

ANNALS OF MATHEMATICS

**Cubic structures, equivariant Euler
characteristics and lattices of modular
forms**

By TED CHINBURG, GEORGIOS PAPPAS, and MARTIN J. TAYLOR



SECOND SERIES, VOL. 170, NO. 2

September, 2009

ANMAAH

Cubic structures, equivariant Euler characteristics and lattices of modular forms

By TED CHINBURG, GEORGIOS PAPPAS, and MARTIN J. TAYLOR

Abstract

We use the theory of cubic structures to give a fixed point Riemann-Roch formula for the equivariant Euler characteristics of coherent sheaves on projective flat schemes over \mathbb{Z} with a tame action of a finite abelian group. This formula supports a conjecture concerning the extent to which such equivariant Euler characteristics may be determined from the restriction of the sheaf to an infinitesimal neighborhood of the fixed point locus. Our results are applied to study the module structure of modular forms having Fourier coefficients in a ring of algebraic integers, as well as the action of diamond Hecke operators on the Mordell-Weil groups and Tate-Shafarevich groups of Jacobians of modular curves.

1. Introduction
2. Cubic structures and categories
3. The main calculation and result
4. Galois structure of modular forms

Proof of Theorem 1.3

5. An equivariant Birch and Swinnerton-Dyer relation

Proof of Theorem 1.4

Acknowledgments

References

Chinburg is supported by NSF Grants #DMS00-70433 and #DMS05-00106. Pappas is supported by NSF Grants #DMS05-01049 and #DMS01-11298 (via the Institute for Advanced Study). Taylor is an EPSRC Senior Research Fellow and recipient of the Royal Society Wolfson Merit award.

1. Introduction

Let X be a projective flat scheme over $\text{Spec}(\mathbb{Z})$ having an action of a finite group G . Let \mathcal{F} be a G -equivariant coherent locally free sheaf on X . One then has an Euler characteristic

$$\chi(X, \mathcal{F}) = \sum_i (-1)^i [\mathrm{H}^i(X, \mathcal{F})]$$

in the Grothendieck group $G_0(\mathbb{Z}[G])$ of all finitely generated G -modules. If the action of G on X is tame, as we shall assume for most of the article, there is a refinement $\chi^P(X, \mathcal{F})$ of $\chi(X, \mathcal{F})$ in the Grothendieck group $K_0(\mathbb{Z}[G])$ of all finitely generated projective $\mathbb{Z}[G]$ -modules. Let X^g be the subscheme of X fixed by the action of $g \in G$, and let $X' = \bigcup_{e \neq g \in G} X^g$. The goal of this paper is to compute $\chi^P(X, \mathcal{F})$ and $\chi(X, \mathcal{F})$ when G is abelian via the restriction of \mathcal{F} to an infinitesimal neighborhood of X' . In view of the fact that $G_0(\mathbb{Z}[G])$ and $K_0(\mathbb{Z}[G])$ generally have nontrivial torsion subgroups, this can be viewed as a variant of the problem of finding a Lefschetz-Riemann-Roch formula for $\chi^P(X, \mathcal{F})$ and $\chi(X, \mathcal{F})$ with an explicit, bounded denominator. Some applications we will discuss are to the Galois module structure of lattices of modular forms, and of the Tate-Shafarevich groups and Mordell-Weil groups of Jacobians of modular curves, with respect to the action of diamond Hecke operators.

Our main tool is the theory of cubic structures. These were first studied in detail by Breen in [Bre83], building on work of Mumford and Grothendieck concerning biextensions and the Theorem of the Cube for abelian varieties. The use of cubic structures in our context was first introduced by one of us in [Pap98], [Pap05] and [Pap08] to prove refined Galois module structure results when the action of the group G on X is free, that is, when $X' = \emptyset$. Here we present a reformulation of cubic structures that is tailored to proving Lefschetz-Riemann-Roch results when G is abelian and the action of G on X is tame. Our results improve on what can be shown using the localization techniques of Thomason from [Tho86] and [Tho92], which generally determine Euler characteristics in $G_0(\mathbb{Z}[G])$ or $K_0(\mathbb{Z}[G])$ only up to classes of finite order dividing an unspecified power of the order $\#G$ of G . (Thomason actually deals with the case in which the action is by a multiplicative group scheme, but some of his techniques extend to the case of constant groups G ; see [CEPT97b].)

We will study the image $\bar{\chi}^P(X, \mathcal{F})$ of $\chi^P(X, \mathcal{F})$ in the classgroup $\text{Cl}(\mathbb{Z}[G])$ of $\mathbb{Z}[G]$, which is the quotient of $K_0(\mathbb{Z}[G])$ by the subgroup generated by the class of $\mathbb{Z}[G]$. The projection $K_0(\mathbb{Z}[G]) \rightarrow \text{Cl}(\mathbb{Z}[G])$ identifies the torsion subgroup of $K_0(\mathbb{Z}[G])$ with the finite abelian group $\text{Cl}(\mathbb{Z}[G])$. With this identification, $\chi^P(X, \mathcal{F}) = \bar{\chi}^P(X, \mathcal{F}) + r(\mathbb{Z}[G])$, where

$$r = \frac{1}{\#G} \text{rank}_{\mathbb{Z}}(\chi^P(X, \mathcal{F})) = \frac{1}{\#G} \sum_i (-1)^i \dim_{\mathbb{Q}}(\mathrm{H}^i(X_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}}))$$

is an integer that can be determined using a nonequivariant Riemann-Roch formula for the Euler characteristic of the restriction $\mathcal{F}_{\mathbb{Q}}$ of \mathcal{F} to the general fiber $X_{\mathbb{Q}}$ of X .

The results we obtain, along with those of [Pap08], support the following conjecture about $\bar{\chi}^P(X, \mathcal{F})$. Define \hat{X}' to be the formal completion of X' in X , and let $\mathcal{F}|_{\hat{X}'}$ be the G -sheaf on \hat{X}' that is the restriction of \mathcal{F} . Since X is projective and the action of G on X is tame, the image of X' in $\text{Spec}(\mathbb{Z})$ is a proper closed set.

LOCALIZATION CONJECTURE 1.1. *There is an integer $N \geq 1$, which depends only on $\dim(X)$, and an integer $\delta \geq 1$, which depends only on $\#G$, for which the following is true. Let $m = \text{g.c.d.}(N, \#G)^\delta$.*

- (a) (Input localization) *The class $m \cdot \bar{\chi}^P(X, \mathcal{F})$ depends only on the pair $(\hat{X}', \mathcal{F}|_{\hat{X}'})$.*
- (b) (Output localization) *The class $m \cdot \bar{\chi}^P(X, \mathcal{F})$ is the class of a projective ideal $I \subset \mathbb{Z}[G]$ such that the order of $\mathbb{Z}[G]/I$ is supported on the image of X' in $\text{Spec}(\mathbb{Z})$.*

The integer m can be thought of as the denominator in a Riemann-Roch formula for $\bar{\chi}^P(X, \mathcal{F})$. We have referred to the two parts of this conjecture in this way because part (a) restricts the amount of input information needed to determine $m \cdot \bar{\chi}^P(X, \mathcal{F})$, while part (b) restricts the complexity of the output information needed to express $m \cdot \bar{\chi}^P(X, \mathcal{F})$.

We would like to thank L. Illusie for pointing out a parallel between Localization Conjecture 1.1 and results of Deligne, Abbès and Vidal concerning the equality of l -adic étale Euler characteristics of constructible sheaves having the same rank and wild ramification at infinity. For the precise statement of these results, see [Ill79], [Abb00] and [Vid04].

The methods of [Tho86], [Tho92] and [CEPT97b] lead to a proof of Localization Conjecture 1.1(a) modulo classes annihilated by a power of $\#G$. Our main results imply the following theorem.

THEOREM 1.2. *Let G be abelian, and suppose that the branch locus of $X \rightarrow X/G = Y$ is supported off the prime divisors of $\#G$. Suppose also that Y is regular and X is normal.*

- (a) *The input localization conjecture is true for X . If $\dim(X) \leq 4$, one can let $m = 1$ if $\#G$ is odd, and $m = 2$ if $\#G$ is even.*
- (b) *If m is the integer used in part (a), then the class $2(\#G)^{\dim(X)} m \cdot \bar{\chi}^P(X, \mathcal{F})$ is the class of a projective ideal $I \subset \mathbb{Z}[G]$ such that the order of $\mathbb{Z}[G]/I$ is supported on the image of X' in $\text{Spec}(\mathbb{Z})$.*

We will in fact show that for abelian G , the output localization conjecture holds for the image of $\chi^P(X, \mathcal{F})$ under a map Θ_n described below.

We now explain the appearance of the integers N and δ in Localization Conjecture 1.1. Suppose the action of G on X is free, so that $X' = \emptyset$. Then each

part of the conjecture is separately equivalent to $m \cdot \bar{\chi}^P(X, \mathcal{F}) = 0$. This statement for an explicit N and δ depending only on $\dim(X)$ and $\#G$, respectively, was proved in [Pap08] when all the Sylow subgroups of G are abelian. However, in [Pap08] it was suggested that Vandiver’s conjecture (which claims that the class number of $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ is prime to p) fails for exactly those primes p for which there is an X with free $\mathbb{Z}/p\mathbb{Z}$ -action and $p > \dim(X)$ with $\bar{\chi}^P(X, \mathcal{F}) \neq 0$. We have introduced the integer N in the conjecture because if one thought the conjecture were true with $N = 1$, then [Pap08] suggests that this would imply the truth of Vandiver’s conjecture, and this seems too strong.

Theorem 1.2 is a consequence of the explicit formula in Theorem 3.13, which is the main result of this paper. We now show some consequences of this formula concerning modular forms and Mordell-Weil and Tate-Shafarevich groups.

Let $p \equiv 1 \pmod{24}$ be a prime, and let $\Gamma = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$. Suppose

$$\chi : \Gamma \rightarrow \mu_r \subset \mathbb{Z}[\zeta_r]^*$$

is a character of prime order $r \mid (p - 1)$ with $r > 3$. Let $S_2(\Gamma_1(p), \mathbb{Z}[\zeta_r])_\chi$ be the $\mathbb{Z}[\zeta_r]$ -module of cusp forms $F(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$ of weight 2, level p and Nebentypus character χ whose Fourier coefficients a_n belong to $\mathbb{Z}[\zeta_r]$. Then $S_2(\Gamma_1(p), \mathbb{Z}[\zeta_r])_\chi$ is a locally free $\mathbb{Z}[\zeta_r]$ -module and of rank $n(\chi) = (p - 25)/12$ by the classical Chevalley-Weil theorem. (See [CW34]; this was one of the first coherent Lefschetz-Riemann-Roch theorems.) For $a \in (\mathbb{Z}/r\mathbb{Z})^*$ let $\{a\}$ be the unique integer in the range $0 < \{a\} < r$ having residue class a , and let $\sigma_a \in \text{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ be the automorphism for which $\sigma_a(\zeta_r) = \zeta_r^a$. Define $\omega_r : (\mathbb{Z}/r\mathbb{Z})^* \rightarrow \mathbb{Z}_r^*$ to be the Teichmüller character. We embed \mathbb{Z} (resp. \mathbb{Z}_r) into the profinite completion $\hat{\mathbb{Z}} = \prod_{l \text{ prime}} \mathbb{Z}_l$ of \mathbb{Z} diagonally (resp. via the factor $l = r$). We then have a modified quadratic Stickelberger element of $\hat{\mathbb{Z}}[\text{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})]$ defined by

$$\theta_2 = \sum_{a \in (\mathbb{Z}/r\mathbb{Z})^*} \frac{(p - 1)}{24r^2} (\{a\}^2 - \omega_r(a)^2) \sigma_a^{-1}.$$

(For the standard definition of the quadratic Stickelberger element, see for example [KL81, p. 115], also [CNT91, p. 308].) Since the ideal class group $\text{Cl}(\mathbb{Z}[\zeta_r])$ is finite, θ_2 acts on this group. Let \mathcal{P}_χ be the prime ideal of $\mathbb{Z}[\zeta_r]$ over (p) with the property that the reduction of χ modulo \mathcal{P}_χ is the $(p - 1)/r$ power of the identity character $(\mathbb{Z}/p\mathbb{Z})^* \rightarrow \mathbf{F}_p^*$.

Let $X_1(p)_\mathbb{Q}$ be the canonical model over \mathbb{Q} of the modular curve, and identify Γ with the group of diamond Hecke operators acting on $X_1(p)_\mathbb{Q}$. For each subgroup $H \subset \Gamma$, let $X_{H,\mathbb{Q}} = X_1(p)_\mathbb{Q}/H$. By applying Theorem 3.13 to a suitable integral model $X_H \rightarrow X_0$ of the $G = \Gamma / \ker(\chi)$ -cover given by $X_{\ker(\chi),\mathbb{Q}} \rightarrow X_{\Gamma,\mathbb{Q}} = X_0(p)_\mathbb{Q}$, we will prove the following result:

THEOREM 1.3. *Suppose $\mathfrak{A} \subset \mathbb{Z}[\zeta_r]$ is an ideal with ideal class $\theta_2 \cdot [\mathcal{P}_\chi]$. Then*

$$S_2(\Gamma_1(p), \mathbb{Z}[\zeta_r])_\chi \simeq \mathbb{Z}[\zeta_r]^{n(\chi)-1} \oplus \mathfrak{A} \quad \text{as } \mathbb{Z}[\zeta_r]\text{-modules.}$$

This theorem verifies Localization Conjecture 1.1 for $X_H \rightarrow X_0$ for the integers $N = 1 = \delta$.

Let J_H be the Jacobian of $X_{H, \mathbb{Q}}$, and define $J_H(\mathbb{Q})$ and $\text{III}(J_H)$ to be the Mordell-Weil and Tate-Shafarevich groups of J_H over \mathbb{Q} . We will assume that $\text{III}(J_H)$ is finite. For $G = \Gamma / \ker(\chi)$ as above, tensoring $\mathbb{Z}[G]$ -modules with the ring homomorphism $\mathbb{Z}[G] \rightarrow \mathbb{Z}[\zeta_r, 1/r]$ induced by χ gives a Steinitz class homomorphism

$$\begin{aligned} s_\chi : G_0(\mathbb{Z}[G]) &\rightarrow G_0(\mathbb{Z}[\zeta_r, 1/(2r)]) / \{\text{free modules}\} = \text{Cl}(\mathbb{Z}[\zeta_r, 1/(2r)]) \\ &= \text{Cl}(\mathbb{Z}[\zeta_r, 1/2]). \end{aligned}$$

In Section 5, we will describe an equivariant version of the Birch and Swinnerton-Dyer conjecture. This should follow from the equivariant Tamagawa number conjecture resulting from the work of Bloch and Kato, Fontaine and Perrin-Riou and of Burns and Flach (see [Fla04] and Remark 5.2).

THEOREM 1.4. *If the Birch and Swinnerton-Dyer conjecture of Section 5 holds then*

$$(1.1) \quad \overline{\theta_2[\mathcal{P}_\chi]} = s_\chi([\text{III}(J_H)]) - \overline{s_\chi([J_H(\mathbb{Q})])} - s_\chi([J_H(\mathbb{Q})])$$

in $\text{Cl}(\mathbb{Z}[\zeta_r, \frac{1}{2}])$, where $\overline{\mathfrak{D}}$ is the complex conjugate of an ideal class \mathfrak{D} .

Now let $C(p)$ be the subgroup of $J_1(p) = J_{\{e\}}$ generated by differences of \mathbb{Q} -rational cusps of $X_1(p)_\mathbb{Q}$; these are the cusps over the cusp ∞ of $X_0(p)_\mathbb{Q}$. (Conrad, Edixhoven and Stein [CES03] conjectured that $C(p) = J_1(p)(\mathbb{Q})_{\text{tor}}$.) Let $J'_H(\mathbb{Q})$ be the image of $J_H(\mathbb{Q})$ in $J_1(p)(\mathbb{Q})/C(p)$ under the pullback homomorphism associated to the quotient morphism $X_1(p)_\mathbb{Q} \rightarrow X_{H, \mathbb{Q}}$. Using a result of Kubert and Lang, we show that $s_\chi([J_H(\mathbb{Q})]) = s_\chi([J'_H(\mathbb{Q})])$. This implies the following.

COROLLARY 1.5. *Suppose $\theta_2[\mathcal{P}_\chi] \neq 0$ in $\text{Cl}(\mathbb{Z}[\zeta_r, \frac{1}{2}])$. Then at least one of $s_\chi([J'_H(\mathbb{Q})])$ or $s_\chi([\text{III}(J_H)])$ is nontrivial if the Birch and Swinnerton-Dyer conjecture of Section 5 holds. Suppose also that $C(p) = J_1(p)(\mathbb{Q})_{\text{tor}}$, as conjectured in [CES03]. Then either the χ -eigenspace of $J_1(p)(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{C}$ is nontrivial, or $s_\chi([\text{III}(J_H)])$ is nontrivial.*

Whether or not $\theta_2[\mathcal{P}_\chi]$ is trivial in $\text{Cl}(\mathbb{Z}[\zeta_r, \frac{1}{2}])$ depends only on p and not on the choice of the character χ of order r . If r is fixed and $\theta_2(\text{Cl}(\mathbb{Z}[\zeta_r, \frac{1}{2}])) \neq 0$, e.g. if $r = 191$, the set of p for which $\theta_2[\mathcal{P}_\chi] \neq 0$ has a positive Dirichlet density. It is not clear, though, whether the alternatives in Corollary 1.5 for such p should occur with a well-defined density, or whether one should occur asymptotically more often than the other. Nevertheless, one should mention the following: The L-functions of

cuspidal eigenforms of weight 2, level p and Nebentypus character χ are not self-dual. According to the expectations of Katz-Sarnak [KS99], the fraction of such forms of level $p \leq x$ whose L-functions do not vanish at $s = 1$ should approach 1 as $x \rightarrow \infty$. Hence, it is plausible that the χ -eigenspace of $J_1(p)(\mathbb{Q}) \otimes \mathbb{C}$ is trivial for many primes p . However, since there could be many Galois orbits of such eigenforms at each level p , the heuristics of [KS99] do not actually imply that this is the case for an infinite number of primes.

The principle behind Theorem 1.4 is that Birch and Swinnerton-Dyer conjectures predict identities in $G_0(\mathbb{Z}[\Gamma])$ involving classes constructed from L-series derivatives, height pairings, period maps, de Rham and Betti cohomology groups, Mordell-Weil groups and Tate-Shafarevich groups. Combinations of these classes that can be defined by Galois-equivariant functions from the characters of Γ to the complex numbers should be trivial or nearly so in $G_0(\mathbb{Z}[\Gamma])$. (This accounts for the fact that values of L-series derivatives, regulators and period maps do not appear in Theorem 1.4.) Determining some of the other classes, e.g. the ones associated to de Rham and Betti cohomology over \mathbb{Z} , then leads to predictions about the rest, as in Theorem 1.4. It would be of great interest to find a motivic cohomology formalism that would make such predictions directly, without the intervention of L-value conjectures; such conjectures play no role in Theorems 3.13 and 1.3.

Let us remark here that the methods used to show Theorem 1.3 apply in many other situations. We plan to consider generalizations to other spaces of weight two modular forms and covers of modular curves in future work. An extension to higher weight modular forms is considered by E. Gurel in his thesis. Shimura varieties more general than modular curves also provide many examples of Galois covers to which our techniques can be applied. This way one can hope to obtain information on the Galois module structure of lattices of automorphic forms for other groups besides $GL(2)/\mathbb{Q}$.

We now explain our approach to computing $\bar{\chi}^P(X, \mathcal{F})$ for general X and abelian G , assuming that the quotient $Y = X/G$ is a regular scheme that is projective and flat over $\text{Spec}(\mathbb{Z})$ and of relative dimension d . Our main tools are the existence of cubic structures on the determinant of cohomology and a localized Riemann-Roch theorem. For simplicity, we will restrict our discussion to the case $\mathcal{F} = \mathcal{O}_X$. The image of $\bar{\chi}^P(X, \mathcal{O}_X) \in \text{Cl}(\mathbb{Z}[G])$ maps, under the natural isomorphism $\text{Cl}(\mathbb{Z}[G]) \xrightarrow{\sim} \text{Pic}(\mathbb{Z}[G])$ given by the determinant of $\mathbb{Z}[G]$ -modules, to the class of the locally free, rank one $\mathbb{Z}[G]$ -module which is given by the determinant of cohomology $\det \mathbb{R}\Gamma(X, \mathcal{O}_X)$ in $\text{Pic}(\mathbb{Z}[G])$.

Suppose that M is a locally free, rank one left $\mathbb{Z}[G]$ -module. To each subset $I \subset \{1, \dots, n\}$ we associate a homomorphism $I : G \rightarrow G^n$, from G to the product G^n of n copies of G , that sends $g \in G$ to the element of G^n having g in the i -th component for $i \in I$ and the identity element of G in all other components. To

obtain information about M , one can consider the base change homomorphisms

$$\Delta_I(M) = \mathbb{Z}[G^n] \otimes_{\mathbb{Z}[G]} M,$$

where the algebra homomorphism $\Delta_I : \mathbb{Z}[G] \rightarrow \mathbb{Z}[G^n]$ is induced by the homomorphism $I : G \rightarrow G^n$. The statement that certain tensor products over $\mathbb{Z}[G^n]$ of these $\Delta_I(M)$ have trivializations puts a constraint on M , which in some cases one might hope would force M to have a trivial class in $\text{Cl}(\mathbb{Z}[G])$.

The particular tensor product that we consider is the so-called Θ_n -product defined by

$$\Theta_n(M) = \bigotimes_{I \subset \{1, \dots, n\}} \Delta_I(M)^{(-1)^{n-\#I}}.$$

We define an “ n -cubic structure” on M to be a trivialization of $\Theta_n(M)$ that satisfies certain “cubic” compatibility conditions. (The terminology and notation come from the theorem of the cube for line bundles on abelian varieties and the associated theta functions; this classical case corresponds to $n = 3$.)

We want to apply the above to $M = \det \mathbb{R}\Gamma(X, \mathcal{O}_X)^{\otimes 2}$ and $n = d + 2$. To extract the maximal amount of information about M , it is important to consider in addition to M a generator s of $M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M$ as a $\mathbb{Q}[G]$ -module. Such an s gives rise to generators $\Delta_I(s) = 1 \otimes_{\mathbb{Z}[G]} s$ and $\Theta_n(s) = \prod_I \Delta_I(s)^{(-1)^{n-\#I}}$ of $\Delta_I(M_{\mathbb{Q}})$ and $\Theta_n(M_{\mathbb{Q}})$, respectively.

In general, this M does not support an n -cubic structure. However, $M_{\mathbb{Q}}$ does because the cover $X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$ is unramified. When $d = \dim(X_{\mathbb{Q}}) = 1$ this follows from the Deligne-Riemann-Roch theorem [Del87]; for general d , it follows from work of Breen [Bre83], Ducrot [Duc05] and Pappas [Pap08]. The resulting trivialization of $\Theta_n(M_{\mathbb{Q}})$ enables us to regard $\Theta_n(s)$ as an element $c_n(s) \in \mathbb{Q}[G^n]^*$. One can view $c_n(s)$ as a kind of secondary class, analogous to the Bott-Chern secondary characteristic class arising from discrepancies between metrics on vector bundles lying in an exact sequence of vector bundles.

