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By Stephen Lichtenbaum

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By STEPHEN LICHTENBAUM

Abstract

There should be a Grothendieck topology for an arithmetic scheme X such that the Euler characteristic of the cohomology groups of the constant sheaf $\mathbb{Z}$ with compact support at infinity gives, up to sign, the leading term of the zeta-function of X at $s = 0$. We construct a topology (the Weil-étale topology) for the ring of integers in a number field whose cohomology groups $H^i(\mathbb{Z})$ determine such an Euler characterstic if we restrict to $i \leq 3$.

Introduction

The purpose of this paper is to serve as the first step in the construction of a new Grothendieck topology (the Weil-étale topology) for arithmetic schemes X (schemes of finite type over $\operatorname{Spec} \mathbb{Z}$), which should be in many ways better suited than the étale topology for the study of arithmetical invariants and of zeta-functions. The Weil-étale cohomology groups of "motivic sheaves" or "motivic complexes of sheaves" should be finitely generated abelian groups, and the special values of zeta-functions should be very closely related to Euler characteristics of such cohomology groups.

As an example of the above philosophy, let $\overline{X}$ be a compactification of X. This involves first completing X to obtain a scheme X_1 such that X is dense in X_1 and $f : X_1 \to \operatorname{Spec} \mathbb{Z}$ is proper over its image, and then, if f is dominant, adding fibers over the missing points of $\operatorname{Spec} \mathbb{Z}$ and the archimedean place of $\mathbb{Q}$ to obtain $\overline{X}$.

Let ϕ be the natural inclusion of X into $\overline{X}$. The following should be true:

(a) The Weil-étale hypercohomology groups of compact support $H^q(\overline{X}, \phi_! \mathbb{Z})$ are finitely generated abelian groups that are equal to 0 for all but finitely many q, and independent of the choice of compactification $\overline{X}$. We will denote them by $H^q_c(X, \mathbb{Z})$.

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(b) If $\widetilde{\mathbb{R}}$ denotes the "sheaf of real-valued functions" on X, then the cohomology groups $H^q(\overline{X}, \phi_! \widetilde{\mathbb{R}})$ are independent of the compactification, and we denote them by $H^q_c(X, \widetilde{\mathbb{R}})$. The natural map from $H^q_c(X, \mathbb{Z}) \otimes \mathbb{R}$ to $H^q_c(X, \widetilde{\mathbb{R}})$ is an isomorphism. (Note that this is not at all a formality, and would for instance be false if we considered cohomology on all of $\overline{X}$.)

(c) There is an element ψ in $H^1(\overline{X}, \widetilde{\mathbb{R}})$ such that the complex $(H^*_c(X, \widetilde{\mathbb{R}}), \cup\psi)$ resulting from taking the Yoneda product with ψ is exact.

Then the Euler characteristic $\chi_c(X)$ of the complex $H^q_c(X, \mathbb{Z})$ is well defined (see §7), and we can describe how the zeta-function $\zeta_X(s)$ behaves at $s = 0$ by the formula $\zeta^*_X(0) = \pm\chi_c(X)$, where $\zeta^*_X(0) = \lim_{s \to 0} \zeta_X(s)s^{-a}$ when a is the order of the zero of $\zeta_X(s)$ at $s = 0$.

Defining $\zeta^*(X, -n)$ in the analogous fashion, and taking advantage of the formula $\zeta_X(s) = \zeta_{X \times \mathbb{A}^n}(s + n)$, we can conjecturally describe the behavior of the zeta-function of any arithmetic scheme at any nonpositive integer $-n$ by the formula $\zeta^*(X, -n) = \pm\chi_c(X \times \mathbb{A}^n)$,

There should exist motivic complexes $\mathbb{Z}(-n)$ whose Euler characteristics give the values of $\zeta^*(X, -n)$ directly, and the above conjectural formula should give a guide to a possible definition.

In this paper we only define the Weil-étale topology in the case when F is a global number field and $X = \operatorname{Spec} O_F$. We then compute the cohomology groups $H^q_c(X, \mathbb{Z})$ for $q = 0, 1, 2, 3$, and verify that our conjectured formula holds true if we arbitrarily set the groups $H^q_c(X, \mathbb{Z})$ to be zero for $q > 3$.

Flach has shown [Fla08] that the above cohomology groups are in fact not zero for q odd and greater than 3. This seems in some sense due to bad behavior at the infinite primes and is an indication that our rather ad hoc definition is not yet the right one. We could of course just work with the truncated complex, and we do that in this paper, but this is not fully satisfactory.

It is not hard to guess possible extensions of the definition given here to arbitrary X, once we have defined Weil groups and Weil maps for higher-dimensional fields, both local and global. Kato has made a very plausible suggestion of such a definition, and we hope to return to this question in subsequent papers.

We close the introduction with two remarks:

First, the definition given here would work for any open subscheme of a smooth projective curve over a finite field. Do the cohomology groups thus obtained agree with the ones defined in our earlier paper [Lic05]? This seems highly likely, but we have not checked it.

Second, what is the relation of these conjectures to the celebrated Bloch-Kato conjectures? In general, they are not even about the same objects. The Bloch-Kato conjectures concern the Hasse-Weil zeta-function of a variety over a number field, and our conjectures concern the scheme zeta-function of a scheme over Spec $\mathbb{Z}$. If the scheme is smooth and proper over Spec $\mathbb{Z}$, then the zeta-function of the scheme is the same as the Hasse-Weil zeta-function of the generic fiber, so then we can ask if the conjectures are compatible. Even this seems far from obvious, although presumably true.

In a forthcoming paper, we will address this question in more detail, explaining what the Weil-étale cohomology groups of 1-motives ought to be, what the relevant zeta-function conjecture is, and what relation this bears to Bloch-Kato.

1. Cohomology of topological groups

Let G be a topological group. We define a Grothendieck topology $T(G)$ as follows: Let $\mathrm{Cat}(T(G))$ be the category of G-spaces and G-morphisms. A collection of maps $\{\pi_i : X_i \to X\}$ will be called a covering (so an element of $\mathrm{Cov}(T(G))$) if it admits local sections: for every $x \in X$ there exists an open neighborhood V of x, an index i, and a continuous map $s_i : V \to X_i$ such that $\pi_i s_i = 1$. We verify easily that $\mathrm{Cat}(T(G))$ has fibered products. It is immediate that $T(G)$ satisfies the axioms for a Grothendieck topology, and we call $T(G)$ the "local-section topology".

Let A be a topological G-module. We define a presheaf of abelian groups $\tilde{A}$ on $T(G)$ by putting $\tilde{A}(X) = \mathrm{Map}_G(X, A)$ (the set of continuous G-equivariant maps from X to A).

PROPOSITION 1.1. *$\tilde{A}$ is a sheaf.*

Proof. We have to show $\tilde{A}$ verifies the sheaf axiom: Let $\{\pi_i : X_i \to X\}$ be a cover. Let θ_1 and θ_2 be the maps $\prod \mathrm{Map}_G(X_i, A) \to \prod \mathrm{Map}_G(X_i \times_X X_j, A)$ induced by the two projections, and let ψ be the natural map from $\mathrm{Map}_G(X, A)$ to $\prod \mathrm{Map}_G(X_i, A)$. We must to check that if f is in $\prod \mathrm{Map}_G(X_i, A)$ and $\theta_1(f) = \theta_2(f)$, there is a unique g in $\mathrm{Map}_G(X, A)$ such that $f = \psi(g)$.

Clearly g exists and is unique as a map of sets; we need only show that g is continuous. This follows immediately from the existence of local sections.

Define $C^p(G, A)$ to be $\mathrm{Map}_G(G^{p+1}, A)$, where G acts diagonally on G^{p+1}. Let δ_p map $C^p(G, A)$ to $C^{p+1}(G, A)$ by the standard formula

$$\delta_p f(g_0, \ldots, g_{p+1}) = \sum_0^{p+1} (-1)^i f(g_0, \ldots, \widehat{g_i}, \ldots, g_{p+1}).$$

Then the cohomology $H_c^p(G, A)$ of this complex is the continuous (homogeneous) cochain cohomology of G with values in A. □

Remark. By the usual computation, this cohomology is the same as the inhomogeneous continuous cochain complex of G with values in A.

Let $*$ denote a point, with trivial G-action.

Definition 1.2. We define the cohomology groups $H^i(G, A)$ to be equal to $H^i(T(G), *, \tilde{A})$.

PROPOSITION 1.3. *Let* $0 \to A \to B \to C \to 0$ *be an exact* (*as abelian groups*) *sequence of* G*-maps of topological* G*-modules . Assume that the topology of* A *is induced from that of* B *and that the map from* B *to* C *admits local sections as a map of topological* G*-sets. Then the sequence of sheaves* $0 \to \tilde{A} \to \widetilde{B} \to \widetilde{C} \to 0$ *on* $T(G)$ *is also exact, and consequently there is a long exact sequence of cohomology*

$$0 \to H^0(G, A) \longrightarrow H^0(G, B) \longrightarrow H^0(G, C) \longrightarrow H^1(G, A) \longrightarrow \cdots .$$

Proof. It is immediate that the sequence of sheaves is left exact. Let X be a G-space, and let $f : X \to C$ be a continuous G-map. Then the projection on the first factor makes the fibered product $X \times_C B$ a local section cover of X. Let p_1 and p_2 be the projections from $X \times_C B$ to X and B, respectively, and let λ be the map from B to C. Then $f \circ p_1 = p_1^* f = \lambda_* p_2 = \lambda \circ p_2$, so the map from $\widetilde{B}$ to $\widetilde{C}$ is surjective. □

PROPOSITION 1.4. *There is a functorial isomorphism between the Čech cohomology groups* $\check{H}^p(*, \tilde{A}) = \check{H}^p(T(G), *, \tilde{A})$ *and* $H_c^p(G, A)$.

Proof. By definition, $\check{H}^p(*, \tilde{A})$ is the direct limit of the groups $\check{H}^p(\mathcal{U}, \tilde{A})$, where $\mathcal{U}$ runs through the set of coverings of $*$. It is immediate that the map from G to $*$ is an initial object in the category of covers, so $\check{H}^p(*, \tilde{A}) = \check{H}^p(\{G\}, \tilde{A})$. But, by definition, this is the cohomology of the complex

$$\mathrm{Map}_G(G, A) \longrightarrow \mathrm{Map}_G(G \times G, A) \longrightarrow \mathrm{Map}_G(G \times G \times G, A) \longrightarrow \cdots ,$$

which is just the homogeneous continuous cochain complex. □

COROLLARY 1.5. *Let* A *be a topological* G*-module with trivial* G*-action. Then there is natural isomorphism between the cohomology group* $H^1(G, A)$ *and* $\mathrm{Hom}_{\mathrm{cont}}(G, A)$.

Proof. In any Grothendieck topology, $H^1(F) = \check{H}^1(F)$ for any sheaf F. On the other hand, the continuous cochain cohomology group $H^1_{\mathrm{cont}}(G, A)$ (with G acting trivially) is well known to be the group of continuous homomorphisms from G to A. □

Our next goal is to relate the cohomology of G to the Čech cohomology of the underlying topological spaces of G and its products.

