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By Frederick W. Gehring and Gaven J. Martin

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Minimal co-volume hyperbolic lattices, I: The spherical points of a Kleinian group

By FREDERICK W. GEHRING and GAVEN J. MARTIN

Abstract

We identify the two minimal co-volume lattices of the isometry group of hyperbolic 3-space that contain a finite spherical triangle group. These two groups are arithmetic and are in fact the two minimal co-volume lattices. Our results here represent the key step in establishing this fact, thereby solving a problem posed by Siegel in 1945. As a consequence we obtain sharp bounds on the order of the symmetry group of a hyperbolic 3-manifold in terms of its volume, analogous to the Hurwitz $84g-84$ theorem of 1892.

The finite spherical subgroups of a Kleinian group give rise to the vertices of the singular graph in the quotient orbifold. We identify the small values of the discrete spectrum of hyperbolic distances between these vertices and show these small values give rise to arithmetic lattices. Once vertices are sufficiently separated, one obtains volume bounds by studying equivariant sets.

1. Introduction

In 1945, Siegel [Sie43] [Sie45] posed the problem of identifying the minimal co-volume lattices of isometries of hyperbolic n-space, or more generally rank-1 symmetric spaces. He solved the problem in two dimensions, identifying the $(2,3,7)$-triangle group as the unique lattice of minimal co-area. Siegel in fact proved what has come to be known as the signature formula from which one may deduce the complete spectrum of co-areas of lattices of the hyperbolic plane. Každan and Margulis [KM68] showed that for each n, the infimum of the co-volume of lattices is positive, answering a question of Selberg.

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At the time of Siegel's result, the theory of covering spaces was not well developed, and he could only suggest a connection between minimal co-area lattices and Hurwitz's $84g-84$ theorem of 1892 [Hur92], which bounds the order of the symmetry group of a Riemann surface in terms of its genus. This connection was confirmed by Macbeath [Mac61]. Selberg's lemma [Sel60] established the existence of torsion-free subgroups of finite index in hyperbolic lattices, among other things. As a consequence of the Mostow rigidity theorem [Mos68], the $84g-84$ theorem takes its expression in terms bounding the order of the symmetry group of a hyperbolic manifold by its volume. The hyperbolic manifolds with maximal symmetry groups are the quotients of the minimal co-volume lattice by its torsion-free normal subgroups. These manifolds and the associated factor groups are studied in [CMT06].

This paper is part of a series that solves Siegel's problem in three dimensions: the unique minimal co-volume lattice of $\mathbb{H}^3$ is the orientation-preserving subgroup of the $\mathbb{Z}_2$ extension of the tetrahedral reflection group with Coxeter diagram 3−5−3. (Note too that it follows that this latter group is uniquely minimal co-volume if we give up the orientation-preserving hypothesis.)

In fact, the key case of Kleinian groups with finite spherical subgroups is dealt with here, see Theorems 10.1, 10.2 and 10.3 where we obtain the sharp bounds in the case of the icosahedral, octahedral and tetrahedral groups, respectively. The case of Kleinian groups with simple elliptic elements of order at least 4 is dealt with in [GM98]. The elliptic elements of a lattice are those of finite order, and "simple" means that the orbit of their fixed-point set (axis) forms a disjoint family of hyperbolic lines; any elliptic of order 7 or more is automatically simple. In the torsion-free case for our purposes, only elementary co-volume bounds are needed. However, it is worth pointing out that in that case remarkable new bounds have been obtained by Agol and Dunfield (see [AST07] for a proof) utilizing Perelman's work. They give 0.67 as a lower bound on the co-volume of any hyperbolic manifold, which is quite close to the conjectured sharp bound 0.9427 attained by the Weeks manifold. In the non-uniform case (that is, for lattices that are not co-compact) the sharp bound 0.0846 attained in $\mathrm{PGL}(2,\mathbb{O}_3)$ is due to Meyerhoff [Mey86]. The only remaining problem, dealt with in the sequel [MM08], is essentially to obtain lower bounds on the co-volume of lattices with only simple elliptics of order 2 and 3. In fact, these lattices are expected to have much larger co-volume than the lattices identified in this paper, yet only marginally larger bounds are currently known. Further, the proof in that case is completely different from the largely geometric approach given here. It is much more closely allied with the results of Gabai, Meyerhoff and Thurston on the topological rigidity theorem [GMT03], as it involves a significant computational component.

Before getting down to details, many involving quite complex calculations in hyperbolic trigonometry, let us sketch the basic idea of the proof. Definitions of the terms used appear below in the body of the paper.

We start with a Kleinian group Γ containing a spherical triangle subgroup G isomorphic to either the icosahedral, octahedral or tetrahedral group. This subgroup G stabilizes a point $P \in \mathbb{H}^3$, and the axes of rotation of the elements in G are in general position. Let Q be the closest translate of P under $\Gamma \setminus G = \{f \in \Gamma : f \notin G\}$. Then Q is stabilized by an isomorphic spherical triangle group whose axes are also in general position. In [GM94], [GMMR97], we show that the spectrum of distances between the axes of rotation of elements of finite order in a Kleinian group is initially discrete; either the axes intersect or they are definite distances apart. Thus either some of the axes emanating from P and Q meet (possibly coinciding), or they are uniformly separated. Those that meet do so at specific angles, and these facts together force P and Q to be a definite distance apart. The sharp bounds on these distances in each case is given in Theorem 9.1. The proof of this result occupies the main body of the paper. This theorem then provides us with a hyperbolic ball $\mathbb{B}_\delta$ of a definite size about P which is disjoint from all of its translates under $\Gamma \setminus G$, and so the volume of the quotient $\mathbb{H}^3/\Gamma$ exceeds that of $\mathbb{B}_\delta/G$. Because G is finite and stabilizes $\mathbb{B}_\delta$, we have $\mathrm{Vol}(\mathbb{B}_\delta/G) = \mathrm{Vol}(\mathbb{B}_\delta)/|G|$. What we must do is carefully analyze the geometry of the situation described to ensure that $\mathbb{B}_\delta$ is sufficiently large so that the volume of $\mathbb{B}_\delta/G$ exceeds that of a known example. Of course this is not possible in general. For instance, the inequality

$$\mathrm{Vol}(\mathbb{B}_\delta/G) \geq \mathrm{Vol}(\mathbb{H}^3/\Gamma)$$

cannot hold for the minimal co-volume lattice since the translates of $\mathbb{B}_\delta$ cannot fill hyperbolic space. To proceed from here, we show that configurations of spherical points (points stabilized by spherical subgroups) in a discrete group that give rise to a small ball $\mathbb{B}_\delta$ imply arithmeticity of the group Γ. This is possible since we have shown [GMMR97] that two-generator discrete groups are arithmetic if they are generated by elements of finite order whose axes are sufficiently close. Once we have determined a group to be arithmetic, we may calculate the minimal co-volume group in which it embeds [GMMR97]. We therefore eliminate small configurations using arithmetic criteria until $\mathbb{B}_\delta$ is sufficiently large. In this way, we not only identify the minimal example but are also able to identify all sufficiently small co-volume examples as arithmetic.

It is somewhat of a curiosity to note that the small co-volume examples we identify are arithmetic Kleinian groups generated by two elements of finite order with trace fields of small discriminant. This is somewhat analogous to the two dimensional situation with arithmetic triangle groups.

Remarks. In a very nice geometric paper, Derevnin and Mednykh [DM88] found the minimal distances between icosahedral points in a Kleinian group. Unfortunately, their methods do not appear to carry over to the case of the other spherical triangle groups, and for our applications we need to know a little more about the spectrum of distances between icosahedral points. However their paper certainly served as an inspiration for our work here. We would also like to thank the referee for carefully reading the paper and making many valuable suggestions and corrections.

2. **Kleinian groups**

A *Kleinian* group Γ is a discrete nonelementary subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$, the group of orientation-preserving isometries of hyperbolic 3-space $\mathbb{H}^3$. In this setting, *nonelementary* means that Γ does not contain an abelian subgroup of finite index. A *lattice* is a Kleinian group Γ for which $\mathbb{H}^3/\Gamma$ has finite volume.

The discrete finite subgroups of isometries of $\mathbb{H}^3$ are classified; see for instance [Bea83], [Mas88], [Rat94]. They are the spherical triangle groups and the cyclic groups: the *tetrahedral group* A_4, the *octahedral group* S_4, the *icosahedral group* A_5 and the cyclic and dihedral groups $\mathbb{Z}_n$ and $\mathbb{D}_n$. Apart from the cyclic groups, each such group stabilizes a unique point in $\mathbb{H}^3$.

