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Abstract

We prove new theorems that are higher-dimensional generalizations of the classical theorems of Siegel on integral points on affine curves and of Picard on holomorphic maps from $\mathbb{C}$ to affine curves. These include results on integral points over varying number fields of bounded degree and results on Kobayashi hyperbolicity. We give a number of new conjectures describing, from our point of view, how we expect Siegel's and Picard's theorems to optimally generalize to higher dimensions.

1. Introduction

In this article we prove new theorems that are higher-dimensional generalizations of the classical theorems of Siegel on integral points on affine curves and of Picard on holomorphic maps from $\mathbb{C}$ to affine curves. In Section 2, we will give the statements of Siegel's and Picard's theorems, and we will recall how these two theorems from such seemingly different areas of mathematics are related. We will then proceed to give a number of new conjectures describing, from our point of view, how we expect Siegel's and Picard's theorems to optimally generalize to higher dimensions. These include conjectures on integral points over varying number fields of bounded degree and conjectures addressing hyperbolic questions. These conjectures appear to be fundamentally new.

We will then summarize our progress on these conjectures. We have been able to get results in all dimensions, with best-possible results in many cases for surfaces. Our technique is based on the new proof of Siegel's theorem given by Corvaja and Zannier in [CZ02]. They showed how one may use the Schmidt subspace theorem to obtain a very simple and elegant proof of Siegel's theorem. More recently, they have used this technique to obtain other results on integral points (see [CZ03], [CZ04a], and [CZ04b]) and Ru has translated the approach to Nevanlinna theory

[Ru04]. We will use the Schmidt subspace theorem approach to get results on integral points on higher-dimensional varieties, and analogously, we will use Vojta's version of Cartan's second main theorem to obtain results on holomorphic curves in higher-dimensional complex varieties, generalizing Picard's theorem.

As an application of our results, we show how to improve a result of Faltings on integral points on the complements of certain singular plane curves, proving a statement about hyperbolicity as well. We end with a discussion of our conjectures, relating them to previously known results and conjectures, and giving examples limiting any improvement to their hypotheses and conclusions.

2. Theorems of Siegel and Picard

It has been observed by Osgood, Vojta, Lang, and others that there is a striking correspondence between statements in Nevanlinna theory and in Diophantine approximation (see [Ru01] and [Voj87]). This correspondence has been extremely fruitful, influencing results and conjectures in both subjects considerably. The correspondence can be formulated in both a qualitative and quantitative way. In this section, we will concentrate on the simplest case of the qualitative correspondence, Siegel's and Picard's theorems.

Let $V \subset \mathbb{A}^n$ be an affine variety defined over a number field k. We will also view V as a complex analytic space. Then it has been noticed that $V(\mathbb{O}_{L,S})$ (the set of points with all coordinates in $\mathbb{O}_{L,S}$, the S-integers of L) seems to be infinite for sufficiently large number fields L and sets of places S if and only if there exists a non-constant holomorphic map $f : \mathbb{C} \to V$. When $V = C$ is a curve (i.e., a one-dimensional variety), this correspondence has been proved to hold exactly, and it is known precisely for which curves C the two statements hold. On the number theory side, Siegel's theorem is the fundamental theorem on integral points on curves. On the analytic side, the analogue is a theorem of Picard. We now give the following formulations of these two theorems.

THEOREM 2.1A (Siegel). *Let k be a number field. Let S be a finite set of places of k containing the archimedean places. Let C be an affine curve defined over k embedded in affine space $\mathbb{A}^m$. Let $\widetilde{C}$ be a projective closure of C. If $\#\widetilde{C} \setminus C > 2$ (over $\bar{k}$), then C has finitely many points in $\mathbb{A}^m(\mathbb{O}_{k,S})$.*

THEOREM 2.1B (Picard). *Let $\widetilde{C}$ be a compact Riemann surface. Let $C \subset \widetilde{C}$. If $\#\widetilde{C} \setminus C > 2$, then all holomorphic maps $f : \mathbb{C} \to C$ are constant.*

In other words, Siegel's and Picard's theorems state that if D consists of many distinct points on a curve X, then any set of integral points on $X \setminus D$ is finite and any holomorphic map $f : \mathbb{C} \to X \setminus D$ is constant. We will thus view as generalizing Siegel's or Picard's theorem any theorem that asserts that if D

has "enough components" then there is some limitation on the integral points on $X \setminus D$ or on the holomorphic maps $f : \mathbb{C} \to X \setminus D$. In Picard's theorem it may also be shown that the curves C in question satisfy the stronger condition of being Kobayashi hyperbolic. We will frequently be able to generalize this fact to higher dimensions as well.

Siegel's theorem is usually stated with the extra information that the hypothesis $\#\widetilde{C} \setminus C > 2$ is unnecessary for nonrational affine curves C. However, it may be shown that this stronger version of Siegel's theorem may be derived from Siegel's theorem as we have stated it by using étale coverings of the curve C; see [CZ02]. A similar statement holds for Picard's theorem. It is Siegel's and Picard's theorems in the form we have given above that we will generalize.

We note that when the geometric genus of C is greater than one, Siegel's theorem follows from the much stronger theorem of Faltings that C has only finitely many k-rational points. Similarly, it is a theorem of Picard that there are no nonconstant holomorphic maps $f : \mathbb{C} \to \widetilde{C}$ when $\widetilde{C}$ is a projective curve of geometric genus greater than one.

3. **Some preliminary definitions**

In order to state our conjectures and results we will need a few definitions. In Vojta's Nevanlinna-Diophantine dictionary [Voj87], the Diophantine object corresponding to a holomorphic map $f : \mathbb{C} \to X \setminus D$ is a set of (D, S)-integral points on X. We now sketch the definition of a set of (D, S)-integral points on X in terms of Weil functions.

Let k be a number field. Let $\mathbb{O}_k$ be the ring of integers of k. As usual, we have a set M_k of absolute values (or places) of k consisting of one place for each prime ideal $\mathfrak{p}$ of $\mathbb{O}_k$, one place for each real embedding $\sigma : k \to \mathbb{R}$, and one place for each pair of conjugate embeddings $\sigma, \overline{\sigma} : k \to \mathbb{C}$. Let k_v denote the completion of k with respect to v. We normalize our absolute values so that $|p|_v = p^{-[k_v:\mathbb{Q}_p]/[k:\mathbb{Q}]}$ if v corresponds to $\mathfrak{p}$ and $\mathfrak{p} \mid p$, and $|x|_v = |\sigma(x)|^{[k_v:\mathbb{R}]/[k:\mathbb{Q}]}$ if v corresponds to an embedding σ (in which case we say that v is archimedean). If v is a place of k and w is a place of a field extension L of k, then we say that w lies above v, or $w \mid v$, if w and v define the same topology on k.

Let D be a Cartier divisor on a projective variety X, both defined over a number field k. Let $v \in M_k$. Extend $|\cdot|_v$ to an absolute value on $\overline{k}_v$. We define a local Weil function for D relative to v to be a function $\lambda_{D,v} : X(\overline{k}_v) \setminus D \to \mathbb{R}$ such that if D is represented locally by (f) on an open set U then

$$\lambda_{D,v}(P) = -\log |f(P)|_v + \alpha_v(P),$$

where α_v is a continuous function on $U(\overline{k}_v)$ (in the v-topology).

By choosing embeddings $k \to \overline{k}_v$ and $\overline{k} \to \overline{k}_v$, we may also think of $\lambda_{D,v}$ as a function on $X(k) \setminus D$ or $X(\overline{k}) \setminus D$. A global Weil function consists of a collection of local Weil functions, $\lambda_{D,v}$, for $v \in M_k$, where the α_v above satisfy certain reasonable boundedness conditions as v varies. We refer the reader to [Lan83] and [Voj87] for a further discussion of this.

Definition 3.1. Let D be an effective Cartier divisor on a projective variety X, both defined over a number field k. Let S be a finite set of places in M_k containing the archimedean places. Let $R \subset X(\overline{k}) \setminus D$. Then R is defined to be a (D, S)-integral set of points if there exists a global Weil function $\lambda_{D,v}$ such that for all $v \in M_k \setminus S$ and all embeddings $\overline{k} \to \overline{k}_v$, the inequality

$$\lambda_{D,v}(P) \leq 0 \quad \text{holds for all } P \text{ in } R.$$

For us, the key property of a set of (D, S)-integral points is given by the following theorem.

THEOREM 3.2. *Let $R \subset X(\overline{k}) \setminus D$ be a set of (D, S)-integral points on X. Then for any regular function f on $X \setminus D$ (defined over $\overline{k}$), there exists a constant $a \in k^*$ such that $af(P)$ is S-integral for all P in R, that is, $af(P)$ lies in the integral closure of $\mathcal{O}_{k,S}$ in $\overline{k}$ for all P in R.*

In fact, in what follows, most of our results hold, and our conjectures should hold, for any k-rational set R satisfying the conclusion of Theorem 3.2. However, we will prefer to work with sets of (D, S)-integral points because they are better geometrically behaved (e.g., under pullbacks) and because they are the right objects to use so that the Diophantine exceptional set we are about to define matches (conjecturally) the holomorphic exceptional set we will define.

We will frequently just say D-integral, omitting the reference to S, when S has been fixed or when the statement is true for all possible S. Except where explicitly stated otherwise, we will also require from now on that a set of (D, S)-integral points be k-rational, i.e., $R \subset X(k)$. We note that sets of D-integral points are also essentially the same as the sets of scheme-theoretic integral points one would get from working with models of $X \setminus D$ over $\mathcal{O}_{k,S}$; see [Voj87, Prop. 1.4.1].

It will be necessary to define various exceptional sets of a variety; see also [Lan91].

Definition 3.3A. Let X be a projective variety, and let D be an effective Cartier divisor on X, both defined over a number field k. Let L be a number field with $L \supset k$, and let S be a finite set of places of L containing the archimedean places. We define the Diophantine exceptional set of $X \setminus D$ with respect to L and S to be

$$\mathrm{Exc}_{\mathrm{Dio},L,S}(X \setminus D) = \bigcup_R \dim_{>0}(\overline{R}),$$

where the union runs over all sets R of L-rational (D,S)-integral points on X and $\dim_{>0}(\overline{R})$ denotes the union of the positive-dimensional irreducible components of the Zariski-closure of R. We define the absolute Diophantine exceptional set of $X \setminus D$ to be

$$\mathrm{Exc}_{\mathrm{Dio}}(X \setminus D) = \bigcup_{L \supset k, S} \mathrm{Exc}_{\mathrm{Dio},L,S}(X \setminus D),$$

with L ranging over all number fields containing k and S ranging over all sets of places of L as above.

These definitions depend only on $X \setminus D$ and not on the choices of X and D.

Definition 3.3B. Let X be a complex variety. We define the holomorphic exceptional set $\mathrm{Exc}_{\mathrm{hol}}(X)$ of X to be the union of all images of non-constant holomorphic maps $f : \mathbb{C} \to X$.

Conjecturally, it is expected that $\mathrm{Exc}_{\mathrm{Dio}}(X \setminus D) = \mathrm{Exc}_{\mathrm{hol}}(X \setminus D)$ (it may also be necessary to take the Zariski-closures of both sides first).

Definition 3.4A. Let X be a projective variety defined over a number field k. Let D be an effective Cartier divisor on X. Then we define $X \setminus D$ to be Mordellic if $\mathrm{Exc}_{\mathrm{Dio}}(X \setminus D)$ is empty. We define $X \setminus D$ to be quasi-Mordellic if $\mathrm{Exc}_{\mathrm{Dio}}(X \setminus D)$ is not Zariski-dense in X.

Definition 3.4B. Let X be a complex variety. We define X to be Brody hyperbolic if $\mathrm{Exc}_{\mathrm{hol}}(X)$ is empty. We define X to be quasi-Brody hyperbolic if $\mathrm{Exc}_{\mathrm{hol}}(X)$ is not Zariski-dense in X.

Note that X being quasi-Brody hyperbolic is a stronger condition than the non-existence of holomorphic maps $f : \mathbb{C} \to X$ with Zariski-dense image. Similarly, $X \setminus D$ being quasi-Mordellic is stronger than the non-existence of Zariski-dense sets of D-integral points on X.

We will use $\mathcal{O}_X(D)$, or simply $\mathcal{O}(D)$ when there is no ambiguity, to denote the invertible sheaf associated to a Cartier divisor D on X, and $h^i(D)$ to denote the dimension of the vector space $H^i(X, \mathcal{O}(D))$. When $h^0(D) > 0$, we will frequently use the notation Φ_D to denote the rational map (unique up to projective automorphisms) from X to $\mathbb{P}^{h^0(D)-1}$ corresponding to a basis of $H^0(X, \mathcal{O}(D))$. The following definition gives a convenient measure of the size of a divisor.

Definition 3.5. For D a divisor on a nonsingular projective variety X, we define the dimension of D to be the integer $\kappa(D)$ such that there exist positive constants c_1 and c_2 such that

$$c_1 n^{\kappa(D)} \le h^0(nD) \le c_2 n^{\kappa(D)}$$

for all sufficiently divisible $n > 0$. If $h^0(nD) = 0$ for all $n > 0$, then we let $\kappa(D) = -\infty$.

If the divisor D is effective, then "sufficiently divisible n" can be replaced by "sufficiently large n" in the definition. As an alternative definition, if $\kappa(D) \geq 0$, one can show that

$$\kappa(D) = \max\{\dim \Phi_{nD}(X) \mid n > 0,\ h^0(nD) > 0\}.$$

If D is a Cartier divisor on a singular complex projective variety X, we define $\kappa(D) = \kappa(\pi^* D)$, where $\pi : X' \to X$ is a desingularization of X. It is easy to show that this is independent of the chosen desingularization. For more properties of $\kappa(D)$, see [Iit82, Ch. 10].

Definition 3.6. We say a Cartier divisor D on X is big if $\kappa(D) = \dim X$.

4. General setup and notation

Throughout, we will use the following general setup and notation.

General setup. For X a complex projective variety, let $D = \sum_{i=1}^{r} D_i$ be a divisor on X with the D_i effective Cartier divisors for all i. Suppose that at most m of the D_i meet at a point, so that the intersection of any $m+1$ distinct D_i is empty.

In the Diophantine setting, we will also assume that X and D are defined over a number field k, and we let S be a finite set of places of k containing the archimedean places.

Note that we do not require the D_i to be irreducible, and moreover, the D_i may have irreducible components in common. From now on, we will freely use the notation X, D, D_i, r, m, k, and S as above without further explanation.

5. Siegel and Picard-type conjectures

This section gives conjectures generalizing Siegel's theorem and Picard's theorem in various directions.

5.1. *Main conjectures.* Some special cases of the conjectures given in this section are related to Vojta's main conjecture [Voj87, Conj. 3.4.3]. In the next section we will also give conjectures related to Vojta's general conjecture [Voj87, Conj. 5.2.6], hence our terminology in this section and the next (see §14.2 for details). We remind the reader that throughout we are using the general setup of the last section.

CONJECTURE 5.1A (Main Siegel-type conjecture). *Suppose that*

$$\kappa(D_i) \geq \kappa_0 > 0 \quad \textit{for all } i.$$

If $r > m + m/\kappa_0$, then there does not exist a Zariski-dense set of k-rational (D, S)-integral points on X.

CONJECTURE 5.1B (Main Picard-type conjecture). *Suppose that*

$$\kappa(D_i) \geq \kappa_0 > 0 \quad \textit{for all } i.$$

If $r > m + m/\kappa_0$, then there does not exist a holomorphic map $f : \mathbb{C} \to X \setminus D$ with Zariski-dense image.

As mentioned earlier, we will usually just say D-integral, omitting k and S from the notation. Siegel's theorem (respectively Picard's theorem) is the case $m = \kappa_0 = \dim X = 1$ of Conjecture 5.1A (respectively Conjecture 5.1B). We note that the dimension of X does not appear in the conjectures, but $\kappa(D_i)$ is bounded by $\dim X$. We will now discuss some consequences and special cases of these conjectures which seem important enough in their own right to be listed separately as new conjectures. At the two extremes of κ_0 we have this:

CONJECTURE 5.2A. *If $\kappa(D_i) > 0$ for all i and $r > 2m$, then there does not exist a Zariski-dense set of D-integral points on X.*

CONJECTURE 5.2B. *If $\kappa(D_i) > 0$ for all i and $r > 2m$, then there does not exist a holomorphic map $f : \mathbb{C} \to X \setminus D$ with Zariski-dense image.*

CONJECTURE 5.3A. *If D_i is big for all i and $r > m + m/\dim X$, then there does not exist a Zariski-dense set of D-integral points on X.*

CONJECTURE 5.3B. *If D_i is big for all i and $r > m + m/\dim X$, then there does not exist a holomorphic map $f : \mathbb{C} \to X \setminus D$ with Zariski-dense image.*

We note that when the D_i are in some sort of general position, so that $m = \dim X$, the inequalities in the last two conjectures above take the nicer form $r > \dim X + 1$.

