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Abstract

In this paper we prove the existence of Kähler metrics of constant scalar curvature on the blow up at finitely many points of a compact manifold that already carries a constant scalar curvature Kähler metric. In the case where the manifold has nontrivial holomorphic vector fields with zeros, we give necessary conditions on the number and locations of the blow up points for the blow up to carry constant scalar curvature Kähler metrics.

1. **Introduction and statement of the results**

1.1. *Introduction.* Letting (M, J, g, ω) be a Kähler manifold of complex dimension $m \geq 2$, we recall that the metric g, the complex structure J and the Kähler form ω are related by

$$\omega(X, Y) = g(J\,X, Y)$$

for all $X, Y \in TM$. Assume that the scalar curvature of g is constant. Given n distinct points $p_1, \ldots, p_n \in M$, the question we would like to address is whether the blow up of M at the points $p_1, \ldots, p_n$ can be endowed with a constant scalar curvature Kähler metric. In the case where the answer to this question is positive, we would like to characterize the Kähler classes on the blown up manifold for which we are able to find such a metric. In [2], we have already given a positive answer to these questions in the case where the manifold M has no nontrivial holomorphic vector field with zeros (this condition is for example fulfilled when the group of automorphisms of M is discrete). Under this condition, we have obtained the following:

THEOREM 1.1 ([2]). *Assume that (M, J, g, ω) is a constant scalar curvature compact Kähler manifold and further assume that (M, J) does not have any nontrivial holomorphic vector field with zeros. Given finitely many points $p_1, \ldots, p_n$ $\in M$ and positive numbers $a_1, \ldots, a_n > 0$, there exists $\varepsilon_0 > 0$ such that, for all*

$\varepsilon \in (0, \varepsilon_0)$, *the blow up of* M *at* $p_1, \ldots, p_n$ *carries a constant scalar curvature Kähler metric* g_ε *which is associated to the Kähler form*

$$\omega_\varepsilon \in \pi^* [\omega] - \varepsilon^2 \, (a_1^{\frac{1}{m-1}} \, \mathrm{PD}[E_1] + \ldots + a_n^{\frac{1}{m-1}} \, \mathrm{PD}[E_n]),$$

where the $\mathrm{PD}[E_j]$ *are the Poincaré duals of the* $(2m-2)$*-homology classes of the exceptional divisors of the blow up at* p_j. *Moreover, as* ε *tends to* 0, *the sequence of metrics* $(g_\varepsilon)_\varepsilon$ *converges to* g (*in smooth topology*) *on compact subsets away from the exceptional divisors.*

If the scalar curvature of g is not zero then the scalar curvatures of g_ε and of g have the same signs. Also, if the scalar curvature of g is zero and the first Chern class of M is nonzero, then one can arrange so that the scalar curvature of g_ε is also equal to 0. This last result complements in any dimension previous constructions which have been obtained in complex dimension $m = 2$ and for zero scalar curvature metrics by Kim-LeBrun-Pontecorvo [11], LeBrun-Singer [16] and Rollin-Singer [23]. Indeed, using twistor theory, Kim-LeBrun-Pontecorvo and LeBrun-Singer have been able to construct such metrics by desingularizing some quotients of minimal ruled surfaces and, more recently, Rollin-Singer [23] have shown that, keeping the scalar curvature zero, one can desingularize compact orbifolds of zero scalar curvature with cyclic orbifold groups by solving, on the desingularization, the hermitian anti-selfdual equation (which implies the existence of a zero scalar curvature Kähler metric).

Theorem 1.1 is obtained using a connected sum of the Kähler form ω at each p_j with a zero scalar curvature Kähler metric g_0 which is defined on $\widetilde{\mathbb{C}}^m$, the blow up of $\mathbb{C}^m$ at the origin. This metric g_0 is associated to a Kähler form η_0 and has been discovered, inspired by previous work of Calabi [6], by Burns when $m = 2$ (and first described by Lebrun in [13]) and by Simanca [25] when $m \geq 3$. Since it is at the heart of our construction, we will briefly describe it in Section 2.

In the present paper, we focus our attention on the case where M has nontrivial holomorphic vector fields with zeros (this condition implies in particular that M has a nontrivial continuous automorphism group).

Given $n \geq 1$, we define

$$M^n_\triangle := \{(p_1, \ldots, p_n) \in M^n \quad : \quad p_a \neq p_b \quad \forall a \neq b\}.$$

A consequence of our main result states that the blow up of M at sufficiently many carefully chosen points can be endowed with a constant scalar curvature Kähler metric.

THEOREM 1.2. *Assume that* (M, J, g, ω) *is a constant scalar curvature compact Kähler manifold. There exists* $n_g \geq 1$ *and for all* $n \geq n_g$ *there exists a nonempty* open *subset* $V_n \subset M^n_\triangle$, *such that for all* $(p_1, \ldots, p_n) \in V_n$ *the blow up of*

M at $p_1, \dots, p_n$ carries a family of constant scalar curvature Kähler metrics $(g_\varepsilon)_\varepsilon$ converging to g (in smooth topology) on compact subsets away from the exceptional divisors, as the parameter ε tends to 0.

This result is a consequence of Theorem 1.3, Lemma 1.1 and Lemma 1.2 below. In particular Theorem 1.3 gives more details about the structure of the Kähler classes on the blow up in which the constant scalar curvature Kähler forms can be found.

In the case where (M, J) does not have any nontrivial holomorphic vector fields with zeros, Theorem 1.2 reduces to Theorem 1.1 with $n_g = 1$ and $V_n = M^n_\triangle$. We emphasize that, in the presence of a nontrivial holomorphic vector field with zeros, the number and position of the blow up points are not arbitrary anymore.

1.2. *The main result.* The determination of the least value of n_g for which the result holds, the location of the points which can be blown up as well as the Kähler classes obtained on the blow up, are rather delicate issues. To describe these we need to digress slightly. Now assume that (M, J, g, ω) is a constant scalar curvature compact Kähler manifold. Thanks to the Matsushima-Lichnerowicz Theorem, the space of holomorphic vector fields with zeros is also the complexification of the real vector space of holomorphic vector fields Ξ that can be written as

$$\Xi = X - i\, J\, X,$$

where X is a Killing vector field with zeros. We denote by $\mathfrak{h}$, the space of Killing vector fields with zeros and by

$$\xi : M \longmapsto \mathfrak{h}^*,$$

the *moment map* which is defined by requiring that, for all $X \in \mathfrak{h}$, the (real-valued) function $f := \langle \xi, X \rangle$ is a Hamiltonian for the vector field X. Namely it is the unique solution of

$$-df = \omega(X, -),$$

which is normalized by

$$\int_M f \,\mathrm{dvol}_g = 0.$$

Equivalently the function f is a solution of

$$-\overline{\partial} f = \tfrac{1}{2}\, \omega(\Xi, -).$$

where $\Xi = X - i\, J\, X$ is the holomorphic vector field associated to X. The map ξ is nothing but the moment map for the action of the hamiltonian isometry group.

Our main result is a consequence of the following sequence of results. The first one gives a sufficient condition on the number and location of the blow up points as well as on the Kähler classes on the blow up manifold for Theorem 1.2 to hold:

THEOREM 1.3. *Assume that (M, J, g, ω) is a compact Kähler manifold with constant scalar curvature and that $(p_1, \dots, p_n) \in M^n_\Delta$ are chosen so that*:

$$\text{(1)} \qquad \xi(p_1), \dots, \xi(p_n) \quad \textit{span} \quad \mathfrak{h}^*,$$

and

$$\text{(2)} \qquad \textit{there exist} \quad a_1, \dots, a_n > 0 \quad \textit{such that} \quad \sum_{j=1}^{n} a_j\, \xi(p_j) = 0 \in \mathfrak{h}^*.$$

Then, there exist $c > 0$, $\varepsilon_0 > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$, there exists on the blow up of M at $p_1, \dots, p_n$ a constant scalar curvature Kähler metric g_ε which is associated to the Kähler form

$$\omega_\varepsilon \in \pi^* [\omega] - \varepsilon^2 \, (a_{1,\varepsilon}^{\frac{1}{m-1}} \, \mathrm{PD}[E_1] + \cdots + a_{n,\varepsilon}^{\frac{1}{m-1}} \, \mathrm{PD}[E_n]),$$

where the $\mathrm{PD}[E_j]$ are the Poincaré duals of the $(2m-2)$-homology classes of the exceptional divisors of the blow up at p_j and where

$$|a_{j,\varepsilon} - a_j| \leq c \, \varepsilon^{\frac{2}{2m+1}}.$$

Finally, the sequence of metrics $(g_\varepsilon)_\varepsilon$ converges to g (in smooth topology) on compacts, away from the exceptional divisors.

Therefore, in the presence of nontrivial holomorphic vector fields with zeros, the number of points which can be blown up, their location, as well as the possible Kähler classes on the blown up manifold have to satisfy some constraints. We emphasize the *Riemannian nature of the result* which is reflected first in the hypothesis since conditions (1) and (2) do depend on the choice of the metric g and second in the conclusion since the metrics we construct on the blow up of M are, away from the exceptional divisors of the blown up points, small perturbations of the initial metric g.

Remark 1.1. A fundamental result concerning Kähler constant scalar curvature metrics is their uniqueness up to automorphisms in their Kähler class, as recently proved (even for the more general class of extremal metrics) by Chen-Tian [7]. This implies that, up to automorphisms, all the constant scalar curvature Kähler metrics produced in this paper are uniquely representative of their Kähler class. Nonetheless Theorem 1.3 can be applied to any of the metrices obtained from a fixed one by moving it with an automorphism of M. We shall exploit this fact in a forthcoming paper [1] to analyze the behaviour of the conditions (1) and (2) which appear in the statement of Theorem 1.3 under the action of the automorphism group.

From a Kählerian point of view we can interpret the role of the a_j's as giving a direction in the Kähler cone of the blown up manifold in which one can deform the Kähler class $\pi^*[\omega]$ (which of course lies on the boundary of this Kähler cone) to find a family of constant scalar curvature Kähler metrics on the blow up of M.

As will be explained in Section 6, the first condition (1) is easily seen to be generic (and open) in the sense that:

LEMMA 1.1. *With the above notation, assume that* $n \geq \dim \mathfrak{h}$ *then, the set of points* $(p_1, \ldots, p_n) \in M^n_\triangle$ *such that condition* (1) *is fulfilled is an open and dense subset of* $M^n_\triangle$.

When $d \geq \dim \mathfrak{h}$, it is well known that, for a choice of blow up points $(p_1, \ldots, p_d)$ in some open and dense subset of $M^d_\triangle$, the group of automorphisms of M blown up at $p_1, \ldots, p_d$ is trivial (observe that $\dim \mathfrak{h}$ is also equal to the dimension of the identity component of the automorphisms group of M). In view of all these results, one is tempted to conjecture that condition (1) is equivalent to the fact that the group of automorphisms of M blown up at $p_1, \ldots, p_n$ is trivial. However, this is not the case since these two conditions turn out to be of a different nature. The role of the zeros of the elements of $\mathfrak{h}$ will be clarified in [1]. For example, let us assume that $\mathfrak{h} = \operatorname{Span}\{X\}$ for $X \neq 0$. If we denote by $f := \langle \xi, X \rangle$, it is enough to choose $p_1, \ldots, p_n$ not all in the zero set of f for condition (1) to hold, while the group of automorphisms of M blown up at $p_1, \ldots, p_n$ is trivial if and only if one of the p_j is chosen away from the zero set of X, which corresponds to the set of critical points of function f !

Condition (2) is more subtle and more of a nonlinear nature. We will prove, in Section 6, that this condition is always fulfilled for some careful choice of the points, provided their number n is chosen larger than some value $n_g \geq \dim \mathfrak{h} + 1$.

LEMMA 1.2. *With the above notation, assume that* $n \geq \dim \mathfrak{h} + 1$, *then the set of points* $(p_1, \ldots, p_n) \in M^n_\triangle$ *for which* (1) *and* (2) *hold is an open* (*possibly empty*) *subset of* $M^n_\triangle$. *Moreover, there exists* $n_g \geq \dim \mathfrak{h} + 1$ *such that, for all* $n \geq n_g$ *the set of points* $(p_1, \ldots, p_n) \in M^n_\triangle$ *for which* (1) *and* (2) *hold is a nonempty open subset of* $M^n_\triangle$.

The proof of the Lemma 1.2 is due to E. Sandier, and we are very grateful to him for allowing us to present it here. In contrast with condition (1), it is easy to convince oneself that condition (1) does not hold for generic choice of the points. For example, assuming that $\mathfrak{h} = \operatorname{Span}\{X\}$ for $X \neq 0$, we say $f := \langle \xi, X \rangle$ and we choose $n \geq 2$. Then (1) holds provided $f(p_1), \ldots, f(p_n)$ are not all equal to 0 and (2) holds provided $f(p_1), \ldots, f(p_n)$ do not all have the same sign. Clearly, the set of such points is a nonempty open subset of $M^n_\triangle$ which is not dense.

Remark 1.2. By definition n_g is larger than $\dim \mathfrak{h}$, which also corresponds to the dimension of the space of holomorphic vector fields with zeros on (M, J) (in particular, dim $\mathfrak{h}$ does not depend on the metric !). It is interesting to determine the least value of n_g, i.e. the minimal number of points for which the two conditions (1) and (2) are fulfilled for a given constant scalar curvature Kähler metric g. Even in explicit examples, the determination of n_g seems to be a hard exercise.

Remark 1.3. We believe that (2) is a necessary condition for the result of Theorem 1.3 to hold. To give further credit to this belief, we refer to the discussion of this issue by Thomas [28, pp. 27, 28] and also to the recent preprint by Stoppa [27] where some partial result is obtained in this direction.

We now give a number of explicit examples to which our result can be applied. If we take $M = \mathbb{P}^m$ endowed with a Fubini-Study metric g_{FS}, we have:

PROPOSITION 1.1. *When* $M = \mathbb{P}^m$ *and* $g = g_{\mathrm{FS}}$, *then* $\dim \mathfrak{h} = m^2 + 2\,m$ *and* $n_{g_{\mathrm{FS}}} \leq 2\,m\,(m+1)$.

This result yields the existence of constant scalar curvature Kähler metrics on the blow up of $\mathbb{P}^m$ at n points which belong to some nonempty open set of $(\mathbb{P}^m)^n_\Delta$, provided $n \geq n_{g_{\mathrm{FS}}}$.

1.3. *The equivariant setting.* As already mentioned, n_g is (by definition) larger than the dimension of the space of holomorphic vector fields vanishing somewhere on M. Nevertheless, in some explicit cases, one can make use of the symmetries of the manifold M and work equivariantly to construct constant scalar curvature Kähler forms on the blow up of M at fewer points than the number n_g given in Theorem 1.1. At first glance, there seems to be some apparent contradiction in this statement; however, one should keep in mind that since one requires the sequence of metrics $(g_\varepsilon)_\varepsilon$ to converge to g as ε tends to 0 away from the exceptional divisors, these equivariant constructions do not hold anymore for choices of the blow up points in some *open* subset of M^n_Δ.

To state the equivariant version of Theorem 1.3, we assume that we are given Γ, a finite subgroup of isometries of (M, J, g, ω), we denote by $\mathfrak{h}^\Gamma \subset \mathfrak{h}$ the Lie subalgebra consisting of elements of $\mathfrak{h}$ which are Γ-invariant and we denote by

$$\xi^\Gamma : M \longrightarrow \mathfrak{h}^{\Gamma\,*},$$

the corresponding moment map. Observe that Γ acting on M will lift as a discrete subgroups of isometries $\tilde{\Gamma}$ on the blow up of M at a finite number of points $p_1, \ldots, p_n$ provided the set of blow up points is closed under the action of Γ (i.e. for all p_j and all $\sigma \in \Gamma$, $\sigma(p_j) \in \{p_1, \ldots, p_n\}$). We then have the equivariant version of Theorem 1.3:

THEOREM 1.4. *Assume that* Γ *is a finite subgroup of isometries of* (M, J, g, ω), *a constant scalar curvature Kähler manifold. Then*, *the result of Theorem* 1.3 *holds* *provided* $p_1, \dots, p_n \in M$ *are chosen so that*:

(3) *the set* $\{p_1, \dots, p_n\}$ *is closed under the action of* Γ,

(4) $\xi^\Gamma(p_1), \dots, \xi^\Gamma(p_n)$ *span* $\mathfrak{h}^{\Gamma\,*}$,

and there exist $a_1, \dots, a_n > 0$ *with*:

(5) $$a_j = a_{j'} \text{ if } p_j = \sigma(p_{j'}) \text{ for some } \sigma \in \Gamma \text{ and } \sum_{j=1}^{n} a_j\, \xi^\Gamma(p_j) = 0 \in \mathfrak{h}^{\Gamma\,*}.$$

Moreover, the constant scalar curvature metrics g_ε *on the blow up of* M *at* $p_1, \dots, p_n$ *are invariant under the action of* $\widetilde{\Gamma}$ (*the lift of* Γ *to the blow up of* M).

On the one hand, working equivariantly with respect to a large finite group of isometries Γ certainly decreases the dimension of the space of Killing vector fields which are invariant under the action of Γ and hence weakens the hypothesis which are needed for the construction to work. On the other hand, observe that the set of points which can be blown up has to be closed under the action of Γ and in general this substantially increases the number of points that *have to* be blown up. There is therefore some delicate balancing between the size of the finite group Γ and the number of blow up points.

We illustrate this fact in Section 7 where we once more consider the case of the projective space $\mathbb{P}^m$. Working equivariantly, we obtain the:

COROLLARY 1.1. *Given* $q_1, \dots, q_{m+1}$ *linearly independent points on* $\mathbb{P}^m$, *there exists* $\varepsilon_0 > 0$ *and for all* $\varepsilon \in (0, \varepsilon_0)$, *there exists a constant* (*positive*) *scalar curvature Kähler metric* g_ε *on the blow up of* $\mathbb{P}^m$ *at* $q_1, \dots, q_{m+1}$ *with associated Kähler form*

$$\omega_\varepsilon \in \pi^*[\omega_{\mathrm{FS}}] - \varepsilon^2\,(\mathrm{PD}[E_1] + \cdots + \mathrm{PD}[E_{m+1}]),$$

where the $\mathrm{PD}[E_j]$ *are the Poincaré duals of the* $(2m-2)$*-homology classes of the exceptional divisors of the blow up at* q_j.

Observe that all volumes of the exceptional divisors are identical. Moreover, the above result is optimal in the number of points because $\mathbb{P}^m$ blown up at $n \leq m$ points is known not to carry any constant scalar curvature Kähler metric since it violates the Matsushima-Lichnerowicz obstruction. Finally, observe that $\mathbb{P}^m$ blown up at $p_1, \dots, p_{m+1}$ still has holomorphic vector fields vanishing somewhere.

It is well known that on $\mathbb{P}^m$, $m+2$ points forming a projective frame are enough to kill all holomorphic vector fields after blow up, and we can prove that this condition also guarantees the existence of a Kähler constant scalar curvature metric. Indeed, working equivariantly, we also have:

COROLLARY 1.2. *Given $n \geq m+2$ and points $q_1, \ldots, q_n \in \mathbb{P}^m$ such that $q_1, \ldots, q_{m+2}$ form a projective frame, the blow up of $\mathbb{P}^m$ at $q_1, \ldots, q_n$ carries constant scalar curvature Kähler metrics and no holomorphic vector fields. Moreover $q_{m+3}, \ldots, q_n$ can be chosen arbitrarily on $\mathbb{P}^m$ blown up at $q_1, \ldots, q_{m+2}$.*

Let us emphasize that, even though the choice of $m+1$ linearly independent points (resp. the choice of a projective frame) ranges into an open and dense subset of $(\mathbb{P}^m)^{m+1}_{\triangle}$ (resp. $(\mathbb{P}^m)^{m+2}_{\triangle}$), Corollary 1.1 and Corollary 1.2 do not show that the constant $n_{g_{\mathrm{FS}}}$ in Proposition 1.1 can be taken to be equal to $m+1$. For example, as will be explained in Section 8, given two different sets of linearly independent points $p_1, \ldots, p_{m+1}$ and $q_1, \ldots, q_{m+1}$, the Kähler metrics on the blow up of $\mathbb{P}^m$ at these different sets of points are in general close to two different Fubini-Study metrics, one of which is the pull back of the other one by a biholomorphic transformation that sends the points p_j into the points q_j. Again, this reflects the Riemannian nature of Theorem 1.3 and Theorem 1.4 while the statements of Corollary 1.1 and 1.2 are more of a Kählerian flavor.

Recall that, for 2-dimensional complex manifolds, Kähler metrics with zero scalar curvature have been obtained by Rollin-Singer [23] on blow ups of $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ (for $n \geq 10$) or $\mathbb{T}^1 \times \mathbb{P}^1$ (for $n \geq 4$) using a different approach based on both algebraic tools and a connected-sum result. Moreover in [22] they have been able to find constant (nonzero) scalar curvature Kähler metrics also on the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $n \geq 6$ points. Also, the existence of zero scalar curvature Kähler metrics on blow ups of $\mathbb{P}^1 \times \Sigma$, when Σ is a Riemann surface of genus greater than or equal to 2, is due to LeBrun-Singer [16], using twistor theory. Our results also help to complement these constructions and we obtain constant scalar curvature metrics on the blow up (at carefully chosen points) of $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$ endowed with the product of Fubini-Study metrics and also on the blow up of $\mathbb{P}^{m_1} \times M$ endowed with the product metric, where (M, J, g, ω) is any m_2-dimensional Kähler manifold with a constant scalar curvature metric and without any a nontrivial holomorphic vector field with zeros. We refer to Section 8 for detailed statements.

1.4. *Relation with* GIT. To end this rather long introduction, let us briefly comment on the relation between our result and the different stability notions which arise in GIT. Recent years have seen some spectacular works, inspired by the analogy with the Hitchin-Kobayashi correspondence for vector bundles, relating the existence of canonical metrics to different stability notions of manifolds [29], [8], [21], [18]. It is then natural to try to interpret our results in terms of algebraic stability of the underlying manifold. Particular care must be taken since such algebraic notions (as Hilbert, Chow or K-stability) need what is called a polarizing class, i.e. a rational Kähler class. Clearly, the results of Theorem 1.3 and Theorem 1.4 do not guarantee the rationality of all $\varepsilon^2 \, a_{j,\varepsilon}^{\frac{1}{m-1}}$ for some ε and for all $j = 1, \ldots, n$.

