\theta\theta\Big)^\sigma = \sum_{\theta\in G} a_\theta \sigma^{-1}\theta = \sum_{\theta\in G} a_{\sigma\theta}\theta, \qquad \sigma\in G.$$

The action of G commutes with the addition and the multiplication in N.

For each $i \in I$ consider the following Q_i-subalgebra of N:

$$N_i = \mathrm{Ind}_{G_i}^G P_i = \Big\{ \sum_{\theta\in G} a_\theta\theta \in N \mid a_\theta \in P_i,\ a_\theta^\tau = a_{\theta\tau} \text{ for all } \theta\in G,\ \tau\in G_i \Big\}.$$

By [Par08, Lem. 1.5], $F = \bigcup_{i\in I} N_i$ is an E-algebra which is G-invariant. We call F the *pre-compound* of the generalized patching data $\mathcal{E}$.

PROPOSITION 1.3 [Par08, Prop. 1.8]. *Assume that*:

(COM) *There is a linear basis of N over Ω that is also a basis of N_i over Q_i for all $i \in I$.*

Then $F = \bigcap N_i$ is a field and F/E is a Galois extension with group G.

For the rest of this section assume that E is a finite Galois extension of a field E_0 with group Γ. In order to solve FSEPs defined by an action of Γ on G we need the following notion:

Definition 1.4. A *proper action* of Γ on the generalized patching data $\mathcal{E}$ is a triple that consists of an action of Γ on the group G, an action of Γ on the field Ω, and an action of Γ on the set I such that the following conditions hold:

(2a) The action of Γ on Ω extends the action of Γ on E.

(2b) $F_i^\gamma = F_{i^\gamma}$, $Q_i^\gamma = Q_{i^\gamma}$, and $G_i^\gamma = G_{i^\gamma}$, for all $i \in I$ and $\gamma \in \Gamma$.

(2c) $(a^\tau)^\gamma = (a^\gamma)^{\tau^\gamma}$ for all $a \in F_i$, $\tau \in G_i$, $i \in I$, and $\gamma \in \Gamma$.

The action of Γ on G defines a semidirect product $G \rtimes \Gamma$ such that $\tau^\gamma = \gamma^{-1}\tau\gamma$ for all $\tau \in G$ and $\gamma \in \Gamma$. Let $\pi: G \rtimes \Gamma \to \Gamma$ be the canonical projection.

The proof of the next proposition is verbally the same as of [HJ98a, Prop. 1.5] (replacing Q there with Ω here).

PROPOSITION 1.5. *Suppose that Γ properly acts on the patching data $\mathcal{E}$ and that $\mathcal{E}$ satisfies (COM). Then F/E_0 is Galois and there is an isomorphism $\psi: G \rtimes \Gamma \to \mathrm{Gal}(F/E_0)$ such that $\mathrm{res}_E \circ \psi = \pi$.*

Definition 1.6. Consider the Ω-algebras homomorphism $\phi : N \to \Omega$ given by $\sum a_\theta \theta \mapsto a_1$. Then $\phi|_F$ is a monomorphism. Since E is invariant under ϕ, $\phi(F)$ is a Galois extension of E with group isomorphic to G. We call $F' = \phi(F)$ the *compound* of $\mathcal{E}$. Then by Proposition 1.5 F'/E_0 is Galois and there is an isomorphism $\psi': G \rtimes \Gamma \to \mathrm{Gal}(F'/E_0)$ such that $\mathrm{res}_E \circ \psi' = \pi$.

2. **Rings of convergent power series**

The rings Q_i that we use in this work for the patching data will be localizations of complete rings under a norm.

Definition 2.1 (Normed ring). Let R be an associative ring with 1. A *norm* on R is a function $|\cdot|: R \to \mathbb{R}$ that satisfies the following conditions for all $a, b \in R$:

(a) $|a| \geq 0$, and $|a| = 0$ if and only if $a = 0$; further $|1| = |-1| = 1$.

(b) $|a + b| \leq \max(|a|, |b|)$.

(c) $|ab| \leq |a| \cdot |b|$.

If $|\cdot|$ satisfies the following stronger condition

(c′) $|ab| = |a| \cdot |b|$,

we say that $|\cdot|$ is an *absolute value* on R.

We sometimes prefer to use the additive notation for a norm $v(a) = -\log(a)$. That is, instead of $|\cdot|$ we shall have a map $v\colon R \to \mathbb{R} \cup \{\infty\}$ such that following conditions hold:

(a) $v(a) = \infty$ if and only if $a = 0$; further $v(1) = v(-1) = 0$.

(b) $v(a+b) \geq \min(v(a), v(b))$.

(c) $v(ab) \geq v(a) + v(b)$.