Of particular interest is the case where D_i is ample for all i. In this case, one easily deduces the following conjectures as consequences of Conjectures 5.3A and 5.3B.

CONJECTURE 5.4A (Main Siegel-type conjecture for ample divisors). *Suppose that D_i is ample for all i. Let R be a set of D-integral points on X.*

(a) *If $r > m + m/\dim X$, then* $\dim R \leq m/(r - m)$.

(b) *In particular, if $r > 2m$, then $X \setminus D$ is Mordellic.*

CONJECTURE 5.4B (Main Picard-type conjecture for ample divisors). *Suppose that D_i is ample for all i. Let $f : \mathbb{C} \to X \setminus D$ be a holomorphic map.*

(a) *If $r > m + m/\dim X$, then* $\dim f(\mathbb{C}) \leq m/(r - m)$.

(b) *If $r > 2m$, then $X \setminus D$ is complete hyperbolic and hyperbolically imbedded in X. In particular, $X \setminus D$ is Brody hyperbolic.*

The main conjectures for ample divisors might possibly extend to big divisors as follows. Let $D = \sum_{i=1}^{r} D_i$ be a sum of big divisors on a projective variety X. Let $n > 0$ be large enough such that the map $\Phi = \Phi_{nD}$, corresponding to nD, is birational onto its image. It is then quite plausible that Conjectures 5.4A and 5.4B extend to the big divisor D if we state things in terms of Φ, i.e., if we replace $\dim R$ and $\dim f(\mathbb{C})$ by $\dim \Phi(R)$ and $\dim \Phi(f(\mathbb{C}))$, respectively, in the conjectures. In particular, this would imply that if D_i is big for all i and $r > 2m$, then $X \setminus D$ is quasi-Mordellic and quasi-Brody hyperbolic (this might even be true with $r \geq 2m$).

5.2. *General conjectures.* We now consider the situation where the field that the integral points are defined over is allowed to vary over all fields of degree less than or equal to d over some fixed field k. So in this section we do not require that integral points be k-rational.

Definition 5.5. Let $R \subset X(\bar{k})$. We define the degree of R over k to be

$$\deg_k R = \sup_{P \in R} [k(P) : k].$$

The next conjecture generalizes the main Siegel-type conjecture of the last section.

CONJECTURE 5.6 (General Siegel-type conjecture). *Suppose that*

$$\kappa(D_i) \geq \kappa_0 > 0 \quad \textit{for all } i.$$

Let d be a positive integer. If $r > m + m(2d-1)/\kappa_0$, then there does not exist a Zariski-dense set of D-integral points on X of degree d over k.

We will also want to define a degree d Diophantine exceptional set for a variety V. With the notation from our earlier definition for $\mathrm{Exc}_{\mathrm{Dio}}$, we make the following definition.

Definition 5.7. Let X be a projective variety and D an effective Cartier divisor on X, both defined over a number field k. Let L be a number field with $L \supset k$, and let S be a finite set of places of L containing the archimedean places. We define the degree d Diophantine exceptional set of $X \setminus D$ with respect to L and S to be

$$\mathrm{Exc}_{\mathrm{Dio},\deg d,L,S}(X \setminus D) = \bigcup_{R} \dim_{>0}(\overline{R}),$$

where the union runs over all sets R of (D,S)-integral points on X of degree d over L. We define the degree d absolute Diophantine exceptional set of $X \setminus D$ to be

$$\mathrm{Exc}_{\mathrm{Dio},\deg d}(X \setminus D) = \bigcup_{L \supset k, S} \mathrm{Exc}_{\mathrm{Dio},\deg d,L,S}(X \setminus D),$$

with L ranging over all number fields containing k and S ranging over all sets of places of L as above.

Similarly, we define $X \setminus D$ to be degree d Mordellic (resp. degree d quasi-Mordellic) if $\mathrm{Exc}_{\mathrm{Dio,deg}d}(X \setminus D)$ is empty (resp. not Zariski-dense in X).

Conjecture 5.6 implies the following conjecture for ample divisors.

CONJECTURE 5.8 (General Siegel-type conjecture for ample divisors). *Suppose that D_i is ample for all i. Let R be a set of D-integral points on X of degree d over k.*

(a) *If $r > m + m(2d-1)/\dim X$, then $\dim R \le m(2d-1)/(r-m)$.*

(b) *In particular, if $r > 2dm$, then $X \setminus D$ is degree d Mordellic.*

This conjecture might possibly be extended to big divisors as in the comments after Conjectures 5.4A and 5.4B.

6. **Overview of results**

Sections 8–12 will be concerned with proving special cases of the above conjectures. In this section we highlight some of our results. Along the lines of the main conjectures, we prove the following theorems.

THEOREM 6.1A. *Suppose $r > 2m \dim X$.*

(a) *If D_i is big for all i, then $X \setminus D$ is quasi-Mordellic.*

(b) *If D_i is ample for all i, then $X \setminus D$ is Mordellic.*

THEOREM 6.1B. *Suppose $r > 2m \dim X$.*

(a) *If D_i is big for all i, then $X \setminus D$ is quasi-Brody hyperbolic.*

(b) *If D_i is ample for all i, then $X \setminus D$ is complete hyperbolic and hyperbolically imbedded in X. In particular, $X \setminus D$ is Brody hyperbolic.*

With some additional mild hypotheses (see Theorems 10.4A and 10.4B), both parts (a) above can be improved to $r > 2[(m+1)/2] \dim X$, where $[x]$ denotes the greatest integer in x.

For any X, the $m = 1$ cases of the main conjectures follow essentially from Siegel's and Picard's theorems (see Theorem 9.14). When X is a nonsingular surface, $m = 2$, and the D_i have no irreducible components in common, we are able to prove the main conjectures, Conjectures 5.1A through 5.4B.

THEOREM 6.2A. *Suppose X is a nonsingular surface and the D_i have no irreducible components in common.*

(a) *If $m = 1$, $\kappa(D_i) > 0$ for all i, and $r > 2$, then there does not exist a Zariski-dense set of D-integral points on X.*

(b) *If $m = 2$, $\kappa(D_i) > 0$ for all i, and $r > 4$, then there does not exist a Zariski-dense set of D-integral points on X.*

(c) *If* $m = 2$, D_i *is big for all* i, *and* $r > 3$, *then* $X \setminus D$ *is quasi-Mordellic.*

(d) *If* $m = 2$, D_i *is ample for all* i, *and* $r > 4$, *then* $X \setminus D$ *is Mordellic.*

THEOREM 6.2B. *Suppose* X *is a nonsingular surface and the* D_i *have no irreducible components in common.*

(a) *If* $m=1$, $\kappa(D_i)>0$ *for all* i, *and* $r>2$, *then there does not exist a holomorphic map* $f:\mathbb{C}\to X\setminus D$ *with Zariski-dense image.*

(b) *If* $m=2$, $\kappa(D_i)>0$ *for all* i, *and* $r>4$, *then there does not exist a holomorphic map* $f:\mathbb{C}\to X\setminus D$ *with Zariski-dense image.*

(c) *If* $m = 2$, D_i *is big for all* i, *and* $r > 3$, *then* $X \setminus D$ *is quasi-Brody hyperbolic.*

(d) *If* $m = 2$, D_i *is ample for all* i, *and* $r > 4$, *then* $X \setminus D$ *is complete hyperbolic and hyperbolically imbedded in* X. *In particular*, $X \setminus D$ *is Brody hyperbolic.*

As to the general conjectures, when the integral points are allowed to vary over fields of a bounded degree, we prove this:

THEOREM 6.3. *Let* d *be a positive integer. If* D_i *is ample for all* i *and* $r > 2d^2 m \dim X$, *then* $X \setminus D$ *is degree* d *Mordellic* (*all sets of* D*-integral points on* X *of degree* d *over* k *are finite*).

As an application of our results, we will discuss an improvement to a result of Faltings, who recently [Fal02] showed how theorems on integral points on the complements of divisors with many components may occasionally be used to prove theorems on integral points on the complements of irreducible divisors. He shows how to do this with certain very singular curves on $\mathbb{P}^2$ by reducing the problem to a covering surface and applying the method of [FW94]. In [Zan05], Zannier uses the subspace theorem approach instead of [FW94] to prove a result similar to Faltings. In Section 13, we will prove a theorem that generalizes both results. As a bonus, we also prove the theorem in the case of holomorphic curves.

7. Preliminaries

7.1. *Diophantine approximation.* Let k be a number field with canonical set of places M_k (see Section 3). A basic identity is the product formula

$$\prod_{v\in M_k} |x|_v = 1 \quad \text{for all } x \in k^*.$$

For a point $P = (x_0, \dots, x_n) \in \mathbb{P}^n(k)$, we define the height to be

$$H(P) = \prod_{v\in M_k} \max(|x_0|_v, \dots, |x_n|_v).$$

It follows from the product formula that $H(P)$ is independent of the choice of homogeneous coordinates for P. It is also easy to see that the height is independent of k. We define the logarithmic height to be

$$h(P) = \log H(P).$$

At the core of our Diophantine results is Vojta's version of Schmidt's subspace theorem [Voj89].

THEOREM 7.1A. *Let k be a number field. Let S be a finite set of places in M_k containing the archimedean places. Let $H_1, \ldots, H_m$ be hyperplanes in $\mathbb{P}^n$ defined over $\bar{k}$ with corresponding Weil functions $\lambda_{H_1}, \ldots, \lambda_{H_m}$. Then there exists a finite union of hyperplanes Z, depending only on $H_1, \ldots, H_m$ (and not k or S), such that for any $\varepsilon > 0$,*

$$\sum_{v \in S} \max_I \sum_{i \in I} \lambda_{H_i,v}(P) \leq (n+1+\varepsilon)h(P) \tag{1}$$

holds for all but finitely many points P in $\mathbb{P}^n(k) \setminus Z$, where the maximum is taken over subsets $I \subset \{1, \ldots, m\}$ such that the linear forms defining H_i for $i \in I$ are linearly independent.

Explicitly, if H is a hyperplane on $\mathbb{P}^n$ defined by the linear form $L(x_0, \ldots, x_n)$, then a Weil function for H is given by

$$\lambda_{H,v}(P) = \log \max_i \frac{|x_i|_v}{|L(P)|_v}, \tag{2}$$

where $P = (x_0, \ldots, x_n)$ and we have chosen embeddings $\bar{k} \to \bar{k}_v$ for each v.

For easy reference, we now collect various properties of D-integral points that we will use (sometimes implicitly) throughout the paper; see also [Voj87].

LEMMA 7.2. *Let k be a number field and S a finite set of places in M_k containing the archimedean places. Let D be an effective Cartier divisor on a projective variety X, both defined over k.*

(a) *Let L be a finite extension of k, and let T be the set of places of L lying over places in S. If R is a set of (D, S)-integral points, then it is a set of (D, T)-integral points.*

(b) *Let E be an effective Cartier divisor on X. If R is a set of $(D+E)$-integral points, then R is a set of D-integral points.*

(c) *The D-integrality of a set is independent of the multiplicities of the components of D.*

(d) *Let Y be a projective variety defined over k. Let $\pi : Y \to X$ be a morphism defined over k with $\pi(Y) \not\subset D$ and $\pi|_{Y \setminus \pi^* D} : Y \setminus \pi^* D \to X \setminus D$ a finite étale map. If R is a set of (D, S)-integral points on X, then there exists a number* *field L such that $\pi^{-1}(R) \subset Y(L)$ (the Chevalley-Weil theorem). Furthermore, $\pi^{-1}(R)$ is a set of $(\pi^* D, T)$-integral points on Y, where T is the set of places of L lying above places of S.*

7.2. *Nevanlinna theory and Kobayashi hyperbolicity.* We will be interested in Nevanlinna theory as it applies to holomorphic maps $f : \mathbb{C} \to \mathbb{P}^n$ and hyperplanes on $\mathbb{P}^n$. Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic map. Then we may choose a representation $\boldsymbol{f} = (f_0, \dots, f_n)$ of f, where $f_0, \dots, f_n$ are entire functions without common zeros. Let us define $\|\boldsymbol{f}\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$. Then we define a characteristic function $T_f(r)$ of f to be

$$T_f(r) = \int_0^{2\pi} \log\|\boldsymbol{f}(re^{i\theta})\| \frac{d\theta}{2\pi}.$$

Note that by Jensen's formula, this function is well defined up to a constant. Let H be a hyperplane in $\mathbb{P}^n$ defined by a linear form L. Then we define a Weil function $\lambda_H(f(z))$ of f with respect to H by

$$\lambda_H(f(z)) = -\log \frac{|L(\boldsymbol{f}(z))|}{\|\boldsymbol{f}(z)\|}. \tag{3}$$

We note that this is independent of the choice of $\boldsymbol{f}$ and depends on the choice of L only up to a constant. The analogue of Schmidt's subspace theorem that we will need is the following version of Cartan's second main theorem.

THEOREM 7.1B (Vojta [Voj97]). *Let $H_1, \dots, H_m$ be hyperplanes in $\mathbb{P}^n$ with corresponding Weil functions $\lambda_{H_1}, \dots, \lambda_{H_m}$. Then there exists a finite union of hyperplanes Z such that for any $\varepsilon > 0$ and any non-constant holomorphic map $f : \mathbb{C} \to \mathbb{P}^n$ with $f(\mathbb{C}) \not\subset Z$, the inequality*

$$\int_0^{2\pi} \max_I \sum_{i \in I} \lambda_{H_i}(f(re^{i\theta})) \frac{d\theta}{2\pi} \le (n+1+\varepsilon) T_f(r) \tag{4}$$

holds for all r outside a set of finite Lebesgue measure, where the maximum is taken over subsets $I \subset \{1, \dots, m\}$ such that the linear forms defining H_i for $i \in I$ are linearly independent.

Closely connected to questions about holomorphic curves is the Kobayashi pseudo-distance and Kobayashi hyperbolicity. See [Lan87] for the definitions of the Kobayashi pseudo-distance, Kobayashi hyperbolic, complete hyperbolic, and hyperbolically imbedded. It is trivial that Kobayashi hyperbolic implies Brody hyperbolic. We will want a criterion for proving the converse in special cases. On projective varieties, this is given by Brody's theorem. More generally, we will use the following theorem; see [Gre77] and [Lan87].

THEOREM 7.3 (Green). *Let X be a complex projective variety. Suppose $Y = \bigcup_{i\in I} D_i$ is a finite union of effective Cartier divisors D_i on X. Suppose that for every subset $\varnothing \subset J \subset I$,*

$$\bigcap_{j\in J} D_j \setminus \bigcup_{i\in I\setminus J} D_i$$

is Brody hyperbolic, where $\bigcap_{j\in\varnothing} D_j = X$. Then $X \setminus Y$ is complete hyperbolic and hyperbolically imbedded in X.

7.3. *Nef and big divisors.* We now recall some basic definitions and facts regarding nef and big divisors. We will assume a basic familiarity with intersection theory (see [Ful98] for a thorough modern account). We will use the notation D^n to denote the intersection number of the n-fold intersection of D with itself. In what follows, X will be a projective variety over an algebraically closed field of characteristic 0.

Definition 7.4. A Cartier divisor D (or invertible sheaf $\mathcal{O}(D)$) on X is said to be numerically effective, or nef, if $D\,.\,C \geq 0$ for any closed integral curve C on X.

The next lemma summarizes some basic properties of nef divisors; see [Kle66].

LEMMA 7.5. *Nef divisors satisfy the following*:

(a) *Let $n = \dim X$. If $D_1, \ldots, D_n$, are nef divisors on X, then*

$$D_1\,.\,D_2\,.\,\ldots\,.\,D_n \geq 0.$$

(b) *Let D be a nef divisor and A an ample divisor on X. Then $A + D$ is ample.*

(c) *Let $f : X \to Y$ be a morphism and let D be a nef divisor on Y. Then $f^*\mathcal{O}(D)$ is nef on X.*

Recall that in Definitions 3.5 and 3.6 we defined $\kappa(D)$ and what it means for a Cartier divisor to be big. If D is big then there exists an $n > 0$ such that Φ_{nD} is birational onto its image. It is always true that $\kappa(D) \leq \dim X$, so D is big if and only if it has the largest possible dimension for a divisor on X. For nef divisors it is possible to give a more numerical criterion for a divisor to be big. It is also possible in this case to get an asymptotic formula for $h^0(nD)$. We have the following lemma, due to Sommese, as it appears in [Kaw82].