The orbit space of a Kleinian group $\mathfrak{Q} = \mathbb{H}^3/\Gamma$ is a *hyperbolic* 3-*orbifold* (or *manifold* if Γ is torsion-free). The orbifold $\mathfrak{Q}$ is a manifold away from the singular locus that consists of the projection to $\mathfrak{Q}$ of the fixed points of elements of Γ. The singular locus is a trivalent graph whose vertices are the projection to $\mathfrak{Q}$ of points in $\mathbb{H}^3$ stabilized by a finite spherical subgroup. Thus we define a *spherical point* of a Kleinian group Γ to be a point $x_0 \in \mathbb{H}^3$ stabilized by one of the spherical triangle subgroups A_4, S_4 or A_5 of Γ. We refer to such a point as a tetrahedral, octahedral or icosahedral point, respectively.

In this paper, we give sharp lower bounds for the hyperbolic distance between spherical points in a Kleinian group. Our results here are summarized in Theorem 9.1. This theorem yields sharp lower bounds for edge lengths in the singular set of a hyperbolic 3-fold. We then use these estimates to construct equivariant sets about the orbit of a spherical point that project to the quotient in a simple way. From this we obtain co-volume estimates. Let us discuss the extremals for a moment; see [GMMR97] for details.

- Γ_0 is the arithmetic Kleinian group obtained as a $\mathbb{Z}_2$-extension of the index-two orientation-preserving subgroup of the group generated by reflection in the faces of the 3−5−3-hyperbolic Coxeter tetrahedron. Γ_0 is a two-generator group, generated by elements g of order 2 and f of order 5. The invariant trace field of this group is $\mathbb{Q}(\gamma)$ of discriminant -275, where γ is a complex root of $z^4+5z^3+7z^2+3z+1$. The associated quaternion algebra is unramified at finite places. If f is chosen to be primitive ($\mathrm{tr}(f)=\pm 2\cos(\pi/5)$), then the group is uniquely defined up to conjugacy if we choose representatives in $\mathrm{PSL}(2,\mathbb{C})$ so that $\mathrm{tr}[f,g]-2=\gamma$.

- Γ_1 is a two-generator arithmetic Kleinian group generated by elliptic elements of order 2 and 3. The invariant trace field of this group is $\mathbb{Q}(\gamma)$ of discriminant -283, where γ is a complex root of $z^4+5z^3+2z^2+z+1$. The associated quaternion algebra is unramified at finite places. The group is uniquely defined up to conjugacy if we choose representatives in $\mathrm{PSL}(2,\mathbb{C})$ so that $\mathrm{tr}[f,g]-2=\gamma$.

THEOREM 2.1. *Let Γ be a Kleinian group with a tetrahedral, octahedral or icosahedral subgroup. Then*

$$\begin{aligned} &\mathrm{Vol}(\mathbb{H}^3/\Gamma)=\mathrm{Vol}(\mathbb{H}^3/\Gamma_0)=0.03905 \quad \textit{and} \quad \Gamma=\Gamma_0, \quad \textit{or}\\ &\mathrm{Vol}(\mathbb{H}^3/\Gamma)=\mathrm{Vol}(\mathbb{H}^3/\Gamma_1)=0.0408 \quad \textit{and} \quad \Gamma=\Gamma_1, \quad \textit{or}\\ &\mathrm{Vol}(\mathbb{H}^3/\Gamma)>0.042. \end{aligned}$$

For completeness we also recall earlier results from [GM05], [GM99] and [GM98].

THEOREM 2.2. *Let Γ be a Kleinian group and suppose that Γ has a torsion-free subgroup of index ≤ 4 or that Γ contains an element of order $n\geq 4$. Then*

$$\mathrm{Vol}(\mathbb{H}^3/\Gamma)\geq 0.041>\mathrm{Vol}(\mathbb{H}^3/\Gamma_1). \tag{2.3}$$

Together, Theorems 2.1 and 2.2 imply that the only missing case necessary for determining the Kleinian group of minimal co-volume is the case that the Kleinian group does not have a torsion-free subgroup of index 4 and all the torsion is of order 2 and 3. This is the problem solved in [MM08].

3. **Preliminaries**

We denote the hyperbolic metric of $\mathbb{H}^3$ by $\rho(x,y)$ and view hyperbolic 3-space $\mathbb{H}^3$ as the upper-half-space of $\mathbb{R}^3$:

$$\mathbb{H}^3=\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3>0\}.$$

Other than the identity, there are three types of orientation-preserving isometries of $\mathbb{H}^3$:

- *parabolic*: f is conjugate to the translation $z\to z+1$;
- *elliptic*: f is conjugate to the rotation $z\to\lambda z$ with $|\lambda|=1$; and
- *loxodromic*: f is conjugate to the dilation $z\to\lambda z$ with $|\lambda|\neq 1$.

Loxodromic and elliptic transformations have two fixed points on the Riemann sphere $\overline{\mathbb{C}} = \partial\mathbb{H}^3$. For such g, the *axis* is the closed hyperbolic line joining these two fixed points. If g is elliptic, then every point on its axis is fixed by g. If g is loxodromic, then the axis is only setwise fixed.

We next recall some simple but needed facts about the spherical triangle groups. Their proof is by elementary calculations in spherical trigonometry.

LEMMA 3.1. *Let $t_{2,3}$ and $t_{3,3}$ denote, respectively, the angles subtended at the origin between the axes of order* 2 *and* 3 *and the axes of order* 3 *of a spherical* (2,3,3)*-triangle group. Then*

$$\cos(t_{2,3}) = 1/\sqrt{3} \quad \textit{and} \quad \cos(t_{3,3}) = 1/3.$$

LEMMA 3.2. *Let $o_{2,3}$, $o_{2,4}$ and $o_{3,4}$ denote the angles subtended at the origin between the axes of order* 2 *and* 3, *the axes of order* 2 *and* 4, *and the axes of order* 3 *and* 4, *respectively, of a spherical* (2,3,4)*-triangle. Then*

$$\cos(o_{2,3}) = \sqrt{2/3}, \quad \cos(o_{2,4}) = 1/\sqrt{2}, \quad \cos(o_{3,4}) = 1/\sqrt{3}.$$

LEMMA 3.3. *Let $i_{2,3}$, $i_{2,5}$ and $i_{3,5}$ denote the angles subtended at the origin between the axes of order* 2 *and* 3, *the axes of order* 2 *and* 5, *and the axes of order* 3 *and* 5, *respectively, of a spherical* (2,3,5)*-triangle. Then*

$$\cos(i_{2,3}) = \frac{2}{\sqrt{3}}\cos(\pi/5), \quad \cos(i_{2,5}) = \frac{1}{2}\csc(\pi/5), \quad \cos(i_{3,5}) = \frac{1}{\sqrt{3}}\cot(\pi/5).$$

4. **Geometric lemmas**

We next establish three results which will be useful in what follows. In particular we establish the important formula (4.3) below giving a relation between the lengths and angles at which two hyperbolic segments meet a pair of hyperbolic lines in $\mathbb{H}^3$. It was pointed out to us that this formula could also be proved, with some care, using the three-dimensional hyperbolic trigonometric formula for a right-angled hexagon given by Fenchel, [Fen89, p. 82].

The second and third results below allow us to analyze the relation between the geometric quantities that occur in the first lemma so as to identify extremal configurations.