The equivalence of these two definitions is standard. If $|\cdot|$ is an absolute value, then v is a rank-1 valuation. We say that R is *complete* if every Cauchy sequence in R converges.

In this section we study rings of convergent power series over a complete normed domain D. These rings have been studied in [Par08, §2, §3, §4]. However, some of the properties proven in that paper required the norm of D to be an absolute value. Here we replace this assumption by a weaker one — we will assume that D is equipped with both a norm *and* an absolute value, with some compatibility condition between the two. However, D is assumed to be complete only with respect to the norm. This technique enables us to embed rings of convergent (with respect to the norm) power series over D in rings of convergent (with respect to the absolute value) power series over $\widehat{K}$, where $\widehat{K}$ is the completion of K with respect to its absolute value. We use the strong properties of the latter rings to gain information about the former rings.

Fix a domain D, complete with respect to a nontrivial norm $|\cdot|$ and let I be a finite set. For each $i \in I$ let $r, c_i \in D$ such that $r, c_i - c_j \in D^\times$ if $i \neq j$. We assume that

$$\left| \frac{r}{c_i - c_j} \right| \leq 1 \quad \text{for all } i \neq j. \tag{1}$$

Let $K = \mathrm{Quot}(D)$ and let $E = K(x)$ be the field of rational functions over K. For each $i \in I$ let $w_i = r/(x - c_i) \in K(x)$.

Consider the subset $R_0 = \sum_{i \in I} D[w_i]$ of the field $E = K(x)$. By [Par08, §4] this is a ring and each element of R_0 has a unique presentation of the form $a_0 + \sum_{i \in I} \sum_{n \geq 1} a_{in} w_i^n$, where $a_{in} \in D$ are almost all zero. Moreover, we can define a norm on R_0 by $\|a_0 + \sum_{i \in I} \sum_{n \geq 1} a_{in} w_i^n\| = \max_{i,n}\{|a_0|, |a_{in}|\}$. Let $R = D\{w_i \mid i \in I\}$ be the completion of R_0 with respect to $\|\cdot\|$, and extend $\|\cdot\|$ to R. By [Par08, Lem. 4.2] we have:

LEMMA 2.2. *Each element f of R has a unique presentation as a multiple power series*:

$$f = a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} w_i^n, \tag{2}$$

where $a_0, a_{in} \in D$, *and* $|a_{in}| \to 0$ *as* $n \to \infty$. *Moreover,*

$$\|f\| = \max_{i,n}\{|a_0|, |a_{in}|\}.$$

We call the partial sum $\sum_{n=1}^{\infty} a_{in} w_i^n$ in (2) the i*-component* of f.

For each $J \subseteq I$ we denote the completion R_J of $D[w_j \mid j \in J]$ by $D\{w_j \mid j \in J\}$. By the preceding lemma R_J is contained in R_I. By [Par08, Prop. 4.7] we have:

PROPOSITION 2.3. *Suppose* $D = K$ *and* $|\cdot|$ *is an absolute value. Then the ring* $R = K\{w_i \mid i \in I\}$ *is a principal ideal domain.*

For the rest of this section assume that, in addition to the norm $|\cdot|$, D is also equipped with a nontrivial absolute value $|\cdot|'$, such that $|a|' \le |a|$ for each $a \in D$ (but D need not be complete with respect to $|\cdot|'$). We extend the absolute value $|\cdot|'$ to the quotient field K. Let $\widehat{K}$ be the completion of K with respect to $|\cdot|'$. Then, $\left|r/(c_i - c_j)\right|' \le 1$ for all $i \ne j$. Thus we may consider the ring $\widehat{K}\{w_i \mid i \in I\}$, and its subrings $\widehat{K}\{w_i \mid i \in J\}$, $J \subseteq I$.

Remark 2.4 (Embedding of R in $\widehat{K}\{w_i \mid i \in I\}$). We distinguish between two types of infinite sums. One with respect to the norm $|\cdot|$, and one with respect to $|\cdot|'$. We denote the first type by $\sum$ (as we have done so far) and the latter by $\sum'$. The assumption $|x|' \le |x|$ implies that whenever a sum $\sum a_i$ is well defined, then so is $\sum' a_i$ and we have $\sum a_i = \sum' a_i$. Therefore, we may consider the ring

$$R = \{a_0 + \sum_{i \in I} \sum_{n=1}^{\infty} a_{in} w_i^n \mid a_{in} \in D,\ |a_{in}| \to 0 \text{ for each } i \in I\}$$

as a subring of

$$\widehat{K}\{w_i \mid i \in I\} = \{a_0 + \sum_{i \in I} {\sum_{n=1}^{\infty}}' a_{in} w_i^n \mid a_{in} \in \hat{K},\ |a_{in}|' \to 0 \text{ for each } i \in I\}.$$

Moreover, the inclusion of R with the $|\cdot|$-topology into $\widehat{K}\{w_i \mid i \in I\}$ with the $|\cdot|'$-topology is continuous.

