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Abstract

We prove that the Julia set $J(f)$ of at most finitely renormalizable unicritical polynomial $f : z \mapsto z^d + c$ with all periodic points repelling is locally connected. (For $d = 2$ it was proved by Yoccoz around 1990.) It follows from *a priori* bounds in a modified Principal Nest of puzzle pieces. The proof of *a priori* bounds makes use of new analytic tools developed in [KL09] that give control of moduli of annuli under maps of high degree.

1. Introduction

1.1. *Statement of the results.* About 15 years ago Yoccoz proved that the Julia set of at most finitely many renormalizable quadratic polynomials $f : z \mapsto z^2 + c$ with all periodic points repelling is locally connected (see [Hub93], [Mil00a]). In this paper, we generalize this result to higher degree unicritical polynomials:

THEOREM A. *The Julia set $J(f)$ of at most finitely renormalizable unicritical polynomials $f : z \mapsto z^d + c$ with all periodic points repelling is locally connected.*

This result follows from *a priori* bounds in an appropriate "Modified Principal Nest" of puzzle pieces,

$$E^0 \Supset E^1 \Supset \cdots \ni 0 :$$

THEOREM B. *The (modified) principal moduli stay away from zero*:

$$\operatorname{mod}(E^{i-1} \smallsetminus E^i) \geq \mu > 0.$$

These *a priori* bounds imply that the puzzle pieces E^i shrink to the critical point, which yields Theorem A by a standard argument.

1.2. *Techniques.* As usual in holomorphic dynamics, our proof has two sides: combinatorial and analytic. Our combinatorial tool is a refined Principal Nest technique of [Lyu97], while the analytic tool is a recently established Quasi-Invariance Law (Covering Lemma) in conformal geometry [KL09]. Let us briefly comment on both sides.

The puzzle machinery was introduced to holomorphic dynamics by Branner and Hubbard [BH92] (in the context of cubic polynomials with one escaping critical point) and Yoccoz [Hub93], [Mil00a] (in the context of quadratic polynomials). The idea is to tile shrinking neighborhoods of the Julia set into topological disks called *puzzle pieces*, and to translate the dynamics on $J(f)$ to the combinatorics of these tilings.

An efficient way to describe these combinatorics is given by the *Principal nest* of puzzle pieces around the origin, $V^0 \supset V^1 \supset \dots V^n \dots \ni 0$, which is inductively constructed so that the first return maps $f^{n_i} : V^i \to V^{i-1}$ are unicritical branched coverings [Lyu97]. It turns out that this nest is not quite suitable for our purposes, so we modify it slightly to obtain a *dynasty of kingdom maps*; see Section 2.

We then observe that since the return times in the dynasty grow exponentially, one can send some puzzle piece E^{i-1} to the top level by an appropriate composition Ψ of the kingdom maps, while the next puzzle piece, E^i, will go at most five levels up (*time inequality*). Thus, the map $\Psi | E^i$ has a bounded degree, which puts us in a position to apply the analytic techniques of [KL09].

The puzzle bears complete information about the Julia set only if the puzzle pieces shrink to points, and so this is a key geometric issue of the theory. To handle this issue, Branner & Hubbard and Yoccoz made use of the Series Law from conformal geometry.[1] It was immediately realized, however, that this method would not work for higher degree polynomials, so that in the higher degree case the problem has remained open since then.

A new analytic tool that we exploit is a Covering Lemma (Quasi-Invariance Law) in conformal geometry [KL09] which roughly asserts that given a branched covering $g : U \to V$ of degree N which restricts to a branched covering $g : A \to B$ of degree d such that $\mathrm{mod}(U \smallsetminus A)$ is small (depending on N), then, under a certain “Collar Assumption”, $\mathrm{mod}(V \smallsetminus B)$ is comparable to $d^2 \,\mathrm{mod}(U \smallsetminus A)$ (independently of N) – see Section 3 for the precise statement.

The Covering Lemma allows us to transfer moduli information from deep levels of the dynasty to shallow ones, and to argue that if on some deep levels the moduli are small, then they must be even smaller on shallow ones. This certainly implies that, in fact, the moduli can never be too small (Theorem B).

[1] also called the Grötzsch Inequality.

Note that for real c, Theorem A was proved before by Levin and van Strien [LvS98]. The method used in [LvS98] exploited real symmetry in a substantial way.

In the forthcoming notes (joint with A. Avila and W. Shen) our *a priori* bounds will be used to prove *rigidity* of the unicritical polynomials under consideration.

1.3 *Terminology and notation.* A *topological disk* means a simply connected domain in $\mathbb{C}$.

We let $\mathrm{orb}(z) \equiv \mathrm{orb}_g(z) = (g^n z)_{n=0}^{\infty}$ be the *orbit* of z under a map g.

