ANNALS OF MATHEMATICS

The zero locus of an admissible normal function

By Patrick Brosnan and Gregory J. Pearlstein

SECOND SERIES, VOL. 170, NO. 2

September, 2009

ANMAAH

Annals of Mathematics, **170** (2009), 883–897

The zero locus of an admissible normal function

By PATRICK BROSNAN and GREGORY J. PEARLSTEIN

Abstract

We prove that the zero locus of an admissible normal function over an algebraic parameter space S is algebraic in the case where S is a curve.

1. Introduction

Let S be a smooth, complex projective variety. Following Morihiko Saito in [Sai96], we define an admissible normal function on S to be an admissible variation of graded-polarized mixed Hodge structure [SZ85] [Kas86] $\mathcal{V}$ over a Zariski open subset $S^* = S - D$ of S that is an extension of the trivial variation $\mathbb{Z}(0)$ by a variation of pure (polarized) Hodge structure $\mathcal{H}$ of weight $w < 0$.

Henceforth, we assume that $w = -1$. In this case, an admissible normal function corresponds to the usual notion of a horizontal normal function on $S - D$ with moderate growth near D together with the existence of a suitable relative weight filtration along each irreducible component of D. In this article's Theorem 4.5, we settle the following conjecture communicated to us by M. Green and P. Griffiths in the case where S is a curve.

CONJECTURE 1.1. *Let ν be an admissible normal function on S. Then the zero locus $\mathcal{Z}$ of ν is an algebraic subvariety of S.*

A rough outline of our proof is as follows: Let $\mathcal{U}$ be a subset of S that is open in the analytic topology and does not intersect D. Then the zero locus $\mathcal{Z}$ of ν on $\mathcal{U}$ is complex analytic since the restriction of ν to $\mathcal{U}$ is a holomorphic section of associated bundle of intermediate Jacobians. Thus, to prove that the zero locus of ν is algebraic, it is sufficient to show that

($*$) for each point $p \in D$ there exists an analytic open neighborhood $\mathcal{U}_p \subset S$ of p on which $\mathcal{Z}$ has only finitely many components.

We verify ($*$) using the orbit theorems of the second author and results of P. Deligne.

The canonical real grading $Y(s)$ (described below) of the mixed Hodge structure $\mathcal{V}_s$ at a point $s \in S - D$ will play an important role in our proof. The central idea is that ν is 0 at s if and only if $Y(s)$ is integral. It is therefore crucial to understand the asymptotics of $Y(s)$ as s tends to a point $s_0 \in D$. In Theorem 3.9, we use Pearlstein's SL_2-orbit theorem [Pea06] to show that $Y^{\ddagger} := \lim_{s \to s_0} Y(s)$ exists when the limit is taken along any angular sector for $s_0 \in D$. Now, it is clear that ν can only vanish in a neighborhood of s_0 if $Y^{\ddagger}$ is integral. Knowing that the limit exists allows us to concentrate on the case where $Y^{\ddagger}$ is integral. This case can then be handled by a rather explicit computation of the zero locus in the neighborhood of s_0.

2. The zero locus at a smooth point

As a preliminary step in our analysis of the zero locus of ν at infinity, we derive the local defining equations of $\mathcal{Z}$ at an interior point of S. To this end, we begin with a review of mixed Hodge structures and their gradings, following [CKS86].

Gradings. Let V be a finite dimensional vector space over a field K of characteristic zero. A grading of an increasing filtration W of V is a semisimple endomorphism Y of V with integral eigenvalues such that

$$W_k = \bigoplus_{j \le k} E_j(Y),$$

where $E_j(Y)$ is the j-eigenspace of Y. Conversely, given a direct sum decomposition

$$V = \bigoplus_{j \in \mathbb{Z}} V_j$$

one has an associated increasing filtration $W_k = \bigoplus_{j \le k} V_j$ that is graded by the semisimple endomorphism that acts as multiplication by k on V_k. If V and W are defined over a subring $R \subset K$, then a grading Y is said to be *defined over* R if $Y \in \mathrm{End}(V_R)$.

Given an increasing filtration W of V, the subgroup $\mathrm{GL}(V)^W$ consisting of all elements $g \in \mathrm{GL}(V)$ that preserve W acts transitively upon the set $\mathcal{Y}(W)$ of all gradings of W by the rule

$$g.Y = \mathrm{Ad}(g)Y. \tag{2.1}$$

The set $\mathcal{Y}(W)$ is also an affine space upon which the nilpotent Lie algebra

$$\mathrm{Lie}_{-1}(W) = \{\alpha \in \mathfrak{gl}(V) \mid \alpha(W_k) \subseteq W_{k-1}\}$$

acts simply transitively upon via the rule $(\alpha, Y) \mapsto Y + \alpha$. In the computations below, we freely mix these two points of view, as illustrated in (2.13).

By a theorem of Deligne, [Del71, Lemme 1.2.8], a mixed Hodge structure (F, W) induces a unique functorial bigrading

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q} \tag{2.2}$$

of the underlying complex vector space $V_{\mathbb{C}}$ such that

(1) $F^p = \bigoplus_{a \geq p} I^{a,b}$;

(2) $W_k = \bigoplus_{a+b \leq k} I^{a,b}$;

(3) $\bar{I}^{p,q} \equiv I^{q,p} \mod \bigoplus_{r<q,s<p} I^{r,s}$.

As such, a mixed Hodge structure (F, W) induces a grading of W via the semisimple endomorphism

$$Y_{(F,W)} : V_{\mathbb{C}} \to V_{\mathbb{C}} \tag{2.3}$$

that acts as multiplication by $(p+q)$ on $I^{p,q}$. We will call this grading *Deligne's grading*.