By Proposition 2.3, $\widehat{K}\{w_i \mid i \in I\}$ is an integral domain, hence so is R. Denote the quotient ring of R by Ω. For each $J \subseteq I$ consider the rings $O_J = D[w_i \mid i \in J]$ and $Q_J = (O_J \smallsetminus \{0\})^{-1} R_J = \left\{\frac{f}{a} \mid f \in R_J,\ a \in O_J \smallsetminus \{0\}\right\}$.

In the rest of this section we describe crucial properties of these rings. The following claims generalize Lemma 4.8, Proposition 4.9, Corollary 4.10, Lemma 6.4 and Lemma 6.5 of [Par08], respectively. Note that [Par08] makes stronger assumptions on the ring D (namely, that $|\cdot|$ itself is an absolute value), yet the proofs remain verbally the same, and we omit them.

LEMMA 2.5. *Let J be a non-empty subset of I. Then*:

(a) $E = \mathrm{Quot}(O_J)$.

(b) *The ring Q_J is the compositum of E and R_J in Ω.*

(c) *If $j \in J$ then $Q_J = (O_{\{j\}} \smallsetminus \{0\})^{-1} R_J$.*

For each $J \subseteq I$, we denote the integral closure of Q_J inside its quotient field by $C(Q_J)$. View $C(Q_J)$ as contained in Ω.

THEOREM 2.6. *Let J, J' be non-empty subsets of I.*

(a) *If $J \cap J' \neq \varnothing$, then $Q_J \cap Q_{J'} = Q_{J \cap J'}$.*

(b) *If $J \cap J' = \varnothing$, then $Q_J \cap C(Q_{J'}) = E$.*

For each $i \in I$, let $Q_i = Q_{I \smallsetminus \{i\}}$, $Q'_i = Q_{\{i\}}$. Note that by this notation, $Q_i \neq Q_{\{i\}}$.

COROLLARY 2.7. $\bigcap_{i \in I} Q_i = E$.

For simplicity, assume $I = \{1, 2, \ldots, k\}$.

LEMMA 2.8. *Let c be an element of D such that $c - c_i \in D^\times$, $|r/(c - c_i)| \leq 1$ for each $1 \leq i \leq k$. Consider the evaluation homomorphism $\phi_c \colon R \to D$ given by $w_1 \mapsto r/(c - c_1), \ldots, w_k \mapsto r/(c - c_k)$. Denote $p = w_1 - r/(c - c_1) \in R$. Then*:

(a) $\mathrm{Ker}(\phi_c)$ *is a principal ideal of R, generated by p.*

(b) *The localization $R_p = \{\frac{a}{b} \in \Omega \mid a \in R,\ b \in R \smallsetminus pR\}$ is a valuation ring of Ω.*

LEMMA 2.9. *Let $\{F_i\}_{i \in I}$ be fields, and let $G, \{G_i\}_{i \in I}$ be groups such that $\mathcal{E} = (E, F_i, Q_i, \Omega; G_i, G)_{i \in I}$ is a generalized patching data (Definition 1.1). Assume that for each $i \in I$ we have $F_i = E(\beta_i)$, where β_i and its conjugates over E are in R, and $\mathrm{discr}_E(\mathrm{irr}(\beta_i, E)) \in R^\times$. Then*:

(a) *Condition (COM) of Section 1 holds for $\mathcal{E}$.*

(b) *Suppose that there is an element $c \in D$ such that $c - c_i \in D^\times$ and $|r/(c - c_i)| \leq 1$ for each $i \in I$. Then the compound F' of $\mathcal{E}$ has a K-rational place.*

3. Galois covers of complete domains

Let A be an integral domain with quotient field K_0. Let K be a finite Galois extension of K_0, and let z be a primitive element of K/K_0 that is integral over A. Denote $f = \mathrm{discr}(\mathrm{irr}(z, K_0))$. Suppose f is not invertible in A, and let $\mathfrak{a}$ be the ideal generated by f. Suppose A is complete and Hausdorff with respect to the $\mathfrak{a}$-adic topology.