Now suppose that we choose a generator s_v of $\mathbb{Z}_v \otimes_{\mathbb{Z}} M$ as a $\mathbb{Z}_v[G]$ -module at each finite place v of \mathbb{Q} . The above procedure (performed now over \mathbb{Q}_v) allows us to obtain elements $c_n(s_v) \in \mathbb{Q}_v[G^n]^*$. The elements $c_n(s_v)$ are our substitutes of the local (abelian) resolvents that appear in Fröhlich’s theory of resolvents for actions of finite groups on rings of algebraic integers. Our main calculation amounts to showing that, under certain assumptions, we can compute $c_n(s_v)$ up to an element of $\mathbb{Z}_v[G^n]^*$ using resolvent theory on the codimension 1 points of Y and a localized Riemann-Roch theorem. (Note that in the main text we mainly work with the $\mathbb{Z}[G^n]$ -ideal in $\mathbb{Q}[G^n]$ given by the idèle $(c_n(s_v))_v$; it is denoted by E in §3.)

Our results are actually expressed in terms of a functor

$$\Theta_n : \mathcal{P}ic^n(\mathbb{Z}[G]) \rightarrow \mathcal{C}_{\mathbb{Z}}(G; n), \quad (N, t) \mapsto (\Theta_n(N), \Theta_n(t)),$$

where $\mathcal{P}ic^n(\mathbb{Z}[G])$ is the Picard category (see [AGV73, XVIII]) of locally free, rank one $\mathbb{Z}[G]$ -modules N with a generator t of $N_{\mathbb{Q}}$, and $\mathcal{C}_{\mathbb{Z}}(G; n)$ is a similar Picard category of locally free, rank one $\mathbb{Z}[G^n]$ -modules where the morphisms are restricted according to certain “cubic” conditions (which will be specified in detail in the next section). Considering isomorphism classes of objects in Picard categories leads to a homomorphism

$$\Theta_n : \text{Pic}(\mathbb{Z}[G]) \rightarrow C_{\mathbb{Z}}(G; n)$$

whose kernel is the group of classes in $\text{Pic}(\mathbb{Z}[G])$ having an n -cubic structure. By the results in [Pap05], this kernel is small (see Theorem 2.7; it is actually trivial when $n \leq 5$). Our main result, Theorem 3.13, gives an explicit “branch locus” formula for $\Theta_n(M)$ when $n = d + 2$, $M = \det \mathbb{R}\Gamma(X, \mathcal{O}_X)^{\otimes 2}$ and the ramification of $\pi : X \rightarrow Y = X/G$ is domestic, in the sense that it is supported away from the prime divisors of $\#G$.

Let us conclude with some remarks. We first note that the approach outlined above allows us to provide a new explanation of the “cubic” conditions on the trivializations of $\Theta_n(M)$ which were first considered by Breen [Bre83] when $n = 3$. We show that Breen’s conditions are among those satisfied by all “ratios” $\Theta_n(s)/\Theta_n(s') \in \mathbb{Q}[G^n]^*$ where s and s' are two generators of $\mathbb{Q} \otimes_{\mathbb{Z}} M$ (see Remark 2.3(b)). We can consider a bigger set of conditions and obtain variants of the notion of cubic structure and of the functor Θ_n above (see Remarks 2.3(d) and 2.6). In fact, by taking the maximal set of such conditions we can obtain a possibly more natural variant of the notion of cubic structure. When $n = 2$, the homomorphism Θ_2 is very closely related to the functor rag of McCulloh [McC87]. Our formulas then essentially specialize to the formulas in loc. cit.

A major remaining problem is to understand the nature of the “cubic relations” when G is non-abelian. Some progress has been made on that problem by the second author when $n = 3$ [Pap03]. (This pertains to covers of arithmetic surfaces.) Also, since the paper was submitted there has been some progress by the authors in developing a refined Riemann-Roch theorem for non-abelian actions on arithmetic surfaces. However, we have not, as yet, been able to establish a comparably precise result about Euler characteristics even in the surface case. The situation for $n > 3$ remains completely unclear at the moment. One of the main themes of [Pap00] and [Pap03] is that, because the γ -filtration on the Grothendieck group of vector bundles on $Y = X/G$ terminates after the $(d + 1)$ -st step, there are relations between the Chern classes of vector bundles over Y which are obtained from various representations of G using the cover $X \rightarrow Y$. Such relations alone can sometimes be used to deduce information about an equivariant Euler characteristic $\bar{\chi}^P(X, \mathcal{F})$. In this paper we have taken advantage of the fact that the γ -filtration on the representation ring of an abelian group has a very simple form. The γ -filtration

on the representation ring of a general finite group has no simple and uniform explicit description; this is at least one reason why it is hard to extend our results to arbitrary finite groups.

2. Cubic structures and categories

2.a. Let G be a finite abelian group, and let R be a Dedekind ring with fraction field K . Let $n \geq 2$ be an integer. We will denote by $G_R^D = \text{Spec}(R[G])$ the Cartier dual of the constant group scheme given by G . We will start by rephrasing the notion of n -cubic structure on a line bundle $\mathcal{L} = \widetilde{M}$ over G_R^D (see [Pap05, § 3]) in terms of the corresponding locally free, rank one $R[G]$ -module $M = \Gamma(G_R^D, \mathcal{L})$.

Suppose r and s are nonnegative integers. Denote by G^r the direct sum of r copies of G , where G^0 is taken to mean the group with one element. For $\mathcal{F} \in C_{r,s} = \text{Hom}(G^r, G^s)$, we will denote by $\Delta_{\mathcal{F}} : R[G^r] \rightarrow R[G^s]$ the induced R -algebra homomorphism. These homomorphisms have the property that

$$(2.1) \quad \Delta_{\mathcal{F}' \circ \mathcal{F}} = \Delta_{\mathcal{F}'} \circ \Delta_{\mathcal{F}}$$

if $\mathcal{F}' \in C_{s,s'}$ for some integer s' . Also, we will denote by $\Delta_{\mathcal{F}}^D : (G_R^D)^s \rightarrow (G_R^D)^r$ the R -scheme morphism given by $\Delta_{\mathcal{F}}$.

Let $\Sigma_{r,s} = \mathbb{Z}[C_{r,s}]$. Define the bilinear map

$$(2.2) \quad \Sigma_{r,s} \times \Sigma_{r',s'} \rightarrow \Sigma_{r',s}$$

to be induced by the composition of homomorphisms $C_{r,s} \times C_{r',s'} \rightarrow C_{r',s}$ when $s' = r$ and to be the zero map otherwise. These maps make $\Sigma = \bigoplus_{r,s \geq 0} \Sigma_{r,s}$ into a ring without unit.

Now let M be a locally free, rank one $R[G^r]$ -module. We will denote by M^{-1} the dual locally free, rank one $R[G^r]$ -module $\text{Hom}_{R[G^r]}(M, R[G^r])$. Let t be a generator of $M_K = K \otimes_R M$ as a free rank one $K[G^r]$ -module. Define

$$(2.3) \quad \Delta_{\mathcal{F}}(M) = R[G^s] \otimes_{\Delta_{\mathcal{F}, R[G^r]}} M \quad \text{and} \quad \Delta_{\mathcal{F}}(t) = 1 \otimes t \in \Delta_{\mathcal{F}}(M)_K$$

so that $\Delta_{\mathcal{F}}(M)$ is a locally free, rank one $R[G^s]$ -module, and $\Delta_{\mathcal{F}}(t)$ is a generator of $\Delta_{\mathcal{F}}(M)_K$. In general, suppose $z = \sum_{\mathcal{F} \in C_{r,s}} z(\mathcal{F}) \cdot \mathcal{F} \in \Sigma_{r,s}$ for some integers $z(\mathcal{F})$. We define

$$(2.4) \quad \Delta_z(M) = \bigotimes_{\mathcal{F}} \Delta_{\mathcal{F}}(M)^{z(\mathcal{F})} \quad \text{and} \quad \Delta_z(t) = \bigotimes_{\mathcal{F}} \Delta_{\mathcal{F}}(t)^{z(\mathcal{F})},$$

where the tensor products are over $R[G^s]$ and $K[G^s]$ respectively. We also have a homomorphism of multiplicative groups $\lambda_z : K[G^r]^* \rightarrow K[G^s]^*$ defined by

$$(2.5) \quad \lambda_z(\alpha) = \prod_{\mathcal{F}} \Delta_{\mathcal{F}}(\alpha)^{z(\mathcal{F})}.$$

If $z' \in \Sigma_{s,s'}$ for some integer s' , then the composition law (2.1) gives a canonical isomorphism

$$(2.6) \quad \Delta_{z'} \circ \Delta_z \simeq \Delta_{z' \cdot z} .$$

In particular, if $z' \cdot z = 0$ in the ring Σ , then there is a canonical isomorphism

$$(2.7) \quad \Delta_{z'}(\Delta_z(M)) \simeq R[G^{s'}].$$

The corresponding isomorphism over K sends $\Delta_{z'}(\Delta_z(t))$ to $1 \in K[G^{s'}]$.

Any subset $I \subset \{1, \dots, n\}$ determines a homomorphism $I \in C_{1,n} = \text{Hom}(G, G^n)$ by defining $I(g)$ for $g \in G$ to have i -th component g for $i \in I$ and i -th component the identity element e of G if $i \notin I$. In this way, we will view subsets I of $\{1, \dots, n\}$ as elements of $C_{1,n}$. Notice that the scheme morphism

$$m_I = \Delta_I^D : (G_R^D)^n = \text{Spec}(R[G^n]) \rightarrow G_R^D = \text{Spec}(R[G])$$

induced by Δ_I is given on points by $m_I(x_1, \dots, x_n) = \sum_{i \in I} x_i$. (Note that we use additive notation for the group operation.) Now consider the element of $\Sigma_{1,n}$ given by

$$s_n = \sum_{I \subset \{1, \dots, n\}} (-1)^{n-\#I} \cdot I .$$

If M is a locally free, rank 1 $R[G]$ -module, for simplicity we will denote $\Delta_{s_n}(M)$ and $\Delta_{s_n}(t)$ by $\Theta_n(M)$ and $\Theta_n(t)$, respectively. Hence, we have

$$\Theta_n(M) = \bigotimes_{I \subset \{1, \dots, n\}} \Delta_I(M)^{(-1)^{n-\#I}} .$$

We now define three conditions on an element $a \in K[G^n]^*$.

CONDITION 1. Consider the trivial homomorphism

$$e : g \rightarrow e \quad \text{in } C_{n,0} = \text{Hom}(G^n, G^0) \subset \Sigma_{n,0} .$$

Since $n \geq 1$, $e \cdot s_n = 0$ in $\Sigma_{1,0}$. The homomorphism $\lambda_e : K[G^n]^* \rightarrow K^*$ is the one induced by the trivial character of G^n . An $a \in K[G^n]^*$ such that $\lambda_e(a) = 1$ will be said to be *rigid*.

CONDITION 2. Suppose that σ is a permutation of $\{1, \dots, n\}$. We then have an element

$$\mathcal{F}_\sigma : (g_1, \dots, g_n) \rightarrow (g_{\sigma(1)}, \dots, g_{\sigma(n)})$$

of $C_{n,n}$. Let id be the identity permutation of $\{1, \dots, n\}$, and let $z_\sigma = \mathcal{F}_\sigma - \mathcal{F}_{\text{id}}$ in $\Sigma_{n,n}$. One checks readily that $z_\sigma \cdot s_n = 0$ in $\Sigma_{1,n}$. An element $a \in K[G^n]^*$ such that $\lambda_{z_\sigma}(a) = 1$ for all permutations σ of $\{1, \dots, n\}$ will be said to be *symmetric*.

CONDITION 3. Define four elements of $C_{n,n+1} = \text{Hom}(G^n, G^{n+1})$ by

$$\mathcal{F}_0 : (g_1, \dots, g_n) \rightarrow (g_1, g_1, g_2, \dots, g_n),$$

$$\mathcal{F}_1 : (g_1, \dots, g_n) \rightarrow (e, g_1, g_2, \dots, g_n),$$

$$\mathcal{F}_2 : (g_1, \dots, g_n) \rightarrow (g_1, g_2, e, \dots, g_n),$$

$$\mathcal{F}_3 : (g_1, \dots, g_n) \rightarrow (g_1, g_2, g_2, \dots, g_n).$$

Set $z = \mathcal{F}_0 - \mathcal{F}_1 + \mathcal{F}_2 - \mathcal{F}_3$ in $\Sigma_{n,n+1}$. One has $z \cdot s_n = 0$ in $\Sigma_{1,n+1}$ since $n \geq 2$. Elements $a \in K[G^n]^*$ such that $\lambda_z(a) = 1$ will be said to satisfy the *cocycle condition*.

Definition 2.1 [Pap05, 3.3]. An element $a \in K[G^n]^*$ that is rigid, symmetric and satisfies the cocycle condition (see Conditions 1, 2 and 3 above) will be said to be *n-cubic*.

Definition 2.2. Suppose that R' is a subring of K that contains R . An *n-cubic structure* on a locally free, rank 1 $R'[G]$ -module M is a trivialization

$$c : R'[G^n] \xrightarrow{\sim} \Theta_n(M)$$

such that for any choice of generator t of $M \otimes_{R'} K$ as a $K[G]$ -module, we have $c(1) = a \cdot \Theta_n(t)$ with $a \in K[G^n]^*$ that is *n-cubic*.

Remark 2.3.

- (a) In the above definition, we are mainly interested in the cases $R' = R$ or $R' = K$.
- (b) Suppose that t' is another choice of $K[G]$ -generator of $M \otimes_{R'} K$ such that $t' = \alpha \cdot t$ with $\alpha \in K[G]^*$. Then $\Theta_n(t')/\Theta_n(t) = \lambda_{s_n}(\alpha)$. Now if z is as in Conditions 1, 2 or 3 above, we have (as in (2.7)) $\lambda_z(\lambda_{s_n}(\alpha)) = \lambda_{z \cdot s_n}(\alpha) = \lambda_0(\alpha) = 1$. Therefore, the element $\lambda_{s_n}(\alpha)$ is *n-cubic*. Hence, it is enough to check the property of the definition for a single choice of $K[G]$ -generator of $K \otimes_{R'} M$.
- (c) We can readily see that the notion of *n-cubic element* coincides with the corresponding notion as defined in [Pap05]. There the conditions are expressed in terms of characters of G , i.e. of points of G_R^D . Also, *n-cubic structures* on M , as defined above, uniquely correspond to *n-cubic structures* (as defined in [Pap05, § 3.a]) on the corresponding line bundle \tilde{M} over $G_{R'}^D$.
- (d) We can consider the following variants of the notion of *n-cubic structure*: Let V be a set of elements $z \in \bigcup_{t \geq 0} \Sigma_{n,t}$ that satisfy $z \cdot s_n = 0$. Then, we can consider “*V-cubic elements*”: By definition, these are elements $a \in K[G^n]^*$ such that $\lambda_z(a) = 1$ for all $z \in V$. The notion of “*V-cubic structure*” is defined as in Definition 2.2 above by replacing “*n-cubic*” by “*V-cubic*”. Notice that the argument in Remark 2.3(b) shows that $\lambda_{s_n}(a)$ is *V-cubic* for $a \in K[G]^*$. Therefore, this remark applies also to *V-cubic structures*.

2.b. The reader is referred to [AGV73, XVIII] for the definition of a Picard category. We now define two such categories:

The category $\mathcal{P}ic_R^\eta(G)$ with objects pairs (M, t) , where M is a locally free, rank 1 $R[G]$ -module and t a generator of $K \otimes_R M$ as a $K[G]$ -module, and morphisms $\phi : (M, t) \rightarrow (M', t')$ given by $R[G]$ -isomorphisms $\phi : M \rightarrow M'$ (with no condition on the generators t and t').

The category $\mathcal{C}_R(G; n)$ with objects pairs (P, γ) , where P is a locally free, rank 1 $R[G^n]$ -module and γ a generator of $K \otimes_R P$ as a $K[G^n]$ -module, and morphisms $\psi : (P, \gamma) \rightarrow (P', \gamma')$ given by $R[G^n]$ -isomorphisms $\psi : P \rightarrow P'$ such that $(\text{id}_K \otimes_R \psi)(\gamma) = a \cdot \gamma'$, where a is an n -cubic element of $K[G^n]^*$.

We see that both $\mathcal{P}ic_R^\eta(G)$ and $\mathcal{C}_R(G; n)$ are strictly commutative Picard categories with product defined by $(M, t) \otimes (M', t') = (M \otimes_{R[G]} M', t \otimes t')$ and $(P, \gamma) \otimes (P', \gamma') = (P \otimes_{R[G^n]} P', \gamma \otimes \gamma')$. The group of isomorphism classes of $\mathcal{P}ic_R^\eta(G)$ is the Picard group $\text{Pic}(R[G])$; we will denote by $C_R(G; n)$ the (abelian) group of isomorphism classes of objects of $\mathcal{C}_R(G; n)$.

LEMMA 2.4. *There is an additive functor $\Theta_n : \mathcal{P}ic_R^\eta(G) \rightarrow \mathcal{C}_R(G; n)$ given by*

$$\Theta_n((M, t)) = (\Theta_n(M), \Theta_n(t)).$$

Proof. It follows from the argument in Remark 2.3(b) that the functor Θ_n is well-defined. The rest is left to the reader. □

The functor Θ_n induces a group homomorphism

$$(2.8) \quad \Theta_n : \text{Pic}(R[G]) \rightarrow C_R(G; n).$$

When the integer n is fixed by the context we will simply write Θ instead of Θ_n .

Remark 2.5. Note that by Definitions 2.1 and 2.2, isomorphisms $(R[G^n], 1) \xrightarrow{\sim} \Theta((M, t))$ in the category $\mathcal{C}_R(G; n)$ correspond to n -cubic structures on the $R[G]$ -module M . Hence, the kernel of (2.8) is the set (which is actually a group under tensor product) of isomorphism classes of locally free, rank 1 $R[G]$ -modules that support an n -cubic structure.

Remark 2.6. Suppose V is a set as in Remark 2.3(d). Then there are obvious variants $\mathcal{C}_R(G; V)$ and $\Theta_V : \mathcal{P}ic_R^\eta(G) \rightarrow C_R(G; V)$, of $\mathcal{C}_R(G; n)$ and Θ_n respectively. They are given by replacing “ n -cubic element” by “ V -cubic element” in the above definitions.

2.c. Assume that R is the ring of integers of a number field K . For $m \geq 0$, let us denote by $\mathbf{A}_{f,K[G^m]}^*$ the finite idèles of $K[G^m]$. Then

$$\mathbf{A}_{f,K[G^m]}^* = \prod'_v K_v[G^m]^*,$$

where the (restricted) product is over all finite places v of K . A finite idèle $(a_v)_v$ for $K[G^m]$ gives a locally free, rank 1 $R[G^m]$ -module $Q((a_v)_v)$ by

$$Q((a_v)_v) = \bigcap_v (R_v[G^m]a_v \cap K[G^m]) \subset K[G^m].$$

The “generic fiber” $Q((a_v)_v) \otimes_R K = K[G^m]$ has 1 as a distinguished generator, and we have $Q((a_v)_v) \otimes_R R_v = R_v[G^m]a_v$. Therefore, the idèle $(a_v)_v$ provides us with an object $(Q((a_v)_v), 1)$ of the category $\mathcal{C}_R(G; m)$ if $m \geq 2$ or of $\mathcal{P}ic_R^\eta(G)$ if $m = 1$.

For each finite place v of K , fix an algebraic closure \bar{K}_v of the completion K_v . Let us denote by $\text{Ch}(G^n)_v$ the (additive) ring of virtual characters of G^n with values in \bar{K}_v . There is a group isomorphism

$$(2.9) \quad K_v[G^n]^* \xrightarrow{\sim} \text{Hom}_{\text{Gal}(\bar{K}_v/K_v)}(\text{Ch}(G^n)_v, \bar{K}_v^*)$$

given by $a_v \mapsto (\phi \mapsto \phi(a_v))$ (cf. [Frö83, II, § 1]).

Notice that for every element \mathcal{J} of $C_{r,s}$, we can obtain an additive map $\Delta_{\mathcal{J}}^D : \text{Ch}(G^s)_v \rightarrow \text{Ch}(G^r)_v$ by composing a character of G^s with the ring homomorphism $\bar{K}_v[G^r] \rightarrow \bar{K}_v[G^s]$ given by $\Delta_{\mathcal{J}}$ and by then extending linearly to virtual sums of such characters. This definition generalizes to elements $z = \sum_{\mathcal{J} \in C_{r,s}} z(\mathcal{J}) \cdot \mathcal{J}$ of $\Sigma_{r,s}$ by linearity, and we obtain additive maps $\Delta_z^D : \text{Ch}(G^s)_v \rightarrow \text{Ch}(G^r)_v$.

There is a commutative diagram

$$(2.10) \quad \begin{array}{ccc} \prod' \text{Hom}_{\text{Gal}(\bar{K}_v/K_v)}(\text{Ch}(G)_v, \bar{K}_v^*) & \xrightarrow{\sim} & \mathbf{A}_{f,K[G]}^* \xrightarrow{Q} \text{Pic}(R[G]) \\ \prod' (\Theta_n^D)^* \downarrow & & \prod' \lambda_{sn} \downarrow \quad \quad \quad \downarrow \Theta_n \\ \prod' \text{Hom}_{\text{Gal}(\bar{K}_v/K_v)}(\text{Ch}(G^n)_v, \bar{K}_v^*) & \xrightarrow{\sim} & \mathbf{A}_{f,K[G^n]}^* \xrightarrow{Q} C_R(G; n), \end{array}$$

where $(\Theta_n^D)^*$ is the map dual to $\Theta_n^D := \Delta_{s_n}^D : \text{Ch}(G^n)_v \rightarrow \text{Ch}(G)_v$. Explicitly, we have

$$(2.11) \quad \Theta_n^D(\phi_1 \otimes \cdots \otimes \phi_n) = \sum_{I \subset \{1, \dots, n\}} (-1)^{n-\#I} \prod_{i \in I} \phi_i = \prod_{i=1}^n (\phi_i - 1).$$

Notice that the image of Θ_n^D is the n -th power of the augmentation ideal of $\text{Ch}(G)_v$.