LEMMA 7.6. *Suppose D is a nef divisor on a nonsingular projective variety X. Let $q = \dim X$. Then $h^0(nD) = (D^q/q!)n^q + O(n^{q-1})$. In particular, $D^q > 0$ if and only if D is big.*

Proof. Let K_X denote the canonical divisor on X. Let L be an ample divisor on X such that $L + K_X$ is very ample. Since D is nef, $nD + L$ is ample, and so by Kodaira's vanishing theorem we have

$$H^i(X, \mathcal{O}(nD + L + K_X)) = 0 \quad \text{for } i > 0.$$

Therefore,

$$h^0(nD + L + K_X) = \chi(\mathcal{O}(nD + L + K_X)) = \frac{D^q}{q!} n^q + O(n^{q-1})$$

by Riemann-Roch. Let Y be a general member of the linear system $|L + K_X|$, so that Y is nonsingular and irreducible. Then we have an exact sequence

$$0 \to H^0(X, \mathcal{O}(nD)) \to H^0(X, \mathcal{O}(nD + L + K_X)) \to H^0(Y, i^*\mathcal{O}(nD + L + K_X))$$

where $i : Y \to X$ is the inclusion map. But since $\dim Y = q - 1$, we have

$$\dim H^0(Y, i^*\mathcal{O}(nD + L + K_X)) \leq O(n^{q-1}).$$

It follows that $h^0(nD) = (D^q/q!)n^q + O(n^{q-1})$. □

Since we will use it multiple times, we state the exact sequence used above as a lemma.

LEMMA 7.7. *Let D be an effective Cartier divisor on X with inclusion map $i : D \to X$. Let $\mathcal{L}$ be an invertible sheaf on X. Then we have exact sequences*

$$0 \to \mathcal{L} \otimes \mathcal{O}(-D) \longrightarrow \mathcal{L} \to i_*(i^*\mathcal{L}) \longrightarrow 0,$$
$$0 \to H^0(X, \mathcal{L} \otimes \mathcal{O}(-D)) \longrightarrow H^0(X, \mathcal{L}) \longrightarrow H^0(D, i^*\mathcal{L}).$$

Proof. If D is an effective Cartier divisor, then a fundamental exact sequence is

$$0 \to \mathcal{O}(-D) \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_D \to 0.$$

Tensoring with $\mathcal{L}$ and using the projection formula, we get the first exact sequence. Taking global sections then gives the second exact sequence. □

On surfaces, Lemma 7.6 can be expanded to include effective divisors with positive self-intersection.

LEMMA 7.8. *Let D be an effective divisor on a nonsingular projective surface X. If $D^2 > 0$, then $h^0(nD) \geq \frac{1}{2}n^2D^2 + O(n)$ and D is big.*

Proof. By Riemann-Roch,

$$h^0(nD) - h^1(nD) + h^0(K - nD) = \tfrac{1}{2}n^2D^2 - \tfrac{1}{2}nD.K + 1 + p_a.$$

Since D is effective and $D \neq 0$, $h^0(K - nD) = 0$ for $n \gg 0$ (for example, choose $n > K.H$, where H is an ample divisor). We also have $h^1(nD) \geq 0$, so $h^0(nD) \geq \frac{1}{2}n^2D^2 + O(n)$ and D is big. □

Unlike when E is effective, it is not always true that if E is nef, then $h^0(D - E) \leq h^0(D)$. We will therefore find the following lemma useful.

LEMMA 7.9. *Let X be a nonsingular projective variety of dimension q. Let D and E be any divisors on X, and let F be a nef divisor on X. Then*

$$h^0(nD + E - mF) \le h^0(nD) + O(n^{q-1}) \quad \textit{for all } m, n \ge 0,$$

where the implied constant is independent of m and n.

Proof. We first claim that if B is any nef divisor, then there exists a divisor C, independent of B, such that $h^0(B + C) > 0$. Explicitly, we may take $C = (q+2)A + K_X$, where A is a very ample divisor on X. We prove this by induction on the dimension q. The case $q = 1$ is easy. For the inductive step, we have an exact sequence

$$\begin{aligned} 0 &\to H^0(X, \mathcal{O}((q+1)A + K_X + B)) \to H^0(X, \mathcal{O}((q+2)A + K_X + B)) \\ &\to H^0(Y, i^*(\mathcal{O}((q+2)A + K_X + B))) \to H^1(X, \mathcal{O}((q+1)A + K_X + B)), \end{aligned}$$

where Y is an irreducible nonsingular element of $|A|$ whose inclusion map is $i : Y \to X$. Since $(q+1)A + B$ is ample, by Kodaira vanishing, the last term above is 0. Since $\omega_Y \cong i^*(\mathcal{O}(A + K_X))$, by induction we get that

$$\dim H^0(Y, i^*(\mathcal{O}((q+2)A + K_X + B))) > 0.$$

Since the penultimate map in the exact sequence above is surjective, we therefore also have $h^0((q+2)A + K_X + B) = h^0(B + C) > 0$, proving our claim. Therefore,

$$\begin{aligned} h^0(nD + E - mF) \le h^0(nD + E - mF + (C + mF)) &= h^0(nD + C + E) \\ &\le h^0(nD) + O(n^{q-1}) \end{aligned}$$

independently of m, where the last inequality follows from Lemma 7.7 as in the proof of Lemma 7.6. □

For completeness, we mention that there exist examples showing that Lemma 7.9 is false if $O(n^{q-1})$ is replaced by $O(n^{q-2})$.

8. **Fundamental theorems on large divisors**

In this section we prove a slightly expanded version of a theorem of Corvaja and Zannier and its analogue for holomorphic curves. These theorems will be fundamental to our future results.

Let D be a divisor on a nonsingular projective variety X defined over a field k. Let $\bar{k}(X)$ denote the function field of X over $\bar{k}$. We will write $D \ge E$ if $D - E$ is effective. Let $\operatorname{div}(f)$ denote the principal divisor associated to f. Let $L(D)$ be the $\bar{k}$-vector space $L(D) = \{f \in \bar{k}(X) \mid \operatorname{div}(f) \ge -D\}$, and let $l(D) = \dim L(D) = h^0(D)$. If E is a prime divisor, we let $\operatorname{ord}_E f$ denote the coefficient of E in $\operatorname{div}(f)$. We make the following definition.

Definition 8.1. Let D be an effective divisor on a nonsingular projective variety X defined over a field k. Then we define D to be a very large divisor on X if for every $P \in D(\bar{k})$ there exists a basis B of $L(D)$ such that $\operatorname{ord}_E \prod_{f \in B} f > 0$ for every irreducible component E of D such that $P \in E$. We define D to be a large divisor if some nonnegative integral linear combination of its irreducible components is very large on X.

Note that in the definition of very large, a basis function $f \in B$ may have a high-order pole along E. We just require that (after cancellation) the product $\prod_{f \in B} f$ has a zero along E.

Remark 8.2. Suppose D is very large. Let $P \in D$, and let $\mathscr{E}$ be the set of irreducible components E of D such that $P \in E$. If B is a basis of $L(D)$ that has the property in the definition of very large with respect to P, then B also works as a basis with respect to any suitably generic $Q \in \bigcap_{E \in \mathscr{E}} E$. Thus, it is easily seen that in the definition of very large, one only needs to use bases $B \in \mathscr{B}$ for some finite set of bases $\mathscr{B}$ for any very large divisor D.

We will see (Theorem 9.9) for example that on any nonsingular projective variety X, the sum of sufficiently many ample effective divisors in general position is large. On the other hand, it is obvious from the definition that if D is an irreducible effective divisor on X, then D cannot be large. Roughly speaking, large divisors have a lot of irreducible components of high D-dimension. With this definition we have the following theorems.

THEOREM 8.3A (Corvaja–Zannier). *Let X be a nonsingular projective variety defined over a number field k. Let $S \subset M_k$ be a finite set of places of k containing the archimedean places. Let D be a large divisor on X defined over k. Then there does not exist a Zariski-dense set of D-integral points on X. Furthermore, if D is very large and Φ_D is a rational map to projective space corresponding to D, then there exists a proper Zariski-closed subset $Z \subset X$ depending only on D (and not k or S) such that $\Phi_D(R \setminus Z)$ is finite for any set R of D-integral points on X. In particular, if Φ_D is birational onto its image, $X \setminus D$ is quasi-Mordellic.*

THEOREM 8.3B. *Let X be a nonsingular complex projective variety. Let D be a large divisor on X. Then there does not exist a holomorphic map $f : \mathbb{C} \to X \setminus D$ with Zariski-dense image. Furthermore, if D is very large and Φ_D is a rational map to projective space corresponding to D, then there exists a proper Zariski-closed subset $Z \subset X$ such that for all holomorphic maps $f : \mathbb{C} \to X \setminus D$, either $f(\mathbb{C}) \subset Z$ or $\Phi_D \circ f$ is constant. In particular, if Φ_D is birational onto its image, $X \setminus D$ is quasi-Brody hyperbolic.*

Theorem 8.3A appears, essentially, in the proof of [CZ04b, Main Theorem], and for curves in [CZ02]. We have added the last two statements to the theorem by using Vojta's result on the exceptional hyperplanes in the Schmidt subspace theorem.

Given Theorems 8.3A and 8.3B, many of our results mentioned in the introduction reduce to showing that certain divisors are large. Let us prove Theorem 8.3A first. Before proving this theorem, we need a lemma.

LEMMA 8.4. *Let X be a projective variety defined over a number field k. Let $R \subset X(k)$ be a Zariski-dense subset of X. Let $v \in M_k$. Then there exists a point P in $X(k_v)$ and a sequence $\{P_i\}$ in R such that $\{P_i\} \to P$ in the v-topology on $X(k_v)$ and $\bigcup\{P_i\}$ is Zariski-dense in X.*

Proof. We will work throughout the proof in the v-topology on $X(k_v)$. Let R^v denote the closure of R in $X(k_v)$ in the v-topology. First we claim that there exists a P in $R^v \subset X(k_v)$ such that for every neighborhood U of P in $X(k_v)$, $U \cap R$ is Zariski-dense in X. Indeed, suppose there is no such P. Then for each P in R^v, let U_P be a neighborhood of P such that $U_P \cap R$ is not Zariski-dense in X. Since $X(k_v)$ is compact because X is projective, R^v is compact, so we may cover R^v by finitely many open sets $U_{P_1}, \ldots, U_{P_n}$. But then $R = (U_{P_1} \cap R) \cup \cdots \cup (U_{P_n} \cap R)$ is not Zariski-dense in X, a contradiction.

Now pick some P as in the claim above. Embed X in $\mathbb{P}^n_k$ for some n. Since k is countable, the set of hypersurfaces in $\mathbb{P}^n_k$ not containing X is countable. Let $\{H_i\}$ be an enumeration of these. There also exists a countable collection of neighborhoods $\{U_i\}$ of P in $X(k_v)$ such that $U_i \subset U_j$ for $i > j$ and $\bigcap U_i = \{P\}$. Since $U_i \cap R$ is Zariski-dense in X, for all i there exists a $P_i \in U_i \cap R$ such that $P_i \notin H_i$. Then $\{P_i\} \to P$ in $X(k_v)$, and $\bigcup\{P_i\}$ is Zariski-dense in X since it is not contained in any hypersurface. $\square$

Proof of Theorem 8.3A. Let D be a large divisor, and let S and X be as in Theorem 8.3A. Since our first assertion depends only on the support of D, we may assume without loss of generality that D is very large on X. Extending k if necessary and enlarging S, we may assume without loss of generality that every irreducible component of D is defined over k and that all of the finitely many functions in $L(D)$ we use (see Remark 8.2) are defined over k. Let $\{\phi_1, \ldots, \phi_{l(D)}\}$ be a basis of $L(D)$ over k. Let R be a (D, S)-integral set of points on X. It suffices to prove the theorem in the case that $\overline{R}$ is irreducible. By repeatedly applying Lemma 8.4, we see that there exists a sequence P_i in R such that for each v in S, $\{P_i\}$ converges to a point $P_v \in X(k_v)$ and $\bigcup\{P_i\}$ is Zariski-dense in $\overline{R}$.

Let S' be the set of places $v \in S$ such that $P_v \in D(k_v)$, and let $S'' = S \setminus S'$. Since D is very large, for each $v \in S'$ we may let L_{iv} for $i = 1, \ldots, l(D)$ be a basis for $L(D)$ such that $\operatorname{ord}_E \prod_{i=1}^{l(D)} L_{iv} > 0$ for all irreducible components E of D such that $P_v \in E(k_v)$. Of course, each L_{iv} is a linear form in the ϕ_j over k. For $v \in S''$, we set $L_{jv} = \phi_j$ for $j = 1, \ldots, l(D)$. Let $\phi(P) = (\phi_1(P), \ldots, \phi_{l(D)}(P))$

for $P \in X \setminus D$. Let H_{jv} denote the hyperplane in $\mathbb{P}^{l(D)-1}$ determined by L_{jv} with respect to the basis $\phi_1, \dots, \phi_{l(D)}$. Let $\lambda_{H_{jv},v}$ be the Weil function for H_{jv} given in (2). We will now show that there exists $\varepsilon > 0$ and a constant C such that for all i,

$$\sum_{v\in S}\sum_{j=1}^{l(D)} \lambda_{H_{jv},v}(\phi(P_i)) > (l(D)+\varepsilon)h(\phi(P_i)) + C. \tag{5}$$

Since R is a set of (D, S)-integral points, we have

$$h(\phi(P_i)) < \sum_{v\in S} \log \max_j |\phi_j(P_i)|_v + O(1).$$

Using this, it suffices to prove that

$$\sum_{v\in S}\sum_{j=1}^{l(D)} \log \max_{j'} \frac{|\phi_{j'}(P_i)|_v}{|L_{jv}(P_i)|_v} > (l(D)+\varepsilon)\sum_{v\in S} \log \max_{j'} |\phi_{j'}(P_i)|_v + C'$$

for some C', or rearranging things, simplifying, and exponentiating,

$$\prod_{v\in S} \max_{j'} |\phi_{j'}(P_i)|_v^{\varepsilon} \prod_{j=1}^{l(D)} |L_{jv}(P_i)|_v$$

is bounded for some $\varepsilon > 0$. Let

$$M = \max\{-\mathrm{ord}_E \phi_j \mid E \text{ is an irreducible component of } D,\ j = 1, \dots, l(D)\}.$$

Let $\varepsilon = 1/M$. For $v \in S''$, both $|\phi_{j'}(P_i)|_v$ and $|L_{jv}(P_i)|_v$ are bounded for all i since $P_v \notin D(k_v)$ and $\phi_{j'}$ and L_{jv} have poles lying only in the support of D. Let $v \in S'$. So $P_v \in D(k_v)$. It follows from the definition of M and the fact that $\mathrm{ord}_E \prod_{i=1}^{l(D)} L_{iv} > 0$ for any irreducible component E of D such that $P_v \in E(k_v)$ that

$$\mathrm{ord}_E \phi_{j'} \left(\prod_{i=1}^{l(D)} L_{iv}\right)^M \geq -M + M \geq 0$$

for any irreducible component E of D such that $P_v \in E(k_v)$. Since the $\phi_{j'}$ and L_{iv} have poles only in the support of D, it follows from the previous order computation that $\max_{j'} |\phi_{j'}(P_i)|_v^{\varepsilon} \prod_{j=1}^{l(D)} |L_{jv}(P_i)|_v$ is bounded for all i and all $v \in S$ when $\varepsilon = 1/M > 0$. This proves (5).

Note that either $h(\phi(P_i)) \to \infty$ as $i \to \infty$ or $\phi(P_i) = \phi(\overline{R})$ and $\phi(P_i)$ is constant for all i. In the latter case the theorem is proved, so we may assume the former. Therefore, making ε smaller, we see that (5) holds with $C = 0$ for all but finitely many i. So by Schmidt's subspace theorem, there exists a finite union of hyperplanes $Z \subset \mathbb{P}^{l(D)-1}$ such that all but finitely many of the points in the set $\{\phi(P_i) = (\phi_1(P_i), \dots, \phi_{l(D)}(P_i)) \mid i \in \mathbb{N}\}$ lie in Z. Using Remark 8.2 we see that we may choose the hyperplanes H_{iv} used above from a finite set of hyperplanes independent of R. Therefore, using the statement on the exceptional hyperplanes in Schmidt's subspace theorem, we see that Z may be chosen to depend only on D and not R, k, or S. Since it was assumed that $\overline{R}$ is irreducible and $\phi(\overline{R})$ is not a point, it follows that $\phi(R) \subset Z$. Since $\phi_1, \dots, \phi_d$ are linearly independent functions in $\bar{k}(X)$ and Z is a finite union of hyperplanes, it follows that $\phi^{-1}(Z)$ is a finite union of proper closed subvarieties of X. So $R \subset \phi^{-1}(Z)$ and the theorem is proved. $\square$

The next proof is very similar.