Let $|\cdot|_f$ be the norm corresponding to $\mathfrak{a}$. That is, $|x|_f = \min(e^{-i} \mid x \in \mathfrak{a}^i)$ for each $x \in A$, where e is the base of the natural logarithms. In particular, since f is not invertible, $f^k \in \mathfrak{a}^k \smallsetminus \mathfrak{a}^{k+1}$, so $|f^k|_f = e^{-k}$ for each $k \in \mathbb{N}$. This norm need not be an absolute value, so we may not be able to extend it to a norm of K_0. However, let A_f be the localization $A[f^{-1}]$. Then:

LEMMA 3.1. *Every element of A_f can be uniquely presented as $f^n g$, where $n \in \mathbb{Z}$ and $g \in A \smallsetminus \mathfrak{a}$, and $|\cdot|_f$ extends to a norm on A_f by $|f^n g|_f = e^{-n}$.*

Proof. The only nontrivial part in proving the first assertion is to show that a nonzero element cannot be divided by f infinitely many times. This follows as A is Hausdorff with respect to $\mathfrak{a}$, so $\bigcap_{i=1}^{\infty} \mathfrak{a}^i = \bigcap_{i=1}^{\infty} \langle f \rangle^i = \bigcap_{i=1}^{\infty} \langle f^i \rangle = 0$.

We define $|f^n g|_f = e^{-n}$. Then $|f^n g_1 + f^m g_2|_f \le \max(e^{-n}, e^{-m})$ and $|(f^n g_1)\cdot(f^m g_2)|_f \le e^{-n} \cdot e^{-m}$ for all $g_1, g_2 \in A$. □

PROPOSITION 3.2. *The ring A_f is complete with respect to $|\cdot|_f$.*

Proof. Let $\{g_i\}_{i=1}^{\infty}$ be a Cauchy sequence in A_f. Then either $|g_i|_f \to 0$ or for each sufficiently large index i we have $|g_i|_f = |g_{i+1}|_f = |g_{i+2}|_f = \cdots$. In the latter case there exists $m \in \mathbb{N}$ with $g_i f^m \in A$ for all i. Then $\{g_i f^m\}$ is a Cauchy sequence in A, hence $\{g_i f^m\}$ converges in A, since A is complete with respect to $|\cdot|_f$. Consequently, $\{g_i\}$ also converges in A_f. □

Denote the ring $A_f[z]$ by B.

PROPOSITION 3.3. *The norm $|\cdot|_f$ extends to a norm $|\cdot|_f$ on the ring B, and B is complete with respect to this norm.*

Proof. The ring B is a free A_f-module with basis $1, z, \dots, z^{d-1}$ ($d = [K : K_0]$). Define $|\sum_{i=0}^{d-1} a_i z^i|_f = \max_i |a_i|_f$ for $a_0, \dots, a_{d-1} \in A_f$. We show that this defines a norm on B. The only nontrivial part is to prove that $|ab|_f \le |a|_f \cdot |b|_f$. Let $a = \sum_{i=0}^{d-1} a_i z^i, b = \sum_{i=0}^{d-1} b_i z^i, a_i, b_i \in A_f$. Denote $ab = \sum_{i=0}^{d-1} c_i z^i$. Since the coefficients of the minimal polynomial of z belong to A, each c_i is a sum of elements of the form $\alpha \sum_{i+j=l} a_i b_j$ for some $l \ge 0$ and $\alpha \in A$. For each such summand, we have $|\alpha|_f \le 1$, and so $|\alpha \sum_{i+j=l} a_i b_j|_f \le |\alpha|_f \cdot |\sum_{i+j=l} a_i b_j|_f \le |\sum_{i+j=l} a_i b_j|_f \le \max_i(|a_i|_f \cdot |b_{l-i}|_f) \le \max_i |a_i|_f \cdot \max_i |b_i|_f = |a|_f \cdot |b|_f$. It follows by the norm properties that $|c_i|_f \le |a|_f \cdot |b|_f$ for each i, hence $|ab|_f \le |a|_f \cdot |b|_f$.

By definition, a sequence in B is Cauchy if and only if it is Cauchy coefficientwise. Since A_f is complete, so is B. □

In [Par08, §6] a ring D is said to be *large* if it satisfies the following condition:

(Large) For each $n \in \mathbb{N}$ there exist $b_1, \dots, b_n \in D$ such that $b_i - b_j \in D^{\times}$ for all $i \ne j$.

It follows from the next proposition that the ring B is large in this sense. This will be needed in the proof of our main result in the next section.

PROPOSITION 3.4. *There exists a bounded (with respect to $|\cdot|$) series $c_1, c_2, \ldots \in B$ such that $c_i^\delta - c_j^\epsilon \in B^\times$ for all distinct $(i,\delta), (j,\epsilon) \in \mathbb{N} \times \mathrm{Gal}(K/K_0)$.*

Proof. The primitive element z satisfies $z^\delta - z^\varepsilon \in B^\times$ for all distinct $\delta, \varepsilon \in \mathrm{Gal}(K/K_0)$, since the discriminant f of $\mathrm{irr}(z, K_0)$ is invertible in $A_f \subseteq B$.