Given a map $g : U \to V$ and a domain $D \subset V$, components of $g^{-1}(D)$ are called *pullbacks* of D under g. Given a connected set $X \subset g^{-1}(D)$, we let $g^{-1}(D)|X$ be the pullback of D containing X.

Given a subset $W \subset V$, the *first landing map* H to W is defined (on the set of points z whose orbits intersect W) as follows: $H(z) = f^l z$, where $l \geq 0$ is the first moment for which $f^l z \in W$.

We say that a map $g : U \to V$ is *unicritical* if it has one critical point (of arbitrary local degree)

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2. **Modified Principal Nest**

2.1. *Generalized polynomial-like maps.* A *generalized polynomial-like map* (GPL map) is a holomorphic map $g \colon \cup W_i \to V$, where $V \subset \mathbb{C}$ is a topological disk and $W_i \Subset V$ are topological disks with disjoint closures such that the restrictions $g : W_i \to V$ are branched coverings, and moreover, all but finitely many of them have degree one.

Remark. To prove Theorem B in full generality, we need to allow infinitely many disks W_i. However, in the "persistently recurrent" case that interests us most it is enough to consider GPL maps defined on finitely many disks W_i.

We let $K_g = \bigcap_{n=0}^{\infty} g^{-n} V$ be the set of points of V on which g is infinitely iterable (the "filled Julia set").

A GPL map g is called *unicritical* if it has a single critical point. *In what follows we consider only unicritical* GPL *maps, and will always put their critical points at* 0. Let d be the local degree of g near 0. We let $W_0 \equiv W$ be the "central domain", that is, the one containing 0.

The *postcritical set* $\mathbb{O}_g$ of a (unicritical) GPL map is the closure of the orbit $\{g^n 0\}_{n=0}^{\infty}$.

Puzzle pieces of depth n of a GPL map g are components of $g^{-n}(V)$. Puzzle pieces containing 0 are called *critical*.

If the critical point returns to some critical puzzle piece A, then the first return map h to A is also GPL. Let $\cup B_i$ be its domain of definition. Restricting h to the union of those components B_i that intersect the postcritical set, we obtain a GPL map called the *generalized renormalization* $r_A(g)$ of g on A.

If we do not specify the domain A of the generalized renormalization, then it is assumed to be W, so $r(g) \equiv r_W(g)$.

2.2. *Dynasty of kingdoms.* Let us introduce a modified notion of (unicritical) GPL map called a kingdom map.

We consider three topological disks, $W \supset U \Supset A \ni 0$, called the *kingdom domain*, the *castle*, and the *king* respectively. Let us consider a family of topological disks $D_j \Subset W \smallsetminus \bar{A}$ ("king's subjects") such that $\bar{D}_j \cap \partial U = \varnothing$. Finally, let $M_k \Subset U \smallsetminus \bar{A}$ be another family of topological disks ("king's men"). A map

$$G : A \cup \bigcup_j D_j \cup \bigcup_k M_k \to W$$

is called a *kingdom map* (of *local degree* d) if

- The closures $\bar{A}$, $\bar{D}_j$ and $\bar{M}_k$ are pairwise disjoint;
- $G : A \to W$ is a d-to-1 branched covering ramified only at 0;
- Each $G : D_j \to W$ is a biholomorphic isomorphism;
- Each $G : M_k \to U$ is a biholomorphic isomorphism.

We let $\mathbb{O}_G$ be the postcritical set of the kingdom map G.

When $U = W$, kingdom maps become GPL maps.

Let us now consider a (unicritical) GPL map $g : \cup W_i \to V$, $W \equiv W_0$. Let us define the *kingdom renormalization* $G = R(g)$ of g whose result will be a kingdom map G.

If $g(0) \in W$ then we say that the *central return* occurs. If $g^k(0) \in W$ for $k = 0, \dots, N-1$ but $g^N(0) \notin W$, then we have a nest of topological disks

$$V \equiv \Omega_0 \Supset W \equiv \Omega_1 \Supset \cdots \Supset \Omega_N \equiv U \tag{2.1}$$

such that $g : \Omega_{k+1} \to \Omega_k$ is a unicritical branched covering of degree d and $g(0) \in \Omega_{N-1} \smallsetminus \Omega_N$. This nest is called a *central cascade* (*of length* N). Note that the non-central-return event corresponds to the cascade of length 1.

In the kingdom renormalization Rg, W will be the kingdom domain and U will be the castle.