Remark 2.4. A mixed Hodge structure (F, W) on V induces a mixed Hodge structure on $\mathfrak{gl}(V)$ with associated bigrading

$$\mathfrak{gl}(V_{\mathbb{C}})^{r,s} = \{\alpha \in \mathfrak{gl}(V_{\mathbb{C}}) \mid \alpha(I^{p,q}) \subseteq I^{r+p,s+q}\}. \tag{2.5}$$

Clearly, each summand $\mathfrak{gl}(V_{\mathbb{C}})^{r,s}$ of $\mathfrak{gl}(V_{\mathbb{C}})$ is closed under the action of ad Y, where $Y = Y_{(F,W)}$.

A mixed Hodge structure (F, W) is *split over* $\mathbb{R}$ if

$$\overline{Y}_{(F,W)} = Y_{(F,W)}.$$

In this case, $Y_{(F,W)}$ may be characterized as the unique real grading of W that preserves F; furthermore [CKS86],

$$I^{p,q} = F^p \cap \overline{F}^q \cap W_{p+q}. \tag{2.6}$$

By [CKS86, Prop. (2.20)], given a mixed Hodge structure (F, W) there exists a unique real element

$$\delta \in \Lambda^{-1,-1} = \bigoplus_{r,s<0} \mathfrak{gl}(V)^{r,s} \tag{2.7}$$

such that $(\hat{F}, W) := (e^{-i\delta}.F, W)$ is split over $\mathbb{R}$. Moreover, δ commutes with every (r,r)-morphism of (F, W).

Normal functions. Returning now to the normal function setting, let S be a smooth, projective complex variety of dimension n. Then, an admissible normal function ν on S corresponds to an extension

$$0 \to \mathcal{H} \longrightarrow \mathcal{V} \longrightarrow \mathbb{Z}(0) \to 0 \tag{2.8}$$

in the category of admissible variations of mixed Hodge structure defined on a Zariski open subset $S - D$ of S, where $\mathcal{H}$ is a variation of pure Hodge structure of weight -1.

Let $p \in S - D$, and let $(s_1, \dots, s_n)$ be local holomorphic coordinates on a polydisk $\Delta^n \subseteq S - D$ that vanish at p. Then, since Δ^n is simply connected, we can parallel translate the data of $\mathcal{V}$ back to the reference fiber $V = \mathcal{V}_p$. The Hodge filtration $\mathcal{F}$ of $\mathcal{V}$ then corresponds to a holomorphic, horizontal decreasing filtration $F(s)$ of $V_{\mathbb{C}}$. The weight filtration W of $\mathcal{V}$ corresponds to a constant filtration W of $V_{\mathbb{Z}}$ with weight-graded quotients

$$\mathrm{Gr}_0^W(V_{\mathbb{Z}}) = \mathbb{Z}(0), \qquad \mathrm{Gr}_{-1}^W(V_{\mathbb{Z}}) = H_{\mathbb{Z}}$$

and $\mathrm{Gr}_k^W = 0$ for $k \neq 0, -1$. Similarly, the graded polarizations of W correspond to constant polarizations of Gr^W.

On account of the short length of W, $(F(s), W)$ is split over $\mathbb{R}$ and hence Deligne's grading

$$Y(s) = Y_{(F(s),W)} \tag{2.9}$$

is the unique real grading of W that preserves $F(s)$. If $Y_{\mathbb{Z}}$ is any integral grading of W, then the image of $1 \in Gr_0^W(V_{\mathbb{Z}}) = \mathbb{Z}(0)$ under the induced map

$$Y(s) - Y_{\mathbb{Z}} : \mathbb{Z}(0) \to H_{\mathbb{R}}/H_{\mathbb{Z}}$$

gives the point in the Griffiths intermediate Jacobian corresponding to the fiber of the extension (2.8) at s via the isomorphism

$$H_{\mathbb{R}}/H_{\mathbb{Z}} \cong \frac{H_{\mathbb{C}}}{F^0(s) + H_{\mathbb{Z}}}.$$

Accordingly, p belongs to $\mathcal{Z}$ if and only if $Y(p)$ is an integral grading of W.

Suppose now that $p \in \mathcal{Z}$. Then, since $Y(s)$ is real analytic in s and the set of integral gradings of W is a discrete subset of the affine space of $\mathbb{R}$-gradings of W, there exists a neighborhood of p on which $\mathcal{Z}$ is given by the equation

$$Y(s) = Y(p).$$

The filtration $F(s)$ takes its values in a classifying space $\mathcal{M}$ of graded-polarized mixed Hodge structure [Pea00], [Usu84]. Let $G_{\mathbb{C}}$ denote the Lie group consisting of all automorphisms of $V_{\mathbb{C}}$ that preserve W and act by complex isometries on

Gr^W. Then, for each point $F \in \mathcal{M}$ there exists a neighborhood $U_{\mathbb{C}}$ of zero in the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ such that the map

$$u \mapsto e^u . F \tag{2.10}$$

is a holomorphic submersion from $U_{\mathbb{C}}$ onto a neighborhood of F in $\mathcal{M}$. If $g \in G_{\mathbb{C}}$ and F is a filtration of V, we use the notation $g . F$ to denote the filtration of V defined by $(g . F)^p = g(F^p)$.

As in (2.5), each point $F \in \mathcal{M}$ induces a mixed Hodge structure $(F^\bullet\mathfrak{g}_{\mathbb{C}}, W_\bullet\mathfrak{g}_{\mathbb{C}})$ on $\mathfrak{g}_{\mathbb{C}}$ with associated bigrading

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{r+s \le 0} \mathfrak{g}^{r,s} \tag{2.11}$$

defined by $\mathfrak{g}^{r,s} = \mathfrak{gl}(V)^{r,s} \cap \mathfrak{g}_{\mathbb{C}}$. Accordingly, the nilpotent subalgebra

$$\mathfrak{q}_F = \bigoplus_{r<0, r+s \le 0} \mathfrak{g}^{r,s}$$

is a vector space complement to the isotopy algebra $\mathfrak{g}_{\mathbb{C}}^F$ of F in $\mathfrak{g}_{\mathbb{C}}$. Consequently, the map (2.10) restricts to a biholomorphism from a neighborhood of zero in $\mathfrak{q}_F$ onto a neighborhood of F in $\mathcal{M}$. Furthermore, by Remark 2.4, $\mathfrak{g}^{r,s}$ is stable under the action of $\operatorname{ad} Y_{(F,W)}$. Hence, $\mathfrak{q}_F$ is also stable under this action.