Proof of Theorem 8.3B. Since our first assertion depends only on the support of D, we may assume without loss of generality that D is very large on X. By Remark 8.2 there exists a finite set J of elements in $L(D)$ such that for any $P \in D$ there exists a subset $I \subset J$ that is a basis of $L(D)$ such that $\operatorname{ord}_E \prod_{g \in I} g > 0$ for every irreducible component E of D such that $P \in E$. Let $\phi_1, \dots, \phi_{l(D)}$ be a basis for $L(D)$. Let $\phi = (\phi_1, \dots, \phi_{l(D)}) : X \setminus D \to \mathbb{P}^{l(D)-1}$. Let J' be the set of linear forms L in $l(D)$ variables over $\mathbb{C}$ such that $L \circ \phi \in J$. If L is a linear form, let H_L be the corresponding hyperplane. Let $f : \mathbb{C} \to X \setminus D$ be a holomorphic map. We will now show that there exists $\varepsilon > 0$ and a constant C such that

$$\int_0^{2\pi} \max_I \sum_{L \in I} \lambda_{H_L}(\phi \circ f(re^{i\theta}))\frac{d\theta}{2\pi} > (l(D)+\varepsilon)T_{\phi\circ f}(r) - C \tag{6}$$

for all $r > 0$, where the maximum is taken over subsets $I \subset J'$ such that I consists of exactly $l(D)$ linearly independent linear forms. Substituting the definition of the Weil function in (3) and the definition of $T_{\phi\circ f}$, we find after some manipulation that the inequality in (6) becomes

$$\int_0^{2\pi} \varepsilon \log\|\phi \circ f(re^{i\theta})\| + \min_I \sum_{L \in I} \log|L \circ \phi \circ f(re^{i\theta})| \frac{d\theta}{2\pi} < C$$

with I as before. Since $\|\phi \circ f(re^{i\theta})\| \le \sqrt{l(D)} \max_j |\phi_j \circ f(re^{i\theta})|$, it clearly suffices to show that

$$\max_j |\phi_j \circ f(re^{i\theta})|^{\varepsilon} \min_I \prod_{L \in I} |L \circ \phi \circ f(re^{i\theta})| \tag{7}$$

is bounded independently of r and θ for some $\varepsilon > 0$. Let $D_1, \dots, D_m$ be the irreducible components of D. Let

$$M = \max\{-\operatorname{ord}_{D_i} \phi_j \mid i = 1, \dots, m,\ j = 1, \dots, l(D)\}.$$

We will work in the classical topology. Let $P \in D$. Then there exists a neighborhood U of P such that for all $Q \in \overline{U}$, if $Q \in D_i$ for some i, then $P \in D_i$. Let $I' \subset J'$ be a subset of J' such that $\operatorname{ord}_{D_i} \prod_{L \in I'} L \circ \phi > 0$ for all i such that $P \in D_i$. If $P \in D_i$, then by the definition of M we have $\operatorname{ord}_{D_i} \phi_j (\prod_{L \in I'} L \circ \phi)^M \ge 0$ for all j. By the definition of U we see that $|\phi_j (\prod_{L \in I'} L \circ \phi)^M|$ is bounded for all

j on the compact set $\overline{U}$. Since D is compact and may be covered by such sets, we see that $\max_j |\phi_j| \min_I \prod_{L\in I} |L\circ\phi|^M$ is bounded on $X\setminus D$ (using also that away from D everything is obviously bounded since the ϕ_j have poles only in D). Therefore the function in (7) is bounded independently of r and θ for $\varepsilon = 1/M$.

If $\phi\circ f$ is constant then there is nothing to prove, so assume otherwise. Then $T_{\phi\circ f}(r)\to\infty$ as $r\to\infty$, and so making ε smaller, we see that we have proved the inequality (6) with $C=0$ for all sufficiently large r. Therefore by Cartan's second main theorem, there exists a finite union of hyperplanes $Z\subset\mathbb{P}^{l(D)-1}$ depending only on D (the H_L depended only on D) such that $\phi(f(\mathbb{C}))\subset Z$. Since the ϕ_j are linearly independent and Z is a finite union of hyperplanes, $\phi^{-1}(Z)$ is a finite union of proper closed subvarieties of X, and $f(\mathbb{C})\subset\phi^{-1}(Z)$. □

Remark 8.5. If D is very large and one can explicitly compute the map ϕ and the hyperplanes used in the above proofs, then one can explicitly compute the closed set Z in the theorems above. This follows from the explicit description of the exceptional hyperplanes in [Voj89] and [Voj97].

9. Results on higher-dimensional varieties

For an effective divisor $D=\sum_{i=1}^r D_i$ on X and $P\in D(\bar{k})$, we define $D_P=\sum_{i:P\in D_i} D_i$.

LEMMA 9.1. *Let $D=\sum_{i=1}^r D_i$ be a divisor on a nonsingular projective variety X with D_i effective for each i. For $P\in D$, let*

$$f_P(m,n)=l(nD-mD_P)-l(nD-(m+1)D_P).$$

If there exists $n>0$ such that $\sum_{m=0}^\infty (m-n)f_P(m,n)>0$ for all $P\in D$, then nD is very large.

Proof. Let $n>0$ be such that $\sum_{m=0}^\infty (m-n)f_P(m,n)>0$ for all $P\in D$. This sum is clearly finite for all $P\in D$, and we let $M_P(n)$ be the largest integer such that $f_P(M_P(n),n)>0$. Let $P\in D$, $M=M_P(n)$, and $V_j=L(nD-jD_P)$. So $\dim V_j/V_{j+1}=f_P(j,n)$. We have $L(nD)=V_0\supset V_1\supset\cdots\supset V_M\neq 0$. Choose a basis of V_M and successively complete it to bases of $V_{M-1},V_{M-2},\ldots,V_0$ to obtain a basis $f_1,\ldots,f_{l(nD)}$. Let E be an irreducible component of D such that $P\in E$. If $f_j\in V_m$, then $\operatorname{ord}_E f_j\geq (m-n)\operatorname{ord}_E D$. So we get that

$$\operatorname{ord}_E \prod_{i=1}^{l(nD)} f_i \geq (\operatorname{ord}_E D)\sum_{m=0}^{M}(m-n)f_P(m,n)>0.$$

Therefore nD is very large. □

THEOREM 9.2. *Suppose X is a nonsingular projective variety, and let $q=\dim X$. Let $D=\sum_{i=1}^r D_i$ be a divisor on X such that D_i is effective and nef for* *each i. Suppose also that every irreducible component of D is nonsingular. If*

$$D^q > 2qD^{q-1}.D_P \quad \text{for all } P \in D,$$

then nD is very large for $n \gg 0$. In particular, D is large.

Proof. Let $P \in D$. Let $D_P = \sum_{j=1}^k a_j E_j$, where each E_j is a distinct prime divisor. Repeatedly applying Lemma 7.7, we obtain

$$\begin{aligned}\dim H^0(X, \mathcal{O}(nD - mD_P)) &- \dim H^0(X, \mathcal{O}(nD - (m+1)D_P)) \\ &\le \textstyle\sum_{j=1}^k \sum_{l=0}^{a_j-1} \dim H^0(E_j, i_{E_j}^* \mathcal{O}(nD - mD_P - \sum_{j'=1}^{j-1} a_{j'}E_{j'} - lE_j)).\end{aligned}$$

Since D_P is nef, $i_{E_j}^*\mathcal{O}(D_P)$ is nef. Thus, setting D, E, F in Lemma 7.9 to divisors associated to

$$i_{E_j}^*\mathcal{O}(D), \qquad i_{E_j}^*\mathcal{O}(-\textstyle\sum_{j'=1}^{j-1} a_{j'}E_{j'} - lE_j), \qquad i_{E_j}^*\mathcal{O}(D_P),$$

respectively, we obtain

$$\begin{aligned}\dim H^0(E_j, i_{E_j}^*\mathcal{O}(nD - mD_P &- \textstyle\sum_{j'=1}^{j-1} a_{j'}E_{j'} - lE_j)) \\ &\le \dim H^0(E_j, i_{E_j}^*\mathcal{O}(nD)) + O(n^{q-2}).\end{aligned}$$

Therefore,

$$\begin{aligned}\dim H^0(X, \mathcal{O}(nD - mD_P)) &- \dim H^0(X, \mathcal{O}(nD - (m+1)D_P)) \\ &\le \sum_{j=1}^k a_j \dim H^0(E_j, i_{E_j}^*\mathcal{O}(nD)) + O(n^{q-2}).\end{aligned}$$

Since D is nef, $l(nD) = (n^q/q!)D^q + O(n^{q-1})$. Since $i_{E_j}^*\mathcal{O}(D)$ is also nef, we have

$$\dim H^0(E_j, i_{E_j}^*\mathcal{O}(nD)) = \frac{n^{q-1}}{(q-1)!}D^{q-1}.E_j + O(n^{q-2}).$$

So

$$f_P(m,n) \le \frac{n^{q-1}}{(q-1)!}\sum_{j=1}^k a_j D^{q-1}.E_j + O(n^{q-2}) = \frac{n^{q-1}}{(q-1)!}D^{q-1}.D_P + O(n^{q-2}).$$

To use this estimate, we borrow a lemma from [CZ04b].

LEMMA 9.3. *Let h and R be integers with $R \le h$, and let $x_1, \ldots, x_h$ and $U_1, \ldots, U_R$ be real numbers. If $0 \le x_i \le U_i$ for $i = 1, \ldots, R$ and $\sum_{j=1}^R U_j \le \sum_{j=1}^h x_j$, then $\sum_{j=1}^h jx_j \ge \sum_{j=1}^R jU_j$.*

Proof. We have

$$\sum_{j=1}^{R} jU_j + \sum_{j=1}^{h}(R+1-j)x_j \le \sum_{j=1}^{R} jU_j + \sum_{j=1}^{R}(R+1-j)x_j$$
$$\le \sum_{j=1}^{R} jU_j + \sum_{j=1}^{R}(R+1-j)U_j = (R+1)\sum_{j=1}^{R} U_j.$$

So, rearranging things,

$$\sum_{j=1}^{h} jx_j \ge \sum_{j=1}^{R} jU_j + (R+1)\Big(\sum_{j=1}^{h} x_j - \sum_{j=1}^{R} U_j\Big),$$

and the last term is nonnegative by assumption. □

Let $R_n = (n^q/q!)D^q$ and $S_n = (n^{q-1}/(q-1)!)D^{q-1}.D_P$. In the notation of Lemma 9.1, we have

$$\sum_{m=0}^{M_P(n)} f_P(m,n) = l(nD) = R_n + O(n^{q-1})$$

and $f_P(m,n) \le S_n + O(n^{q-2})$. We will assume from now on that $S_n \ne 0$ (the case $S_n = 0$ is similar). Then using our estimate, we have

$$M_P(n) \ge R_n/S_n + O(1) \quad \text{and} \quad \sum_{m=0}^{R_n/S_n+O(1)} (S_n + O(n^{q-2})) \le \sum_{m=0}^{M_P(n)} f_P(m,n).$$

So using Lemma 9.3 with $x_i = f_P(i,n)$ and $U_i = S_n + O(n^{q-2})$, for $n \gg 0$ we get the estimate

$$\sum_{m=0}^{M_P(n)} (m-n) f_P(m,n) \ge \sum_{m=0}^{R_n/S_n+O(1)} m(S_n + O(n^{q-2})) - n\sum_{m=0}^{M_P(n)} f_P(m,n)$$
$$\ge \frac{R_n^2}{2S_n} - nR_n + O(n^q)$$
$$\ge \frac{R_n}{S_n}\Big(\frac{n^q}{2q!}\big(D^q - 2qD^{q-1}.D_P\big) + O(n^{q-1})\Big).$$

So for $n \gg 0$, $\sum_{m=0}^{M_P}(m-n) f_P(m,n) > 0$ if $D^q > 2qD^{q-1}.D_P$. Then we are done by Lemma 9.1. □

The result for $q = 1$ is this:

COROLLARY 9.4. *Let D be an effective divisor on a nonsingular projective curve X. If D is a sum of more than 2 distinct points on X then D is large.*

By Theorems 8.3A and 8.3B we then recover two corollaries.

COROLLARY 9.5A. *Siegel's theorem* (*Theorem* 2.1A).

COROLLARY 9.5B. *Picard's theorem* (*Theorem* 2.1B).

Actually, we have only proved these theorems for nonsingular curves $\widetilde{C}$, but the general case follows from this case by looking at the normalization of $\widetilde{C}$.

Suppose that we have a divisor $D = \sum_{i=1}^r D_i$ satisfying the hypotheses of Theorem 9.2. We would like to modify D to a divisor $D' = \sum_{i=1}^r a_i D_i$ so that we may optimally apply the theorem. When each D_i is ample, this amounts to choosing the a_i so that in the embedding given by nD' for $n \gg 0$ the degree of each $a_i D_i$ is the same. In terms of intersection theory, we would like $a_i D_i \,.\, (D')^{q-1}$ to be the same for each i. We make the following definition:

Definition 9.6. Suppose X is a q-dimensional nonsingular projective variety. Let $D = \sum_{i=1}^r D_i$ be a divisor on X with $D_1, \dots, D_r$ effective. We say D has equidegree with respect to $D_1, \dots, D_r$ if $D_i \,.\, D^{q-1} = D^q/r$ for $i = 1, \dots, r$. We say that D is equidegreelizable (with respect to $D_1, \dots, D_r$) if there exist real numbers $a_i > 0$ such that if $D' = \sum_{i=1}^r a_i D_i$ then D' has equidegree with respect to $a_1 D_1, \dots, a_r D_r$ (where we extend intersections to $\operatorname{Div} X \otimes \mathbb{R}$ in the canonical way).

We will frequently just say D is equidegreelizable, omitting the reference to the D_i when it is clear what we mean.

LEMMA 9.7. *Let X be a nonsingular projective variety. Let $q = \dim X$. Let $D_1, \dots, D_r$ be divisors on X with $D_i^q > 0$ for all i. Suppose that all q-fold intersections of the D_i are nonnegative. Then $\sum_{i=1}^r D_i$ is equidegreelizable with respect to $D_1, \dots, D_r$.*

Proof. Consider the function $f(a_1, \dots, a_r) = (\sum_{i=1}^r e^{a_i} D_i)^q$ on $\mathbb{R}^r$ subject to the constraint $g(a_1, \dots, a_r) = \sum_{i=1}^r a_i = 0$. Since all q-fold intersections of the D_i are nonnegative, $f(a_1, \dots, a_r) \geq e^{qa_i} D_i^q$ for any i. Since $D_i^q > 0$ for all i, as $\max\{a_i\} \to \infty$ we have $f(a_1, \dots, a_r) \to \infty$. It follows that f attains a minimum on the plane $\sum_{i=1}^r a_i = 0$. Therefore there exists a solution $\lambda, a_1, \dots, a_r$ to the Lagrange multiplier equations

$$g = 0, \qquad \frac{\partial f}{\partial a_i} = qe^{a_i} D_i \,.\, \Big(\sum_{i=1}^r e^{a_i} D_i\Big)^{q-1} = \lambda \frac{\partial g}{\partial a_i} = \lambda, \quad \text{for } i = 1, \dots, r.$$

So $D' = \sum_{i=1}^r e^{a_i} D_i$ has equidegree with respect to $e^{a_1} D_1, \dots, e^{a_r} D_r$, and trivially $e^{a_i} > 0$ for all i. □

We give an example to show that not all divisor sums are equidegreelizable.

Example 9.8. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Let

$$D_1 = P_1 \times \mathbb{P}^1, \qquad D_2 = P_2 \times \mathbb{P}^1, \qquad D_3 = \mathbb{P}^1 \times Q,$$

where P_1, P_2, and Q are points in the various $\mathbb{P}^1$. So

$$D_1 . D_2 = D_1^2 = D_2^2 = D_3^2 = 0 \quad \text{and} \quad D_1 . D_3 = D_2 . D_3 = 1.$$

Let $D = a_1 D_1 + a_2 D_2 + a_3 D_3$. Since $a_3 D_3 . D = a_1 D_1 . D + a_2 D_2 . D$, it is clear that there do not exist $a_1, a_2, a_3 > 0$ such that $a_i D_i . D = D^2/3$ for $i = 1, 2, 3$. So $D = D_1 + D_2 + D_3$ is not equidegreelizable with respect to D_1, D_2, and D_3.

With this definition, we have the following theorem.