Nevertheless, if we blow up enough points to kill all the automorphisms of the base manifold and if we succeed in applying Theorem 1.3, then the rationality of the Kähler class with a canonical representative can be achieved using the implicit function theorem [15], and one can conclude that the resulting polarized manifold is, for example, asymptotically Chow stable using the result of Donaldson [8] or K-semistable as proved by Chen-Tian [7] and Donaldson [9]. We shall return to this question in [1], where we study in detail the margins of freedom in choosing the weights and the position of points for which the present construction works.

With the help of symmetries in particular cases, rationality of the Kähler class can be achieved even in the presence of a continuous automorphism group (as is for example the case of $(\mathbb{P}^m,\ g_{\mathrm{FS}})$ blown up at $m+1$ points in general position). Again when this happens one can conclude, using Donaldson [9] and Chen-Tian's results [7], that the resulting polarized manifolds are K-semistable. In the remaining cases one should recall that Tian [30] conjectured that the existence of a Kähler constant scalar curvature metric should be equivalent to the analytic G-stability of the manifold $(M, [\omega])$ (a notion independent of the rationality of the Kähler class) for some maximal compact subgroup G of the automorphism group.

Finally, as already mentioned, condition (2) that arises in the statement of our main result can be understood as a *balancing condition* and we believe that it should be related to some suitable stability property of the blown up manifold. Again, we refer to the recent survey by Thomas [28, pp. 27–28] for a discussion of how this condition can be interpreted geometrically and to the recent preprint by Stoppa [27]. The fact that some positivity condition must hold is present in all known examples in different veins, and has been deeply investigated in the case of complex surfaces with zero scalar curvature by LeBrun-Singer [16], Rollin-Singer [23], and for Del Pezzo surfaces by Rollin-Singer [22].

1.5. *Plan of the paper.* In Section 2, we describe weighted Hölder spaces which constitute the key tool for our perturbation result. This will also be the opportunity to give some details about Burns-Simanca's metric g_0 defined on $\widetilde{\mathbb{C}}^m$, the blow up of $\mathbb{C}^m$ at the origin. In Section 3, we explain the structure of the scalar curvature operator under some perturbation of the Kähler metric in a given Kähler class. Section 4 is devoted to the study of the mapping properties of the linearized scalar curvature operators either about the manifold (M, J, g, ω) with finitely many points removed or about the complete noncompact manifold $(\widetilde{\mathbb{C}}^m, J_0, g_0, \eta_0)$. In Section 5 we construct infinite-dimensional families of constant scalar curvature Kähler metrics on the complement of finitely many small balls in M or in some large ball in $\widetilde{\mathbb{C}}^m$. These families are parametrized by the boundary data of the Kähler potential. We finally explain at the end of this section how the boundary data on the different summands can be chosen so that the different Kähler metrics can be connected together. This will complete the proof of Theorem 1.3. In Section 6, we give the proofs of Lemma 1.1 and Lemma 1.2. Finally, the last two sections are devoted to the study of the examples to which our result applies.

2. **Weighted spaces**

In this section, we describe weighted Hölder spaces on the noncompact (not complete) open manifold $(M^* := M \setminus \{p_1, \ldots, p_n\}, g)$, as well as weighted Hölder spaces on the noncompact complete manifold $(\widetilde{\mathbb{C}}^m, g_0)$, the blow up of $\mathbb{C}^m$ at the origin endowed with a scalar flat Kähler metric.

For all $r > 0$, we agree that

$$B_r := \{z \in \mathbb{C}^m \quad : \quad |z| < r\},$$

denotes the open ball of radius $r > 0$ in $\mathbb{C}^m$, $\bar{B}_r$ denotes the corresponding closed ball and

$$\bar{B}_r^* := \bar{B}_r \setminus \{0\},$$

the punctured closed ball. We will also define

$$C_r := \mathbb{C}^m \setminus \bar{B}_r \qquad \text{and} \qquad \bar{C}_r := \mathbb{C}^m \setminus B_r,$$

to be respectively the complement in $\mathbb{C}^m$ of the closed the ball and the open ball of radius $r > 0$.

Definition 2.1. Assume that $\ell \in \mathbb{N}$ and $\alpha \in (0, 1)$ are fixed. Given $\bar{r} > 0$ and a function $f \in \mathscr{C}^{\ell,\alpha}_{loc}(\bar{B}^*_{\bar{r}})$, we define

$$\|f\|_{\mathscr{C}^{\ell,\alpha}_\delta(\bar{B}^*_{\bar{r}})} := \sup_{0<r\leq\bar{r}} r^{-\delta} \, \|f(r\,\cdot)\|_{\mathscr{C}^{\ell,\alpha}(\bar{B}_1\setminus B_{1/2})},$$

and, for any function $f \in \mathscr{C}^{\ell,\alpha}_{loc}(\bar{C}_{\bar{r}})$, we define

$$\|f\|_{\mathscr{C}^{\ell,\alpha}_\delta(\bar{C}_{\bar{r}})} := \sup_{r\geq\bar{r}} r^{-\delta} \, \|f(r\,\cdot)\|_{\mathscr{C}^{\ell,\alpha}(\bar{B}_2\setminus B_1)}.$$

The norm $\|\cdot\|_{\mathscr{C}^{\ell,\alpha}_\delta(\bar{B}^*_{\bar{r}})}$ (resp. $\|\cdot\|_{\mathscr{C}^{\ell,\alpha}_\delta(\bar{C}_{\bar{r}})}$) measures the polynomial rate of blow up or decay of functions at 0 (resp. at ∞).

2.1. *Weighted spaces on* M^*. Assume that (M, J, g, ω) is a m-dimensional Kähler manifold and that we are also given n distinct points $p_1, \ldots, p_n \in M$. Near each p_j, the manifold M is biholomorphic to a neighborhood of 0 in $\mathbb{C}^m$ and we can choose complex coordinates $z := (z^1, \ldots, z^m)$ in a neighborhood of 0 in $\mathbb{C}^m$, to parametrize a neighborhood of p_j in M. In order to distinguish between the different neighborhoods and coordinate systems, we agree that, for all r small enough, say $r \in (0, r_0)$, $B_{j,r}$ (resp. $\bar{B}_{j,r}$ and $\bar{B}^*_{j,r}$) denotes the open ball (resp. the closed and closed punctured ball) of radius r in the coordinates z parametrizing a fixed neighborhood of p_j. We assume that r_0 is chosen small enough so that the $\bar{B}_{j,r_0}$ do not intersect each other. Without loss of generality, we can assume that near p_j, the coordinates we choose are normal coordinates and it follows from the $\partial\,\bar\partial$-Lemma (see [10, 107–108]) that the Kähler form ω can be expanded as

$$\omega := i\,\partial\,\bar\partial\,(\tfrac{1}{2}\,|z|^2 + \zeta_j), \tag{6}$$

for some function $\zeta_j \in \mathscr{C}^{3,\alpha}_4(\bar{B}^*_{j,r_0})$. This in particular implies that, in these coordinates, the Euclidean metric on $\mathbb{C}^m$ and the metric g induced by ω agree up to order 2.

For all $r \in (0, r_0)$, we set

$$\bar{M}_r := M \setminus \cup_j\, B_{j,r}. \tag{7}$$

We have already mentioned that

$$M^* := M \setminus \{p_1, \dots, p_n\}. \tag{8}$$

The weighted spaces of functions defined on the noncompact (not complete) manifold (M^*, g) is then defined as the set of functions whose decay or blow up near any p_j is controlled by a power of the distance to p_j. More precisely, we have:

Definition 2.2. Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, we define the *weighted space* $\mathscr{C}^{\ell,\alpha}_\delta(M^*)$ to be the *space of functions* $f \in \mathscr{C}^{\ell,\alpha}_{loc}(M^*)$ for which the following norm is finite:

$$\|f\|_{\mathscr{C}^{\ell,\alpha}_\delta(M^*)} := \|f_{|\bar{M}_{r_0/2}}\|_{\mathscr{C}^{\ell,\alpha}(\bar{M}_{r_0/2})} + \sup_{j=1,\dots,n} \|f_{|\bar{B}^*_{j,r_0}}\|_{\mathscr{C}^{\ell,\alpha}_\delta(\bar{B}^*_{j,r_0})}.$$

Burns-Simanca's metric and weighted spaces on the blow up of $\mathbb{C}^m$. We now turn to the description of weighted space on $(\tilde{\mathbb{C}}^m, J_0, g_0, \eta_0)$, the blow up at the origin of $\mathbb{C}^m$ endowed with Burns-Simanca's metric. As already mentioned in the introduction, the scalar curvature of the Kähler form η_0 is equal to 0. By construction, the Kähler form η_0 is invariant under the action of $U(m)$. If $u = (u^1, \dots, u^m)$ are complex coordinates in $\mathbb{C}^m \setminus \{0\}$, the Kähler form η_0 can be written as

$$\eta_0 = i\,\partial\,\bar\partial\,\left(\tfrac{1}{2}|u|^2 + E_m(|u|)\right). \tag{9}$$

More precisely

$$\eta_0 = i\,\partial\,\bar\partial\,\left(\tfrac{1}{2}|u|^2 + \log|u|\right), \tag{10}$$

in dimension $m = 2$. In dimension $m \geq 3$, even though there is no explicit formula, we have

$$\eta_0 = i\,\partial\,\bar\partial\,\left(\tfrac{1}{2}|u|^2 - |u|^{4-2m} + \mathcal{O}(|u|^{2-2m})\right). \tag{11}$$

These expansions follow from the analysis in [2]. Observe that there is some flexibility in the definition of η_0 since, for all $a > 0$, the metric associated to $a^2\, \eta_0$ is still a zero scalar curvature Kähler metric on $\widetilde{\mathbb{C}}^m$. In the expansion of η_0, the effect of this scaling amounts, after a change of variables, to modifying the coefficient in front of $-\log|u|$ into $-a^2 \log|u|$, when $m = 2$ or the coefficient in front of $|u|^{4-2m}$ into $a^{2m-2}\,|u|^{4-2m}$, in higher dimensions. We have chosen to normalize these coefficients to be equal to 1.

An important property which will be crucial for our construction is that, in the expansion of η_0, the coefficient in front of $\log|u|$, in dimension $m = 2$ or the coefficient in front of $-|u|^{4-2m}$, in dimension $m \geq 3$ are positive. Another property the reader should keep in mind is that, for any choice of complex coordinates (modulo $U(m)$) on $\mathbb{C}^m$, one can construct Burns-Simanca's metric. This flexibility will play an important role in Section 5 where the coordinates u must be adapted to the action of Γ, a compact group of isometries.

To simplify the notation, we set

$$N := \widetilde{\mathbb{C}}^m,$$

and, for all $R > 1$, we define

$$\bar{N}_R := N \setminus C_R. \tag{12}$$

We will denote by g_0 the metric associated to the Kähler form η_0. We are now in a position to define weighted spaces on the noncompact complete manifold (N, g_0). This time, we are interested in functions which decay or blow up at infinity at a rate which is controlled by a power of the distance to a fixed point in N. More precisely:

Definition 2.3. Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, we define the *weighted space* $\mathscr{C}^{\ell,\alpha}_\delta(N)$ to be the *space of functions* $f \in \mathscr{C}^{\ell,\alpha}_{loc}(N)$ for which the norm

$$\|f\|_{\mathscr{C}^{\ell,\alpha}_\delta(N)} := \|f_{|\bar{N}_2}\|_{\mathscr{C}^{\ell,\alpha}(\bar{N}_2)} + \|f_{|\bar{C}_1}\|_{\mathscr{C}^{\ell,\alpha}_\delta(\bar{C}_1)}$$

is finite.

3. **The geometry of the equation**

If (M, J, g, ω) is a Kähler manifold, we will indicate by Ric_g the Ricci tensor and by either $\boldsymbol{s}(g)$ or $\boldsymbol{s}(\omega)$ the scalar curvature of the metric g associated to ω.

Following LeBrun-Simanca [15], we want to understand the behavior of the scalar curvature under deformations of the Kähler form given by

$$\tilde{\omega} := \omega + i\,\partial\,\overline{\partial}\, f,$$

where f is a (real-valued) function defined on M. In local coordinates $(v^1, \ldots, v^m)$, if we write

$$\widetilde{\omega} = \frac{i}{2} \sum_{a,b} \widetilde{g}_{a\bar{b}}\, dv^a \wedge d\overline{v}^b,$$

then the scalar curvature of $\widetilde{\omega}$ is given by

$$s(\widetilde{\omega}) = -\sum_{a,b} \widetilde{g}^{a\bar{b}}\, \partial_{v^a}\, \partial_{\overline{v}^b} \, \log\left(\det\left(\widetilde{g}\right)\right), \tag{13}$$

where $\widetilde{g}^{a\bar{b}}$ are the coefficients of the inverse of $(\widetilde{g}_{a\bar{b}})$, the matrix of the coefficients of the metric $\widetilde{g}$ associated to $\widetilde{w}$. The following result is proven in [15] (or [4, Lemma 2.158] and [6, p. 271]:

PROPOSITION 3.1. *The scalar curvature of $\widetilde{\omega}$ can be expanded in powers of f and its derivatives as*

$$s\,(\widetilde{\omega}) = s\,(\omega) - \frac{1}{2}\left(\Delta_g^2 + 2Ric_g \cdot \nabla_g^2\right) f + Q_g(\nabla^2 f),$$

where Q_g is a second order nonlinear differential operator that collects all the nonlinear terms.

We shall return to the structure of the nonlinear operator Q_g later and for the time being, let us concentrate on the operator

$$\mathbb{L}_g := \Delta_g^2 + 2\,\mathrm{Ric}_g \cdot \nabla_g^2, \tag{14}$$

which will play a key role in our construction. To analyze this operator, we define a second order operator P_g by

$$\begin{aligned} P_g : \mathscr{C}^\infty(M) &\longrightarrow \Lambda^{0,1}(M, T^{1,0}), \\ f &\longmapsto \tfrac{1}{2}\overline{\partial}\, \Xi_f, \end{aligned} \tag{15}$$

where

$$\Xi_f := J\,\nabla f + i\,\nabla f.$$

Following [15], we find that

$$P_g^*\, P_g = \frac{1}{2}\Delta_g^2 + \mathrm{Ric}_g \cdot \nabla_g^2 + \frac{1}{2}(\nabla s + i\, J\, \nabla s), \tag{16}$$

where s is the scalar curvature of the metric g. The key observation is that, when the scalar curvature of g is constant we simply have

$$P_g^*\, P_g = \frac{1}{2}\Delta_g^2 + \mathrm{Ric}_g \cdot \nabla_g^2. \tag{17}$$

In particular, to any element f of $\mathrm{Ker}\,\mathbb{L}_g$ one can associate $\Xi_f := J\,\nabla f + i\,\nabla f$ a holomorphic vector field with zeros and $X_f := J\,\nabla f \in \mathfrak{h}$ a Killing field with zeros. For constant scalar curvature Kähler metrics, more is true and the following result provides the crucial relation between the kernel of the linearized scalar curvature operator $\mathbb{L}_g$ and the space of holomorphic vector fields with zeros or Killing vector fields with zeros:

THEOREM 3.1 ([15]). *Assume that* (M, J, g, ω) *is a compact constant scalar curvature Kähler manifold. Then the complexification of the subspace of the kernel of* $\mathbb{L}_g$ *spanned by functions whose mean over* M *is* 0, *is in one to one correspondence with the* (*complex*) *vector space of holomorphic vector fields with zeros and also with* $\mathfrak{h}$ *the* (*real*) *vector space of Killing vector fields with zeros.*

The previous considerations extend to $(\widetilde{\mathbb{C}}^m, J_0, g_0, \eta_0)$ and this implies the following important result which states that there are no elements in the kernel of the operator $\mathbb{L}_{g_0}$ which decay at infinity.

PROPOSITION 3.2 ([2]). *There are no nontrivial solution to* $\mathbb{L}_{g_0} f = 0$, *which belong to* $\mathscr{C}^{4,\alpha}_\delta(N)$, *for some* $\delta < 0$.

The proof of this result is given in [2] and borrows idea from a proof of a similar, more general, result proved in [12]. Since it is a key element of our construction we briefly sketch the proof here.

Proof. Assume that for $\delta < 0$ we have some real-valued function $f \in \mathscr{C}^{4,\alpha}_\delta(N)$ satisfying $\mathbb{L}_{g_0} f = 0$. Multiplying this equation by f and integrating by parts over N, we find that the vector field Ξ defined by $-\overline{\partial} f = \frac{1}{2}\eta_0(\Xi, -)$ is a holomorphic vector field which tends to 0 at infinity. Using Hartogs' Theorem, the restriction of Ξ to C_r, for $r > 0$, can be extended to a holomorphic vector field on $\mathbb{C}^m$. Since this vector field decays at infinity, it has to be identically equal to 0. This implies that Ξ is identically equal to 0 on C_r and this vector, field being holomorphic, is identically equal to 0 on N. However f being a real-valued function, this implies that $\partial f = \overline{\partial} f = 0$. Hence the function f is constant and decays at infinity. This implies that f is identically equal to 0 in N. □

4. **Mapping properties**

We collect some mapping properties for the linearized scalar curvature operators defined between weighted Hölder spaces.

4.1. *Analysis of the operator defined on* M^*. The results we want to obtain are based on the fact that, near each p_j, and in suitable coordinates, the metric g on M is asymptotic to the Euclidean metric. This implies that, in each B_{j,r_0} the operator $\mathbb{L}_g$ is close to the operator Δ^2, where Δ denotes the Laplacian in $\mathbb{C}^m$ when endowed with the Euclidean metric.

Let L be some elliptic operator (with smooth coefficients) acting on functions defined in the ball $\overline{B}_r \subset \mathbb{C}^m$. The *indicial roots* of L at $0 \in \mathbb{C}^m$ are the real numbers $\delta \in \mathbb{R}$ for which there exists a function $v \in \mathscr{C}^\infty(S^{2m-1})$, $v \neq 0$, with

$$L(|z|^\delta\, v(\theta)) = \mathbb{O}(|z|^{\delta+1})$$

near 0.

Let ψ be an eigenfunction of the Laplace-Beltrami operator $-\Delta_{S^{2m-1}}$ that is associated to the eigenvalue $\lambda = a\,(2m-2+a)$ for some $a \in \mathbb{N}$. We have

$$\Delta^2\,(r^\delta\,\psi) = 0,$$

in $\mathbb{C}^m \setminus \{0\}$ if

$$\delta = 2-2\,m-a, 4-2\,m-a, a, a+2.$$

This, together with the fact that the eigenfunctions of $-\Delta_{S^{2m-1}}$ constitute a Hilbert basis of $L^2(S^{2m-1})$, shows that the set of indicial roots of Δ^2 at 0 is given by $\mathbb{Z} - \{5-2\,m, \ldots, -1\}$ when $m \geq 3$ and is given by $\mathbb{Z}$ when $m = 2$. Using the normal coordinates near p_j as defined in Section 2, it is easy to check that the indicial roots of $\mathbb{L}_g$ at p_j are the same as the indicial roots of Δ^2 at 0.

The mapping properties of $\mathbb{L}_g$ when defined between weighted spaces are very sensitive to the choice of the weight parameter and the indicial roots play here a crucial role. We refer to [17], [20] and [19] for further details on the general theory of these operators defined between weighted function spaces.

We define the function G by

$$G(z) := -\log|z| \quad \text{when } m = 2 \quad \text{and} \quad G(z) := |z|^{4-2\,m} \quad \text{when } m \geq 3.$$

Observe that, unless the metric g is the Euclidean metric, these functions are not solutions of the homogeneous problem associated to $\mathbb{L}_g$ in the punctured ball $\bar{B}^*_{j,r_0}$. However, reducing r_0 if this is necessary, they can be perturbed into $\tilde{G}_j$ solutions of the homogeneous problem $\mathbb{L}_g\,\tilde{G}_j = 0$ in $\bar{B}^*_{j,r_0}$. Indeed, we have the following:

LEMMA 4.1. *There exist $r_0 > 0$ and functions $\tilde{G}_j$ which are solutions of $\mathbb{L}_g\,\tilde{G}_j = 0$ in $\bar{B}_{j^*,r_0}$ and which are asymptotic to G in the sense that $\tilde{G}_j - G \in \mathscr{C}^{4,\alpha}_{6-2m}(\bar{B}^*_{j,r_0})$ when $m \geq 4$ and $\tilde{G}_j - G \in \mathscr{C}^{4,\alpha}_{\delta}(\bar{B}^*_{\ell,r_0})$ for any $\delta < 6-2m$, when $m = 2, 3$.*

Proof. When $m \geq 3$, observe that

$$\Delta\,|z|^\delta = \delta\,(2m-2+\delta)\,|z|^{\delta-2} \qquad \text{and} \qquad \Delta\,|z|^{\delta-2} = (\delta-2)(2m-4+\delta)\,|z|^{\delta-4}.$$

When $\delta \in (4-2m, 0)$ the coefficients on the right-hand side are negative and the maximum principle yields, for all $\psi \in \mathscr{C}^{0,\alpha}_{\delta-4}(\bar{B}^*_{r_0})$ the existence of $\varphi \in \mathscr{C}^{4,\alpha}_{\delta}(\bar{B}_{r_0})$ the solution of

$$\Delta^2\,\varphi = \psi,$$

in $B^*_{r_0}$, with $\varphi{=}\Delta\varphi{=}0$ on ∂B_{r_0}. Schauder's estimates then imply that $\|\varphi\|_{\mathscr{C}^{4,\alpha}_{\delta}(\bar{B}_{r_0})} \leq c\,\|\psi\|_{\mathscr{C}^{0,\alpha}_{\delta-4}(\bar{B}_{r_0})}$ for some constant independent of r_0. Thanks to the expansion given in (6), a simple perturbation argument shows that a similar result is true when B_{r_0} is replaced by B_{j,r_0} and Δ^2 is replaced by $\mathbb{L}_g$, provided r_0 is chosen small enough.