LEMMA 1.6. *Let* G *be a topological group and* X *a topological space. Let* $W = G \times X$, *and let* G *act on* W *by* $g(h, w) = (gh, w)$. *In the local section topology on* W, *every cover* $\{\pi_i : U_i \to W\}$ *has a refinement by a cover of the form* $\{G \times V_x\}$, *where* V_x *is a topological neighborhood of the point* x *in* X.

Proof. Let $x \in X$. There is an open neighborhood V_x of x, an open neighborhood T_x of the identity e of G, an index i, and a section $\lambda_x : T_x \times V_x \to U_i$. Let i_x be the inclusion of V_x in X. Define a map $\rho_x : G \times V_x \to U_i$ by $\rho_x(g, v) = g(\lambda_x(e, v))$. Clearly $\{G \times V_x\}$ is a local section cover of $G \times X$.

We have

$$\pi_i \rho_x(g, v) = \pi_i g \lambda_x(e, v) = g \pi_i \lambda_x(e, v) = g(e, v) = (g, v).$$

This shows that $\pi_i \rho_x = id \times i_x$, and hence that $\{G \times V_x\}$ refines $\{U_i\}$. □

If F is a sheaf on X, let $H^q(X, F)$ denote $H^q(T()e, X, F)$, where e is the trivial group acting on X. We note that this is the same as the usual cohomology of sheaves $H^q_{\text{top}}(X, F)$, since usual topological covers are cofinal in local-section covers.

COROLLARY 1.7. (a) *Let E be a local-section sheaf on $G \times X$. Define a local-section presheaf $\alpha_* E$ on X by $\alpha_* E(Y) = E(G \times Y)$. Then $\alpha_* E$ is a sheaf for the local-section topology on X, and α_* is exact.*

(b) *$H^q(T(G), G \times X, E)$ is isomorphic to $H^q(X, \alpha_* E)$.*

Proof. To prove (a), let $\{U_i \to Y\}$ be a local-section cover of Y. Then $\{G \times U_i \to G \times Y\}$ is a local-section G-cover of $G \times Y$ and $G \times (U_i \times_Y U_j)$ is naturally isomorphic to $(G \times U_i) \times_{G \times Y} (G \times U_j)$, so $\alpha_* F$ is a sheaf.

We note that α_* is clearly left exact. Let $E \to F$ be a surjective map of sheaves on $G \times X$. Let $x \in \alpha_* F(Y) = F(G \times Y)$. There exists a local-section cover $\{\pi_i : U_i \to G \times Y\}$ such that $\pi_i^*(x) \in F(U_i)$ lifts to $E(U_i)$. By Lemma 1.6, we may assume that U_i is $G \times V_i$, where the V_i are an open cover of Y, and $\pi_i = (id, \lambda_i)$, where λ_i is the inclusion of V_i in Y. Clearly $\lambda_i^* x$ comes from $\alpha_* E(V_i)$, so $\alpha_* E \to \alpha_* F$ is surjective, and α_* is exact.

It is immediate that α^{-1}, defined by $\alpha^{-1}(Y) = Y \times G$, is a map of topologies, so we have a Leray spectral sequence for α_*. This spectral sequence degenerates because α_* is exact, yielding the desired isomorphism. □

2. **An alternate definition**

Let G be a topological group with identity element e. We construct a simplicial G-space S_n as follows:

Let $S_n = G^{n+1}$, and let $g \in G$ act on S_n by mapping $(g_0, g_1, \dots, g_n)$ to $(gg_0, g_1, \dots, g_n)$.

Now define face maps $\rho_i : S_n \to S_{n-1}$ by

$$\rho_i(g_0, \dots, g_n) = \begin{cases} (g_0, \dots, g_i g_{i+1}, \dots, g_n) & \text{for } 0 \le i < n, \\ (g_0, \dots, g_{n-1}) & \text{for } i = n. \end{cases}$$

The maps ρ_i are maps of G-spaces, and a straightforward verification shows that $\rho_i \rho_j = \rho_{j-1} \rho_i$ if $i < j$.

Define degeneracy maps

$$s_i : S_n \to S_{n+1}, \quad (g_0, \ldots, g_n) \mapsto (g_0, \ldots, g_i, e, \ldots, g_n).$$

Now let $\widetilde{S}_n = G^{n+1}$, but with $g \in G$ acting diagonally on $\widetilde{S}_n$, so that $(g_0, \ldots, g_n)$ is taken to $(gg_0, \ldots, gg_n)$. Let $\pi_i : \widetilde{S}_n \to \widetilde{S}_{n-1}$ by $\pi_i(g_0, \ldots, g_n) = (g_0, \ldots, \widehat{g_i}, \ldots, g_n)$. Computation shows, first, that π_i is a G-map, and second, that $\pi_i \pi_j = \pi_{j-1} \pi_i$ if $i < j$.

Let $\phi : \widetilde{S}_n \to S_n$ by $\phi(g_0, \ldots, g_n) = (g_0, g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n)$. We verify that ϕ is a G-map and that $\rho_i \phi = \phi \pi_i$.

Let F be a local-section sheaf on the site $T(G)$. Let F_n be the sheaf on G^n (as topological space) defined by $F_n(U) = F(G \times U)$, where G acts on $G \times U$ by acting by left translation on G and trivially on U.

Define $\rho_{-1} : S_n \to S_{n-1}$ by $\rho_{-1}(g_0, \ldots, g_n) = (g_1, \ldots, g_n)$ Let ψ be the bijection from S_n to G^{n+1} defined by $\psi(g_0, \ldots, g_n) = (h_1, \ldots, h_{n+1})$ with $h_{i+1} = g_i$ for $0 \le i \le n$. Then define $\bar{\rho}_i$ to be $\psi \circ \rho_{i-1} \circ \psi^{-1}$ for $0 \le i \le n+1$.

Let p_n be the natural projection from $S_n = (G \times G^n)$ to G^n. We check that $p_{n-1} \rho_i = \bar{\rho}_i p_n$, and so automatically $\bar{\rho}_i \bar{\rho}_j = \bar{\rho}_{j-1} \bar{\rho}_i$.

Now take the (second) canonical flabby resolution $T_{j,n}$ of F_n on G^n. Some words are in order. If X is a topological space and F is a sheaf on X, the usual canonical flabby resolution is obtained by defining $C^0(F)$ to be $\prod_{x \in X} (i_x)_* F_x$, embedding F in $C^0(F)$, taking the quotient G, embedding G in $C^0(G)$, and continuing this process to obtain a flabby resolution $0 \to F \to C^0(F) \to C^0(G) \to \cdots$. On the other hand, the second canonical flabby resolution looks like $0 \to F \to C^0(F) \to C^0(C^0(F)) \to \cdots$, after defining suitable coboundary maps (see [God58, §6.4] for details). We have to use this construction to compare our definition with Wigner's definition of $\hat{H}^*(G, A)$ from [Wig73, p. 91]. Note that when Wigner says "canonical semisimplicial resolution", he means this one. By construction we have for each i a map from F_{n-1} to $(\bar{\rho}_i)_* F_n$, which is easily seen to induce inductively a map from $T_{j,n-1}$ to $(\bar{\rho}_i)_* T_{j,n}$, and hence a map from $\Gamma(G^{n-1}, T_{j,n-1})$ to $\Gamma(G^n, T_{j,n})$. By taking the alternating sum of these maps as i varies we get a map $\delta_{j,n} : \Gamma(G^{n-1}, T_{j,n-1}) \to \Gamma(G^n, T_{j,n})$, and thus a double complex. We define $\widetilde{H}^*(G, F)$ to be the hypercohomology of this double complex.

PROPOSITION 2.1. *The cohomology groups $H^i(T(G), *, F)$ are functorially isomorphic to the cohomology groups $\widetilde{H}^i(G, F)$.*

Proof. We need only check that $H^0 = \widetilde{H}^0$, that the $\widetilde{H}^i$ form a cohomological functor, and that the $\widetilde{H}^i$ vanish on injectives for $i > 0$.

Because the canonical flabby resolution takes short exact sequences of sheaves into short exact sequences of complexes, the $\widetilde{H}^i$ form a cohomological functor. (Recall that Corollary 1.7 implies that an exact sequence of sheaves on $T(G)$ gives rise to an exact sequence of sheaves on G^n for every n.)

If $F = I$ is injective, I restricts to an injective sheaf J_n on $G \times G_n$. (If f is the map from $G \times G_n$ to a point, f^* takes injectives to injectives, since it has the exact left adjoint $f_!$.) We know that $I_n = \alpha_* J_n$ is flabby, hence acyclic, and so the homology of the flabby resolution of I_n reduces to $H^0(G^n, I_n)$; the spectral sequence of a double complex shows that our hypercohomology is the cohomology of the complex $H^0(G_n, I_n) = I_n(G_n) = I(G \times G_n)$. The equality $\rho_i \phi = \phi \pi_i$ shows that the homology of $I(G \times G_n)$ is the same as the homology of $I(G^{n+1})$ with diagonal action, which is the Čech cohomology $\check{H}^i(G, I)$. Since the Čech cohomology vanishes for injectives for $i > 0$, so does $\widetilde{H}^i(G, F)$.

Finally, it follows again from the formula $\rho_i \phi = \phi \pi_i$ that if F is any presheaf on the category of G-spaces, the cohomology of the complex $F(S_n)$ is naturally isomorphic to that of $\alpha_* F(G^n) = F(\widetilde{S}_n)$, where the coboundary maps are the alternating sums of the maps induced by ρ_i and π_i, respectively. It follows that the cohomology $\widetilde{H}^0(G, F)$ is naturally isomorphic to the Čech cohomology $\check{H}^0(G, F)$, which in turn is $H^0(T(G), *, F)$. □

Remark 2.2. We observe that if F is a sheaf of the form $\tilde{A}$, then our cohomology groups are exactly the cohomology groups denoted by $\hat{H}^*(G, A)$ by David Wigner [Wig73, p. 91]. We then obtain as a corollary of Wigner's Theorem 2 [Wig73, p. 91], that if G is locally compact, σ-compact, finite dimensional, and A is separable and has Wigner's "property F", then our $H^*(G, A)$ are naturally isomorphic to the groups $H^*(G, A)$ defined by Wigner in [Wig73], (which we will call $H^*_{\mathrm{Wig}}(G, A)$). We further point out that under the same conditions Wigner's groups are naturally isomorphic to the groups (which we will call $H^*_M(G, A)$) defined by Calvin Moore in [Mil80] and used by C. S. Rajan in [Raj04]. (Wigner's Theorem 2 does not explicitly require separability, but his proof that certain categories of modules are quasi-abelian is not valid without it.) To apply this result, we recall that [Wig73, Prop. 3] tells us that any locally connected complete metric topological group (for instance, $\mathbb{Z}$, S^1, or $\mathbb{R}$) has property F.