Let us consider the first return map to $W = \Omega_1$:

$$h : X_0 \cup \bigcup_{i>0} X_i \to W,$$

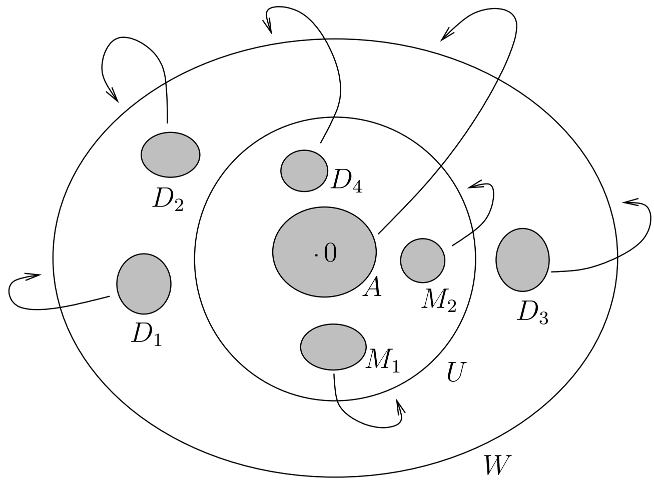

Figure 2.1. Kingdom map

where $X_0 \ni 0$ (in case when $N \geq 2$, $X_0 = \Omega_2$).

Consider the domain X_s, $s > 0$, containing $g^{N-1}(0)$. Then the pullback $A = g^{-(N-1)}(X_s)|\,0$ is the king of Rg. The kingdom map G on A is defined as $h \circ g^{N-1} : A \to W$. Notice that it is a unicritical d-to-1 branched covering.

Let us define *king's subjects* D_j as non-critical pullbacks of the domains X_i ($i \neq 0$) under the maps

$$g^{k-1} : \Omega_k \to W, \quad k = 1, 2, \dots, N,$$

that intersect the postcritical set. Thus, each subject D_j is univalently mapped onto some X_i, $i > 0$, by an appropriate map $g^{k-1} : \Omega_k \to W$, $k \in [1, N]$. On this subject, we define the kingdom map $G : D_j \to W$ as $G|D_j = h \circ g^{k-1}|D_j$. Obviously, it is a biholomorphic isomorphism.

Finally, we define king's men M_k as the pullbacks of U under $g : U \to g(U) \supset U$ that intersect the postcritical set. There are at most d king's men, and g univalently maps each of them onto U. Let $G|M_k = g|M_k$.

Thus, we have defined the desired kingdom renormalization

$$G = R(g) : A \cup \bigcup_j D_j \cup \bigcup_k M_k \to W.$$

Let $g^N(0) \in W_j$, $j > 0$. For $G = R(g)$, define the *king's apartment* Δ as $g^{-N}(W_j)|\,0$. Then $A \Subset \Delta \subset U$ and the map $g^{N+1} : \Delta \to V$ is a unicritical d-to-1 branched covering. This creates a collar $\Delta \smallsetminus A$ around the king.

Remark. If $N = 1$ (i.e., the non-central return occurs under g), then the kingdom renormalization $G = R(g)$ coincides with the generalized renormalization defined in Section 2.1.

Given a kingdom map G, let us define its renormalization $g = r(G)$ as the first return map $g : \cup B_i \to A$ to the king A restricted to those domains B_i that intersect the postcritical set $\mathcal{O}_G$. It is a unicritical GPL map.

Beginning with some GPL map $g \equiv g_0$, we construct in the above way a *dynasty* of kingdoms, that is, a sequence (g_n, G_n) such that g_n is a GPL map, G_n is a kingdom map, $G_n = R(g_n)$ and $g_{n+1} = r(G_n)$. This dynasty terminates if and only if:

- The map g is combinatorially non-recurrent; that is, the critical point does not return to some critical puzzle piece; or
- Some map g_n has an infinite central cascade, i.e., it is a Douady-Hubbard polynomial-like map [DH85] with non-escaping critical point. In this case g is called *renormalizable in the sense of Douady and Hubbard*.

When we consider a dynasty of kingdoms (g_n, G_n), the associated domains will be marked with superscript n (e.g., V^n, W^n, etc.) However, we usually skip the label when we are concerned with a single kingdom.

Remark. It is easy to see that the maps g_n coincide with the generalized renormalizations of g on domains V^n as defined in Section 2.1, i.e., $g_n = r_{V^n}(g)$.

The nest

$$V^0 \supset W^0 \supset \cdots \supset W^{n-1} \supset V^n \supset W^n \supset V^{n+1} \supset \ldots$$

is called the *Modified Principal Nest*. Sometimes it is convenient to relabel it in a uniform way:

$$E^0 \supset E^1 \supset \cdots \supset E^i \supset E^{i+1} \supset \ldots, \tag{2.2}$$

so that $V^n = E^{2n}$, $W^n = E^{2n+1}$. The consecutive E-domains are dynamically related: $E^{i-1} = \psi_i(E^i)$, where ψ_i is a unicritical d-to-1 branched covering which is an appropriate iterate of g.