Now, $\mathbb{L}_g\, G = (\mathbb{L}_g - \Delta^2)\, G$ and, thanks to the expansion given in (6), we conclude that

$$\mathbb{L}_g\, G \in \mathscr{C}^{0,\alpha}_{2-2m}(\bar{B}^*_{j,r_0}).$$

When $m \geq 4$, we fix $\delta = 6-2m$ and $\delta \in (-2,0)$ when $m = 3$. According to the above discussion, we can define $\varphi_j \in \mathscr{C}^{4,\alpha}_{\delta}(\bar{B}^*_{j,r_0})$ to be the solution of

$$\mathbb{L}_g\, \varphi_j = \mathbb{L}_g\, G, \tag{18}$$

with $\varphi_j = \Delta\varphi_j = 0$ on $\partial B_{j,r_0}$. In this case, we simply take $\tilde{G}_j := G + \varphi_j$.

When $m = 2$, one shows that there exists φ_j solution of (18) which is the sum of an affine function $z \longmapsto \ell_j(z)$ and a function belonging to $\mathscr{C}^{4,\alpha}_{\delta}(\bar{B}^*_{j,r_0})$ for any $\delta \in (1,2)$. Since any affine function is annihilated by the operator $\mathbb{L}_g$, this time we define $\tilde{G}_j := G + \varphi_j - \ell_j$. □

With the functions $\tilde{G}_j$ at hand, we define the *deficiency spaces*

$$\mathscr{D}_0 := \mathrm{Span}\{\chi_1, \dots, \chi_n\}, \qquad \text{and} \qquad \mathscr{D}_1 := \mathrm{Span}\{\chi_1\, \tilde{G}_1, \dots, \chi_n\, \tilde{G}_n\},$$

where χ_j is a cutoff function which is identically equal to 1 in $B_{j,r_0/2}$ and identically equal to 0 in $M - B_{j,r_0}$.

When $m \geq 3$, we fix $\delta \in (4-2m, 0)$ and define the operator

$$\begin{array}{rccl} L_\delta : & (\mathscr{C}^{4,\alpha}_{\delta}(M^*) \oplus \mathscr{D}_1) \times \mathbb{R} & \longrightarrow & \mathscr{C}^{0,\alpha}_{\delta-4}(M^*) \\ & (f, \beta) & \longmapsto & \mathbb{L}_g\, f + \beta, \end{array}$$

Whereas, when $m = 2$, we fix $\delta \in (0, 1)$ and define the operator

$$\begin{array}{rccl} L_\delta : & (\mathscr{C}^{4,\alpha}_{\delta}(M^*) \oplus \mathscr{D}_0 \oplus \mathscr{D}_1) \times \mathbb{R} & \longrightarrow & \mathscr{C}^{0,\alpha}_{\delta-4}(M^*) \\ & (f, \beta) & \longmapsto & \mathbb{L}_g\, f + \beta. \end{array}$$

To keep notation short, it will be convenient to set $\mathscr{D} := \mathscr{D}_1$ when $m \geq 3$ and $\mathscr{D} := \mathscr{D}_0 \oplus \mathscr{D}_1$ when $m = 2$. The main result of this section reads:

PROPOSITION 4.1. *Assume that the points $p_1, \dots, p_n \in M$ are chosen so that $\xi(p_1), \dots, \xi(p_n)$ span $\mathfrak{h}^*$, then the operator L_δ defined above is surjective* (*and has an* $(n+1)$*-dimensional kernel*).

Proof. The proof of this result follows from the general theory described in [17], [20] and [19] (see also the corresponding proof in [2]). However, we choose here to describe an almost self contained proof. Recall that the kernel of $\mathbb{L}_g$ is spanned by the functions $f_0 \equiv 1$ and the functions

$$f_1 := \langle \xi, X_1 \rangle, \quad \dots, \quad f_d := \langle \xi, X_d \rangle,$$

where $X_1, \dots, X_d$ is a basis of $\mathfrak{h}$ and $d = \dim \mathfrak{h}$. Recall, that, by construction the functions f_j, for $j = 1, \dots, d$ have mean 0. We use the fact that, thanks to (17), the operator $\mathbb{L}_g$ is self-adjoint and hence, for $\varphi \in L^1(M)$, the problem

$$\mathbb{L}_g\, f = \varphi,$$

is solvable if and only if φ satisfies

$$\int_M \varphi\, f_j \, d\mathrm{vol}_g = 0,$$

for $j = 0, \dots, d$.

Observe that $\mathscr{C}^{0,\alpha}_{\delta-4}(M^*) \subset L^1(M)$ when $\delta > 4-2m$. Now, given $\varphi \in L^1(M)$, we choose

$$\beta = \tfrac{1}{\mathrm{vol}(M)} \int_M \varphi\, d\mathrm{vol}_g,$$

and, since $\xi(p_1), \dots, \xi(p_n)$ span $\mathfrak{h}^*$,we also choose $a_1, \dots, a_n \in \mathbb{R}$ so that

$$\int_M \varphi\, \xi\, d\mathrm{vol}_g = \sum_{j=1}^n a_j\, \xi(p_j).$$

Applying this equality to any of the $X_{j'}$, we can also be write

$$\int_M \varphi\, f_{j'}\, d\mathrm{vol}_g = \sum_{j=1}^n a_j\, f_{j'}(p_j),$$

for $j' = 1, \dots, d$. Then, the problem

$$\mathbb{L}_g\, f + \beta = \varphi - \sum_{j=1}^n a_j\, \delta_{p_j},$$

is solvable in $W^{3,p}(M)$ for all $p \in [1, \frac{2m}{2m-1})$ and uniqueness of the solution is guarantied if we impose in addition that f be orthogonal to the functions $f_0, f_1, \dots, f_d$. To complete the proof, we invoke regularity theory [19] which implies that $f \in \mathscr{C}^{4,\alpha}_{\delta}(M^*) \oplus \mathscr{D}_1$ when $m \geq 3$ and $\mathscr{C}^{4,\alpha}_{\delta}(M^*) \oplus \mathscr{D}_0 \oplus \mathscr{D}_1$ when $m = 2$. The estimate of the dimension of the kernel will not be used in the paper and is left to the reader. □

Observe that, when solving the equation $\mathbb{L}_g\, f + \beta = \varphi$ in M^*, the constant β is determined by

$$\beta = \frac{1}{\mathrm{vol}(M)} \int_M \varphi\, d\mathrm{vol}_g.$$

4.2. *Analysis of the operator defined on N.* We denote by g_0 Burns-Simanca's metric associated to the Kähler form η_0.

Let L be some elliptic operator (with smooth coefficients) acting on functions defined in $\overline{C}_r \subset \mathbb{C}^m$. The *indicial roots* of L at infinity are the real numbers $\delta \in \mathbb{R}$

for which there exists a function $v \in \mathcal{C}^\infty(S^{2m-1})$, $v \neq 0$, with

$$L(|z|^\delta \, v(\theta)) = \mathcal{O}(|z|^{\delta-1}),$$

at infinity.

As above, we use the fact that g_0 is asymptotic to the Euclidean metric, as the expansions given in (10) and (11) show. This implies that, in $\bar{C}_1$, the operator $\mathbb{L}_{g_0}$ is close to the operator Δ^2 and, at infinity, they have the same indicial roots. Using the analysis at the beginning of the previous subsection, one checks that this set is equal to $\mathbb{Z} - \{5-2m, \dots, -1\}$ when $m \geq 3$ and to $\mathbb{Z}$ when $m = 2$.

Given $\delta \in \mathbb{R}$, we define the operator

$$\begin{array}{rccc} \tilde{L}_\delta : & \mathcal{C}^{4,\alpha}_\delta(N) & \longrightarrow & \mathcal{C}^{0,\alpha}_{\delta-4}(N) \\ & f & \longmapsto & \mathbb{L}_{g_0} f, \end{array}$$

and recall the following result from [2]:

PROPOSITION 4.2. *Assume that* $\delta \in (0,1)$. *Then the operator* $\tilde{L}_\delta$ *defined above is surjective and has a one-dimensional kernel spanned by a constant function.*

Proof. The result of Proposition 3.2 precisely states that the operator $\tilde{L}_{\delta'}$ is injective when $\delta' < 0$. This implies that the operator $\tilde{L}_\delta$ is surjective when $\delta > 4 - 2m$. When $\delta \in (0,1)$, this also implies that the operator $\tilde{L}_\delta$ has a one-dimensional kernel, spanned by a constant function. □

4.3. *Bi-harmonic extensions.* Two results concerning the bi-harmonic extensions of boundary data will be needed.

PROPOSITION 4.3. *There exists* $c > 0$ *and given* $h \in \mathcal{C}^{4,\alpha}(\partial B_1)$, $k \in \mathcal{C}^{2,\alpha}(\partial B_1)$ *there exists a function* $H^i_{h,k} \in \mathcal{C}^{4,\alpha}(\bar{B}_1)$ *such that*

$$\Delta^2 \, H^i_{h,k} = 0 \qquad in \qquad B_1,$$

with

$$H^i_{h,k} = h \quad and \quad \Delta H^i_{h,k} = k \qquad on \qquad \partial B_1.$$

Moreover,

$$\|H^i_{h,k}\|_{\mathcal{C}^{4,\alpha}(\bar{B}_1)} \leq c \, (\|h\|_{\mathcal{C}^{4,\alpha}(\partial B_1)} + \|k\|_{\mathcal{C}^{2,\alpha}(\partial B_1)}).$$

We will also need the following result which differs slightly from the corresponding result used in [2].

PROPOSITION 4.4. *There exists* $c > 0$ *and given* $h \in \mathcal{C}^{4,\alpha}(\partial B_1)$, $k \in \mathcal{C}^{2,\alpha}(\partial B_1)$ *such that*

$$\int_{\partial B_1} k = 0,$$

there exists a function $H^o_{h,k} \in \mathscr{C}^{4,\alpha}_{3-2m}(\bar{C}_1)$ *such that*

$$\Delta^2 \, H^o_{h,k} = 0, \qquad in \qquad C_1,$$

with

$$H^o_{h,k} = h \quad and \quad \Delta H^o_{h,k} = k \qquad on \qquad \partial B_1.$$

Moreover,

$$\| H^o_{h,k} \|_{\mathscr{C}^{4,\alpha}_{3-2m}(\bar{C}_1)} \leq c \, (\|h\|_{\mathscr{C}^{4,\alpha}(\partial B_1)} + \|k\|_{\mathscr{C}^{2,\alpha}(\partial B_1)}).$$

The proof of this result follows the proof of Proposition 5.6 in [2], the rationale being that there exists a bi-harmonic extension of the boundary data (h, k) which is defined on the complement of the unit ball and decays at infinity (at least when $m \geq 3$). Moreover, this function is bounded by a constant times the distance from the origin to the power $4-2m$ (when $m \geq 3$). In the case where the function k is assumed to have mean 0, then the rate of decay can be improved and estimated as the distance to the origin to the power $3-2m$. To see this we decompose both functions h and k over eigenfunctions of the Laplacian on the sphere. Namely

$$h = \sum_{a=0}^{\infty} h^{(a)} \qquad \text{and} \qquad k = \sum_{a=0}^{\infty} k^{(a)},$$

where the functions $h^{(a)}$ and $h^{(a)}$ satisfy

$$-\Delta_{S^{2m-1}} \, h^{(a)} = a \, (2m-2+a) \, h^{(a)},$$

and

$$-\Delta_{S^{2m-1}} \, k^{(a)} = a \, (2m-2+a) \, k^{(a)}.$$

Since we have assumed that $k^{(0)} = 0$, the function $H^o_{h,k}$ is explicitly given by

$$(19) \quad H^o_{h,k} = h^{(0)} \, |z|^{2-2m} + \sum_{a=1}^{\infty} \left(\left(h^{(a)} + \tfrac{1}{4(a+m-2)} \, k^{(a)} \right) \, |z|^{2-2m-a} - \tfrac{1}{4(a+m-2)} \, k^{(a)} \, |z|^{4-2m-a} \right).$$

At least, one can check that the series converges for all $|z| > 1$ and has the correct decay at infinity.

5. **Perturbation results**

Building on the results of the previous section we perturb the Kähler form ω on M with small ball centered at the points p_j excised and we also perturb the Kähler form η_0 on a large ball of $\widetilde{\mathbb{C}}^m$. These perturbation results will lead to the existence of infinite dimensional families of constant scalar curvature Kähler metrics parametrized by their boundary data.

5.1. *Perturbation of ω.* We consider the Kähler metric $\tilde{g}$ associated to the Kähler form

$$\tilde{\omega} = \omega + i\,\partial\,\overline{\partial}\,\zeta. \tag{20}$$

As mentioned in Proposition 3.1, the scalar curvature of $\tilde{g}$ can be expanded in powers of the function ζ and its derivatives as

$$\mathbf{s}(\tilde{g}) = \mathbf{s}(g) - \tfrac{1}{2}\,\mathbb{L}_g\,\zeta + Q_g(\nabla^2\zeta), \tag{21}$$

where the operator $\mathbb{L}_g$ is the one defined in (14) and where Q_g collects all the nonlinear terms. The structure of Q_g is quite complicated however; it follows from the explicit computation of the scalar curvature of $\tilde{g}$ in normal coordinates as given by the formula (13), that, near p_j, the nonlinear operator Q_g can be decomposed as

$$\begin{aligned} Q_g(\nabla^2 f) = & \sum_q B_{q,4,2}(\nabla^4 f, \nabla^2 f)\, C_{q,4,2}(\nabla^2 f) \\ & + \sum_q B_{q,3,3}(\nabla^3 f, \nabla^3 f)\, C_{q,3,3}(\nabla^2 f) \\ & + |z| \sum_q B_{q,3,2}(\nabla^3 f, \nabla^2 f)\, C_{q,3,2}(\nabla^2 f) \\ & + \sum_q B_{q,2,2}(\nabla^2 f, \nabla^2 f)\, C_{q,2,2}(\nabla^2 f), \end{aligned} \tag{22}$$

where the sum over q is finite, the operators $(U, V) \longmapsto B_{q,a,b}(U, V)$ are bilinear in their entries and have coefficients that are smooth functions on $\overline{B}_{j,r_0}$. The nonlinear operators $W \longmapsto C_{q,a,b}(W)$ have Taylor expansions (with respect to W) whose coefficients are smooth functions on $\overline{B}_{j,r_0}$.

Assume we are given $a_0, a_1, \ldots, a_n > 0$ such that there exists a solution of

$$\mathbb{L}_g\, H_{\boldsymbol{a}} = a_0 - c_m \sum_{j=1}^{n} a_j\, \delta_{p_j}, \tag{23}$$

where the constant c_m is defined by

$$c_m := 8\,(m-2)\,(m-1)\,\mathrm{Vol}(S^{2m-1}) \ \text{ when } m \geq 3 \quad \text{and} \ \ c_2 := 4\,\mathrm{Vol}(S^3).$$

Here we have set

$$\boldsymbol{a} := (a_0, \ldots, a_n).$$

Observe that such a function $H_{\boldsymbol{a}}$ exists if and only if a_0 is given by

$$a_0 = c_m \sum_{j=1}^{n} a_j,$$

and the coefficients $a_1, \dots, a_n$ are solutions of the system

$$\sum_{j=1}^{n} a_j \, \xi(p_j) = 0.$$

(Simply test (23) with the constant function and the functions $\langle \xi, X \rangle$ with $X \in \mathfrak{h}$.) It is not hard to check that:

LEMMA 5.1. *Near each p_j, the function $H_{\boldsymbol{a}}$ satisfies*

$$H_{\boldsymbol{a}} + a_j \, \tilde{G}_j + b_j \in \mathscr{C}^{4,\alpha}_1(\bar{B}^*_{j,r_0}),$$

for some constant $b_j \in \mathbb{R}$.

We fix

$$r_\varepsilon := \varepsilon^{\frac{2m-1}{2m+1}}. \tag{24}$$

We would like to find a function ζ defined in $\bar{M}_{r_\varepsilon}$ and a constant $\nu \in \mathbb{R}$ so that

$$\boldsymbol{s}(\tilde{g}) = \boldsymbol{s}(g) + \nu, \tag{25}$$

where $\tilde{g}$ is the metric associated to the Kähler form $\tilde{\omega} = \omega + i\,\partial\bar{\partial}\zeta$.

This equation is a fourth order nonlinear elliptic equation and boundary data are required to define a solution. Assume that we are given $h_j \in \mathscr{C}^{4,\alpha}(\partial B_1)$ and $k_j \in \mathscr{C}^{2,\alpha}(\partial B_1)$, for $j = 1, \dots, n$, satisfying

$$\|h_j\|_{\mathscr{C}^{4,\alpha}(\partial B_1)} + \|k_j\|_{\mathscr{C}^{2,\alpha}(\partial B_1)} \le \kappa \, r_\varepsilon^4, \tag{26}$$

where $\kappa > 0$ will be fixed later on. Further assume that

$$\int_{\partial B_1} k_j = 0. \tag{27}$$

It will be convenient to set

$$\boldsymbol{h} := (h_1, \dots, h_n) \qquad \text{and} \qquad \boldsymbol{k} := (k_1, \dots, k_n).$$

We define in $\bar{M}_{r_\varepsilon}$ the function

$$H_{\boldsymbol{h},\boldsymbol{k}} := \sum_{j=1}^{n} \chi_j \, H^o_{h_j,k_j}(\cdot / r_\varepsilon), \tag{28}$$

where, for each $j = 1, \dots, n$, the cutoff function χ_j is identically equal to 1 in $B_{j,r_0/2}$ and identically equal to 0 in $M \setminus B_{j,r_0}$.

The idea is to find the solution ζ of (25) as a perturbation of the function $\varepsilon^{2m-2} H_{\boldsymbol{a}} + H_{\boldsymbol{h},\boldsymbol{k}}$. The result we obtain reads:

PROPOSITION 5.1. *There exist $\gamma > 0$, $c > 0$ and $\varepsilon_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$, there exists a constant scalar curvature Kähler metric $g_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ defined in $\bar{M}_{r_\varepsilon}$, such that, for all $j = 1, \ldots, n$, the Kähler form associated to $g_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ can be written as*

$$\omega_{\varepsilon,\boldsymbol{h},\boldsymbol{k}} := i\, \partial\, \bar{\partial} \left(\tfrac{1}{2}\, |z|^2 + \zeta^{(j)}_{\varepsilon,\boldsymbol{h},\boldsymbol{k}} \right),$$

in $\bar{B}_{j,r_0} \setminus B_{j,r_\varepsilon}$ for some function $\zeta^{(j)}_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ satisfying the following estimates:

$$\|\zeta^{(j)}_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}(r_\varepsilon\, \cdot) + \varepsilon^{2m-2}\, r_\varepsilon^{4-2m}\, a_j\, G - H^o_{h_j,k_j}\|_{\mathscr{C}^{4,\alpha}(\bar{B}_2 \setminus B_1)} \leq c\, r_\varepsilon^4, \tag{29}$$

$$\begin{aligned} \|(\zeta^{(j)}_{\varepsilon,\boldsymbol{h},\boldsymbol{k}} - \zeta^{(j)}_{\varepsilon,\boldsymbol{h}',\boldsymbol{k}'})(r_\varepsilon\, \cdot) - H^o_{h_j-h'_j,k_j-k'_j}\|_{\mathscr{C}^{4,\alpha}(\bar{B}_2 \setminus B_1)} \\ \leq c\, \varepsilon^\gamma\, \|(\boldsymbol{h}-\boldsymbol{h}', \boldsymbol{k}-\boldsymbol{k}')\|_{(\mathscr{C}^{4,\alpha})^n \times (\mathscr{C}^{2,\alpha})^n}. \end{aligned} \tag{30}$$

Moreover the scalar curvature of $g_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ satisfies

$$|\boldsymbol{s}(g_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}) - \boldsymbol{s}(g)| \leq c\, \varepsilon^{2m-2} \tag{31}$$

and

$$|\boldsymbol{s}(g_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}) - \boldsymbol{s}(g_{\varepsilon,\boldsymbol{h}',\boldsymbol{k}'})| \leq c\, \varepsilon^\gamma\, \|(\boldsymbol{h}-\boldsymbol{h}', \boldsymbol{k}-\boldsymbol{k}')\|_{(\mathscr{C}^{4,\alpha})^n \times (\mathscr{C}^{2,\alpha})^n}. \tag{32}$$

Before we proceed with the proof of this result, we would like to mention, and this is an essential point, that the constant c which appears in the statement of the result does not depend on κ, provided ε is small enough. Also, the constant γ can be made explicit even though this is not useful. The remaining of the section is devoted to the proof of this result.

Proof of Proposition 5.1. We change variables:

$$\zeta := \varepsilon^{2m-2}\, H_{\boldsymbol{a}} + H_{\boldsymbol{h},\boldsymbol{k}} + f,$$

and

$$\nu := \tfrac{1}{2}\, (\beta - \varepsilon^{2m-2}\, a_0).$$

Now, (25), now reads

$$\boldsymbol{s}\left(\omega + i\, \partial\bar{\partial}\, (\varepsilon^{2m-2}\, H_{\boldsymbol{a}} + H_{\boldsymbol{h},\boldsymbol{k}} + f)\right) = \boldsymbol{s}\,(\omega) + \tfrac{1}{2}\, (\beta - \varepsilon^{2m-2}\, a_0), \tag{33}$$

where f and $\beta \in \mathbb{R}$ have to be determined. Thanks to (21), this amounts to solving the equation

$$\mathbb{L}_g\, f + \beta = 2\, Q_g(\nabla^2(\varepsilon^{2m-2}\, H_{\boldsymbol{a}} + H_{\boldsymbol{h},\boldsymbol{k}} + f)) - \mathbb{L}_g\, H_{\boldsymbol{h},\boldsymbol{k}}, \tag{34}$$

in $\bar{M}_{r_\varepsilon}$.