LEMMA 4.1. *Suppose that λ_1 and λ_2 are hyperbolic lines in $\mathbb{H}^3$ with disjoint pairs of endpoints z_1, w_1 and z_2, w_2, and suppose that γ is a hyperbolic segment of length ℓ with endpoints $p_1 \in \lambda_1$ and $p_2 \in \lambda_2$. If γ forms angles of ψ_1 and ψ_2 with the half lines of λ_1 and λ_2 from p_1 to z_1 and p_2 to z_2, respectively, then*

$$\sin(\psi_1)\sin(\psi_2)\cosh(\ell + i\phi) - \cos(\psi_1)\cos(\psi_2) = 2\,(z_1, z_2, w_2, w_1) - 1, \tag{4.2}$$

where ϕ is the dihedral angle formed by the hyperbolic planes determined by the segment γ and the above half lines of λ_1 and λ_2 and

$$(z_1, w_1, w_2, z_2) = \frac{(z_1 - w_2)(w_1 - z_2)}{(z_1 - z_2)(w_1 - w_2)} \quad \textit{is the cross ratio.}$$

Proof. By performing a preliminary Möbius transformation, we may assume that γ lies in the j-axis (the set of points $\{(0,0,x_3) : x_3 > 0\}$) and that $z_1 = a$, $w_1 = -b$, $z_2 = c\, e^{i\phi}$ and $w_2 = -d\, e^{i\phi}$, where a, b, c, d are real and positive. By relabeling, we may also assume that $ab < cd$. Then $p_1 = (0, 0, \sqrt{ab})$ and $p_2 = (0, 0, \sqrt{cd})$, a consequence of the fact that the foot of the altitude of a right triangle divides the hypotenuse into segments, the product of whose lengths is the square of the length of the altitude. Hence

$$\ell = \log \frac{\sqrt{cd}}{\sqrt{ab}}, \quad \sin(\psi_1) = \frac{2\sqrt{ab}}{a+b}, \quad \sin(\psi_2) = \frac{2\sqrt{cd}}{c+d},$$

from which it follows that

$$\cos(\psi_1) = \frac{a-b}{a+b} \quad \text{and} \quad \cos(\psi_2) = -\frac{c-d}{c+d}.$$

Next

$$(z_1, z_2, w_2, w_1) = \frac{cd\, e^{i\phi} + (ac+bd) + ab\, e^{-i\phi}}{(a+b)(c+d)}$$

and

$$\begin{aligned} 2(z_1,w_1,w_2,z_2) - 1 &= \frac{2cd\,e^{i\phi} + (a-b)(c-d) + 2ab e^{-i\phi}}{(a+b)(c+d)} \\ &= +\frac{2\sqrt{ab}}{a+b}\frac{2\sqrt{cd}}{c+d}\frac{1}{2}\Big(\frac{\sqrt{cd}}{\sqrt{ab}}e^{i\phi} + \frac{\sqrt{ab}}{\sqrt{cd}}e^{-i\phi}\Big) + \frac{a-b}{a+b}\frac{c-d}{c+d} \\ &= \sin(\psi_1)\sin(\psi_2)\cosh(\ell + i\phi) - \cos(\psi_1)\cos(\psi_2). \qquad \square \end{aligned}$$

Remarks. If γ^* is another hyperbolic segment of length ℓ^* that forms angles ψ_1^* and ψ_2^* with the lines λ_1 and λ_2 as above, then

$$\begin{aligned} &\sin(\psi_1^*)\sin(\psi_2^*)\cosh(l + i\phi) - \cos(\psi_1^*)\cos(\psi_2^*) \\ &= \sin(\psi_1)\sin(\psi_2)\cosh(\ell + i\phi) - \cos(\psi_1)\cos(\psi_2), \end{aligned}$$

where ℓ^* and ϕ^* are the corresponding length and dihedral angle for γ^*. In particular, if $\psi_1^* = \psi_2^* = \pi/2$, we obtain the formula

$$\sin(\psi_1)\sin(\psi_2)\cosh(\ell + i\phi) - \cos(\psi_1)\cos(\psi_2) = \cosh(\delta + i\theta), \tag{4.3}$$

where δ is the distance between the lines λ_1 and λ_2 and θ is the dihedral angle determined by the lines λ_1 and λ_2 and their common perpendicular. If $\delta = 0$, then λ_1 and λ_2 determine a hyperbolic plane, ϕ is equal to zero, θ is the angle at which

λ_1 and λ_2 meet in this plane, and (4.3) reduces to

$$\cosh(\ell) = \frac{\cos(\psi_1)\cos(\psi_2)+\cos(\theta)}{\sin(\psi_1)\sin(\psi_2)},$$

the second cosine law of plane hyperbolic geometry; see [Bea83].

We call $\delta + i\theta$ the *complex distance* between the lines λ_1 and λ_2.

The next two lemmas will allow us to derive estimates for ℓ in terms of δ from (4.3).

LEMMA 4.4. *Suppose that*

$$s\ \cosh(u+iv) - c = \cosh(x+iy), \tag{4.5}$$

where $0 < x, u < \infty$, $0 \le y, v \le \pi/2$ *and* $0 < s, c < 1$. *Then*

(1) $v \le y$;

(2) $\sinh(x) \le s\ \sinh(u)$, *whence* $x < u$;

(3) u *is increasing in* x *and increasing in* v *for fixed* y, s *and* c;

(4) u *is decreasing in* s *and decreasing in* c *for fixed* x, y *and* v.

Proof. Let $z = \cosh(x+iy)$ and $w = \cosh(u+iv)$. Then

$$\begin{aligned} |z+1| - |z-1| &= |\cosh(x+iy)+1| - |\cosh(x+iy)-1| \\ &= 2|\cosh^2((x+iy)/2)| - 2|\sinh^2((x+iy)/2)| \\ &= 2\cos(y) \end{aligned} \tag{4.6}$$

and similarly

$$|w+1| - |w-1| = 2\cos(v). \tag{4.7}$$

Thus to establish conclusion (1), it suffices to show that

$$|w+1| - |w-1| \ge |z+1| - |z-1| \tag{4.8}$$

whenever

$$sw - c = z \quad \text{for } \mathrm{Re}(w) \ge 0 \text{ and } 0 \le s, c \le 1. \tag{4.9}$$

We do this in two steps.

Suppose first that $c = 0$, fix $w \ne 0$, and let

$$f(s) = |s\,w+1| - |s\,w-1| \quad \text{for } 0 \le s \le 1.$$

Then

$$f(s) = \frac{4\,\mathrm{Re}(sw)}{|sw+1|+|sw-1|} \le \frac{2\,\mathrm{Re}(w)}{|w|} \tag{4.10}$$

and

$$\begin{aligned} f'(s) &= \frac{s|w|^2 + \mathrm{Re}(w)}{|sw+1|} - \frac{s|w|^2 - \mathrm{Re}(w)}{|sw-1|} \\ &= \frac{-s|w|^2 f(s) + \mathrm{Re}(w)(|sw+1| + |sw-1|)}{|sw+1|\,|sw-1|} \\ &\geq \frac{-s|w|^2 (2\,\mathrm{Re}(w)/|w|) + 2s\,\mathrm{Re}(w)|w|}{|sw+1|\,|sw-1|} = 0 \end{aligned}$$

by (4.10). Thus $f(s)$ is non-decreasing in s and

$$|z+1| - |z-1| = f(s) \leq f(1) = |w+1| - |w-1|$$

for $0 < s < 1$. This together with (4.6) and (4.7) implies conclusion (1) for the case where $c = 0$.

Suppose next that $0 < c \leq 1$, $s = 1$ and

$$w = \cosh(u+iv) = \cosh(x+iy) + c = z + c \quad \text{for } \mathrm{Re}(w) \geq 0.$$

Then

$$|w+1| - |w-1| \geq |z+1| - |z-1|,$$

and we again obtain conclusion (1) from (4.6) and (4.7). Now the inequality in (1) follows for $z = sw - c$ with $0 \leq c, s \leq 1$ from the two cases considered above by replacing w with sw: in the second case, $|z+1| - |z-1| \leq |sw+1| - |sw-1|$, whereas the first case gives $|sw+1| - |sw-1| \leq |w+1| - |w-1|$.

Next (4.5) gives the two equations

$$\begin{aligned} s\cosh(u)\cos(v) &= \cosh(x)\cos(y) + c, \\ s\sinh(u)\sin(v) &= \sinh(x)\sin(y). \end{aligned} \tag{4.11}$$

Hence $\sinh(x) = s(\sin(v)/\sin(y))\sinh(u) \leq s\,\sinh(u) < \sinh(u)$, and we get conclusion (2).

Fix the variables y, s, c. Then (4.11) defines u as a function of x and v. We differentiate each equation implicitly with respect to x and solve for $\partial u/\partial x$ to obtain

$$\frac{\partial u}{\partial x} = \frac{\cosh(u)\cosh(x)\sin(v)\sin(y) + \sinh(u)\sinh(x)\cos(v)\cos(y)}{s(\cosh^2(u) - \cos^2(v))} > 0$$

for $x > 0$. Thus u is increasing in x.