2.3. *First king.* We will describe in this section how to associate to a unicritical polynomial $f : z \mapsto z^d + c$ (or, more generally, polynomial-like map) a dynasty of kingdom maps. Our standing assumption is that *the Julia set J_f is connected and all periodic points of f are repelling*. Then f has $d-1$ *non-dividing* fixed points β_i (landing points of the external rays[2] with angles $2\pi/(d-1)$), and one *dividing*

[2] In the case of a polynomial-like map, external rays are defined by means of straightening.

fixed point α. There are $q > 1$ external rays $\mathcal{R}_i^0$ landing at α which are cyclically permuted by the dynamics; see [Mil00b].

Let us select some equipotential E^0; it bounds some topological disk Q^0. The rays $\mathcal{R}_i^0$ divide Q^0 into q disks Y_i^0 called the *Yoccoz puzzle pieces* of depth 0. Let $Y^0 \equiv Y_0^0$ stand for the critical puzzle piece, i.e., the one containing 0.

The equipotential $E^1 = f^{-1}E^0$ bounds some topological disk Q^1. Let us consider dq rays of $f^{-1}(\cup\mathcal{R}_i^0)$. They divide Q^1 into $(q-1)d+1$ topological disks called Yoccoz puzzle pieces of depth 1. Let Y^1 stand for the critical puzzle piece of depth 1. There are also $q-1$ puzzle pieces Y_i^1 of depth 1 contained in the corresponding off-critical pieces of depth 0. All other puzzle pieces of depth 1 will be denoted Z_j^1. They are attached to the f-preimages of α that are different from α itself.

The map f is called *satellite renormalizable* (or *immediately renormalizable*) if

$$f^{lq}(0) \in Y^0, \quad l = 0, 1, 2 \dots .$$

In this case, we let $Y^{lq} = f^{-lq}(Y^0)|0$ and consider the unicritical branched covering $f^q : Y^q \to Y^0$ of degree d. By a slight "thickening" of the domain of this map (see [Mil00a]), it can be turned into a unicritical GPL map called the (*satellite*) *renormalization* Rf of f.[3]

In the satellite renormalizable case, f does not originate any dynasty. Otherwise, there exists an $l \in \mathbb{N}$ such that $f^{lq}(0)$ belongs to some puzzle piece Z_j^1. In this case, we let $V^0 = f^{-lq}(Z_j^1)|0$ be the *first kingdom*, and we let $g \equiv g_0 : \cup W_i^0 \to V^0$ be the first return map to V^0. It is easy to check that $W_i^0 \Subset V^0$. Let G_0 be the associated kingdom map, which originates the dynasty (g_n, G_n) associated with f.

The map f is called *primitively renormalizable* if its dynasty contains a quadratic-like map $g_n : W^n \to V^n$ with connected Julia set. This quadratic-like map is called the (*primitive*) *renormalization* Rf of f. In this case, we cannot construct the next kingdom map G_n, so the dynasty terminates. It also terminates if the map g is combinatorially non-recurrent. Otherwise, the process can be continued indefinitely, and the dynasty (g_n, G_n) is eternal.

If the map f is renormalizable (either in the satellite or in the primitive sense), we can take its renormalization Rf and consider its dynasty. If Rf is renormalizable, we can pass to the second renormalization $R^2 f$, and so on. If the map f is at most finitely renormalizable, in the end we obtain a non-renormalizable quadratic-like map $R^m f$. This is the map we will be working with. So, *in what follows we will assume that f itself is non-renormalizable.*

[3] In the context of GPL maps we use the term "DH renormalization" to distinguish it from the generalized renormalization. In the polynomial case, we refer to it as just "renormalization", which should not lead to confusion.

From now on, we can forget about the original polynomial $f : z \mapsto z^d + c$, and replace it with the first map $g : \cup W_i^0 \to V^0$ of the associated dynasty.

2.4. *Extensions.* Let us begin with a trivial but useful observation:

LEMMA 2.1 (Telescope). *Let X_k be a sequence of topological disks, $k = 0, 1, \dots, n$, and let $\phi_k : X_k \to \phi(X^k)$ be branched coverings of degree d_k such that $\phi(X_k) \supset X_{k+1}$. Let $\Phi = \phi_{n-1} \circ \cdots \circ \phi_0$ (wherever it is defined), and let $P \subset X_0$ be a component of its domain of definition. Then $\Phi : P \to V_n$ is a branched covering of degree at most $d_0 \cdots d_{n-1}$.*

LEMMA 2.2. *Let $g^m z \in A$ be the first landing of* orb(z) *at A. Then there exists a puzzle piece $P \ni z$ such that g^m univalently maps P onto U.*