Definition 5.1. Given $\bar r \in (0, r_0/2)$, $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, the *weighted space* $\mathscr{C}^{\ell,\alpha}_\delta(\bar M_{\bar r})$ is defined to be the *space of functions* $f \in \mathscr{C}^{\ell,\alpha}(\bar M_{\bar r})$ *endowed with the norm*

$$\|f\|_{\mathscr{C}^{\ell,\alpha}_\delta(\bar M_{\bar r})} := \|f|_{\bar M_{r_0/2}}\|_{\mathscr{C}^{\ell,\alpha}(\bar M_{r_0/2})} + \sum_{j=1}^n \sup_{2\bar r \le r \le r_0} r^{-\delta} \, \|f|_{\bar B_{j,r_0}\setminus B_{j,\bar r}}(r\,\cdot)\|_{\mathscr{C}^{\ell,\alpha}(\bar B_{j,1}\setminus B_{j,1/2})}.$$

For each $\bar r \in (0, r_0/2)$, it will be convenient to define an *extension* (linear) operator

$$\mathscr{E}_{\bar r} : \mathscr{C}^{0,\alpha}_{\delta'}(\bar M_{\bar r}) \longrightarrow \mathscr{C}^{0,\alpha}_{\delta'}(M^*),$$

as follows:

(i) In $M_{\bar r}$, $\mathscr{E}_{\bar r}(f) = f$,

(ii) in each $\bar B_{j,\bar r} - B_{j,\bar r/2}$

$$\mathscr{E}_{\bar r}(f)(z) = \chi\left(|z|/\bar r\right) \, f\left(\bar r \, z/|z|\right),$$

(iii) in each $\bar B_{j,\bar r/2}$, $\mathscr{E}_{\bar r}(f) = 0$,

where $t \longmapsto \chi(t)$ is a smooth cutoff function identically equal to 0 for $t < 5/8$ and identically equal to 1 for $t > 7/8$. It is easy to check that there exists a constant $c > 0$, depending on δ' but independent of $\bar r \in (0, r_0/2)$, such that

$$\|\mathscr{E}_{\bar r}(f)\|_{\mathscr{C}^{0,\alpha}_{\delta'}(M^*)} \le c \, \|f\|_{\mathscr{C}^{0,\alpha}_{\delta'}(\bar M_{\bar r})}. \tag{35}$$

Instead of solving (34) in $\bar M_{r_\varepsilon}$, we prefer to solve the equation

$$\mathbb{L}_g \, f + \beta = \mathscr{E}_{r_\varepsilon}\Big(\big(2\, Q_g(\nabla^2(\varepsilon^{2m-2}\, H_{\boldsymbol a} + H_{\boldsymbol h,\boldsymbol k} + f)) - \mathbb{L}_g \, H_{\boldsymbol h,\boldsymbol k}\big)\,|_{\bar M_{r_\varepsilon}}\Big), \tag{36}$$

in M^*. We fix

$$\delta \in (4-2m, 5-2m),$$

and we make use of the analysis of Section 5 that allows us to find $\mathscr{G}_\delta$ a right inverse for the operator L_δ. We can then rephrase the solvability of (36) as a fixed point problem

$$(f, \beta) = \mathscr{N}(\varepsilon, \boldsymbol h, \boldsymbol k; f),$$

where the nonlinear operator $\mathscr{N}$ is defined by

$$\mathscr{N}(\varepsilon, \boldsymbol h, \boldsymbol k; f) := \mathscr{G}_\delta\left(\mathscr{E}_{r_\varepsilon}\Big(\big(2\, Q_g(\nabla^2(\varepsilon^{2m-2}\, H_{\boldsymbol a} + H_{\boldsymbol h,\boldsymbol k} + f)) - \mathbb{L}_g \, H_{\boldsymbol h,\boldsymbol k}\big)\,|_{\bar M_{r_\varepsilon}}\Big)\right).$$

To keep notation short, it will be convenient to denote

$$\mathscr{F} := (\mathscr{C}^{4,\alpha}_\delta(M^*) \oplus \mathscr{D}) \times \mathbb{R}.$$

This space is naturally endowed with the product norm.

The existence of a fixed point to this nonlinear problem is based on the following technical Lemma. Let us agree that c_κ is a constant that depends on κ, whereas c is a constant that does not depend on κ provided ε is chosen small enough. These constants do not depend on ε and may vary from line to line. This being understood, we have the:

LEMMA 5.2. *There exist* $c > 0$, $c_\kappa > 0$ *and* $\varepsilon_\kappa > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_\kappa)$

$$\|\mathcal{N}(\varepsilon, \boldsymbol{h}, \boldsymbol{k}; 0)\|_{\mathcal{F}} \leq c_\kappa \, (r_\varepsilon^{2m+1} + \varepsilon^{4m-4} \, r_\varepsilon^{6-4m-\delta}), \tag{37}$$

and

$$\|\mathcal{N}(\varepsilon, \boldsymbol{h}, \boldsymbol{k}; f) - \mathcal{N}(\varepsilon, \boldsymbol{h}, \boldsymbol{k}; f')\|_{\mathcal{F}} \leq c_\kappa \, \varepsilon^{2m-2} \, r_\varepsilon^{6-4m-\delta} \, \|f - f'\|_{\mathcal{F}}. \tag{38}$$

Finally,

$$\begin{aligned} &\|\mathcal{N}(\varepsilon, \boldsymbol{h}, \boldsymbol{k}; f) - \mathcal{N}(\varepsilon, \boldsymbol{h}', \boldsymbol{k}'; f)\|_{\mathcal{F}} \\ &\qquad \leq c_\kappa \, (r_\varepsilon^{2m-3} + \varepsilon^{2m-2} \, r_\varepsilon^{2-2m-\delta}) \, \|(\boldsymbol{h} - h', \boldsymbol{k} - k')\|_{(\mathscr{C}^{4,\alpha})^n \times (\mathscr{C}^{2,\alpha})^n}, \end{aligned} \tag{39}$$

provided $f, f' \in \mathscr{C}^{4,\alpha}_\delta(M^*) \oplus \mathscr{D}$ *satisfy*

$$\|f\|_{\mathscr{C}^{4,\alpha}_\delta(M^*) \oplus \mathscr{D}} + \|f'\|_{\mathscr{C}^{4,\alpha}_\delta(M^*) \oplus \mathscr{D}} \leq 4 \, c_\kappa \, (r_\varepsilon^{2m+1} + \varepsilon^{4m-4} \, r_\varepsilon^{6-4m-\delta}),$$

and the all the components of $\boldsymbol{h}, \boldsymbol{h}', \boldsymbol{k}, \boldsymbol{k}'$ *satisfy* (26) *and* (27).

Proof. The proof of these estimates follows what is already done in [2] with minor modifications. We briefly recall how the proof of the first estimate is obtained and leave the proof of the second and third estimates to the reader.

First, we use the result of Proposition 4.4 to estimate

$$\|H_{\boldsymbol{h},\boldsymbol{k}}\|_{\mathscr{C}^{4,\alpha}_{3-2m}(\bar{M}_{r_\varepsilon})} \leq c_\kappa \, r_\varepsilon^{2m+1}. \tag{40}$$

Observe that, by construction, $H_{\boldsymbol{h},\boldsymbol{k}} = 0$ in $\bar{M}_{r_0}$ and $\Delta^2 \, H_{\boldsymbol{h},\boldsymbol{k}} = 0$ in each $\bar{B}_{j,r_0/2} \setminus B_{j,r_\varepsilon}$, hence

$$\mathbb{L}_g \, H_{\boldsymbol{h},\boldsymbol{k}} = (\mathbb{L}_g - \Delta^2) \, H_{h,k},$$

in this set. Making use of the expansion (6) which reflects the fact that the metric g is, in each $\bar{B}_{j,r_0}$, asymptotic to the Euclidean metric, we get the estimate

$$\|\mathbb{L}_g \, H_{\boldsymbol{h},\boldsymbol{k}}\|_{\mathscr{C}^{0,\alpha}_{\delta-4}(\bar{M}_{r_\varepsilon})} \leq c_\kappa \, r_\varepsilon^{2m+1}(1 + r_\varepsilon^{5-2m-\delta}) \leq 2 \, c_\kappa \, r_\varepsilon^{2m+1},$$

provided ε is chosen small enough. This is where we implicitly use the fact that $\delta < 5 - 2m$.

Next, we use the structure of the nonlinear operator Q_g as described in (22), the estimate (40) and also the fact that

$$\|\nabla^2 H_{\boldsymbol{a}}\|_{\mathscr{C}^{2,\alpha}_{2-2m}(\bar{M}_{r_\varepsilon})} \leq c,$$

to evaluate the term $Q_g\,(\nabla^2(\varepsilon^{2m-2}\,H_{\boldsymbol{a}} + H_{\boldsymbol{h},\boldsymbol{k}}))$. Roughly speaking, in an expression of the form $Q_g(\nabla^2\,\psi)$, the most relevant terms (as far as estimates are concerned) are the ones of the form $B_{q,4,2}(\nabla^4\,\psi, \nabla^2\psi)$ and $B_{q,3,3}(\nabla^3\psi, \nabla^3\,\psi)$ provided the second derivatives of ψ remain bounded (which is precisely our case). We find

$$\|Q_g\,(\nabla^2(\varepsilon^{2m-2}\,H_{\boldsymbol{a}} + H_{\boldsymbol{h},\boldsymbol{k}}))\|_{\mathscr{C}^{0,\alpha}_{\delta-4}(\bar{M}_{r_\varepsilon})} \le c\,\varepsilon^{4m-4}\,(1 + r_\varepsilon^{6-4m-\delta}) \le c\,\varepsilon^{4m-4}\,r_\varepsilon^{6-4m-\delta},$$

for some constant $c > 0$ which does not depend on κ provided ε is chosen small enough. The last inequality implicitly uses the fact that $6-4m-\delta < 2-2m < 0$ since $\delta > 4-2m$. The first estimate then follows at once. The proof of the other estimates follows the same lines. One should keep in mind that the function space we are working with is $\mathscr{C}^{4,\alpha}_\delta(M^*) \oplus \mathscr{D}$ and not $\mathscr{C}^{4,\alpha}_\delta(M^*)$. □

Reducing $\varepsilon_\kappa > 0$ if necessary, we can assume that,

$$c_\kappa\,\varepsilon^{2m-2}\,r_\varepsilon^{6-4m-\delta} \le \tfrac{1}{2}, \tag{41}$$

for all $\varepsilon \in (0, \varepsilon_\kappa)$. Then, the estimates (37) and (38) in the above lemma are enough to show that

$$(\varphi, \beta) \longmapsto \mathscr{N}(\varepsilon, \boldsymbol{h}, \boldsymbol{k}; \varphi),$$

is a contraction from

$$\{(\varphi, \beta) \in \mathscr{F} \quad : \quad \|(\varphi, \beta)\|_{\mathscr{F}} \le 2\,c_\kappa\,(r_\varepsilon^{2m+1} + \varepsilon^{4m-4}\,r_\varepsilon^{6-4m-\delta})\},$$

into itself and hence has a unique fixed point $(f_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}, \beta_{\varepsilon,\boldsymbol{h},\boldsymbol{k}})$ in this set. This fixed point yields a solution $f_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ of (33) in $\bar{M}_{r_\varepsilon}$, with $\beta = \beta_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ and hence provides a Kähler metric $g_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ on $\bar{M}_{r_\varepsilon}$ associated to the Kähler form

$$\omega_{\varepsilon,\boldsymbol{h},\boldsymbol{k}} = \omega + i\,\partial\bar{\partial}\left(\varepsilon^{2m-2}\,H_{\boldsymbol{a}} + H_{\boldsymbol{h},\boldsymbol{k}} + f_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}\right).$$

(Reducing ε_κ if necessary, it is easy to check that the associated metric $g_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ is indeed positive in $\bar{M}_{r_\varepsilon}$.) By construction, the scalar curvature of this metric is constant and equal to

$$\boldsymbol{s}(\omega_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}) = \boldsymbol{s}(\omega) + \tfrac{1}{2}\,(\beta_{\varepsilon,\boldsymbol{h},\boldsymbol{k}} - \varepsilon^{2m-2}\,a_0). \tag{42}$$

Since

$$|\beta_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}| \le 2\,c_\kappa\,(r_\varepsilon^{2m+1} + \varepsilon^{4m-4}\,r_\varepsilon^{6-4m-\delta}) \le c\,\varepsilon^{2m-2},$$

for all ε small enough, we immediately get (31).

The Kähler potential $\zeta^{(j)}_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ which appears in the statement of Proposition 5.1 is defined as follows: In dimension $m \ge 3$ we consider the function

$$\zeta^{(j)}_{\varepsilon,\boldsymbol{h},\boldsymbol{k}} := \zeta^{(j)} + \varepsilon^{2m-2}\,H_{\boldsymbol{a}} + H_{\boldsymbol{h},\boldsymbol{k}} + f_{\varepsilon,\boldsymbol{h},\boldsymbol{k}},$$

which is defined in $\overline{B}_{j,r_0} \setminus B_{j,r_\varepsilon}$ and where $\zeta^{(i)}$ is the potential defined in (6). When $m = 2$ minor modifications are needed and we define

$$\zeta^{(j)}_{\varepsilon,\boldsymbol{h},\boldsymbol{k}} := \zeta^{(j)} + \varepsilon^2\, H_{\boldsymbol{a}} + H_{\boldsymbol{h},\boldsymbol{k}} + f_{\varepsilon,\boldsymbol{h},\boldsymbol{k}} + \varepsilon^2\,(b_j + a_j\ \log r_\varepsilon),$$

where the constant b_j is the one which appears in Lemma 5.1. Observe that, locally, adding a constant to the Kähler potential does not alter the corresponding Kähler metric.

The estimate (29) derives at once from the the following ingredients: $\zeta^{(j)} \in \mathcal{C}^{4,\alpha}_4(\overline{B}^*_{j,r_0})$, the results of Lemma 4.1 and Lemma 5.1 used to estimate $H_{\boldsymbol{a}} - a_j\, G$ and finally (37) used to estimate $f_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$. The estimates (30) and (32) follow from (39) together with the fact that $r_\varepsilon = \varepsilon^{\frac{2m-1}{2m+1}}$. We also find that the constant $\gamma > 0$ can be chosen to be

$$\gamma < \min\Big(\frac{2m-1}{2m+1}, 2m-2+\frac{2m-1}{2m+1}\,(6-2m-\delta)\Big).$$

(Be careful to see that $f_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ belongs to $\mathcal{C}^{4,\alpha}_\delta(M^*) \oplus \mathcal{D}$ and not to $\mathcal{C}^{4,\alpha}_\delta(M^*)$.) Notice that the restriction $\delta \in (0, 2/3)$ is needed in dimension $m = 2$ in order to obtain (29) and (30). This completes the proof of Proposition 5.1. □

5.2. *Perturbation of* η_0. We perform an analysis similar to the one in the previous subsection starting from the blow up of $\mathbb{C}^m$ at the origin endowed with Burns-Simanca's metric g_0. We keep the notation of Section 2.

Given $a > 0$, we consider on $N = \tilde{\mathbb{C}}^m$, the perturbed Kähler form

$$\tilde{\eta} = a^2\,\eta_0 + i\,\partial\,\bar\partial\,\tilde{\zeta}.$$

Everything we will do will be uniform in a as long as this parameter remains both bounded from above and bounded away from 0. Therefore, we will assume that

$$a \in [a_{\min}, a_{\max}], \tag{43}$$

where $0 < a_{\min} < a_{\max}$ are fixed.

Using the fact that

$$\boldsymbol{s}(a^2\,\eta_0 + i\,\partial\,\bar\partial\,\zeta) = \boldsymbol{s}(a^2\,(\eta_0 + i\,a^{-2}\,\partial\,\bar\partial\,\zeta)) = a^{-2}\,\boldsymbol{s}(\eta_0 + i\,a^{-2}\,\partial\,\bar\partial\,\zeta),$$

we see that the scalar curvature of $\tilde\eta$ can be expanded as

$$\boldsymbol{s}(\tilde\eta) = -\tfrac{1}{2}\,a^{-4}\,\mathbb{L}_{g_0}\,\zeta + a^{-2}\,Q_{g_0}(a^{-2}\,\nabla^2\zeta). \tag{44}$$

Observe that we have used the fact that the scalar curvature of η_0 is identically equal to 0 ! Again, the structure of the nonlinear operator Q_{g_0} is also quite involved but, in $\overline{C}_1$, it enjoys a decomposition similar to the one described in the previous section. Indeed, using the expansions (10) and (11) we see that we can decompose

$$
\begin{aligned}
Q_{g_0}(\nabla^2 f) = & \sum_q B_{q,4,2}(\nabla^4 f, \nabla^2 f)\, C_{q,4,2}(\nabla^2 f) \\
& + \sum_q B_{q,3,3}(\nabla^3 f, \nabla^3 f)\, C_{q,3,3}(\nabla^2 f) \\
& + \sum_q |u|^{1-2m}\, B_{q,3,2}(\nabla^3 f, \nabla^2 f)\, C_{q,3,2}(\nabla^2 f) \\
& + \sum_q |u|^{-2m}\, B_{q,2,2}(\nabla^2 f, \nabla^2 f)\, C_{q,2,2}(\nabla^2 f)
\end{aligned}
$$

where the sum over q is finite, the operators $(U, V) \longmapsto B_{q,j,j'}(U, V)$ are bilinear in the entries and have coefficients which are bounded in $\mathscr{C}^{0,\alpha}(\bar{C}_1)$. The nonlinear operators $W \longmapsto C_{q,a,b}(W)$ have Taylor expansions (with respect to W) whose coefficients are bounded in $\mathscr{C}^{0,\alpha}(\bar{C}_1)$.

We define

$$R_\varepsilon := \frac{r_\varepsilon}{\varepsilon}, \tag{45}$$

where r_ε is given by (24). We would like to find a function $\tilde{\zeta}$ defined in $N_{R_\varepsilon/a}$, a solution of the equation

$$\mathbf{s}\,(\tilde{\eta}) = \varepsilon^2\, \nu, \tag{46}$$

where $\nu \in \mathbb{R}$ is a given constant satisfying

$$\nu \in [\nu_{\min}, \nu_{\max}].$$

The estimates we obtain will not depend on ν provided ν remains in this range and $\nu_{\min} < \nu_{\max}$ are fixed.

Again, (46) is a fourth order nonlinear elliptic equation which has to be complemented with boundary data. Given $h \in \mathscr{C}^{4,\alpha}(\partial B_1)$ and $k \in \mathscr{C}^{2,\alpha}(\partial B_1)$ satisfying

$$\|h\|_{\mathscr{C}^{4,\alpha}(\partial B_1)} + \|k\|_{\mathscr{C}^{2,\alpha}(\partial B_1)} \leq \kappa\, R_\varepsilon^{3-2m}, \tag{47}$$

where $\kappa > 0$ will be fixed later on, we define

$$\tilde{H}_{h,k} := \tilde{\chi}\,(H^i_{h,k}(a\,\cdot/R_\varepsilon) - H^i_{h,k}(0)) + H^i_{h,k}(0), \tag{48}$$

where $\tilde{\chi}$ is a cutoff function which is identically equal to 1 in C_2 and identically equal to 0 in N_1.

We would like to find $\tilde{\zeta}$ a solution of (46) as a perturbation of the function $\tilde{H}_{h,k}$. As in the previous analysis, let us agree that c_κ is a constant which depends on κ, whereas c is a constant that does not depend on κ provided ε is chosen small enough. These constants do not depend on ε and may vary from line to line. The result we obtain parallels the result obtained in the previous subsection.

PROPOSITION 5.2. *There exist $c > 0$ and $\varepsilon_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$, there exists a constant scalar curvature Kähler metric $g_{\varepsilon,a,h,k,\nu}$ defined in $N_{R_\varepsilon/a}$, whose Kähler form can be written as*

$$\eta_{\varepsilon,a,h,k,\nu} := i\,\partial\bar\partial\,(\tfrac{1}{2}a^2\,|u|^2 + \tilde\zeta_{\varepsilon,a\,h,k,\nu}),$$

in $\bar N_{R_\varepsilon/a} - N_{R_\varepsilon/2a}$ for some function $\tilde\zeta_{\varepsilon,a\,h,k,\nu}$. Moreover the scalar curvature of $g_{\varepsilon,a,h,k,\nu}$ is equal to $\varepsilon^2\,\nu$ and the function $\tilde\zeta_{\varepsilon,a\,h,k,\nu}$ satisfies

$$\|\tilde\zeta_{\varepsilon,a,h,k,\nu}(R_\varepsilon\cdot/a) + a^{2m-2}\,R_\varepsilon^{4-2m}\,G - H^i_{h,k}\|_{\mathscr{C}^{4,\alpha}(B_1\setminus B_{1/2})} \leq c\,R_\varepsilon^{3-2m}, \tag{49}$$

$$\begin{aligned}\|(\tilde\zeta_{\varepsilon,a,h,k,\nu} - \tilde\zeta_{\varepsilon,a',h',k',\nu'})(R_\varepsilon\cdot/a) - H^i_{h-h',k-k'}\|_{\mathscr{C}^{4,\alpha}(\bar B_1\setminus B_{1/2})}& \\ \leq c_\kappa\,(R_\varepsilon^{\delta-1}\,\|(h-h',k-k')\|_{\mathscr{C}^{4,\alpha}\times\mathscr{C}^{2,\alpha}} + R_\varepsilon^{3-2m}\,(|\nu-\nu'| + |a-a'|)).&\end{aligned} \tag{50}$$

Again, and this is an essential point, we would like to emphasize that the constant c which appears in the statement of the result does not depend on κ, provided ε is small enough.

The remainder of the section is devoted to the proof of this technical result.

Proof of Proposition 5.2. Replacing in (44) the function $\tilde\zeta$ by $\tilde H_{h,k} + f$, we see that (46) can be written as

$$\mathbb{L}_{g_0}\,(\tilde H_{h,k} + f) = 2\,a^2\,Q_{g_0}(a^{-2}\,\nabla^2(\tilde H_{h,k} + f)) - 2\,\varepsilon^2\,a^4\,\nu, \tag{51}$$

equation which we would like to solve in $N_{R_\varepsilon/a}$. Here ν and a are given and f has to be determined.