THEOREM 9.9. *Let X be a nonsingular projective variety*; *let $q = \dim X$. Let $D = \sum_{i=1}^r D_i$ be a big divisor on X equidegreelizable with respect to $D_1, \ldots, D_r$, with $D_1, \ldots, D_r$ nef and effective. Suppose that every irreducible component of D is nonsingular. Suppose that the intersection of any $m+1$ distinct D_i is empty. If $r > 2mq$, then D is large. Furthermore, there exists a very large divisor E with the same support as D such that Φ_E is birational onto its image.*

Proof. Since D is equidegreelizable, we may find positive integers a_i such that if $D' = \sum_{i=1}^r a_i D_i$ then $(a_i D_i . (D')^{q-1}/(D')^q)$ is arbitrarily close to $1/r$ for each i. Note that D' is again big. Since for any $P \in D(\bar{k})$, P belongs to at most m divisors D_i, and $r > 2mq$, we have

$$2q(D')^{q-1} . (D')_P = 2q \sum_{i : P \in D_i(\bar{k})} a_i D_i . (D')^{q-1} < (D')^q.$$

Therefore by Theorem 9.2, nD' is very large for $n \gg 0$. The last statement follows from the fact that D' is big. □

LEMMA 9.10. *Let X be a complex projective variety. Let $D = \sum_{i=1}^r D_i$ be a sum of effective Cartier divisors on X. Then there exists a nonsingular projective variety X', a birational morphism $\pi : X' \to X$, and a divisor $D' = \sum_{i=1}^r D_i'$ on X' such that* $\operatorname{Supp} D_i' \subset \operatorname{Supp} \pi^* D_i$ *for all i, every irreducible component of D' is nonsingular, $|D_i'|$ is base-point free for all i (in particular D_i' is nef), and $\kappa(D_i') = \kappa(D_i) = \dim \Phi_{D_i'}(X')$ for all i. Also, if X and D are defined over a number field, then X', D', and π are defined over some number field.*

Proof. Taking a resolution of the singularities of X and of the embedded singularities of the irreducible components of D, we may assume that X and every irreducible component of D are nonsingular. For each i, let $m_i > 0$ be such that $\dim \Phi_{m_i D_i}(X) = \kappa(D_i)$. Let $\pi : X' \to X$ be the map obtained by blowing up the schemes of base-points [Har77, pp. 168–169] of all the linear systems $|m_i D_i|$. Then $\pi^*(m_i D_i) = D_i' + F_i$ for each i, where $|D_i'|$ is base-point free and F_i is the fixed part of $|\pi^*(m_i D_i)|$. We have, trivially from the definition, $\kappa(D_i) = \kappa(m_i D_i)$. Further,

$\kappa(m_i D_i) = \kappa(\pi^*(m_i D_i))$ (in fact $l(mD_i) = l(\pi^*(mD_i))$ for all m follows easily from $\pi_* \mathcal{O}_{X'} = \mathcal{O}_X$ and the projection formula). Finally, $\kappa(\pi^*(m_i D_i)) = \kappa(D_i')$ since by construction $\kappa(D_i') = \max_{n>0} \dim \Phi_{nD_i'}(X') \geq \kappa(D_i) = \kappa(\pi^*(m_i D_i))$ (the other inequality being trivial). So $\kappa(D_i') = \kappa(D_i)$ for all i and therefore X', π, and $D' = \sum_{i=1}^r D_i'$ satisfy the requirements of the lemma. □

We now obtain one of our main results.

THEOREM 9.11A. *Let X be a projective variety defined over a number field k. Let $q = \dim X$. Let $D = \sum_{i=1}^r D_i$ be a divisor on X defined over k such that the D_i are effective Cartier divisors and the intersection of any $m+1$ distinct D_i is empty.*

(a) *If D_i is big for each i and $r > 2mq$, then $X \setminus D$ is quasi-Mordellic.*

(b) *If D_i is ample for each i and $r > 2mq$, then $X \setminus D$ is Mordellic.*

THEOREM 9.11B. *Let X be a complex projective variety. Let $q = \dim X$. Let $D = \sum_{i=1}^r D_i$ be a divisor on X such that the D_i are effective Cartier divisors and the intersection of any $m+1$ distinct D_i is empty.*

(a) *If D_i is big for each i and $r > 2mq$, then $X \setminus D$ is quasi-Brody hyperbolic.*

(b) *If D_i is ample for each i and $r > 2mq$, then $X \setminus D$ is complete hyperbolic and hyperbolically imbedded in X. In particular, $X \setminus D$ is Brody hyperbolic.*

Aside from the statement about being complete hyperbolic and hyperbolically imbedded, the same proof works for both Theorems 9.11A and 9.11B.

Proof. We will first prove parts (a) for both theorems. Let $\pi : X' \to X$ and D' be as in Lemma 9.10 with respect to X and $D = \sum_{i=1}^r D_i$. Since $\operatorname{Supp} D' \subset \operatorname{Supp} \pi^* D$, it is easily seen that if the conclusions of parts (a) of the theorems hold for $D' = \sum_{i=1}^r D_i'$ and X', then they hold for $D = \sum_{i=1}^r D_i$ and X. Therefore, replacing $(X, D, D_1, \ldots, D_r)$ by $(X', D', D_1', \ldots, D_r')$, we can assume (extending k in the Diophantine case if necessary) that X is nonsingular, every irreducible component of D is nonsingular, and D_i is nef for all i. The desired statements then follow from Lemma 9.7, Theorem 9.9, and Theorems 8.3A and 8.3B.

For part (b) of Theorem 9.11A (resp. Theorem 9.11B), we note that by (a) any set of D-integral points (resp. the image of any holomorphic map $f : \mathbb{C} \to X \setminus D$) is not Zariski-dense. Let R be a set of D-integral points (resp. the image of a holomorphic map $f : \mathbb{C} \to X \setminus D$). Let Y be an irreducible component of the Zariski-closure of R. Suppose $\dim Y > 0$. Then D pulls back to a sum of r ample (hence big) divisors on Y such that the intersection of any $m+1$ of them is empty. But $R \cap Y$ is a Zariski-dense set of $D|_Y$-integral points on Y (resp. the image of a holomorphic map $f : \mathbb{C} \to Y \setminus D$), contradicting part (a) proved above since $r > 2mq \geq 2m \dim Y$. Therefore $\dim Y = 0$.

To prove the extra hyperbolicity statements in (b) in the analytic case, we use Theorem 7.3. Let $\varnothing \subset J \subset \{1, \dots, r\}$. Let $s = \#J$. Let X' be an irreducible component of $\bigcap_{j \in J} D_j$. In checking the hypotheses of Theorem 7.3, we can clearly assume that $X' \not\subset D_i$ for any $i \in I \setminus J$ and that $\dim X' > 0$. Let $D' = \sum_{i \in I \setminus J} D_i|_{X'}$. Then D' is a sum of $r-s$ ample divisors on X' and the intersection of any $m-s+1$ of the ample divisors is empty since X' is already contained in an intersection of s of the D_i. Since $r > 2mq$ implies that $r - s > 2(m-s)\dim X'$, by what we have proved above, $X' \setminus D'$ is Brody hyperbolic. So by Theorem 7.3, $X \setminus D$ is complete hyperbolic and hyperbolically imbedded in X. □

We end this section by showing that our main conjectures in the simple case $m = 1$ can be proved by reducing to Siegel's and Picard's theorems. We will need the following Bertini theorem; see [Iit82, Th. 7.19].

THEOREM 9.12. *Let $|D|$ be a base-point free linear system on a nonsingular projective variety X with* $\dim \Phi_D(X) \geq 2$. *Then every member of $|D|$ is connected, and a general member of $|D|$ is nonsingular and irreducible.*

LEMMA 9.13. *Suppose $D = D_1 + D_2$ is an effective Cartier divisor on a projective variety X with $\kappa(D_1) > 0$, $\kappa(D_2) > 0$, and $D_1 \cap D_2 = \varnothing$. Then $\kappa(D) = \kappa(D_1) = \kappa(D_2) = 1$.*

Proof. By Lemma 9.10, we may assume that X is nonsingular and $|D|$ is base-point free. If $\kappa(D) \geq 2$, then $\dim \Phi_{nD}(X) \geq 2$ for some $n > 0$. But by Theorem 9.12, every divisor in $|nD|$ is connected, which contradicts that $D_1 \cap D_2 = \varnothing$. □

THEOREM 9.14. *The main conjectures*, *Conjectures* 5.1A *through* 5.4B, *are true if $m = 1$ (i.e., $D_i \cap D_j = \varnothing$ for all $i \neq j$).*

Proof. By Lemma 9.13, it is sufficient to prove the conjectures when $D = \sum_{i=1}^r D_i$ with $r > 2$ and $\kappa(D) = 1$. By Lemma 9.10, we may assume that X is nonsingular and D is base-point free. For $n \gg 0$, $\Phi_{nD}(X)$ is a nonsingular curve C and Φ_{nD} has connected fibers. Therefore, since $D_i \cap D_j = \varnothing$ for $i \neq j$, we have $\Phi_{nD}(X \setminus D) \subset C \setminus \{r \text{ points}\}$. Since $r > 2$, we are done by Siegel's and Picard's theorems. □

10. **A filtration lemma**

We now show how some of the results in the last section may be improved by use of a linear algebra lemma on filtrations. The idea of using this lemma, as well as its statement and proof, are taken from the paper [CZ04b]. Corvaja and Zannier used it to prove a result on integral points on surfaces, and it will be essential for our results on surfaces in the next section also.

LEMMA 10.1. *Let V be a vector space of finite dimension d over a field k. Let $V = W_1 \supset W_2 \supset \cdots \supset W_h$ and $V = W_1^* \supset W_2^* \supset \cdots \supset W_{h^*}^*$ be two filtrations on V. There exists a basis $v_1, \ldots, v_d$ of V that contains a basis of each W_j and W_j^*.*

Proof. The proof will be by induction on d. The case $d = 1$ is trivial. By refining the first filtration, we may assume without loss of generality that W_2 is a hyperplane in V. Let $W_i' = W_i^* \cap W_2$ for $i = 1, \ldots, h^*$. By the inductive hypothesis, there exists a basis $v_1, \ldots, v_{d-1}$ of W_2 containing a basis of each of $W_3, \ldots, W_h$ and $W_1', \ldots, W_h'$. Let l be the maximal index with $W_l^* \not\subset W_2$, and let $v_d \in W_l^* \setminus W_l'$. We claim that $B = \{v_1, \ldots, v_d\}$ is a basis of V with the required property. It clearly contains a basis of W_i for each i. Let $i \in \{1, \ldots, h^*\}$. If $i > l$, then $W_i^* = W_i'$, and so by construction B contains a basis of W_i^*. If $i \le l$, then $v_d \in W_l^* \setminus W_l' \subset W_i^* \setminus W_i'$. Since B contains a basis B_i' of W_i' and W_i' is a hyperplane in W_i^*, we see that $B_i' \cup \{v_d\}$ is a basis of W_i^*. □

Using our notation from the last section, suppose that for $P \in D$ we have $D_P = D_{P,1} + D_{P,2}$, where $D_{P,1}$ and $D_{P,2}$ are effective divisors with no irreducible components in common. We may then prove the following versions of Lemma 9.1 and Theorem 9.2.

LEMMA 10.2. *Let $D = \sum_{i=1}^r D_i$ be a nonzero divisor on a nonsingular projective variety X with D_i effective for each i. Let $P \in D$. Let*

$$f_{P,j}(m,n) = l(nD - mD_{P,j}) - l(nD - (m+1)D_{P,j})$$

for $j = 1, 2$. If there exists $n > 0$ such that for every $P \in D$ and $j = 1, 2$ either $\sum_{m=0}^{\infty}(m-n) f_{P,j}(m,n) > 0$ or $D_{P,j} = 0$, then nD is very large.

THEOREM 10.3. *Suppose X is a q-dimensional nonsingular projective variety. Let $D = \sum_{i=1}^r D_i$ be a divisor on X such that $D_{P,j}$ is effective and nef for all $P \in D$ and $j = 1, 2$. Suppose also that every irreducible component of D is nonsingular. If*

$$D^q > 2qD^{q-1}.D_{P,j} \quad \text{for all } P \in D \text{ and } j = 1, 2,$$

then nD is very large for $n \gg 0$.

The proofs are similar to the proofs of Lemma 9.1 and Theorem 9.2. The only difference is that in the proof of Lemma 10.2, we look at the two filtrations of $L(nD)$ given by $W_j = L(nD - jD_{P,1})$ and $W_j^* = L(nD - jD_{P,2})$, and we use the filtration lemma to construct a basis $f_1, \ldots, f_{l(nD)}$ that contains a basis for each W_j and W_j^*.

Suppose now that $D = \sum_{i=1}^r D_i$, where the D_i are effective divisors and the intersection of any $m+1$ distinct D_i is empty. We may then write $D_P = D_{P,1} + D_{P,2}$, where $D_{P,1}$ and $D_{P,2}$ are each not a sum of more than $[(m+1)/2]$

of the D_i, where $[x]$ denotes the greatest integer in x. Using this, we get the following improvements to Theorems 9.11A(a) and 9.11B(a).

THEOREM 10.4A. *Let X be a nonsingular projective variety defined over a number field k. Let $q = \dim X$. Let $D = \sum_{i=1}^r D_i$ be a divisor on X defined over k such that the D_i are effective divisors with no irreducible components in common and such that the intersection of any $m+1$ distinct D_i is empty. Suppose also that every irreducible component of D is nonsingular. If D_i is nef and big for each i and $r > 2[(m+1)/2]q$, then $X \setminus D$ is quasi-Mordellic.*

THEOREM 10.4B. *Let X be a nonsingular complex projective variety. Let $q = \dim X$. Let $D = \sum_{i=1}^r D_i$ be a divisor on X such that the D_i are effective divisors with no irreducible components in common and such that the intersection of any $m+1$ distinct D_i is empty. Suppose also that every irreducible component of D is nonsingular. If D_i is nef and big for each i and $r > 2[(m+1)/2]q$, then $X \setminus D$ is quasi-Brody hyperbolic.*

11. **Surfaces**

When X is a surface, the results of the last two sections can be made more precise. With regards to integral points, this section builds on some of the work in [CZ04b]. Corvaja and Zannier prove, essentially, Theorem 11.2 [CZ04b, Main Theorem], and they prove Theorem 11.5A when $m = 2$ and the D_i have multiples that are all numerically equivalent. The Nevanlinna-theoretic analogues of the results in [CZ04b] were proved by Ru and Liu in [LR05]. Our results overlap with their results as well.

We first prove a consequence of the Hodge index theorem.

LEMMA 11.1. *Let D be a divisor on a nonsingular projective surface X with $D^2 > 0$. Then $(D^2)(E^2) \le (D\,.\,E)^2$ for any divisor E on X.*

Proof. Using the Hodge index theorem, we can diagonalize the intersection pairing on $\operatorname{Num} X \bigotimes \mathbb{R}$ with one $+1$ on the diagonal and all other diagonal entries -1. We will identify elements of $\operatorname{Pic} X$ as elements of $\operatorname{Num} X \bigotimes \mathbb{R}$ in the canonical way. Extend D to an orthogonal basis B of $\operatorname{Num} X \bigotimes \mathbb{R}$. Let E be any divisor on X. Writing E in the basis B, it is apparent from the Hodge index theorem that $(D^2)(E^2) \le (D\,.\,E)^2$. □

For surfaces, Theorem 10.3 can be improved as follows:

THEOREM 11.2 (Corvaja and Zannier). *Let X be a nonsingular projective surface. Let $D = \sum_{i=1}^r D_i$ be a nef divisor on X with effective divisors D_i and $D^2 > 0$. For $P \in D$, let*

$$D_P = \sum_{i : P \in D_i} D_i = D_{P,1} + D_{P,2},$$

where $D_{P,1}$ and $D_{P,2}$ are effective divisors with no irreducible components in common. Suppose that for all $P \in D$, $j = 1, 2$, and $m, n > 0$, we have either $l(nD - mD_{P,j}) = 0$ or

$$l(nD - mD_{P,j}) - l(nD - (m+1)D_{P,j}) \leq (nD - mD_{P,j}).D_{P,j} + O(1),$$

where the implied constant does not depend on m or n. For $j = 1, 2$, let

$$A_{P,j} = D_{P,j}.D_{P,j}, B_{P,j} = D.D_{P,j} \quad \text{and} \quad C = D.D.$$

If for all $P \in D$ and $j = 1, 2$, either we have $D_{P,j} = 0$ or we have

$$\begin{aligned} &A_{P,j} > 0 \text{ implies } B_{P,j}^2 - 2A_{P,j}C + 3A_{P,j}B_{P,j} \\ &\qquad\qquad\qquad\qquad + (3A_{P,j} - B_{P,j})(B_{P,j}^2 - A_{P,j}C)^{1/2} < 0, \\ &A_{P,j} = 0 \text{ implies } C > 4B_{P,j}, \\ &A_{P,j} < 0 \text{ implies } B_{P,j}^2 - 2A_{P,j}C + 3A_{P,j}B_{P,j} \\ &\qquad\qquad\qquad\qquad + (3A_{P,j} - B_{P,j})(B_{P,j}^2 - A_{P,j}C)^{1/2} > 0, \end{aligned}$$

then nD is very large for $n \gg 0$ (note that by Lemma 11.1, $B_{P,j}^2 - A_{P,j}C > 0$).