THEOREM 2.3. *There is a spectral sequence*

$$E_1^{p,q} = H^q_{\mathrm{top}}(G^p, \alpha_* F) \Rightarrow H^{p+q}(T(G), *, F).$$

Proof. This is just the spectral sequence of the double complex defining $\widetilde{H}^*(G, F)$. □

COROLLARY 2.4. *Let G be* (a) *a profinite group or* (b) *the Weil group of a global function field*, *and let A be a topological G-module. Then the cohomology* *groups $H^i(G, A)$ are canonically isomorphic to the usual groups $H^i_{\mathrm{cont}}(G, A)$ given by complexes of continuous cochains.*

Proof. We show first that the cohomological dimension of a profinite space X is zero. To do this it suffices to show (by using alternating cochains) that every open cover has a refinement by a disjoint cover. It is immediate that X has a base for its topology consisting of sets U_i that are both open and closed. By compactness, any cover has a refinement $\{U_1, \dots, U_n\}$ consisting of finitely many such U_i . Let $C(U) = X - U$. Then

$$\{U_1, U_2 \cap C(U_1), U_3 \cap C(U_2) \cap C(U_3), \dots\}$$

is a further refinement which is disjoint.

In case (a) each G^q is profinite and so has cohomological dimension zero; in case (b) the Weil group G is the topological product of a profinite group and a discrete group, so G^q is the disjoint union of open profinite spaces, so again has cohomological dimension zero. So in each case the spectral sequence degenerates to yield that $H^*(G, F)$ is the cohomology of the complex $F(G \times G^p)$. We see, using $\rho_i \phi = \phi \pi_i$ as in the proof of Proposition 2.1, that this is the same as the cohomology of the complex $F(G^{p+1})$, with G acting diagonally, which is just the homogeneous continuous cochain complex of the G-module A if $F = \tilde{A}$. □

LEMMA 2.5. *Let X be the product of a compact space and a metrizable space, and let E be a sheaf of modules over the sheaf of continuous real-valued functions on X. Then $H^q(X^p, E) = 0$ for all $p, q > 0$.*

Proof. The hypothesis implies that for any p, X^p is again the product of a compact space and a metrizable space, and so paracompact. We recall from [God58, p. 157] that any sheaf of modules over the sheaf of continuous real-valued functions on a paracompact space is fine, so "mou", so acyclic. □

COROLLARY 2.6. *Let G be a topological group which is, as a topological space, the product of a compact space and a metrizable space (e.g. the Weil group of a global or local field, where the metrizable space is either $\mathbb{R}$ or $\mathbb{Z}$), and let $\mathbb{R}$ also denote the real numbers with their usual topology and trivial G-action. Then the cohomology groups $H^p(G, \mathbb{R})$ are given by the cohomology of the complex of homogeneous continuous cochains from G to $\mathbb{R}$.*

Proof. Let $F = \widetilde{\mathbb{R}}$. We first observe that $\alpha_*(F)(U) = F(G \times U) = \mathrm{Map}_G(G \times U, \mathbb{R})$, which is naturally isomorphic to $\mathrm{Map}(U, \mathbb{R}) = \widetilde{\mathbb{R}}(U)$. Then Lemma 2.5 implies that the spectral sequence of Theorem 2.3 degenerates, so that the cohomology $H^p(G, \mathbb{R})$ is given by the cohomology of the complex $H^0_{\mathrm{top}}(G^p, \alpha_*\widetilde{\mathbb{R}}) = \mathrm{Map}_G(G \times G^p, \mathbb{R}) = \mathrm{Map}_G(S_p, \mathbb{R})$. As above, this is just the cohomology of the homogeneous cochain complex $\mathrm{Map}_G(\widetilde{S}_p, \mathbb{R})$. □

3. Cohomology of the Weil group

Let F be a number field (resp. a local field), $\overline{F}$ an algebraic closure of F, and G_F the Galois group of $\overline{F}$ over F. Let K be a finite Galois extension of F, and let C_K denote the idèle class group of K (resp. K^*).

Now fix a Weil group W_F associated with the topological class formation $\mathrm{Lim}(C_K)$, where the limit is taken over fields K that are finite and Galois over F. We recall that W_F is equipped with a continuous homomorphism $g : W_F \to G_F$. Let $W_K = g^{-1}(G_K)$, and let W_K^c be the closure of the commutator subgroup of W_K in W_F. Then it is shown in [AT68] that W_F / W_K^c is a Weil group for the pair $(G(K/F), C_K)$ (resp. $(G(K/F), K^*)$). So having fixed a Weil group W_F, we have canonical maps from it to $W_{K/F} = W_F / W_K^c$. The standard construction of the Weil group W_F (see [AT68]) shows that W_F is the projective limit of the groups $W_{K/F}$.

Now let F be a number field and S a finite set of valuations of F including the archimedean valuations and all valuations that ramify in K except for the trivial valuation. Let $U_{K,S}$ be the subgroup of the idèle group I_K consisting of those idèles that are 1 at valuations lying over S, and units at valuations not lying over S. It is well known (see [NSW00, p. 393]) that $U_{K,S}$ is a cohomologically trivial $G(K/F)$-module. The natural map from $U_{K,S}$ to the idèle class group C_K is obviously injective, and we identify $U_{K,S}$ with its image. Let the S-idèle class group $C_{K,S}$ be defined by $C_{K,S} = C_K / U_{K,S}$. Then the natural maps from the Tate cohomology groups $\hat{H}^i(G(K/F), C_K)$ to $\hat{H}^i(G(K/F), C_{K,S})$ are isomorphisms for all i.

Let α be the fundamental class in $\hat{H}^2(G(K/F), C_K)$ and β its image in $\hat{H}^2(G(K/F), C_{K,S})$. It follows immediately from the fact that C_K is a class formation that for all i cup-product with β defines an isomorphism between $\hat{H}^i(G(K/F), \mathbb{Z})$ and $\hat{H}^{i+2}(G(K/F), C_{K,S})$. We then define the S-Weil group $W_{K/F,S}$ to be the extension of $G(K/F)$ by $C_{K,S}$ determined by β. There is clearly a natural surjection p_S from $W_{K/F}$ to $W_{K/F,S}$, and it follows from the arguments in [AT68, p. 238] that there is a natural isomorphism from $W_{K/F,S}^{\mathrm{ab}}$ to $C_{F,S}$.

Let $N_{K,S}$ be the kernel of the natural map from W_F to $W_{K/F,S}$. Let A be a topological W_F-module, and let $A_{K,S}$ be the topological $W_{K/F,S}$ module consisting of the invariant elements $A^{N_{K,S}} \subseteq A$. Assume that $A = \bigcup A_{K,S}$.

LEMMA 3.1. *The Weil group W_F is the projective limit over K and S of the groups $W_{K/F,S}$.*

Proof. It suffices to show that the relative Weil group $W_{K/F}$ is the projective limit over S of the groups $W_{K/F,S}$. The maps p_S induce a map p from $W_{K/F}$ to the projective limit. Let $W_{K/F}^1$ (resp. $W_{K/F,S}^1$) be the kernel of the absolute value map of $W_{K/F}$ (resp. $W_{K/F,S}$) to $\mathbb{R}^*$. Since $W_{K/F}^1$ is compact and the maps p_S are surjective, p is surjective as a map from $W^1_{K/F}$ to the projective limit of $W^1_{K/F,S}$, and hence p is surjective. The proof that p is injective immediately reduces to showing that the map from C_K to the projective limit of the $C_{K/F,S}$ is injective, which in turn follows from the corresponding fact for the idèle groups. □

Definition 3.2. We define the cohomology group $H^q(W_F, A)$ to be the direct limit of the cohomology groups $H^q(W_{K/F,S}, A_{K,S})$.

The cohomology groups of $W_{K/F,S}$-modules are the ones defined in Section 1. We observe that $W_{K/F,S}$ is locally compact, σ-compact and finite-dimensional, so Wigner's comparison theorem (see Remark 2.2) applies and the cohomology of these groups with coefficients in $\mathbb{Z}$, $\mathbb{R}$ or S^1 are the same as Wigner's cohomology groups and therefore also Moore's cohomology groups.

We now calculate the cohomology groups $H^q(W_F, \mathbb{Z})$: It is evident that $H^0(W_F, \mathbb{Z}) = \mathbb{Z}$. Since $H^1(W_{K/F,S}, \mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(W_{K/F,S}, \mathbb{Z}) = 0$ (because $W_{K/F,S}$ is an extension of a compact group by a connected group), we have $H^1(W_F, \mathbb{Z}) = 0$.

We have the following result of Moore [Moo76, Th. 9, p. 29], as quoted by Rajan [Raj04, Prop. 5], concerning the Moore cohomology groups $H^q_M(G, *)$:

LEMMA 3.3. *Let G be a locally compact group. Let N be a closed normal subgroup of G and let A be a locally compact*, *complete metrizable topological G-module. Then there is a spectral sequence*

$$E_2^{p,q} \Rightarrow H_M^{p+q}(G, A)$$

with the property that $E_2^{p,q} = H^p_M(G/N, H^q_M(N, A))$ *if* $q = 0$, $q = 1$, *or when* $H^q_M(N, A) = 0$.

LEMMA 3.4. *We have*

$$H^q(W_{K/F,S}, \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } q = 0, \\ \mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R} & \text{if } q = 1, \\ 0 & \text{if } q > 1. \end{cases}$$

Proof. We know by Corollary 2.6 that these cohomology groups are given by the continuous cochain cohomology. It is well known (see [BW80]) that if G is compact, then $H^q(G, \mathbb{R}) = 0$ for $q > 0$. We also know $H^0(\mathbb{R}, \mathbb{R}) = \mathbb{R}$, $H^1(\mathbb{R}, \mathbb{R}) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}, \mathbb{R})$, and $H^q(\mathbb{R}, \mathbb{R}) = 0$ for $q > 1$. The result then follows from Lemma 3.3 and the fact that we have the exact sequence

$$1 \to W^1_{K/F,S} \longrightarrow W_{K/F,S} \longrightarrow \mathbb{R} \to 1, \text{ with } W^1_{K/F,S} \text{ compact.}$$ □

LEMMA 3.5. $H^q_M(\mathbb{R}, \mathbb{Z}) = 0$ *for* $q > 0$.

Proof. The group $H^q_M(\mathbb{R},\mathbb{Z})$ is equal to $H^q_{\mathrm{Wig}}(\mathbb{R},\mathbb{Z})$, which by [Wig73, Th. 4] is equal to $H^q(B_{\mathbb{R}},\mathbb{Z})$, which is itself 0 for $q>0$ because $\mathbb{R}$ is contractible. □

We see from Lemma 3.4 and the exact sequence $0\to\mathbb{Z}\to\mathbb{R}\to S^1\to 0$ that $0\to\mathrm{Hom}_{\mathrm{cont}}(\mathbb{R},\mathbb{R})\to H^1(W_{K/F,S},S^1)\to H^2(W_{K/F,S},\mathbb{Z})\to 0$. Since there is a natural isomorphism between abelianized Weil group $(W_{K/F,S})^{\mathrm{ab}}$ and $C_{F,S}$, we have that $H^1(W_{K/F,S},S^1)$ is the Pontriagin dual $C^D_{F,S}$ of $C_{F,S}$, which yields that $H^2(W_{K/F,S},\mathbb{Z})$ is the Pontriagin dual $(C^1_{F,S})^D$ of the idéle class group of norm one. By taking limits over K and S, we obtain that $H^2(W_F,\mathbb{Z})=(C^1_F)^D$.