Letting $F = F(p)$, it then follows by the remarks of the previous paragraphs that near p we can write $F(s) = e^{\Gamma(s)} . F$ relative to a unique holomorphic function $\Gamma(s)$ with values in $\mathfrak{q}_F$ that vanishes at p. Let $Y = Y(p)$, and let $\Gamma(s) = \Gamma_0(s) + \Gamma_{-1}(s)$ denote the decomposition of $\Gamma(s)$ into $\mathfrak{q}_F$-valued functions according to the eigenvalues of $\operatorname{ad} Y$.

LEMMA 2.12. *Let $\mathfrak{n} \subset \mathfrak{gl}(V_{\mathbb{C}})$ be a nilpotent Lie algebra, and let $I \subset \mathfrak{n}$ be an ideal such that $[I, I] = 0$. Let $\Psi(t) = \sum_{n \ge 0} t^n/(n+1)!$ be the Taylor series of $(e^t - 1)/t$. Let $u \in \mathfrak{n}$ and $v \in I$. Then $e^{u+v}e^{-u} = e^{\Psi(\operatorname{ad} u)v}$.*

Proof. The Campbell-Baker-Hausdorff formula implies that

$$e^{x+y}e^{-x} = e^{\Phi(y, (\operatorname{ad} x)y, (\operatorname{ad} x)^2 y, \dots)}$$

for some universal Lie power series $\Phi(t_0, t_1, \dots)$ with constant term 0. (See [Bou72, Ch. 2, §4].) Therefore

$$\Phi(y, (\operatorname{ad} x)y, (\operatorname{ad} x)^2 y, \dots) = \sum_{j>0} \Phi_j(y, (\operatorname{ad} x)y, (\operatorname{ad} x)^2 y, \dots),$$

where $\Phi_j(y, (\operatorname{ad} x)y, (\operatorname{ad} x)^2 y, \dots)$ is homogeneous of degree j in y. Set $x = u$ and $y = v$. Then $\Phi(y, (\operatorname{ad} x)y, (\operatorname{ad} x)^2 y, \dots)$ converges by the nilpotence of $\mathfrak{n}$. Also, since I is an ideal and $[I, I] = 0$, we have

$$\Phi_j(y, (\operatorname{ad} x)y, (\operatorname{ad} x)^2 y, \dots) = 0 \quad \text{for } j > 1.$$

As such,

$$e^{u+v}e^{-u} = e^{\Phi(v,(\operatorname{ad}u)v,(\operatorname{ad}u)^2v,\dots)} = e^{\Phi_1(v,(\operatorname{ad}u)v,(\operatorname{ad}u)^2v,\dots)}$$

It then follows from [Bou72, Ch. 2, Prop. (5.5)] that

$$\Phi_1(v,(\operatorname{ad}u)v,(\operatorname{ad}u)^2v,\dots) = \Psi(\operatorname{ad}u)v. \qquad \square$$

Setting $\mathfrak{n} = \mathfrak{q}_F$, $I = \mathfrak{q}_F \cap \mathrm{Lie}_{-1} W$, $u = \Gamma_0(s)$ and $v = \Gamma_{-1}(s)$, it then follows from the previous lemma that

$$\begin{aligned} e^{\Gamma(s)}.Y &= e^{\Gamma_0(s)+\Gamma_{-1}(s)}e^{-\Gamma_0(s)}e^{\Gamma_0(s)}.Y \\ &= e^{\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s)}.Y = Y + \Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) \end{aligned} \tag{2.13}$$

is a holomorphic grading of the weight filtration (over $\mathbb{C}$); this grading preserves $F(s)$. Thus there is a real analytic section $\zeta(s)$ of $\mathfrak{g}_{\mathbb{C}}^{F(s)} \cap \mathrm{Lie}_{-1}(W)$ such that

$$Y(s) = Y + \Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) + \zeta(s),$$

and hence the equation $Y(s) = Y(p)$ is equivalent to

$$\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) + \zeta(s) = 0. \tag{2.14}$$

Accordingly, near p

$$\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) \in \mathfrak{g}_{\mathbb{C}}^{F(s)} \cap \mathrm{Lie}_{-1}(W) \tag{2.15}$$

on $\mathscr{Z}$. Conversely, whenever (2.15) holds, $Y = Y(p)$ is a real grading of W that preserves $F(s)$. Because these two properties specify $Y(s)$ uniquely, it then follows that whenever (2.15) holds, $Y(s) = Y(p)$. Thus $\mathscr{Z}$ is given by (2.15) on a neighborhood of p.

Applying $e^{-\operatorname{ad}\Gamma(s)}$ to both sides of (2.15), it then follows that this relation for $\mathscr{Z}$ near p is

$$e^{-\operatorname{ad}\Gamma(s)}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) \in \mathfrak{g}_{\mathbb{C}}^{F} \cap \mathrm{Lie}_{-1}(W). \tag{2.16}$$

To simplify this relation, note that the left side (2.16) is a $\mathfrak{q}_F$-valued function since $\Gamma(s)$, $\Gamma_0(s)$ and $\Gamma_{-1}(s)$ take values in $\mathfrak{q}_F$. Consequently,

$$e^{-\operatorname{ad}\Gamma(s)}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) = 0 \tag{2.17}$$

is an equation for $\mathscr{Z}$ since $\mathfrak{g}_{\mathbb{C}}^{F} \cap \mathfrak{q}_F = 0$.