2.d. Suppose $\phi : G \rightarrow H$ is a homomorphism of finite abelian groups; this induces ring homomorphisms $R[G] \rightarrow R[H]$ and $\phi^{(n)} : R[G^n] \rightarrow R[H^n]$. If $c \in K[G^n]^*$ is an n -cubic element, then so is its image $\phi^{(n)}(c) \in K[H^n]^*$. Tensoring with $R[H]$ over $R[G]$, resp. with $R[H^n]$ over $R[G^n]$, gives additive functors

$\mathcal{P}ic_R^n(G) \rightarrow \mathcal{P}ic_R^n(H)$, resp. $\mathcal{C}_R(G; n) \rightarrow \mathcal{C}_R(H; n)$. The corresponding diagram

$$(2.12) \quad \begin{array}{ccc} \text{Pic}(R[G]) & \xrightarrow{\Theta} & C_R(G; n) \\ \downarrow & & \downarrow \\ \text{Pic}(R[H]) & \xrightarrow{\Theta} & C_R(H; n) \end{array}$$

commutes. There is also a commutative diagram

$$(2.13) \quad \begin{array}{ccccc} \prod' \text{Hom}_{\text{Gal}(\bar{K}_v/K_v)}(\text{Ch}(G^n)_v, \bar{K}_v^*) & \xrightarrow{\sim} & \mathbf{A}_{f,K[G^n]}^* & \xrightarrow{\mathcal{Q}} & C_R(G; n) \\ \prod' C_\phi^* \downarrow & & \prod' \phi^{(n)} \downarrow & & \downarrow \phi \\ \prod' \text{Hom}_{\text{Gal}(\bar{K}_v/K_v)}(\text{Ch}(H^n)_v, \bar{K}_v^*) & \xrightarrow{\sim} & \mathbf{A}_{f,K[H^n]}^* & \xrightarrow{\mathcal{Q}} & C_R(H; n), \end{array}$$

where C_ϕ^* is dual to the map $C_\phi : \text{Ch}(H^n)_v \rightarrow \text{Ch}(G^n)_v$ given by composing characters with $\phi^n : G^n \rightarrow H^n$.

2.e. Let us now assume that $R = \mathbb{Z}$. Let \mathcal{M}_G be the normalization of $\mathbb{Z}[G]$ in $\mathbb{Q}[G]$. Then \mathcal{M}_G is the maximal order in $\mathbb{Q}[G]$. Tensoring with \mathcal{M}_G over $\mathbb{Z}[G]$ induces a homomorphism $\text{Pic}(\mathbb{Z}[G]) \rightarrow \text{Pic}(\mathcal{M}_G)$. Denote its kernel by $D(\mathbb{Z}[G])$; by Rim’s theorem it is trivial when G is of prime order.

Now recall that the k -th Bernoulli number B_k is defined by the power series identity: $t/(e^t - 1) = \sum_{k=0}^\infty B_k t^k/k!$. For a prime p we denote by $|\cdot|_p$ the usual p -adic absolute value with $|p|_p = p^{-1}$. Let $e(1) = 1$ and for $k \geq 2$, let us set

$$e(k) = \begin{cases} \text{numerator}(B_k/k) & \text{if } k \text{ is even,} \\ \prod_{p|h_p^+} |\#\mathbf{K}_{2k-2}(\mathbb{Z})|_p^{-1} & \text{if } k \text{ is odd,} \end{cases}$$

where $\mathbf{K}_{2k-2}(\mathbb{Z})$ is the Quillen K-group (a finite group for $k > 1$) and we have $h_p^+ = \#\text{Cl}(\mathbb{Z}[\zeta_p + \zeta_p^{-1}])$. Note that, according to Vandiver’s conjecture, p does not divide h_p^+ , which implies $e(k) = 1$ for k odd. The following is essentially one of the main results of [Pap05] (or [Pap98] when $n = 3$).

THEOREM 2.7. *Suppose that $R = \mathbb{Z}$ and that $\Theta_n : \text{Pic}(\mathbb{Z}[G]) \rightarrow C_{\mathbb{Z}}(G; n)$ is the homomorphism (2.8).*

- (a) $\ker(\Theta_n)$ is annihilated by $\prod_{k=1}^{n-1} \prod_{p|e(k)} |\#G|_p^{-1}$.
- (b) If $2 \leq n \leq 5$, then $\ker(\Theta_n)$ is trivial.
- (c) If all the prime divisors of $\#G$ are greater than or equal to n and satisfy Vandiver’s conjecture, then $\ker(\Theta_n) \subset D(\mathbb{Z}[G])$. In particular, if $\#G = p$ is a prime number $\geq n$ that satisfies Vandiver’s conjecture, then $\ker(\Theta_n)$ is trivial.

Proof. Recall that by Remark 2.5, elements in $\ker(\Theta_n) \subset \text{Pic}(\mathbb{Z}[G])$ are represented by invertible sheaves that support an n -cubic structure. Therefore, parts (a) and (b) are given by [Pap05, Th. 1.1]. Part (c) follows from [Pap05, Th. 1.2]. \square

3. The main calculation and result

We continue to assume that R is a Dedekind ring with field of fractions K . Suppose that $h : Y \rightarrow \text{Spec}(R)$ is a regular flat projective scheme, equidimensional of (absolute) dimension $d + 1$. Let $\pi : X \rightarrow Y$ be a G -cover, where G is a finite abelian group. By definition, this means that X supports a (right) action of G , that π is finite and identifies Y with the quotient X/G , and that π is generically on Y a G -torsor. Denote by $f : X \rightarrow \text{Spec}(R)$ the structure morphism. We assume that X is normal and the following hypothesis:

HYPOTHESIS T. The action of G on X is *tame*, i.e., for every point x of X , the order of the inertia subgroup I_x of x is relatively prime to the residue field characteristic of x . In addition, we assume that the cover $\pi_K : X_K \rightarrow Y_K$ over $\text{Spec}(K)$ is unramified (i.e., a G -torsor; then X_K is also regular). This last condition follows from the assumption on the tameness of the action, if $\text{Spec}(R) \rightarrow \text{Spec}(\mathbb{Z})$ is surjective.

Remark 3.1. (a) It follows from the above assumptions that $\pi : X \rightarrow Y$ is flat. To show this, let x and y be points of X and Y with $\pi(x) = y$ and let \hat{Y} , resp. \hat{X} , be the spectrum of the strict henselization of the local ring $\mathbb{O}_{Y,y}$, resp. $\mathbb{O}_{X,x}$. By descent, it is enough to show that $\hat{\pi} : \hat{X} \rightarrow \hat{Y}$ is flat. By [Ray70] we have

$$(3.1) \quad X \times_Y \hat{Y} \simeq \hat{X} \times^{I_x} G := (\hat{X} \times G)/I_x$$

as schemes with G -action. Therefore, $\hat{X}/I_x \simeq \hat{Y}$. By our assumptions, $\#I_x$ is relatively prime to the residue characteristic, \hat{Y} is regular and \hat{X} is normal. The argument in [Rob80] now shows that $\hat{\pi}$ is flat.

(b) To construct examples in which Hypothesis T holds, one can start with a regular projective model Y of Y_K and a G -torsor $\pi_K : X_K \rightarrow Y_K$. One then considers whether the normalization X of Y in the total quotient ring $K(X_K)$ of X_K is tame over Y , using for example Abhyankar’s lemma [GM71, 2.3.2]. Suppose, for instance, that Y' is a regular curve over $\text{Spec}(R[1/N])$ for some integer N with $(N, \#G) = 1$, and that $\pi' : X' \rightarrow Y'$ is a G -torsor. Resolution of a $\text{Spec}(R)$ -model for Y' then leads to a G -cover $X \rightarrow Y$ over $\text{Spec}(R)$ which satisfies all of our assumptions and has general fiber π'_K . In fact, this cover has at worst “domestic ramification” (see Section 3.f).

Under the above conditions, the sheaf $\pi_*(\mathbb{O}_X)$ of $\mathbb{O}_Y[G]$ -modules on Y is locally $\mathbb{O}_Y[G]$ -free of rank 1. (Since by Remark 3.1(a) π is flat, this follows from [CEPT96, Th. 2.6 and Prop. 2.7].) Hence, we may think of $\pi_*(\mathbb{O}_X)$ as an invertible

sheaf (line bundle) on the scheme $Y \times_{\text{Spec}(R)} G_R^D = G_Y^D$. For simplicity, we will denote this invertible sheaf by \mathcal{L} . Similarly, if \mathcal{G} is a locally free coherent \mathcal{O}_Y -sheaf on Y , we can consider the \mathcal{O}_X -sheaf $\mathcal{F} = \pi^*(\mathcal{G})$ with compatible G -action and regard $\pi_*(\mathcal{F})$ as a locally free coherent $\mathcal{O}_{G_Y^D}$ -sheaf on G_Y^D . We then have

$$\pi_*(\mathcal{F}) \simeq \mathcal{G} \otimes_{\mathcal{O}_Y} \pi_*(\mathcal{O}_X) = \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{L}.$$

In general, if \mathcal{F} is any locally free coherent G -sheaf on X (i.e., a locally free coherent \mathcal{O}_X -module with G action compatible with the action of G on X), then $\pi_*(\mathcal{F})$ can also be thought of as a locally free coherent $\mathcal{O}_{G_Y^D}$ -sheaf on G_Y^D .

Denote by $\tilde{h} : G_Y^D \rightarrow G_R^D$ the base change of $h : Y \rightarrow \text{Spec}(R)$ by $G_R^D \rightarrow \text{Spec}(R)$. Suppose that \mathcal{H} is a locally free coherent $\mathcal{O}_{G_Y^D}$ -module. The total derived image $\mathbb{R}\tilde{h}_*(\mathcal{H})$ in the derived category of complexes of sheaves of $\mathcal{O}_{G_R^D}$ -modules on G_R^D that are bounded below is “perfect” (i.e., it is locally quasi-isomorphic to a bounded complex of locally free coherent $\mathcal{O}_{G_R^D}$ -modules; see [BG171, § III]). Hence, by [KM76], we can associate to $\mathbb{R}\tilde{h}_*(\mathcal{H})$ a graded invertible sheaf

$$\det_* \mathbb{R}\tilde{h}_*(\mathcal{H}) = (\det \mathbb{R}\tilde{h}_*(\mathcal{H}), \text{rank}(\mathbb{R}\tilde{h}_*(\mathcal{H})) \bmod 2)$$

on G_R^D (the “determinant of cohomology”). In what follows, we will mostly consider situations in which the second term, $\text{rank}(\mathbb{R}\tilde{h}_*(\mathcal{H})) \bmod 2$ (a Zariski locally constant $\mathbb{Z}/2\mathbb{Z}$ -function on G_R^D), is trivial. Then we will call $\det \mathbb{R}\tilde{h}_*(\mathcal{H})$ “the determinant of cohomology”.

Suppose now that R is the ring of integers of a number field. The tameness assumption allows us to define the equivariant projective Euler characteristic $\chi^P(X, \mathcal{F})$ in the Grothendieck group of finitely generated projective $R[G]$ -modules $K_0(R[G])$; see [CEPT96, Th. 8.3] and also [Chi94]. We denote by $\bar{\chi}^P(X, \mathcal{F})$ the class of $\chi^P(X, \mathcal{F})$ in the quotient

$$\text{Cl}(R[G]) := K_0(R[G])^{\text{red}} = K_0(R[G]) / \pm \{\text{free } R[G]\text{-modules}\}$$

(the “projective class group of $R[G]$ ”). Since G is abelian and R has Krull dimension 1, there is a natural identification

$$\text{Pic}(G_R^D) = \text{Pic}(R[G]) = \text{Cl}(R[G]).$$

Under this identification, we have [Pap98, § 2.c]

$$(3.2) \quad [\det \mathbb{R}\tilde{h}_*(\pi_*(\mathcal{F}))] = \bar{\chi}^P(X, \mathcal{F}).$$

3.a. *Resolvent theory over Y .* Let $\chi : G \rightarrow \Gamma(S', \mathcal{O}_{S'}^*) = \mathbf{G}_m(S')$ be a character with $S' \rightarrow \text{Spec}(R)$ an R -scheme; then χ corresponds to an S' -point of G_R^D . Set $\pi' : X' = X \times_R S' \rightarrow Y' = Y \times_R S'$. Under Hypothesis T, π' identifies Y' with the quotient X'/G (see [CEPT96, Th. 5.1] or see [KM85, Prop. A7.1.3] combined with (3.1)). Recall G acts on $(\pi')_*(\mathcal{O}_{X'}) = \pi_*(\mathcal{O}_X) \otimes_R \mathcal{O}_{S'}$ via its action on \mathcal{O}_X .

Consider the subsheaf of $(\pi')_*(\mathbb{O}_{X'})$ given by local sections $b \in \mathbb{O}_{X'} = \mathbb{O}_X \otimes_R \mathbb{O}_{S'}$ that satisfy $g \cdot b = \chi(g)b$. We will denote this subsheaf by $\mathbb{O}_{X',\chi}$. We can see that, under our assumptions, $\mathbb{O}_{X',\chi}$ is an invertible $\mathbb{O}_{Y'}$ -sheaf on Y' . (In accordance with the classical theory of Fröhlich, one might call $\mathbb{O}_{X',\chi}$ a *resolvent* invertible sheaf over Y' .)

PROPOSITION 3.2. *Let $\chi_1, \dots, \chi_i : G \rightarrow \Gamma(S', \mathbb{O}_{S'}^*)$ be characters of G , where S' is an R -scheme. With notations as above, there is a canonical homomorphism*

$$\mu : \mathbb{O}_{X',\chi_1} \otimes_{\mathbb{O}_{Y'}} \cdots \otimes_{\mathbb{O}_{Y'}} \mathbb{O}_{X',\chi_i} \rightarrow \mathbb{O}_{X',\chi_1 \cdots \chi_i}$$

of invertible sheaves over Y' that is an isomorphism over the complement of the branch locus of $\pi' : X' \rightarrow Y'$. In particular, the base change μ_K is an isomorphism over the generic fiber Y_K .

Proof. The homomorphism μ is given by restricting the i -fold multiplication

$$\mathbb{O}_{X'} \otimes_{\mathbb{O}_{Y'}} \cdots \otimes_{\mathbb{O}_{Y'}} \mathbb{O}_{X'} \rightarrow \mathbb{O}_{X'}$$

of the sheaf of rings $\mathbb{O}_{X'}$. (To simplify the notation we write $\mathbb{O}_{X'}$ instead of $(\pi')_*(\mathbb{O}_{X'})$.) Over the complement \mathcal{V}' of the branch locus of π' the morphism $\pi'|_{\mathcal{V}'} : \mathcal{U}' = \pi'^{-1}(\mathcal{V}') \rightarrow \mathcal{V}'$ is a G -torsor, and therefore

$$\pi'^{-1}(\mathcal{V}') \times_{\mathcal{V}'} \pi'^{-1}(\mathcal{V}') \simeq G \times \pi'^{-1}(\mathcal{V}').$$

A standard argument using base change by the finite étale morphism $\pi'|_{\mathcal{V}'}$ and descent shows that the restriction of μ on \mathcal{V}' is an isomorphism (see for example, [Pap08, (2.3)]). □

Now consider $I \subset \{1, \dots, n\}$. Recall that the scheme morphism $m_I : (G_R^D)^n \rightarrow G_R^D$ induced by the algebra homomorphism Δ_I of Section 2 is given by $(x_1, \dots, x_n) \mapsto \sum_{i \in I} x_i$. (Recall that we are using additive notation for the group operation on the points of G_R^D . However, such points correspond to characters of G , and the group operation is actually given by multiplication of characters.) Also the i -th projection morphism $p_i : (G_R^D)^n \rightarrow G_R^D$ is induced by $\Delta_{\{i\}}$. Now set $\mathcal{L}_I = \bigotimes_{i \in I} p_i^*(\mathcal{L})$. Suppose now that, in the construction of Proposition 3.2, we take $S' = (G_R^D)^n$, and for $i \in I$ we let χ_i be the $(G_R^D)^n$ -valued character of G that is the “universal” character $\chi_u : G \rightarrow R[G]^*$ with $\chi_u(g) = g$ on the i -th factor and the trivial character on all the other factors. Then Proposition 3.2 applied to this situation has a corollary:

COROLLARY 3.3. *For each $n \geq 1$ and $I \subset \{1, \dots, n\}$, there is a canonical homomorphism*

$$\mu^I : \mathcal{L}_I \rightarrow m_I^*(\mathcal{L})$$

of invertible sheaves over $(G_R^D)_Y^n = Y \times_R (G_R^D)^n$ that is an isomorphism over the base change of the complement of the branch locus of $\pi : X \rightarrow Y$. In particular, the base change μ_K^I is an isomorphism over the generic fiber $Y_K \times_K (G_K^D)^n$.

Proposition 3.2 has another corollary:

COROLLARY 3.4. *Let $\chi : G \rightarrow \Gamma(S', \mathbb{O}_{S'})$ be a character of G . Then there is a canonical homomorphism*

$$v : \mathbb{O}_{X', \chi}^{\otimes \#G} \rightarrow \mathbb{O}_{X', \chi^{\#G}} = \mathbb{O}_{X', 1} = \mathbb{O}_{Y'}$$

of invertible sheaves over $Y' = Y \times_R S'$ that is an isomorphism over the complement of the branch locus of $\pi' : X' \rightarrow Y'$.

In the remainder of this subsection, we assume (in addition to the hypotheses imposed in the beginning of §3) that R is a complete discrete valuation ring with perfect residue field of characteristic prime to $\#G$, and that R contains a primitive $\#G$ -th root of unity ζ .

Suppose that x is a codimension 1 point of X with $\pi(x) = y$. Hypothesis T implies that the inertia subgroup I_x is trivial unless y maps to the closed point of $\text{Spec}(R)$. The action of I_x on the cotangent space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ defines a faithful character $\phi_x : I_x \rightarrow k(x)^*$ with values in the roots of unity of the residue field $k(x)$. Since R^* contains a root of unity ζ of order $\#G$ by assumption and $\#G$ is prime to the characteristic of $k(x)$, we may view ϕ_x as taking values in the subgroup $\langle \zeta \rangle$ generated by ζ . If $\chi : G \rightarrow \langle \zeta \rangle \subset R^*$ is another character of G , the restriction of χ to I_x has the form $\phi_x^{n(\chi, x)}$ for a unique integer $n(\chi, x)$ in the range $0 \leq n(\chi, x) < \#I_x$. Since all points of X over y are conjugate under the action of G , we can see that $I_x \subset G$ and $n(\chi, x)$ depend only y . Often, we will denote them by I_y and $n(\chi, y)$ respectively. Set

$$(3.3) \quad g(\chi, y) = -\frac{n(\chi, y)}{\#I_y}.$$

LEMMA 3.5. *Under the above assumptions, the map $v : \mathbb{O}_{X, \chi}^{\otimes \#G} \rightarrow \mathbb{O}_Y$ identifies $\mathbb{O}_{X, \chi}^{\otimes \#G}$ with $\mathbb{O}_Y(F(\chi))$, where*

$$F(\chi) = \sum_y \#G \cdot g(\chi, y) \cdot y$$

is the divisor with y running over the finite set of codimension 1 points of Y that are contained in the special fiber of $Y \rightarrow \text{Spec}(R)$.

Proof. Let y be a codimension 1 point of Y that is contained in the special fiber. It is enough to prove the statement after replacing Y by an étale neighborhood of $y \in Y$. Then, by using (3.1), we can assume that for every $x \in X$ with $\pi(x) = y$, the decomposition subgroup of x is equal to the inertia subgroup I_x . Suppose that a is a local section of \mathbb{O}_X in a neighborhood of $\pi^{-1}(y)$ such that $g \cdot a = \chi(g)a$ for $g \in G$, so that a defines a local section of $\mathbb{O}_{X, \chi} \subset \pi_*(\mathbb{O}_X)$ in a neighborhood of y . The subgroup I_x acts on a via the character $\chi|_{I_x} = \phi_x^{n(\chi, x)}$. Let ϖ_x , resp. ϖ_y , be uniformizers of the discrete valuation rings $\mathbb{O}_{X,x}$, resp. $\mathbb{O}_{Y,y}$. Since I_x acts

on the quotient $\varpi_x^j \mathbb{O}_{X,x} / \varpi_x^{j+1} \mathbb{O}_{X,x}$ via the character ϕ_x^j for $j \geq 0$, we see that a is in $\varpi_x^{n(\chi,x)} \mathbb{O}_{X,x}$. On the other hand, we can choose a local section a' of \mathbb{O}_X in a neighborhood of $\pi^{-1}(y)$ that is congruent to $\varpi_x^{n(\chi,x)} \bmod \varpi_x^{n(\chi,x)+1} \mathbb{O}_{X,x}$ and that has very high valuation at every other point of X lying over y . Define $\alpha = e_\chi \cdot a'$, where

$$e_\chi = \frac{1}{\#G} \sum_{g \in G} \chi(g)^{-1} g$$

is the idempotent of χ . Then α satisfies $g \cdot \alpha = \chi(g)\alpha$, and we have

$$\alpha \equiv \frac{1}{\#G} \sum_{g \in I_x} \chi(g)^{-1} (g \cdot \varpi_x^{n(\chi,x)}) \equiv \frac{\#I_x}{\#G} \varpi_x^{n(\chi,x)} \bmod \varpi_x^{n(\chi,x)+1} \mathbb{O}_{X,x},$$

since $g \cdot \varpi_x \equiv \phi_x(g) \varpi_x \bmod \varpi_x^2 \mathbb{O}_{X,x}$ and $\chi(g) = \phi_x(g)^{n(\chi,x)}$ for $g \in I_x$. By the definition of the map ν , its image is the \mathbb{O}_Y -ideal sheaf with local sections generated by the $\#G$ -th powers of the local sections of $\mathbb{O}_{X,\chi} \subset \pi_*(\mathbb{O}_X)$. Since $\varpi_y \mathbb{O}_{X,x} = \varpi_x^{\#I_x} \mathbb{O}_{X,x}$, the above considerations now show that the stalk of this ideal sheaf at y is $\varpi_y^{-g(\chi,y) \cdot \#G} \mathbb{O}_{Y,y}$. \square

COROLLARY 3.6. *Let $\phi : G^n \rightarrow R^*$ be a character and denote by ϕ_i the restriction of ϕ to the i -th factor of G^n . For a subset $I \subset \{1, \dots, n\}$ denote by $\phi^*(\mu^I)$ the base change of μ^I of Corollary 3.3 by the morphism $Y \rightarrow Y \times_R (G_R^D)^n = (G_R^D)_Y^n$ induced by base change via $\phi : R[G^n] \rightarrow R$. Then the $\#G$ -th tensor power $\phi^*(\mu^I)^{\otimes \#G}$ of $\phi^*(\mu^I)$ is identified with the natural injection of invertible sheaves*

$$\mathbb{O}_Y(\sum_{i \in I} F(\phi_i)) \rightarrow \mathbb{O}_Y(F(\prod_{i \in I} \phi_i)).$$

3.b. The determinant of cohomology. We continue with the assumptions and notations of the beginning of Section 3. In particular, we assume Hypothesis T. Denote by $\tilde{h} : G_Y^D \rightarrow G_R^D$ the base change of h by $G_R^D \rightarrow \text{Spec}(R)$. Recall that we set $\mathcal{L} = \pi_*(\mathbb{O}_X)$ (an invertible sheaf on G_Y^D). If \mathcal{G} is a locally free coherent \mathbb{O}_Y -module, we can consider the square of the determinant of cohomology

$$\delta(\mathcal{G} \otimes_{\mathbb{O}_Y} \mathcal{L}) = (\det \mathbb{R}\tilde{h}_*(\mathcal{G} \otimes_{\mathbb{O}_Y} \mathcal{L}))^{\otimes 2};$$

this is a line bundle on G_R^D . (If the function $\text{rank}(\mathbb{R}\tilde{h}_*(\mathcal{G} \otimes_{\mathbb{O}_Y} \mathcal{L}))$ is always even on G_R^D , we need not consider the square $(\det \mathbb{R}\tilde{h}_*(\mathcal{G} \otimes_{\mathbb{O}_Y} \mathcal{L}))^{\otimes 2}$: In this case, all the arguments below can be carried out for $\det \mathbb{R}\tilde{h}_*(\mathcal{G} \otimes_{\mathbb{O}_Y} \mathcal{L})$ instead of $\delta(\mathcal{G} \otimes_{\mathbb{O}_Y} \mathcal{L})$.) In order to simplify the exposition, we will now identify in our notation invertible sheaves (line bundles) over G_R^D with the corresponding locally free rank 1 $R[G]$ -modules of their sections; this should not cause any confusion. We will also use the same notation δ to denote the functor given by the square of the determinant of cohomology over various base schemes without distinction. Also, in a further

attempt to lighten the presentation, we will first concentrate our discussion to the case $\mathcal{G} = \mathbb{O}_Y$ for simplicity; the general case, which is similar, will be discussed later.