We next wish to show that $H^3(W_F,\mathbb{Z})=0$, and to do this it is, by Lemma 3.4, enough to show that $H^2(W_F,S^1)=0$. We first observe that Rajan's proof in [Raj04] that the Moore cohomology groups $H^2_M(W_F,S^1)=0$ works equally well to show that $H^2_M(W^1_F,S^1)=0$. Since for Moore cohomology, the cohomology of the projective limit of compact groups is the direct limit of the cohomologies, (by [Mil80] or [Raj04]) we have that

$$\begin{aligned}0=H^2_M(W^1_F,S^1)&=\varinjlim H^2_M(W^1_{K/F,S},S^1)\\&=\varinjlim H^2(W^1_{K/F,S},S^1)\quad\text{(by Remark 2.2)}\\&=\varinjlim H^3(W^1_{K/F,S},\mathbb{Z})\quad\text{(by Lemma 3.4).}\end{aligned}$$

It is easy to see that the Weil group $W_{K/F,S}$ is the direct product (in both the algebraic and topological senses) of $W^1_{K/F,S}$ and $\mathbb{R}$. Applying the Hochschild-Serre spectral sequence (from Lemma 3.3) coming from the exact sequence $1\to\mathbb{R}\to W_{K/F,S}\to W^1_{K/F,S}\to 1$, and using Lemma 3.5, we conclude that

$$H^q(W_{K/F,S},\mathbb{Z})=H^q(W^1_{K/F,S},\mathbb{Z}).$$

So,

$$\begin{aligned}H^3(W_F,\mathbb{Z})&=\varinjlim H^3(W_{K/F,S},\mathbb{Z})\quad\text{(by definition)}\\&=\varinjlim H^3(W^1_{K/F,S},\mathbb{Z})=0.\end{aligned}$$

We sum up what we have shown in the following theorem:

THEOREM 3.6. *The cohomology groups $H^q(W_F,\mathbb{Z})$ for $q\le 3$ are*

$$H^0(W_F,\mathbb{Z})=\mathbb{Z},\qquad H^2(W_F,\mathbb{Z})=(C^1_F)^D\quad(\textit{the Pontriagin dual of } C^1_F),$$
$$H^1(W_F,\mathbb{Z})=0,\qquad H^3(W_F,\mathbb{Z})=0.$$

Unfortunately so far we have not succeeded in computing the cohomology groups $H^q(W_F,\mathbb{Z})$ for $q>3$.

4. The Weil-étale topology for number rings

Let F be a number field, and choose an algebraic closure $\overline{F}$ of F. Let $G_F = G(\overline{F}/F)$ be the Galois group of $\overline{F}/F$.

Let v be a valuation of F, and F_v the completion of F at v. Choose an algebraic closure $\overline{F}_v$ of F_v, and an embedding of $\overline{F}$ in $\overline{F}_v$. Choose a global Weil group W_F and a local Weil group W_{F_v}. For each finite extension E of F in $\overline{F}$, let $E_v = EF_v$ be the induced completion of E. Let w be a valuation of $\overline{F}$ lying over v, and let i_w^* be the natural inclusion of G_{F_v} in G_F whose image is the decomposition group of w.

Definition 4.1. A Weil map θ_v is a continuous homomorphism from W_{F_v} to W_F such that there exists a valuation w of $\overline{F}$ such that the diagrams

$$\begin{array}{ccc} W_{F_v} & \longrightarrow & G_{F_v} \\ {\scriptstyle\theta_v}\downarrow & & \downarrow{\scriptstyle i_w^*} \\ W_F & \longrightarrow & G_F \end{array} \qquad\qquad \begin{array}{ccc} E_v^* & \longrightarrow & W_{E_v}^{\mathrm{ab}} \\ {\scriptstyle n_v}\downarrow & & \downarrow \\ C_E & \longrightarrow & W_E^{\mathrm{ab}} \end{array}$$

are commutative for all finite extension fields E of F, where n_v maps $a \in E_v^*$ to the class of the idéle whose v-component is a and whose other components are 1, and the map from $W_{E_v}^{\mathrm{ab}}$ to W_E^{ab} is induced by θ_v.

It is an easy consequence of [Tat79] that Weil maps always exist, and are unique up to an inner automorphism of W_F.

The local Weil group $W_v = W_{F_v}$ maps to ${W_v}^{\mathrm{ab}} = F_v^*$, which in turn maps to $\mathbb{Z}$ by the valuation map v. Let I_v be the kernel of the composite map from W_v to $\mathbb{Z}$.

We choose once and for all a set of Weil maps $\theta_v : W_v \to W_{\mathbb{Q}}$ for all valuations v of $\mathbb{Q}$. If w is any valuation of a number field $F$and w lies over the valuation v of $\mathbb{Q}$, the inclusion of W_w in W_v and θ_v induce a Weil map $\theta_w : W_w \to W_F$.

Let $\overline{Y} = \overline{Y}_F$ be the set of all valuations of F. We require the trivial valuation v_0 to be in $\overline{Y}$, corresponding to the generic point of Spec O_F, where O_F is the ring of integers of F. Let $W_{\kappa(v)}$ be $\mathbb{Z}$ if v is non-archimedean, $\mathbb{R}$ if v is archimedean, and W_F if $v = v_0$. We say that v is a specialization of w if w is v_0 and v is not. In each case there is a natural map π_v from the local Weil group W_v to $W_{\kappa(v)}$, and we let I_v be its kernel. It is an easy exercise to verify that if K_w is a finite Galois extension of F_v, then the map π_v factors through W_{K_w/F_v}.

Let K be a finite Galois extension of F. Let S be a finite set of nontrivial valuations of F, containing all the valuations of F that ramify in K. We now define a Grothendieck topology $T()K, S, \overline{Y}$:

We first define a category Cat $T()K, S, \overline{Y}$. The objects of Cat $T()K, S, \overline{Y}$ are collections $((X_v), (f_v))$, where v runs through all points of $\overline{Y}$, X_v is a $W_{\kappa(v)}$-space, and if v is a specialization of w, then $f_v : X_v \to X_w$ is a map of W_v-spaces. (We regard X_v as a W_v-space via π_v, and X_w as a W_v-space via the Weil map θ_v.) If $v = v_0$, we require that the action of W_F on X_v factor through $W_{K/F,S}$.

A morphism g from $\mathscr{X} = ((X_v), (f_v))$ to $\mathscr{X}' = ((X'_v), (f'_v))$ is a collection of W_v-maps $g_v : X_v \to X'_v$ such that $g_{v_0} f_v = f'_v g_v$.

We say that g is a local section morphism if the maps g_v from X_v to $g_v(X_v)$ admit local sections.

We define the fibered product of $((X_{1,v}), (g_{1,v}))$ and $((X_{2,v}), (g_{2,v}))$ over $((X_{3,v}), (g_{3,v}))$ by $((X_{1,v} \times_{X_{3,v}} X_{2,v}), ((g_{1,v} \times g_{2,v}))$.

We define the coverings $\operatorname{Cov}(T()K, S, \overline{Y})$ by saying a family of morphisms in our category $\{((X_{i,v}), (f_{i,v})) \to ((X_v), (f_v))\}$ is a cover if $\{X_{i,v} \to X_v\}$ is a local section cover for all v.

Our category clearly has a final object $*_{(K,S)}$ whose components are the one-point space for each v in $\overline{Y}$.

If E is a sheaf on $T()K, S, \overline{Y}$, we define

$$H^i(\overline{Y}_{K,S}, E) = H^i(T()K, S, \overline{Y}, *_{(K,S)}, E).$$

We now define a topology $T()\overline{Y}$ by letting the category $\operatorname{Cat} T()\overline{Y}$ be the union of the categories $T()K, S, \overline{Y}$, and we let $\operatorname{Cov} T()\overline{Y}$ be the union of the covers $\operatorname{Cov} T()K, S, \overline{Y}$.

If E is a sheaf on $T()\overline{Y}$, we let $E_{K,S}$ be its restriction to $T()K, S, \overline{Y}$, and we define $H^i(\overline{Y}, E)$ to be the direct limit over K and S of the $H^i(\overline{Y}_{K,S}, E_{K,S})$.

Let w be a valuation of F. If $w \neq v_0$, we define a morphism of topologies i_w^{-1} from $T()K, S, \overline{Y}$ (resp. $T()\overline{Y}$) to $T()W_{\kappa(w)}$ by $i_w^{-1}((X_v), (f_v)) = X_w$. If $w = v_0$, we define similarly a morphism of topologies i_w^{-1} from $T()K, S, \overline{Y}$ (resp. $T()\overline{Y}$) to $T()W_{K/F,S}$ (resp. $T()W_F$). We have the corresponding direct image maps $(i_w)_*$ from sheaves on $T()W_{\kappa(v)}$ or sheaves on $T()W_{K/F,S}$ to sheaves on $T()K, S, \overline{Y}$ by $(i_w)_*(E)((X_v), (f_v)) = E(X_w)$. For psychological reasons we define $j_{K,S}$ to be i_{v_0}. It is clear that i_v^{-1} preserves covers and fibered products, and so is a morphism of topologies.

Definition 4.2. Let $\theta : H \to G$ be a morphism of topological groups, and let X be an H-space. Define an equivalence relation $\sim$ on $X \times G$ by $(x, g) \sim (x', g')$ if and only if there exists a $\tau \in H$ such that $x' = \tau x$ and $g' = g\theta(\tau^{-1})$. Define $X \times^H G$ to be the quotient (with the quotient topology) of $X \times G$ by $\sim$. We make $X \times^H G$ into a G-space by $g(x, g') = (x, gg')$.

Remark. The functor that takes an H-space X to the G-space $X \times^H G$ is easily seen to be left adjoint to the forgetful functor from G-spaces to H-spaces, where we regard a G-space as an H-space via θ.

LEMMA 4.3. *Let G be a topological group, and let I be a closed subgroup such that the projection ρ from G to G/I admits local sections. Then the category of G-spaces with maps to G/I is equivalent to the category of I-spaces, and the covers in the respective categories correspond.*

Proof. If X is an I-space, let $\alpha(X) = (X \times^I G, \lambda)$, where

$$\lambda : X \times^I G \to G/I, \quad (x, \sigma) \mapsto \text{the coset } \sigma I.$$

If Z is a G-space with a map $\pi : Z \to G/I$, let $\beta(Z, \pi) = \pi^{-1}(I)$. It is straightforward to verify that α and β are inverse functors. □

We now claim that the covers correspond.

LEMMA 4.4. *If $\rho : G \to G/I$ and the cover $\{X_i \to X\}$ both admit local sections, then the cover $\{X_i \times^I G \to X \times^I G\}$ admits local sections.*

Proof. Let $y = [x, \sigma]$ be the class of (x, σ) in $X \times^I G$. Let U be a neighborhood of $\rho(\sigma)$ such that there exists a continuous section $s : U \to G$ of ρ. Let $V = \rho^{-1}(U)$. Let $U^* = s(U)$. We claim that $X \times^I V$ is functorially isomorphic to $X \times U^*$. It is immediate that given $[x, v]$ in $X \times^I V$, there exists a unique pair (x', v') in $X \times U^*$ such that $[x, v] = [x', v']$. In fact $(x', v') = ((s\rho(v))^{-1} v x, s\rho(v))$.