THEOREM 2.18. *Near p, the zero locus of ν is given by the equation $\Gamma_{-1}(s)=0$.*

Proof. Applying $e^{\operatorname{ad}\Gamma(s)}$ to (2.17) implies that the zero locus is given by the equation

$$\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) = 0. \tag{2.19}$$

By 2.12,

$$\Psi(u)v = v + \sum_{j>0} \frac{(\operatorname{ad} u)^j v}{(j+1)!}$$

and hence

$$\Psi(\operatorname{ad}\Gamma_0)\Gamma_{-1} = \Gamma_{-1} + \sum_{j>0} \frac{(\operatorname{ad}\Gamma_0)^j \Gamma_{-1}}{(j+1)!}. \tag{2.20}$$

Consequently, if

$$\Gamma_0 = \sum_{k>0} \Gamma^{-k,k} \quad \text{and} \quad \Gamma_{-1} = \sum_{\ell>0} \Gamma^{-\ell,\ell-1}$$

denote the decomposition of Γ_0 and Γ_{-1} into Hodge components with respect to the bigrading (2.11), then

$$\Psi(\operatorname{ad}\Gamma_0)\Gamma_{-1} \equiv \Gamma^{-1,0} \mod \bigoplus_{r\geq 2} \mathfrak{g}^{-r,r-1}.$$

As such, (2.19) then implies that $\Gamma^{-1,0} = 0$. Proceeding by induction, assume that $\Gamma^{-\ell,\ell-1} = 0$ for $\ell < n$. Then

$$\Psi(\operatorname{ad}\Gamma_0)\Gamma_{-1} \equiv \Gamma^{-n,n-1} \mod \bigoplus_{r\geq n+1} \mathfrak{g}^{-r,r-1},$$

and hence (2.19) implies $\Gamma^{-n,n-1} = 0$. Thus, $\Gamma_{-1} = 0$ is the local defining equation for $\mathcal{Z}$. □

Remark 2.21. Theorem 2.18 implies the following estimate for the codimension of $\mathcal{Z}$ at p: Let $\alpha = (d\Gamma_0)(p)$ and

$$U = \{\, \beta \in \operatorname{Hom}(T_p(S), \mathfrak{g}^{-1,0}) \mid \alpha \wedge \beta + \beta \wedge \alpha = 0 \,\}.$$

Then, $\operatorname{codim}_p(\mathcal{Z}) \leq \max\{\operatorname{rank}(\beta) \mid \beta \in U\,\} \leq \dim \mathfrak{g}^{-1,0} = \dim I^{-1,0}$.

3. **Limiting grading**

In this section, we prove that when S is a curve, the grading (2.9) has a well-defined limit $Y^{\ddagger}$ as s approaches a puncture $p \in S$. Simple examples show that in higher dimensions, the limiting value of (2.9) depends not only on the point in the boundary divisor but also the direction of approach.

Let $\Delta \subset S$ be a disk containing the puncture p. By passing to a finite cover if necessary, we can assume that the local monodromy of the restriction of $\mathcal{V}$ to the punctured disk $\Delta^* = \Delta - \{p\}$ is unipotent. Let s be a local coordinate on Δ that vanishes at p, let A be an angular sector of Δ^*, and let s_o be a point in A. Then, over A, we can parallel translate the Hodge filtration of $\mathcal{V}$ back to a single valued filtration $F(s)$ on $V = \mathcal{V}_{s_o}$. Analytic continuation of $F(s)$ to all of Δ^* then gives the period map $\varphi : \Delta^* \to \Gamma\backslash\mathcal{M}$ of $\mathcal{V}$. By local liftability, there exists a holomorphic, horizontal lifting of φ to a map $\tilde{F}$ from the upper half-plane U into $\mathcal{M}$ making the diagram below commute.

$$\begin{tikzcd} U \arrow[r, "\tilde{F}"] \arrow[d, "s=e^{2\pi i z}"'] & \mathcal{M} \arrow[d] \\ \Delta^* \arrow[r, "\varphi"] & \Gamma\backslash\mathcal{M}. \end{tikzcd}$$

Furthermore, upon picking a branch of $\log(s)$ on A and letting

$$z = x + iy = \frac{1}{2\pi i}\log(s),$$

there is unique lifting $\tilde{F}(z)$ such that $\tilde{F}(z) = F(s)$ for $s \in A$. By unipotent monodromy, we have $\tilde{F}(z+1) = e^N . \tilde{F}(z)$ and hence $\tilde{\varphi}(z) = e^{-zN} . \tilde{F}(z)$ drops to a map $\tilde{\varphi}$ from Δ^* to the "compact dual" $\check{\mathcal{M}} \cong G_{\mathbb{C}}/G_{\mathbb{C}}^{F_o}$ of $\mathcal{M}$, where $F_o \in \mathcal{M}$ is an arbitrary base point (cf. [Pea00]). The admissibility of $\mathcal{V}$ then asserts that

(a) $F_\infty = \lim_{s\to 0}\tilde{\varphi}(s)$ exists;

(b) the relative weight filtration M of W and N exists.

From these properties, together with Schmid's orbit theorems, Deligne then deduces [SZ85] that the pair (F_∞, M) is a mixed Hodge structure relative to which N is a $(-1,-1)$-morphism.

Remark 3.1. The definition of an admissible variation of mixed Hodge structure over a curve was formulated by Steenbrink and Zucker in [SZ85]. In place of the existence of the limiting value of the period map $\tilde{\varphi}$, they require the extendability of the Hodge bundles with respect to Deligne's canonical extension of $\mathcal{V}$. The definition of admissibility in several variables via a curve test was given by Kashiwara in [Kas86].