Proof. Let $P = g^{-m}(U)|z$. Then $g^m = h^k \circ g^s$, where $g^s(z)$ is the first landing of orb z at U, $h = h_U : \cup B_i \to U$ is the generalized renormalization on U, and k is the first landing moment of orb$_h(g^s(z))$ at A. It is a simple exercise to show that g^s is univalent on $g^{-s}(U)|z$. Moreover, h univalently maps each non-central component B_i, $i > 0$, onto U. Now the assertion follows from the Telescope Lemma. □

COROLLARY 2.3. *Let $z \in A$, and let $g^m z \in A$ be the first return of* orb(z) *to A. Let $P = g^{-m}(A)|z$. If P is not critical then the map $g^m : P \to A$ is univalent. Otherwise $g^m : P \to A$ is a unicritical branched covering of degree d.*

Proof. Decompose $g^m : P \to A$ as $g : P \to g(P)$ and the first landing map $g^{m-1} : g(P) \to A$. □

Applying this to the first return of the critical point to $A^{n-1} = V^n$, we obtain:

COROLLARY 2.4. *The map $g_n : W^n \to V^n$ admits an analytic extension to a puzzle piece $\tilde{W}^n \Supset W^n$ such that $g_n : \tilde{W}^n \to U^{n-1}$ is a unicritical branched covering of degree d. Moreover, $\tilde{W}^n \subset V^n$.*

Let us now construct similar extensions for kingdom maps:

LEMMA 2.5. *There is puzzle piece $\tilde{A} \Supset A$ such that the map $G : A \to W$ admits a unicritical degree d extension to a map $\tilde{A} \to V$. Moreover, $\tilde{A} \subset \Delta$ where Δ is the king's apartment.*

Proof. The map $G : A \to W$ can be decomposed as $g^k \circ g^N$ where N is the length of the central cascade of g, and k is the first entry time of orb$_g(g^N(A))$ to W (recall that $g^N(A) \subset V \setminus W$). The map $g^N : A \to g^N(A)$ admits an analytic extension to a unicritical d-to-1 branched covering $g^N : \Delta \to W_j$ for some $j > 0$, while $g : W_i \to V$ is a biholomorphic isomorphism for any $i > 0$. Now the conclusion follows by the Telescope Lemma. □

Let us define *enlargements* $\hat{E}^i$ of domains E^i of the Modified Principal Nest (2.2) as follows: $\hat{W}^n = V^n$ and $\hat{V}^n = U^{n-1}$. We also have the *buffers* $\tilde{E}^i \subset \hat{E}^i$ constructed in Corollary 2.4 and Lemma 2.5. These lemmas tell us that any map ψ_i analytically extends to a unicritical d-to-1 branched covering $\psi_i : \tilde{E}^i \to \hat{E}^{i-1}$. For $i < k$, let

$$\Phi_{i,k} = \psi_{i+1} \circ \cdots \circ \psi_k : E^k \to E^i.$$

By the Telescope Lemma, we have:

LEMMA 2.6. *For* $0 < i < k$, *the map* $\Phi_{i,k}$ *admits an analytic extension to a* d^{k-i}*-to-*1 *branched covering from some puzzle piece* $F^k \supset E^k$ *onto* $\hat{E}^i$.

2.5. *Travel times.* Consider two puzzle pieces P and Q for some GPL or kingdom map F. If $F^l P = Q$, we let $\mathrm{Time}_F(P, Q) = l$ (note that time l is uniquely determined). For the "absolute time" measured with respect to the initial map g, we use notation $\mathrm{Time}(P, Q) \equiv \mathrm{Time}_g(P, Q)$.

Let

- T_n= $\mathrm{Time}(A^n, W^n)$, i.e., $G_n|A^n = g^{T_n}|A^n$;
- $t_n = \mathrm{Time}(W^n, V^n)$, i.e., $g_n|W^n = g^{t_n}$;
- $s_n = \mathrm{Time}(W^n, W^{n-1}) = t_n + T_{n-1}$ for $n \geq 1$; $s_0 = t_0 = \mathrm{Time}(W^0, V^0)$.

LEMMA 2.7. *The travel times satisfy the following inequalities*:

$$t_n \geq T_{n-1}\,; \quad T_n \geq s_n\,; \quad s_n \geq 2s_{n-1}.$$

Proof. By definition, $g_n(W^n)$ is the first return of W^n to $V^n = A^{n-1}$ under iterates of G_{n-1}, so that $g_n|W^n = G^k_{n-1}|W^n$ for some $k \geq 1$. Hence

$$g^{t_n}|W^n = g_n|W^n = G_{n-1}^{\circ(k-1)} \circ G_{n-1}|W^n = g^s \circ g^{T_{n-1}}|W^n$$

for some $s \geq 0$, and the first inequality follows.