Let $m = \max_{\delta \neq \epsilon}(1, \log|(z^\delta - z^\epsilon)^{-1}|_f)$. For each $i \in \mathbb{N}$ let $\alpha_i = f^{m+1} + f^{m+2} + \cdots + f^{m+i}$. If $1 \le i < j$, then

$$\alpha_j - \alpha_i = f^{m+i+1} + f^{m+i+2} + \cdots + f^{m+j} = f^{m+i+1}(1 + f + \cdots + f^{j-i-1}),$$

so $|\alpha_j - \alpha_i|_f < e^{-m}$. Moreover, $\alpha_j - \alpha_i \in A_f^\times$, because A_f is complete with respect to $|\cdot|_f$ (Proposition 3.2) and $|f + \cdots + f^{j-i-1}|_f < 1$ (so $(1 + f + \cdots + f^{j-i-1})^{-1} = \sum_{l=0}^\infty (-f - \cdots - f^{j-i-1})^l$). Define $c_i = z + \alpha_i$. Then for each $\delta \in \mathrm{Gal}(K/K_0)$ we have $c_i^\delta - c_j^\delta = \alpha_i - \alpha_j \in B^\times$. If $\delta \neq \varepsilon$, then $|(\alpha_j - \alpha_i)/(z^\delta - z^\varepsilon)|_f \le |\alpha_j - \alpha_i|_f \cdot |(z^\delta - z^\varepsilon)^{-1}|_f < 1$, hence $1 + (\alpha_j - \alpha_i)/(z^\delta - z^\varepsilon)$ is invertible in B, therefore so is $(z^\delta - z^\varepsilon) + (\alpha_j - \alpha_i) = c_j^\delta - c_i^\varepsilon$. Note that the series $\{|c_i|\}_{i=1}^\infty$ is bounded by $\max\{|z|, 1\}$. □

PROPOSITION 3.5. *Suppose that $\mathrm{Gal}(K/K_0)$ acts on B, that is, $B^\sigma = B$ for each $\sigma \in \mathrm{Gal}(K/K_0)$. Then the action is continuous with respect to $|\cdot|_f$.*

Proof. Let $\{x_i\}_{i=0}^\infty$ be a convergent sequence in B, and let x be its limit (with respect to $|\cdot|_f$). We must prove that for each $\sigma \in \mathrm{Gal}(K/K_0)$ the sequence x_i^σ converges to x^σ. Without loss of generality we may assume that $x = 0$, and prove that x_i^σ converges to 0. Let $x_i = \sum_{j=0}^{d-1} a_{ij} z^j$ with $a_{ij} \in A_f$. Then $|x_i|_f = \max_j |a_{ij}|_f \to 0$ as $i \to \infty$, so for all $0 \le j \le d-1$ we have $|a_{ij}|_f \to 0$ as $i \to \infty$. Now, $|x_i^\sigma|_f = \left|\sum_{j=0}^{d-1} a_{ij} (z^\sigma)^j\right|_f \le \max_j |a_{ij}(z^\sigma)^j|_f \le \max_j(|a_{ij}|_f \cdot |z^\sigma|_f^j)$. Since $|a_{ij}|_f \to 0$ as $i \to \infty$, also $|a_{ij}| \cdot |z^\sigma|_f^j \to 0$ as $i \to \infty$, for all $0 \le j \le d-1$. Thus $|x_i^\sigma|_f \to 0$ as $i \to \infty$. □

PROPOSITION 3.6. *Let $\mathfrak{p}$ be an ideal of A which contains f. For each $x \in A$ let $|x| = \min\{e^{-i} \mid x \in \mathfrak{p}^i\}$. Suppose that $|\cdot|$ extends to an absolute value on K_0. Extend $|\cdot|$ arbitrarily to an absolute value on K. Then for each $x \in B$ we have $|x| \le |x|_f$.*

Proof. Since $f \in \mathfrak{p}$, we have $|f| \le e^{-1}$. Each $0 \neq x \in B$ may be written as $f^k a$, with $a \in A[z]$, $|a|_f = 1$, $k \in \mathbb{Z}$. Indeed, let $x = \sum_{i=0}^{d-1} a_i z^i$ with $a_i \in A_f$. Let $k = \max_i(-\log(|a_i|_f))$, $a = f^{-k} x$. Then $x = f^k a$, $a \in A[z]$ and $|a|_f = 1$. Since $A[z]$ is an integral extension of A, $|a| \le 1$, and so $|x| = |f|^k \cdot |a| \le e^{-k} \cdot 1 = e^{-k} = |x|_f$. □

4. Solution of split embedding problems

Let A be a Noetherian integrally closed domain and let $0 \neq \mathfrak{p} \subset A$ be a proper ideal of A, such that A is complete with respect to the $\mathfrak{p}$-adic topology. Moreover, suppose that the order function of $\mathfrak{p}$ extends to a discrete valuation of $K_0 = \mathrm{Quot}(A)$. Equivalently, the map $A \to \mathbb{R}$ given by $|x|' = \min\{e^{-i} \mid x \in \mathfrak{p}^i\}$ extends to an absolute value on K_0. Let K be a finite Galois extension of K_0.