In analogy with Section 2, the limit mixed Hodge structure (F_∞, M) induces a mixed Hodge structure on $\mathfrak{g}_{\mathbb{C}}$ with Deligne bigrading

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{r,s}\mathfrak{g}^{r,s}. \tag{3.2}$$

Likewise, the nilpotent subalgebra

$$\mathfrak{q}_\infty = \bigoplus_{r<0}\mathfrak{g}^{r,s}$$

is a vector space complement to the isotopy algebra $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$. Reasoning as in Section 2 (cf. [Pea00]), it then follows that near the puncture $s = 0$ we can write $\tilde{\varphi}(s) = e^{\Gamma(s)} . F_\infty$ relative to a unique holomorphic function $\Gamma(s)$ that takes values in $\mathfrak{q}_\infty$

and vanishes at $s = 0$. Untwisting the definition of $\tilde{\varphi}$, it then follows that

$$F(s) = e^{\log(s)N/(2\pi i)} e^{\Gamma(s)} . F_\infty \tag{3.3}$$

over the angular sector A.

To determine the asymptotic behavior of the grading

$$Y(s) = Y_{(F(s),W)}$$

on A, we shall use (3.3) together with the SL_2-orbit theorem of [Pea06] and a result of Deligne that constructs a grading Y of the weight filtration W that is well adapted to both N and the limiting mixed Hodge structure (F_∞, M).

More precisely, suppose that $\mathrm{Gr}^W_k = 0$ for $k \neq 0, -1$ and Y_M is a grading of M that preserves W and satisfies $[Y_M, N] = -2N$. Then, Deligne, in [Del93] and [KP03, Appendix], shows that there exists a unique, functorial grading

$$Y' = Y'(N, Y_M) \tag{3.4}$$

of W such that Y' commutes with both N and Y_M. Furthermore,

(a) if Y_M is defined over $\mathbb{R}$, then so is Y';

(b) if (F, M) is a mixed Hodge structure for which N is a $(-1,-1)$-morphism and induces sub-mixed Hodge structures on W, then the grading Y' produced from N and the grading of M by the $I^{p,q}$'s of (F, M) preserves F.

LEMMA 3.5. *Let (F, M) be the limiting mixed Hodge structure of an admissible variation $\mathcal{V} \to \Delta^*$ as above. Let $Y_M = Y_{(F,M)}$, and let Y' be the grading of W defined by application of Deligne's construction to the pair (N, Y_M). Then each summand $\mathfrak{g}^{r,s}$ of (3.2) — and therefore $\mathfrak{q}_\infty$ — is closed under the action of $\mathrm{ad}\, Y'$.*

Proof. Suppose that (F, M) is split over $\mathbb{R}$. Then, since Y' is defined over $\mathbb{R}$ by part (a) of Deligne's result and preserves F by part (b), it follows by (2.6) that Y' preserves the $I^{p,q}$'s of (F, M), and hence $\mathrm{ad}\, Y'$ preserves the summands of (3.2). The general case (cf. [Pea06]) follows using Deligne's splitting (2.7) and the functoriality of Deligne's construction. □

To show the existence of $\lim_{s\to 0} Y(s)$ we now recall the following SL_2-orbit theorem of the second author:

THEOREM 3.6 ([Pea06, Th. 4.2]). *Let $(\hat{F}, M) = (e^{-i\delta} . F_\infty, M)$ as in (2.7) of the limiting mixed Hodge structure of $\mathcal{V}$ and*

$$\Lambda^{-1,-1} = \bigoplus_{r,s<0} \mathfrak{g}^{r,s}_{(\hat{F},M)}.$$

Define $G_\mathbb{R} = G_\mathbb{C} \cap \mathrm{GL}(V_\mathbb{R})$, and let $\mathfrak{g}_\mathbb{R}$ denote the Lie algebra of $G_\mathbb{R}$. Then, there exists a distinguished, real analytic function $g : (a, \infty) \to G_\mathbb{R}$ and an element $\zeta \in \mathfrak{g}_\mathbb{R} \cap \ker(\mathrm{ad}\, N) \cap \Lambda^{-1,-1}$ such that

(a) $e^{iyN}.F_\infty = g(y)e^{iyN}.\hat{F}$;

(b) *$g(y)$ and $g^{-1}(y)$ have convergent series expansions about ∞ of the form*

$$g(y) = e^{\zeta}(1 + g_1 y^{-1} + g_2 y^{-2} + \cdots)$$
$$g^{-1}(y) = (1 + f_1 y^{-1} + f_2 y^{-2} + \cdots)e^{-\zeta}$$

with g_k, $f_k \in \ker(\operatorname{ad} N)^{k+1}$;

(c) *δ, ζ and the coefficients g_k are related by the formula*

$$e^{i\delta} = e^{\zeta}\Big(1 + \sum_{k>0} \frac{1}{k!}(-i)^k (\operatorname{ad}\, N_0)^k\, g_k\Big).$$

Remark 3.7. A several variable version of the SL_2-orbit theorem has been recently obtained by Kato, Nakayama and Usui in [KNU08].

Combining the SL_2-orbit theorem with (3.3), we obtain the following asymptotic formula for $F(s)$ over the angular sector A:

$$\begin{aligned} F(s) = e^{zN}e^{\Gamma(s)}.F_\infty &= e^{xN}e^{\Gamma_1(s)}e^{iyN}.F_\infty \\ &= e^{xN}e^{\Gamma_1(s)}g(y)e^{iyN}.\hat{F} = e^{xN}g(y)e^{\Gamma_2(s)}e^{iyN}.\hat{F}, \end{aligned}$$

where $\Gamma_1(s) = \operatorname{Ad}(e^{iyN})\Gamma(s)$ and $\Gamma_2(s) = \operatorname{Ad}(g^{-1}(y))\Gamma_1(s)$.