For the second inequality, notice that T_n is the first return time of the critical orbit to W^n after the first entry to the annulus $V^n \smallsetminus W^n$. The first entry to $V^n \smallsetminus W^n$ occurs at time $\geq t_n$ (since t_n is the first return time of 0 to V^n). Return back to W^n from $V^n \smallsetminus W^n$ occurs at time $\geq T_{n-1}$ (since T_{n-1} is the first moment T when $g^T(V^n) \cap V^n \neq \varnothing$).

Now the third inequality follows:

$$s_n = t_n + T_{n-1} \geq 2T_{n-1} \geq 2s_{n-1}. \qquad \square$$

COROLLARY 2.8. *For any* g, *we have* $t_n = \mathrm{Time}(W^n, V^n) \geq 2^{n-1}$.

LEMMA 2.9. $\mathrm{Time}(W^n, W^{n-2}) \geq \mathrm{Time}(V^n, V^0)$.

Proof. We have:

$$\mathrm{Time}(W^n, W^{n-2}) = s_n + s_{n-1} = t_n + T_{n-1} + s_{n-1},$$

while

$$\begin{aligned}\mathrm{Time}(V^n, V^0) &= \mathrm{Time}(A^{n-1}, W^{n-1}) + \mathrm{Time}(W^{n-1}, V^0)\\ &= T_{n-1} + s_{n-1} + \cdots + s_0.\end{aligned}$$

Thus, the desired inequality is reduced to:

$$t_n \geq s_{n-2} + \cdots + s_0.$$

Now the first two inequalities of Lemma 2.7 imply that $t_n \geq s_{n-1}$, and the last one implies that $s_{n-1} \geq s_{n-2} + \cdots + s_0$. □

Take some W^n, and let l_0 be the smallest $l \geq \mathrm{Time}(V^n, V^0)$ such that $g^l(W^n) \subset W^0$.

LEMMA 2.10. $l_0 \leq \mathrm{Time}(W^n, W^{n-2})$.

Proof. Let $p = \mathrm{Time}(V^n, V^0)$, $l = \mathrm{Time}(W^n, W^{n-2})$. By Lemma 2.9, $l \geq p$. Moreover, $g^l(W^n) = W^{n-2} \subset W^0$. Hence $l \geq l_0$ by definition of l_0. □

We will now make some combinatorial choices.

Fix some (big) m. Let h be the generalized renormalization of g on W^0. Assuming $\mathrm{Time}_h(g^{l_0}(W^n)), W^0) \geq m$, let $l_0 < l_1 < l_2 < \cdots < l_m$ be the m consecutive return moments of $\mathrm{orb}(g^{l_0}(W^n))$ to W^0. In other words,

$$g^{l_k}(W^n) = h^k(g^{l_0}(W^n)).$$

Let $n > \log_2 m + 5$. Then by Corollary 2.8 and Lemma 2.10,

$$\begin{aligned}\mathrm{Time}_h(g^{l_0}(W^n), W^0) &\geq \mathrm{Time}_h(W^{n-2}, V^{n-2})\\ &> \mathrm{Time}_h(W^{n-3}, V^{n-3})\\ &\geq \mathrm{Time}_{g_1}(W^{n-3}, V^{n-3}) > 2^{n-5} > m.\end{aligned} \tag{2.3}$$

It follows that the moments l_k are well defined and

$$l_m - l_0 < \mathrm{Time}(W^{n-3}, V^{n-3}) < \mathrm{Time}(W^{n-2}, V^{n-2}). \tag{2.4}$$

Putting this estimate together with Lemma 2.10, we conclude:

LEMMA 2.11. $l_m < \mathrm{Time}(W^n, V^{n-2})$.

2.6. *Degrees.* Let $O = (g^{l_k}(W^n))_{k=0}^m$. By Lemma 2.11, O is contained in the piece $\mathcal{T}$ of $\mathrm{orb}_h W^n$ beginning with W^n and ending with V^{n-2}. Let us split $\mathcal{T}$ into five pieces. Namely, let $\mathcal{T}_i$ be the pieces of $\mathcal{T}$ between two consecutive domains, E^i and E^{i-1}, of the sequence

$$W^n \equiv E^{2n+1}, E^{2n}, \ldots, E^{2n-4} \equiv V^{n-2}. \tag{2.5}$$

Let $O_i = \mathcal{T}_i \cap O$.

By (2.3), each $\mathcal{T}_i$ has length bigger than m. Hence at most two of the pieces O_i are non-empty, and so one of them contains at least $m/2$ elements. Now, let O_i stand for such a piece.