Next, differentiating the equations in (4.11) implicitly with respect to v yields

$$\frac{\partial u}{\partial v} = \frac{\cosh(u)\sinh(x)\sin(v)\cos(y) - \sinh(u)\cosh(x)\cos(v)\sin(y)}{\cosh(u)\cosh(x)\sin(v)\sin(y) + \sinh(u)\sinh(x)\cos(v)\cos(y)} = -\frac{D}{N}.$$

Then

$$\cosh(u)\sinh(x) = \sinh(x-u) + \sinh(u)\cosh(x) < \sinh(u)\cosh(x)$$

since $x - u < 0$, and

$$\sin(v)\cos(y) = \sin(v-y) + \cos(v)\sin(y) \le \cos(v)\sin(y)$$

since $v - y \le 0$. Thus $N < 0 < D$ for $v > 0$, and u is increasing in v.

Similarly, if we fix the variables x, y and v and if we then differentiate the second of the equations at (4.11), we find $\partial u/\partial s = -\tanh(u)/s < 0$. Finally, with a bit of manipulation (4.11) yields

$$\tanh(u)(\cosh(x)\cos(y) + c) = \cot(v)\sinh(x)\sin(y)$$

so that the right side is constant and $\partial u/\partial c = -\sinh(u)/(s\cos(v)) < 0$. Thus u is decreasing in s and decreasing in c. □

LEMMA 4.12. *Suppose that* $0 < a \le \pi/2$ *is a constant and that*

$$\begin{aligned} &s_1\cosh(u + iv_1) - c_1 = \cosh(x_1 + iy_1),\\ &s_2\cosh(u + iv_2) - c_2 = \cosh(x_2 + iy_2), \quad v_1 + v_2 = a, \end{aligned} \tag{4.13}$$

where $0 < x_j, u < \infty$, $0 < y_j, v_j \le \pi/2$ *and* $0 < s_j, c_j < 1$ *for* $j = 1, 2$. *Then* u *is increasing in* x_1 *and in* x_2 *for fixed* s_1, s_2, c_1 *and* c_2.

Proof. We fix x_2 and let y_1, y_2 vary with x_1. Lemma 4.4 applied to the equations in (4.13) implies that

$$v_1 \le y_1, \quad v_2 \le y_2, \quad x_1 < u, \quad x_2 < u. \tag{4.14}$$

Next, if we differentiate the real and imaginary parts of the equations in (4.13) with respect to x_1, we obtain

$$\begin{aligned} s_1\Big(\alpha_1\frac{\partial u}{\partial x_1} - \beta_1\frac{\partial v_1}{\partial x_1}\Big) &= \gamma_1 - \delta_1\frac{\partial y_1}{\partial x_1},\\ s_1\Big(\beta_1\frac{\partial u}{\partial x_1} + \alpha_1\frac{\partial v_1}{\partial x_1}\Big) &= \delta_1 + \gamma_1\frac{\partial y_1}{\partial x_1},\\ s_2\Big(\alpha_2\frac{\partial u}{\partial x_1} + \beta_2\frac{\partial v_1}{\partial x_1}\Big) &= -\delta_2\frac{\partial y_2}{\partial x_1},\\ s_2\Big(\beta_2\frac{\partial u}{\partial x_1} - \alpha_2\frac{\partial v_1}{\partial x_1}\Big) &= \gamma_2\frac{\partial y_2}{\partial x_1}, \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} \alpha_j &= \sinh(u)\cos(v_j), & \beta_j &= \cosh(u)\sin(v_j),\\ \gamma_j &= \sinh(x_j)\cos(y_j), & \delta_j &= \cosh(x_j)\sin(y_j). \end{aligned} \tag{4.16}$$

The equations in (4.15) imply that $\partial u/\partial x_1 = N/D$, where

$$\begin{aligned}4N &= (\alpha_2\delta_2-\beta_2\gamma_2)(\gamma_1^2+\delta_1^2) = N_1N_2,\\ D/s_1 &= (\alpha_1\gamma_1+\beta_1\delta_1)(\alpha_2\delta_2-\beta_2\gamma_2)+(\alpha_2\gamma_2-\beta_2\delta_2)(\alpha_1\delta_1-\beta_1\gamma_1)\\ &= D_1\,D_2+D_3\,D_4.\end{aligned}$$

Then from (4.14) and (4.16) we see that

$$\begin{aligned}N_1 &= \sin(v_2+y_2)\sinh(u-x_2)-\sin(v_2-y_2)\sinh(u+x_2)>0,\\ N_2 &= \cosh(2x_1)-\cos(2y_1)>0,\end{aligned}$$

and

$$\begin{aligned}D_1 &= \cosh(u+x_1)\cos(v_1-y_1)-\cosh(u-x_1)\cos(v_1+y_1)>0,\\ D_2 &= \sinh(u-x_2)\sin(v_2+y_2)-\sinh(u+x_2)\sin(v_2-y_2)>0,\\ D_3 &= \cosh(u+x_2)\cos(v_2-y_2)-\cosh(u-x_2)\cos(v_2+y_2)>0,\\ D_4 &= \sinh(u-x_1)\sin(v_1+y_1)-\sinh(u+x_1)\sin(v_1-y_1)>0.\end{aligned}$$

Thus $N>0$, $D>0$ and u is increasing in x_1 with x_2 fixed. Of course, by symmetry we must also have u is increasing in x_2 with fixed x_1. □

5. (p,q,r)-Kleinian groups

A (p,q,r)-Kleinian group is a Kleinian group generated by three rotations of order p,q,r about the edges of a hyperbolic triangle. In [GM05] we determined all of these groups and found various geometric constraints associated with them, in particular the vertex structure of the Kleinian group and the edge lengths possible for the hyperbolic triangle.

The relevance of these results is as follows. Suppose that Γ is a Kleinian group and suppose that two spherical points P and Q in $\mathbb{H}^3$ are stabilized by subgroups Γ_P and Γ_Q of Γ, respectively, where $\Gamma_P\cap\Gamma_Q\neq \mathrm{Id}$. Geometrically, this means that P and Q both lie on some common elliptic axis of Γ_P and Γ_Q. In this case we will say that P and Q have a *common axis*. We shall abuse notation and refer to an axis of P or Q when we mean the axis of an elliptic element in Γ_P or Γ_Q, respectively.

Now if there is an axis of P that meets an axis of Q other than the common axis, then the three elliptic axes form a hyperbolic triangle and the corresponding elliptics generate a (p,q,r)-Kleinian group. Hence the distance $\rho(P,Q)$ is bounded below by the corresponding possible edge-lengths of the associated hyperbolic triangle as computed in [GM05].

We record the various possibilities in the following theorem. In all instances, with the notable exception of S_4, we shall see that these distances are the sharp bounds.

The reader is encouraged to work out a few simple examples of (p, q, r)-groups. The key idea in determining discreteness is to analyze an obviously associated group generated by reflection in the sides of a hyperbolic tetrahedron naturally formed from the vertex stabilizers. The difficult cases occur when the dihedral angles are not submultiples of π. The following tables occur as [GM05, Tabs. 4–9, §10], where, in some cases, we have gleaned just a little more information from the body of that paper. The values in these and subsequent tables are approximate, with the implied accuracy. The same applies for values reported in theorems and text.

THEOREM 5.1. *Suppose that P and Q are spherical triangle points lying on a common axis η of order n, and suppose that an axis of P meets an axis of Q, possibly on the sphere at infinity, other than the axis η. If P is isomorphic to A_4 and if Q is isomorphic to A_4, S_4 or A_5, then the distance $\rho(P, Q)$ is either one of the 5 entries in the appropriate table below and η has the corresponding order n, or $\rho(P, Q)$ exceeds the 4th tabulated value or the 5th tabulated value in case the orders of the 4th and 5th entry are the same.*

(A_4, A_4)

n	$\rho(P, Q)$
3	0.69314
3	0.76914
3	0.92905
3	1.0050
2	1.06128

Table 5.1

(A_4, S_4)

n	$\rho(P, Q)$
3	1.01481
3	1.31696
3	1.43364
2	1.43796
3	1.49279

Table 5.2

(A_4, A_5)

n	$\rho(P, Q)$
3	1.22646
3	1.62669
2	1.76110
3	1.87988
3	1.98339

Table 5.3

If P is isomorphic to S_4 or A_5 and if Q is isomorphic to S_4 or A_5, then the distance $\rho(P, Q)$ is either one of the first 7 entries in the appropriate table below and η has the corresponding order n, or $\rho(P, Q)$ exceeds the 6th tabulated value or the 7th tabulated value in case the orders of the 6th and 7th entry are the same.