Let $\hat{Y}_M = Y_{(\hat{F},M)}$, and let $\hat{Y}$ be the grading of W produced by the application of Deligne's construction to the pair $(N, \hat{Y}_M)$. Then by [Pea06], $H = \hat{Y}_M - \hat{Y}$ belongs to $\mathfrak{g}_{\mathbb{R}}$ and satisfies $[H, N] = -2N$. Furthermore, since $\hat{Y}_M$ and $\hat{Y}$ preserve $\hat{F}$, so does H. Therefore $e^{iyN}.\hat{F} = y^{-H/2}.F_o$, where $F_o = e^{iN}.\hat{F}$. By the SL_2-orbit theorem [CKS86], F_o belongs to $\mathcal{M}$. Consequently,

$$F(s) = e^{xN}g(y)e^{\Gamma_2(s)}y^{-H/2}.F_o = e^{xN}g(y)y^{-H/2}e^{\Gamma_3(s)}.F_o,$$

where

$$\Gamma_3(s) = \operatorname{Ad}(y^{H/2})\Gamma_2(s) = \operatorname{Ad}(y^{H/2}g(y)e^{iyN})\Gamma(s).$$

To continue, observe that, since $y = -(1/2\pi)\log|s|$ and H has only finitely many eigenvalues (all of which are integral), the action of $\operatorname{Ad}(y^{H/2})$ on $\mathfrak{g}_{\mathbb{C}}$ is bounded by an integral power of $y^{1/2}$. Similarly, since $g(y)$ is bounded as $s \to 0$, so is the action of $\operatorname{Ad}(g(y))$. Likewise, since N is nilpotent, the action of $\operatorname{Ad}(e^{iyN})$ on $\mathfrak{g}_{\mathbb{C}}$ is bounded by a power of y. Therefore, since $\Gamma(s)$ is a holomorphic function of s that vanishes at $s = 0$, $\Gamma_3(s)$ is a real analytic function on A satisfying the growth condition $\Gamma_3(s) = O((\log|s|)^b s)$ for some half integral power b. In particular, near $s = 0$,

$$Y_{(e^{\Gamma_3(s)}.F_o,W)} = Y_{(F_o,W)} + \gamma_4(s)$$

for some real analytic function $\gamma_4(s)$ that is again of order $(\log|s|)^b s$. By [Del93] and [KP03, Appendix], $Y_{(F_o,W)} = \hat{Y}$. Therefore

$$\begin{aligned} Y(s) &= e^{xN} g(y) y^{-H/2} . Y_{(e^{\Gamma_3(s)}.F_o,W)} = e^{xN} g(y) y^{-H/2} . (Y_{(F_o,W)} + \gamma_4(s)) \\ &= e^{xN} g(y) . (\hat{Y} + \gamma_5(s)), \end{aligned}$$

where $\gamma_5(s) = \mathrm{Ad}(y^{-H/2})\gamma_4(s)$ is again of order $\log|s|^{b'} s$ for some half-integral power b'.

Define

$$\begin{aligned} \tilde{g}(s) &= e^{xN} g(y) e^{-xN} \\ &= e^{\zeta}\Big(1 + \sum_{k>0} (\mathrm{Ad}(e^{xN} g_k)) y^{-k}\Big). \end{aligned}$$

Then, since $x = (1/2\pi)\mathrm{Arg}(s)$ is bounded on the angular sector A,

$$\lim_{s\to 0} \tilde{g}(s) = e^{\zeta}.$$

Consequently, because N commutes with $\hat{Y}$,

$$Y(s) = \tilde{g}(s) . (\hat{Y} + Ad(e^{xN})\gamma_5(s)). \tag{3.8}$$

Therefore, since $\gamma_5(s)$ is order $(\log(s))^{b'} s$, we can take the limit of (3.8) along any angular sector to obtain this:

THEOREM 3.9. *We have*

$$Y^{\ddagger} := \lim_{s\to 0} Y(s) = e^{\zeta} . \hat{Y} \tag{3.10}$$

when the limit is taken along any angular sector.

Remark 3.11. Since the right-hand side of (3.10) depends only on the triple (F_∞, W, N), $Y^{\ddagger}$ is independent of choice of angular sector A. Likewise, a change of local coordinate s changes F_∞ to $e^{\lambda N} . F_\infty$. Therefore, due to the functorial nature of Deligne's construction of the grading Y' and the fact that $[Y', N] = 0$, the right side of (3.10) is independent of the choice of coordinate s. Likewise, since the right side of (3.10) commutes with N, it is well defined independent of the choice of reference fiber.

4. **Zero locus at infinity**

To verify Conjecture 1.1 in the case where S is a curve, we now note that the finiteness condition $(*)$ in Section 1 is preserved under passage to finite covers. Therefore, we may assume as in Section 3 that the associated variation of mixed Hodge structure $\mathcal{V}$ has unipotent monodromy about each point $p \in D$. The requirement that the zero locus of ν has only finitely many components on a neighborhood of $p \in D$ is then equivalent to the existence of a disk $\Delta \subset S$ such that $\Delta \cap D = \{p\}$ on which the zero locus of ν is either

(a) the empty set;

(b) all of Δ, in which case $\mathcal{V}$ is the trivial extension of $\mathbb{Z}(0)$ by $\mathcal{H}$.

Applying Deligne's construction (3.4) to the limiting mixed Hodge structure (F_∞, M), we get a grading Y_∞ of W that preserves F_∞. Therefore

$$Y_\infty(s) = e^{\log(s)N/(2\pi i)} e^{\Gamma(s)}.Y_\infty$$

is a (complex) grading of W that preserves the Hodge filtration of $F(s)$ near $s = 0$ over the angular sector A. By Lemma 3.5, $\mathfrak{q}_\infty$ is closed under the action of $\operatorname{ad} Y_\infty$. Therefore $\Gamma(s)$ decomposes into a sum $\Gamma(s) = \Gamma_0(s) + \Gamma_{-1}(s)$ of $\mathfrak{q}_\infty$-valued functions according to the eigenvalues of $\operatorname{ad} Y_\infty$. Consequently,

$$\begin{aligned} Y_\infty(s) &= e^{\log(s)N/(2\pi i)} e^{\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s)}.Y_\infty \\ &= e^{\log(s)N/(2\pi i)}.(Y_\infty + \Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s)) \\ &= Y_\infty + e^{\log(s)\operatorname{ad}N/(2\pi i)}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s). \end{aligned} \tag{4.1}$$