So if $\{X_i \to X\}$ admits local sections, so does $\{X_i \times U^* \to X \times U^*\}$, and then so does $\{X_i \times^I V \to X \times^I V\}$ and therefore also $\{X_i \times^I G \to X \times^I G\}$. □

LEMMA 4.5. *If I is a locally compact subgroup of a Hausdorff topological group G, the natural projection from G to G/I admits local sections.*

Proof. This is proved in [Mil80]. □

So we have proved this:

THEOREM 4.6. *Let G be a Hausdorff topological group, I a locally compact subgroup, and A a continuous G-module. Then $H^i(T(G), G/I, \tilde{A})$ is naturally isomorphic to $H^i(I, A) = H^i(T()I, *, \tilde{A})$ (see Definition 1.2).*

THEOREM 4.7. *Let $j = j_{\overline{Y}}$, and let A be a topological W_F-module. There exists a spectral sequence*

$$E_2^{p.q} = H^p(\overline{Y}, R^q j_* \tilde{A}) \Rightarrow H^{p+q}(W_F, A).$$

Proof. This follows from [Art62, p. 44] by applying his Theorem 4.11 to $j = j_{K,S}$ and taking direct limits over K and S. □

The rest of this section will be devoted to computing the sheaves $R^q j_* \tilde{A}$. Let v be in $\overline{Y}$. Our goal is to prove the following:

THEOREM 4.8. *Let A be a $W_{K/F,S}$-module, let $q > 0$, and let $B = B_q$ be $R^q(j_{K,S})_* \tilde{A}$. Then the natural map from B to $\coprod_{v \in S} i_{v*} i_v^* B$ given by adjointness is an isomorphism of sheaves.*

LEMMA 4.9. *Let E be a Weil-étale sheaf on $T(_{}K, S, \overline{Y}$. Suppose $i_v^* E = 0$ for all $v \in \overline{Y}$. Then $E = 0$.*

Proof. We know that $i_v^* E$ is the sheafification of the presheaf inverse image $i_v^p E$. If X_v is a $W_{\kappa(v)}$-space, $i_v^p E(X_v)$ is the direct limit of $E(U)$, where $U = ((X_v'), (f_v))$ is an object of $\mathrm{Cat}(T(_{}K, S, \overline{Y})$ such that there is a map from X_v to $i_v^{-1}(U) = X_v'$. Since there exists a U (for example the object which has X_v at v, has $X_v \times^{W_v} W_{K/F,S}$ at the generic point, and has the empty set elsewhere) with $i_v^{-1}(U) = X_v$, we may always assume that $i_v^{-1}(U) = X_v$, i.e., that $X_v' = X_v$.

More generally, if $h_v : Z_v \to X_v$ and $f_v : X_v \to X_{v_0}$ are maps of $W_{\kappa(v)}$- and W_v-spaces, respectively, then the map $h_v' : X_{v_0}' = Z_v \times^{W_v} W_{K/F,S} \to X_{v_0}$, $(z, w) \mapsto w f_v h_v(z)$ is well defined and so we get a map of $(X_{v_0}', Z_v, \phi, \ldots \phi)$ to $(X_{v_0}, X_v, (X_w))$ that induces the original map h_v.

If $U = (X_{v_0}, (X_v), (f_v))$ is an object of $\mathrm{Cat}\, T(_{}K, S, \overline{Y}$, and $\alpha \in E(U)$, then $i_v^* E = 0$ implies there is a covering $\{X_{v_i}\}$ of X_v such that α goes to zero in each $i^p E(X_{v_i}) = E(U_i)$, where the v component of U_i is $X_{v,i}$. By the argument in the preceding paragraph, we can induce these coverings from families of maps to U, and the collection of all these families will be a covering of U in which α goes to zero, thus making $\alpha = 0$. □

LEMMA 4.10.

(a) *i_v^* is exact.*

(b) *$i_v^* i_{v*} i_v^* E$ is canonically isomorphic to $i_v^* E$.*

(c) *$i_w^* i_{v*} = 0$ if $v \neq w$.*

(d) *i_{v*} is exact.*

Proof. (a) Since i_v^* is a left adjoint, it is right exact. Suppose that the sheaf E injects into the sheaf E', and that $\alpha \in i_v^* E(X_v)$ goes to zero in $i_v^* E'(X_v)$. There exists a Weil-étale cover $(X_{v,i})$ of X_v and objects $\mathscr{X}_i$ of $\mathrm{Cat}\, T(_{}K, S, \overline{Y}$ such that for each i, α restricted to $X_{v,i}$ comes from an element β_i in $E(\mathscr{X}_i)$, $(\mathscr{X}_i)_v = X_{v,i}$, and the image of β_i in $E'(\mathscr{X}_i)$ is equal to zero. Hence $\beta_i = 0$ and since α goes to zero in a Weil-étale cover, we have $\alpha = 0$.

(b) This is a formal consequence of the fact that i_v^* is left adjoint to i_{v*}.

(c) Note that $(i_w^p F)(X_w)$ is the direct limit of $F((X_{w_0}, X_w, \phi, \ldots, \phi), f_w)$, where $f_w : X_w \to X_{w_0}$. If $F = (i_v)_* E$ and $v \neq w$, then

$$F((X_{w_0}, X_w, \phi, \ldots, \phi), f_w) = E(\phi) = 0.$$

(d) This follows immediately from the fact that v is a specialization of v_0, and if $\mathscr{X} = (X_{v_0}, X_v, (X_w (w \neq v, v_0))$, then any covering $X_{v,i}$ of $X_v = i_v^{-1}(\mathscr{X})$ comes from the covering $\mathscr{X}_i = (X_{v_0}, X_{v,i}, (X_w))$ of $\mathscr{X}$. □

LEMMA 4.11. *If v is not in S, the natural map induced by θ_v from W_v to $W_{K/F,S}$ annihilates the kernel I_v of the natural map from W_{F_v} to $\mathbb{Z}$.*

Proof. Let w be the valuation lying over v determined by θ_v, so that w is unramified over v. The diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_w^* & \longrightarrow & W_{K_w/F_v} & \longrightarrow & G(K_w/F_v) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \pi_v} & & \downarrow{\scriptstyle i} & & \\ 1 & \longrightarrow & \mathbb{Z} & \xrightarrow{f} & \mathbb{Z} & \longrightarrow & G(\kappa(w)/\kappa(v)) & \longrightarrow & 1 \end{array}$$

is commutative because the fundamental classes of the two extensions correspond. (Here i is the natural isomorphism and f is the residue field degree.) It follows that the image of I_v in W_{K_w/F_v} is isomorphic to the unit group $\mathrm{Ker}(w)$, which goes to zero in $C_{K/F,S}$ and so *a fortiori* in $W_{K/F,S}$. □

LEMMA 4.12. *Let E be the sheaf $R^q(j_{K,S})_*\tilde{A}$, with $q > 0$. If v is not in S, then $i_v^* E = 0$.*

Proof. Given a $W_{\kappa(v)}$-space X_v and an element α in $i_v^p(E)(X_v)$, we will produce a cover $\{X_{v,i}\}$ of X_v so that the restriction of α vanishes on each $X_{v,i}$. By Lemma 4.11, if v is not in S, the Weil map θ_v from W_v to $W_{K/F,S}$ factors through $W_{\kappa(v)}$. So let us define X_{v_0} to be $X_v \times^{W_{\kappa(v)}} W_{K/F,S}$. By using the definition of i_v^p, we see easily that $i_v^p(E)(X_v)$ is $E(X_{v_0}, X_v, \dots, \phi, \dots)$, where all the spaces X_w in which $w \neq v, v_0$ are empty. By passing to a cover, we may assume that α comes from an element β in $H^q(X_{v_0}, \tilde{A})$. Since $q > 0$ and higher cohomology dies in a cover (for instance, apply [Mil80, Ch. III, Prop. 1.13] to the identity map), we may choose a cover $X_{v_0,i}$ of X_{v_0} such that β goes to zero in $H^q(X_{v_0,i}, \tilde{A})$. Letting $X_{v,i} = X_{v_0,i} \times_{X_{v_0}} X_v$, we see that α goes to zero on each $X_{v,i}$. □

Proof of Theorem 4.8. By Lemma 4.12,

$$\prod_{v \in \overline{Y}} (i_v)_* i_v^* B = \prod_{v \in S} (i_v)_* i_v^* B.$$

By Lemma 4.10, the map from B to $\prod_{v \in S} (i_v)_* i_v^* B$ induces an isomorphism on stalks, and hence is an isomorphism by Lemma 4.9. □

Let $j = j_{K/F,S}$. We obtain a corollary from Theorem 4.8, Lemma 4.10(b), and the fact that cohomology commutes with direct products:

COROLLARY 4.13. *If $q > 0$, then*

$$H^p(\overline{Y}, R^q j_* \tilde{A}) = \coprod_{v \in S} H^p(W_{\kappa(v)}, i_v^* R^q j_* \tilde{A}).$$

The next section will be devoted to computing these cohomology groups for small values of p and q.

In any Grothendieck topology, we have the presheaf $\mathbb{Z}'$, defined by assigning the group $\mathbb{Z}$ to any object and the identity to any map. We define $\mathbb{Z}$ to be the sheaf associated with the presheaf $\mathbb{Z}'$.

LEMMA 4.14. *For any morphism of Grothendieck topologies, the inverse image of the sheaf* $\mathbb{Z}$ *is again* $\mathbb{Z}$.

Proof. This is an easy exercise. □

If the topology is $T(G)$, we also have the sheaf $\tilde{\mathbb{Z}}$, which corresponds to the trivial G-module $\mathbb{Z}$ and is characterized by $\tilde{\mathbb{Z}}(X) = \mathrm{Map}_G(X, \mathbb{Z})$. We can define a map from the presheaf $\mathbb{Z}$ to $\tilde{\mathbb{Z}}$ by sending n to the map with the constant value n. This is clearly injective and induces an injection from $\mathbb{Z}$ to $\tilde{\mathbb{Z}}$.

This map is also surjective. Let $f : X \to \mathbb{Z}$, and let $X_n = f^{-1}(n)$. The X_n form a disjoint open cover of X, and the Cech cohomology H^0 of the presheaf $\mathbb{Z}'$ with respect to this cover contains an element g that is n on X_n. Then g determines an element of $\mathbb{Z}$ that maps onto f. An easy extension of these arguments yields the following proposition.