Let us consider the enlargement $\hat{E}^{i-1}$ of E^{i-1}. Notice that it is contained in W^{n-3}. Pull $\hat{E}^{i-1}$ back along the h-orbit of W^n. This inscribes every domain of this orbit, $W^n, h(W^n), \dots, h^s(W^n) = E^{i-1}$, into a bigger *buffer* domain $F, h(F), \dots, h^s(F) = \hat{E}^{i-1}$.

By Lemma 2.6, we have:

LEMMA 2.12. *The map $h^s : F \to \hat{E}^{i-1}$ has degree at most d^5.*

Moreover,

LEMMA 2.13. *The domains $h^k(F)$ enclosing the domains of O_i are pairwise disjoint.*

Proof. Otherwise there would be two nested domains $h^k(F) \subset h^s(F)$, $k < s$. Let $L = s - k$. Pushing $h^k(F)$ forward to $\hat{E}^{i-1}$ we see that $h^L(\hat{E}^{i-1}) \supset \hat{E}^{i-1}$. All the more, $h^L(W^{n-3}) \supset W^{n-3}$, so that $L \geq \mathrm{Time}_h(W^{n-3}, V^{n-3})$.

On the other hand, by (2.3),

$$L < \mathrm{Time}_h(W^{n-3}, V^{n-3}),$$

a contradiction. □

Let us now consider some domain $\Lambda = g^{l_k}(W^n) \in O_i$, and let $\Lambda' = g^{l_k}(F)$ be its buffer. Since there is a biholomorphic push-forward $(\Lambda', \Lambda) \to (\hat{E}^{i-1}, E^{i-1})$, we have:

LEMMA 2.14. $\mathrm{mod}(\Lambda' \smallsetminus \Lambda) = \mathrm{mod}(\hat{E}^{i-1} \smallsetminus E^{i-1})$.

Let $\Upsilon = g^{-l_k}(V^0)|0$.

LEMMA 2.15. *We have $W^n \subset \Upsilon \subset V^n$ and*

$$\deg(g^{l_k} : \Upsilon \to V^0) \leq d^{2n+m}.$$

Proof. The first inclusion is trivial. The second inclusion, $\Upsilon \subset V^n$, follows from $l_k \geq l_0 \geq \mathrm{Time}(V^n, V^0)$.

Let us estimate the degree. Let $s = \mathrm{Time}(V^n, V^0)$. Then

$$\deg(g^s : V^n \to V^0) = d^{2n}.$$

Let us now consider the first landing map H to W^0. It is easy to see that each component Q_j of the domain of H is mapped biholomorphically onto W^0 and, moreover, $H : Q_j \to W^0$ admits an extension to a biholomorphic isomorphism $\tilde{Q}_j \to V^0$. Let $\Upsilon_i = g^{l_i}(\Upsilon)$. Then we have:

$$\Upsilon_0 = H \circ g^s(\Upsilon)$$

and

$$\Upsilon_{i+1} = H \circ (g|W^0)\,(\Upsilon_i), \quad i = 0, 1, \dots, k-1 \le m-1,$$

and the Telescope Lemma concludes the proof. □

2.7. *Summary.* We fix an arbitrary m and let $n > \log_2 m + 5$. Then for any domain $\Lambda = \Lambda_k = g^{l_k}(W^n) \in O_i$, the map $\Psi = \Psi_k = g^{l_k} : W^n \to \Lambda$ admits a holomorphic extension to a branched covering

$$\Psi : (\Upsilon, F, W^n) \to (V^0, \Lambda', \Lambda) \tag{2.6}$$

such that:

(P1) $\deg(\Psi : \Upsilon \to V^0) \le d^{2n+m}$;

(P2) $\deg(\Psi : F \to \Lambda') \le d^5$;

and

(P3) $\Upsilon \subset V^n$;

(P4) $\operatorname{mod}(\Lambda' \smallsetminus \Lambda) = \operatorname{mod}(\hat{E}^{i-1} \smallsetminus E^{i-1})$.

Moreover, there are at least $m/2$ domains Λ_k in the orbit O_i, and their buffers Λ'_k are pairwise disjoint.

3. **Quasi-Additivity Law and Covering Lemma**

QUASI-ADDITIVITY LAW [KL09, §10.3]. *Fix some $\eta > 0$. Let $W \Subset V$ and $\Lambda_i \Subset \Lambda'_i \Subset W$, $i = 1, \dots, m$, be topological disks such that the closures of Λ'_i are pairwise disjoint. Then there exists a $\delta_0 > 0$ (depending on η and m) such that: If for some $\delta \in (0, \delta_0)$,* $\operatorname{mod}(V \smallsetminus \Lambda_i) < \delta$ *while* $\operatorname{mod}(\Lambda'_i \smallsetminus \Lambda_i) > \eta\delta$, *then*

$$\operatorname{mod}(V \smallsetminus W) < \frac{2\eta^{-1}\delta}{m}.$$

QUASI-INVARIANCE LAW/COVERING LEMMA [KL09]. *Fix some $\eta > 0$. Let $U \Supset A' \Supset A$ and $V \Supset B' \Supset B$ be two nests of topological disks. Let $g : (U, A', A) \to (V, B', B)$ be a branched covering between the respective disks. Let $d = \deg(A' \to B')$ and $D = \deg(U \to V)$. Assume*

$$\operatorname{mod}(B' \smallsetminus B) > \eta \operatorname{mod}(U \smallsetminus A).$$