As in Section 2, we then have

$$Y(s) = Y_\infty(s) + \zeta(s) \tag{4.2}$$

for some section $\zeta(s)$ of $\mathfrak{g}_{\mathbb{C}}^{F(s)} \cap \mathrm{Lie}_{-1}(W)$. In principle, $\zeta(s)$ may have singularities at $s = 0$. To see that this is not the case, observe that since $\Gamma(s)$ is holomorphic and vanishes at $s = 0$ and N is nilpotent,

$$\lim_{s\to 0} e^{\log(s)\operatorname{ad}(N)/(2\pi i)}\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) = 0. \tag{4.3}$$

Therefore, since the limit $Y^\ddagger = \lim_{s\to 0} Y(s)$ exists by Theorem 3.9, Equations (4.1), (4.2) and (4.3) imply that $\zeta(s)$ also has a continuous extension to 0 in the angular sector A.

By continuity, if $Y^\ddagger$ is not an integral grading of W, then there is a neighborhood of zero in angular sector A on which $Y(s)$ is not integral, and hence ν has no zeros on this neighborhood. Thus, it remains to consider the case where $Y^\ddagger$ is integral. By the functoriality of Deligne's construction (cf. [Pea06]), $\hat{Y} = e^{-i\delta}.Y_\infty$ and hence by (3.10)

$$Y^\ddagger = e^{\zeta}.\hat{Y} = e^{\zeta}e^{-i\delta}.Y_\infty.$$

By the Campbell-Baker-Hausdorff formula, $e^{\zeta}e^{-i\delta} = e^{\xi}$ for some (unique)

$$\xi \in \ker(\operatorname{ad} N) \cap \Lambda^{-1,-1}_{(\hat{F},M)}$$

since both ζ and δ belong to the Lie subalgebra $\ker(\operatorname{ad} N) \cap \Lambda^{-1,-1}_{(\hat{F},M)}$.

To continue, note that

$$\mathfrak{g}^{r,s}_{(F_\infty,M)} = e^{i\,\mathrm{ad}\,\delta}(\mathfrak{g}^{r,s}_{(\hat{F},M)})$$

and hence

$$\Lambda^{-1,-1}_{(F_\infty,M)} = e^{i\,\mathrm{ad}\,\delta}\Lambda^{-1,-1}_{(\hat{F},M)} = \Lambda^{-1,-1}_{(\hat{F},M)}$$

since $\Lambda^{-1,-1}_{(\hat{F},M)}$ is closed under $\mathrm{ad}\,\delta$. As such,

$$\xi \in \ker(\mathrm{ad}\,N) \cap \Lambda^{-1,-1}_{(\hat{F},M)} = \ker(\mathrm{ad}\,N) \cap \Lambda^{-1,-1}_{(F_\infty,M)}.$$

Consequently, upon decomposing $\xi = \xi_0 + \xi_{-1}$ relative to $\mathrm{ad}\,Y_\infty$, we have

$$Y^\ddagger = e^{\xi}.Y_\infty = Y_\infty + \Psi(\mathrm{ad}\,\xi_0)\xi_{-1}.$$

Furthermore, since Y_∞ commutes with N and preserves each $\mathfrak{g}^{r,s}_{(F_\infty,M)}$,

$$\xi_0, \xi_{-1} \in \ker(\mathrm{ad}\,N) \cap \Lambda^{-1,-1}_{(F_\infty,M)} \subseteq \mathfrak{q}_\infty.$$

Returning now to Equations (4.1) and (4.2), it then follows that

$$Y(s) = Y^\ddagger - \Psi(\mathrm{ad}\,\xi_0)\xi_{-1} + e^{\log(s)\,\mathrm{ad}\,N/(2\pi i)}\Psi(\mathrm{ad}\,\Gamma_0(s))\Gamma_{-1}(s) + \zeta(s),$$

where $\zeta(s)$ is a real analytic section of $\mathfrak{g}^{F(s)}_{\mathbb{C}} \cap \mathrm{Lie}_{-1}(W)$. In particular, since $\lim_{s\to 0} Y(s) = Y^\ddagger$ is integral, it then follows from the continuity of $Y(s)$ that near $s = 0$, the zeros of ν occur where

$$-\Psi(\mathrm{ad}\,\xi_0)\xi_{-1} + e^{\log(s)\,\mathrm{ad}\,N/(2\pi i)}\Psi(\mathrm{ad}\,\Gamma_0(s))\Gamma_{-1}(s) + \zeta(s) = 0.$$

Equivalently,

$$\begin{aligned}(4.4)\quad &\mathrm{Ad}(e^{\log(s)N/(2\pi i)}e^{\Gamma(s)})^{-1}\\ &\qquad\times\big(\Psi(\mathrm{ad}\,\xi_0)\xi_{-1} - e^{\log(s)\,\mathrm{ad}\,N/(2\pi i)}\Psi(\mathrm{ad}\,\Gamma_0(s))\Gamma_{-1}(s)\big)\\ &\qquad\qquad = \mathrm{Ad}(e^{\log(s)N/(2\pi i)}e^{\Gamma(s)})^{-1}\zeta(s).\end{aligned}$$

Because N, $\Gamma(s)$, $\Gamma_0(s)$, $\Gamma_{-1}(s)$, ξ_0 and ξ_{-1} take values in the subalgebra $\mathfrak{q}_\infty$, so does the left side of (4.4). Likewise, since $\zeta(s)$ takes values in $\mathfrak{g}^{F(s)}_{\mathbb{C}} \cap \mathrm{Lie}_{-1}(W)$ and $F(s) = e^{\log(s)N/(2\pi i)}e^{\Gamma(s)}F_\infty$, the right side of (4.4) takes values in $\mathfrak{g}^{F_\infty}_{\mathbb{C}}$. Therefore, since $\mathfrak{q}_\infty \cap \mathfrak{g}^{F_\infty}_{\mathbb{C}} = 0$, it then follows that the zeros of ν occur exactly where

$$e^{\log(s)\,\mathrm{ad}\,N/(2\pi i)}\Psi(\mathrm{ad}\,\Gamma_0(s))\Gamma_{-1}(s) = \Psi(\mathrm{ad}\,\xi_0)\xi_{-1}.$$