THEOREM 4.1. *Let x be a free variable over K, and put $E_0 = K_0(x)$, $E = K(x)$. Suppose that $\Gamma = \mathrm{Gal}(K/K_0)$ acts (from the right) on a finite group G. Let $G \rtimes \Gamma$ be the corresponding semidirect product and let $\pi\colon G \rtimes \Gamma \to \Gamma$ be the canonical projection. Then the constant split embedding problem $\pi\colon G \rtimes \Gamma \to \Gamma = \mathrm{Gal}(E/E_0)$ has a rational (hence regular) solution. That is, there is an extension F of E such that*

(a) *F/E_0 is Galois,*

(b) *there is an isomorphism $\psi\colon \mathrm{Gal}(F/E_0) \to G \rtimes \Gamma$ such that $\pi \circ \psi = \mathrm{res}_E$, and*

(c) *F has a K-rational place (hence F/K is regular).*

Proof. We attach a generalized patching data $\mathscr{E}$ to the embedding problem and define a proper action of Γ on $\mathscr{E}$. Then by Proposition 1.5 we conclude that the pre-compound F of $\mathscr{E}$ gives a solution to the embedding problem. Our proof is similar to that of [HJ98a, Prop. 5.2], however here we must use a generalized patching data defined over a suitable subring of K, instead of the patching data of [HJ98a] that is defined over a complete field.

Fix a finite set I on which Γ acts from the right and a system of generators $T = \{\tau_i \mid i \in I\}$ of G such that for each $i \in I$

(5a) $\{\gamma \in \Gamma \mid i^\gamma = i\} = \{1\}$,

(5b) $\tau_i^\gamma = \tau_{i^\gamma}$, for every $\gamma \in \Gamma$, and

(5c) $|I| \geq 2$.

(For example, assuming $G \neq 1$, let $I = G \times \Gamma$, let Γ act on I by $(\sigma, \gamma)^\delta = (\sigma, \gamma\delta)$, and let $\tau_{(\sigma,\gamma)} = \sigma^\gamma$.) Let G_i be the subgroup generated by τ_i. Then $G_i^\gamma = G_{i^\gamma}$ for all $\gamma \in \Gamma$ and $G = \langle G_i \mid i \in I\rangle$. This establishes Condition (1d) of Definition 1.1. Choose a system of representatives J for the Γ-orbits of I. Then every $i \in I$ can be uniquely written as $i = j^\gamma$ with $j \in J$ and $\gamma \in \Gamma$.

Let w be a primitive element w for K/K_0 which is integral over A, and take an arbitrary element $0 \neq \alpha \in \mathfrak{p}$. Then $z = \alpha w$ is a primitive element for K/K_0 integral over A, and $f = \mathrm{discr}(\mathrm{irr}(z, K_0)) = \alpha^{d(d-1)} \mathrm{discr}(\mathrm{irr}(w, K_0)) \in \mathfrak{p}$.

Since A is integrally closed, so is $A_f = A[f^{-1}]$. Let D be the integral closure of A_f in K. Since f is in $(A_f)^\times$, $D = A_f[z]$, by [FJ05, Lem. 6.1.2].

The ring A is complete with respect to the $\mathfrak{p}$-adic topology, and since it Noetherian, it is also Hausdorff with respect to this topology. By [ZS60, Th. VIII.5.14], A is complete and Hausdorff in the $\mathfrak{b}$-adic topology for every ideal $\mathfrak{b}$ contained in $\mathfrak{p}$. In particular, for $\mathfrak{b} = \langle f \rangle$. Thus D is complete with respect to the nontrivial norm $|\cdot|$ that corresponds to f, given by Proposition 3.3. Extend $|\cdot|'$ (the absolute value that $\mathfrak{p}$ defines on K_0) arbitrarily to an absolute value on K. By Proposition 3.6, $|x|' \le |x|$ for each $x \in D$.

Construction A: *Choice of* (*dependent*) *variables.* We choose the variables w_i that are used to define the rings in our patching data.