Let s be any generator of the $K[G]$ -module $\delta(\mathcal{L})_K$. Our plan is to calculate the isomorphism class of the image $\Theta_{d+2}((\delta(\mathcal{L}), s)) = (\Theta_{d+2}(\delta(\mathcal{L})), \Theta_{d+2}(s))$ in the category $\mathcal{C}_R(G; d + 2)$ in terms of data obtained from the branch locus of the cover π . Under our assumptions, the cover of generic fibers π_K is unramified and hence a G -torsor. Hence, it follows from the results of [Duc05] and [Pap08] (see below) that the $K[G]$ -module $\delta(\mathcal{L})_K$ supports a canonical n -cubic structure γ with $n = d + 2$. Therefore, there is a distinguished $K[G^n]$ generator $\gamma(1)$ of $\Theta(\delta(\mathcal{L})_K) \simeq \Theta(\delta(\mathcal{L}))_K$. (From hereon we will omit the subscript $n = d + 2$.) We start with the following fundamental observation:

It is enough to calculate the isomorphism class of the pair $(\Theta(\delta(\mathcal{L})), \gamma(1))$ in the group $C_R(G; n)$ in terms of data obtained from the branch locus of the cover π .

Indeed, because γ defines an n -cubic structure on $\Theta(\delta(\mathcal{L}))_K$, if s is a $K[G]$ -generator of $\delta(\mathcal{L})_K$, then by definition, the “ratio” $\Theta(s)/\gamma(1) \in K[G^n]^*$ is an n -cubic element. Therefore, the pair $(\Theta(\delta(\mathcal{L})), \gamma(1))$ is isomorphic to $\Theta((\delta(\mathcal{L}), s)) = (\Theta(\delta(\mathcal{L})), \Theta(s))$ in the category $\mathcal{C}_R(G; n)$.

Set

$$\mathfrak{D} := \bigotimes_{I \subset \{1, \dots, n\}} \delta(\mathcal{L}_I)^{(-1)^{n-\#I}} .$$

[Duc05, Th. 4.2], that paper’s main result, gives a canonical trivialization $b : R[G^n] \xrightarrow{\sim} \mathfrak{D}$. (This follows from the existence of a $(d + 2)$ -cubic structure on the functor from line bundles on G_Y^D to line bundles on G_R^D given by the square of the determinant of cohomology; see also [Pap08, § 4, especially Def. 4.2 and § 4.e]. In fact, after wrestling with signs, a harder result about the determinant of the cohomology — without squaring and without a condition on the rank of $\mathbb{R}\tilde{h}_*(\mathcal{L})$ — is shown in [Duc05]. We are not going to use this more complicated result. When $d = 1$, this trivialization can also be obtained directly using the Deligne-Riemann-Roch theorem and the bilinearity of the Deligne pairing [Del87].) Using this we obtain an isomorphism

$$\Theta(\delta(\mathcal{L})) \otimes \mathfrak{D}^{-1} \xrightarrow{\text{id} \otimes b^{-1}} \Theta(\delta(\mathcal{L})) .$$

Now notice that by using μ_K^I in Corollary 3.3 and the functoriality of the determinant of cohomology we can obtain an isomorphism

$$\mathfrak{D}_K \xrightarrow{\sim} \bigotimes_I (\delta((\mathcal{L}_I)_K))^{(-1)^{n-\#I}} \xrightarrow{\sim} \bigotimes_I (\delta(m_I^*(\mathcal{L})_K))^{(-1)^{n-\#I}} \xrightarrow{\sim} \Theta(\delta(\mathcal{L})_K),$$

where the latter isomorphism comes from the identification

$$\Theta(\delta(\mathcal{L})) = \bigotimes_I m_I^*(\delta(\mathcal{L}))^{(-1)^{n-\#I}} \simeq \bigotimes_I \delta(m_I^*(\mathcal{L}))^{(-1)^{n-\#I}} .$$

Hence, we obtain a trivialization

$$(3.4) \quad K[G^n] \xrightarrow{\sim} \Theta(\delta(\mathcal{L})_K) \otimes \mathcal{D}_K^{-1}.$$

Consider the composition

$$(3.5) \quad \gamma : K[G^n] \xrightarrow{\sim} \Theta(\delta(\mathcal{L})_K) \otimes \mathcal{D}_K^{-1} \xrightarrow{\text{id} \otimes b_K^{-1}} \Theta(\delta(\mathcal{L})_K).$$

By [Pap08, Th. 4.7] and its proof, the trivialization γ is an n -cubic structure on $\delta(\mathcal{L})_K$.

Now let us consider the image E of the $R[G^n]$ -submodule

$$\Theta(\delta(\mathcal{L})) \otimes \mathcal{D}^{-1} \subset (\Theta(\delta(\mathcal{L})) \otimes \mathcal{D}^{-1})_K \simeq \Theta(\delta(\mathcal{L})_K) \otimes \mathcal{D}_K^{-1}$$

under the inverse of the isomorphism (3.4) above. It follows that E is a locally free $R[G^n]$ -submodule of $K[G^n]$. There is a commutative diagram

$$(3.6) \quad \begin{array}{ccccc} K[G^n] & \xrightarrow{\sim} & \Theta(\delta(\mathcal{L})_K) \otimes \mathcal{D}_K^{-1} & \xrightarrow{\text{id} \otimes b_K^{-1}} & \Theta(\delta(\mathcal{L})_K) \\ \cup & & \uparrow & & \uparrow \\ E & \xrightarrow{\sim} & \Theta(\delta(\mathcal{L})) \otimes \mathcal{D}^{-1} & \xrightarrow{\text{id} \otimes b^{-1}} & \Theta(\delta(\mathcal{L})), \end{array}$$

where the two vertical arrows are given by the compositions

$$\begin{aligned} \Theta(\delta(\mathcal{L})) \otimes \mathcal{D}^{-1} &\subset (\Theta(\delta(\mathcal{L})) \otimes \mathcal{D}^{-1})_K \xrightarrow{\sim} \Theta(\delta(\mathcal{L})_K) \otimes \mathcal{D}_K^{-1}, \\ \Theta(\delta(\mathcal{L})) &\subset \Theta(\delta(\mathcal{L}))_K \xrightarrow{\sim} \Theta(\delta(\mathcal{L})_K). \end{aligned}$$

LEMMA 3.7. *Let s be a $K[G]$ -generator of the module $\delta(\mathcal{L})_K$. Then the element $(E, 1)$ is isomorphic to $\Theta((\delta(\mathcal{L}), s))$ in the category $\mathcal{C}_R(G; n)$.*

Proof. An isomorphism is given by the composition of the second row of the diagram (3.6). As remarked above, $\gamma(1)$ is the image of 1 under the composition of the first row. Hence $(E, 1)$ is isomorphic to $(\Theta(\delta(\mathcal{L})), \gamma(1))$, which we showed earlier to be isomorphic to $\Theta((\delta(\mathcal{L}), s))$ in the category $\mathcal{C}_R(G; n)$. \square

From the definitions, we have canonical isomorphisms

$$E \xrightarrow{\sim} \Theta(\delta(\mathcal{L})) \otimes \mathcal{D}^{-1} \xrightarrow{\sim} \bigotimes_I (\delta(m_I^*(\mathcal{L})) \otimes \delta(\mathcal{L}_I)^{-1})^{(-1)^{n-\#I}} = \prod_I E_I^{(-1)^{n-\#I}}$$

when $E_I = \delta(m_I^*(\mathcal{L})) \otimes \delta(\mathcal{L}_I)^{-1}$. Here E_I is a locally free rank 1 $R[G^n]$ -module. We also have a canonical isomorphism

$$E_I \xrightarrow{\sim} \delta(\mathcal{L}_I \xrightarrow{\mu^I} m_I^*(\mathcal{L}))$$

in which we think of μ^I as a perfect complex of $\mathbb{O}_Y[G^n]$ -modules with terms at positions -1 and 0 . Since μ_K^I is an isomorphism, $\delta(\mu_K^I)$ gives a trivialization $K[G^n] \xrightarrow{\sim} (E_I)_K$ that we can use to identify E_I with a locally free $R[G^n]$ -submodule of $K[G^n]$. Using these identifications, we obtain that, as locally free $R[G^n]$ -submodules (“fractional ideals”) of $K[G^n]$,

$$E = \prod_I E_I^{(-1)^{n-\#I}}.$$

We will obtain our main result by calculating E_I and E under some additional assumptions.

Remark 3.8. The above arguments readily extend from the case $\mathcal{G} = \mathbb{O}_Y$ to the case of a general coherent locally free \mathbb{O}_Y -module \mathcal{G} . Indeed, by [Duc05, § 4.7], the functor from line bundles on G_Y^D to line bundles on G_R^D given by $\mathcal{M} \mapsto \delta(\mathcal{G} \otimes_{\mathbb{O}_Y} \mathcal{M})$ has a canonical n -cubic structure. This provides a canonical trivialization of

$$(3.7) \quad \mathfrak{D}(\mathcal{G}) := \bigotimes_I \delta(\mathcal{G} \otimes_{\mathbb{O}_Y} \mathcal{L}_I)^{(-1)^{n-\#I}}$$

over $(G_R^D)^n$. Now, in the same way as we have seen above, this trivialization combined with the isomorphisms μ_K^I provides an n -cubic structure on the generic fiber $\delta(\mathcal{G} \otimes_{\mathbb{O}_Y} \mathcal{L})_K$. The rest of the argument is also identical: In the end we obtain locally free $R[G^n]$ -submodules $E_I(\mathcal{G})$ and $E(\mathcal{G}) \subset K[G^n]$ such that $(E(\mathcal{G}), 1)$ is isomorphic to the image $\Theta((\delta(\mathcal{G} \otimes_{\mathbb{O}_Y} \mathcal{L}), s))$ in $\mathcal{C}_R(G; d + 2)$.

3.c. Branch divisors and Riemann-Roch. We continue with the assumptions and notations given at the beginning of Section 3. However, in this subsection, we will assume in addition that R is a complete discrete valuation ring that has perfect residue field k of characteristic prime to $\#G$ and that contains a primitive root of unity ζ of order equal to $\#G$. Recall also $n = d + 2$.

Let $\phi : G^n \rightarrow \langle \zeta \rangle \subset R^*$ be a character, and denote by ϕ_i the restriction of ϕ to the i -th factor of G^n . Recall that for $I \subset \{1, \dots, n\}$ we denote by $\phi^*(\mu^I)$ the base change of μ^I by $Y \rightarrow (G_Y^D)^n$ given by using $\phi : R[G^n] \rightarrow R$. The morphism $\phi^*(\mu^I)$ then identifies with the “multiplication”

$$\bigotimes_{i \in I} \mathbb{O}_{X, \phi_i} \rightarrow \mathbb{O}_{X, \prod_{i \in I} \phi_i},$$

which was considered in Proposition 3.2. By functoriality, we have

$$\phi(E_I) = \delta(\phi^*(\mu^I)) \subset K \quad \text{and} \quad \phi(E) = \prod_I \delta(\phi^*(\mu^I))^{(-1)^{n-\#I}} \subset K.$$

We will calculate the R -fractional ideals $\phi(E_I)$ and $\phi(E)$ in K using a localized Riemann-Roch theorem for the morphism $h : Y \rightarrow \text{Spec}(R)$. Denote by $h_s : Y_s \rightarrow$

$\text{Spec}(k)$ the special fiber of h . Let $\mathcal{F}_{I,\phi}$ be the cokernel of $\phi^*(\mu^I)$; it is a coherent sheaf of \mathcal{O}_Y -modules that is supported on the special fiber Y_s . Therefore, it gives a class $[\mathcal{F}_{I,\phi}]$ in the Grothendieck group $G_0(Y_s)$ of coherent \mathcal{O}_{Y_s} -modules on Y_s . The following can be deduced from the definition and basic properties of the determinant of cohomology. (Note that the functor δ in this statement is given by the square of the determinant of cohomology; this explains the appearance of the factor 2 in the exponent below.)

LEMMA 3.9. *Denote by ϖ a uniformizer of R . We have*

$$\phi(E_I) = \varpi^{-2 \cdot \chi(\mathcal{F}_{I,\phi})} \cdot R,$$

where $\chi(\mathcal{F}_{I,\phi}) = (h_s)_*([\mathcal{F}_{I,\phi}]) \in G_0(\text{Spec}(k)) = \mathbb{Z}$ is the Euler characteristic of $[\mathcal{F}_{I,\phi}]$.

In what follows we borrow heavily from [Ful98], to which we refer the reader for notations and more details. Suppose that D_1, \dots, D_q are effective divisors on Y with supports $|D_1|, \dots, |D_q|$ contained in Y_s . Then we can define, following [Ful98, §§ 2 and 17], a bivariant class $[D_1] \cap \dots \cap [D_q] \in A^q(Y_s \rightarrow Y)$ that induces homomorphisms $A_*(Y) \rightarrow A_{*-q}(Y_s)$ denoted by $x \mapsto ([D_1] \cap \dots \cap [D_q]) \cap x$. These homomorphisms are given by the composition

$$A_*(Y) \rightarrow A_{*-q}(|D_1| \cap \dots \cap |D_q|) \rightarrow A_{*-q}(Y_s).$$

Here, the first map is defined by [Ful98, § 2]; the second is obtained from the inclusion of the set-theoretic intersection of supports $|D_1| \cap \dots \cap |D_q|$ in Y_s . The bivariant class $[D_1] \cap \dots \cap [D_q]$ is multiadditive in the D_i and independent of their order [Ful98, §§ 2.3 and 2.4]. (If $z = [Z] \in A_k(Y)$ with Z integral of Krull dimension k , then $[D] \cap z$ is the class of the $(k-1)$ -cycle on $|D| \subset Y_s$ obtained by restricting the line bundle $\mathcal{O}_Y(D)$ on Z and taking a corresponding divisor supported on $|D| \cap Z$. The general definition follows from this by using an inductive procedure.) If $D_1 = \dots = D_q = D$, we will denote this class by $[D]^q$.

Now let \mathcal{E}^\bullet be a finite complex of locally free \mathcal{O}_Y -modules that is exact off Y_s . Denote by $\text{ch}_{Y_s}^Y(\mathcal{E}^\bullet)$ the localized Chern character of \mathcal{E}^\bullet defined by the MacPherson graph construction following [Ful98, § 18]. (Strictly speaking, the reference [Ful98] only covers schemes over a base field; the extension to the situation we want is described in [GS92] or [Rob98, § 11], but see also [Ful98, § 20].) By definition, this is a bivariant class in $A^*(Y_s \rightarrow Y)_{\mathbb{Q}}$, which in particular induces

$$\text{ch}_{Y_s}^Y(\mathcal{E}^\bullet) \cap \cdot : A_*(Y)_{\mathbb{Q}} \rightarrow A_*(Y_s)_{\mathbb{Q}}.$$

Suppose that c is a complex of the form $\mathcal{L} \rightarrow \mathcal{M}$ (at degrees -1 and 0) with \mathcal{L} and \mathcal{M} two line bundles on Y . Suppose c is exact off Y_s , and set

$$\text{ch}_{Y_s}^Y(c) = \sum_{q=0}^{d+1} \text{ch}_{Y_s}^{Y,q}(c) \in A^*(Y_s \rightarrow Y)_{\mathbb{Q}} = \bigoplus_{q=0}^{d+1} A^q(Y_s \rightarrow Y)_{\mathbb{Q}}.$$

From [Rob98, Thms. 11.4.6 and 12.3.1], we have this:

PROPOSITION 3.10. (a) $\text{ch}_{Y_s}^{Y,0}(c) = 0$.

(b) *Suppose that c is the inclusion $\mathbb{O}_Y \rightarrow \mathbb{O}_Y(D)$ for D an effective divisor with support $|D| \subset Y_s$. Then*

$$\text{ch}_{Y_s}^Y(c) = \sum_{q=1}^{d+1} \frac{[D]^q}{q!}.$$

(c) *Similarly, if c is the inclusion $\mathbb{O}_Y(-D) \rightarrow \mathbb{O}_Y$ for D an effective divisor with support $|D| \subset Y_s$, then*

$$\text{ch}_{Y_s}^Y(c) = - \sum_{q=1}^{d+1} \frac{(-1)^q [D]^q}{q!}.$$

(d) *If $c^m : \mathcal{L}^{\otimes m} \rightarrow \mathcal{M}^{\otimes m}$ is the m -fold tensor product of c , then*

$$\text{ch}_{Y_s}^{Y,q}(c^m) = m^q \text{ch}_{Y_s}^{Y,q}(c).$$

(e) *If \mathcal{G} is a locally free coherent \mathbb{O}_Y -module, $\text{ch}_{Y_s}^Y(\mathcal{G} \otimes_{\mathbb{O}_Y} c) = \text{ch}(\mathcal{G}) \cap \text{ch}_{Y_s}^Y(c)$, where $\text{ch}(\mathcal{G}) \in A^*(Y)_{\mathbb{Q}}$ is the usual Chern character of \mathcal{G} .*

Now let us return to calculating the fractional ideals $\phi(E_I)$ and $\phi(E)$. Since Y is assumed regular and h projective, an embedding $i : Y \rightarrow \mathbf{P}_R^m$ produces an exact sequence of coherent \mathbb{O}_Y -modules

$$0 \rightarrow N_{Y|\mathbf{P}_R^m}^{\vee} \rightarrow i^* \Omega_{\mathbf{P}_R^m/R}^1 \rightarrow \Omega_{Y/R}^1 \rightarrow 0$$

with $N_{Y|\mathbf{P}_R^m}^{\vee}$ and $i^* \Omega_{\mathbf{P}_R^m/R}^1$ locally free. (Here $N_{Y|\mathbf{P}_R^m}^{\vee}$ is the dual of the normal bundle $N_{Y|\mathbf{P}_R^m}$ of the embedding.) Let $\text{Td}(h) = \text{td}(h) \cap [Y]$ be the Todd class in $A_*(Y)_{\mathbb{Q}}$ of the relative tangent complex $[(i^* \Omega_{\mathbf{P}_R^m/R}^1)^{\vee} \rightarrow N_{Y|\mathbf{P}_R^m}]$ (at degrees 0 and 1).

THEOREM 3.11. *With the above assumptions and notations, consider the function T_{π} on R -valued 1-dimensional characters of G given by*

$$\psi \mapsto T_{\pi}(\psi) := \sum_{q=1}^{d+1} \sum_{(y_1, \dots, y_q)} \frac{\prod_{j=1}^q g(\psi, y_j)}{q!} \text{deg} \left(\left(\prod_{j=1}^q [y_j] \right) \cap \text{Td}_q(h) \right) \in \mathbb{Q},$$

where we write deg for the degree of zero cycles over k and (y_1, \dots, y_q) runs over all q -tuples of codimension 1 points of Y that lie in Y_s . (We use the same symbol for the corresponding effective divisors.) Extend linearly the function T_{π} to the

ring of R -valued characters of G , i.e., to integral linear combinations of R -valued 1-dimensional characters. Then we have

$$\phi(E) = \varpi^{-2 \cdot T_\pi(\Theta^D(\phi))} \cdot R \subset K$$

where for $\phi = \phi_1 \otimes \dots \otimes \phi_{d+2}$ a 1-dimensional character of G^{d+2} , we have

$$\Theta^D(\phi) = (\phi_1 - 1) \cdots (\phi_{d+2} - 1) = \sum_I (-1)^{d+2-\#I} \prod_{i \in I} \phi_i.$$

Proof. We will see that this follows from Lemma 3.9 and the following:

THEOREM 3.12 (Riemann-Roch theorem).

$$\chi(\mathcal{F}_{I,\phi}) = (h_s)_* \left((\text{ch}_{Y_s}^Y(\phi^*(\mu^I)) \cap \text{Td}(h))_0 \right).$$

In this equality, the map $(h_s)_*$ on the right side is the push forward of zero-cycles $A_0(Y_s)_\mathbb{Q}$ to $A_0(\text{Spec}(k))_\mathbb{Q} = \mathbb{Q}$; i.e., it is given by the degree of zero cycles over k . This special case of a “localized” Riemann-Roch theorem follows from [Rob98, Thms. 12.5.1 and 12.6.1]. It can also be derived following the proof of [Ful98, Th. 18.2(1)] by considering the morphism $h_s : Y_s \rightarrow \text{Spec}(k)$ as a morphism of schemes over $S = \text{Spec}(R)$. ([Ful98] gives a similar result for schemes over a base S that is a nonsingular scheme over a field.)