Proof. Let $P \in D$ and $j \in \{1, 2\}$ with $D_{P,j} \neq 0$. Let $A = A_{P,j}$ and $B = B_{P,j}$. By assumption, in the notation of Lemma 10.2, for all $m, n > 0$ we have either $f_{P,j}(m, n) = 0$ or

$$f_{P,j}(m, n) = l(nD - mD_{P,j}) - l(nD - (m+1)D_{P,j}) \leq nB - mA + O(1),$$

where the implied constant in the $O(1)$ does not depend on m or n. Note that

$$l(nD) = \tfrac{1}{2}D^2n^2 + O(n) = \tfrac{1}{2}Cn^2 + O(n).$$

Solving

$$\sum_{m=0}^{M(n)} nB - mA + O(1) = \tfrac{1}{2}Cn^2 + O(n) = l(nD)$$

for $M(n)$, we obtain

$$\begin{aligned} &M(n) = ((B \pm \sqrt{B^2 - AC})/A)n + O(1) && \text{if } A \neq 0, \\ &M(n) = (C/2B)n + O(1) && \text{if } A = 0 \text{ and } B \neq 0, \\ &M(n) \gg n^2 && \text{if } A = 0 \text{ and } B = 0. \end{aligned}$$

From now on, we will always choose the minus sign in the first expression above. Note that $nB - mA \geq O(1)$ for $0 \leq m \leq M(n)$ in the above three cases. So the estimate $f_{P,j}(m, n) \leq nB - mA + O(1)$ is valid for $0 \leq m \leq M(n)$. We also have

$\sum_{m=0}^{\infty} f_{P,j}(m,n) = l(nD)$. Therefore by Lemma 9.3,

$$\sum_{m=0}^{\infty}(m-n)f_{P,j}(m,n) \geq \sum_{m=0}^{M(n)} m(nB - mA + O(1)) - nl(nD). \tag{8}$$

Let $t = (B - \sqrt{B^2 - AC})/A$. If $A \neq 0$, then substituting t into (8) yields

$$\sum_{m=0}^{\infty}(m-n)f_{P,j}(m,n) \geq \left(-\tfrac{1}{3}At^3 + \tfrac{1}{2}Bt^2 - \tfrac{1}{2}C\right)n^3 + O(n^2).$$

So if $-\frac{1}{3}At^3 + \frac{1}{2}Bt^2 - \frac{1}{2}C > 0$, then by Lemma 10.2 nD will be very large for $n \gg 0$. Algebraic simplification then gives the theorem in the case $A \neq 0$. The other cases are similar. □

LEMMA 11.3. *Let X be a nonsingular projective surface. Let C be an irreducible curve on X and D any divisor on X. Then*

$$h^0(D) - h^0(D - C) \leq \max\{0, 1 + C.D\}.$$

Proof. The statement depends only on the linear equivalence class of D. So replacing D by an appropriate divisor linearly equivalent to D, we may assume that the support of D does not contain any possible singularity of C. By Lemma 7.7,

$$h^0(D) - h^0(D - C) \leq \dim H^0(C, \mathcal{O}(D)|_C).$$

Since the support of D does not contain any singularity of C, $\mathcal{O}(D)|_C$ has degree $C.D$ on C, and $\dim H^0(C, \mathcal{O}(D)|_C) \leq \max\{0, 1 + C.D\}$. □

LEMMA 11.4. *Let X be a nonsingular projective surface. Let D be a nef divisor on X. Let E be an effective divisor on X such that either E is linearly equivalent to an irreducible curve or $C.E \leq 0$ for every irreducible component C of E. Then for all $m, n > 0$, either $l(nD - mE) = 0$ or*

$$l(nD - mE) - l(nD - (m+1)E) \leq (nD - mE).E + O(1), \tag{9}$$

where the implied constant is independent of m and n.

Proof. Suppose that E is linearly equivalent to an irreducible curve C. If $(nD - mE).E \geq 0$, then (9) holds by Lemma 11.3. If

$$(nD - mE).E = nD.C - mC.C < 0,$$

then since D is nef we must have $C.C > 0$. But if $l(nD - mE) > 0$, then $nD - mE$ is linearly equivalent to an effective divisor $F = G + m'C$, where $m' \geq 0$ and G is an effective divisor not containing C. Since clearly $G.C \geq 0$, $F.C = (nD - mE).E < 0$ implies $C.C < 0$, a contradiction. So either $l(nD - mE) = 0$ or (9) holds in this case.

Now suppose we are in the second case, where $C \,.\, E \le 0$ for every irreducible component C of E. Let $E = \sum_{j=1}^{k} a_j C_j$, where each C_j is a distinct prime divisor. Then as in the proof of Theorem 9.2 we have

$$l(nD - mE) - l(nD - (m+1)E) \le \sum_{j=1}^{k} \sum_{l=0}^{a_j - 1} \dim H^0(C_j, i_{C_j}^* \mathcal{O}(nD - mE - \textstyle\sum_{j'=1}^{j-1} a_{j'} C_{j'} - lC_j)).$$

But

$$\begin{aligned}\dim H^0\big(C_j, i_{C_j}^* \mathcal{O}\big(nD - mE - \textstyle\sum_{j'=1}^{j-1} a_{j'} C_{j'} - lC_j\big)\big) &\le \dim H^0(C_j, i_{C_j}^* \mathcal{O}(nD - mE)) + O(1) \\ &\le (nD - mE) \,.\, C_j + O(1),\end{aligned}$$

where the implied constant is independent of m and n. The second inequality follows since $(nD - mE) \,.\, C_j \ge nD \,.\, C_j \ge 0$ as D is nef and $E \,.\, C_j \le 0$. Combining the above inequalities, we then see that (9) always holds in this case. □

Going back to the general setup of Section 4, we prove these theorems:

THEOREM 11.5A. *Let X be a nonsingular projective surface. Suppose the D_i have no irreducible components in common.*

(a) *If D_i is big for all i and $r \ge 4[(m+1)/2]$, then $X \setminus D$ is quasi-Mordellic.*

(b) *If D_i is ample for all i and either m is even and $r > 2m$ or m is odd and $r > 2m+1$, then $X \setminus D$ is Mordellic.*

THEOREM 11.5B. *Let X be a nonsingular projective surface. Suppose the D_i have no irreducible components in common.*

(a) *If D_i is big for all i and $r \ge 4[(m+1)/2]$, then $X \setminus D$ is quasi-Brody hyperbolic.*

(b) *If D_i is ample for all i and either m is even and $r > 2m$ or m is odd and $r > 2m+1$, then $X \setminus D$ is complete hyperbolic and hyperbolically imbedded in X. In particular, $X \setminus D$ is Brody hyperbolic.*

Proof. We first prove parts (a). It suffices to prove these in the case $r = 4[(m+1)/2]$. For any effective divisor E on the surface X, there exists an effective divisor E' on X with $\operatorname{Supp} E' \subset \operatorname{Supp} E$, $|E'|$ base-point free, and $\dim \Phi_{E'}(X) = \kappa(E)$ (this follows, for instance, from [Zar62, Th. 6.1]). Therefore we can reduce to the case where $|D_i|$ is base-point free for all i and $\dim \Phi_{D_i}(X) = 2$ for all i. So $D_i^2 > 0$ and D_i is nef for each i. By Lemma 9.7, D is equidegreelizable. So we may find positive integers $a_1, \dots, a_r$ such that if $D' = \sum_{i=1}^{r} a_i D_i$ then $a_i D_i \,.\, D'/(D')^2$ is arbitrarily close to $1/r$ for all i. Since at most m of the D_i meet at any given point, D'_P is a sum of at most m of the $a_i D_i$ for any $P \in D'$. Therefore we may write $D'_P = D'_{P,1} + D'_{P,2}$, where each $D'_{P,j}$ is a sum of at most $[(m+1)/2]$ of the $a_i D_i$, and $D'_{P,1}$ and $D'_{P,2}$ have no irreducible components in common. Note that when $D'_{P,j} \neq 0$, from our assumptions on the D_i we have $|D'_{P,j}|$ is base-point free and $\dim \Phi_{D'_{P,j}}(X) = 2$. So by Theorem 9.12, $D'_{P,j}$ is linearly equivalent to an irreducible curve. Therefore, by Lemma 11.4, we will be able to apply Theorem 11.2 to D'.

The hardest case is clearly when $D'_{P,j}$ is a sum of the maximum $[(m+1)/2]$ of the $a_i D_i$. For simplicity, we will now restrict to this case. It follows that, in the notation of Theorem 11.2, for all such P and j,

$$\left|\frac{C}{B_{P,j}} - \frac{r}{[\frac{1}{2}(m+1)]}\right| = \left|\frac{C}{B_{P,j}} - 4\right| < \varepsilon,$$

where by adjusting the a_i in D', ε may be made arbitrary close to 0, while at the same time $A_{P,j}/B_{P,j}$ is positive and bounded away from 0. Furthermore, by Lemma 11.1, $A_{P,j}/B_{P,j} \leq B_{P,j}/C$. Let $a = A_{P,j}/B_{P,j}$ and $c = C/B_{P,j}$. Then by Theorem 11.2, we must show that

$$1 - 2ac + 3a + (3a-1)\sqrt{1-ac} < 0, \tag{10}$$

where $0 < a \leq 1/c$. When $c = 4$, the left side of the inequality becomes $1 - 5a + (3a-1)\sqrt{1-4a}$. This is easily seen to have a root only at $a = 0$ for $0 \leq a \leq 1/4$, and is negative for $0 < a \leq 1/4$, since putting $a = 1/4$ gives $-1/4$. So when $c = 4 + \varepsilon$, since a is bounded away from zero as $\varepsilon \to 0$, we see that (10) is satisfied for small enough ε. Therefore by Theorem 11.2, for an appropriate choice of D', nD' is very large for $n \gg 0$. Since D' is big, $\Phi_{nD'}$ is a birational map onto its image for some arbitrarily large n. By Theorems 8.3A and 8.3B we are done, as D and D' have the same support.

Assume the hypotheses in parts (b). Let Y be the Zariski-closure of a set of D-integral points (resp. image of a holomorphic map $f : \mathbb{C} \to X \setminus D$). By what we have proved above, $\dim Y \leq 1$. If $\dim Y = 1$, let C be an irreducible component of Y with $\dim C > 0$. Since each D_i is ample, D_i must intersect C in some point. Since at most m of the D_i meet at a point and $r > 2m$, we see that $D|_C$ contains at least three distinct points. Therefore by Siegel's (resp. Picard's) theorem, we get a contradiction since we have a Zariski-dense set of $D|_C$-integral points on C (resp. a holomorphic map $\mathbb{C} \to C \setminus D|_C$ with Zariski-dense image). This same argument and Theorem 7.3 show that in the analytic case, $X \setminus D$ is complete hyperbolic and hyperbolically embedded in X. □

It is possible to make certain minor improvements to these theorems.

THEOREM 11.6A. *Let X be a nonsingular projective surface. Suppose $m = 2$, $D = \sum_{i=1}^{4} D_i$, $D_i . D_j > 0$ for $i \neq j$, $D_1^2 > 0$, D_i is nef for all i, and the D_i have no irreducible components in common. Suppose also that the conclusion of*

Lemma 11.4 holds with D any positive integral linear combination of the D_i and $E = D_j$ for $j = 1, 2, 3, 4$. Then $X \setminus D$ is quasi-Mordellic.

THEOREM 11.6B. *With the same hypotheses as above, in the analytic setting, $X \setminus D$ is quasi-Brody hyperbolic.*

Proof. We first show that for any $\varepsilon > 0$,

$$\Big(\sum_{i=1}^{4} e^{a_i} D_i\Big)^2 \geq \exp(\tfrac{2}{3} \max_i \{a_i\})$$

on the real plane $(1+\varepsilon)a_1 + \sum_{i=2}^{4} a_i = 0$. If $\max_i\{a_i\} = a_1$ then

$$\Big(\sum_{i=1}^{4} e^{a_i} D_i\Big)^2 \geq e^{2a_1} D_1^2 \geq e^{2a_1}.$$

Otherwise, if $\max_i\{a_i\} = a_j$, $j > 1$, then clearly we must have $a_k \geq -a_j/3$ for some $j \neq k$. Then

$$\Big(\sum_{i=1}^{4} e^{a_i} D_i\Big)^2 \geq e^{a_j + a_k} D_j \,.\, D_k \geq e^{2a_j/3}$$

since $D_j \,.\, D_k \geq 1$. Therefore $(\sum_{i=1}^{4} e^{a_i} D_i)^2$ takes a minimum on the plane $(1+\varepsilon)a_1 + \sum_{i=2}^{4} a_i = 0$. Looking at the Lagrange multiplier equations as in Lemma 9.7, we see that there exist real numbers $b_i > 0$ and $\lambda > 0$ (depending on ε) such that if $D' = \sum_{i=1}^{4} b_i D_i$, then $b_1 D_1 \,.\, D' = (1+\varepsilon)\lambda$ and $b_i D_i \,.\, D' = \lambda$ for $i = 2, 3, 4$. Written differently,

$$\frac{(D')^2}{b_1 D_1 \,.\, D'} = \frac{4+\varepsilon}{1+\varepsilon} \quad \text{and} \quad \frac{(D')^2}{b_i D_i \,.\, D'} = 4 + \varepsilon > 4 \quad \text{for } i = 2, 3, 4.$$

Note also that it follows from the first inequality we proved that in terms of $a_1, \ldots$ $\ldots, a_4$, the region where $(\sum_{i=1}^{4} e^{a_i} D_i)^2$ takes a minimum may be bounded independently of $\varepsilon < 1$. Therefore there exist positive constants K and K', independent of $\varepsilon < 1$, such that we may choose $K < b_i < K'$ for all i, and in particular, as $\varepsilon \to 0$, $(b_1 D_1)^2/(b_1 D_1 \,.\, D')$ is bounded away from zero.

We now choose positive integers c_i such that c_i/c_j closely approximates b_i/b_j, and let $E = \sum_{i=1}^{4} c_i D_i$. Having chosen ε small enough and the integers c_i correctly, we will then have $E^2 > 4c_i D_i \,.\, E$ for $i = 2, 3, 4$, and we will have $E^2/(c_1 D_1 \,.\, E)$ close enough to 4 (see the proof of Theorems 11.5A and 11.5B) so that the inequalities in Theorem 11.2 hold for E and $E_{P,j} = c_i D_i$ for any i. Since $m = 2$, we may always take $E_{P,j} = 0$ or $E_{P,j} = c_i D_i$ for some i. By our hypotheses, we may apply Theorem 11.2, so nE is very large for $n \gg 0$. Since

$D_1^2 > 0$, E is big. So we are done by Theorems 8.3A and 8.3B, as D and E have the same support. ☐

Example 11.7. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Let $D_1 = \{0\} \times \mathbb{P}^1$, $D_2 = \mathbb{P}^1 \times \{0\}$, and let D_3 and D_4 be ample effective divisors on X. Suppose also that the intersection of any three of the D_i is empty. Let $D = \sum_{i=1}^4 D_i$. Then the hypotheses of Theorems 11.6A and 11.6B are satisfied and $X \setminus D$ is quasi-Mordellic and quasi-Brody hyperbolic. Note also that

$$X \setminus D_1 \cup D_2 \cong \mathbb{A}^2 \cong \mathbb{P}^2 \setminus \{\text{a line}\}.$$

Therefore, we can also prove many theorems for $\mathbb{P}^2 \setminus D$, where D is a sum of three prime divisors on $\mathbb{P}^2$.

Recently, Corvaja and Zannier [CZ06] have shown another way how their methods may get results on $\mathbb{P}^2 \setminus D$, where D is a sum of three prime divisors satisfying certain hypotheses.

We have the following corollaries to Theorems 11.6A and 11.6B.