Proposition 3.4 gives an element $c \in D$ and a subset $\{c_j \mid j \in J\} \subseteq D$ such that $c - c_j^\delta \in D^\times$ and $c_i^\delta - c_j^\gamma \in D^\times$ for all distinct $(i,\delta),(j,\gamma) \in J \times \Gamma$. For each $i = j^\delta \in I$ we define $c_i = c_j^\delta$. Then $c_i^\delta = c_{i^\delta}$ for all $i \in I$, $\delta \in \Gamma$ and $c - c_i$, $c_i - c_j \in D^\times$ if $i \ne j$. Since $f \in \mathfrak{p}$, we have $|f| < 1$, so we may choose a positive integer t with $|f^t/(c_i - c_j)| \le 1$ and $|f^t/(c - c_i)| \le 1$ for $i \ne j$. Let $r = f^t$ and define $w_i = r/(x - c_i)$ for all $i \in I$. Then condition (5) of Section 2 is satisfied.

Construction B: *Construction of the rings* Q_i. By Remark 2.4, the ring $R = D\{w_i \mid i \in I\}$ is an integral domain. Let Ω be its quotient field. For each $i \in I$ let

$$Q_i = Q_{I \smallsetminus \{i\}} = (D[w_j \mid j \ne i] \smallsetminus \{0\})^{-1} R_{I \smallsetminus \{i\}} \quad \text{and} \quad Q_i' = Q_{\{i\}}$$

(we use the notation of §2). By Corollary 2.7, $\bigcup_{i \in I} Q_i = E$, which establishes Condition (1e) of Definition 1.1. Moreover, by Theorem 2.6(a), $Q_i' = \bigcup_{j \ne i} Q_j$, for all $i \in I$.

The group $\Gamma = \mathrm{Gal}(K/K_0)$ leaves $D = A_f[z]$ invariant, because D is the integral closure of A_f in K. By Proposition 3.5, Γ acts continuously on D with respect to the norm $|\cdot|$. Moreover, Γ lifts isomorphically to $\mathrm{Gal}(E/E_0)$. For all $\gamma \in \Gamma$ and $i \in I$ we have $w_i^\gamma = w_{i^\gamma}$ (by Construction A). Hence, Γ acts continuously on $D[w_i \mid i \in I]$. Therefore, Γ lifts to a group of automorphisms of the completion $R = D\{w_i \mid i \in I\}$ of $D[w_i \mid i \in I]$ with respect to $|\cdot|$. Finally, Γ extends to a group of automorphisms of $\blacksquare = \mathrm{Quot}(R)$. Under this action, $Q_i^\gamma = Q_{i^\gamma}$ and $(Q_i')^\gamma = Q_{i^\gamma}'$ for all $\gamma \in \Gamma$ and $i \in I$.

For each $j \in J$, [Par08, Prop. 6.1] gives a cyclic extension F_j/E with group $G_j = \langle \tau_j \rangle$ such that F_j/K has a prime divisor of degree 1. Moreover, F_j is an unramified extension of E.

By [Par08, Rem. 4.3(a)] the map $w_j \mapsto x$ extends to a K-isomorphism of $K((w_j))$ onto $K((x))$ which maps $R_{\{j\}}$ onto $D\{x\}$. By Proposition 3.4, D satisfies Condition 3 preceding [Par08, Lem. 6.3]. Hence, by [Par08, Lem. 6.3] we may replace F_j/E by an isomorphic extension such that $F_j = E(\beta_j)$, where β_j and its conjugates over E belong to $R_{\{j\}}$, and $\mathrm{discr}_E(\mathrm{irr}(\beta_j, E)) \in R_{\{j\}}^\times$. In particular, $F_j \subseteq Q_j'$.

For an arbitrary $i \in I$ there exist unique $j \in J$ and $\gamma \in \Gamma$ such that $i = j^\gamma$ (by (5a)). Let $F_i = F_j^\gamma = E(\beta_j^\gamma)$. Since γ acts on Ω and leaves E invariant, F_i is a Galois extension of E and $F_i \subseteq Q_i'$. This establishes Condition (1b) of Definition 1.1.

Construction C: *The action of* Γ *on the rings* Q_i. The isomorphism $\gamma\colon F_j \to F_i$ gives an isomorphism $\mathrm{Gal}(F_j/E) \cong \mathrm{Gal}(F_i/E)$ which maps each $\tau \in \mathrm{Gal}(F_j/E)$ onto $\gamma^{-1} \circ \tau \circ \gamma \in \mathrm{Gal}(F_i/E)$ (notice that the elements of the Galois groups act from the right). In particular, it maps τ_j onto $\gamma^{-1} \circ \tau_j \circ \gamma$. We can therefore identify G_i with $\mathrm{Gal}(F_i/E)$ such that τ_i coincides with $\gamma^{-1} \circ \tau_j \circ \gamma$. This means that $(a^\tau)^\gamma = (a^\gamma)^{\tau^\gamma}$ for all $a \in F_j$ and $\tau \in G_j$.