(S_4, S_4)

n	$\rho(P, Q)$
4	1.06128
4	1.12838
3,4	1.31696
4	1.38433
4	1.48710
3	1.56680
2	1.70004

Table 5.4

(S_4, A_5)

n	$\rho(P, Q)$
3	1.22646
3	1.98339
3	2.13275
2	2.27311
3	2.34868
2	2.35576
2	2.83641

Table 5.5

(A_5, A_5)

n	$\rho(P, Q)$
5	1.38257
5	1.61692
3	1.90285
5	2.04442
5	2.16787
5	2.22404
2	2.82643

Table 5.6

Remarks. The last entry in each table is either the next possible distance or the smallest distance possible on a common axis with the given order. Most of the distances are achieved in the orientation-preserving subgroups of groups generated by reflection in the faces of a hyperbolic tetrahedron. We shall use this data later, basically only in the cases (A_4, A_4), (S_4, S_4), and (A_5, A_5), and then only the first few distances. The additional data is presented for completeness and also, should the reader desire, it can be used to improve volume bounds in other special cases.

6. **Complex distances between elliptic axes**

In previous studies of Kleinian groups we described various portions of the parameter spaces for discrete groups generated by two elliptic transformations f and g. These results will be central to what follows. See, for example, [GMMR97], [GM96], [GM99], [Mar98]. The complex hyperbolic distance $\delta + i\theta$ between the axes of f and g is a natural parameter in this space. It satisfies the relation

$$\sinh^2(\delta \pm i\theta) = \frac{4\gamma(f,g)}{\beta(f)\beta(g)}. \tag{6.1}$$

Here the complex numbers $\beta(f)$, $\beta(g)$ and $\gamma(f,g)$ are the trace and commutator parameters for f and g, and are given by

$$\beta(f) = \mathrm{tr}(f)^2 - 4, \quad \beta(g) = \mathrm{tr}(g)^2 - 4, \quad \gamma(f,g) = \mathrm{tr}(fgf^{-1}g^{-1}) - 2, \tag{6.2}$$

where $\mathrm{tr}(h)$ denotes the trace of the matrix $C \in \mathrm{PSL}(\mathbb{C})$ that represents the Möbius transformation h. See [GM96]. In Figure 1, we illustrate the disk-covering technique

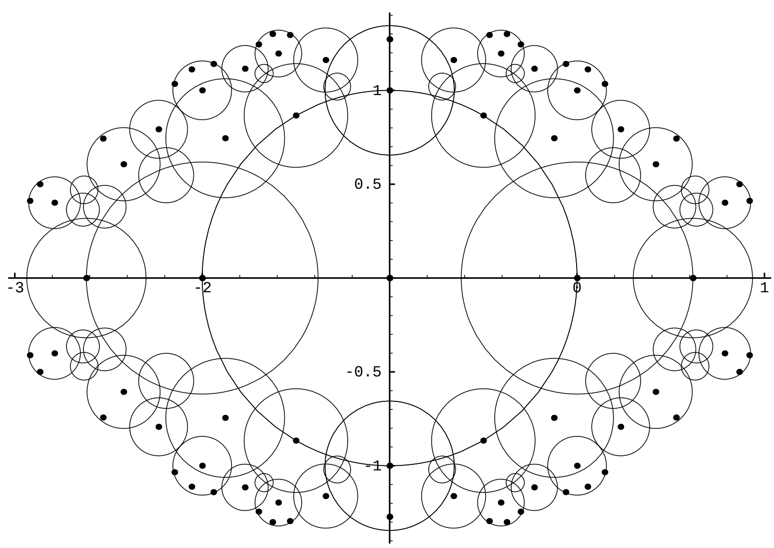

Figure 1. Possible values for the commutator parameter $\gamma(f,g)$ when f has order 4 and g has order 2.

used to identify all discrete groups generated by elliptics of order 2 and 4 whose axes are sufficiently close. For other similar examples, see the diagrams in [GMMR97, §11].

What we see in Figure 1 is part of a slice through the one-complex-dimensional space of discrete groups generated by two elliptics of the given order; it is somewhat akin to the Reilly slice for groups with two parabolic generators. The disks contain no commutator parameters for Kleinian groups other than those identified. For instance, a disk about 0 most often arises from application of a classical inequality such as Jørgensen's inequality [Jør76]. It is from such coverings that the following lists are generated. We will want to improve this particular case subsequently, and from this the reader should get a clear idea of how such pictures are generated. Note that the symmetry in Figure 1 is about the axis $\{x = -1\}$.

We will need the following result.

THEOREM 6.3. *Let* $\Gamma = \langle f, g\rangle$ *be a Kleinian group generated by elliptic elements of orders* p, *and* q *and let* $\delta + i\theta$ *be the complex distance between the axes of* f *and* g. *Then for each pair* (p, q), *where* $p = 3, 4$ *and* $q = 2, 3, 4$, *the complex distance* $\delta + i\theta$ *either appears in the corresponding table* (*on this page or the next*) *or the distance* δ *is greater than the last real entry in the table.*

$(p = 3, q = 2)$
$\delta \pm i\,\theta$
$0.19707 + i\ 0.78539$
$0.21084 + i\ 0.33189$
$0.23371 + i\ 0.49318$
$0.24486 + i\ 0.67233$
$0.24809 + i\ 0.40575$
$0.27407 + i\ 0.61657$
$0.27465 + i\ 0.78539$
$0.27702 + i\ 0.56753$
$0.27884 + i\ 0.22832$
0.28088

Table 6.1

$(p = 3, q = 3)$
$\delta \pm i\,\theta$
$0.39415 + i\ 1.57079$
$0.42168 + i\ 0.66379$
$0.46742 + i\ 0.98637$
$0.48973 + i\ 1.34468$
$0.49619 + i\ 0.81150$
$0.54814 + i\ 1.23135$
$0.54930 + i\ 1.57079$
$0.55404 + i\ 1.13507$
$0.55769 + i\ 0.45665$
0.56177

Table 6.2

The corresponding commutator values then follow from (6.1) and these are the identified points of Figure 1. Most of the complex distances $\delta + i\,\theta$ in the above tables occur when the group $\langle f, g\rangle$ is arithmetic. See the tables in [GMMR97, §8].

Notice that there is an elliptical region in Figure 1 bounded by the curve given by setting δ to be a fixed constant and letting θ vary. This ellipse is symmetric across the real line and also across the line $\{z \in \mathbb{C} : \mathrm{Re}(z) = -1\}$. In general, for two-generator groups with one generator of order two, there will be such a symmetry in the space of discrete groups. We will use this symmetry to reduce the space we must describe. For instance, suppose $\Gamma = \langle f, g\rangle$ with g of order two and f not parabolic. Let h be an elliptic of order two whose axis is perpendicular to the axis of g and passes through the point of the axis of g closest to the axis of f. Then $gfg^{-1} = hfh^{-1}$. So $\langle f, hfh^{-1}\rangle = \langle f, gfg^{-1}\rangle$ is discrete, and since $\langle f, h\rangle$ contains this group with index 2, it is also discrete. Thus $\langle f, h\rangle$ is Kleinian. We may compute, using (6.1), that

$$\gamma(f,h) = \beta(f) - \gamma(f,g),$$

since the axes of f and g are at the same distance of the axes of f and h, but the angle has been changed by $\pi/2$. This equation shows the space of discrete groups with one generator of order two is symmetric about the point $\beta(f)/2$. Of course complex conjugation is another obvious symmetry in our situation.

The reader will get an idea of how these values are identified as we now seek to improve bounds in the case of lattices generated by elements of order 2 and 4 in a special case. Namely, for a later application we wish to extend this for all complex distances $\delta + i\theta$ with $\delta < 0.57$ and $|\theta| < \pi/8$.