PROPOSITION 4.15. *The natural map* ϕ *from the sheaf* $\mathbb{Z}$ *on* $T()K, S, \overline{Y}$ *to the sheaf* $j_* j^* \mathbb{Z} = j_* \mathbb{Z}$ *is an isomorphism.*

5. The computation of $H^p(\overline{Y}, R^q j_* \mathbb{Z})$ and $H^p(\overline{Y}, R^q j_* \mathbb{R})$

LEMMA 5.1. *Let* G *be a discrete group and* E *a sheaf on* $T(G)$. *Then the canonical map from* $\widetilde{E(G)}$ *to* E *induces an isomorphism in cohomology.*

Proof. Because G is discrete, any covering of a discrete G-space X by G-spaces X_i has a refinement consisting of the X_i with the discrete topology. It then follows by a standard comparison theorem in the theory of Grothendieck topologies that the $T(G)$-cohomology of any discrete G-space X is the same as the cohomology of X in the standard topology of discrete G-sets and families of surjective morphisms. But sheaves in this topology may be identified with G-modules by making a sheaf F correspond to the G-module $F(G)$. (G is a left G-space by left multiplication, and the G-action on $F(G)$ is induced by letting $\sigma \in G$ act on G by right multiplication by σ^{-1}.) □

In view of all this, we may identify the cohomology groups $H^p(T(G), *, E)$ with the groups $H^p(G, E(G))$, where the cohomology is defined by the usual cochain definition.

LEMMA 5.2. *Let* v *be a finite place, let* A *be a continuous* W_F*-module, let* $E = i_v^* R^q j_{K,S*} \tilde{A}_{K,S}$, *and let* $G = \mathbb{Z} = W_{\kappa(v)}$. *Let* $\theta_{v,K,S}$ *be the map obtained by composing* θ_v *with the natural projection from* W_F *to* $W_{K/F,S}$. *Then*:

(a) $E(G) = H^q(\theta_{v,K,S}(I_v), A_{K,S})$, *and hence*

$$H^p(W_{\kappa(v)}, i_v^* R^q j_{K,S*}\tilde{A}_{K,S}) = H^p(W_{\kappa(v)}, H^q(\theta_{v,K,S}(I_v), A_{K,S}));$$

(b) $\varinjlim_{K,S} H^p(W_{\kappa(v)}, i_v^* R^q j_{K,S*}\tilde{A}_{K,S}) = H^p(W_{\kappa(v)}, H^q(I_v, A))$.

Proof. Let i_v^p denote the presheaf inverse image. Recall that, if C is a presheaf on $T()K, S, \overline{Y}$, $i_v^p(C)(G)$ is given by the direct limit of those $C(\mathscr{X})$ for which there is a map $\phi : G \to i_v^{-1}(\mathscr{X})$. Since the object $\mathscr{X}'$ defined by

$$X_{v_0} = W_{K/F,S}/\theta_{v,K,S}(I_v), \quad X_v = W_{\kappa(v)}, \quad X_w = \phi \quad \text{for } w \neq v, v_0$$

is a final object in the set over which the limit defining i^p is taken, we see that if L is any sheaf on $T()K/F, S$, then $i^p L(G) = L(\mathscr{X}')$. Since G has no nontrivial covers, it is immediate that $i_v^*(L)$ is the G-module $L(G)$. Since $\mathscr{X}$ has no nontrivial covers, if E' is any sheaf on $T()W_{K/F,S}$, then

$$i_v^* R^q j_{K,S*} E'(G) = H^q(T()W_{K/F,S}, W_{K/F,S}/\theta_{v,K,S}(I_v), E').$$

Hence $E(G)$ is naturally isomorphic to $H^q(T()W_{K/F,S}, W_{K/F,S}/\theta_v(I_v), A_{K,S})$.

By Theorem 4.6, this is just $H^q(\theta_{v,K,S}(I_v), A_{K,S})$, and an application of Lemma 5.1 completes the proof of (a). Now observe that since $W_{\kappa(v)} = \mathbb{Z}$,

$$\varinjlim_{K,S}(H^q(W_{\kappa(v)}, H^q(\theta_{v,K,S}(I_v), A_{K,S})) = H^p(W_{\kappa(v)}, \varinjlim_{K,S} H^q(\theta_{v,K,S}(I_v), A_{K,S})).$$

Since $\theta_{v,K,S}(I_v)$ is compact and Moore cohomology commutes with limits for compact groups, this is in turn equal to

$$H^p(W_{\kappa(v)}, H^q(\varinjlim_{K,S} \theta_{v,K,S}(I_v), A_{K,S})),$$

which, by Lemma 3.1, is equal to $H^p(W_{\kappa(v)}, H^q(I_v, A))$, which shows (b). □

LEMMA 5.3. *Let v be a finite place. Then*:

(a) $H^1(I_v, \mathbb{Z}) = H^1(I_v, \mathbb{R}) = 0$,

(b) $H^0(W_{\kappa(v)}, H^2(I_v, \mathbb{Z}))$ *is naturally isomorphic to the Pontriagin dual U_v^D of the local units U_v in the completion F_v of the field F at v, and*

(c) $H^0(W_{\kappa(v)}, H^2(I_v, \mathbb{R})) = 0$.

Proof. If $A = \mathbb{Z}$ or $\mathbb{R}$, then $H^1(I_v, A) = \mathrm{Hom}(I_v, A) = 0$. From the exact sequence $1 \to I_v \to G_v \to \hat{\mathbb{Z}} \to 1$, we get the Hochschild-Serre spectral sequence $H^p(\hat{\mathbb{Z}}, H^q(I_v, \mathbb{Z})) \Rightarrow H^{p+q}(G_v, \mathbb{Z})$. This spectral sequence yields the short exact sequence $0 \to H^2(\hat{\mathbb{Z}}, \mathbb{Z}) \to H^2(G_v, \mathbb{Z}) \to H^0(\hat{\mathbb{Z}}, H^2(I_v, \mathbb{Z})) \to 0$. By local class field theory $H^2(G_v, \mathbb{Z})$ is naturally isomorphic to $\mathrm{Hom}(F_v^*, \mathbb{Q}/\mathbb{Z})$, so the above exact sequence shows that $H^0(\hat{\mathbb{Z}}, H^2(I_v, \mathbb{Z}))$ is naturally isomorphic to

$\mathrm{Hom}(U_v, \mathbb{Q}/\mathbb{Z})$, which (since U_v is profinite) is the Pontriagin dual of U_v. But since $W_{\kappa(v)} = \mathbb{Z}$ is dense in $\hat{\mathbb{Z}}$, (b) follows immediately. Since I_v is compact, $H^2(I_v, \mathbb{R}) = 0$, which proves (c). ☐

LEMMA 5.4. *Let $\theta : H \to G$ be a map of topological groups, so that we may regard any G-space as an H-space via θ. Let I be a topological subgroup of H. Let Z be any topological space, regarded as a G-space with trivial G-action, and let X be any G-space. Then, given any H-map ϕ from $H/I \times Z$ to X, there is a unique G-map ψ from the G-space $G/\theta(I) \times Z$ to X such that ϕ factors through ψ.*

Proof. This follows immediately from the remark after Definition 4.2. ☐

LEMMA 5.5. *Let G be a connected topological group, and let X be a topological space on which G acts trivially. Then $\check{H}^q(T(G), X, \mathbb{Z})$ is naturally isomorphic to $\check{H}^q_{\mathrm{top}}(X, \mathbb{Z})$.*

Proof. We first claim that any local-section G-cover $\rho_i : \{X_i \to X\}$ has a refinement by a cover of the form $\{G \times V_i\}$, where $\{V_i\}$ is an open cover of X, and G acts on $G \times V_i$ by left multiplication on the first factor. Given $x \in X$, let U_x be an open neighborhood of x such that $s_x : U_x \to X_{i(x)}$ is a section of $\rho_{i(x)}$. Define $\phi_x : G \times U_x \to X_{i(x)}$ by $\phi_x(g, u) = g s_x(u)$, and verify first that ϕ_x is a G-map and next that $\mathrm{pr}_2 = \rho_i(x)\phi_x$, thus showing that $\{G \times U_x\}$ refines $\{X_i\}$.

We next claim that the Čech complex for the sheaf $\mathbb{Z}$ of the G-cover $\{G \times V_i\}$ is the same as the Čech complex of the cover $\{V_i\}$ of X. This follows immediately because any map from a power G^n of the connected group G to the discrete group $\mathbb{Z}$ is constant. ☐

LEMMA 5.6. *Let Z be a contractible topological space. Let v be a fixed archimedean place of $\overline{Y}_{K/F,S}$, and let $H = W_{\kappa(v)}$. Let H act on $H \times Z$ by left multiplication on the first factor. We claim that*

(a) $(i_v^p R^1 j_* \mathbb{Z})(H \times Z) = 0$,

(b) $(i_v^p R^2 j_* \mathbb{Z})(H) = H^2(I_v, \mathbb{Z})$,

(c) $(i_v^* R^2 j_* \mathbb{Z})(H) = H^2(I_v, \mathbb{Z})$.

Proof. Let E be any sheaf on $\overline{Y}_{K/F,S}$. Then by definition, $(i_v^p(E))(H \times Z)$ is equal to the direct limit of the $E((X_w), (f_w))$, where

$$H \times Z \to i_v^{-1}((X_w), (f_w)) = X_v.$$

Now let $E = R^q j_* \mathbb{Z}$. It is immediate that we may assume in the direct limit that $X_v = H \times Z$ and that X_w is the empty set if w is neither v nor the generic point v_0. Lemma 5.4 shows that we may assume that X_{v_0} is $W_{K/F}/\theta_v(I_v)$ and hence that

$$R^q j_* \mathbb{Z}((X_w), (f_w)) = H^q(T() W_{K/F}, W_{K/F}/\theta_v(I_v) \times Z, \mathbb{Z}).$$

By an easy generalization of Theorem 4.6, this is the same as $H^q(T()I_v, Z, \mathbb{Z})$. If $q = 1$, this is equal to $\check{H}^1(I_v, Z, \mathbb{Z})$, which in turn is equal by Lemma 5.5 to $\check{H}^1_{\mathrm{top}}(Z, \mathbb{Z})$, which is zero since Z is contractible.

If $q = 2$ and Z is a point, we have

$$i_v^p(R^2 j_* \mathbb{Z})(H) = H^2(T()I_v, *, \mathbb{Z}) = H^2(I_v, \mathbb{Z}).$$

Part (c) then follows immediately because H has no nontrivial covers. □

LEMMA 5.7. *Let G be a topological group, and let n be a positive integer. Then G^n, regarded as a G-space with G acting diagonally, is isomorphic to $G \times G^{n-1}$, where G acts by left multiplication on the first factor and trivially on G^{n-1}.*

Proof. Let $\phi : G^n \to G \times G^{n-1}$ by $\phi(g_1, \dots, g_n) = (g_1, g_1^{-1} g_2, \dots, g_{n-1}^{-1} g_n)$. It is easy to see that ϕ is a G-isomorphism. □

PROPOSITION 5.8. *Let v be an archimedean place.*

(a) $H^p(W_{\kappa(v)}, i_v^* R^1 j_* \mathbb{Z}) = 0$ *for* $p = 0, 1,$ *and* 2.

(a′) $H^p(W_{\kappa(v)}, i_v^* R^1 j_* \widetilde{\mathbb{R}}) = 0$ *for* $p = 0, 1,$ *and* 2.

(b) $H^0(W_{\kappa(v)}, i_v^* R^2 j_* \mathbb{Z}) = H^2(I_v, \mathbb{Z})^{W_{\kappa(v)}}$.