COROLLARY 11.8A. *Let X be a nonsingular projective surface. Suppose $m = 2$, $D = \sum_{i=1}^4 D_i$, D_1, D_2, and D_3 are big, $\kappa(D_4) > 0$, and the D_i have no irreducible components in common. Then $X \setminus D$ is quasi-Mordellic.*

COROLLARY 11.8B. *Let X be a nonsingular projective surface. Suppose $m = 2$, $D = \sum_{i=1}^4 D_i$, D_1, D_2, and D_3 are big, $\kappa(D_4) > 0$, and the D_i have no irreducible components in common. Then $X \setminus D$ is quasi-Brody hyperbolic.*

Proof. As in the proofs of Theorems 11.5A and 11.5B, we can reduce to the case where $|D_i|$ is base-point free for all i, $\dim \Phi_{D_i}(X) = 2$ for $i = 1, 2, 3$, and $\dim \Phi_{D_4}(X) = \kappa(D_4)$. So in particular, D_i is nef for all i, $D_1^2, D_2^2, D_3^2 > 0$, and $D_4^2 \geq 0$. By Lemma 9.13, $D_i . D_j > 0$ for $i \neq j$. For $i = 1, 2, 3$, D_i is linearly equivalent to an irreducible curve by Theorem 9.12, since by our reductions $|D_i|$ is base-point free and $\dim \Phi_{D_i}(X) = 2$. The same holds for D_4 if $D_4^2 > 0$. If $D_4^2 = 0$, then for every irreducible component C of D_4 we must have $C . D_4 = 0$ since D_4 is nef. This verifies the hypotheses of Lemma 11.4 with $E = D_i$ for $i = 1, 2, 3, 4$. Therefore, we may apply Theorems 11.6A and 11.6B to X and D. ☐

We note that one can construct examples where $m = 2$, $D = \sum_{i=1}^4 D_i$, D_1 and D_2 are big, $\kappa(D_3) = \kappa(D_4) = 1$, the D_i have no irreducible components in common, and there exist Zariski-dense sets of D-integral points. We now prove a theorem in the case where we only have $\kappa(D_i) > 0$ for all i.

THEOREM 11.9A. *Let X be a nonsingular projective surface. Suppose the D_i have no irreducible components in common. If $\kappa(D_i) > 0$ for all i and $r > 4[(m+1)/2]$, then there does not exist a Zariski-dense set of D-integral points on X.*

THEOREM 11.9B. *Let X be a nonsingular projective surface. Suppose the D_i have no irreducible components in common. If $\kappa(D_i) > 0$ for all i and $r > 4[(m+1)/2]$, then there does not exist a holomorphic map $f : \mathbb{C} \to X \setminus D$ with Zariski-dense image.*

Proof. As usual, we can reduce to the case where $|D_i|$ is base-point free for all i. In this case, for any subset $I \subset \{1, \dots, r\}$, if $D_I = \sum_{i \in I} D_i$ is big, then there exists $n_I > 0$ such that $\dim \Phi_{n_I D_I}(X) = 2$. Since the D_i are nef, this happens if and only if $D_i \,.\, D_j > 0$ for some $i, j \in I$. Let $N = \prod_I n_I$, where I ranges over subsets such that D_I is big. Let

$$D' = ND \quad \text{and} \quad D'_i = ND_i.$$

We see that for any nonnegative integral linear combination E of the D'_i, if E is big, then E is linearly equivalent to an irreducible divisor since $|E|$ is base-point free and $\dim \Phi_E(X) = 2$, and otherwise, for every irreducible component C of E we have $C \,.\, E = 0$. Therefore, by Lemma 11.4, replacing D by D', we may assume that we can apply Theorem 11.2 to any nonnegative integral linear combination of the D_i.

By Theorem 9.14, we are done if any three of the D_i have pairwise empty intersection. So suppose that this is not the case. Then we have $m \geq 2$ and $r \geq 5$. We now show that D is equidegreelizable. As in the proof of Lemma 9.7, it is sufficient to show that $(\sum_{i=1}^r e^{a_i} D_i)^2$ attains a minimum on the plane $\sum_{i=1}^r a_i = 0$. For this, it will suffice to show that

$$\Big(\sum_{i=1}^r e^{a_i} D_i\Big)^2 \geq \exp\big(\tfrac{1}{3} \max_i\{a_i\}\big).$$

Suppose $\max_i\{a_i\} = a_j$ for $j \in \{1, \dots, r\}$. Let a_k and a_l be some choice of the next largest a_i. Clearly, since $\sum_{i=1}^r a_i = 0$ and $r \geq 5$, we must have $a_k, a_l \geq -2a_j/(r-2) \geq -2a_j/3$. We now show that either $D_j \,.\, D_k \geq 1$ or $D_j \,.\, D_l \geq 1$. Suppose otherwise. Then by an earlier assumption, we must have $D_k \,.\, D_l \geq 1$. But then $D_k + D_l$ is big, and so we must have $(D_k + D_l) \,.\, D_j \geq 1$ by Lemma 9.13, a contradiction. So if, say, $D_j \,.\, D_k \geq 1$ then

$$\Big(\sum_{i=1}^r e^{a_i} D_i\Big)^2 \geq e^{a_j + a_k} D_j \,.\, D_k \geq \exp\big(\tfrac{1}{3} \max_i\{a_i\}\big),$$

as was to be shown. Since D is equidegreelizable, there exist positive integers c_i such that if $D' = \sum_{i=1}^r c_i D_i$, then $c_i D_i \,.\, D'/(D')^2$ is as close as we like to $1/r$. Since we may choose $D'_{P,j}$ to consist of a sum of at most $[(m+1)/2]$ of the $c_i D_i$ and $r > 4[(m+1)/2]$, we may choose the c_i so that we always have $C > 4B_{P,j}$ in Theorem 11.2 (applied to D'). We also have $A_{P,j} \geq 0$. But then, as we have seen previously, the needed inequalities of Theorem 11.2 will be satisfied. $\square$

12. **Integral points over varying number fields**

We now consider the case where the integral points are allowed to vary over number fields of a bounded degree over some number field k. As an application of their results on surfaces in [CZ04b], Corvaja and Zannier prove the following theorem.

THEOREM 12.1. *Let X be a projective curve defined over a number field k. Let $D=\sum_{i=1}^{r} P_i$ be a divisor on X defined over k such that the P_i are distinct points. If $r>4$, then all sets of D-integral points on X quadratic over k are finite.*

This theorem can also be obtained as a consequence of a result of Vojta (see §14.3). Using the same technique Corvaja and Zannier used, looking at symmetric powers of X, our higher-dimensional results give the following theorem.

THEOREM 12.2. *Let $n=\dim X$. If D_i is ample for all i and $r>2d^2mn$, then all sets of D-integral points on X of degree d over k are finite.*

Proof. Suppose $r>2d^2mn$, and let $R\subset X(\bar{k})$ be a set of D-integral points on X of degree d over k. It suffices to prove the finiteness of R in the case where for every $P\in R$ we have $[k(P):k]=d$. Let X^d be the d-fold product of X with itself, and let $\pi_i: X^d\to X$ be the i-th projection map for $i=1,\dots,d$. Let $\operatorname{Sym}^d X$ denote the d-fold symmetric product of X with itself, and let $\phi: X^d\to \operatorname{Sym}^d X$ be the natural map. Let $E_i=\phi(\pi_1^* D_i)$ and $E=\sum_{i=1}^{r} E_i$. Since $\phi^* E_i=\sum_{j=1}^{d}\pi_j^* D_i$ is ample on X^d and ϕ is a finite surjective morphism, it follows that E_i is ample. By looking at the corresponding statement on X^d, we see that the intersection of any $dm+1$ distinct E_i is empty. We also have $\dim \operatorname{Sym} X^d=dn$. Since $r>2(dm)(dn)$, by Theorem 9.11A(b) all sets of k-rational E-integral points on $\operatorname{Sym}^d X$ are finite. For $P\in R$, let $P^{(1)},\dots,P^{(d)}$ denote the d conjugates of P over k. Then $R'=\{(P^{(1)},\dots,P^{(d)})\in X^d \mid P\in R\}$ is a set of $\sum_{i=1}^{d}\pi_i^* D$-integral points on X^d. So $\phi(R')$ is a set of E-integral points on $\operatorname{Sym}^d X$. Note that $\phi(R')$ is actually a set of k-rational points on $\operatorname{Sym}^d X$. Therefore, from above, $\phi(R')$ must be finite, and so clearly R must be finite. □

13. **A result of Faltings**

In [Fal02], Faltings proves the finiteness of integral points on the complements of certain irreducible singular curves in $\mathbb{P}^2$. Recently, a similar result has also been obtained by Zannier in [Zan05]. We show, as a corollary of our work on surfaces, how we may improve both results on integral points, and at the same time we will prove an analogous statement for holomorphic curves.

Let X be a nonsingular projective surface over an algebraically closed field k of characteristic 0. Let $\mathscr{L}=\mathcal{O}_X(L)$ be an ample line bundle on X with K_X+3L ample. Assume that the global sections $\Gamma(X,\mathscr{L})$ generate

(a) $\mathscr{L}_x/\mathfrak{m}_x^4\mathscr{L}_x$ for all points $x \in X$,

(b) $\mathscr{L}_x/\mathfrak{m}_x^3\mathscr{L}_x \oplus \mathscr{L}_y/\mathfrak{m}_y^3\mathscr{L}_y$ for all pairs $\{x, y\}$ of distinct points, and

(c) $\mathscr{L}_x/\mathfrak{m}_x^2\mathscr{L}_x \oplus \mathscr{L}_y/\mathfrak{m}_y^2\mathscr{L}_y \oplus \mathscr{L}_z/\mathfrak{m}_z^2\mathscr{L}_z$ for all triples $\{x, y, z\}$ of distinct points.

A three-dimensional subspace $V \subset \Gamma(X, \mathscr{L})$ that generates $\mathscr{L}$ gives a morphism $f_V : X \to \mathbb{P}^2$. Faltings studies this map when V is suitably generic.

Definition 13.1. Let $V \subset \Gamma(X, \mathscr{L})$ be a three-dimensional subspace. We call V generic if

(a) V generates $\mathscr{L}$,

(b) the discriminant locus $Z \subset X$ of f_V is nonsingular,

(c) the restriction of f_V to Z is birational onto its image $D \subset \mathbb{P}^2$, and

(d) D has only cusps and nodes as singularities.

Three-dimensional subspaces $V \subset \Gamma(X, \mathscr{L})$ are naturally parametrized by a Grassmannian G. Let $n = L^2$. Faltings proves the following theorem.

THEOREM 13.2. *Let the notation be as above.*

(a) *Generic V form a dense open subset G' of G.*

(b) *For generic V, let $\pi : Y \to X \to \mathbb{P}^2$ denote the associated normal Galois covering. Then Y is nonsingular, Z is irreducible, and the covering group* $\operatorname{Aut}(Y/\mathbb{P}^2)$ *is the full symmetric group S_n.*

From now on we assume that we are in situations associated to a generic V. Faltings gives the following description of the geometry of $\pi^* D$.

THEOREM 13.3. *Let $\pi^* D$ be the pullback of D to Y. Then*

$$\pi^* D = 2 \sum_{1 \le i < j \le n} Z_{ij} = \sum_{i=1}^{n} E_i,$$

where Z_{ij} is effective and nonsingular for every i and j and $E_i = \sum_{j \ne i} Z_{ij}$ is the pullback of Z under the i-th projection map $Y \to X$. Furthermore, let $P \in \pi^* D$. Then one of the following holds:

(a) *$\pi(P)$ is a smooth point of D, and $P \in Z_{ij}$ for exactly one pair $\{i, j\}$.*

(b) *$\pi(P)$ is a node of D, and exactly two components Z_{ij} and Z_{kl} of $\pi^* D$ for disjoint pairs $\{i, j\}$ and $\{k, l\}$ intersect at P.*

(c) *$\pi(P)$ is a cusp of D, and exactly three components Z_{ij}, Z_{ik}, Z_{jk} intersect at P for some i, j, k.*

Let $d = \deg D$, and assume that everything above is defined over a number field. The main result of [Fal02] is this:

THEOREM 13.4 (Faltings). *If $dL-\alpha Z$ is ample on X for some $\alpha > 12$, then $\mathbb{P}^2 \setminus D$ is Mordellic.*

Zannier proves this without the ampleness condition if the Kodaira number of X is nonnegative, and more generally he gives a numerical condition replacing the condition on L and Z above. We will be able to completely remove the ampleness condition from Theorem 13.4. We also prove the analogue for holomorphic curves.

THEOREM 13.5A. *$\mathbb{P}^2 \setminus D$ is Mordellic.*

THEOREM 13.5B. *$\mathbb{P}^2 \setminus D$ is complete hyperbolic. In particular, $\mathbb{P}^2 \setminus D$ is Brody hyperbolic.*

Proof. Let Z_{ij} and E_i be as in Theorem 13.3. Note that since we assumed that $Z \sim K_X + 3L$ is ample, E_i is ample for all i. Let N be such that NE_i is very ample for all i. Let $D' = N\pi^*D$. Since $\pi : Y \setminus D' \to \mathbb{P}^2 \setminus D$ is a finite étale covering, we are reduced to proving the theorems for $Y \setminus D'$; see Lemma 7.2(d) and [Lan87, Ch. 1, Prop. 3.6]. We now use Theorem 11.2 to show that $D' = 2N \sum_{1\le i<j\le n} Z_{ij}$ is large. Let $P \in D'$. By Theorem 13.3, for $l = 1, 2$, we can always choose $D'_{P,l} = 2NZ_{ij}$, $D'_{P,l} = 2NZ_{ij} + 2NZ_{ik}$, or $D'_{P,l} = 0$ for some i, j, k. An examination of the proof of Theorem 11.2 shows that in checking its hypotheses, we may replace $D'_{P,l}$ by any divisor F with $F \ge D'_{P,l}$, so in particular, we can replace $D'_{P,l}$ by $2NE_i$ for some i.

Since $2NE_i$ is very ample, $2NE_i$ is linearly equivalent to a prime divisor, and so the technical hypothesis related to Lemma 11.4 in Theorem 11.2 is satisfied. Since $D' = N \sum_{i=1}^n E_i$, by symmetry among the E_i, we have $(D')^2/(D'.(2NE_i)) = n/2$. The assumption (a) on L given at the beginning of the section implies that $n = L^2 \ge 9$, so $(D')^2/(D'.(2NE_i)) \ge 9/2$. Since E_i is ample, $(2NE_i)^2 > 0$. Thus, in checking the hypotheses of Theorem 11.2, we have $A_{P,j} > 0$ and $C/B_{P,j} \ge 9/2$. It then follows from the proof of Theorems 11.5A and 11.5B that the relevant inequality in Theorem 11.2 is satisfied (in fact, $C/B_{P,j} \ge 4$ is sufficient for this). So D' is large. It follows from Theorem 13.3 that at most four E_i meet at a point. Since $n \ge 9$, any curve on Y must therefore meet $D' = N \sum_{i=1}^n E_i$ in at least three distinct points. Thus, combining Theorems 8.3A and 8.3B with Siegel's and Picard's theorems, we find that $Y \setminus D'$ is Mordellic and Brody hyperbolic. A simple application of Theorem 7.3 shows that furthermore, in the analytic case, $Y \setminus D'$ is complete hyperbolic. □

14. Remarks on the Siegel and Picard-type conjectures

In this section we will discuss the sharpness of the inequalities and the necessity of certain hypotheses in many of the conjectures, how our conjectures relate to

Vojta's conjectures, and what special cases of the conjectures are known by previous work.

14.1. *Examples constraining improvements to the conjectures.* As is well known, the algebraic tori $\mathbb{G}_m^n$ admit both a Zariski-dense set of integral points (over number fields k with $\mathbb{O}_k^*$ infinite) and, over the field $\mathbb{C}$, a holomorphic map $f : \mathbb{C} \to \mathbb{G}_m^n$ with Zariski-dense image. As natural compactifications of $\mathbb{G}_m^n$, we have $\mathbb{P}^n$ with $n+1$ ample divisors at infinity and $(\mathbb{P}^1)^n$ with $2n$ divisors at infinity of D-dimension $\kappa = 1$. Taking slightly more general compactifications, the following example shows that the inequalities in the main Siegel and Picard-type conjectures, Conjectures 5.1A and 5.1B, are sharp for all values of m and $\kappa_0 > 0$.

Example 14.1. Let $X = (\mathbb{P}^n)^q$, and let π_j be the j-th projection map from X to $\mathbb{P}^n$ for $j = 1, \dots, q$. Let k be a number field with $\mathbb{O}_k^*$ infinite, and let S be the set of archimedean places of k. Let H_i be the hyperplane on $\mathbb{P}^n$ defined by $x_i = 0$ for $i = 0, \dots, n$. Let

$$D_{i,j} = \pi_j^* H_i \quad \text{for } 0 \le i \le n \text{ and } 1 \le j \le q.$$

Let $1 \le m \le nq$. Let $r = [m + m/n]$ and $r' = [r/(n+1)] = [m/n]$. Let

$$D = \sum_{j=1}^{r'} \sum_{i=0}^{n} D_{i,j} + \sum_{i=1}^{r-r'(n+1)} D_{i,r'+1}.$$

Then $\mathbb{G}_m^{nq}$ embeds as a Zariski-dense subset of $X \setminus D$. Therefore there exists a Zariski-dense set of (D, S)-integral points on $X \setminus D$ and a holomorphic map $f : \mathbb{C} \to X \setminus D$ with Zariski-dense image. Furthermore, there are at most $nr' + r - r'(n+1) = r - r' = m$ of the $D_{i,j}$ in D meeting at a given point, and D is a sum of $r = [m + m/n]$ of the $D_{i,j}$ with $\kappa(D_{i,j}) = n$ for all i and j.