Definition 5.2. Given $\bar R > 1$, $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, the *weighted space* $\mathscr{C}^{\ell,\alpha}_\delta(\bar N_{\bar R})$ is defined to be the *space of functions* $f \in \mathscr{C}^{\ell,\alpha}(\bar N_{\bar R})$ *endowed with the norm*

$$\|f\|_{\mathscr{C}^{\ell,\alpha}_\delta(\bar N_{\bar R})} := \|f_{|_{\bar N_1}}\|_{\mathscr{C}^{\ell,\alpha}(\bar N_1)} + \sup_{1\leq R\leq \bar R} R^{-\delta}\,\|f_{|_{\bar C_{R/2}\setminus C_R}}(R\,\cdot)\|_{\mathscr{C}^{\ell,\alpha}(\bar B_1\setminus B_{1/2})}.$$

For each $\bar R \geq 1$, it will be convenient to define an "extension" (linear) operator

$$\tilde{\mathscr{E}}_{\bar R} : \mathscr{C}^{0,\alpha}_{\delta'}(\bar N_{\bar R}) \longrightarrow \mathscr{C}^{0,\alpha}_{\delta'}(N),$$

as follows:

(i) In $N_{\bar R}$, $\tilde{\mathscr{E}}_{\bar R}\,(f) = f$,

(ii) in $\bar C_{2\bar R} \setminus C_{\bar R}$

$$\tilde{\mathscr{E}}_{\bar R}\,(f)(u) = \bar\chi\left(\frac{|u|}{\bar R}\right)\,f\left(\bar R\,\frac{u}{|u|}\right),$$

(iii) in $\bar{C}_{2\,\bar{R}}$, $\widetilde{\mathscr{E}}_{\bar{R}}\,(f) = 0$,

where $t \longmapsto \overline{\chi}(t)$ is a smooth cutoff function identically equal to 1 for $t < 5/4$ and identically equal to 0 for $t > 7/4$. It is easy to check that there exists a constant $c > 0$, depending on δ' but independent of $\bar{R} \geq 2$, such that

$$\|\widetilde{\mathscr{E}}_{\bar{R}}(f)\|_{\mathscr{C}^{0,\alpha}_{\delta'}(N)} \leq c\, \|f\|_{\mathscr{C}^{0,\alpha}_{\delta'}(\bar{N}_{\bar{R}})}. \tag{52}$$

Instead of solving (51) in $N_{R_\varepsilon/a}$ we would rather solve

$$\mathbb{L}_{g_0}\, f = \widetilde{\mathscr{E}}_{R_\varepsilon/a}\Big(\Big(2\,a^2\, Q_{g_0}(a^{-2}\,\nabla^2(\tilde{H}_{h,k} + f)) - \mathbb{L}_{g_0}\,\tilde{H}_{h,k} - 2\,\varepsilon^2\,a^4\,\nu\Big)\,|_{\bar{N}_{R_\varepsilon/a}}\Big), \tag{53}$$

in N. To this aim, we fix

$$\delta \in (0,1),$$

and use the result of Proposition 4.2. This provides a right inverse $\widetilde{\mathscr{G}}_\delta$ for the operator $\mathbb{L}_{g_0}$ and we can rephrase the solvability of (53) as a fixed point problem:

$$f = \widetilde{\mathscr{N}}(\varepsilon, a, h, k, \nu; f) \tag{54}$$

where the nonlinear operator $\widetilde{\mathscr{N}}$ is defined by

$$\begin{aligned}\widetilde{\mathscr{N}}&(\varepsilon, a, h, k, \nu; f)\\ &:= \widetilde{\mathscr{G}}_\delta\,\Big(\widetilde{\mathscr{E}}_{R_\varepsilon/a}\Big(2\,a^2\,Q_{g_0}(a^{-2}\,\nabla^2(\tilde{H}_{h,k} + f)) - \mathbb{L}_{g_0}\,\tilde{H}_{h,k} - 2\,\varepsilon^2\,a^4\,\nu\Big)\Big).\end{aligned}$$

To keep notation short, it will be convenient to define

$$\widetilde{\mathscr{F}} := \mathscr{C}^{4,\alpha}_{\delta}(N).$$

The existence of a fixed point for this nonlinear problem will follow from the:

LEMMA 5.3. *There exist* $c > 0$ (*independent of* κ), $c_\kappa > 0$ *and* $\varepsilon_\kappa > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_\kappa)$

$$\|\widetilde{\mathscr{N}}(\varepsilon, a, h, k, \nu; 0)\|_{\widetilde{\mathscr{F}}} \leq c\, R_\varepsilon^{3-2m-\delta}, \tag{55}$$

Moreover,

$$\|\widetilde{\mathscr{N}}(\varepsilon, a, h, k, \nu; f) - \widetilde{\mathscr{N}}(\varepsilon, a, h, k, \nu; f')\|_{\widetilde{\mathscr{F}}} \leq c_\kappa\, R_\varepsilon^{3-2m-\delta}\, \|f - f'\|_{\widetilde{\mathscr{F}}}, \tag{56}$$

and

$$\begin{aligned}&\|\widetilde{\mathscr{N}}(\varepsilon, a, h, k, \nu; f) - \widetilde{\mathscr{N}}(\varepsilon, a, h', k', \nu'; f)\|_{\widetilde{\mathscr{F}}}\\ &\qquad \leq c_\kappa\,(R_\varepsilon^{-1}\, \|(h - h', k - k')\|_{\mathscr{C}^{4,\alpha}\times\mathscr{C}^{2,\alpha}} + R_\varepsilon^{3-2m-\delta}\, |\nu' - \nu|),\end{aligned} \tag{57}$$

provided $f, f' \in \widetilde{\mathscr{F}}$, *satisfy*

$$\|f\|_{\widetilde{\mathscr{F}}} + \|f'\|_{\widetilde{\mathscr{F}}} \leq 4\,c\, R_\varepsilon^{3-2m-\delta},$$

and h, h' *and* k, k' *satisfy* (47).

Proof. Again, the proof of these estimates follows the corresponding proof in [2]. First, we use the result of Proposition 4.3 together with (47) to estimate

$$\|\nabla^2 H^i_{h,k}\|_{\mathscr{C}^{2,\alpha}(\bar{B}_1)} \leq c_\kappa \, R_\varepsilon^{3-2m}. \tag{58}$$

In $\bar{N}_{R_\varepsilon/a} \setminus N_2$ observe that $\widetilde{H}_{h,k} = H^i_{h,k}(a \cdot / R_\varepsilon) - H^i_{h,k}(0)$ and the derivatives of $\widetilde{H}_{h,k}$ can be computed using (58). In $\bar{N}_2 \setminus N_1$ we need to take into account the effect of the cutoff function $\widetilde{\chi}$ and we have

$$\|\nabla^2 \widetilde{H}_{h,k}\|_{\mathscr{C}^{2,\alpha}(\bar{N}_2)} \leq c_\kappa \, R_\varepsilon^{2-2m}. \tag{59}$$

Observe that we can write

$$\mathbb{L}_{g_0} \widetilde{H}_{h,k} = \left(\mathbb{L}_{g_0} - \Delta^2\right) \widetilde{H}_{h,k},$$

in $\bar{N}_{R_\varepsilon/a} \setminus N_2$.

The expansions given in (10) and (11) show that the coefficients of Ric_{g_0} belong to $\mathscr{C}^{0,\alpha}_{-2m}(N)$, moreover, they also show that $\mathbb{L}_{g_0} - \Delta^2$ is a fourth order differential operator such that the coefficients of ∇^{j+2} belong to $\mathscr{C}^{0,\alpha}_{j-2m}(N)$, for $j = 0, 1, 2$. This, together with (58) and (59), implies that

$$\|\mathbb{L}_{g_0} \widetilde{H}_{h,k}\|_{\mathscr{C}^{0,\alpha}_{\delta-4}(N_{R_\varepsilon/a})} \leq c \, R_\varepsilon^{2-2m}.$$

Next, we use the structure of Q_{g_0} as described above together with (58) and (59) to estimate

$$\|a^2 \, Q_{g_0}(a^{-2} \, \nabla^2 \, \widetilde{H}_{h,k})\|_{\mathscr{C}^{0,\alpha}_{\delta-4}(\bar{N}_{R_\varepsilon/a})} \leq c_\kappa \, R_\varepsilon^{4-4m}.$$

Finally, we estimate

$$\|\varepsilon^2 \, a^4 \, \nu\|_{\mathscr{C}^{0,\alpha}_{\delta-4}(\bar{N}_{R_\varepsilon/a})} \leq c \, R_\varepsilon^{3-2m-\delta},$$

for some constant $c > 0$ which does not depend on ε. Here we have used the fact that

$$\varepsilon^{-2} = R_\varepsilon^{2m+1}.$$

This completes the proof of the first estimate. We leave the derivation of the other estimates to the reader. □

Reducing $\varepsilon_\kappa > 0$ if necessary, we can assume that,

$$c_\kappa \, R_\varepsilon^{3-2m-\delta} \leq \tfrac{1}{2}, \tag{60}$$

for all $\varepsilon \in (0, \varepsilon_\kappa)$. Then, the estimates (55) and (56) in the above lemma are enough to show that

$$f \longmapsto \widetilde{\mathscr{N}}(\varepsilon, a, h, k, \nu; f),$$

is a contraction from

$$\{f \in \widetilde{\mathscr{F}} \quad : \quad \|f\|_{\widetilde{\mathscr{F}}} \leq 2 \, c \, R_\varepsilon^{3-2m-\delta}\},$$

into itself and hence has a unique fixed point $\tilde{f}_{\varepsilon,a,h,k,\nu}$ in this set. This fixed point is a solution of (51) and hence provides a Kähler metric $g_{\varepsilon,a,h,k,\nu}$ on $\bar{N}_{R_\varepsilon/a}$ which is associated to the Kähler form

$$\omega_{\varepsilon,a,h,k,\nu} = a^2\,\eta_0 + i\,\partial\bar{\partial}\left(H_{h,k} + \tilde{f}_{\varepsilon,a,h,k}\right).$$

(Reducing ε_κ if necessary, it is easy to check that $g_{\varepsilon,a,h,k,\nu}$ is indeed positive). The scalar curvature of this metric is constant equal to

$$s(g_{\varepsilon,a,h,k,\nu}) = \varepsilon^2\,\nu. \tag{61}$$

The Kähler potential $\tilde{\zeta}_{\varepsilon,a,h,k,\nu}$ which appears in the statement of Proposition 5.2 is then defined as follows: In dimension $m \geq 3$, we consider the function

$$\tilde{\zeta}_{\varepsilon,a,h,k} := a^2\,E_m + \tilde{H}_{h,k} + \tilde{f}_{\varepsilon,a,h,k,\nu},$$

which is defined in $\bar{C}_{R_\varepsilon/2a} \setminus B_{R_\varepsilon/a}$ and where E_m is the potential defined in (9). When $m = 2$ minor modifications are needed and we define

$$\tilde{\zeta}_{\varepsilon,a,h,k} := a^2\,E_m + \tilde{H}_{h,k} + \tilde{f}_{\varepsilon,a,h,k,\nu} - a^2\,\log(R_\varepsilon/a).$$

Again, observe that, locally, adding a constant to the Kähler potential does not alter the Kähler metric.

The estimate (29) follows at once from (10), (11) together with (55) that can be used to estimate $\tilde{f}_{\varepsilon,a,h,k,\nu}$. The other estimate follows from Lemma 5.3 when $a = a'$. While, when $a \neq a'$ one can check from the construction of $\tilde{f}_{\varepsilon,a,h,k,\nu}$ that

$$\|\tilde{f}_{\varepsilon,a,h,k,\nu}(R_\varepsilon\,\cdot/a) - \tilde{f}_{\varepsilon,a',h,k,\nu}(R_\varepsilon\,\cdot/a')\|_{\mathscr{C}^{4,\alpha}(\bar{B}_1\setminus B_{1/2})} \leq c_\kappa\,R_\varepsilon^{3-2m}\,|a-a'|.$$

The proof of this estimate can be obtained by consideration of a family of diffeomorphisms $\vartheta_a : N \longrightarrow N$ depending smoothly on a such that

$$\vartheta_a(u) = a\,u,$$

in $N \setminus N_2$. We then write

$$\tilde{f}_{\varepsilon,a,h,k,\nu} = \tilde{f}'_{\varepsilon,a,h,k,\nu} \circ \vartheta_a,$$

so that $\tilde{f}'_{\varepsilon,a,h,k,\nu}$ is a solution of

$$\tilde{f}'_{\varepsilon,a,h,k,\nu} = \tilde{\mathscr{N}}(\varepsilon, a, h, k, \nu;\, \tilde{f}'_{\varepsilon,a,h,k,\nu} \circ \vartheta_a) \circ \vartheta_a^{-1}.$$

Using this, it is now a simple exercise to estimate $\tilde{f}'_{\varepsilon,a,h,k,\nu} - \tilde{f}'_{\varepsilon,a',h,k,\nu}$, and we leave the details to the reader. This completes the proof of the result. □

5.3. *Cauchy data matching*: *The proof of Theorem* 1.3. Building on the analysis of the previous sections we complete the proof of Theorem 1.3. We will explain the modifications required to complete the proof of Theorem 1.4 in the next subsection.

Granted the results of Proposition 5.1, we choose boundary data

$$\boldsymbol{h} := (h_1, \dots, h_n), \qquad \boldsymbol{k} := (k_1, \dots, k_n),$$

whose components satisfy (26) and we assume that all the components of $\boldsymbol{k}$ have mean zero. Applying Proposition 5.1, we obtain a Kähler metric $g_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ defined on $\bar{M}_{r_\varepsilon}$.

Granted the result of Proposition 5.2, we choose boundary data

$$\tilde{\boldsymbol{h}} := (\tilde{h}_1, \dots, \tilde{h}_n), \qquad \tilde{\boldsymbol{k}} := (\tilde{k}_1, \dots, \tilde{k}_n),$$

whose components satisfy (47) and real parameters $\tilde{\boldsymbol{a}} := (\tilde{a}_1, \dots, \tilde{a}_n)$ whose components satisfy (43). Applying Proposition 5.2 for each $j = 1, \dots, n$, we obtain a Kähler metric $\varepsilon^2 \, g_{\varepsilon,\hat{a}_j,h_j,k_j,\nu}$ defined on $\bar{N}_{R_\varepsilon/\hat{a}_j}$, with

$$\hat{a}_j := \tilde{a}_j^{\frac{1}{2(m-1)}} \qquad \text{and} \qquad \nu := \boldsymbol{s}(g_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}).$$

Observe that we have scaled the metric by a factor ε^2 !

We are now in a position to describe the generalized connected-sum construction. The manifold

$$M_\varepsilon := M \sqcup_{p_1,\varepsilon} N_1 \sqcup_{p_2,\varepsilon} \cdots \sqcup_{p_n,\varepsilon} N_n,$$

is obtained by connecting $\bar{M}_{r_\varepsilon}$ with the truncated spaces $N_{R_\varepsilon/\hat{a}_1}, \dots, N_{R_\varepsilon/\hat{a}_n}$. The identification of the boundary $\partial B_{j,r_\varepsilon}$ of $\bar{M}_{r_\varepsilon}$ with the boundary $\partial N_{R_\varepsilon/\hat{a}_j}$ of $N_{R_\varepsilon/\hat{a}_j}$ is performed using the change of variables

$$(z^1, \dots, z^m) = \varepsilon \, \hat{a}_j \, (u^1, \dots, u^m),$$

where $(z^1, \dots, z^m)$ are the coordinates in B_{j,r_0} and $(u^1, \dots, u^m)$ are the coordinates in C_1.

To keep notation short, we set

$$\psi_j^o := \zeta^{(j)}_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}(r_\varepsilon \,\cdot) \in \mathscr{C}^{4,\alpha}(\bar{B}_2 \setminus B_1),$$
$$\psi_j^i := \varepsilon^2 \, \tilde{\zeta}^{(j)}_{\varepsilon,\hat{a}_j,h_j,k_j,\nu}(R_\varepsilon \cdot /\hat{a}_j) \in \mathscr{C}^{4,\alpha}(\bar{B}_1 \setminus B_{1/2}).$$

The problem is now to determine the boundary data and parameters in such a way that, the metric $g_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ on $\bar{M}_{r_\varepsilon}$ and, for each $j = 1, \dots, n$, the metric $\varepsilon^2 \, g_{\varepsilon,\hat{a}_j,h_j,k_j,\nu}$ on $\bar{N}_{R_\varepsilon/\hat{a}_j}$ agree on the boundaries we have identified. This amounts to finding the boundary data and parameters of the construction so that the functions ψ_j^o and ψ_j^i have their partial derivatives up to order 3 which coincide on ∂B_1.

It turns out that it is enough to solve the following system of equations

$$\psi_j^o = \psi_j^i, \quad \partial_r \, \psi_j^o = \partial_r \, \psi_j^i, \quad \Delta \, \psi_j^o = \Delta \, \psi_j^i, \quad \partial_r \, \Delta \, \psi_j^o = \partial_r \, \Delta \, \psi_j^i, \tag{62}$$

on ∂B_1 where $r = |v|$ and $v = (v^1, \dots, v^m)$ are coordinates in $\mathbb{C}^m$. Indeed, let us assume that we have already solved this problem. The first identity in (62) implies that ψ_j^o and ψ_j^i as well as all their k-th order partial derivatives with respect to any vector field tangent to ∂B_1, with $k \leq 4$, agree on ∂B_1. The second identity in (62) shows that $\partial_r \psi_j^o$ and $\partial_r \psi_j^i$ as well as all their k-th order partial derivatives with respect to any vector field tangent to ∂B_1, with $k \leq 3$, agree on ∂B_1. Using the decomposition of the Laplacian in polar coordinates, it is easy to check that the third identity implies that $\partial_r^2 \psi_j^o$ and $\partial_r^2 \psi_j^i$ as well as all their k-th order partial derivatives with respect to any vector field tangent to ∂B_1, with $k \leq 2$, agree on ∂B_1. And finally, the last identity in (62) implies that $\partial_r^3 \psi_j^o$ and $\partial_r^3 \psi_j^i$ as well as all their first order partial derivative with respect to any vector field tangent to ∂B_1, agree on ∂B_1. Therefore, any k-th order partial derivatives of the functions ψ_j^o and ψ_j^i, with $k \leq 3$, coincide on ∂B_1.

Moreover, by construction, the Kähler form

$$i\, \partial \bar{\partial} \left(\tfrac{1}{2}\, |v|^2 + \psi_j^o\right),$$

defined in $\bar{B}_2 \setminus B_1$ and the Kähler form

$$i\, \partial \bar{\partial} \left(\tfrac{1}{2}\, |v|^2 + \psi_j^i\right),$$

defined in $\bar{B}_1 \setminus B_{1/2}$, both have the same constant scalar curvature equal to $\boldsymbol{s}(\omega_{\varepsilon,\boldsymbol{h},\boldsymbol{k}})$. We conclude that the function ψ defined by $\psi := \psi_j^o$ in $\bar{B}_2 \setminus B_1$ and $\psi := \psi_j^i$ in $\bar{B}_1 \setminus B_{1/2}$ is $\mathscr{C}^{3,\alpha}$ (and in fact $\mathscr{C}^{3,1}$) in $\bar{B}_2 \setminus B_{1/2}$ and is a (weak) solution of the nonlinear elliptic partial differential equation

$$\boldsymbol{s}\, \left(i\, \partial \bar{\partial} (\tfrac{1}{2}\, |v|^2 + \psi)\right) = \boldsymbol{s}(\omega_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}) = \text{constant}.$$

It then follows from elliptic regularity theory together with a bootstrap argument (see for example [14] or [3]) that the function ψ is in fact smooth. Hence, by gluing the Kähler metrics $\omega_{\varepsilon,\boldsymbol{h},\boldsymbol{k}}$ defined on $\bar{M}_{r_\varepsilon}$ with the metrics $\varepsilon^2\, \eta_{\varepsilon,\hat{a}_j,\tilde{h}_j,\tilde{k}_j,\nu}$ defined on $N_{R_\varepsilon/\hat{a}_j}$, we produce a Kähler metric on M_ε that has constant scalar curvature. This will complete the proof of Theorem 1.3.

It remains to explain how to find the boundary data

$$\boldsymbol{h} = (h_1, \dots, h_n), \;\; \boldsymbol{k} = (k_1, \dots, k_n), \;\; \tilde{\boldsymbol{h}} = (\tilde{h}_1, \dots, \tilde{h}_n) \;\text{ and }\; \tilde{\boldsymbol{k}} = (\tilde{k}_1, \dots, \tilde{k}_n),$$

as well as the parameters $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \dots, \tilde{a}_n)$.

It follows from the result of Propositions 5.1 and 5.2 that, for each $j = 1, \dots, n$ the following expansion holds in $\bar{B}_2 \setminus B_1$:

$$\psi_j^o = -a_j\, \varepsilon^{2m-2}\, r_\varepsilon^{4-2m}\, G + H^o_{h_j,k_j} + \mathbb{O}_{\mathscr{C}^{4,\alpha}}(r_\varepsilon^4).$$

Similarly, it follows from the result of Proposition 5.2 together with (10) and (11) that, for each $j = 1, \dots, n$, the following expansion holds in $\bar B_1 \setminus B_{1/2}$

$$\psi_j^i = -\tilde a_j \, \varepsilon^{2m-2} \, r_\varepsilon^{4-2m} \, G + \varepsilon^2 \, H^i_{\tilde h, \tilde k} + \mathcal{O}_{\mathcal{C}^{4,\alpha}}(r_\varepsilon^4).$$

The functions $\mathcal{O}_{\mathcal{C}^{4,\alpha}}(r_\varepsilon^4)$ depend nonlinearly on $\boldsymbol{h}, \boldsymbol{k}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{k}}$ and $\tilde{\boldsymbol{a}}$, but they are bounded by a constant (independent of κ) times $r_\varepsilon^4 = \varepsilon^2 \, R_\varepsilon^{3-2m}$ in $\mathcal{C}^{4,\alpha}(\bar B_2 - B_1)$ or in $\mathcal{C}^{4,\alpha}(\bar B_1 - B_{1/2})$ topology. We shall be more specific about this once some changes of variables are performed.

We change the boundary data functions h_j and k_j into h_j' and k_j' defined by

$$\begin{aligned} h_j' &:= (\tilde a_j - a_j) \, r_\varepsilon^{4-2m} \, \varepsilon^{2m-2} + h_j, \\ k_j' &:= 4\,(m-2)(a_j - \tilde a_j) \, \varepsilon^{2m-2} \, r_\varepsilon^{4-2m} + k_j \end{aligned}$$

when $m \geq 3$ and

$$h_j' := h_j, \qquad k_j' := 2\,(\tilde a_j - a_j)\,\varepsilon^2 + k_j$$

when $m = 2$. Recall that the functions k_j are assumed to satisfy (27) while the functions k_j' do not satisfy such a constraint anymore. The role of the scalar $\tilde a_j - a_j$ is precisely to recover this lost degree of freedom in the assignment of the boundary data.