$(p=4, q=2)$
$\delta \pm i\,\theta$
$0.41572 + i\ 0.59803$
$0.42698 + i\ 0.44303$
$0.44068 + i\ 0.78539$
$0.50495 + i\ 0.67478$
$0.52254 + i\ 0.34470$
$0.52979 + i\ 0.24899$
$0.52979 + i\ 0.53640$
0.53063
$0.53063 + i\ 0.45227$
$0.53063 + i\ 0.78539$
0.53264

Table 6.3

$(p=4, q=3)$
$\delta \pm i\,\theta$
$0.54930 + i\ 1.57079$
0.61759

Table 6.4

$(p=4, q=4)$
$\delta \pm i\,\theta$
$0.83147 + i\ 1.19606$
$0.85397 + i\ 0.88606$
$0.88137 + i\ 1.57079$
$1.00991 + i\ 1.34957$
$1.04509 + i\ 0.68940$
$1.05959 + i\ 0.49798$
$1.05959 + i\ 1.07281$
1.06128
$1.06128 + i\ 0.90455$
$1.06128 + i\ 1.57079$
1.06528

Table 6.5

The first step in proving this extension is to identify all commutator parameters from the identity at (6.1) corresponding to axial distances less than 0.53264 from Table 6.3 with $p = 4$ and $q = 2$. As $\gamma = 2\sinh^2(\delta \pm i\,\theta)$ in this case, these are given in Table 6.6. There we have used the symmetries described above to report only those γ-values for which $\mathrm{Re}(\gamma) \geq -1$ and $\mathrm{Im}(\gamma) \geq 0$.

$p = 4,\ q = 2,\ \gamma = \mathrm{tr}[f, g] - 2$

$\delta \pm i\,\theta$	γ
$0.41572 + i\,0.59803$	$-0.5 + i\,0.8660$
$0.42698 + i\,0.44303$	$-0.1225 + i\,0.7448$
$0.44068 + i\,0.78539$	$-1 + i$
$0.50495 + i\,0.67478$	$-0.6588 + i\,1.1615$
$0.52254 + i\,0.34470$	$0.2327 + i\,0.7925$
$0.52979 + i\,0.53640$	$-0.2281 + i\,1.1151$
$0.52979 + i\,0.24899$	$0.4196 + i\,0.6062$
0.53063	0.6180
$0.53063 + i\,0.78539$	$-1 + i\,1.2720$
$0.53063 + i\,0.45227$	i

Table 6.6

Next we have the following polynomial trace identities, some of which can be found in [GM92], [GM94]. The remainder can be obtained from directly multiplying out the matrix representatives as in [GM94] by machine.

LEMMA 6.4. *Let f and g be elliptic transformations of order* 4 *and* 2, *respectively. Set $\gamma = \gamma(f, g)$. Then*

$$\gamma((gf)^4 g, f) = p_1(\gamma) = \gamma(-1 + \gamma + \gamma^2)^2,$$
$$\gamma((gf)^3 g, f) = p_2(\gamma) = \gamma^3(2 + \gamma),$$
$$\gamma((gf)^3(gf^{-1})^3 g, f) = -2 - p_3(\gamma) = -2 + (2 + \gamma)(1 + \gamma^2 + \gamma^3)^2,$$
$$\gamma((gf)^3(gf^{-1})^3(gf)^3 g, f) = p_4(\gamma) = \gamma(2 + \gamma)(1 + 2\gamma + \gamma^2 + 2\gamma^3 + \gamma^4)^2.$$

The following lemma from [GM92] is a specialized case of a much more general result.

LEMMA 6.5. *Let f and h generate a discrete group*, *and suppose that f is elliptic of order* 4. *Then there are elliptics g_1, g_2 of order* 2 *such that $\langle f, g_i \rangle$ is discrete and*

$$\gamma(f, g_1) = \gamma(f, h), \qquad \gamma(f, g_2) = \beta(f) - \gamma(f, h) = -2 - \gamma(f, h).$$

The proof of this lemma consists in identifying two different $\mathbb{Z}_2$ extensions of the group $\langle f, hfh^{-1} \rangle$ generated by two elements with the same trace. These two extensions will be two-generator groups with one generator of order two, namely g_1 or g_2 whose axes will be perpendicular. Further, $\gamma(f, g_i)$ will exhibit the symmetry about $-1 = \beta(f)/2$, as discussed earlier. We deduce the following corollary.

COROLLARY 6.6. *Let $\gamma = \gamma(f, g)$ be the commutator parameter for a discrete group generated by elliptics of orders* 4 *and* 2, *respectively. Then $p_i(\gamma)$ for $i = 1, 2, 3, 4$ is also the commutator parameter for such a discrete group.*

In what follows, all calculations are carried out to 10 digits and typically we report just the first four or five. We can now prove this:

THEOREM 6.7. *Let $\delta + i\theta$ be the complex distance between the axes of elliptic transformations f and g of orders* 4 *and* 2, *respectively, in a discrete nonelementary group. If $\delta < 0.57$ and $|\theta| < \pi/8$, then $\delta + i\theta$ is one of the four values*

$$0.5306, \quad 0.52254 + i\,0.34470, \quad 0.52979 + i\,0.24899, \quad 0.56419 + i\,0.39269.$$

Moreover, only in the second case does the complex distance $\delta + i(\pi/4 - \theta)$ not occur for a discrete group.

Proof. The last statement as well as the theorem follows for $\delta < 0.532$ from our tables. In the γ plane, the curves of constant distance δ are ellipses and the curves of constant angle θ are hyperbolas. Table 6.6 identifies all the commutators inside the upper-right quarter of the ellipse

$$\mathscr{E}_1 = \{z : |z + 2| + |z| < 2\cosh(2 \times 0.53264)\}.$$

We want to identify all commutator values inside the ellipse

$$\mathscr{E}_2 = \{z : |z + 2| + |z| < 2\cosh(2 \times 0.57)\}$$

with the proviso that $|\theta| < \pi/8$. These two ellipses and the hyperbola are identified in Figure 2 along with the commutator values.

We now describe what the disks are in that figure.

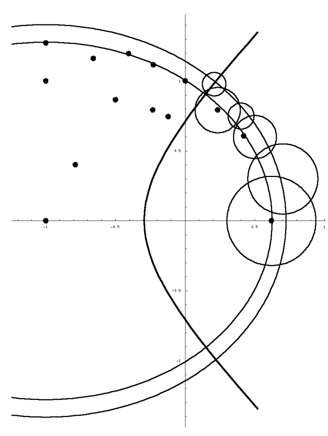

Figure 2. Enlarged ellipse with hyperbola $|\theta| < \pi/8$.

We claim that for each disk $\mathbb{B}_i$ and associated polynomial p_j listed above at Table 6.7 and given in Lemma 6.4, we have $p_j(\mathbb{B}_i) \in \mathscr{E}_1$. To see this, one verifies the inequality $|p_j(z)+2|+|p_j(z)| < 3.24628 = 2\cosh(2\times 0.53264)$ for $z \in \partial\mathbb{B}_i$, a one dimensional calculus problem which is computationally quite simple. These polynomials are open maps and thus the image of the disk is bounded by the image of its boundary. If this boundary is in the ellipse, then so is the image disk.

According to Corollary 6.6, only the preimages of 0, -1, -2 (the commutator values of the elementary discrete groups) together with the preimages of the commutator values already listed in Table 6.6 (and their symmetric points) could be the commutator values for discrete groups. Therefore we compute all the preimages of these points and decide which of these are in $\mathscr{E}_2 \setminus \mathscr{E}_1$ with $0 \le \theta \le \pi/8$. Most are not, but occasionally some such values occur. Let us examine the case of the disk $\mathbb{B}_3$. The image of $\mathbb{B}_3$ under the polynomial p_2 contains six points,

$$-1,\ -2,\ -1+i,\ -1.5+i\,0.8660,\ -1.8775+i\,0.7448,\ -1.3412+i\,1.1615.$$

Each of these points has a symmetric image appearing in Table 6.6. For example, $-1.5+i\,0.8660$ is the complex conjugate of $-2-(-0.5+i\,0.8660)$, and $-1.3412+i\,1.1615$ is conjugate to $-2-(-0.6588+i\,1.1615)$. The points $-0.5+i\,0.8660$ and $-0.6588+i\,1.1615$ lie in that table. We must examine which of the preimages in $\mathbb{B}_3$ of these six points lies in $\mathscr{E}_2$. For instance, of the four preimages of -1, only $z_0 = 0.419643+i\,0.606291$ lies in the disk, and this is a point already identified in Table 6.6 as lying in $\mathscr{E}_1$. The point $z' = 0.643309+i\,0.583691$ lies in $\mathbb{B}_3$ as the preimage of a point symmetric to $-0.6588+i\,1.1615$. We see that $|z'+2|+|z'| = 3.57563$ is slightly outside of $\mathscr{E}_2$, as are all other such points. Thus $(\mathscr{E}_2 \setminus \mathscr{E}_1) \cap \mathbb{B}_3$ contains no additional γ-values corresponding to discrete groups.