(b′) $H^0(W_{\kappa(v)}, i_v^* R^2 j_* \widetilde{\mathbb{R}}) = 0$.

(c) $H^2(I_v, \mathbb{Z})^{W_{\kappa(v)}} = U_v{}^D$.

Proof. We have the standard spectral sequence

$$E_2^{p,q} = \check{H}^p(W_{\kappa(v)}, \underline{H}^q(i_v^* R^1 j_* \tilde{A})) \Rightarrow H^{p+q}(W_{\kappa(v)}, i_v^* R^1 j_* \tilde{A})$$

from the Čech to the derived functor cohomology, where we know that $E_2^{0,q} = 0$ for $q > 0$.

We begin with the case $p = 2$. The spectral sequence immediately gives the exact sequence

$$0 \to \check{H}^2(W_{\kappa(v)}, i_v^* R^1 j_* \tilde{A}) \to H^2(W_{\kappa(v)}, i_v^* R^1 j_* \tilde{A}) \to \check{H}^1(W_{\kappa(v)}, \underline{H}^1(i_v^* R^1 j_* \tilde{A})).$$

So it suffices to show that the first and third terms in this exact sequence are zero. We begin with the first:

We first let $A = \mathbb{Z}$ and show that, more generally, $\check{H}^p(W_{\kappa(v)}, i_v^* R^1 j_* \mathbb{Z}) = 0$ for any p. Since the covering $\{H\}$ of $*$ is initial, it is enough to show that $(i_v^* R^1 j_* \mathbb{Z})(H^n) = 0$. Due to Lemma 5.7, this is equivalent to showing that $(i_v^* R^1 j_* \mathbb{Z})(H \times H^{n-1}) = 0$, where H acts trivially on H^{n-1}. But this is an immediate consequence of Lemmas 1.6 and 5.5, since H is contractible and locally contractible.

We next let $A = \mathbb{R}$. If E is any sheaf of $\mathbb{R}$ -vector spaces on $H \times H^{n-1}$ and $q > 0$, then Corollary 1.7 shows that $H^q_H(H \times H^{n-1}, E)$ is isomorphic to $H^q(H^{n-1}, \alpha_* E)$, which is equal to zero by Lemma 2.5.

Now we look at the third term. Since $H^1 = \check{H}^1$, we have to show that $\check{H}^1(H \times H^{n-1}, i_v^* R^1 j_* \tilde{A})$ is equal to 0. A typical term in a coinitial cover of $H \times H^{n-1}$ is $H^r \times X$, with X contractible, locally contractible, and metrizable. But rewriting this as $H \times (H^{r-1} \times X)$ and again using Lemma 5.5 in the case when $A = \mathbb{Z}$ and Lemma 2.5 when $A = \mathbb{R}$ enables us to copy the arguments of the preceding paragraph, since $H^{r-1} \times X$ is also contractible, locally contractible, and metrizable.

The case when $p = 1$ is similar but easier.

Now cases (b) and (b′) follow immediately from Lemma 5.6(c).

If v is complex, $H^2(I_v, \mathbb{Z}) = \mathbb{Z}$, and v is real, we have the exact sequence $1 \to S^1 \to I_v \to \mathbb{Z}/2\mathbb{Z} \to 1$. The Hochschild-Serre spectral sequence shows that $H^2(I_v, \mathbb{Z}) = H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. In both cases the Weil group $W_{\kappa(v)}$ acts trivially, and we get $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$, the respective duals of S^1 and ± 1. □

THEOREM 5.9. *Let A be either* $\mathbb{Z}$ *or* $\mathbb{R}$.

(a) $H^p(\overline{Y}_{K,S}, R^1(j_{K,S})_* \tilde{A}) = 0$ *for* $p = 0, 1, 2$.

(b) $H^p(\overline{Y}, R^1 j_* \tilde{A}) = 0$ *for* $p = 0, 1, 2$.

(c) $H^0(\overline{Y}_{K,S}, R^2(j_{K,S})_* \mathbb{Z}) = \coprod_{v \in S} (U_v)^D$.

(c′) $H^0(\overline{Y}, R^2(j_{K,S})_* \widetilde{\mathbb{R}}) = 0$.

(d) $H^0(\overline{Y}, R^2 j_* \mathbb{Z}) = \coprod_{v \neq v_0} (U_v)^D$.

(d′) $H^0(\overline{Y}, R^2 j_* \widetilde{\mathbb{R}}) = 0$.

Proof. Parts (a) and (c) follow immediately from Corollary 4.13, Lemmas 5.2 and 5.3, and Proposition 5.8. Parts (b) and (d) follow from (a) and (c) by taking limits.

Let E be either $\mathbb{R}$ or S^1. We will also use E (i.e., either $\mathbb{R}$ or S^1 as the case may be) to denote the sheaf on $\overline{Y}$ determined by defining $E(X_{v_0}, (X_v), (f_v))$ to be the set of compatible continuous W_v-maps from X_v to E, where W_v acts trivially on E. It is clear both that this is a sheaf and that such a set is determined by giving a W_{v_0}-map from X_{v_0} to E. It is also clear that this is the same sheaf as the sheaf $j_* \widetilde{E}$.

The Leray spectral sequence for the map j_* yields

$$0 \to H^1(\overline{Y}, \mathbb{R}) \longrightarrow H^1(W_F, \mathbb{R}) \longrightarrow H^0(\overline{Y}, R^1 j_* \widetilde{\mathbb{R}}) \longrightarrow H^2(\overline{Y}, \mathbb{R}) \longrightarrow H^2(W_F, \mathbb{R}) = 0,$$

where $H^2(W_F, \mathbb{R}) = 0$ by Lemma 3.4.

But $R^1 j_* \widetilde{\mathbb{R}}$ is isomorphic to $\coprod (i_v)_* i_v^* R^1 j_* \widetilde{\mathbb{R}}$, and so we conclude easily that $H^1(\overline{Y}, R^1 j_* \mathbb{R})$ is isomorphic to $\coprod H^1(I_v, \mathbb{R})$, where the sums are taken over all nontrivial valuations of F. But whether v is archimedean or non-archimedean,

I_v is compact, so $H^1(I_v, \mathbb{R}) = \mathrm{Hom}(I_v, \mathbb{R}) = 0$. We conclude that $H^1(\overline{Y}, \mathbb{R}) = H^1(W_F, \mathbb{R}) = \mathrm{Hom}(W_F, \mathbb{R})$ and that $H^2(\overline{Y}, \mathbb{R}) = 0$. □

We have proved this:

THEOREM 5.10.

(a) $H^0(\overline{Y}, j_*\widetilde{\mathbb{R}}) = \mathbb{R}$.

(b) $H^1(\overline{Y}, j_*\widetilde{\mathbb{R}}) = \mathbb{R}$.

(c) $H^2(\overline{Y}, j_*\widetilde{\mathbb{R}}) = 0$.

Proof. Part (a) is clear, and (b) and (c) follow from the Leray spectral sequence, using Theorem 5.9. □

We observe that the sequence of sheaves $0 \to j_*\mathbb{Z}(= \mathbb{Z}) \to \mathbb{R} \to S^1$ on $\overline{Y}$ is exact. It is clearly left exact, and the argument in the proof of Proposition 1.3 shows right exactness.

Let $\mathrm{Pic}(\overline{Y})$ be the Arakelov class group of F, i.e., the group obtained by taking the idèle group of F and dividing by the principal idèles and the unit idèles (a unit idèle (u_v) is defined by $|u_v|_v = 1$ for all v). Let $\mathrm{Pic}^1(\overline{Y})$ be the kernel of the absolute value map from $\mathrm{Pic}(\overline{Y})$ to $\mathbb{R}^*$. Let $\mu(F)$ denote the group of roots of unity in F.

THEOREM 5.11.

(a) $H^0(\overline{Y}, \mathbb{Z}) = \mathbb{Z}$.

(b) $H^1(\overline{Y}, \mathbb{Z}) = 0$.

(c) $H^2(\overline{Y}, \mathbb{Z}) = (\mathrm{Pic}^1(\overline{Y})^D$.

(d) $H^3(\overline{Y}, \mathbb{Z}) = \mu(F)^D$.

Proof. We begin by recalling that $j_*\mathbb{Z} = \mathbb{Z}$ by Proposition 4.15. Then (a) is clear. The Leray spectral sequence for j_* gives $H^1(\overline{Y}, \mathbb{Z}) = H^1(W_F, \mathbb{Z}) = 0$, which proves (b). It also gives (using Theorem 5.9) the exact sequence

$$0 \to H^2(\overline{Y}, \mathbb{Z}) \to H^2(W_F, \mathbb{Z}) \to \coprod_{v \neq v_0} (U_v)^D \to H^3(\overline{Y}, \mathbb{Z}) \to H^3(W_F, \mathbb{Z}) = 0.$$

This is easily seen (using Theorem 3.6) to be the Pontriagin dual of the sequence

$$0 \to H^3(\overline{Y}, \mathbb{Z})^D \to \prod_{v \neq v_0} U_v \to C_F^1 \to H^2(\overline{Y}, \mathbb{Z})^D \to 0,$$

which completes the proof, since the roots of unity are the kernel of the map from the unit idèles to $C^1(F)$ and $\mathrm{Pic}^1(F)$ is defined to be the cokernel. □

6. Cohomology with compact support

Let Y be Spec O_F, and let φ be the natural inclusion of Y in $\overline{Y}$. Let E be any sheaf on Y. We define the sheaf $\varphi_! E$ on $\overline{Y}$ to be the sheaf associated with the presheaf P defined by $P((X_v),(f_v)) = E((X_v))$ if $X_v = \phi$ for all v not in Y, and $P((X_v),(f_v)) = 0$ otherwise. We note that $\varphi_!$ is exact.

PROPOSITION 6.1. *Let F be any sheaf on $\overline{Y}$. There is an exact sequence*

$$0 \to \varphi_!\varphi^* F \to F \to i_* i^* F \to 0$$

of sheaves on $\overline{Y}$, where $i_* i^* F = \prod_{v \in Y_\infty} (i_v)_* i_v^* F$.

Proof. We first show that for all v in $\overline{Y}$, there exists an exact sequence

$$0 \to i_v^* \varphi_! \varphi^* F \to i_v^* F \to i_v^* i_* i^* F \to 0.$$

We first see easily that if v is non-archimedean that $i_v^* \varphi_! \varphi^* F = i_v^* F$, and $i_v^*(i_* i^* F) = 0$ by Lemma 4.10(c), so we get exactness. If v is archimedean, $i_v^* \varphi_! \varphi^* F = 0$ and $i_v^*(i_* i^* F) = i_v^* i_* F$ by Lemma 4.10(b), so we get exactness again.

The exactness of the above exact sequences implies the proposition, using Lemma 4.9 and the fact that i_v^* is exact (from Lemma 4.10(a)). □

LEMMA 6.2. *Let v be an archimedean valuation. Then*:

(a) $H^i(W_{\kappa(v)}, \mathbb{Z}) = 0$ *for* $i > 0$, *and*

(b) $H^i(\overline{Y}, i_* \mathbb{Z}) = 0$ *for* $i > 0$.