Finally, we set

$$\tilde h_j' := \varepsilon^2 \tilde h_j, \qquad \tilde k_j' := \varepsilon^2 \, \tilde k_j$$

If k is a constant function of ∂B_1, we extend the definition of $H^o_{h,k}$ by setting

$$H^o_{0,k}(z) := \tfrac{k}{4(m-2)} \, k \, (|z|^{2-2m} - |z|^{4-2m}),$$

when $m \geq 3$ and by

$$H^o_{0,k}(z) = \tfrac{1}{2} \, k \, \log|z|,$$

when $m = 2$.

With these new variables, the expansions for both ψ_j^o and $\varepsilon^2 \, \psi_j^i$ can now be written as

$$\begin{aligned} \psi_j^o &= -\tilde a_j \, r_\varepsilon^{4-2m} \, \varepsilon^{2m-2} \, G + H^o_{h_j', k_j'} + \mathcal{O}_{\mathcal{C}^{4,\alpha}}(r_\varepsilon^4), \\ \psi_j^i &= -\tilde a_j \, r_\varepsilon^{4-2m} \, \varepsilon^{2m-2} \, G + H^i_{\tilde h_j', \tilde k_j'} + \mathcal{O}_{\mathcal{C}^{4,\alpha}}(r_\varepsilon^4). \end{aligned}$$

As usual, the boundary data $\boldsymbol{h}' := (h_1', \dots, h_m')$, $\tilde{\boldsymbol{h}}' := (\tilde h_1', \dots, \tilde h_m')$ are assumed to be bounded by a constant κ times r_ε^4 in $\mathcal{C}^{4,\alpha}(\partial B_1)$ and the boundary data $\boldsymbol{k}' := (k_1', \dots, k_m')$, $\tilde{\boldsymbol{k}}' := (\tilde k_1', \dots, \tilde k_m')$ are assumed to be bounded by a constant κ times r_ε^4 in $\mathcal{C}^{2,\alpha}(\partial B_1)$. The terms $\mathcal{O}_{\mathcal{C}^{4,\alpha}}(r_\varepsilon^4)$ depend nonlinearly on $\boldsymbol{h}', \tilde{\boldsymbol{h}}', \boldsymbol{k}', \tilde{\boldsymbol{k}}'$ and are bounded, in $\mathcal{C}^{4,\alpha}(\bar B_2 \setminus B_1)$ or in $\mathcal{C}^{4,\alpha}(\bar B_1 \setminus B_{1/2})$ topology, by a constant (independent of κ) times r_ε^4, provided ε is chosen small enough. We can make this statement more precise by saying that

$$\|\psi_j^o + \tilde{a}_j\, r_\varepsilon^{4-2m}\, G - H^o_{h'_j,k'_j}\|_{\mathscr{C}^{4,\alpha}(\bar{B}_2\setminus B_1)} \leq c\, r_\varepsilon^4, \tag{63}$$

and also that

$$\|\psi_j^i + +\tilde{a}_j\, r_\varepsilon^{4-2m}\, G - \varepsilon^2\, H^i_{\tilde{h}'_j,\tilde{k}'_j}\|_{\mathscr{C}^{4,\alpha}(\bar{B}_1\setminus B_{1/2})} \leq c\, \varepsilon^2\, R_\varepsilon^{3-2m} = c\, r_\varepsilon^4, \tag{64}$$

for some constant $c > 0$ that does not depend on κ, provided ε is chosen small enough, say $\varepsilon \in (0, \varepsilon_\kappa)$. These two estimates follow at once from the estimates in Proposition 5.1, Proposition 5.2 and also from the choice of r_ε.

The system (62) we have to solve can now be written as follows: For all $j = 1, \ldots, n$

$$\left\{\begin{array}{rcll} H^o_{h'_j,k'_j} & = & H^i_{\tilde{h}'_j,\tilde{k}'_j} & +\ \mathscr{O}_{\mathscr{C}^{4,\alpha}(\partial B_1)}(r_\varepsilon^4) \\ \partial_r H^o_{h'_j,k'_j} & = & \partial_r H^i_{\tilde{h}'_j,\tilde{k}'_j} & +\ \mathscr{O}_{\mathscr{C}^{3,\alpha}(\partial B_1)}(r_\varepsilon^4) \\ \Delta H^o_{h'_j,k'_j} & = & \Delta H^i_{\tilde{h}'_j,\tilde{k}'_j} & +\ \mathscr{O}_{\mathscr{C}^{2,\alpha}(\partial B_1)}(r_\varepsilon^4) \\ \partial_r \Delta H^o_{h'_j,k'_j} & = & \partial_r \Delta H^i_{\tilde{h}'_j,\tilde{k}'_j} & +\ \mathscr{O}_{\mathscr{C}^{1,\alpha}(\partial B_1)}(r_\varepsilon^4), \end{array}\right. \tag{65}$$

on ∂B_1 where $\mathscr{O}_{\mathscr{C}^{\ell,\alpha}(\partial B_1)}(r_\varepsilon^4)$ are functions that depend nonlinearly on $\boldsymbol{h}', \boldsymbol{k}', \tilde{\boldsymbol{h}}'$ and $\tilde{\boldsymbol{k}}'$ and that are bounded in $\mathscr{C}^{\ell,\alpha}(\partial B_1)$ topology by a constant (independent of κ) times r_ε^4, provided ε is small enough, say $\varepsilon \in (0, \varepsilon_\kappa)$.

By definition of $H^o_{h,k}$ and $H^i_{h,k}$, the first and third equations reduce to

$$\left\{\begin{array}{rcl} h'_j & = & \tilde{h}'_j + \mathscr{O}_{\mathscr{C}^{4,\alpha}(\partial B_1)}(r_\varepsilon^4) \\ k'_j & = & \tilde{k}'_j + \mathscr{O}_{\mathscr{C}^{2,\alpha}(\partial B_1)}(r_\varepsilon^4). \end{array}\right. \tag{66}$$

Inserting these into the second and third sets of equations and using the linearity of the mappings $(h, k) \longmapsto H^o_{h,k}$ and $(h, k) \longmapsto H^i_{h,k}$, the second and third equations become

$$\left\{\begin{array}{rcll} \partial_r H^o_{h'_j,k'_j} & = & \partial_r H^i_{h'_j,k'_j} & +\ \mathscr{O}_{\mathscr{C}^{3,\alpha}(\partial B_1)}(r_\varepsilon^4) \\ \partial_r \Delta H^o_{h'_j,k'_j} & = & \partial_r \Delta H^i_{h'_j,k'_j} & +\ \mathscr{O}_{\mathscr{C}^{1,\alpha}(\partial B_1)}(r_\varepsilon^4), \end{array}\right. \tag{67}$$

for all $j = 1, \ldots, n$. We now make use of the following result whose proof can be found in [2]:

LEMMA 5.4. *The mapping*

$$\begin{array}{rccc} \mathscr{P}: & \mathscr{C}^{4,\alpha}(\partial B_1) \times \mathscr{C}^{2,\alpha}(\partial B_1) & \longrightarrow & \mathscr{C}^{3,\alpha}(\partial B_1) \times \mathscr{C}^{1,\alpha}(\partial B_1) \\ & (h, k) & \longmapsto & (\partial_r\, (H^i_{h,k} - H^o_{h,k}), \partial_r\, \Delta\, (H^i_{h,k} - H^o_{h,k})), \end{array}$$

is an isomorphism.

By Lemma 5.4, (67) reduces to

$$
\left\{
\begin{array}{rcl}
h'_j &=& \mathbb{O}_{\mathscr{C}^{4,\alpha}(\partial B_1)}(r_\varepsilon^4) \\
k'_j &=& \mathbb{O}_{\mathscr{C}^{2,\alpha}(\partial B_1)}(r_\varepsilon^4),
\end{array}
\right.
\tag{68}
$$

for all $j = 1, \dots, n$. This, together with (66), yields a fixed point problem that can be written as

$$(\boldsymbol{h}', \tilde{\boldsymbol{h}}', \boldsymbol{k}', \tilde{\boldsymbol{k}}') = S_\varepsilon(\boldsymbol{h}', \tilde{\boldsymbol{h}}', \boldsymbol{k}', \tilde{\boldsymbol{k}}),$$

and we know from (63) and (64) that the nonlinear operator S_ε satisfies

$$\| S_\varepsilon(\boldsymbol{h}', \tilde{\boldsymbol{h}}', \boldsymbol{k}', \tilde{\boldsymbol{k}}')\|_{(\mathscr{C}^{4,\alpha})^{2n} \times (\mathscr{C}^{2,\alpha})^{2n}} \leq c_0 \, r_\varepsilon^4,$$

for some constant $c_0 > 0$ that does not depend on κ, provided $\varepsilon \in (0, \varepsilon_\kappa)$. We finally choose

$$\kappa = 2\, c_0,$$

and $\varepsilon \in (0, \varepsilon_\kappa)$. We have therefore proved that S_ε is a map from

$$A_\varepsilon := \big\{ (\boldsymbol{h}', \tilde{\boldsymbol{h}}', \boldsymbol{k}', \tilde{\boldsymbol{k}}') \in (\mathscr{C}^{4,\alpha})^{2n} \times (\mathscr{C}^{2,\alpha})^{2n} \; : \; \|(\boldsymbol{h}', \tilde{\boldsymbol{h}}', \boldsymbol{k}', \tilde{\boldsymbol{k}}')\|_{(\mathscr{C}^{4,\alpha})^{2n} \times (\mathscr{C}^{2,\alpha})^{2n}} \leq \kappa \, r_\varepsilon^4 \big\}$$

into itself. It follows from (30), (32) and (50) that, by reducing ε_κ if this is necessary, S_ε becomes a contraction mapping from A_ε into itself for all $\varepsilon \in (0, \varepsilon_\kappa)$. Therefore, S_ε has a unique fixed point in this set. This completes the proof of the existence of a solution to (62). The proof of the existence, for all ε small enough, of a constant scalar curvature Kähler metric g_ε defined on M_ε is therefore complete. Observe that the scalar curvature of g and g_ε are close since the estimate

$$|\boldsymbol{s}(\omega_\varepsilon) - \boldsymbol{s}(\omega)| \leq c \, \varepsilon^{2m-2},$$

follows directly from (31). We also know that the coefficients $\tilde{a}_j$ are close to the coefficients a_j since they satisfy $r_\varepsilon^{4-2m} \, \varepsilon^{2m-2} \, |\tilde{a}_j - a_j| \leq c \, r_\varepsilon^4$ and this implies the estimate

$$|\tilde{a}_j - a_j| \leq c \, r_\varepsilon^{2m} \, \varepsilon^{2-2m} = c \, \varepsilon^{\frac{2}{2m+1}},$$

which is precisely the last estimate which appears in the statement of Theorem 1.3.

Remark 5.1. Observe that our construction of the Kähler form ω_ε is obtained through the application of successive fixed point theorems for contraction mapping. Therefore, reducing the range in which ε can be chosen if this is necessary, we can assume that ω_ε depends continuously on the parameters of the construction (such as the Kähler class we start with, the parameter ε, the points we blow up, the weights $a_j, \dots$).

5.4. *The equivariant setting.* It should be clear from the construction that everything we have done works in the equivariant setting provided the isometries of the group Γ extend to isometries on the different summands of the manifold M_ε when they are endowed either with the metric g or one of the Burns-Simanca metrics $\varepsilon^2 g_0$.

Care must then be taken to find the best coordinates to be used to construct a well adapted Burns-Simanca's metric for the action of Γ. Since $U(m)$ is the isometry group of any of Burns-Simanca metric, it is enough to impose the condition that $\Gamma \subset U(m)$ on the neighborhood of the point $p \in M$ that will be blown up. This amounts to linearizing, on a small neighborhood of p, the action of Γ. The following result is borrowed from [5]:

PROPOSITION 5.3. *Let D be a domain of a complex manifold and $\Gamma \subset \mathrm{Aut}\,(D, J)$ be a compact subgroup with a fixed point $p \in D$. In a neighborhood of p, there exist complex coordinates centered at p such that in these coordinates the action of Γ is given by linear transformations.*

Proof. Let $A_a \colon T_p D \longrightarrow T_p D$ be the infinitesimal action at p induced by the action of $a \in \Gamma$, and let (V, w) be a coordinate system centered at p with $w \colon V \longrightarrow T_p D$ the associated coordinate functions. Without loss of generality we can assume that V is Γ-invariant and that $D\, w|_p = \mathrm{Id}$.

Let $\Psi \colon V \times \Gamma \longrightarrow T_p D$ be the map defined by

$$\Psi(q, a) := A_a(w(a^{-1}(q))),$$

and $z \colon V \longrightarrow T_p D$ the map given by

$$z(q) := \frac{1}{|\Gamma|} \int_\Gamma \Psi(q, a')\, d\mu(a'),$$

where $d\mu$ denotes Haar's measure on Γ.

Clearly, the map z is Γ-equivariant, in the sense that $z(a(q)) = A_a(z(q))$. Moreover, $Dz|_p = \mathrm{Id}$; hence z defines a coordinate system in which the action of Γ is linear. □

We will refer to these coordinates as Γ-*linear coordinates*. In order to use all the analytical results proved up to now, we have to find Γ-linear coordinates which are also normal about any fixed point p of Γ. To wit:

PROPOSITION 5.4. *Assume that $\Gamma \subset \mathrm{Isom}\,(M, g)$ is compact and p is a fixed point of Γ. Then there exist $(z^1, \dots, z^m)$, Γ-linear coordinates centered at p such that*

$$\omega = i\, \partial\bar\partial(\tfrac{1}{2}\, |z|^2 + \zeta),$$

where the function ζ is Γ invariant and $\zeta = \mathcal{C}^{3,\alpha}_4(B^*_{r_0})$.

Proof. Let us first consider any set of normal coordinates $w=(w^1,\dots,w^m)$ centered at p (see for example [10, 107–108]). We then use the above averaging construction to construct, starting from this system of coordinates, a new system of coordinates $z=(z^1,\dots,z^m)$ that are Γ-linear. We claim that $(z^1,\dots,z^m)$ are still normal coordinates centered at p. Indeed, since each $a\in\Gamma$ is an isometry for the Kähler metric g and since p is invariant under the action of Γ, the distance from p to $a(q)$ is equal to the distance from p to q. Moreover, since z is now assumed to be a system of normal coordinates and since A_a is unitary (remember that Γ is made of isometries), it is immediate to check that

$$|\Psi(q,a)-w(q)|=\mathcal{O}(|w(q)|^3).$$

Averaging over G we find that

$$|z(q)-w(q)|=\mathcal{O}(|w(q)|^3).$$

This implies that in the expansion of the z-coordinates with respect to the w-coordinates, no quadratic terms appear. This is precisely the condition that ensures that the coordinates z are normal coordinates (the linear part in the change of coordinates is controlled by the conditions $Dw_p=Dz_p=\mathrm{Id}$).

Applying the $\partial\overline{\partial}$-Lemma we find that there exists a Kähler potential such that

$$\omega=i\,\partial\overline{\partial}(\tfrac{1}{2}\,|z|^2+f),$$

where $f=\mathcal{O}(|z|^4)$. To obtain a Γ-invariant Kähler potential we average f over Γ and define

$$\zeta(z):=\frac{1}{|\Gamma|}\int_\Gamma f(A_a z)\,d\mu(a).$$

Clearly $\zeta=\mathcal{O}(|z|^4)$ and, the Kähler form ω being Γ-invariant, we have

$$\omega=i\,\partial\overline{\partial}(\tfrac{1}{2}\,|z|^2+\zeta),$$

and this time the function ζ is Γ-invariant. □

In our construction of K invariant constant scalar curvature Kähler metrics on blow ups, this proposition will be used in the following way : we apply the previous result to the subgroup Γ of elements of K which fix a point $p_j\in M$ and we obtain normal coordinates, in D a neighborhood of p_j, for which the action of Γ is linear. We then consider Burns-Simanca's metric constructed using Γ-linear normal coordinates. The linear and nonlinear analysis applies *verbatim* in this equivariant setting.

6. **Understanding the constraints**

In this section, we give the proof of Lemmas 1.1 and 1.2.

Recall that $\mathfrak{h}$ denotes the (real) vector space of Killing vector fields with zeros. Given $p_1, \dots, p_n \in M$, we define the mapping

$$\begin{array}{rcl} \Lambda_{p_1,\dots,p_n} : \ \mathfrak{h} & \longrightarrow & \mathbb{R}^n \\ X & \longmapsto & (\langle \xi(p_1), X\rangle, \dots, \langle \xi(p_n), X\rangle). \end{array}$$

We start with the proof of the:

LEMMA 6.1. *Assume that* $n \geq \dim \mathfrak{h}$. *t Then the set of points* $(p_1, \dots, p_n) \in M^n_\triangle$ *such that*

$$\xi(p_1), \dots, \xi(p_n) \qquad \textit{span} \qquad \mathfrak{h}^*,$$

is an open and dense subset of $M^n_\triangle$.

Proof. Observe that it is sufficient to check that the result is true when $n = d := \dim \mathfrak{h}$, since the result will remain true if we increase the number of points ! When $n = d$, the property that $\xi(p_1), \dots, \xi(p_d)$ span $\mathfrak{h}^*$ is equivalent to the fact that $\Lambda_{p_1,\dots,p_d}$ has full rank (equal to d).

We prove by induction on $k = 1, \dots, d$ that, if E is a k-dimensional subspace of $\mathfrak{h}$, the set of $(p_1, \dots, p_k) \in M^k_\triangle$ for which

$$\det (\Lambda_{p_1,\dots,p_k}|_E) \neq 0,$$

is open and dense in $M^k_\triangle$. When $k = 1$, $E = \mathrm{Span}\{X\}$ where $X \in \mathfrak{h} - \{0\}$, the result is straightforward since the condition reduces to the fact that p is not in the zero set of the function $f := \langle \xi, X\rangle$. The function f being a solution of $\mathbb{L}_g\, f = 0$, we can write

$$-\bar\partial f = \tfrac{1}{2}\, \omega(\Xi, -),$$

where Ξ is a holomorphic vector field. Since Ξ is holomorphic, f cannot be constant on some open subset unless it it identically equal to 0. Therefore, the zero set of f is closed and has empty interior, and hence its complement is open and dense in M. Now, let us assume that the result is true for $k-1$ and let $X_1, \dots, X_k$ be a basis of E. Using the expansion of the determinant of a matrix with respect to the last column, we write the function

$$(p_1, \dots, p_k) \longmapsto \det (\Lambda_{p_1,\dots,p_k}|_E),$$

as

$$\det (\Lambda_{p_1,\dots,p_k}|_E) = m_1\, \langle \xi(p_k), X_1\rangle + \cdots + m_k\, \langle \xi(p_k), X_k\rangle,$$

where we have set

$$m_j := (-1)^{k+j} \det (\Lambda_{p_1,\dots,p_{k-1}}|_{E_j}),$$

with

$$E_j := \mathrm{Span}\{X_a \quad : \quad a = 1, \dots, k \quad a \neq j\} \subset E.$$

By assumption, the set of $(p_1, \ldots, p_{k-1}) \in M_\triangle^{k-1}$ for which

$$(\det(\Lambda_{p_1,\ldots,p_{k-1}}|_{E_1}), \ldots, \det(\Lambda_{p_1,\ldots,p_{k-1}}|_{E_k})) \neq 0,$$

is open and dense in $M_\triangle^{k-1}$. Let $(p_1, \ldots, p_{k-1}) \in M_\triangle^{k-1}$ be such a point. The vector fields $X_1, \ldots, X_k$ being linearly independent, the function

$$f'(p) := \det(\Lambda_{p_1,\ldots,p_{k-1},p}\,|_{E_k}) = \langle \xi(p), m_1 X_1 + \cdots + m_k X_k \rangle,$$

is not identically equal to 0. Again, arguing as above we find that the zero set of f' (which belongs to the kernel of the operator $\mathbb{L}_g$) has empty interior (and is closed). Therefore its complement is open and dense. This completes the proof of the result. □

The second condition for our construction to work asks for the existence $p_1, \ldots, p_n \in M$, with $n \geq \dim \mathfrak{h} + 1$, for which it is possible to find $a_1, \ldots, a_n > 0$ satisfying

$$\sum_{j=1}^{n} a_j \, \xi(p_j) = 0.$$

This amounts to asking for the existence of points $p_1, \ldots, p_n \in M$ such that the image of $\Lambda_{p_1,\ldots,p_n}$ is included in a hyperplane of $\mathbb{R}^n$ whose normal belongs to the positive cone in $\mathbb{R}^n$. This later condition is equivalent to the requirement that all nonzero elements of $\operatorname{Im} \Lambda_{p_1,\ldots,p_n}$ have entries which change sign. This discussion can be summarized as follows:

LEMMA 6.2. *The n-tuple $(p_1, \ldots, p_n) \in M_\triangle^n$ satisfies condition* (2) *for some $a_1, \ldots, a_n > 0$ if and only if, for all $X \in \mathfrak{h}$, the entries of $\Lambda_{p_1,\ldots,p_n}(X)$ do not all have the same sign.*

Observe that it is enough to check this last property for all X in the unit sphere of $\mathfrak{h}$. Moreover, it should now be clear that once we have found $p_1, \ldots, p_n$ satisfying this condition, then the condition remains fulfilled after any adjunction of points to this list since the property that the entries of $\Lambda_{p_1,\ldots,p_n}(X)$ do not have the same sign remains true.