Now, let us deduce Theorem 3.11 from Theorem 3.12 by calculating the 0-th component of $\sum_I (-1)^{n-\#I} \text{ch}_{Y_s}^Y(\phi^*(\mu^I)) \cap \text{Td}(h)$: Using Proposition 3.10(a) and (d), we can write

$$\begin{aligned} (3.8) \quad (\text{ch}_{Y_s}^Y(\phi^*(\mu^I)) \cap \text{Td}(h))_0 &= \sum_{q=1}^{d+1} \text{ch}_{Y_s}^{Y,q}(\phi^*(\mu^I)) \cap \text{Td}_q(h) \\ &= \sum_{q=1}^{d+1} \frac{\text{ch}_{Y_s}^{Y,q}(\phi^*(\mu^I)^{\otimes \#G})}{(\#G)^q} \cap \text{Td}_q(h). \end{aligned}$$

Corollary 3.6 now identifies the complex $\phi^*(\mu^I)^{\otimes \#G}$ with

$$c'_I : \mathbb{O}_Y(\sum_{i \in I} F(\phi_i)) \rightarrow \mathbb{O}_Y(F(\prod_{i \in I} \phi_i)).$$

Let us write $D_I = -F(\prod_{i \in I} \phi_i)$ and $D_I + D'_I = -\sum_{i \in I} F(\phi_i)$ (these are both effective divisors supported on the special fiber Y_s). There is an exact sequence of complexes,

$$\begin{array}{ccccccc} 0 \rightarrow \mathbb{O}_Y(-D_I) & \longrightarrow & \mathbb{O}_Y(-D_I) \oplus \mathbb{O}_Y & \longrightarrow & \mathbb{O}_Y & \rightarrow 0 \\ & & \uparrow \text{id} \oplus i'' & & \uparrow i & & \\ & & c'_I \uparrow & & & & \\ 0 \rightarrow \mathbb{O}_Y(-D_I - D'_I) & \longrightarrow & \mathbb{O}_Y(-D_I) \oplus \mathbb{O}_Y(-D_I - D'_I) & \longrightarrow & \mathbb{O}_Y(-D_I) & \rightarrow 0, \end{array}$$

with c'_I , i and i'' the natural injective homomorphisms. Therefore, by [Ful98, Prop. 18.1], we have

$$(3.9) \quad \text{ch}_{Y_S}^Y(c'_I) + \text{ch}_{Y_S}^Y(\mathbb{O}_Y(-D_I) \xrightarrow{i} \mathbb{O}_Y) = \text{ch}_{Y_S}^Y(\mathbb{O}_Y(-D_I - D'_I) \xrightarrow{i''} \mathbb{O}_Y).$$

Recall $n = d + 2$. Applying Proposition 3.10(c) and telescoping using the identity (for given $q < n$)

$$\sum_{I \subset \{1, \dots, n\}} (-1)^{\#I} \left(\sum_{i \in I} X_i \right)^q = 0$$

gives

$$\sum_{I \subset \{1, \dots, n\}} (-1)^{n-\#I} \text{ch}_{Y_S}^Y(\mathbb{O}_Y(\sum_{i \in I} F(\phi_i)) \rightarrow \mathbb{O}_Y) = 0.$$

This translates to

$$\sum_{I \subset \{1, \dots, n\}} (-1)^{n-\#I} \text{ch}_{Y_S}^Y(\mathbb{O}_Y(-D_I - D'_I) \xrightarrow{i''} \mathbb{O}_Y) = 0.$$

Hence, using (3.9) and Proposition 3.10(c), we obtain

$$\begin{aligned} \sum_I (-1)^{n-\#I} \text{ch}_{Y_S}^Y(c'_I) &= - \sum_I (-1)^{n-\#I} \text{ch}_{Y_S}^Y(\mathbb{O}_Y(-D_I) \rightarrow \mathbb{O}_Y) \\ &= \sum_I (-1)^{n-\#I} \sum_{q=1}^{d+1} \frac{(-1)^q [D_I]^q}{q!}. \end{aligned}$$

This gives

$$\sum_I (-1)^{n-\#I} \text{ch}_{Y_S}^{Y,q}(c'_I) = \sum_I (-1)^{n-\#I+q} \frac{[D_I]^q}{q!}.$$

Therefore, using Proposition 3.10(d) and Corollary 3.6, we now obtain

$$\begin{aligned} \sum_I (-1)^{n-\#I} \text{ch}_{Y_S}^{Y,q}(\phi^*(\mu^I)) &= \sum_I (-1)^{n-\#I} \frac{\text{ch}_{Y_S}^{Y,q}(\phi^*(\mu^I)^{\otimes \#G})}{(\#G)^q} \\ &= \sum_I (-1)^{n-\#I+q} \frac{[D_I]^q}{(\#G)^q q!}. \end{aligned}$$

Since by definition $D_I = -F(\prod_i \phi_i)$, by using Lemma 3.5 we find that this is equal to

$$\begin{aligned} \sum_I (-1)^{n-\#I+q} \sum_{(y_1, \dots, y_q)} \frac{(-\#G)^q g(\prod_{i \in I} \phi_i, y_1) \cdots g(\prod_{i \in I} \phi_i, y_q)}{(\#G)^q q!} [y_1] \cap \cdots \cap [y_q] \\ = \sum_I (-1)^{n-\#I} \sum_{(y_1, \dots, y_q)} \frac{g(\prod_{i \in I} \phi_i, y_1) \cdots g(\prod_{i \in I} \phi_i, y_q)}{q!} [y_1] \cap \cdots \cap [y_q], \end{aligned}$$

where (y_1, \dots, y_q) runs over all q -tuples of codimension 1 fibral points of Y . Altogether, the 0-th component of $\sum_I (-1)^{n-\#I} \text{ch}_Y^Y(\phi^*(\mu^I)) \cap \text{Td}(h)$ is

$$\sum_I (-1)^{n-\#I} \sum_{q=1}^{d+1} \sum_{(y_1, \dots, y_q)} \frac{\prod_{j=1}^q g(\prod_{i \in I} \phi_i, y_j)}{q!} \left(\bigcap_{j=1}^q [y_j] \right) \cap \text{Td}_q(h).$$

Theorem 3.11 now follows from Theorem 3.12 and Lemma 3.9. □

3.d. It is easy to extend the results of Section 3.c from the case $\mathcal{G} = \mathcal{O}_Y$ to the case of a general coherent locally free \mathcal{O}_Y -sheaf \mathcal{G} : We want to calculate

$$\phi(E(\mathcal{G})) = \bigotimes_I \delta(\phi^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{L}_I \xrightarrow{\mathcal{G} \otimes_{\mathcal{O}_Y} \mu^I} \mathcal{G} \otimes_{\mathcal{O}_Y} m_I^*(\mathcal{L})))^{(-1)^{n-\#I}}.$$

The exact same proof now works (one has to also use Proposition 3.10(e)) to obtain that

$$(3.10) \quad \phi(E(\mathcal{G})) = \varpi^{-2 \cdot T_{\pi, \mathcal{G}}(\Theta^D(\phi))} \cdot R \subset K,$$

where $T_{\pi, \mathcal{G}}$ is the \mathbb{Q} -valued function on the R -valued 1-dimensional characters of G given by

$$(3.11) \quad T_{\pi, \mathcal{G}}(\psi) = \sum_{l=1}^{d+1} \sum_{(y_1, \dots, y_l)} \frac{\prod_{j=1}^l g(\psi, y_j)}{l!} \sum_{t=0}^{d+1-l} \text{deg}(\text{ch}^t(\mathcal{G}) \cap \left(\bigcap_{j=1}^l [y_j] \right) \cap \text{Td}_{t+l}(h)).$$

The notation and assumptions here are as in Theorem 3.11.

3.e. Here we assume in addition that X and Y are relative curves over $\text{Spec}(R)$ ($d = 1$). Notice that we have

$$\begin{aligned} \text{Td}_1(h) &= \frac{1}{2} c_1((i^* \Omega_{\mathbf{P}_R^m/R}^1)^\vee) - \frac{1}{2} c_1(N_{Y|\mathbf{P}_R^m}) \\ &= \frac{1}{2} c_1(\det(i^* \Omega_{\mathbf{P}_R^m/R}^1)^{-1} \otimes \det(N_{Y|\mathbf{P}_R^m}^\vee)) = -\frac{1}{2} c_1(\omega), \end{aligned}$$

where $\omega \simeq \det(i^* \Omega_{\mathbf{P}_R^m/R}^1) \otimes \det(N_{Y|\mathbf{P}_R^m}^\vee)^{-1}$ is the canonical sheaf for $Y \rightarrow \text{Spec}(R)$. Hence, in this case, the formula (3.11) specializes to

$$(3.12) \quad T_{\pi, \mathcal{G}}(\psi) = \text{rank}(\mathcal{G}) \sum_{(y_1, y_2)} \frac{1}{2} g(\psi, y_1) \cdot g(\psi, y_2) \text{deg}([y_1] \cap [y_2]) - \sum_y g(\psi, y) \text{deg}(\text{rank}(\mathcal{G})[y] \cap \frac{1}{2} c_1(\omega) - c_1(\mathcal{G}) \cap [y]).$$

Actually, in this case we have $\text{deg}([y_1] \cap [y_2]) = \text{deg}(\mathcal{O}_Y(y_1)|_{y_2})$ (the degree of the line bundle $\mathcal{O}_Y(y_1)$ restricted on y_2 over $\text{Spec}(k)$). For simplicity, we will denote

this intersection number by $y_1 \cdot y_2$ (see [Lan88, III]). Also, using adjunction [Lan88, IV § 4], we find that

$$\deg([y] \cap c_1(\omega)) = -\deg([y] \cap [y]) + \deg(c_1(\omega_{y/k})) = -y \cdot y - 2\chi(y, \mathcal{O}_y)$$

(here $\omega_{y/k}$ is the canonical sheaf of y over $\text{Spec}(k)$; see [Lan88]). Then formula (3.12) for $\mathcal{G} = \mathcal{O}_Y$ can be written as

$$(3.13) \quad T_\pi(\psi) = \sum_{(y_1, y_2)} \frac{1}{2} g(\psi, y_1) \cdot g(\psi, y_2) (y_1 \cdot y_2) + \sum_y \frac{1}{2} g(\psi, y) (y \cdot y + 2\chi(y, \mathcal{O}_y)).$$

3.f. *The main theorem.* Let R be the ring of integers of a number field K . We continue with the assumptions and notations of the beginning of Section 3. In fact, we now make an assumption in addition to Hypothesis T:

HYPOTHESIS D. The residue field characteristic of each point of Y that ramifies in $\pi : X \rightarrow Y$ is relatively prime to the order of the group G .

Denote by S the finite set of rational primes such that the cover $\pi : X \rightarrow Y$ is only ramified at points above S . By our Hypothesis D, $p \in S$ implies that p does not divide $\#G$. Denote by S_K the set of places of K that lie above S .

Let \mathcal{G} be a locally free coherent \mathcal{O}_Y -sheaf on Y , and consider the G -sheaf $\mathcal{F} = \pi^* \mathcal{G}$ on X . Recall the homomorphism

$$\Theta = \Theta_{d+2} : \text{Cl}(R[G]) = \text{Pic}(R[G]) \rightarrow C_R(G; d + 2)$$

of Section 2.b. For a finite place v of K , we denote by ϖ_v a uniformizer of the completion R_v and fix an algebraic closure \bar{K}_v of its fraction field K_v . Recall that, as in Section 2.c, any finite idèle $(a_v)_v \in \mathbf{A}_{f,K}^*[G^{d+2}]$ gives the element $(\bigcap_v (R_v[G^{d+2}]_{a_v} \cap K[G^{d+2}]), 1)$ of $C_R(G; d + 2)$. Now let $v \in S_K$ (then $(v, \#G) = 1$), and denote by R'_v the complete discrete valuation ring $R'_v \subset \bar{K}_v$ obtained by adjoining to R_v a primitive root of unity of order equal to $\#G$. Then ϖ_v is also a uniformizer for R'_v . Let us consider the cover $\pi'_v : X \otimes_R R'_v \rightarrow Y \otimes_R R'_v$ obtained from π by base change. Since R'_v has residue field characteristic prime to $\#G$ and contains a primitive $\#G$ -th root of unity, we can now apply the constructions and results of Sections 3.c and 3.d to the cover π'_v and the sheaf $\mathcal{G} \otimes_R R'_v$. For simplicity, we will denote by $T_{v,\mathcal{G}}$ the function $T_{\pi'_v, \mathcal{G} \otimes_R R'_v} : \text{Ch}(G)_v \rightarrow \mathbb{Q}$ given by (3.11). Recall the isomorphism

$$(3.14) \quad K_v[G^{d+2}]^* \xrightarrow{\sim} \text{Hom}_{\text{Gal}(\bar{K}_v/K_v)}(\text{Ch}(G^{d+2})_v, \bar{K}_v^*)$$

given by evaluating characters of G^{d+2} . Also recall that if $\phi = \phi_1 \otimes \dots \otimes \phi_{d+2}$ is a character of G^{d+2} given by a $(d + 2)$ -tuple $(\phi_i)_i$ of 1-dimensional \bar{K}_v -valued characters of G , we have $\Theta^D(\phi) = (\phi_1 - 1) \cdots (\phi_{d+2} - 1) \in \text{Ch}(G^{d+2})_v$. Let us now

observe that the definition of the function $T_{v,\mathcal{G}}$ implies that, for all $\sigma \in \text{Gal}(\bar{K}_v/K_v)$ and $\psi \in \text{Ch}(G)_v$, we have $T_{v,\mathcal{G}}(\psi) = T_{v,\mathcal{G}}(\psi^\sigma)$. Indeed, this can be derived from the equality $g(\psi, y_j) = g(\psi^\sigma, y_j^\sigma)$, where y_j^σ is the image of the irreducible component y_j under the action of $\sigma \in \text{Gal}(\bar{K}_v/K_v)$. This, in turn, follows from the identity

$$\left(\frac{g(\varpi)}{\varpi}\right)^\sigma = \frac{g(\varpi^\sigma)}{\varpi^\sigma},$$

with $g \in I_{y_j}$ and ϖ, ϖ^σ uniformizers of the local rings at y_j and y_j^σ , respectively, which is true since the G -action is ‘‘defined over R ’’. Hence, the map

$$\phi \mapsto \varpi_v^{-T_{v,\mathcal{G}}(\Theta^D(\phi))}$$

gives a function in $\text{Hom}_{\text{Gal}(\bar{K}_v/K_v)}(\text{Ch}(G^{d+2})_v, \bar{K}_v^*)$.

THEOREM 3.13. *With the above assumptions and notations,*

$$\Theta(2 \cdot \bar{\chi}^P(X, \mathcal{F})) = \left(\prod_v (R_v[G^{d+2}]\lambda_v \cap K[G^{d+2}]), 1\right),$$

where $(\lambda_v)_v \in \mathbf{A}_{f,K[G^{d+2}]}^*$ is the (unique) finite idèle such that

$$(3.15) \quad \phi(\lambda_v) = \begin{cases} 1 & \text{if } v \notin S_K, \\ \varpi_v^{-2 \cdot T_{v,\mathcal{G}}(\Theta^D(\phi))} & \text{if } v \in S_K \end{cases}$$

for all \bar{K}_v -valued characters ϕ of G^{d+2} .

If the ‘‘usual’’ Euler characteristic $\chi(Y, \mathcal{G}) = \sum_i (-1)^i \text{rank}_R[\mathbf{H}^i(Y, \mathcal{G})]$ is even, then we can eliminate both occurrences of the factor 2 from the statement: $\Theta(\bar{\chi}^P(X, \mathcal{F}))$ is then given by the idèle $(\lambda'_v)_v$ with

$$\phi(\lambda'_v) = \begin{cases} 1 & \text{if } v \notin S_K, \\ \varpi_v^{-T_{v,\mathcal{G}}(\Theta^D(\phi))} & \text{if } v \in S_K. \end{cases}$$

Proof. By (3.2) it is enough to consider the image of the class of the determinant of cohomology $\det \mathbb{R}\tilde{h}_*(\pi_*(\mathcal{F})) = \det \mathbb{R}\tilde{h}_*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{L})$ (or of its square) under Θ . Note that if we assume that $\chi(Y, \mathcal{G}) = \sum_i (-1)^i \text{rank}_R[\mathbf{H}^i(Y, \mathcal{G})]$ is even, the function given by $\text{rank}(\mathbb{R}\tilde{h}_*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{L}))$ is always even on G_R^D . Indeed, since this function is Zariski locally constant and G_R^D is connected, it is enough to check this over the generic fiber $\text{Spec}(K)$; there it follows from the Grothendieck-Riemann-Roch theorem and the fact that \mathcal{L}_K is a torsion line bundle (cf. Corollary 3.4), and hence it has trivial Chern character. Hence in this ‘‘even’’ case we do not have to consider the square.

By Remark 3.8 (cf. Lemma 3.7), it is now enough to show that, for each finite place v of K ,

$$E(\mathcal{G})R_v = R_v[G^{d+2}]\lambda_v \quad \text{in } K_v[G^{d+2}].$$

When $v \notin S_K$, $E(\mathcal{G})R_v = R_v[G^{d+2}]$ by Corollary 3.3 and the definition of $E(\mathcal{G})$. Hence, since $\lambda_v = 1$, $E(\mathcal{G})R_v = \lambda_v R_v[G^{d+2}]$. When $v \in S_K$, $(v, \#G) = 1$, and we have an isomorphism

$$R_v[G^{d+2}] \xrightarrow{\sim} \bigoplus_{\phi} R_v(\phi), \quad a \mapsto (\phi(a))_{\phi},$$

with ϕ ranging over the $\text{Gal}(\bar{K}_v/K_v)$ -orbits of \bar{K}_v -valued characters of G^{d+2} and $R_v(\phi) \subset R'_v \subset \bar{K}_v$ the complete discrete valuation ring generated over R_v by the values of ϕ . Therefore, to show the statement for $v \in S_K$, it is enough to verify that $\phi(E(\mathcal{G})R_v(\phi)) = \phi(\lambda_v)R'_v(\phi)$. For that is enough to check $\phi(E(\mathcal{G})R'_v) = \phi(\lambda_v)R'_v$. This now follows from Section 3.d (cf. Theorem 3.11), and the fact that the formation of the determinant of cohomology (and therefore also of the ideals $E(\mathcal{G})$) commutes with base change. \square

Remark 3.14. Theorem 3.13 determines $2 \cdot \bar{\chi}^P(X, \mathcal{F})$ up to an element of $\text{Ker}(\Theta_{d+2})$ from numerical information. The group $\text{Ker}(\Theta_{d+2})$ is bounded above in Theorem 2.7; in particular, it is trivial if $R = \mathbb{Z}$ and $d \leq 3$. Note, though, that this differs from the usual approach in Galois module structure theory of specifying an idèle β in $\mathbf{A}_{f,K[G]}^*$ that represents $2 \cdot \bar{\chi}^P(X, \mathcal{F})$ (e.g. modulo $\text{Ker}(\Theta_{d+2})$ or a larger subgroup of $\text{Cl}(R[G])$). In the modular form example of the next section, we produce such a β under some additional hypotheses. In general, we can produce an idèle in $\mathbf{A}_{f,K[G]}^*$ giving the class $2(\#G)^{d+1} \chi^P(X, \mathcal{F})$ modulo $\text{Ker}(\Theta_{d+2})$; see Section 3.g below.

Remark 3.15. Suppose that \mathcal{F} is a general locally free G -sheaf on X (i.e., a locally free coherent \mathbb{O}_X -module with G action compatible with the action of X) that is not necessarily of the form $\pi^*\mathcal{G}$ for an \mathbb{O}_Y -sheaf \mathcal{G} . In this remark, we explain how our methods can also lead to a calculation of $\bar{\chi}^P(X, \mathcal{F})$. Set $\mathcal{H} = (\pi_*(\mathcal{F}))^G$. Under our assumptions, this is a locally free \mathbb{O}_Y -sheaf and there is an exact sequence

$$(3.16) \quad 0 \rightarrow \pi^*(\mathcal{H}) \rightarrow \mathcal{F} \rightarrow \mathcal{Q} \rightarrow 0$$

of G -sheaves on X with \mathcal{Q} supported on the (fibral) branch locus of π . We have $\bar{\chi}^P(X, \mathcal{F}) = \bar{\chi}^P(X, \pi^*(\mathcal{H})) + \bar{\chi}^P(X, \mathcal{Q})$. The class $\bar{\chi}^P(X, \pi^*(\mathcal{H}))$ can be calculated using Theorem 3.13. On the other hand, the class $\bar{\chi}^P(X, \mathcal{Q})$ naturally lifts to a class in the torsion free group $\bigoplus_v K_0(k(v)[G])$ which can be calculated (at least when X is regular) using the Lefschetz-Riemann-Roch theorem of [BFQ79] (see [CEPT97a] for an example of such a calculation).

3.g. Let us discuss here how we can deduce Theorem 1.2. Observe that under its assumptions, π is flat by Remark 3.1(a) and we can write the short exact sequence (3.16) with \mathcal{Q} supported on the fixed point locus X' . Moreover, we can

see that \mathcal{Q} depends only on the pair $(\hat{X}', \mathcal{F}|\hat{X}')$. As above, we have $\bar{\chi}^P(X, \mathcal{F}) = \bar{\chi}^P(X, \pi^*(\mathcal{H})) + \bar{\chi}^P(X, \mathcal{Q})$. The class $\bar{\chi}^P(X, \mathcal{Q})$ is represented by a finite module of finite projective dimension of order supported on the image of X' in $\text{Spec}(\mathbb{Z})$. Hence, we can reduce the proof to the case that \mathcal{F} is of the form $\pi^*\mathcal{G}$ for \mathcal{G} a locally free coherent sheaf on Y . Then (a) follows by combining Theorem 3.13 with Theorem 2.7. To show (b), observe that the character function $\psi \mapsto (\#G)^{d+1}T_{\pi, \mathcal{G}}(\psi)$ takes values in \mathbb{Z} . (This follows from the definition of $T_{\pi, \mathcal{G}}(\psi)$ and the localized Riemann-Roch theorem.) Therefore, we can use this character function to define by a formula as in (3.15) an idèle in $\mathbf{A}_{f, \mathbb{Q}[G]}^*$ whose image under the map Θ_{d+2} is $2(\#G)^{d+1} \cdot \bar{\chi}^P(X, \mathcal{F})$. By construction, this idèle represents a finite module of finite projective dimension whose order is supported on the primes below the branch locus and whose class in $\text{Cl}(\mathbb{Z}[G])$ agrees with $2(\#G)^{d+1} \cdot \bar{\chi}^P(X, \mathcal{F})$ modulo $\ker(\Theta_{d+2})$. The result now follows from Theorem 2.7.

4. Galois structure of modular forms

The object of this section is to prove Theorem 1.3. With the assumptions of that theorem, let $\mathfrak{M}_{\Gamma_1(p)}$ be the moduli stack classifying triples (E, C, γ) , where $E \rightarrow S$ is a generalized elliptic curve (see [DR73]), C is a locally free rank p subgroup of the smooth locus E^{sm} , and $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \rightarrow C^D$ is a “generator” of the Cartier dual of C ; we ask that C intersects every irreducible component of every geometric fiber of $E \rightarrow S$. (Here generator is meant in the sense of [KM85, Ch. 1]. Notice that a group scheme embedding $\mu_p \hookrightarrow E^{\text{sm}}$ gives data C and γ ; in fact, if p is invertible on S , giving C together with γ as above exactly amounts to giving a group scheme embedding $\mu_p \hookrightarrow E^{\text{sm}}$.) We denote by $X_1 = X_1(p)$ the corresponding coarse moduli scheme over $\text{Spec}(\mathbb{Z})$. The group $(\mathbb{Z}/p\mathbb{Z})^*$ acts on $\mathfrak{M}_{\Gamma_1(p)}$ via

$$(4.1) \quad (a \bmod p) \cdot (E, C, \gamma) = (E, C, \gamma \circ a^{-1}).$$

(When p is invertible, this action sends the corresponding $j : \mu_p \hookrightarrow E^{\text{sm}}$ to the composition $j \circ (z \mapsto z^a) : \mu_p \hookrightarrow E^{\text{sm}}$.) This produces a faithful $\Gamma = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ -action on X_1 . When H is a subgroup of Γ , we let X_H be the quotient X_1/H , and we define $X_H[1/p] = \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} X_H$. Set $X_0 = X_{\Gamma}$.