Similarly, none of the other disks except $\mathbb{B}_6$ contains new γ-values for Kleinian groups either. The fourth value of Theorem 6.7, $0.56419+i\,0.39269$, occurs when we consider the disk $\mathbb{B}_6$. Here $\gamma = 0.2071067+i\,0.97831$, a root of the polynomial p_4. This γ-value is actually that of an arithmetic lattice (according to the criteria of [GMMR97]) whose co-volume is approximately 1.032.

disk	polynomial
$\mathbb{B}_1 = \mathbb{B}(0.6180, 0.319)$	$p_1(z)$
$\mathbb{B}_2 = \mathbb{B}(0.7+i\,0.3, 0.25)$	$p_1(z)$
$\mathbb{B}_3 = \mathbb{B}(0.5+i\,0.6, 0.154)$	$p_2(z)$
$\mathbb{B}_4 = \mathbb{B}(0.4+i\,0.75, 0.093)$	$p_2(z)$
$\mathbb{B}_5 = \mathbb{B}(0.23278+i\,0.79255, 0.162)$	$p_3(z)$
$\mathbb{B}_6 = \mathbb{B}(0.20710+i\,0.97831, 0.084)$	$p_4(z)$

Table 6.7. Disk-covering.

All that remains to be checked is that the union of the preimages of these disks and their complex conjugates covers the region between the ellipses. This is again another computationally straightforward calculation. □

The reader can find a direct application of this result below. It identifies the three closest octahedral points on a common axis of order 4.

7. Spherical points on a common axis

In this section we shall identify the first few values of $\rho(P, Q)$ when P and Q are spherical points on a common axis. We deal with the cases where P and Q are pairs of tetrahedral, octahedral and icosahedral points separately. The most difficult case is where P and Q are tetrahedral points on a common axis of order 3. We consider this case first and give the argument in some detail. The proofs in all the other cases are quite similar and, in the interests of brevity, we leave it to the reader to fill in some of the details.

Tetrahedral points on a common 3 *axis.* If P and Q are tetrahedral points on a common elliptic axis η of order 3, then we can choose another elliptic axis α_3 of order 3 from Q that comes as close as possible to P. The point Q lies in the boundary of some fundamental region for the action of Γ_P on $\mathbb{H}^3$ formed by all hyperbolic rays from P passing through a $(2, 3, 3)$ spherical triangle on a small hyperbolic sphere about P. An initial segment of α_3 lies in one such region whose "edges" consist of the axis η and elliptic axes β_2 of order 2 and β_3 of order 3 from P. The angle between β_2 and β_3 at P is $t_{2,3}$.

Suppose first that α_3 meets β_j for $j = 2$ or 3. Then the elliptics corresponding to α_3, β_j and η generate a (p, q, r)-Kleinian group and the estimates of Table 5.1 imply that

$$\rho(P, Q) = 0.6931 \quad \text{or} \quad \rho(P, Q) \geq 0.7691. \tag{7.1}$$

This is illustrated in Figure 3.

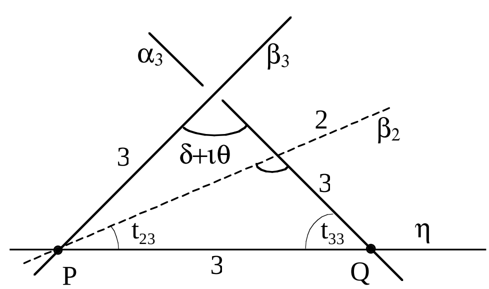

Figure 3. Tetrahedral points on common order 3 axis.

Suppose next that α_3 does not meet β_2 or β_3, and let ϕ_j be the dihedral angle formed by the two hyperbolic planes containing $\eta \cup \alpha_3$ and $\eta \cup \beta_j$. Then $\phi_2 + \phi_3$ is the dihedral angle between the hyperbolic planes containing $\eta \cup \beta_2$ and $\eta \cup \beta_3$, and hence $\phi_2 + \phi_3 = \pi/3$. Next let $\delta_j + i\theta_j$ be the complex distance between α_3 and β_j, let $\gamma \subset \eta$ be the hyperbolic segment joining P and Q, and let $\ell = \rho(P, Q)$. Then we obtain the two equations

$$\begin{aligned} \cosh(\ell + i\phi_2) &= \frac{\cos(t_{2,3})\cos(t_{3,3}) + \cosh(\delta_2 + i\theta_2)}{\sin(t_{2,3})\sin(t_{3,3})}, \\ \cosh(\ell + i\phi_3) &= \frac{\cos(t_{3,3})\cos(t_{3,3}) + \cosh(\delta_3 + i\theta_3)}{\sin(t_{3,3})\sin(t_{3,3})} \end{aligned} \tag{7.2}$$

from (4.3), where, by Lemma 3.1,

$$\cos(t_{2,3}) = \frac{1}{\sqrt{3}}, \quad \sin(t_{2,3}) = \frac{\sqrt{2}}{\sqrt{3}}, \quad \cos(t_{3,3}) = \frac{1}{3}, \quad \sin(t_{3,3}) = \frac{\sqrt{8}}{3}.$$

Either equation in (7.2) yields $\rho(P, Q) = \ell$ if we know the corresponding complex distance $\delta_2 + i\theta_2$ or $\delta_3 + i\theta_3$. These two equations together also determine ℓ if we know both real distances δ_2 and δ_3. From each of the two complex equations (7.2), we may eliminate the angle θ_i to obtain the two real equations

$$\Big(\frac{4\cosh(\ell)\cos(\phi_2) - 1}{\cosh(\delta_2)}\Big)^2 + \Big(\frac{4\sinh(\ell)\sin(\phi_2)}{\sinh(\delta_2)}\Big)^2 = 27,$$
$$\Big(\frac{8\cosh(\ell)\cos(\phi_3) - 1}{\cosh(\delta_3)}\Big)^2 + \Big(\frac{8\sinh(\ell)\sin(\phi_3)}{\sinh(\delta_3)}\Big)^2 = 81.$$

In particular, $\ell = 0.72093$ if $\delta_2 = 0.28088$ and $\delta_3 = 0.56177$. Lemma 4.12 then implies that

$$\rho(P, Q) \geq 0.72093 \quad \text{if } \delta_2 \geq 0.28088 \text{ and } \delta_3 \geq 0.56177.$$

The following tables give the values of ℓ for all possible complex distances with $\delta_2 \leq 0.28088$ or $\delta_3 \leq 0.56177$. See Tables 6.1 and 6.2. Hence in order to conclude that $\ell \geq 0.72093$ in general, we need only show that for each entry with $\ell < 0.72093$ in Tables 7.1 and 7.2, the corresponding group generated by the three elliptics whose axes contain the lines η, α_3 and one of β_2 or β_3 is not discrete. However, sometimes it happens that these three elliptics do generate a discrete group. This will then give another example of tetrahedral points on a common axis of order 3. These lead to the exceptional cases in Theorem 7.4 below. We shall give examples of both types of occurrences.

For this suppose that f, g_2, g_3 and h are respectively the elliptics of orders 3, 2, 3 and 3 with α_3, β_2, β_3 and η as their axes (be aware this differs from the notation used earlier). Then

$$\gamma(f, g_j) = \beta(f)\beta(g_j)\sinh^2(\delta_j + i\,\theta_j), \quad \gamma(f, h) = -2, \quad \gamma(g_j, h) = -2$$

by (6.1), and the group $\langle f, g_j, h\rangle$ is determined up to at most two conjugacy classes by these numbers and the orders of f, g_j and h. See [Cao94]. An important point here is that it is possible that one conjugacy class corresponds to a discrete group while the other does not. This was first observed by Cao in [Cao94] and we will see further examples here.

For each complex distance $\delta_j + i\,\theta_j$ in Tables 7.1 and 7.2 with $\ell \leq 0.72093$, we construct matrix groups corresponding to each conjugacy class. Cao gives the requisite equations to solve, and also identifies the two conjugacy classes arising from the various choices of square roots. We then can find in each of these groups an element that, together with one of the elliptic generators, will not generate a discrete subgroup, or we establish discreteness. First, the entry $\ell = 0.66217$ in Table 7.1 is easily eliminated as we know the group in question to be the 3−5−3 Coxeter group where tetrahedral vertex stabilizers do not occur on a common axis of order 3.