Proof. Part (a) is immediate because $W_{\kappa(v)} = \mathbb{R}$, $\mathbb{R}$ is contractible, and $\mathbb{Z}$ is discrete. Then (b) follows becaue i_* is exact. □

THEOREM 6.3.

(a) $H^0(\overline{Y}, \varphi_! \mathbb{Z}) = 0$.

(b) $H^1(\overline{Y}, \varphi_! \mathbb{Z}) = (\coprod_{S_\infty} \mathbb{Z})/\mathbb{Z})$.

(c) $H^2(\overline{Y}, \varphi_! \mathbb{Z}) = \mathrm{Pic}^1(\overline{Y})^D$.

(d) $H^3(\overline{Y}, \varphi_! \mathbb{Z}) = \mu(F)^D$.

Proof. This follows immediately from Theorem 5.10, Proposition 6.1, and Lemma 6.2. □

PROPOSITION 6.4. *There is a natural exact sequence*

$$0 \to \mathrm{Pic}(Y)^D \longrightarrow \mathrm{Pic}^1(\overline{Y})^D \longrightarrow \mathrm{Hom}(U_F, \mathbb{Z}) \to 0.$$

Proof. Let F_v denote the completion of F at the archimedean valuation V. Then we have a natural inclusion i of $\prod_v F_v^*$ into the idèle group J_F. We then obtain an exact sequence

$$0 \longrightarrow \coprod \mathbb{R}^*_{>0} \xrightarrow{\ \tilde{i}\ } \operatorname{Pic}(\overline{Y}) \longrightarrow \operatorname{Pic}(Y) \longrightarrow 0,$$

where $\tilde{i}$ is induced by i.

Then the logarithmic embedding of the units yields the exact sequence

$$0 \to V/L \to \operatorname{Pic}^1(\overline{Y}) \to \operatorname{Pic}(Y) \to 0,$$

where V is the kernel of the sum map from $\coprod_v \mathbb{R}$ to $\mathbb{R}$, and L is the lattice in V obtained by taking the image of the unit group U_F under the map that sends a unit u to the vector $(\log|u|_v)$.

We now examine the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (V/L)^D & \longrightarrow & V^D & \longrightarrow & L^D & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle\alpha} & & \uparrow & & \\
 & & 0 & \longrightarrow & \operatorname{Hom}(V,\mathbb{R}) & \xrightarrow{\ \beta\ } & \operatorname{Hom}(L,\mathbb{R}) & & \\
 & & & & \uparrow & & \uparrow{\scriptstyle\gamma} & & \\
 & & & & 0 & \longrightarrow & \operatorname{Hom}(L,\mathbb{Z}), & &
\end{array}$$

where α and β are isomorphisms, and γ is injective. This defines an isomorphism of $\operatorname{Hom}(L,\mathbb{Z})$ and $(V/L)^D$, and the proposition follows after we observe that the natural map from $\operatorname{Hom}(L,\mathbb{Z})$ to $\operatorname{Hom}(U_F,\mathbb{Z})$ is an isomorphism. □

7. Euler characteristics

Let $n \geq 1$, and let

$$0 \longrightarrow V_0 \xrightarrow{\ T_0\ } V_1 \xrightarrow{\ T_1\ } \cdots \xrightarrow{\ T_{n-1}\ } V_n \longrightarrow 0$$

be an exact sequence of real vector spaces, and let B_i denote an ordered basis for V_i.

It is well known (see, for example, [Mil66, p. 365] or [Del87, §1]) that one can define the determinant of the above data in $\mathbb{R}^*/\pm 1$ in so that if $V_j = 0$ except for $j = i$ and $j = i+1$, we obtain $(\det M_i)^{(-1)^i}$, where M_i is the matrix of T_i with respect to the bases B_i and B_{i+1}, and if the basis B_i is changed by a matrix N, the determinant changes by $\det(N)^{\pm 1}$.

In particular, if the V_i' are of the form $A_i \otimes \mathbb{R}$ with A_i a finitely generated abelian group, and we choose bases of V_i coming from bases B_i of A_i modulo torsion, then the determinant is independent of the choice of B_i.

Now let A_0, A_1, ..., A_n be finitely generated abelian groups, and define $V_i = A_i \otimes \mathbb{R}$. Assume that there exist $\mathbb{R}$-linear transformations $T_i : V_i \to V_{i+1}$ such that the sequence

$$0 \longrightarrow V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} \cdots \xrightarrow{T_{n-1}} V_n \longrightarrow 0$$

is exact.

We define the Euler characteristic $\chi(A_0, A_1, \dots A_n, T_0, \dots, T_{n-1})$ to be the alternating product

$$\prod_{i=0}^{n} |((A_i)_{\mathrm{tor}})|^{(-1)^i}$$

divided by the determinant of $(V_0, \dots V_n, T_0, \dots, T_{n-1}, B_0, \dots, B_n)$, where the B_i are the images of bases of the free abelian groups $A_i/(A_i)_{\mathrm{tor}}$.

8. **Dedekind zeta-functions at zero**

In this section we wish to verify that the conjecture stated in the introduction is true for Dedekind zeta-functions when we replace the group $H_c^q(Y, \mathbb{Z}) = H^q(\overline{Y}, \varphi_! \mathbb{Z})$ by 0 for $q > 3$. We first define our Euler characteristic. Let F be a number field, let O_F be the ring of integers in F, and let $Y = \operatorname{Spec} O_F$. Let $\overline{Y}$ be Y together with the archimedean primes of F, given the Weil-étale topology as above. Let φ be the inclusion of Y in $\overline{Y}$.

Let ψ in $H^1(\overline{Y}, \mathbb{R})$ be the homomorphism obtained by mapping W_F to its abelianization C_F and then taking the logarithm of the absolute value.

We next observe that first, by standard arguments, the category of sheaves of $\mathbb{R}$-modules has enough injectives, and second, that any injective sheaf of $\mathbb{R}$-modules is injective as a sheaf of abelian groups. These observations imply that taking the Yoneda product with ψ in $H^1(\overline{Y}, \mathbb{R}) = \operatorname{Ext}^1_{\overline{Y}}(\mathbb{R}, \mathbb{R})$ induces a map

$$H^q(\overline{Y}, F) = \operatorname{Ext}^q_{\overline{Y}}(\mathbb{R}, F) \longrightarrow H^{q+1}(\overline{Y}, F) = \operatorname{Ext}^{q+1}_{\overline{Y}}(\mathbb{R}, F),$$

where F is any sheaf of R-modules.

THEOREM 8.1. *Let $\widetilde{H}^q(\overline{Y}, \varphi_! \mathbb{Z})$ be $H^q(\overline{Y}, \varphi_! \mathbb{Z})$ if $q \leq 3$ and zero otherwise. Let ζ_F be the Dedekind zeta-function of F. Then the Euler characteristic $\chi(\widetilde{H}^*(\overline{Y}, \varphi_! \mathbb{Z}))$ is well defined and is equal to $\pm \zeta_F^*(0)$.*

Proof. We first observe that the groups $\widetilde{H}^i(\overline{Y}, \varphi_! \mathbb{Z})$ are finitely-generated by Theorem 6.3 and Proposition 6.4. We must show next that the natural map from $H^i(\overline{Y}, \varphi_! \mathbb{Z}) \otimes \mathbb{R}$ to $H^i(\overline{Y}, \varphi_! \mathbb{R})$ is an isomorphism for $i \leq 3$. Look at the commutative diagram

$$
\begin{array}{ccccccccc}
 & & H^2(\overline{Y},\mathbb{R})=0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & H^2(\overline{Y},\varphi_!\mathbb{R}) & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & H^1(\overline{Y},i_*\mathbb{R}) & \xrightarrow{\alpha} & H^1(\overline{Y},i_*S^1) & \longrightarrow & H^2(\overline{Y},i_*\mathbb{Z})=0 & & \\
 & & \gamma\uparrow & & \uparrow & & \uparrow & & \\
 & & H^1(\overline{Y},\mathbb{R}) & \longrightarrow & H^1(\overline{Y},S^1) & \longrightarrow & H^2(\overline{Y},\mathbb{Z}) & \longrightarrow & H^2(\overline{Y},\mathbb{R})=0 \\
 & & \delta\uparrow & & \uparrow & & \beta\uparrow & & \\
 & & H^1(\overline{Y},\varphi_!\mathbb{R}) & \longrightarrow & H^1(\overline{Y},\varphi_!S^1) & \longrightarrow & H^2(\overline{Y},\varphi_!\mathbb{Z}) & \xrightarrow{\varepsilon} & H^2(\overline{Y},\varphi_!\mathbb{R}).
\end{array}
$$

It is easy to see that γ is injective, so δ is the zero map, so $H^1(\overline{Y},\varphi_!\mathbb{R})$ may be identified with $H^1(\overline{Y},\varphi_!\mathbb{Z})\otimes\mathbb{R}$. We take a basis of $H^1(\overline{Y},\varphi_!\mathbb{R})$ obtained by choosing r_1+r_2-1 archimedean primes of F.

By a tedious but straightforward calculation with injective resolutions, we see that the map ε may be computed by applying β, lifting to $H^1(\overline{Y},S^1)$, mapping to $H^1(\overline{Y},i_*S^1)$, applying α^{-1}, and mapping to $H^2(\overline{Y},\varphi_!\mathbb{R})$.

Now by comparing this diagram with the diagram at the end of Section 6, we may identify $H^2(\overline{Y},\varphi_!\mathbb{R})$ with $\mathrm{Hom}(V_0,\mathbb{R})$, where $V=\coprod_{v\in S_\infty}\mathbb{R}^*_{>0}$ and V_0 is the kernel of the product map to $\mathbb{R}^*_{>0}$. Next, we may take as a basis of this group coming from $H^2(\overline{Y},\varphi_!\mathbb{Z})$ the dual basis of any basis for the units of F modulo torsion, identifying V_0 with $U_F\otimes\mathbb{R}^*_{>0}$ via the map $u\mapsto(|u|_v)$ for the same (r_1+r_2-1) element set of v's that we used above. Finally the Yoneda product with ψ clearly takes 1_v to the map f_v, where $f_v((x_w))=\log x_v$.

It is now easy to see that the determinant of the pair made of $H^*(\overline{Y},\varphi_!\mathbb{Z})$ and Yoneda product with ψ is R^{-1}, where R is the classical regulator.

Since $H^0(\overline{Y},\varphi_!\mathbb{Z})=0$, $(H^1(\overline{Y},\varphi_!\mathbb{Z}))_{\mathrm{tor}}=0$, $|(H^2(\overline{Y},\varphi_!\mathbb{Z}))_{\mathrm{tor}}|=h$, and $|H^3(\overline{Y},\varphi_!\mathbb{Z})|=w$, the Euler characteristic of $H^*(\overline{Y},\varphi_!\mathbb{Z})$ is equal to hR/w, which up to sign is $\zeta^*_F(0)$. □

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E-mail address: Stephen_Lichtenbaum@Brown.edu
DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, BOX 1917, 151 THAYER STREET, PROVIDENCE, RI 02912, UNITED STATES