If $\operatorname{mod}(U \smallsetminus A) < \varepsilon(\eta, D)$ *then*

$$\operatorname{mod}(V \smallsetminus B) < 2\eta^{-1} d^2 \operatorname{mod}(U \smallsetminus A).$$

4. *A priori* bounds

The following lemma tells us that if some principal modulus is very small then it should be even smaller on some preceding level:

LEMMA 4.1. *There exist* $n = n(d) \in \mathbb{N}$ *and* $\varepsilon = \varepsilon(d, n) > 0$ *such that*: *If on some level* $q \geq n$, $\operatorname{mod}(V^q \smallsetminus W^q) < \varepsilon$, *then on some previous level* $p < q$,

$$\operatorname{mod}(V^p \smallsetminus W^p) < \tfrac{1}{2} \operatorname{mod}(V^q \smallsetminus W^q). \tag{4.1}$$

Proof. We will use the set-up of Section 2.7, except that the base GPL map g will not be g_0 but rather g_s on some deeper level. Let us fix some $m > 2^7 d^{23}$. Let $q > n > \log_2 m + 5$. We take $g = g_{q-n}$ as the base map and consider the associated 3-domain branched covering $\Psi = \Psi_k$ (2.6)

$$\Psi : (\Upsilon, F, W^q) \to (V^{q-n}, \Lambda', \Lambda),$$

where $\Lambda = \Lambda_k$ is one of the domains of the orbit O_i. Set $\eta = 1/2d$ for the First Covering Lemma. Let us consider two cases:

Case 1. Assume that for some domain $\Lambda \in O_i$,

$$\operatorname{mod}(\Lambda' \smallsetminus \Lambda) < \frac{1}{2d} \operatorname{mod}(V^q \smallsetminus W^q).$$

By Property (P4), $\operatorname{mod}(\Lambda' \smallsetminus \Lambda) = \operatorname{mod}(\hat{E}^{i-1} \smallsetminus E^{i-1})$, which is equal to either $\operatorname{mod}(V^{(i-2)/2} \smallsetminus W^{(i-2)/2})$ (if i is even) or to

$$\begin{aligned} \operatorname{mod}(U^{(i-3)/2} \smallsetminus A^{(i-3)/2}) &\geq \operatorname{mod}(\Delta^{(i-3)/2} \smallsetminus A^{(i-3)/2}) \\ &\geq \frac{1}{d} \operatorname{mod}(V^{(i-3)/2} \smallsetminus W^{(i-3)/2}) \quad \text{if } i \text{ is odd.} \end{aligned}$$

In both cases we conclude that (4.1) holds for p which is equal to either $(i-2)/2$ or $(i-3)/2$. (Note that $p < q$ since by construction of the buffers, $i - 1 < 2q + 1$.)

Case 2. Assume that for all $\Lambda_k \in O_i$,

$$\operatorname{mod}(\Lambda'_k \smallsetminus \Lambda_k) \geq \frac{1}{2d} \operatorname{mod}(V^q \smallsetminus W^q). \tag{4.2}$$

Then the Covering Lemma is applicable to every map $\Psi = \Psi_k$, provided $\varepsilon = \varepsilon(d, n)$ is sufficiently small. It yields:

$$\operatorname{mod}(V^{q-n} \smallsetminus \Lambda_k) \leq 4d^{11} \operatorname{mod}(\Upsilon \smallsetminus W^q) \leq 4d^{11} \operatorname{mod}(V^q \smallsetminus W^q). \tag{4.3}$$

Estimates (4.2) and (4.3) show that the Quasi-Additivity Law is applicable with $\eta = 1/8d^{12}$. Since there are at least m domains $\Lambda_k \subset \Lambda'_k \subset W^{n-q}$ in the orbit O_i, this implies:

$$\operatorname{mod}(V^{n-q} \smallsetminus W^{n-q}) \leq \frac{2^6 d^{23}}{m} \operatorname{mod}(V^q \smallsetminus W^q) < \frac{1}{2} \operatorname{mod}(V^q \smallsetminus W^q),$$

and we are done. □

Lemma 4.1 immediately yields Theorem B from the introduction.

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