It follows that for all $i \in I$ and $\gamma \in \Gamma$ we have $F_i^\gamma = F_{i^\gamma}$. Moreover, $(a^\tau)^\gamma = (a^\gamma)^{\tau^\gamma}$ for all $a \in F_i$ and $\tau \in G_i$.

Construction D: *Generalized patching data.* Fix $j \in J$, $\gamma \in \Gamma$, and let $i = j^\gamma$, $\beta_i = \beta_j^\gamma$. Then $F_i = E(\beta_i)$. Consider an element $\sigma \in G_i$. There exists $\tau \in G_j$ with $\sigma = \gamma^{-1}\tau\gamma$ and $\beta_i^\sigma = (\beta_j^\gamma)^{\gamma^{-1}\tau\gamma} = (\beta_j^\tau)^\gamma \in (R_{\{j\}})^\gamma = R_{\{i\}}$. Similarly, $\mathrm{discr}(\mathrm{irr}(\beta_i, E)) = (\mathrm{discr}(\mathrm{irr}(\beta_j, E)))^\tau$, so $\mathrm{discr}(\mathrm{irr}(\beta_i, E)) \in R_{\{i\}}^\times$.

If $y \in \mathrm{Quot}(Q_i) \cap Q_i'$ is algebraic over E, then y is integral over Q_i. Hence, by Theorem 2.6(b), $y \in E$. It follows that $F_i \cap \mathrm{Quot}(Q_i) = E$, for all $i \in I$. This proves Condition (1c) of Definition 1.1.

Thus, $\mathscr{E} = (E, F_i, Q_i, \Omega; G_i, G)_{i \in I}$ is a generalized patching data (Definition 1.1) and Γ acts properly on $\mathscr{E}$ (Definition 1.4). By Lemma 2.9(a) $\mathscr{E}$ satisfies Condition (COM). By Proposition 1.5, the pre-compound F of $\mathscr{E}$ satisfies assertions (a) and (b).

Definition 1.6 gives an E-isomorphism of F onto the compound F' of the patching data, which then also satisfies (a) and (b). The element c chosen in Construction A enables us to use Lemma 2.9(b), which proves (c). □

THEOREM 4.2. *Suppose* $K_0 = \mathrm{Quot}(A)$ *is Hilbertian. Then every finite split embedding problem over* K_0 *is solvable.*

Proof. Given an FSEP over K_0, we can solve the problem over $K_0(x)$, by Theorem 4.1, and specialize the solution to a solution over K_0, by [FJ05, Lem. 16.4.2]. □

THEOREM 4.3. *Let* L *be an arbitrary field, and* $n \geq 2$ *an integer. Then*:

(a) *Every constant finite split embedding problem over* $L((X_1, \dots, X_n))(x)$ *has a rational solution.*

(b) *Every finite split embedding problem over* $L((X_1, \dots, X_n))$ *is solvable.*

Proof. The ring $L[[X_1, \dots, X_n]]$ is an integrally closed Noetherian domain [ZS60, VII.§1]. The order function of $\mathfrak{p} = \langle X_1, \dots, X_n \rangle$ extends to a discrete valuation on the quotient field $L((X_1, \dots, X_n))$, and $L[[X_1, \dots, X_n]]$ is complete with respect to this valuation [ZS60, VII.§1]. By a theorem of Weissauer $L((X_1, \dots, X_n))$ is Hilbertian [FJ05, Exa. 15.5.2]. We conclude from Theorem 4.2 that (a) and (b) hold. □

Theorem 4.2 also helps solve FSEPs over the field $\mathrm{Quot}(\mathbb{Z}[[X_1, \dots, X_n]])$, where $n \geq 1$. More generally:

THEOREM 4.4. *Let B be a Noetherian integrally closed domain which is not a field, and let $n \geq 1$ an integer. Then:*

(a) *Every constant finite split embedding problem over*

$$\mathrm{Quot}(B[[X_1, \dots, X_n]])(x)$$

has a rational solution.

(b) *Every finite split embedding problem over*

$$\mathrm{Quot}(B[[X_1, \dots, X_n]])$$

is solvable.

Proof. The ring B is a Krull domain, by [Nag62, Prop. 33.4], so the quotient field of $B[[X_1, \dots, X_n]]$ is Hilbertian, again by Weissauer [FJ05, Exa. 15.5.3]. The rest of the proof is identical to that of Theorem 4.3. □

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