As stated in the introduction, except in special cases, we have not been able to find the explicit value of the minimal number of points for which the above condition is satisfied. Nevertheless, we have the general result:

LEMMA 6.3. *There exists $n_g \geq \dim \mathfrak{h} + 1$ and, for all $n \geq n_g$, there exists a nonempty open set $V_n \subset M_\triangle^n$ such that, for all $(p_1, \ldots, p_n) \in V_n$,*

$$\xi(p_1), \ldots, \xi(p_n) \qquad \textit{span} \qquad \mathfrak{h}^*,$$

and there exist $a_1, \dots, a_n > 0$ *such that*

$$\sum_{j=1}^{n} a_j \, \xi(p_j) = 0.$$

Proof. According to Lemma 6.1, for all $n \geq \dim \mathfrak{h}$, there exists an open and dense subset $W_n \subset M^n_\triangle$ such that for all $(p_1, \dots, p_n) \in W_n$, the image of $\mathfrak{h}$ by $\Lambda_{p_1,\dots,p_n}$ is d-dimensional and varies smoothly as the points change in W_n. This already shows that the set of points $(p_1, \dots, p_n) \in M^n_\triangle$ for which conditions (1) and (2) hold is an open (possibly empty !) subset of $M^n_\triangle$.

Pick X in the unit sphere of $\mathfrak{h}$. By construction, the function $p \longmapsto \langle \xi(p), X \rangle$ has mean 0. Therefore it is possible to find $p_X, \tilde{p}_X \in M$ such that

$$\langle \xi(p_X), X \rangle < 0 < \langle \xi(\tilde{p}_X), X \rangle.$$

By continuity, we also have

$$\langle \xi(p_X), X' \rangle < 0 < \langle \xi(\tilde{p}_X), X' \rangle,$$

for all X' in some open neighborhood O_X of X in the unit sphere of $\mathfrak{h}$. As X varies, the sets O_X constitute an open cover of the unit sphere of $\mathfrak{h}$, and by compactness one can extract from this open cover a finite sub-cover $O_{X^{(1)}}, \dots, O_{X^{(\tilde{n})}}$. We set

$$p_j := p_{X^{(j)}} \qquad \text{and} \qquad p_{j+\tilde{n}} := \tilde{p}_{X^{(j)}},$$

for $j = 1, \dots, \tilde{n}$.

Given any X in the unit sphere of $\mathfrak{h}$, it belongs to some $O_{X^{(j)}}$ and hence the j-th and the $(\tilde{n} + j)$-th entries of the vector

$$(\langle \xi(p_1), X \rangle, \dots, \langle \xi(p_{2\tilde{n}}), X \rangle),$$

do not have the same sign. Therefore, we have found $2\tilde{n}$ points satisfying the required condition. Then $n_g \geq \dim \mathfrak{h} + 1$ is defined to be the least number of points for which both conditions are fulfilled. □

7. The case of $\mathbb{P}^m$

A convenient way to study our problems on the blow up at points of projective spaces is to look at the projective space $\mathbb{P}^m$ endowed with a Fubini-Study metric g_{FS} as the quotient of the unit sphere in $\mathbb{C}^{m+1}$ via the standard S^1-action given by the restriction of complex scalar multiplication. We denote by $z = (z^1, \dots, z^{m+1})$ complex coordinates in $\mathbb{C}^{m+1}$.

It is well known that the automorphism group of $\mathbb{P}^m$ is given by the projectivization of $\mathrm{GL}(m + 1, \mathbb{C})$, whose complex dimension is $d = (m + 1)^2 - 1$. We therefore seek for d real-valued functions whose $(1, 0)$-part of the gradient generate the Lie algebra of the automorphism group. This can be done in two equivalent ways: either by explicit computation on the automorphism group, or by relying on the equivalence described in the previous section between this and the study of the kernel of the operator

$$\mathbb{L}_{g_{\rm FS}} = \Delta^2_{g_{\rm FS}} + 2\,\mathrm{Ric}_{g_{\rm FS}} \cdot \nabla^2_{g_{\rm FS}},$$

which, for $\mathbb{P}^m$ with its Fubini-Study metric $g_{\rm FS}$ induced by the Hopf fibration, becomes

$$\mathbb{L}_{g_{\rm FS}} = \Delta_{g_{\rm FS}}(\,\Delta_{g_{\rm FS}} + 4\,(m+1)).$$

Our problem reduces to seeking a basis of functions with mean zero of the eigenspace of the Laplacian $-\Delta_{g_{\rm FS}}$ associated to the eigenvalue $4\,(m+1)$ (i.e. the eigenspace of the Laplacian $-\Delta_{S^{2m+1}}$ that are associated to the eigenvalue $4(m{+}1)$ and are invariant under the S^1 action), and this is clearly given by the $m^2 + 2\,m$ functions

$$f_{ab}(z) := z^a\,\bar{z}^b + z^b\,\bar{z}^a, \qquad \hat{f}_{ab}(z) := i\;(z^a\,\bar{z}^b - z^b\,\bar{z}^a),$$

for $1 \leq a < b \leq m+1$ and

$$\tilde{f}_a(z) := |z^a|^2 - |z^{a+1}|^2,$$

for $a = 1, \ldots, m$. Here local coordinates $z = (z^1, \ldots, z^{m+1})$ are normalized so that $|z| = 1$. Recall that we should add the constant function $f_0 \equiv 1$ to this list of functions in order to have a basis of the kernel of $\mathbb{L}_{g_{\rm FS}}$.

7.1. *Example* 1: *Proof of Proposition* 1.1. We give an upper bound for the number $n_{g_{\rm FS}}$ which corresponds to the least number of points (larger than or equal to $\dim \mathfrak{h} + 1$) for which conditions (1) and (2) are fulfilled. Given $\alpha, \alpha' \in \mathbb{C}$ such that $|\alpha|^2 + |\alpha'|^2 = 1$, we consider the following set of points:

$$\begin{aligned} p_{jj'} &:= [0 : \cdots : 0 : \alpha : 0 : \cdots : 0 : \alpha' : 0 : \cdots : 0], \\ \tilde{p}_{jj'} &:= [0 : \cdots : 0 : \alpha : 0 : \cdots : 0 : \, -\alpha' : 0 : \cdots : 0], \\ \hat{p}_{jj'} &:= [0 : \cdots : 0 : \alpha' : 0 : \cdots : 0 : i\,\alpha : 0 : \cdots : 0], \\ \check{p}_{jj'} &:= [0 : \cdots : 0 : \alpha' : 0 : \cdots : 0 : -i\,\alpha : 0 : \cdots : 0], \end{aligned}$$

where $1 \leq j < j' \leq m+1$ correspond to the indices of the nonzero entries. There are exactly $n := 2\,m\,(m+1)$ such points which can be labeled $q_1, \ldots, q_n$. In order to be able to apply Theorem 1.3, we need to check that the image of $\mathfrak{h}$ by $\Lambda_{q_1,\ldots,q_n}$ is $(m^2 + 2\,m)$-dimensional and is contained in a hyperplane whose normal vector has positive entries. It is easy to check that (1) is fulfilled provided $|\alpha| \neq |\alpha'|$ and $\Re((\alpha\bar{\alpha}')^2) \neq 0$. As far as the second condition is concerned, observe that

$$\langle \sum_{j=1}^{n} \xi(q_j), X \rangle = 0,$$

for all $X \in \mathfrak{h}$ (it enough to check this formula for Killing vector fields associated to the potentials f_{ab}, $\widehat{f}_{ab}$ and $\widetilde{f}_{ab}$ since these span $\mathfrak{h}$; hence one is left to check that $\sum_{j=1}^{n} f(q_j) = 0$ for $f = f_{ab}, \widehat{f}_{ab}$ or $\widetilde{f}_{ab}$). Details are left to the reader. This completes the proof of Proposition 1.1.

Obviously any explicit calculation will be rather troublesome. It is, hence, very convenient (and giving best results) to introduce symmetries acting on the projective space in order to reduce as much as possible the dimension of the space of elements of the kernel of $\mathbb{L}_{g_{\mathrm{FS}}}$ that is invariant under these symmetries.

7.2. *Example* 2: *Proof of Corollary* 1.1. Let us consider the group Γ_1, acting on $(\mathbb{P}^m, g_{\mathrm{FS}})$, which is generated by the transformations

$$[z^1 : \cdots : z^{m+1}] \longmapsto [\pm z^1 : \cdots : \pm z^{m+1}],$$

as well as by the permutations of the affine coordinates

$$[z^1 : \cdots : z^j : \cdots : z^{j'} : \cdots : z^{m+1}] \longmapsto [z^1 : \cdots : z^{j'} : \cdots : z^j : \cdots : z^{m+1}],$$

for all $1 \leq j < j' \leq m+1$. Of course, the action of any element of the group on $\mathbb{C}^{m+1}$ maps the unit sphere into itself. The space of elements of the kernel of $\mathbb{L}_{g_{\mathrm{FS}}}$ which are invariant under the action of the elements of Γ_1 reduces to the constant functions. Applying either Theorem 1.1 (which works in an equivariant setting as well) or Theorem 1.4, we see that the blow up of $\mathbb{P}^m$ at any set of points which is invariant by the action of the group Γ_1 carries constant scalar curvature Kähler metrics.

Since the following set of points

$$p_1 := [1 : 0 : \cdots : 0], \;\; p_2 := [0 : 1 : \cdots : 0], \; \ldots, \; p_{m+1} := [0 : \cdots : 0 : 1],$$

is closed under the action of Γ_1, Theorem 1.4 implies that, for all $\varepsilon > 0$ small enough (say $\varepsilon \in (0, \overline{\varepsilon}_0)$), that the blow up of $\mathbb{P}^m$ at $p_1, \ldots, p_{m+1}$ carries a constant scale curvature Kähler metric g_ε associated to the Kähler form

$$\omega_\varepsilon \in \pi^*[\omega_{\mathrm{FS}}] - \varepsilon^2 \, (a_\varepsilon^{\frac{1}{m-1}} \, \mathrm{PD}[E_1] + \ldots, + a_\varepsilon^{\frac{1}{m-1}} \, \mathrm{PD}[E_{m+1}]).$$

Observe that symmetry implies that the weights associated to the exceptional divisors must all be equal. Also, since we know that $\varepsilon \longmapsto a_\varepsilon$ is a continuous function of ε, the image of $\varepsilon \longmapsto \varepsilon \, a_\varepsilon^{\frac{1}{2(m-1)}}$ contains an interval of the form $(0, \varepsilon_0)$. This completes the proof of Corollary 1.1 for this special set of linearly independent points.

Now, if $q_1, \ldots, q_{m+1} \in \mathbb{P}^m$ is *any* set of linearly independent points, we can consider an automorphism of the projective space ψ such that

$$\psi(p_j) = q_j,$$

for all $j = 1, \dots, n$. The result we have already obtained shows that one can find constant scalar curvature Kähler metrics on the blow up of $\mathbb{P}^m$ at $q_1, \dots, q_n$ but this time the metric will be close to $\psi^* g_{\mathrm{FS}}$ away from the blow up points. Corollary 1.1 then follows from the fact that $[\psi^* g_{\mathrm{FS}}]$ is independent of ψ and of the choice of the Fubini-Study metric.

Observe that $\varepsilon^2 \, a_\varepsilon^{\frac{1}{m-1}}$ can take the values of any small enough rational number and this shows that:

COROLLARY 7.1. *The blow up of* $\mathbb{P}^m$ *at* $q_1, \dots, q_{m+1}$ *linearly independent points, polarized by* $k\, \pi^*[\omega_{\mathrm{FS}}] - (\mathrm{PD}[E_1] + \cdots + \mathrm{PD}[E_{m+1}])$ *is K-semistable for k sufficiently large.*

It is worth mentioning that this result is optimal in the number of points to be blown up, since for fewer points the manifold would have nonreductive automorphisms group, and hence no Kähler metrics of constant scalar curvature by the Mathushima-Lichnerovicz obstruction. Another interesting aspect of this example is that the manifold obtained still has nontrivial (in fact an m-dimensional) automorphism group. The point being that the automorphisms surviving the blow up procedure are precisely those that are not Γ_1-invariant.

7.3. *Example* 3. We work equivariantly with respect to the action of the group Γ_2 which is generated by the transformations

$$[z^1 : \cdots : z^{m+1}] \longmapsto [\pm z^1 : \cdots : \pm z^{m+1}],$$

as well as the permutation of the first $m-1$ affine complex coordinates of $\mathbb{C}^{m+1}$

$$[z^1 : \cdots : z^j : \cdots : z^{j'} : \cdots : z^{m+1}] \longmapsto [z^1 : \cdots : z^{j'} : \cdots : z^j : \cdots : z^{m+1}],$$

for all $1 \leq j < j' \leq m-1$. We consider the following set of blow up points

$$p_1 := [1 : 0 : \cdots : 0], \quad p_2 := [0 : 1 : 0 : \cdots : 0], \ \dots, \ p_m := [0 : 0 : \cdots : 1 : 0],$$

and

$$p_{m+1} := [0 : \cdots : 0 : \alpha : \alpha'], \qquad p_{m+2} := [0 : 0 : \cdots : 0 : \alpha : -\alpha'],$$

where $\alpha, \alpha' \in \mathbb{C}$ satisfy $|\alpha|^2 + |\alpha'|^2 = 1$. Observe that the only potentials invariant under the action of the group Γ_2 are linear combinations of

$$f(z) := 2 - (m+1)\,(|z^m|^2 + |z^{m+1}|^2) \quad \text{and} \quad f'(z) := |z^m|^2 - |z^{m+1}|^2,$$

where again the coordinates $z = (z^1, \dots, z^{m+1})$ are chosen so that $|z| = 1$. The corresponding Killing vector fields will be denoted by X and X'. In order to be able to apply Theorem 1.4, we need to check that the image of $\mathrm{Span}\{X, X'\}$ by $\Lambda_{p_1,\dots,p_{m+2}}$ is two-dimensional and that there exists $a_1, \dots, a_{m+2} > 0$ such that

$a_1 = \cdots = a_{m-1}$ and $a_{m+1} = a_{m+2}$ and

$$\sum_{j=1}^{m+2} a_j \, \xi^{\Gamma_2}(p_j) = 0.$$

Now, we compute

$$\Lambda_{p_1,\ldots,p_{m+2}}(X) = (2, \ldots, 2, 1-m, 1-m, 1-m),$$

and

$$\Lambda_{p_1,\ldots,p_{m+2}}(X') = (0, \ldots, 0, 1, |\alpha|^2 - |\alpha'|^2, |\alpha|^2 - |\alpha'|^2).$$

Condition (4) in Theorem 1.4 is always satisfied, while the existence of positive weights a_j is immediate provided $|\alpha'|^2 > |\alpha|^2$ since in this case, the weights are given by

$$a_1 = \cdots = a_{m-1} = 1 + |\alpha'|^2 - |\alpha|^2,$$

$$a_m = 2\,(|\alpha'|^2 - |\alpha|^2) \qquad \text{and} \qquad a_{m+1} = a_{m+2} = 1.$$

This proves that, by working equivariantly with respect to the action of the group Γ_2, the blow up of $\mathbb{P}^m$ at the above $m+2$ points carries a constant scalar curvature Kähler form.

It is an easy observation that the addition of points to a list of points satisfying our conditions preserves these conditions (use Lemma 6.2 to see that condition (2) is preserved). In particular we can add the Γ_2-orbit of any point p (which does not initially belong to the list) to the above list, and keep the two conditions fulfilled. For generic choice of the point p the Γ_2-orbit of p has $m!\,(2^m - 1)$ points, so this substantially increases the number of points one has to blow. However, if one can also add to the above list points of the form

$$[0 : \cdots : 0 : \tilde{\alpha} : \pm\tilde{\alpha}'],$$

where $\tilde{\alpha}, \tilde{\alpha}' \in \mathbb{C}$, $|\alpha|^2 + |\alpha'|^2 = 1$ so that the list of points remains invariant under the action of Γ_2. This clearly increases the number of blow up points by 2 when $\tilde{\alpha}, \tilde{\alpha}' \neq 0$ or by 1 when α or $\tilde{\alpha} = 0$. Using this idea, one shows that the blow up of $\mathbb{P}^m$ at $m+2+k$ points carries a constant scalar curvature Kähler form. Therefore, we have obtained:

COROLLARY 7.2. *The blow up of* $\mathbb{P}^m$ *at* $n \geq m+1$ (*carefully chosen*) *points admits constant scalar curvature Kähler metrics.*

7.4. *Example* 4: *Proof of Corollary* 1.2. For general Kähler manifolds, with d-dimensional space of holomorphic vector fields with zeros, one needs to blow up d points in general position to obtain a manifold without any holomorphic vector field. In this respect $\mathbb{P}^m$ is very special since it is easy to observe that $m+2$ points suffice provided they form a so called *projective frame*, namely any choice of $m+1$

of them are linearly independent in $\mathbb{C}^{m+1}$ (such sets of points are often said to be *in generic position with respect to hyperplanes*).

Suppose that, using some equivariant construction (since $m+2 \leq \dim \mathfrak{h}$), we have found a projective frame $p_1, \ldots, p_{m+2}$ for which we can prove that $\mathbb{P}^m$ blown up at $p_1, \ldots, p_{m+2}$ has a Kähler constant scalar curvature metric. The manifold we obtain after blow up has at most discrete automorphisms and we can then apply to it the results of Theorem 1.1 to blow up any other set of points and still get constant scalar curvature Kähler metrics.

For all these reasons we now look for a projective frame for which some equivariant construction works. We consider the group Γ_3 of permutations of the $m+1$ affine complex coordinates, generated by

$$[z^1 : \cdots : z^j : \cdots : z^{j'} : \cdots : z^{m+1}] \longmapsto [z^1 : \cdots : z^{j'} : \cdots : z^j : \cdots : z^{m+1}],$$

for $1 \leq j < j' \leq m+1$. Given $\alpha \in \mathbb{C}$ we define the points

$$p_1 := [1 : \cdots : 1], \quad p_2 := [\alpha : 1 : \cdots : 1], \quad \ldots, \quad p_{m+2} := [1 : \cdots : 1 : \alpha].$$

The set of points $\{p_1, \ldots, p_{m+1}\}$ is invariant under the action of Γ_3. The only Killing fields which are invariant under the action of Γ_3 are associated to a multiple of the potential

$$f := \sum_{a \neq b} z^a \, \overline{z}^b,$$

when coordinates $z = (z^1, \ldots, z^{m+1})$ are normalized by $|z| = 1$. Let us denote by X the Killing vector field associated to this potential. Again, we need to check that $\Lambda_{p_1,\ldots,p_{m+2}}(X)$ is not zero and that there exists $a_1, \ldots, a_{m+2} > 0$ such that $a_2 = \ldots, a_{m+2}$ (to preserve the symmetry) for which (5) (§1) holds. Condition (4) is always fulfilled since

$$\Lambda_{p_1,\ldots,p_{m+2}}(X) = \left(m, \frac{m\,(m-1+2\,\Re\alpha)}{m+|\alpha|^2}, \ldots, \frac{m\,(m-1+2\,\Re\alpha)}{m+|\alpha|^2}\right),$$

and condition (5) holds for

$$a_1 = m + |\alpha|^2 \qquad a_2 = \cdots = a_{m+2} = m(m+1)(1 - 2\,\Re\alpha - m).$$

Therefore the hypotheses of Theorem 1.4 are fulfilled provided we choose $2\,\Re\alpha < 1-m$. It is easy to see that the points $p_1, \ldots, p_{m+2}$ form a projective frame and hence the blow up of $\mathbb{P}^m$ at these points does not carry any holomorphic vector field. This completes the proof of Corollary 1.2 for this special set of blow up points.

Since the blow up of $\mathbb{P}^m$ at another projective frame $q_1, \ldots, q_{m+2}$ is biholomorphic to the blow up of $\mathbb{P}^m$ at $p_1, \ldots, p_{m+2}$, this result ensures the existence of constant scalar curvature Kähler metrics on the blow up of $\mathbb{P}^m$ at *any* projective frame. Recall that the freedom of choices of projective frames in $\mathbb{P}^m$ ranges clearly in an open and dense subset of $(\mathbb{P}^m)^{m+2}$. In addition, since the manifolds obtained after blowing up the points of a projective frame do not have holomorphic vector field and carry constant scalar curvature metrics, we can subsequently apply the result of Theorem 1.1 to get the existence of constant scalar curvature Kähler metrics on the blow up of $\mathbb{P}^m$ at any set of points $p_1, \ldots, p_n$, $n \geq m+2$, provided $m+2$ of them constitute a projective frame. This completes the proof of Corollary 1.2 in full generality.

As in Example 2, we can interpret our result in terms of K-semistability. Since $a_1 \neq a_2$, we cannot conclude directly that there exists constant scalar curvature metrics in rational classes. Nevertheless, since the blown up manifolds do not have any holomorphic vector fields, the application of the implicit function theorem [15] guarantees the existence of constant scalar curvature Kähler metrics in any nearby Kähler class and in particular in a wealth of rational classes too (losing though explicitness on their form). To summarize, we have:

COROLLARY 7.3. *On the blow up of* $\mathbb{P}^m$ *at the points* $p_1, \ldots, p_{m+2}$ *defined above, there exist rational Kähler classes close to*

$$\omega_\varepsilon = \pi^*[\omega_{\mathrm{FS}}] - \varepsilon^2 \left(a_1^{\frac{1}{m-1}} \, \mathrm{PD}[E_1] + \cdots + a_{m+2}^{\frac{1}{m-1}} \, \mathrm{PD}[E_{m+2}] \right)$$

for which the polarized manifold $(M, [\omega_\varepsilon])$ *is* K*-semistable, provided* ε *is small enough.*

7.5. *Comments.* So far we have studied the problem of finding a Kähler constant scalar curvature metric on the blow up of $\mathbb{P}^m$ at a given set of points, regardless of the Kähler classes we obtain. It is also interesting to keep track of the Kähler classes for which such a canonical representative exists. Let us fix $m = 2$ where our understanding is more complete and start with the minimum number of points, $n = 3$, for which the problem is nonvacuous. Siu [26] and Tian-Yau [31] proved that the blow up of $\mathbb{P}^2$ at three points has a constant scalar curvature metric in

$$\pi^*[\omega_{\mathrm{FS}}] - (\mathrm{PD}[E_1] + \mathrm{PD}[E_2] + \mathrm{PD}[E_3]),$$

(hence an Einstein metric) if and only if the points do not lie on a line.

Example 2 shows that

$$\pi^*[\omega_{\mathrm{FS}}] - \varepsilon^2 \, (\mathrm{PD}[E_1] + \mathrm{PD}[E_2] + \mathrm{PD}[E_3]),$$

has a canonical representative if and only if the points are not aligned (the reverse follows from Matsushima-Licherowicz obstruction). Ross-Thomas ([24, Ex. 5.30] have proved that the classes

$$\pi^*[\omega_{\mathrm{FS}}] - \varepsilon^2 \, \mathrm{PD}[E_1] - \varepsilon^4 \, (\mathrm{PD}[E_2] + \mathrm{PD}[E_3])$$

do not have a constant scalar curvature representative independent of the position of the points, provided ε is small enough.