We have not yet discussed the $\kappa_0 = 0$ case. Let D be a nontrivial effective divisor on a projective surface X such that there exists either a Zariski-dense set of D-integral points on X or a holomorphic map $f : \mathbb{C} \to X \setminus D$ with Zariski-dense image. By blowing up points of D on X, we can get a divisor D' on a surface X' such that $X \setminus D \cong X' \setminus D'$ and D' has arbitrarily many components. Note that the exceptional curves E in D' have $\kappa(E) = 0$. So, as is suggested by the κ_0 in the denominators of the inequalities in the conjectures, there is no possible result of the type in the main Siegel and Picard-type conjectures if one allows divisors D_i with $\kappa(D_i) = 0$; see, however, the results in Section 14.3.

There are also examples showing that even if the hypotheses of the main Siegel and Picard-type conjectures are satisfied, the exceptional sets involved may be Zariski-dense. For example, let $X = \mathbb{P}^1 \times \mathbb{P}^1$, and let $D = \sum_{i \in I} P_i \times \mathbb{P}^1$ be a finite sum with $P_i \in \mathbb{P}^1(k)$ for $i \in I$ and some number field k. Then it is easy to show that $\mathrm{Exc}_{\mathrm{Dio}}(X \setminus D) = \mathrm{Exc}_{\mathrm{hol}}(X \setminus D) = X \setminus D$.

We give the following examples related to the main conjectures for ample divisors; see [Fuj72], [Gre72], and [NW02] for the constructions.

Example 14.2A. Let D be the sum of any r hyperplanes in general position (i.e., the intersection of any $n+1$ of them is empty) in $\mathbb{P}^n$ with $n < r \leq 2n$. Assume also that D is defined over a number field. Then one may show that there exists a linear subspace $L \subset \mathbb{P}^n$ with $\dim L = [n/(r-n)]$ such that L contains a Zariski-dense set of $D|_L$-integral points (for some k and S).

Example 14.2B. In the same situation as above, one may also show that there exists a holomorphic map $f : \mathbb{C} \to L \setminus D$ with Zariski-dense image.

In the simplest case, where $r = 2m = 2n$, we may simply take L to be a line that passes through points P and Q, where P is the intersection of, say, the first n hyperplanes and Q is the intersection of the last n hyperplanes. Then $L \setminus L \cap D \cong \mathbb{G}_m$, and so we see that we may not have finiteness or constancy for the objects in question.

We now give two examples related to the general conjectures (so in these two examples we do not require sets of D-integral points to be k-rational). The next example shows that the inequalities in the general conjectures are best possible when X is a curve.

Example 14.3. Let X be a projective curve defined over a number field k with $\mathcal{O}_k^*$ infinite. Let $f : X \to \mathbb{P}^1$ be a morphism of degree d defined over k. Let $P, Q \in \mathbb{P}^1(k)$ be two distinct points over which f is unramified, and let $D = P + Q$. Then there exists an infinite set R of k-rational D-integral points on $\mathbb{P}^1 \setminus D$. Since f has degree d, $f^{-1}(R)$ is a set of f^*D-integral points on $X \setminus f^*D$ of degree d over k, and f^*D is a sum of $2d$ distinct points on X.

Taking products of curves, we can get examples in all dimensions showing that the inequality in the general Siegel-type conjecture cannot be improved in the case $\kappa_0 = 1$. A related example shows that the inequality in part (b) of the general Siegel-type conjecture for ample divisors cannot be improved.

Example 14.4. Let $D = \sum_{i=1}^{2md} H_i$ be a sum of hyperplanes on $\mathbb{P}^n$ defined over a number field k such that the intersection of any $m+1$ of the H_i is empty. Suppose also that $\bigcap_{i=(j-1)m+1}^{jm} H_i = \{P_j\}$ consists of a single point for $j = 1, \dots, 2d$ and the P_j are collinear. Then there exist infinite sets of D-integral points of degree d on $\mathbb{P}^n \setminus D$ over large enough number fields. Indeed, the line L through the P_j intersects D in $2d$ points, and by an appropriate use of Example 14.3, we see that $L \setminus L \cap D$ contains infinite sets of integral points of degree d over large enough number fields.

14.2. *Relations to Vojta's conjectures.* We first show how some special cases of the main conjectures are related to Vojta's main conjecture. If D is a divisor on a nonsingular complex variety X, we say that D has normal crossings if every point $P \in D$ has an analytic open neighborhood in X with analytic local coordinates $z_1, \dots, z_n$ such that D is locally defined by $z_1 \cdot z_2 \cdots z_i = 0$ for some i. Let $m(D, P) = \sum_{v \in S} \lambda_{D,v}(P)$. Inspired by results in equi-dimensional Nevanlinna theory, Vojta made the following conjecture in [Voj87].

CONJECTURE 14.5A (Vojta's main conjecture). *Let X be a nonsingular projective variety with canonical divisor K. Let D be a normal crossings divisor on X, and let k be a number field over which X and D are defined. Let A be a big divisor on X. Let $\varepsilon > 0$. Then there exists a proper Zariski-closed subset $Z = Z(X, D, \varepsilon, A)$ of X such that*

$$m(D, P) + h_K(P) \leq \varepsilon h_A(P) + O(1) \quad \textit{for all points } P \in X(k) \setminus Z.$$

The analogue is conjectured for holomorphic curves:

CONJECTURE 14.5B. *Let X be a nonsingular complex projective variety with canonical divisor K. Let D be a normal crossings divisor on X. Let A be a big divisor on X. Let $\varepsilon > 0$. Then there exists a proper Zariski-closed subset $Z = Z(X, D, \varepsilon, A)$ of X such that for all holomorphic maps $f : \mathbb{C} \to X$ whose image is not contained in Z, the inequality*

$$m(D, r) + T_K(r) \leq \varepsilon T_A(r) + O(1)$$

holds for all r outside a set of finite Lebesgue measure.

See [Voj87] for the definitions and properties of the terms appearing in the conjectures. Qualitatively, these conjectures have the following simple consequences.

CONJECTURE 14.6A. *Let X be a nonsingular projective variety defined over a number field k. Let K be the canonical divisor of X, and let D be a normal crossings divisor on X defined over k. Suppose that $K + D$ is big. Then $X \setminus D$ is quasi-Mordellic.*

CONJECTURE 14.6B. *Let X be a nonsingular complex projective variety. Let K be the canonical divisor of X, and let D be a normal crossings divisor on X. Suppose that $K + D$ is big. Then $X \setminus D$ is quasi-Brody hyperbolic.*

To relate these conjectures to our conjectures, we recall the following theorem, which is a consequence of Mori theory [Mor82, Lemma 1.7].

THEOREM 14.7. *Let X be a nonsingular complex projective variety of dimension n. If $D_1, \dots, D_{n+2}$ are ample divisors on X, then $K + \sum_{i=1}^{n+2} D_i$ is ample.*

So, when X is nonsingular, the D_i are ample, and D has normal crossings, we see that Conjectures 5.3A and 5.3B are consequences of Conjectures 14.6A and 14.6B.

Similarly, we now show that the general Siegel-type conjecture for normal crossings divisors on $\mathbb{P}^n$ follows from Vojta's general conjecture. Let X be a variety defined over a number field k and let $P \in X(\bar{k})$. Let

$$d(P) = \frac{1}{[k(P):\mathbb{Q}]} \log|D_{k(P)/\mathbb{Q}}|,$$

where $D_{k(P)/\mathbb{Q}}$ is the discriminant of $k(P)$ over $\mathbb{Q}$. We call $d(P)$ the absolute logarithmic discriminant of P.

CONJECTURE 14.8 (Vojta's general conjecture). *Let X be a nonsingular projective variety with canonical divisor K. Let D be a normal crossings divisor on X, and let k be a number field over which X and D are defined. Let A be a big divisor on X. Let $\varepsilon > 0$. If ν is a positive integer, then there exists a proper Zariski-closed subset $Z = Z(\nu, X, D, \varepsilon, A)$ of X such that*

$$m(D,P) + h_K(P) \leq d(P) + \varepsilon h_A(P) + O(1) \tag{11}$$

for all points $P \in X(\bar{k})\backslash Z$ such that $[k(P):k] \leq \nu$.

Actually, Vojta's general conjecture as it appears in [Voj87] has the discriminant term as $(\dim X)d(P)$, but it is now believed that the $\dim X$ term is unnecessary; see [Voj99, Conj. 8.7] or the discussion at the end of [Voj89].

THEOREM 14.9. *Assume Vojta's general conjecture. Let $D = \sum_{i=1}^r D_i$ be a sum of r nontrivial effective divisors on $\mathbb{P}^n$ such that D is a normal crossings divisor defined over k. Let ν be a positive integer. If $r > 2\nu + n - 1$, then $X\backslash D$ is degree ν quasi-Mordellic.*

Proof. Let $\varepsilon < 1$ and let $Z = Z(\nu, \mathbb{P}^n, D, \varepsilon, H)$ be the exceptional set in Vojta's general conjecture, where H is a hyperplane on $\mathbb{P}^n$. Let R be a set of D-integral points on $\mathbb{P}^n$ of degree ν over k. If $r \geq 2\nu + n$, then

$$m(D,P) + h_K(P) = h_D(P) + h_K(P) + O(1) \geq (2\nu - 1)h(P) + O(1) \tag{12}$$

for all $P \in R$, where h is the usual absolute logarithmic height on $\mathbb{P}^n$. An elementary inequality relating the discriminant and height on projective space, due to Silverman [Sil84], is

$$d(P) \leq (2\nu - 2)h(P) + O(1) \tag{13}$$

for all $P \in \mathbb{P}^n(\bar{k})$ with $[k(P):k] \leq \nu$. Combining inequalities (12) and (13), and using the fact that there are only finitely many points of bounded height and bounded degree, we see that the inequality (11) in Vojta's general conjecture (with

$A = H$ and $\varepsilon < 1$) is violated for all but finitely many $P \in R$. Therefore $R \setminus Z$ is finite, proving the theorem. □

So assuming Vojta's general conjecture, we see that the general Siegel-type conjecture is true for projective space in the case that D has normal crossings.

14.3. *Previously known results related to the conjectures.* As was discussed earlier, our work builds on previous work of Corvaja and Zannier, who obtained results on surfaces in [CZ04b], and initiated the general method we have used in [CZ02]. The Nevanlinna-theoretic analogues of [CZ04b] were proved by Liu and Ru in [LR05]. We briefly discussed these previous results in Section 11.

Previous results on integral points and holomorphic curves on $X \setminus D$ when D has "lots of components" have been given by, among others, Vojta, Noguchi, and Winkelmann. As a consequence of his work on integral points on subvarieties of semi-abelian varieties, Vojta [Voj96] proved this:

THEOREM 14.10A. *Let X be a projective variety defined over a number field k. Let ρ denote the Picard number of X. Let D be an effective divisor on X defined over k that has more than* $\dim X - h^1(X, \mathcal{O}_X) + \rho$ *irreducible components over $\bar{k}$. Then there does not exist a Zariski-dense set of D-integral points on X.*

Similarly, the analogue for holomorphic curves follows as a special case of work of Noguchi [Nog81].

THEOREM 14.10B. *Let X be a complex projective variety. Let ρ denote the Picard number of X. Let D be an effective divisor on X that has more than* $\dim X - h^1(X, \mathcal{O}_X) + \rho$ *irreducible components. Then there does not exist a holomorphic map $f : \mathbb{C} \to X \setminus D$ with Zariski-dense image.*

We note that it is easily shown that both theorems are sharp in that there are divisors with $\dim X - h^1(X, \mathcal{O}_X) + \rho$ irreducible components for which the conclusions of the theorems are false. For a weaker, but more elementary, theorem along the lines of Theorem 14.10A, see also [Voj87, Th. 2.4.1]. As consequences of Theorems 14.10A and 14.10B, we see that Conjectures 5.3A and 5.3B are true when $m \geq \dim X$ and X is a projective variety with Picard number one (e.g., $X = \mathbb{P}^n$).

From the work of Noguchi and Winkelmann [NW02], we have the following theorems related to our main conjectures for ample divisors (some special cases of these results had been obtained previously by various other people; see [NW02] for the history).

THEOREM 14.11A. *Let X be a projective variety of dimension n defined over a number field k. Let S be a finite set of places of k containing the archimedean places. Let ρ be the Picard number of X. Let $D = \sum_{i=1}^{r} D_i$ be a divisor on X defined over k with the D_i being effective reduced ample Cartier divisors such that the intersection of any $n + 1$ of them is empty.*

(a) *If $r > n+1$ then all sets of D-integral points R have* $\dim R \leq n\rho/(r-n)$.

(b) *If $r > n(\rho+1)$, then $X \setminus D$ is Mordellic.*

(c) *If $X \subset \mathbb{P}^N$, all D_i are hypersurface cuts of X, and $r > 2n$, then $X \setminus D$ is Mordellic.*

THEOREM 14.11B. *Let X be a complex projective variety of dimension n. Let ρ be the Picard number of X. Let $D = \sum_{i=1}^r D_i$ be a divisor on X with the D_i being effective reduced ample Cartier divisors such that the intersection of any $n+1$ of them is empty.*

(a) *If $r > n+1$ then all holomorphic maps $f : \mathbb{C} \to X \setminus D$ have*

$$\dim f(\mathbb{C}) \leq \frac{n}{r-n}\rho.$$

(b) *If $r > n(\rho+1)$, then $X \setminus D$ is complete hyperbolic and hyperbolically imbedded in X. In particular, $X \setminus D$ is Brody hyperbolic.*

(c) *If $X \subset \mathbb{P}^N$, all D_i are hypersurface cuts of X, and $r > 2n$, then $X \setminus D$ is complete hyperbolic and hyperbolically imbedded in X. In particular, $X \setminus D$ is Brody hyperbolic.*

Thus, when $m = \dim X$, the D_i are reduced divisors, and $\rho(X) = 1$, the main conjectures for ample divisors, Conjectures 5.4A and 5.4B, are true. Similarly, parts (c) of the above theorems give special cases of parts (b) of Conjectures 5.4A and 5.4B.

In [Voj92], Vojta proved the following generalization of the Thue-Siegel-Roth-Wirsing theorem and Faltings' theorem on rational points on curves.

THEOREM 14.12. *Let X be a nonsingular projective curve defined over a number field k with canonical divisor K. Let D be an effective divisor on X defined over k with no multiple components and A an ample divisor on X. Let ν be a positive integer, and let $\varepsilon > 0$. Then*

$$m(D,P) + h_K(P) \leq d_a(P) + \varepsilon h_A(P) + O(1)$$

for all $P \in X(\bar{k}) \setminus D$ with $[k(P):k] \leq \nu$, where $d_a(P)$ is an arithmetic discriminant on X.

An arithmetic discriminant on X is determined by the choice of a regular model for X over $\mathcal{O}_k$ and is unique up to $O(1)$ (see [Voj91] for the definition and properties). To obtain a qualitative result about integral points from Theorem 14.12, we use the following discriminant-height inequality of Song and Tucker [ST99, Eq. 2.0.3].

THEOREM 14.13. *Let X be a nonsingular projective curve defined over a number field k with canonical divisor K. Let A be an ample divisor on X. Let ν be* *a positive integer. Let* $\varepsilon > 0$. *Then*

$$d_a(P) \le h_K(P) + (2[k(P):k] + \varepsilon)h_A(P) + O(1)$$

for all $P \in X(\bar{k})$ *with* $[k(P):k] \le \nu$.

Using Theorems 14.12 and 14.13, we easily obtain a qualitative result on integral points on curves.

COROLLARY 14.14. *Let* X *be a nonsingular projective curve defined over a number field* k. *Let* D *be an effective divisor on* X *that is a sum of more than* 2ν *distinct points. Then* $X \setminus D$ *is degree* ν *Mordellic.*

Therefore our general Siegel-type conjectures are true for curves. Of course, for $\mathbb{P}^1$ this was already known from the Thue-Siegel-Roth-Wirsing theorem. As mentioned earlier, the special case $\nu = 2$ of Corollary 14.14 was also proved by Corvaja and Zannier using the Schmidt subspace theorem technique [CZ04b].

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