Since $\Psi(\mathrm{ad}\,\xi_0)\xi_{-1} \in \ker(\mathrm{ad}\,N)$, this last equation can be further reduced to just $\Psi(\mathrm{ad}\,\Gamma_0(s))\Gamma_{-1}(s) = \Psi(\mathrm{ad}\,\xi_0)\xi_{-1}$. Since $\Gamma(s)$ is a holomorphic function which vanishes at zero, so is $\Psi(\mathrm{ad}\,\Gamma_0(s))\Gamma_{-1}(s)$. Hence $\nu = 0$ has solutions near $s = 0$ only if $\Psi(\mathrm{ad}\,\xi_0)\xi_{-1} = 0$ (i.e. $Y^\ddagger = Y_\infty$). In this case, the local equation for $\mathscr{Z}$ is just $\Psi(\operatorname{ad}\Gamma_0(s))\Gamma_{-1}(s) = 0$. Again, because $\Gamma(s)$ is holomorphic at $s = 0$, the solutions to the previous equation are either isolated or all of A.

Thus, we have obtained the following.

THEOREM 4.5. *Let ν be an admissible normal function on a complex, projective curve S smooth outside of a finite set $D \subset S$. Then the zero locus $\mathscr{Z}$ of ν is an algebraic subset of $S - D$.*

Remark 4.6. The theorem was previously known in the following special case: Assume that for all $s \in D$

(1) the monodromy T of $\mathscr{H}$ about s satisfies $(T - \mathrm{id})^2 = 0$;

(2) the vanishing cycle group of $\mathscr{H}$ at s is a direct sum of $\mathbb{Q}(0)$ as a rational Hodge structure.

In this case the theorem follows from [Sai96, Cor. 2.9]. In the case where the normal function arises from a family of null-homologous cycles (the geometric case), the theorem also follows from H. Clemens's results in [Cle83]. There, (1) is listed as restriction (1.9) and (2) as restriction (1.11).

Acknowledgments

We thank P. Deligne, M. Green, P. Griffiths, J. Lewis and S. Usui for helpful conversations. We also thank the Institute for Advanced Study in Princeton, New Jersey for hosting us (in nearby offices) while most of this work was done. Finally, we thank the referee for many useful comments.

References

[Bou72] N. BOURBAKI, *Éléments de mathématique. Fasc.* XXXVII. *Groupes et algèbres de Lie. Chapitre* II: *Algèbres de Lie libres. Chapitre* III: *Groupes de Lie*, *Actualités Scientifiques et Industrielles* **1349**, Hermann, Paris, 1972. MR 58 #28083a Zbl 0244.22007

[CKS86] E. CATTANI, A. KAPLAN, and W. SCHMID, Degeneration of Hodge structures, *Ann. of Math.* **123** (1986), 457–535. MR 88a:32029 Zbl 0617.14005

[Cle83] H. CLEMENS, The Néron model for families of intermediate Jacobians acquiring "algebraic" singularities, *Inst. Hautes Études Sci. Publ. Math.* **58** (1983), 5–18 (1984). MR 86d:14042 Zbl 0529.14025

[Del71] P. DELIGNE, Théorie de Hodge, II, *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5–57. MR 58 #16653a Zbl 0219.14007

[Del93] ———, Letter to E. Cattani and A. Kaplan, 1993.

[KP03] A. KAPLAN and G. PEARLSTEIN, Singularities of variations of mixed Hodge structure, *Asian J. Math.* **7** (2003), 307–336. MR 2007b:14021 Zbl 1076.14018

[Kas86] M. KASHIWARA, A study of variation of mixed Hodge structure, *Publ. Res. Inst. Math. Sci.* **22** (1986), 991–1024. MR 89i:32050 Zbl 0621.14007

[KNU08] K. KATO, C. NAKAYAMA, and S. USUI, SL(2)-orbit theorem for degeneration of mixed Hodge structure, *J. Algebraic Geom.* **17** (2008), 401–479. MR 2395135 Zbl 1144.14005

[Pea06] G. PEARLSTEIN, SL_2-orbits and degenerations of mixed Hodge structure, *J. Differential Geom.* **74** (2006), 1–67. MR 2008f:32016 Zbl 1107.14010

[Pea00] G. J. PEARLSTEIN, Variations of mixed Hodge structure, Higgs fields, and quantum cohomology, *Manuscripta Math.* **102** (2000), 269–310. MR 2001m:32033 Zbl 0973.32008

[Sai96] M. SAITO, Admissible normal functions, *J. Algebraic Geom.* **5** (1996), 235–276. MR 97c:14044 Zbl 0918.14018

[SZ85] J. STEENBRINK and S. ZUCKER, Variation of mixed Hodge structure, I, *Invent. Math.* **80** (1985), 489–542. MR 87h:32050a Zbl 0626.14007

[Usu84] S. USUI, Variation of mixed Hodge structure arising from family of logarithmic deformations. II: Classifying space, *Duke Math. J.* **51** (1984), 851–875. MR 86h:14005 Zbl 0558.14005

(Received May 18, 2006)
(Revised January 22, 2007)

E-mail address: brosnan@math.ubc.ca
DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF BRITISH COLUMBIA, ROOM 121, 1984 MATHEMATICS ROAD, VANCOUVER, BC V6T 1Z2, CANADA

E-mail address: gpearl@math.msu.edu
DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824, UNITED STATES