The Tate curve $\bar{\mathcal{G}}_m/q^{\mathbb{Z}}$ over $\text{Spec}(\mathbb{Z}[[q]])$ together with the embedding $\mu_p \subset \bar{\mathcal{G}}_m/q^{\mathbb{Z}}$ (see [DR73, VII]) gives a morphism $\tau : \text{Spec}(\mathbb{Z}[[q]]) \rightarrow X_H$. We call the support of the corresponding section $\text{Spec}(\mathbb{Z}) \rightarrow \text{Spec}(\mathbb{Z}[[q]]) \rightarrow X_H$ the ∞ cusp. Over \mathbb{C} , provided we trivialize $\mu_p(\mathbb{C})$ via $\zeta_p = e^{2\pi i/p}$, this corresponds to the “usual” ∞ cusp and the parameter q corresponds to $e^{2\pi iz}$ with z in the upper half plane \mathfrak{H} . The morphism τ identifies $\text{Spec}(\mathbb{Z}[[q]])$ with the formal completion of X_H along ∞ .

Remark 4.1. (a) Alternatively, we can define $\mathfrak{M}_{\Gamma_1(p)}$ by considering the non-proper moduli stack of triples (E, C, γ) as above with $E \rightarrow S$ an “honest” (i.e., smooth) elliptic curve and then take its normalization over the affine j -line (compare to the approach in [DR73] and [KM85]).

(b) $\mathfrak{M}_{\Gamma_1(p)}$ is a variant of the following moduli stack, which is used in [CES03]: Let $\mathfrak{M}'_{\Gamma_1(p)}$ be the moduli stack of triples (E, Σ, γ') where $E \rightarrow S$ is a generalized elliptic curve, Σ a locally free rank p subgroup of the smooth locus E^{sm} that intersects every irreducible component of every geometric fiber of $E \rightarrow S$, and $\gamma' : (\mathbb{Z}/p\mathbb{Z})_S \rightarrow \Sigma$ a “generator” of Σ . If (E, C, γ) is a point of $\mathfrak{M}_{\Gamma_1(p)}$ with E an honest elliptic curve, then the Weil pairing induces a canonical isomorphism $C^D \xrightarrow{\iota} E[p]/C$ (here $E[p]$ is the p -torsion subgroup scheme of E). The map $(E, C, \gamma) \mapsto (E/C, E[p]/C, \gamma')$ with $\gamma' := \iota \circ \gamma$ induces an isomorphism of moduli stacks on the complement of the cusps; this extends (by normalization) to an isomorphism $\mathfrak{M}_{\Gamma_1(p)} \xrightarrow{\sim} \mathfrak{M}'_{\Gamma_1(p)}$ which in turn induces an isomorphism between the corresponding coarse moduli schemes (compare [CES03, p. 381]).

Since we have assumed $p \equiv 1 \pmod{24}$, the genus g_0 of $(X_0)_{\mathbb{C}}$ is $(p - 13)/12$. The work of Deligne and Rapoport [DR73], Katz and Mazur [KM85], and Conrad, Edixhoven and Stein [CES03] implies the following two results.

THEOREM 4.2. (a) *The scheme $X_H \rightarrow \text{Spec}(\mathbb{Z})$ is a flat projective curve, X_H is normal and Cohen-Macaulay, and $X_H[1/p] \rightarrow \text{Spec}(\mathbb{Z}[1/p])$ is smooth. The special fiber of X_H over p has two irreducible components D_{∞}^H and D_0^H distinguished by the fact that D_{∞}^H intersects the cuspidal section ∞ ; these have multiplicities 1 and $(p - 1)/(2 \cdot \#H)$, respectively.*

(b) *The scheme X_H has at most two nonregular points that are rational singularities and lie on $D_0^H - (D_0^H \cap D_{\infty}^H)$. Their exact number depends on $\#H \pmod{6}$: In particular, if 6 divides $\#H$, there are no such points and X_H is then regular. If $H = \{1\}$, there are two nonregular points on X_1 . There is a morphism $b : X'_1 \rightarrow X_1$ that is a rational resolution of those two singular points and a morphism $c : X'_1 \rightarrow \mathcal{X}_1$ that is a sequence of blow-downs of exceptional curves such that \mathcal{X}_1 is regular and all the geometric fibers of $\mathcal{X}_1 \rightarrow \text{Spec}(\mathbb{Z})$ are integral. Let $U = X_1 - D_0^{\{1\}} \subset X_1$. Then $U \rightarrow \text{Spec}(\mathbb{Z})$ is smooth, b and c are isomorphisms on $b^{-1}(U)$, and $\mathcal{X}_1 - c(b^{-1}(U))$ has dimension 0.*

(c) *The special fiber of $X_0 = X_{\Gamma}$ over p is reduced with simple normal crossings. Each of the two irreducible components $D_{\infty} = D_{\infty}^{\Gamma}$ and $D_0 = D_0^{\Gamma}$ are isomorphic to $\mathbf{P}_{\mathbb{F}_p}^1$, and $D_0 \cdot D_{\infty} = g_0 + 1 = (p - 1)/12$.*

Proof. Parts (a) and (b) follow from [CES03, Thms. 4.1.1 and 4.2.6]. (See [CES03, § 5.3]; the reader should be aware that [CES03] uses the variant moduli problem of Remark 4.1(b).) Part (c) follows from [DR73, VI.6.16]. □

THEOREM 4.3. *Assume that 6 divides the order #H.*

- (a) *The morphism $\pi_H : X_H \rightarrow X_0$ is a tame $G = \Gamma/H$ cover of regular projective curves, and $\pi_H[1/p] : X_H[1/p] \rightarrow X_0[1/p]$ is a G -torsor.*
- (b) *The morphism π_H is totally ramified over the generic point of D_0 , and unramified over the generic point of D_∞ . The irreducible components D_0^H and D_∞^H of $X_H \otimes_{\mathbb{Z}} \mathbb{F}_p$ are the (reduced) inverse images of D_0 and D_∞ under π_H . The character $\chi_{D_0^H}$ giving the action of G on the cotangent space of the codimension 1 generic point of D_0^H equals $\omega^{-2\#H}$, where $\omega : (\mathbb{Z}/p\mathbb{Z})^* \rightarrow \mathbb{F}_p^*$ is the Teichmüller (identity) character.*

Proof. The fact that $\pi_H[1/p]$ is a G -torsor follows from [Maz77, II.1]; the rest of part (a) then follows from Theorem 4.2. It remains to show part (b). We can see from the references above that the geometric closed points of $D_0^{\{1\}}$, resp. $D_\infty^{\{1\}}$, that correspond to ordinary elliptic curves are given by triples $(E, \mathbb{Z}/p\mathbb{Z} \subset E, 0)$, resp. $(E, \mu_p \subset E, \gamma)$, with $\gamma : \mathbb{Z}/p\mathbb{Z} \xrightarrow{\sim} \mu_p^D = \mathbb{Z}/p\mathbb{Z}$. Since by Theorem 4.2(a), D_∞^H has multiplicity 1, π_H is unramified over the generic point of D_∞ . It remains to consider D_0^H ; our claim is a statement about the completion of the local ring at the generic point of D_0^H . It is enough to show this statement after a base change from \mathbb{Z} to the Witt ring $W = W(\overline{\mathbb{F}}_p)$ of an algebraic closure of \mathbb{F}_p .

Let E be an ordinary elliptic curve over $\overline{\mathbb{F}}_p$. The pair $(E, \mathbb{Z}/p\mathbb{Z} \subset E, 0)$ corresponds to a point s of $D_0^{\{1\}}$. Let \mathcal{R}_s be the formal deformation ring of the point $(E, \mathbb{Z}/p\mathbb{Z} \subset E, 0)$ in the moduli stack $\mathcal{M}_{\Gamma_1(p)}$. Then \mathcal{R}_s supports an action of $\Delta \times (\mathbb{Z}/p\mathbb{Z})^*$ where $\Delta = \text{Aut}(E)$. Let H' be the inverse image of H under the surjection $(\mathbb{Z}/p\mathbb{Z})^* \rightarrow \Gamma$, and let s' and s'' be the images of s on X_H and X_0 , respectively. The completions of the local rings are

$$(4.2) \quad \widehat{\mathcal{O}}_{X_H \otimes_{\mathbb{Z}} W, s'} \simeq (\mathcal{R}_s)^{\Delta \times H'} \quad \text{and} \quad \widehat{\mathcal{O}}_{X_0 \otimes_{\mathbb{Z}} W, s''} \simeq (\mathcal{R}_s)^{\Delta \times (\mathbb{Z}/p)^*}$$

as rings with $G = (\mathbb{Z}/p\mathbb{Z})^*/H'$ -action. Using [CES03, Th. 3.3.3] and the proof of [CES03, Th. 4.1.1] (or alternatively a direct calculation using the description of the scheme of generators of the multiplicative group scheme μ_p given in [KM85, II]), we can deduce the following: There is a $\Delta \times (\mathbb{Z}/p\mathbb{Z})^*$ -isomorphism

$$(4.3) \quad \mathcal{R}_s \simeq W[[v, u]]/(v^{p-1} - p),$$

where the action on the right side is as follows: $\delta \in \Delta$ acts via $\delta \cdot v = \psi(\delta)v$ and $\delta \cdot u = \psi^2(\delta)u$ with ψ a faithful character of Δ , while $a \in (\mathbb{Z}/p\mathbb{Z})^*$ acts via $a \cdot v = \omega^{-1}(a)v$ and $a \cdot u = u$. Assume now that E is an ordinary elliptic curve over $\overline{\mathbb{F}}_p$ with $j \neq 0, 1728$. Then $\Delta = \{\pm 1\}$. Part (b) for D_0^H follows then directly from the above and (4.2). □

PROPOSITION 4.4. *Let H be an arbitrary subgroup of Γ . The group $H^0(X_H, \mathbb{C}_{X_H})$ is isomorphic to \mathbb{Z} with trivial Γ/H -action, and $H^1(X_H, \mathbb{C}_{X_H})$ is \mathbb{Z} -torsion free.*

Proof. We have $H^0(X_H, \mathbb{C}_{X_H}) \simeq \mathbb{Z}$ because $X_H \otimes_{\mathbb{Z}} \mathbb{C}$ is connected and $X_H \rightarrow \text{Spec}(\mathbb{Z})$ is projective and flat. The claim about $H^1(X_H, \mathbb{C}_{X_H})$ follows from Theorem 4.2(a) and [BLR90, 9.7/1] (see Proposition 5.3). \square

If R is a subring of \mathbb{C} , we will denote by $S_2(\Gamma_1(p), R)$ the R -module of cusp forms $F(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$ for the congruence subgroup $\Gamma_1(p) \subset \text{PSL}_2(\mathbb{Z})$ whose Fourier coefficients a_n belong to R . (These are the Fourier coefficients “at the cusp ∞ ”.) If M is a finitely generated $\mathbb{Z}[\Gamma]$ -module, let M^\vee be the $\mathbb{Z}[\Gamma]$ module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ with Γ action $(af)(z) = f(a^{-1}z)$ for $a \in \Gamma$, $f \in M^\vee$, and $z \in M$.

PROPOSITION 4.5. *There are Γ -equivariant isomorphisms*

$$S_2(\Gamma_1(p), \mathbb{Z}) \simeq H^0(X_1, \omega_{X_1/\mathbb{Z}}) \simeq H^1(X_1, \mathbb{C}_{X_1})^\vee,$$

where the Γ -action on $S_2(\Gamma_1(p), \mathbb{Z})$ is via the diamond operators and $\omega_{X_1/\mathbb{Z}}$ denotes the canonical (dualizing) sheaf of $X_1 \rightarrow \text{Spec}(\mathbb{Z})$.

Proof. The Γ -isomorphism

$$H^0(X_1, \omega_{X_1/\mathbb{Z}}) \simeq H^1(X_1, \mathbb{C}_{X_1})^\vee$$

follows from duality [DR73, I (2.1.1)]. Let $G(q) = \sum_{n \geq 1} a_n q^n \in S_2(\Gamma_1(p), \mathbb{Z})$ with $q = e^{2\pi i n z}$ and consider $G(q)dq/q$ as a regular differential over $\text{Spec}(\mathbb{Z}[[q]])$. A standard argument using the Kodaira-Spencer map shows that $G(q)dq/q$ extends to a regular differential over $X_1[1/p]$ (see for example [Maz77, § II.4]). This extension must also be regular in an open neighborhood of the section at ∞ . Hence there is an open subset U' of the set $U \subset X_1$ defined in Theorem 4.2(b) such that $G(q)dq/q$ is regular on U' and $U - U'$ is a finite set of closed points. We obtain an injective Γ -equivariant homomorphism

$$(4.4) \quad \Phi : S_2(\Gamma_1(p), \mathbb{Z}) \rightarrow H^0(U', \omega_{U'/\mathbb{Z}}) = H^0(\mathcal{X}_1, \omega_{\mathcal{X}_1/\mathbb{Z}}) = H^0(X_1, \omega_{X_1/\mathbb{Z}}),$$

where the latter two equalities follow from the fact that $b : X'_1 \rightarrow X_1$ and $c : X'_1 \rightarrow \mathcal{X}_1$ are rational morphisms that are isomorphisms on $b^{-1}(U')$, and that $\mathcal{X}_1 - c(b^{-1}(U'))$ has codimension 2 in \mathcal{X}_1 . The surjectivity of Φ follows from pulling back elements of $H^0(X_1, \omega_{X_1/\mathbb{Z}})$ via $\tau : \text{Spec}(\mathbb{Z}[[q]]) \rightarrow U$ and using the Kodaira-Spencer isomorphism. \square

PROPOSITION 4.6. *We have $G = \Gamma/H$ -isomorphisms*

$$S_2(\Gamma_H(p), \mathbb{Z}) := S_2(\Gamma_1(p), \mathbb{Z})^H \simeq H^1(X_H, \mathbb{C}_{X_H})^\vee.$$

Proof. Denote by M_H the H -coinvariants of a Γ -module M . We then have a Γ/H -isomorphism $(M_H)^\vee \simeq (M^\vee)^H$. In view of Proposition 4.5, it will be enough to exhibit a $\mathbb{Z}[\Gamma/H]$ -isomorphism

$$(4.5) \quad \mathbf{H}^1(X_1, \mathcal{O}_{X_1})_H^\vee \simeq \mathbf{H}^1(X_H, \mathcal{O}_{X_H})^\vee.$$

Let $\mu : X_1 \rightarrow X_H$ be the quotient morphism. Since taking H -coinvariants is a right exact functor and the coherent cohomology groups $\mathbf{H}^i(X_H, \cdot)$ are trivial when $i \geq 2$, we obtain

$$(4.6) \quad \mathbf{H}^1(X_1, \mathcal{O}_{X_1})_H \simeq \mathbf{H}^1(X_H, \mu_*\mathcal{O}_{X_1})_H \simeq \mathbf{H}^1(X_H, (\mu_*\mathcal{O}_{X_1})_H).$$

The element $\text{Tr} = \sum_{h \in H} h$ of the group ring $\mathbb{Z}[H]$ acts on $\mu_*\mathcal{O}_{X_1}$ to give a Γ/H -equivariant morphism of \mathcal{O}_{X_H} -sheaves given by

$$(4.7) \quad (\mu_*\mathcal{O}_{X_1})_H \xrightarrow{\text{Tr}} \mathcal{O}_{X_H} = (\mu_*\mathcal{O}_{X_1})^H.$$

Since the stalks of $\mu_*\mathcal{O}_{X_1}$ are H -cohomologically trivial at points where the ramification of $\mu : X_1 \rightarrow X_H$ is tame (see [CEPT96]), the cokernel \mathcal{C} and kernel \mathcal{K} of Tr are coherent and supported on the subset of X_H over which the cover μ has wild ramification. By [Maz77, § II.2] this is (at most) a finite set of points in characteristics 2 and 3. Hence the cohomology groups of \mathcal{C} and \mathcal{K} vanish in positive dimensions. It follows that Tr induces a surjection $\mathbf{H}^1(X_H, (\mu_*\mathcal{O}_{X_1})_H) \rightarrow \mathbf{H}^1(X_H, \mathcal{O}_{X_H})$ with finite kernel, and so (4.5) follows from (4.6). \square

The following result is a corollary of a theorem of Rim [Rim59].

PROPOSITION 4.7. *Let $\chi : \Gamma \rightarrow \mathbb{Z}[\zeta_r]^*$ be a 1-dimensional character of prime order $r \geq 5$ with kernel H . Let $G = \Gamma/H$ and suppose M is a finitely generated torsion-free $\mathbb{Z}[G]$ -module. Define $M^{\vee, \chi}$ to be the $\mathbb{Z}[\zeta_r]$ -module $(M^\vee \otimes \mathbb{Z}[\zeta_r]\chi^{-1})^G$.*

- (a) *There is a unique homomorphism $e'_\chi : \mathbf{G}_0(\mathbb{Z}[G]) \rightarrow \text{Cl}(\mathbb{Z}[\zeta_r])$ such that for all M as above, either $M^{\vee, \chi} = \{0\}$ and $e'_\chi([M]) = 0$ or $M^{\vee, \chi}$ is isomorphic to $\mathbb{Z}[\zeta_r]^s \oplus \mathfrak{A}$ for some integer $s \geq 0$ and a $\mathbb{Z}[\zeta_r]$ -ideal \mathfrak{A} in the ideal class $e'_\chi([M])$.*
- (b) *There is a unique isomorphism $t_\chi : \mathbf{K}_0(\mathbb{Z}[G]) \rightarrow \mathbb{Z} \oplus \text{Cl}(\mathbb{Z}[\zeta_r])$ such that $t_\chi([P]) = (\text{rank}_{\mathbb{Z}[G]}(P), e_\chi(\overline{[P]}))$ if P is a projective $\mathbb{Z}[G]$ -module, where $\overline{[P]}$ is the image of P in $\text{Cl}(\mathbb{Z}[G])$ and $e_\chi : \text{Cl}(\mathbb{Z}[G]) \rightarrow \text{Cl}(\mathbb{Z}[\zeta_r])$ is the unique homomorphism such that $e_\chi(\overline{[P]}) = e'_\chi(f([P]))$ for all projective P , where $f : \mathbf{K}_0(\mathbb{Z}[G]) \rightarrow \mathbf{G}_0(\mathbb{Z}[G])$ is the forgetful homomorphism.*

Proof of Theorem 1.3

With the notations of the theorem, recall that $S_2(\Gamma_1(p), \mathbb{Z}[\zeta_r])_\chi$ is the $\mathbb{Z}[\zeta_r]$ -submodule of $S_2(\Gamma_1(p), \mathbb{Z}[\zeta_r])$ consisting of cusp forms of weight 2 and of Nebentypus character χ whose Fourier coefficients at ∞ are in $\mathbb{Z}[\zeta_r]$. Proposition 4.5 and

its proof together with the fact that formation of the canonical sheaf commutes with the base change $\mathbb{Z} \rightarrow \mathbb{Z}[\zeta_r]$ implies that

$$S_2(\Gamma_1(p), \mathbb{Z}[\zeta_r]) \simeq S_2(\Gamma_1(p), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_r].$$

Propositions 4.5 and 4.6 now give an isomorphism

$$(4.8) \quad S_2(\Gamma_1(p), \mathbb{Z}[\zeta_r])_{\chi} \simeq H^1(X_H, \mathbb{O}_{X_H})^{\vee, \chi}$$

of (torsion free) $\mathbb{Z}[\zeta_r]$ -modules. (Here we use the notation of Proposition 4.7.)

The projective class $\chi^P(X_H, \mathbb{O}_{X_H}) \in K_0(\mathbb{Z}[G])$ has the property that

$$(4.9) \quad f(\chi^P(X_H, \mathbb{O}_{X_H})) = [H^0(X_H, \mathbb{O}_{X_H})] - [H^1(X_H, \mathbb{O}_{X_H})]$$

where $f : K_0(\mathbb{Z}[G]) \rightarrow G_0(\mathbb{Z}[G])$ is the forgetful homomorphism. If P is a projective $\mathbb{Z}[G]$ -module, then $\mathbb{Q} \otimes_{\mathbb{Z}} P$ is a free $\mathbb{Q}[G]$ -module, so

$$\text{rank}_{\mathbb{Z}[\zeta_r]}(P^{\vee, \chi}) = \text{rank}_{\mathbb{Z}[G]}(P) = \text{rank}_{\mathbb{Z}}(P)/r.$$

Therefore (4.9) implies

$$(4.10) \quad \text{rank}_{\mathbb{Z}[\zeta_r]}(H^0(X_H, \mathbb{O}_{X_H})^{\vee, \chi}) - \text{rank}_{\mathbb{Z}[\zeta_r]}(H^1(X_H, \mathbb{O}_{X_H})^{\vee, \chi}) \\ = (1 - g(X_H))/r,$$

where $g(X_H)$ is the genus of X_H and $H^0(X_H, \mathbb{O}_{X_H}) \simeq \mathbb{Z}$ with trivial G -action by Proposition 4.4. Thus $H^0(X_H, \mathbb{O}_{X_H})^{\vee, \chi} = 0$ since χ is nontrivial. Because the generic fiber of $X_H \rightarrow X_0$ is étale of degree r , we now conclude from (4.8) and the Hurwitz theorem that

$$n(\chi) := \text{rank}_{\mathbb{Z}[\zeta_r]}(S_2(\Gamma_1(p), \mathbb{Z}[\zeta_r])_{\chi}) = (g(X_H) - 1)/r = g(X_0) - 1.$$

Since $p \equiv 1 \pmod{24}$, we have $g(X_0) = (p - 13)/12$, so $n(\chi) = (p - 25)/12$ as stated in Theorem 1.3.

Recall that $\bar{\chi}^P(X_H, \mathbb{O}_{X_H})$ is the image of $\chi^P(X_H, \mathbb{O}_{X_H})$ in $\text{Cl}(\mathbb{Z}[G])$. We conclude from (4.8), (4.9) and Propositions 4.4 and 4.7 that there is an isomorphism of $\mathbb{Z}[\zeta_r]$ -modules

$$(4.11) \quad S_2(\Gamma_1(p), \mathbb{Z}[\zeta_r])_{\chi} \simeq \mathbb{Z}[\zeta_r]^{n(\chi)-1} \oplus \mathfrak{U},$$

where \mathfrak{U} is a $\mathbb{Z}[\zeta_r]$ -ideal having ideal class $-e_{\chi}(\bar{\chi}^P(X_H, \mathbb{O}_{X_H}))$.