For the blow up of $\mathbb{P}^2$ at four points, Tian has proved that a constant scalar curvature Kähler metric exists in

$$\pi^*[\omega_{\mathrm{FS}}] - (\mathrm{PD}[E_1] + \mathrm{PD}[E_2] + \mathrm{PD}[E_3] + \mathrm{PD}[E_4]),$$

if and only if they form a projective frame. Example 4 also tells us that

$$\pi^*[\omega_{\mathrm{FS}}] - \varepsilon^2\, \mathrm{PD}[E_1] - \varepsilon^2\, a\, (\mathrm{PD}[E_2] + \mathrm{PD}[E_3] + \mathrm{PD}[E_4]),$$

for some $a > 0$, has a canonical representative (and hence the same is true for any class in an open subset of the Kähler cone around these classes since there are no more automorphisms surviving the blow up procedure), while again Ross-Thomas' obstruction prevents other classes from having such a representative. It is easy to see that one could also use Siu-Tian-Yau's metric as base metric on the blow up of $\mathbb{P}^2$ at three points to blow up one further point in our construction so as also to get (an open subset of the Kähler cone around) the classes of the form

$$\pi^*[\omega_{\mathrm{FS}}] - (\mathrm{PD}[E_1] + \mathrm{PD}[E_2] + \mathrm{PD}[E_3]) - \varepsilon^2\, \mathrm{PD}[E_4].$$

We also know thanks to Example 3 that for an open set of four points, three on a line, such a representative exists in the classes

$$\pi^*[\omega_{\mathrm{FS}}] - \varepsilon^2\, (\mathrm{PD}[E_1] + \mathrm{PD}[E_2] + \mathrm{PD}[E_3] + \mathrm{PD}[E_4]),$$

but one needs to be careful about the fact that these manifolds are not biholomorphic to the ones obtained blowing up a projective frame.

For more than four points a similar game can be played but this time Tian's Einstein metrics (which exist if no three collinear points are blown up, no 5 of them lie on a quadric and no 8 on a cubic) can be used on different base manifolds also. So we end up with a wealth of open subsets of the Kähler cone.

8. **Further examples**

To show some other applications of our method we look at the problem of blowing up products of Kähler constant scalar curvature manifolds. As we recalled in the introduction such problems have been deeply investigated in complex dimension 2 and we know by now that Kähler metrics with zero scalar curvature exist on blow ups of $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ (at $n \geq 10$ points) or $\mathbb{T}^1 \times \mathbb{P}^1$ (for $n \geq 4$), as proved by Rollin-Singer in [23] . Moreover in [22] they have been able to find constant (nonzero) scalar curvature Kähler metrics also on the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $n \geq 6$ points, while zero scalar curvature Kähler metrics on blow ups of $\mathbb{P}^1 \times \Sigma$, when Σ is a Riemann surface of genus greater than or equal to 2, have been constructed by LeBrun-Singer [16].

We now look at similar situations in arbitrary dimensions.

8.1. *Example* 5: *The case of* $\mathbb{P}^{m_1} \times M$. This type of manifold, when M is taken to be a Riemann surface has attracted particular interest since a complete understanding of these examples leads via algebraic geometric techniques to the relation with stability of rank-two vector bundles over Riemann surfaces. By understanding via a different approach these models in our more general setting, we hope to provide tool in the study of similar approaches to higher rank vector bundles over any Kähler constant scalar curvature manifold.

We assume throughout this example that (M, J, g, ω) is a Kähler manifold of any dimension m_2 and without any holomorphic vector fields vanishing somewhere. We consider on $\mathbb{P}^{m_1} \times M$ the product metric $\hat{g} := g_{\text{FS}} + g$, with a Fubini-Study metric on $\mathbb{P}^{m_1}$ normalized as in the previous section.

With these conventions the kernel of the operator

$$\mathbb{L}_{\hat{g}} = \Delta^2_{\hat{g}} + 2\,\text{Ric}_{\hat{g}} \cdot \nabla^2_{\hat{g}},$$

is naturally identified with the vector space spanned by the constant functions and the functions f_{ab}, $\widehat{f}_{ab}$ and $\widetilde{f}_a$ defined on $\mathbb{P}^{m_1}$, for $1 \le a < b \le m_1 + 1$, as described above.

As in Example 2, we then look at the group acting on $\mathbb{P}^{m_1}$ which reduces the invariant kernel to be generated by the constant function. We set

$$p_1 := [1 : 0 : \cdots : 0], \quad \ldots, \quad p_{m_1+1} = [0 : \cdots : 0 : 1],$$

and consider the points $(p_1, q_1), \ldots, (p_{m_1+1}, q_{m_1+1})$ in $\mathbb{P}^{m_1} \times M$ where $q_1, \ldots, q_{m_1+1}$ are arbitrarily chosen on M and possibly coinciding. It is easy to check that our conditions are fulfilled and hence we have obtained the:

COROLLARY 8.1. *Given any two points* $q_1, \ldots, q_{m_1+1}$ *in* M *(possibly coinciding), the blow up of* $\mathbb{P}^{m_1} \times M$ *at* $(p_1, q_1), \ldots, (p_{m_1+1}, q_{m_1+1})$ *carries constant scalar curvature Kähler metrics.*

When $m_1 = 1$, a stronger version of the above corollary was proved by LeBrun-Singer in [16] when M is a Riemann surface of genus at least 2. In fact Lebrun-Singer proved that the metric on the blow up can be chosen to have zero scalar curvature. When $M = \Sigma$ is a Riemann surface, we have gained the freedom of assigning any sign to the constant scalar curvature obtained. In the case M is a torus, one must recall that all holomorphic vector fields are parallel so that it falls into the category allowed by this construction. In complex dimension 2, Rollin-Singer [23] proved that four suitably choosen points to be blown up suffice to have zero scalar curvature metrics.

The strategy used in either Example 1, Example 3 or Example 4 extends easily to $\mathbb{P}^{m_1} \times M$. Details are left to the reader.

8.2. *Example* 6: *The case of* $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$. The case of $\mathbb{P}^1 \times \mathbb{P}^1$ falls directly in the previous discussion since $\mathbb{P}^2$ blown up at $n+1 \geq 2$ points is biholomorphic to the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at n (suitably chosen) points; the results of Corollary 1.1 and Corollary 1.2 translate directly into the following:

COROLLARY 8.2. *For any $n \geq 2$ there exist points*

$$(p_1, \dots, p_n) \in (\mathbb{P}^1 \times \mathbb{P}^1)^n_\Delta$$

such that the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $p_1, \dots, p_n$ has constant scalar curvature Kähler metrics.

For the general case, as seen in the previous examples, we can either work equivariantly with respect to a symmetry group or apply the general strategy to estimate n_g, the least number for which Theorem 1.3 applies. For this second line, we can easily adapt the construction of Example 1 to show that the blow up of $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$ at the points of the form (p, q) where, with obvious notation, $p \in \{p_{ij}, \tilde{p}_{ij}, \hat{p}_{ij}, \check{p}_{ij}\}$ and $q \in \{q_{ij}, \tilde{q}_{ij}, \hat{q}_{ij}, \check{q}_{ij}\}$, carries a Kähler metric with constant scalar curvature. It is again easy to check that the conditions are fulfilled. This shows that $n_g \leq 4\, m_1\, m_2\, (m_1+1)\, (m_2+1)$ for $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$.

To reduce the number of points that have to be blown up, we must introduce a some symmetry group. Let us indicate by $(z^1, \dots, z^{m_1+1})$ and $(u^1, \dots, u^{m_2+1})$ complex affine coordinates for the two factors $\mathbb{P}^{m_1}$ and $\mathbb{P}^{m_2}$. On the product manifold we consider the product of the Fubini-Study metrics normalized as above.

Consider the group Γ_4 generated by permutations of the first m_1 affine complex coordinates of $(z^1, \dots, z^{m_1+1})$, the permutations of the first m_2 affine complex coordinates of $(u^1, \dots, u^{m_2+1})$ and also by

$$([z^1 : \dots : z^{m_1+1}], [u^1 : \dots : u^{m_2+1}]) \\ \longmapsto ([\pm z^1 : \dots : \pm z^{m_1+1}], [\pm u^1 : \dots : \pm u^{m_2+1}]),$$

acting on $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$. We see that the only potentials, in the kernel of the linearized scalar curvature operator, that are invariant under the action of Γ_4 are linear combinations of

$$f(z,u) := 1 - (m_1+1)\, |z^{m_1+1}|^2 \quad \text{and} \quad f'(z,u) := 1 - (1+m_2)\, |u^{m_2+1}|^2,$$

if, as usual, we agree that the coordinates

$$z = (z^1, \dots, z^{m_1+1}) \quad \text{and} \quad u = (u^1, \dots, u^{m_2+1})$$

are normalized by $|z| = |u| = 1$. We denote by X and X' the associated Killing vector fields. Let us look at the points

$$p_1 := ([1:0:\cdots:0],[0:\cdots:0:1]),\ \ldots,\ p_{m_1} := ([0:\cdots:1:0],[0:\cdots:0:1]),$$

$$\widetilde{p}_1 := ([1:0:\cdots:0],[1:0:\cdots:0]),\ \ldots,\ \widetilde{p}_{m_1m_2} := ([0:\cdots:1:0],[0:\cdots:1:0]),$$

$$\widehat{p}_1 := ([0:\cdots:0:1],[1:0:\cdots:0]),\ \ldots,\ \widehat{p}_{m_2} := ([0:\cdots:0:1],[0:\cdots:1:0]).$$

The points $p_1, \ldots, p_{m_1}$ are obtained by the action of the permutations of the first m_1 coordinates of the first factor. The points $\widetilde{p}_1, \ldots, \widetilde{p}_{m_1m_2}$ are obtained by the action of the permutations of the first m_1 coordinates of the first factor and the permutations of the first m_2 coordinates of the second factor. The points $\widehat{p}_1, \ldots, \widehat{p}_{m_2}$ are obtained by the action of the permutations of the first m_2 coordinates of the second factor. There are exactly $n := m_1 + m_1\, m_2 + m_2$ points that we will label $q_1, \ldots, q_n$ (they are arranged by first listing the points p_j, then the points $\widetilde{p}_j$ and finally the points $\widehat{p}_j$). We have

$$\Lambda_{q_1,\ldots,q_n}(X) = (1, \ldots, 1, 1, \ldots, 1, -m_1, \ldots, -m_1),$$
$$\Lambda_{q_1,\ldots,q_n}(X') = (-m_2, \ldots, -m_2, 1, \ldots, 1, 1, \ldots, 1).$$

We also have

$$m_2(1+m_1)\sum_{j=1}^{m_1} \xi^{\Gamma_4}(p_j) + (m_1m_2-1)\sum_{j=1}^{m_1m_2} \xi^{\Gamma_4}(\widetilde{p}_j) + m_1(1+m_2)\sum_{j=1}^{m_2} \xi^{\Gamma_4}(\widehat{p}_j) = 0.$$

The assumptions of Theorem 1.4 are fulfilled provided $m_1\, m_2 \geq 2$ and we get constant (positive) scalar curvature Kähler metrics on the blow up of $P^{m_1} \times P^{m_2}$ at these points.

COROLLARY 8.3. *Assume that $m_1m_2 \geq 2$. There exists $(m_1+1)(m_2+1)-1$ points such that the blow up of $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$ at those points carries constant scalar curvature Kähler metrics of positive scalar curvature.*

This estimate is certainly not optimal. For example, let us analyze the special case of $\mathbb{P}^1 \times \mathbb{P}^2$ in more detail. In this example we can get a better estimate on the least number of points necessary for Theorem 1.4 to hold by looking at the group Γ_5 generated by the transformations

$$([z^1 : z^2], [u^1 : u^2 : u^3]) \longmapsto ([\pm z^1 : \pm z^2], [\pm u^1 : \pm u^2 : \pm u^3]).$$

The potentials invariant under the action of Γ_5 are now combinations of the functions

$$f_1(z,u) := |z^1|^2 - |z^2|^2, \quad f_2(z,u) := |u^1|^2 - |u^2|^2 \quad \text{and} \quad f_3(z,u) := |u^2|^2 - |u^3|^2.$$

We choose

$$p_1 := ([1:0],[1:0:0]), \quad p_2 := ([1:0],[0:1:0]),$$
$$p_3 := ([0:1],[1:0:0]), \quad p_4 := ([0:1],[0:0:1]).$$

We denote by X_j the Killing vector field associated to the potential f_j. It is easy to check that

$$\begin{aligned} \Lambda_{p_1,\dots,p_4}(X_1) &= (1,1,-1,-1), \\ \Lambda_{p_1,\dots,p_4}(X_2) &= (1,-1,1,0), \\ \Lambda_{p_1,\dots,p_4}(X_3) &= (0,1,0,-1), \end{aligned}$$

and

$$\xi^{\Gamma_5}(p_1) + 2\,\xi^{\Gamma_5}(p_2) + \xi^{\Gamma_5}(p_3) + 2\,\xi^{\Gamma_5}(p_4) = 0,$$

from which it follows at once that the assumptions of Theorem 1.4 are fulfilled. We have thus:

COROLLARY 8.4. *The blow up of* $\mathbb{P}^1 \times \mathbb{P}^2$ *at* $p_1, \dots, p_4$ *carries constant positive scalar curvature Kähler metrics of positive scalar curvature.*

Note that the previous calculation gave five points for the existence of the canonical metric. The examples just described still carry vanishing holomorphic vector fields and so cannot *a priori* be used for iteration of blow ups. Following the line of ideas described in Example 4 we now give an estimate on the least number of points for which the procedure works and no holomorphic vector fields exist on the blown up manifold.

Consider the group Γ_5 generated by the permutations of the $m_1 + 1$ affine coordinates of the first factor and the permutations of the $m_2 + 1$ affine coordinates of the second factor. The invariant potentials are now spanned by the functions

$$f(z,u) := \sum_{a \neq b} z^a\, \overline{z}^b \quad \text{and} \quad f'(z,u) := \sum_{a \neq b} u^a\, \overline{u}^b.$$

We choose $\alpha, \tilde{\alpha} \in \mathbb{C}$ and consider in $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$ the points of the form (p_1, q_1), (p_1, q_2), (p_2, q_1) and (p_2, q_2) where

$$\begin{aligned} p_1 &:= [1 : \cdots : 1], & q_1 &:= [1 : \cdots : 1], \\ p_2 &\in \{[\alpha : 1 : \cdots : 1], \dots, [1 : \cdots : 1 : \alpha]\}, \\ q_2 &\in \{[\tilde{\alpha} : 1 : \cdots : 1], \dots, [1 : \cdots : 1 : \tilde{\alpha}]\}. \end{aligned}$$

There are exactly $n := (m_1 + 2)(m_2 + 2)$ such points which are labeled $r_1, \dots, r_n$ (and we arrange them by first listing the point (p_1, q_1), then points of the form (p_1, q_2), next points of the form (p_2, q_1) and finally points of the form (p_2, q_2)). If X and X' denote the Killing vector fields associated to f and f', we have

$$\Lambda_{r_1,\dots,r_n}(X) = (m_1, m_1, \dots, m_1, A, \dots, A, A, \dots, A),$$

and

$$\Lambda_{r_1,\dots,r_n}(X') = (m_2, B, \dots, B, m_2, \dots, m_2, B, \dots, B),$$

where

$$A := \frac{m_1\,(m_1 - 1 + 2\,\Re\,\alpha)}{m_1 + |\alpha|^2} \qquad \text{and} \qquad B := \frac{m_2\,(m_2 - 1 + 2\,\Re\,\tilde{\alpha})}{m_2 + |\tilde{\alpha}|^2}.$$

The assumptions of Theorem 1.4 are fulfilled provided $A < 0$, $B < 0$ and $AB < m_1 m_2$ and hence the blow up of $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$ at those points has no holomorphic vector fields. This proves the following:

COROLLARY 8.5. *There exist* $(m_1 + 2)(m_2 + 2)$ *points such that the blow up of* $\mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$ *at those points carries constant scalar curvature Kähler metrics and no holomorphic vector field.*

References

[1] C. AREZZO and F. PACARD, On the Kähler classes of constant scalar curvature metrics on blow ups, arXiv:0706.1838.

[2] ———, Blowing up and desingularizing constant scalar curvature Kähler manifolds, *Acta Math.* **196** (2006), 179–228. MR 2007i:32018 Zbl 1123.53036

[3] T. AUBIN, *Nonlinear Analysis on Manifolds. Monge-Ampère Equations*, *Grundl. Math. Wissen.* **252**, Springer-Verlag, New York, 1982. MR 85j:58002 Zbl 0512.53044

[4] A. L. BESSE, *Einstein Manifolds*, *Ergeb. Math. Grenzgeb.* **10**, Springer-Verlag, New York, 1987. MR 88f:53087 Zbl 0613.53001

[5] S. BOCHNER and W. T. MARTIN, *Several Complex Variables*, *Princeton Math. Series* **10**, Princeton Univ. Press, Princeton, NJ, 1948. MR 10,366a Zbl 0041.05205

[6] E. CALABI, Extremal Kähler metrics, in *Seminar on Differential Geometry*, *Ann. of Math. Stud.* **102**, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259–290. MR 83i:53088 Zbl 0487.53057

[7] X. X. CHEN and G. TIAN, Geometry of Kähler metrics and foliations by holomorphic discs, *Publ. Math. Inst. Hautes Études Sci.* (2008), 1–107. MR 2009g:32048

[8] S. K. DONALDSON, Scalar curvature and projective embeddings. I, *J. Differential Geom.* **59** (2001), 479–522. MR 2003j:32030 Zbl 1052.32017

[9] ———, Lower bounds on the Calabi functional, *J. Differential Geom.* **70** (2005), 453–472. MR 2006k:32045 Zbl 1149.53042

[10] P. GRIFFITHS and J. HARRIS, *Principles of Algebraic Geometry*, Wiley-Interscience Publ., New York, 1978. MR 80b:14001 Zbl 0408.14001

[11] J. KIM, C. LEBRUN, and M. PONTECORVO, Scalar-flat Kähler surfaces of all genera, *J. Reine Angew. Math.* **486** (1997), 69–95. MR 98m:32045

[12] A. KOVALEV and M. SINGER, Gluing theorems for complete anti-self-dual spaces, *Geom. Funct. Anal.* **11** (2001), 1229–1281. MR 2002m:53071 Zbl 1012.53026

[13] C. LEBRUN, Counter-examples to the generalized positive action conjecture, *Comm. Math. Phys.* **118** (1988), 591–596. MR 89f:53107 Zbl 0659.53050

[14] C. LEBRUN and S. R. SIMANCA, On the Kähler classes of extremal metrics, in *Geometry and Global Analysis* (*Sendai*, 1993), Tohoku Univ., Sendai, 1993, pp. 255–271. MR 96h:58037 Zbl 0921.53032

[15] C. LEBRUN and S. R. SIMANCA, Extremal Kähler metrics and complex deformation theory, *Geom. Funct. Anal.* **4** (1994), 298–336. MR 95k:58041 Zbl 0801.53050

[16] C. LEBRUN and M. SINGER, Existence and deformation theory for scalar-flat Kähler metrics on compact complex surfaces, *Invent. Math.* **112** (1993), 273–313. MR 94e:53070 Zbl 0793.53067

[17] R. B. LOCKHART and R. C. MCOWEN, Elliptic differential operators on noncompact manifolds, *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* **12** (1985), 409–447. MR 87k:58266 Zbl 0615.58048

[18] T. MABUCHI, An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds. I, *Invent. Math.* **159** (2005), 225–243. MR 2005m:32047 Zbl 1118.53047

[19] R. MAZZEO, Elliptic theory of differential edge operators I, *Comm. in* PDE **16** (1991), 1615–1664. MR 93D:58152 Zbl 0745.58045

[20] R. B. MELROSE, *The Atiyah-Patodi-Singer Index Theorem*, *Research Notes in Math.* **4**, A K Peters Ltd., Wellesley, MA, 1993. MR 96g:58180 Zbl 0796.58050

[21] S. T. PAUL, Geometric analysis of Chow Mumford stability, *Adv. Math.* **182** (2004), 333–356. MR 2005b:32054 Zbl 1050.53061

[22] Y. ROLLIN and M. SINGER, Construction of Kähler surfaces with constant scalar curvature, math.DG/0412405.

[23] ———, Non-minimal scalar-flat Kähler surfaces and parabolic stability, *Invent. Math.* **162** (2005), 235–270. MR 2007b:32035 Zbl 1083.32021

[24] J. ROSS and R. THOMAS, An obstruction to the existence of constant scalar curvature Kähler metrics, *J. Differential Geom.* **72** (2006), 429–466. MR 2007c:32028 Zbl 1125.53057

[25] S. R. SIMANCA, Kähler metrics of constant scalar curvature on bundles over $\mathbf{CP}_{n-1}$, *Math. Ann.* **291** (1991), 239–246. MR 92m:53115

[26] Y. T. SIU, The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group, *Ann. of Math.* **127** (1988), 585–627. MR 89e:58032 Zbl 0876.53044

[27] J. STOPPA, Unstable blow ups, math.AG/0702154.

[28] R. P. THOMAS, Notes on GIT and symplectic reduction for bundles and varieties, in *Surveys in Differential Geometry. Vol. X*, *Surv. Differ. Geom.* **10**, Internat. Press, Somerville, MA, 2006, pp. 221–273. MR 2408226 Zbl 1132.14043

[29] G. TIAN, Kähler-Einstein metrics with positive scalar curvature, *Invent. Math.* **130** (1997), 1–37. MR 99e:53065 Zbl 0892.53027

[30] ———, *Canonical metrics in Kähler Geometry*, *Lectures in Mathematics* ETH *Zürich*, Birkhäuser, Basel, 2000. MR 2001j:32024

[31] G. TIAN and S.-T. YAU, Kähler-Einstein metrics on complex surfaces with $C_1(M) > 0$, *Comm. Math. Phys.* **112** (1987), 175–203. MR 88k:32070

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