We now compute the image of $\bar{\chi}^P(X_H, \mathbb{O}_{X_H})$ under the homomorphism $\Theta = \Theta_3 : \text{Cl}(\mathbb{Z}[G]) \rightarrow C_{\mathbb{Z}}(G, 3)$ by applying our main result, Theorem 3.13, to the cover $\pi_H : X_H \rightarrow X_0$. (Notice that $g(X_0)$ is odd; hence, we can use the version of the theorem that does not include the factor of 2.) Since the index of H in Γ is the prime $r \geq 5$, the order of H is divisible by 6. By Theorem 4.3, π_H is ramified only at the fiber over p . The field \mathbb{Q}_p already contains a primitive $(p - 1)$ -st root of unity. Hence, we may take $R'_{(p)} = \mathbb{Z}_p$. We find that $\Theta(\bar{\chi}^P(X_H, \mathbb{O}_{X_H})) \in C_{\mathbb{Z}}(G; 3)$

is given by the idèle $(b_v)_v \in \mathbf{A}_{f, \mathbb{Q}[G^3]}^*$ that is 1 at all places $v \neq (p)$ and is such that

$$(4.12) \quad (\chi \otimes \phi \otimes \psi)(b_{(p)}) = p^{-T((\chi-1)(\phi-1)(\psi-1))}$$

with $T : \text{Ch}(G)_p \rightarrow \mathbb{Q}$ the function associated to the cover $X_H \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow X_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in Theorem 3.13. For $a \in \mathbb{Z}/r\mathbb{Z}$, let $\{a\}$ be the unique integer in the range $0 \leq \{a\} < r$ having residue class a . By Theorem 4.2(c) and Theorem 4.3, the expression in (3.13) becomes

$$(4.13) \quad \begin{aligned} T(\psi) &= \frac{1}{12}(1-p) \cdot \left(\frac{1}{2}g(\psi, D_0)^2 + \frac{1}{2}g(\psi, D_0)\right) + g(\psi, D_0) \\ &= \frac{1}{12}(1-p) \left(\frac{1}{2}\{a\}_r^2/r^2 - \frac{1}{2}\{a\}_r/r\right) - \{a\}_r/r \end{aligned}$$

when $\psi = \chi_0^{-a}$, $\chi_0 = \omega^{(p-1)/r}$, and $\omega : (\mathbb{Z}/p\mathbb{Z})^* \rightarrow \mathbb{Z}_p^*$ is the Teichmüller character.

For $a \in \mathbb{Z}/r\mathbb{Z}$ define $\omega_r(a) = 0$ if $a = 0$, and otherwise let $\omega_r(a) \in \mathbb{Z}_r \subset \hat{\mathbb{Z}}$ be the Teichmüller character associated to r . Define

$$(4.14) \quad T_1(\psi) = \frac{1}{12}(1-p) \left(\frac{1}{2}\omega_r(a)^2/r^2\right) \quad \text{and} \quad T_2(\psi) = -\{a\}_r/r$$

when $\psi = \chi_0^{-a}$ as above. We extend $\psi \mapsto T_i(\psi)$ to a function on the character ring $\text{Ch}(G)_p$ by additivity. Since $p \equiv 1 \pmod{24}$ and $r \mid \frac{1}{24}(1-p)$, we can define $\beta = (\beta_v)_v$ with $\beta_v \in \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_v[G]^*$ by

$$(4.15) \quad \psi(\beta_v) = \begin{cases} 1 & \text{if } v \neq (p), \\ p^{-T(\psi)+T_1(\psi)+T_2(\psi)} & \text{if } v = (p). \end{cases}$$

Since $\text{Cl}(\mathbb{Z}[G])$ is a torsion group, β defines a unique class $[\beta]$ in $\text{Cl}(\mathbb{Z}[G])$.

We now show

$$(4.16) \quad \bar{\chi}^P(X_H, \mathbb{O}_{X_H}) = [\beta].$$

Define $D = [\beta] - \bar{\chi}^P(X_H, \mathbb{O}_{X_H})$, and let $R = \hat{\mathbb{Z}}$ if $i = 1$ and $R = \mathbb{Z}$ if $i = 2$. From (4.14) one has $rT_i(\psi) \in R$ and $rT_i(\psi) \equiv a \pmod{rR}$. It follows that $T_i(\chi\phi\psi - \chi\phi - \phi\psi - \chi\psi + \chi + \phi + \psi - 1)$ lies in R for all triples (χ, ϕ, ψ) of elements of $\text{Ch}(G)_p$. Hence there are elements $a_i = (a_{i,v})_v \in \prod_{v \text{ finite}} (R \otimes_{\mathbb{Z}} \mathbb{Q}_v[G^3])^*$ for which

$$(4.17) \quad (\chi \otimes \phi \otimes \psi)(a_{i,v}) = \begin{cases} 1 & \text{if } v \neq (p), \\ p^{T_i(\chi\phi\psi - \chi\phi - \phi\psi - \chi\psi + \chi + \phi + \psi - 1)} & \text{if } v = (p). \end{cases}$$

We now conclude from (4.12), (4.13) and (4.15) that

$$(4.18) \quad 1 \otimes \Theta(D) = c_1 + c_2$$

in $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} C_{\mathbb{Z}}(G; 3)$, where c_i is the class associated to the element a_i .

Let us first show

$$(4.19) \quad c_2 = 0.$$

For this it will suffice to show that there is a cubic element $\lambda \in \mathbb{Q}[G^3]^*$ such that λa_2 is a unit idèle of $\mathbb{Q}[G^3]$. Fix a primitive p -th root of unity $\zeta_p \in \overline{\mathbb{Q}}^*$, and let

$$\tau(\psi) = \sum_{j \in (\mathbb{Z}/p)^*} \psi(j) \zeta_p^j$$

be the usual Gauss sum associated to ψ . Let τ be the unique extension of the map $\psi \mapsto \tau(\psi)$ to a homomorphism from R_G to $\overline{\mathbb{Q}}^*$. We let $\tau^{(3)}$ be the element of $\text{Hom}(R_{G^3}, \overline{\mathbb{Q}}^*)$ that sends (χ, ϕ, ψ) to

$$\tau(\chi\phi\psi - \chi\phi - \phi\psi - \chi\psi + \chi + \phi + \psi - 1).$$

From the behavior of Gauss sums under automorphisms of $\overline{\mathbb{Q}}$, and the factorization of the ideals they generate (cf. [Lan94, § IV.3]), it follows that $\tau^{(3)}$ is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -equivariant, and corresponds to an element $\lambda \in \mathbb{Q}[G^3]^*$ of the required kind. This shows (4.19).

Turning now to c_1 , let $\sigma(s)$ be the automorphism of G that sends $g \in G$ to g^s for $s \in (\mathbb{Z}/r\mathbb{Z})^*$. By (2.13) the action of $\text{Aut}(G)$ on $\mathbb{Z}_p[G]^*$ corresponds to the action of $\text{Aut}(G)$ on $f \in \text{Hom}(\text{Ch}(G)_p, \mathbb{Q}_p^*)$ defined by

$$(\sigma(s)(f))(\chi) = f(\sigma(s)^{-1}(\chi)) = f(\chi^s) \quad \text{for } \chi \in \text{Ch}(G)_p.$$

From the definition of the T_1 in (4.14) and the multiplicativity of the Teichmüller character, we have $(\sigma(s)T')(\psi) = \omega_r(s)^2 T'(\psi)$. It follows that the element $\alpha(s) = \sigma(s) - \omega_r(s)^2$ sends a_1 to the identity function, so

$$(4.20) \quad \alpha(s)c_1 = 0.$$

Because Θ is $\text{Aut}(G)$ -equivariant, we can now conclude from (4.18), (4.19) and (4.20) that

$$(4.21) \quad 1 \otimes (\Theta(\alpha(s) \cdot D)) = 0 \quad \text{in } \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} C_{\mathbb{Z}}(G; 3).$$

If $A \rightarrow B$ is an injection of abelian groups and A is finite, then $A = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} A \rightarrow \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} B$ is injective, as one sees by reducing to the case in which $A \rightarrow B$ is the inclusion $n^{-1}\mathbb{Z}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ for some $n \geq 1$. So (4.21) and the injectivity of Θ implies

$$(4.22) \quad \alpha(s) \cdot D = 0 \quad \text{in } \text{Cl}(\mathbb{Z}[G]).$$

Similarly, since $r^2 T_1(\psi)$ is in $\mathbb{Z}_r \subset \hat{\mathbb{Z}}$ and $\text{Cl}(\mathbb{Z}[G])$ is a torsion group, we see from the injectivity of Θ that D is in the r -Sylow subgroup of $\text{Cl}(\mathbb{Z}[G])$.

We now use the fact that $\text{Cl}(\mathbb{Z}[G])$ is isomorphic to $\text{Cl}(\mathbb{Z}[\zeta_r])$. Define C_j to be the group of classes c in the r -Sylow subgroup of $\text{Cl}(\mathbb{Z}[\zeta_r])$ for which $(\sigma(s) - \omega_r(s)^j)(c) = 0$ for all $s \in (\mathbb{Z}/r)^*$. We have shown a_1 corresponds to a

class in C_2 . By the Spiegelungssatz (cf. [Was82, Th. 10.9]), $C_2 = 0$ if $C_{-1} = 0$. Herbrand’s theorem [Was82, Th. 6.17] shows that if $C_{-1} \neq 0$, then the Bernoulli number $B_{r-(r-2)} = B_2$ is congruent to 0 mod $r\mathbb{Z}_r$. This is impossible since $B_2 = \frac{1}{6}$ and $r \geq 5$, so we have shown (4.16).

In view of (4.11) and (4.16), the proof of Theorem 1.3 reduces to showing

$$(4.23) \quad -e_\chi([\beta]) = \theta_2 \cdot [\mathcal{P}_\chi]$$

when θ_2 and \mathcal{P}_χ are as defined in Theorem 1.3. By applying a suitable element of $\Delta = \text{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q})$ to both sides, we can reduce to the case in which $\chi = \chi_0 = \omega^{(p-1)/r}$. From the definitions of e_χ in Proposition 4.7, β in (4.15), and of θ_2 and \mathcal{P}_χ , we see that

$$-e_{\chi_0}([\beta]) = \theta_2 \cdot [\mathcal{P}_{\chi_0}] - \frac{p-1}{24r} \theta_1 \cdot [\mathcal{P}_{\chi_0}],$$

where $\theta_1 = \sum_{a \in (\mathbb{Z}/r)^*} \{a\} \sigma_a^{-1} \in \mathbb{Z}[\Delta]$. By Stickelberger’s theorem, θ_1 annihilates $\text{Cl}(\mathbb{Z}[\zeta_r])$, so the proof is complete. \square

Remark 4.8. Our techniques can be used to obtain a result for cusp forms of (general) square free level N . In the general case, the moduli theoretic model for $X_0(N)$ is not regular over \mathbb{Z} . Hence, as in Remark 3.1(b) we need to deal with the cover obtained by the normalization of the modular curve $(X_1(N))_{\mathbb{Q}}$ over a regular resolution of $X_0(N)$. The details, which are somewhat involved, will appear in future work.

5. An equivariant Birch and Swinnerton-Dyer relation

Suppose that G is a finite abelian group, and let $V \rightarrow W$ be a G -cover of smooth projective, geometrically connected curves of positive genus over \mathbb{Q} . We will denote by $A = \text{Jac}(V)$ the Jacobian variety of V (an abelian variety over \mathbb{Q}); the group G acts on A by (Picard) functoriality. Let $A(\mathbb{Q})$ and $\text{III}(A)$ denote the Mordell-Weil group and the Tate-Shafarevich group of A over \mathbb{Q} . Let $H_1(A(\mathbb{C}), \mathbb{Z})^+$ be the subgroup of the (singular) homology group $H_1(A(\mathbb{C}), \mathbb{Z})$ that is fixed by the action of complex conjugation on $A(\mathbb{C})$. Define \mathcal{A} to be the Néron model of A over $\text{Spec}(\mathbb{Z})$; this supports a natural G -action that extends the action of G on $A = \mathcal{A} \times_{\mathbb{Z}} \mathbb{Q}$. Consider the tangent space $\text{Lie}(\mathcal{A})$ of \mathcal{A} at the origin section $e : \text{Spec}(\mathbb{Z}) \rightarrow \mathcal{A}$. $\text{Lie}(\mathcal{A})$ is a finitely generated $\mathbb{Z}[G]$ -module which is free as a \mathbb{Z} -module. For a prime number l , we denote by $\Phi_l(A)$ the group of \mathbf{F}_l -points of the (finite) group scheme of connected components of the fiber of \mathcal{A} over \mathbf{F}_l . We also denote by $\Phi_\infty(A)$ the group of connected components $A(\mathbf{R})/A(\mathbf{R})^0$ of the Lie group $A(\mathbf{R})$. Suppose R is a Dedekind ring. Denote by $x \mapsto x^\vee$ the involution on $G_0(R[G])$ defined in the following way. If M is a finitely generated $R[G]$ -module,

there is a short exact sequence

$$(5.1) \quad 0 \rightarrow L_1 \longrightarrow L_0 \longrightarrow M \rightarrow 0$$

of $R[G]$ -modules in which L_0 and L_1 are R -torsion free. We set

$$[M]^\vee = [\mathrm{Hom}_R(L_0, R)] - [\mathrm{Hom}_R(L_1, R)].$$

(Here the G -structure on $\mathrm{Hom}_R(L, R)$ is given as usual by $(g \cdot f)(l) = f(g^{-1}l)$.)

CONJECTURE 5.1. *Assume that the Tate-Shafarevich group $\mathrm{III}(A)$ is finite. Then the identity*

$$(5.2) \quad [\mathrm{Lie}(\mathcal{A})] - [\mathrm{H}_1(A(\mathbb{C}), \mathbb{Z})^+] + [\Phi_\infty(A)] \\ + \sum_l [\Phi_l(A)] + [A(\mathbb{Q})]^\vee - [A(\mathbb{Q})] + [\mathrm{III}(A)] = 0$$

is true in the Grothendieck group $G_0(\mathbb{Z}[G])$ of finitely generated $\mathbb{Z}[G]$ -modules.

Remark 5.2. We call this a Birch and Swinnerton-Dyer relation because it should follow from an appropriate “refined” version of the classical Birch and Swinnerton-Dyer conjecture for abelian varieties. A trivialization of the virtual module corresponding to the left side of (5.2) should arise from the leading terms in the Taylor expansion at $s = 1$ of L-series associated to the G action on A , the height pairing on $A(\mathbb{Q})$ and the period map of A . A prototype of such an extension of the Birch and Swinnerton-Dyer conjecture can be found in the works of Gross (see [Gro82], [BG85]). Following the seminal work of Bloch and Kato, very general equivariant Tamagawa number conjectures for motives with endomorphisms have been developed by Fontaine and Perrin-Riou and by Burns and Flach. These include, in particular, refined versions of the Birch and Swinnerton-Dyer conjecture; see [BF01] or the survey [Fla04]. Conjecture 5.1 should be a consequence of more precise conjectural formulas of this kind, though the details have not appeared in the literature; we are grateful to Matthias Flach for a private communication on this matter.

PROPOSITION 5.3. *Let X be a normal flat projective curve over \mathbb{Z} on which G acts and which has rational singularities. Suppose that the general fiber $X_\mathbb{Q}$ is G -isomorphic to V , and that 1 is the greatest common divisor of the multiplicities of the irreducible components of each geometric fiber of $X \rightarrow \mathrm{Spec}(\mathbb{Z})$. Then $\mathrm{H}^1(X, \mathbb{O}_X)$ is G -isomorphic to $\mathrm{Lie}(\mathcal{A})$ (and hence is \mathbb{Z} -torsion free), and $\mathrm{H}^0(X, \mathbb{O}_X) \cong \mathbb{Z}$ with trivial G -action.*

Proof. Since V is smooth and geometrically connected and $X \rightarrow \mathrm{Spec}(\mathbb{Z})$ is projective and flat, we have $\mathrm{H}^0(X, \mathbb{O}_X) \cong \mathbb{Z}$ with trivial G -action. The claim that $\mathrm{H}^1(X, \mathbb{O}_X)$ is G -isomorphic to $\mathrm{Lie}(\mathcal{A})$ follows, under our assumptions, from

[BLR90, 9.7/1] and its proof. (For a more general treatment of the relation between $\text{Lie}(\mathcal{A})$ and $H^1(X, \mathbb{O}_X)$, see [LLR04].) \square

COROLLARY 5.4. *Assume the Tate-Shafarevich group $\text{III}(A)$ is finite. With the hypotheses of Proposition 5.3, Conjecture 5.1 predicts*

$$(5.3) \quad [\mathbb{Z}] - f([\chi^P(X, \mathbb{O}_X)]) = [\text{H}_1(A(\mathbb{C}), \mathbb{Z})^+] - [\Phi_\infty(A)] - \sum_l [\Phi_l(A)] - ([A(\mathbb{Q})]^\vee) + [A(\mathbb{Q})] - [\text{III}(A)]$$

in $G_0(\mathbb{Z}[G])$, where $f : K_0(\mathbb{Z}[G]) \rightarrow G_0(\mathbb{Z}[G])$ is the forgetful homomorphism.

5.a. Modular curves. In this section we specialize Corollary 5.4 to the modular curve case considered in Section 4. Let $p, r \geq 5$ be primes such that $p \equiv 1 \pmod{24r}$, and let $\Gamma = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ be the group of diamond operators acting on the model $X_1(p)$ of the modular curve $X_1(p)_\mathbb{Q}$ described in Section 4. By Theorem 4.2(a), $X = X_1(p)$ satisfies the hypotheses of Proposition 5.3. The next statement follows now from Theorem 4.3.

PROPOSITION 5.5. *Let Sh be the subgroup of order 6 in Γ . For all subgroups H containing Sh , the cover $X_H = X_1(p)/H \rightarrow X_0 = X_1(p)/\Gamma$ is a tame $G = \Gamma/H$ cover of curves over \mathbb{Z} that satisfies the hypotheses of Theorem 3.13. Theorem 3.13 leads to an exact expression for the term $\chi^P(X_H, \mathbb{O}_{X_H}) \in K_0(\mathbb{Z}[G])$ appearing in the prediction (5.3) when $X = X_H$.*

Example 5.6. Suppose $H = Sh$. Then $X_{Sh} = X_1(p)/Sh$ is a model of the Shimura cover of $X_0(p)_\mathbb{Q}$. This is the largest quotient of $X_1(p)_\mathbb{Q}$ that is unramified over $X_0(p)_\mathbb{Q}$. The formula for $\chi^P(X_{Sh}, \mathbb{O}_{X_{Sh}})$ resulting from Theorem 3.13 requires using the injection $\Theta : \text{Cl}(\mathbb{Z}[G]) \rightarrow C_\mathbb{Z}(G; 3)$.

To use the more explicit formula of Theorem 1.3, we will now let H be the subgroup of (prime) index r in Γ . Our goal is to show Theorem 1.4 and Corollary 1.5.

Fix a character $\chi : G \rightarrow \mathbb{C}^*$ of order r . Tensoring $\mathbb{Z}[G]$ -modules with the ring homomorphism $\mathbb{Z}[G] \rightarrow \mathbb{Z}[\zeta_r, \frac{1}{2r}]$ induced by χ gives a Steinitz class homomorphism

$$s_\chi : G_0(\mathbb{Z}[G]) \rightarrow G_0(\mathbb{Z}[\zeta_r, \frac{1}{2r}]) / \{\text{free modules}\} = \text{Cl}(\mathbb{Z}[\zeta_r, \frac{1}{2r}]) = \text{Cl}(\mathbb{Z}[\zeta_r, \frac{1}{2}]).$$

LEMMA 5.7. *Suppose that $V = \mathbb{Q} \otimes_\mathbb{Z} X_H$ in Conjecture 5.1. Then s_χ sends to 0 in $\text{Cl}(\mathbb{Z}[\zeta_r, \frac{1}{2}])$ the classes $[\text{H}_1(A(\mathbb{C}), \mathbb{Z})^+]$, $[\Phi_\infty(A)]$, and $[\Phi_l(A)]$ for all primes l .*

Proof. By [CES03, Th. 1.1.3 and Cor. 1.1.5], $\Phi_p(A)$ is a finite cyclic group with trivial action by G , so $s_\chi(\Phi_p(A)) = 0$. Since X_H has good reduction outside of p , $\Phi_l(A) = \{0\}$ for $l \neq p$. By [CES03, Prop. 6.1.12], Φ_∞ is a finite two-group, so $s_\chi([\Phi_\infty]) = 0$. Finally, $\text{H}_1(A(\mathbb{C}), \mathbb{Z})$ is isomorphic to $\text{H}_1(V(\mathbb{C}), \mathbb{Z})$ as a module for $G \times T$, where $T = \{e, c\}$ is the group generated by the action of complex

conjugation c on $A(\mathbb{C})$ and $V(\mathbb{C})$. Since the action of G on $V(\mathbb{C})$ is free, we can find a finite triangulation of $V(\mathbb{C})$ that is stable under the action of $G \times T$ and is such that the stabilizer of any element of the triangulation lies in $\{e\} \times T$. Computing $H_*(V(\mathbb{C}), \mathbb{Z})$ using this triangulation, we see that the Euler characteristic

$$(5.4) \quad [H_0(V(\mathbb{C}), \mathbb{Z})] - [H_1(V(\mathbb{C}), \mathbb{Z})] + [H_2(V(\mathbb{C}), \mathbb{Z})]$$

in $G_0(\mathbb{Z}[G \times T])$ is an integral combination of classes of permutation modules of the form $\mathbb{Z}[(G \times T)/J]$, where $J \subset \{e\} \times T$. The morphism $M \rightarrow M_2^+ := \mathbb{Z}[\frac{1}{2}] \otimes_{\mathbb{Z}} M^T$ is an exact functor from the category of $\mathbb{Z}[G \times T]$ -modules to the category of $\mathbb{Z}[\frac{1}{2}][G]$ -modules. Applying this functor to (5.4) and using that $H_0(V(\mathbb{C}), \mathbb{Z})$ and $H_2(V(\mathbb{C}), \mathbb{Z})$ have trivial G -action leads to the conclusion that $s_\chi([H_1(A(\mathbb{C}), \mathbb{Z})^+])$ is a sum of classes of the form $s_\chi(M_2^+)$, where M is a permutation module of the form $\mathbb{Z}[(G \times T)/J]$ for some $J \subset \{e\} \times T$. One checks readily that all such $s_\chi(M_2^+)$ are trivial, which completes the proof. \square

Proof of Theorem 1.4

With the notations of Lemma 5.7 and Theorem 1.3, we are to show that Conjecture 5.1 implies

$$(5.5) \quad \overline{\theta_2[\mathcal{P}_\chi]} = s_\chi([\text{III}(A)]) - \overline{s_\chi([A(\mathbb{Q})])} - s_\chi([A(\mathbb{Q})]) \quad \text{in } \text{Cl}(\mathbb{Z}[\zeta_r, 1/2]),$$

where $A = \text{Jac}(X_{H, \mathbb{Q}})$.

By Corollary 5.4, Conjecture 5.1 implies the equality (5.3). The equality (5.3) gives (5.5) in view of (4.16), (4.23), Proposition 4.7, Lemma 5.7, $s_\chi([\mathbb{Z}]) = 0$ and the fact that

$$(5.6) \quad s_\chi(M^\vee) = -\overline{s_\chi(M)}$$

if $M^\vee = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and $\overline{\mathcal{D}}$ is the complex conjugate of an ideal class \mathcal{D} .

Now let $C(p)$ be the subgroup of $J_1(p)(\mathbb{Q})$ that is generated by the differences of \mathbb{Q} -rational cusps on $X_1(p)_{\mathbb{Q}}$. (With our definition of $X_1(p)_{\mathbb{Q}}$, these are the cusps that lie above the cusp ∞ of $X_0(p)_{\mathbb{Q}}$). By the Manin-Drinfeld theorem, $C(p) \subset J_1(p)(\mathbb{Q})_{\text{tor}}$. As in the introduction, denote by $J'_H(\mathbb{Q})$ the image of $A(\mathbb{Q})$ in $J_1(p)(\mathbb{Q})/C(p)$ under the homomorphism induced by the pullback map $\pi^* : A(\mathbb{Q}) \rightarrow J_1(p)(\mathbb{Q})$.

PROPOSITION 5.8. *Under the above assumptions,*

$$s_\chi([J'_H(\mathbb{Q})]) = s_\chi([A(\mathbb{Q})]) \quad \text{in } \text{Cl}(\mathbb{Z}[\zeta_r, 1/2]).$$

Proof. The main ingredient in the proof is the following:

LEMMA 5.9. *The quotient morphism $\pi : X_1(p)_{\mathbb{Q}} \rightarrow V = (X_H)_{\mathbb{Q}}$ induces homomorphisms $\pi^* : A(\mathbb{Q}) \rightarrow J_1(p)(\mathbb{Q})$ and $\pi_* : J_1(p)(\mathbb{Q}) \rightarrow A(\mathbb{Q})$. Let T_H be the trace element $\sum_{h \in H} h$ of the group ring $\mathbb{Z}[H]$.*

- (a) *The composition $\pi^* \circ \pi_*$ (resp. $\pi_* \circ \pi^*$) is multiplication by T_H (resp. $\#H$) on $J_1(p)(\mathbb{Q})$ (resp. $A(\mathbb{Q})$).*
- (b) *Both $\text{kernel}(\pi^*)$ and the Tate cohomology group $\hat{H}^0(H, C(p))$ are finite cyclic groups with trivial action by G .*
- (c) *The G -module $T_H(C(p))$ has trivial class in $G_0(\mathbb{Z}[G])$.*

Proof. Statement (a) is true because π^* (resp. π_*) is induced by the pullback (resp. pushdown) of divisors via the H -cover π . The morphism π is a composition $X_1(p)_{\mathbb{Q}} \rightarrow (X_{Sh})_{\mathbb{Q}} \rightarrow V$ where the first, resp. second morphism, is totally ramified, resp. unramified. It follows that the kernel of $\pi_{\mathbb{Q}}^*$ is contained in the kernel of $(\pi_{Sh}^*)_{\mathbb{Q}} : A(\overline{\mathbb{Q}}) \rightarrow \text{Jac}(X_{Sh})(\overline{\mathbb{Q}})$. By Kummer theory, the field $\overline{\mathbb{Q}}(X_{Sh})$ is isomorphic to $\overline{\mathbb{Q}}(V)(f^{1/\delta})$ for some $f \in \overline{\mathbb{Q}}(V)^*$, where δ is the degree of π_{Sh} . There is a divisor c on $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} V$ such that $\delta c = \text{div}(f)$, and the class $[c]$ of c in $A(\overline{\mathbb{Q}})$ generates the kernel of $(\pi_{Sh}^*)_{\mathbb{Q}}$. Since Γ is abelian, the group G acts trivially on $[c]$. Therefore, $\text{kernel}(\pi^*) \subset \text{kernel}(\pi_{\mathbb{Q}}^*) \subset \text{kernel}((\pi_{Sh}^*)_{\mathbb{Q}})$ has the properties stated in part (b).

Concerning $\hat{H}^0(H, C(p))$ and $T_H(C(p))$, let I_{Γ} be the augmentation ideal of $\mathbb{Z}[\Gamma]$. Define \mathcal{F} to be the kernel of surjection $\mathbb{Z}[\Gamma] \rightarrow \mu_p^{\otimes 2}$ induced by the square of the Teichmüller character $\omega_p : \Gamma \rightarrow \mu_p$. In [KL81, Th. 3.4] it is shown that there is an exact sequence

$$(5.7) \quad 0 \rightarrow I_{\Gamma} \cap \mathcal{F} \xrightarrow{t} I_{\Gamma} \rightarrow C(p) \rightarrow 0$$

of $\mathbb{Z}[\Gamma]$ -modules in which t is multiplication by a certain second Sticklerberger element in $\mathbb{Q}[\Gamma]$, whose definition we will not require. From the definition of \mathcal{F} and $p > 2$, we see there is an exact sequence

$$(5.8) \quad 0 \rightarrow I_{\Gamma} \cap \mathcal{F} \xrightarrow{i} I_{\Gamma} \rightarrow \mu_p^{\otimes 2} \rightarrow 0$$

in which i is inclusion. Suppose that

$$(5.9) \quad 0 \rightarrow I_{\Gamma} \cap \mathcal{F} \xrightarrow{h} I_{\Gamma} \rightarrow M \rightarrow 0$$

is any exact sequence of $\mathbb{Z}[\Gamma]$ -modules. Taking Tate cohomology with respect to H gives an exact sequence

$$(5.10) \quad \hat{H}^0(H, I_{\Gamma}) \rightarrow \hat{H}^0(H, M) \rightarrow H^1(H, I_{\Gamma} \cap \mathcal{F}) \rightarrow H^1(H, I_{\Gamma}).$$

From the cohomology of the exact sequence $0 \rightarrow I_{\Gamma} \rightarrow \mathbb{Z}[\Gamma] \rightarrow \mathbb{Z} \rightarrow 0$, we see that $\hat{H}^0(H, I_{\Gamma}) \cong \hat{H}^{-1}(H, \mathbb{Z}) = (0)$ and $H^1(H, I_{\Gamma}) \cong \hat{H}^0(H, \mathbb{Z})$ is a finite cyclic group with trivial G -action. Hence all submodules of $H^1(H, I_{\Gamma})$ are finite and cyclic with

trivial G -action. When we let $(M, h) = (\mu_p^{\otimes 2}, i)$ in (5.10), we get $\hat{H}^0(H, M) = 0$ since $M^H = (\mu_p^{\otimes 2})^H = (0)$ because ω_p^2 is nontrivial on H . Therefore (5.10) shows $H^1(H, I_\Gamma \cap \mathcal{F})$ is finite and cyclic with trivial G -action.

We apply (5.10) with $(M, h) = (C(p), t)$. Since $\hat{H}^0(H, I_\Gamma) = 0$, $\hat{H}^0(H, C(p))$ is isomorphic to a submodule of $H^1(H, I_\Gamma \cap \mathcal{F})$. Hence $\hat{H}^0(H, C(p))$ is finite and cyclic with trivial G -action, and $[\hat{H}^0(H, C(p))] = 0$ in $G_0(\mathbb{Z}[G])$. This completes the proof of Lemma 5.9(b), and it reduces the proof of part (c) to showing that

$$(5.11) \quad [C(p)^H] = 0 \quad \text{in } G_0(\mathbb{Z}[G])$$

because $\hat{H}^0(H, C(p)) = C(p)^H / T_H(C(p))$.

Taking the H -cohomology of (5.9) gives an exact sequence

$$(5.12) \quad 0 \rightarrow (I_\Gamma \cap \mathcal{F})^H \xrightarrow{h} (I_\Gamma)^H \rightarrow M^H \rightarrow H^1(H, I_\Gamma \cap \mathcal{F}).$$

We have already shown $H^1(H, I_\Gamma \cap \mathcal{F})$ is finite and cyclic with trivial G -action, so every submodule of this module has trivial class in $G_0(\mathbb{Z}[G])$. Therefore (5.12) implies

$$(5.13) \quad [M^H] = [(I_\Gamma)^H] - [(I_\Gamma \cap \mathcal{F})^H] \quad \text{in } G_0(\mathbb{Z}[G]).$$

Applying this with the pairs (M, h) given by $(C(p), t)$ and $(\mu_p^{\otimes 2}, i)$ gives

$$(5.14) \quad [C(p)^H] = [(\mu_p^{\otimes 2})^H] \quad \text{in } G_0(\mathbb{Z}[G]).$$

Hence (5.11) follows from (5.14) and the fact previously observed that $(\mu_p^{\otimes 2})^H$ is trivial. □

Now let us complete the proof of Proposition 5.8. When T_H is the trace element of $\mathbb{Z}[H]$, we have $T_H(C(p)) = \pi^* \pi_* C(p) \subset \pi^* A(\mathbb{Q})$ by Lemma 5.9(a). Hence

$$W := (C(p) \cap \pi^* A(\mathbb{Q})) / T_H(C(p)) \subset \hat{H}^0(H, C(p)) = C(p)^H / T_H(C(p)),$$

so W is cyclic with trivial G action by Lemma 5.9(b). In $G_0(\mathbb{Z}[G])$ we have

$$\begin{aligned} [J'_H(\mathbb{Q})] + [W] + [T_H(C(p))] \\ = [J'_H(\mathbb{Q})] + [C(p) \cap \pi^* A(\mathbb{Q})] = A(\mathbb{Q}) - [\text{kernel}(\pi^*)]. \end{aligned}$$

Since W and $\text{kernel}(\pi^*)$ are finite cyclic with trivial G -action, we have $s_\chi([W]) = s_\chi([\text{kernel}(\pi^*)]) = 0$. Also, by Lemma 5.9(c), $[T_H(C(p))] = 0$. So $s_\chi([J'_H(\mathbb{Q})]) = s_\chi([A(\mathbb{Q})])$. □

Remark 5.10. We can alternatively consider the following variant of the conjectural identity in Theorem 1.4. The $\Gamma = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ -action on

$$A = \text{Jac}(X_1(p)_\mathbb{Q})$$

induces an embedding of $\mathbb{Q}[\Gamma] \simeq \bigoplus_r \mathbb{Q}(\zeta_r)$ in the endomorphism algebra $\text{End}^0(A) = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The summands here correspond to characters $\chi : \Gamma \rightarrow \mathbb{Z}[\zeta_r]^*$ of order r . The component of $\mathbb{Q}[G]$ that corresponds to χ determines an isogeny class $\{A_\chi\}$ of abelian varieties over \mathbb{Q} with the following property: For each prime l the (well-defined) $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -representation $H_{et}^1(A_\chi \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_l)$ is the χ -component V_l^χ of $V_l = H_{et}^1(A \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. The field $\mathbb{Q}(\zeta_r)$ embeds into the endomorphism algebra $\text{End}^0(\{A_\chi\})$. The product $\prod_{f, \epsilon(f)=\chi} A_f$, where A_f is the abelian variety associated by Shimura to the cuspidal Hecke eigenform f of weight 2 and f runs over all such normalized eigenforms with Nebentypus character χ , belongs to this isogeny class. Here we fix another representative A_χ of this isogeny class by picking for each prime l the Galois stable lattice $(H_{et}^1(A \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l[\zeta_r]\chi^{-1})^\Gamma \subset V_l^\chi$. The abelian variety A_χ supports multiplication by $\mathbb{Z}[\zeta_r]$, i.e., an embedding $\mathbb{Z}[\zeta_r] \subset \text{End}(A_\chi)$. Let us denote by \mathcal{A}_χ the Néron model of A_χ over \mathbb{Z} . Then we can show that

$$\text{Lie}(\mathcal{A}_\chi) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p-1}] \simeq (\overline{S_2(\Gamma_1(p), \mathbb{Z}[\zeta_r])}_\chi)^* \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p-1}],$$

where $*$ denotes the \mathbb{Z} -dual of $\mathbb{Z}[\zeta_r]$ -modules. If r is prime we have the conjectural identity

$$\overline{\theta_2[\mathcal{P}_\chi]} = [\text{III}(A_\chi)] + [\Phi_p(A_\chi)] - \overline{[A_\chi(\mathbb{Q})]} - [A_\chi(\mathbb{Q})]$$

in $G_0(\mathbb{Z}[\zeta_r, \frac{1}{p-1}]) / \{\text{free modules}\} = \text{Cl}(\mathbb{Z}[\zeta_r, \frac{1}{p-1}])$.

Acknowledgments

We would like to thank L. Illusie, N. Katz, L. Merel and P. Sarnak for useful conversations, and the referee for helpful comments.

References

- [Abb00] A. ABBES, The Grothendieck-Ogg-Shafarevich formula for arithmetic surfaces, *J. Algebraic Geom.* **9** (2000), 529–576. MR 2001b:14039 Zbl 0986.14015
- [AGV73] M. ARTIN, A. GROTHENDIECK, and J. L. VERDIER (eds.), *Théorie des Topos et Cohomologie Étale des Schémas. Tome 3. Exposés IX à XIX.* (Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 = SGA 4), *Lecture Notes in Math.* **305**, Springer-Verlag, New York, 1973. MR 50 #7132 Zbl 0245.00002
- [BFQ79] P. BAUM, W. FULTON, and G. QUART, Lefschetz-Riemann-Roch for singular varieties, *Acta Math.* **143** (1979), 193–211. MR 82c:14022a Zbl 0454.14009
- [BGI71] P. BERTHELOT, A. GROTHENDIECK, and L. ILLUSIE (eds.), *Théorie des Intersections et Théorème de Riemann-Roch* (Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 = SGA 6), *Lecture Notes in Math.* **225**, Springer-Verlag, New York, 1971. MR 50 #7133 Zbl 0218.14001
- [BLR90] S. BOSCH, W. LÜTKEBOHMERT, and M. RAYNAUD, *Néron Models*, *Ergeb. Math. Grenzgeb.* **21**, Springer-Verlag, New York, 1990. MR 91i:14034 Zbl 0705.14001

- [Bre83] L. BREEN, *Fonctions thêta et théorème du cube*, *Lecture Notes in Math.* **980**, Springer-Verlag, New York, 1983. MR 87d:14036 Zbl 0558.14029
- [BG85] J. P. BUHLER and B. H. GROSS, Arithmetic on elliptic curves with complex multiplication. II, *Invent. Math.* **79** (1985), 11–29. MR 86j:11066 Zbl 0584.14027
- [BF01] D. BURNS and M. FLACH, Tamagawa numbers for motives with (non-commutative) coefficients, *Doc. Math.* **6** (2001), 501–570. MR 2002m:11055 Zbl 1052.11077
- [CNT91] P. CASSOU-NOGUÈS and M. J. TAYLOR, Un élément de Stickelberger quadratique, *J. Number Theory* **37** (1991), 307–342. MR 92e:11125 Zbl 0719.11075
- [CW34] C. CHEVALLEY and A. WEIL, Über das Verhalten der Integrale. I: Gattung bei Automorphismen des Funktionenkörpers, *Abh. Math. Semin. Hamb. Univ.* **10** (1934), 358–361. Zbl 0009.16001 JFM 60.0098.01
- [Chi94] T. CHINBURG, Galois structure of de Rham cohomology of tame covers of schemes, *Ann. of Math.* **139** (1994), 443–490. MR 95h:11125a Zbl 0828.14007
- [CEPT96] T. CHINBURG, B. EREZ, G. PAPPAS, and M. J. TAYLOR, Tame actions of group schemes: integrals and slices, *Duke Math. J.* **82** (1996), 269–308. MR 97d:14066 Zbl 0907.14021
- [CEPT97a] ———, ϵ -constants and the Galois structure of de Rham cohomology, *Ann. of Math.* **146** (1997), 411–473. MR 98j:14025 Zbl 0939.14009
- [CEPT97b] ———, Riemann-Roch type theorems for arithmetic schemes with a finite group action, *J. Reine Angew. Math.* **489** (1997), 151–187. MR 98e:14006 Zbl 0903.19001
- [CES03] B. CONRAD, B. EDIXHOVEN, and W. STEIN, $J_1(p)$ has connected fibers, *Doc. Math.* **8** (2003), 331–408. MR 2004k:11094 Zbl 1101.14311
- [Del87] P. DELIGNE, Le déterminant de la cohomologie, in *Current Trends in Arithmetical Algebraic Geometry* (Arcata, CA, 1985), *Contemp. Math.* **67**, Amer. Math. Soc., Providence, RI, 1987, pp. 93–177. MR 89b:32038 Zbl 0629.14008
- [DR73] P. DELIGNE and M. RAPOPORT, Les schémas de modules de courbes elliptiques, in *Modular Functions of One Variable. II* (Antwerp, 1972) (P. DELIGNE and W. KUYK, eds.), *Lecture Notes in Math.* **349**, Springer-Verlag, New York, 1973, pp. 143–316. MR 49 #2762 Zbl 0281.14010
- [Duc05] F. DUCROT, Cube structures and intersection bundles, *J. Pure Appl. Algebra* **195** (2005), 33–73. MR 2005g:14016 Zbl 1067.14007
- [Fla04] M. FLACH, The equivariant Tamagawa number conjecture: A survey, in *Stark's Conjectures: Recent Work and New Directions*, *Contemp. Math.* **358**, Amer. Math. Soc., Providence, RI, 2004, pp. 79–125. MR 2005k:11132 Zbl 1070.11025
- [Frö83] A. FRÖHLICH, *Galois Module Structure of Algebraic Integers*, *Ergeb. Math. Grenzgeb.* **1**, Springer-Verlag, New York, 1983. MR 85h:11067 Zbl 0501.12012
- [Ful98] W. FULTON, *Intersection Theory*, 2nd ed., *Ergeb. Math. Grenzgeb.* **2**, Springer-Verlag, New York, 1998. MR 99d:14003 Zbl 0885.14002
- [GS92] H. GILLET and C. SOULÉ, An arithmetic Riemann-Roch theorem, *Invent. Math.* **110** (1992), 473–543. MR 94f:14019 Zbl 0777.14008
- [Gro82] B. H. GROSS, On the conjecture of Birch and Swinnerton-Dyer for elliptic curves with complex multiplication, in *Number Theory Related to Fermat's Last Theorem* (Cambridge, MA, 1981) (N. KOBLITZ, ed.), *Progr. Math.* **26**, Birkhäuser, Mass., 1982, pp. 219–236. MR 84e:14020 Zbl 0506.14040

- [GM71] A. GROTHENDIECK and J. P. MURRE, *The Tame Fundamental Group of a Formal Neighbourhood of a Divisor with Normal Crossings on a Scheme*, *Lecture Notes in Math.* **208**, Springer-Verlag, New York, 1971. MR 47 #5000 Zbl 0216.33001
- [Ill79] L. ILLUSIE, Théorie de Brauer et caractéristique d'Euler-Poincaré, *Astérisque* **82–83** (1978–79), 173–219.
- [KM85] N. M. KATZ and B. MAZUR, *Arithmetic Moduli of Elliptic Curves*, *Ann. Math. Studies* **108**, Princeton Univ. Press, Princeton, NJ, 1985. MR 86i:11024 Zbl 0576.14026
- [KS99] N. M. KATZ and P. SARNAK, Zeroes of zeta functions and symmetry, *Bull. Amer. Math. Soc.* **36** (1999), 1–26. MR 2000f:11114 Zbl 0921.11047
- [KM76] F. F. KNUDSEN and D. MUMFORD, The projectivity of the moduli space of stable curves. I: Preliminaries on “det” and “Div”, *Math. Scand.* **39** (1976), 19–55. MR 55 #10465 Zbl 0343.14008
- [KL81] D. S. KUBERT and S. LANG, *Modular Units*, *Grundle. Math. Wissens.* **244**, Springer-Verlag, New York, 1981. MR 84h:12009 Zbl 0492.12002
- [Lan88] S. LANG, *Introduction to Arakelov Theory*, Springer-Verlag, New York, 1988. MR 89m:11059 Zbl 0667.14001
- [Lan94] ———, *Algebraic Number Theory*, 2nd ed., *Grad. Texts Math.* **110**, Springer-Verlag, New York, 1994. MR 95f:11085 Zbl 0811.11001
- [LLR04] Q. LIU, D. LORENZINI, and M. RAYNAUD, Néron models, Lie algebras, and reduction of curves of genus one, *Invent. Math.* **157** (2004), 455–518. MR 2005m:14039 Zbl 1060.14037
- [Maz77] B. MAZUR, Modular curves and the Eisenstein ideal, *Inst. Hautes Études Sci. Publ. Math.* (1977), 33–186. MR 80c:14015 Zbl 0394.14008
- [McC87] L. R. MCCULLOH, Galois module structure of abelian extensions, *J. Reine Angew. Math.* **375/376** (1987), 259–306. MR 88k:11080 Zbl 0619.12008
- [Pap98] G. PAPPAS, Galois modules and the theorem of the cube, *Invent. Math.* **133** (1998), 193–225. MR 99d:11122 Zbl 0923.14030
- [Pap00] ———, Galois module structure and the γ -filtration, *Compositio Math.* **121** (2000), 79–104. MR 2001f:14032 Zbl 0968.14026
- [Pap03] ———, Chow groups and cubic structures, preprint, 2003.
- [Pap05] ———, Cubic structures and ideal class groups, *Ann. Sci. École Norm. Sup.* **38** (2005), 471–503. MR 2006j:11090 Zbl 1135.11033
- [Pap08] ———, Galois module structure of unramified covers, *Math. Ann.* **341** (2008), 71–97. MR 2377470 Zbl 1138.11050
- [Ray70] M. RAYNAUD, *Anneaux Locaux Henséliens*, *Lecture Notes in Math.* **169**, Springer-Verlag, New York, 1970. MR 43 #3252 Zbl 0203.05102
- [Rim59] D. S. RIM, Modules over finite groups, *Ann. of Math.* **69** (1959), 700–712. MR 21 #3474 Zbl 0092.26104
- [Rob80] P. ROBERTS, Abelian extensions of regular local rings, *Proc. Amer. Math. Soc.* **78** (1980), 307–310. MR 81a:13017 Zbl 0436.13011
- [Rob98] P. C. ROBERTS, *Multiplicities and Chern Classes in Local Algebra*, *Cambridge Tracts in Mathematics* **133**, Cambridge Univ. Press, 1998. MR 2001a:13029 Zbl 0917.13007
- [Tho86] R. W. THOMASON, Lefschetz-Riemann-Roch theorem and coherent trace formula, *Invent. Math.* **85** (1986), 515–543. MR 87j:14028 Zbl 0653.14005

- [Tho92] ———, Une formule de Lefschetz en K -théorie équivariante algébrique, *Duke Math. J.* **68** (1992), 447–462. MR 93m:19007 Zbl 0813.19002
- [Vid04] I. VIDAL, Théorie de Brauer et conducteur de Swan, *J. Algebraic Geom.* **13** (2004), 349–391. MR 2005m:14030 Zbl 1070.14020
- [Was82] L. C. WASHINGTON, *Introduction to Cyclotomic Fields*, *Grad. Texts in Math.* **83**, Springer-Verlag, New York, 1982. MR 85g:11001 Zbl 0484.12001

(Received November 18, 2004)

(Revised September 17, 2006)

E-mail address: ted@math.upenn.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA DAVID RITTENHOUSE LAB,
209 SOUTH 33RD STREET, PHILADELPHIA, PA 19104-6395, UNITED STATES

E-mail address: pappas@math.msu.edu

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI 48824,
UNITED STATES

E-mail address: Martin.Taylor@manchester.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANCHESTER, M60 1QD, UNITED KINGDOM