Let us work through two further examples carefully.

Example. $\delta_3 + i\,\theta_3 = 0.39415 + i\ 1.57079$ and $\ell = 0.44329$.

Actually this case again corresponds to the 3−5−3 group, but we deal with it in a more general fashion. Following [Cao94], we construct the matrix groups (with entries reporting the first 3 decimal places)

$$f = \begin{pmatrix} .5 & i\,.866 \\ i\,.866 & .5 \end{pmatrix},\ h = \begin{pmatrix} .5 & -.583 \\ 1.284 & .5 \end{pmatrix},\ g_1 = \begin{pmatrix} .537+i\,.779 & -.267+i\,.180 \\ .267+i\,.396 & .462-i\,.779 \end{pmatrix}$$

$$f = \begin{pmatrix} .5 & i\,.866 \\ i\,.866 & .5 \end{pmatrix},\ h = \begin{pmatrix} .5 & -.583 \\ 1.284 & .5 \end{pmatrix},\ g_2 = \begin{pmatrix} .462+i\,.779 & .267+i\,.180 \\ -.267+i\,.396 & .537-i\,.779 \end{pmatrix}.$$

$p=3,\ q=2$	
$\delta_2 \pm i\,\theta_2$	ℓ
$0.19707 + i\ 0.78539$	0.66217
$0.21084 + i\ 0.33189$	0.97406
$0.23371 + i\ 0.49318$	0.90652
$0.24486 + i\ 0.67233$	0.79369
$0.24809 + i\ 0.40575$	0.95396
$0.27407 + i\ 0.61657$	0.84965
$0.27465 + i\ 0.78539$	0.72227
$0.27702 + i\ 0.56753$	0.88220
$0.27884 + i\ 0.22832$	1.02137

Table 7.1

$p=3,\ q=3$	
$\delta_3 \pm i\,\theta_3$	ℓ
$0.39415 + i\ 1.57079$	0.44329
$0.42168 + i\ 0.66379$	0.64244
$0.46742 + i\ 0.98637$	0.59281
$0.48973 + i\ 1.34468$	0.56579
$0.49619 + i\ 0.81150$	0.66763
$0.54814 + i\ 1.23135$	0.64266
$0.54930 + i\ 1.57079$	0.61382
$0.55404 + i\ 1.13507$	0.66209
$0.55769 + i\ 0.45665$	0.83001

Table 7.2

We then compute that in each case f, g_i and h all have order 3, that

$$\gamma(f,h) = -2.6180 \quad \text{and} \quad \gamma(f,g_i) = \gamma(g_i,h) = -2$$

and, using [Cao94] again, that we have identified distinct conjugacy classes. Notice above that in fact hg_1 is elliptic of order two while hg_2 is elliptic of order three.

Now we want to show both of these groups are not discrete. To do this, we observe that

$$\gamma(h, fg_1) = -2.0354 + i\,0.4921 \quad \text{and} \quad \gamma(h, g_2 f) = -2.0354 - i\,0.4921$$

and that fg_i (and hence $g_i f$) is elliptic of order 2 in both cases. The complex distance between h and fg_1 in the first case and h and $g_2 f$ in the second case is then $0.1708 + i\,0.9572$ by (6.1); this, however, is not possible in a discrete group as shown in Table 7.2.

Example. $\delta_3 + i\,\theta_3 = 0.421686 + i\,0.663791$ and $\ell = 0.64244$.

As above we construct the matrix groups

$$f = \begin{pmatrix} .5 & i\,.866 \\ i\,.866 & .5 \end{pmatrix},$$

$$h_1 = \begin{pmatrix} .5 & -.813 + i\,1.039 \\ .349 + i\,.447 & .5 \end{pmatrix}, \quad g_1 = \begin{pmatrix} .492 + i\,.813 & -.094 + i\,.355 \\ .094 + i\,.221 & .507 - i\,.813 \end{pmatrix}$$

and

$$h_2 = \begin{pmatrix} .5 & -.349 - i\,.447 \\ .813 - i\,1.039 & .5 \end{pmatrix}, \quad g_2 = \begin{pmatrix} .816 + i\,.769 & -.587 - i\,.125 \\ .587 + i\,.703 & .183 - i\,.769 \end{pmatrix}.$$

We again check in each case that f, g_i and h_i have order 3, that

$$\gamma(f,h_i) = -0.75183 + i\,1.03398 \quad \text{and} \quad \gamma(f,g_i) = \gamma(g_i,h_i) = -2 \tag{7.3}$$

and that we have identified distinct conjugacy classes.

Next we check discreteness. First note $f^{-1}g_1$ is elliptic of order 3 and that $\gamma(h_1, f^{-1}g_1) = -2.437 - i\,0.5247$, which is not the commutator value for a discrete group with a generator of order 3 as per Table 7.1. Thus $\langle f, h_1, g_1\rangle$ is not a discrete group, and we turn to consider the second group. Then, running through the first few words in the group and calculating commutator values for the group $\langle f, h_2, g_2\rangle$ does not lead immediately to a contradiction to discreteness. We must therefore consider the possibility this group is discrete.

We have already identified in [GMMR97, Th. 8.2] the group generated by elements u and v of orders 3 and 2 with $\gamma(u,v) = -0.2118 + i\,0.4013$ as a discrete arithmetic lattice. The subgroup of index at most two generated by u and vuv^{-1} has $\gamma(u, vuv^{-1}) = \gamma(u,v)(\gamma(u,v) + 3) = -0.7515 + 1.0339i$. This number, already identified at (7.3), determines the group uniquely up to conjugacy, and so we deduce $\Gamma = \langle f, h\rangle$ is a discrete arithmetic lattice generated by two elements of order 3. We consider the group $\langle u, v\rangle$, since it is easier to use our techniques obtained from the polynomial trace identities in groups with one generator of order 2. We seek $w \in \langle u, v\rangle$ of order 3 so that $\gamma(u, w) = -2$ and $\gamma(v, w) = -3$. Then $\langle u, w\rangle$ is a tetrahedral point, and w meets the axis of v at right angles, so the group $\langle u, vuv^{-1}, w\rangle$ is the three-generator group we seek. The most likely place to look for w is among the conjugates of u. Indeed, put $y = vu^{-1}(vu)^3 v$ and $w = yuy^{-1}$. With the choice of matrix representatives

$$u = \begin{pmatrix} .5 & i\,.866 \\ i\,.866 & .5 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & -.2638 - i\,.7657 \\ .4022 - i\,1.1673 & 0 \end{pmatrix},$$

we compute

$$w = \begin{pmatrix} -0.8164 - i\,0.7698 & -0.5877 - i\,0.1256 \\ 0.5885 + i\,0.7031 & -0.1835 + i\,0.7698 \end{pmatrix}$$

and

$$\gamma(u, v) = -0.2118 + i\,0.4013, \quad \gamma(u, w) = -2, \quad \gamma(v, w) = -3.$$

This shows us that there is a discrete group with tetrahedral points on a common axis of order 3 at a distance 0.642446 and that this distance is uniquely obtained in an arithmetic Kleinian group.

Fortunately, for all the other candidates coming from Tables 7.1 and 7.2 for us to consider, it is not difficult to show that there are no more discrete groups by using essentially the same words to generate a contradiction.

We thus obtain the following result.

Theorem 7.4. *Suppose that P and Q are tetrahedral points in a Kleinian group on a common axis of order* 3. *Then*

(1) $\rho(P, Q) = 0.6424$ *and the extremal is the arithmetic Kleinian group* $\Gamma_{3,4}$ *of* [GMMR97], *or*

(2) $\rho(P, Q) = 0.6931$ *and the extremal is the arithmetic orientation-preserving subgroup of the tetrahedral reflection group* Γ_{27} *of* [GM05], *or*

(3) $\rho(P, Q) > 0.7209$.

Tetrahedral points on a common 2 axis. We consider next the case where we have two tetrahedral points on a common axis of order 2. The bound we obtain is a consequence of results from the next section and the following surprising phenomenon.

THEOREM 7.5. *Suppose F and G are tetrahedral groups that stabilize the points $P, Q \in \mathbb{H}^3$, and suppose that the group that they generate, $\Gamma = \langle F, G\rangle$, is discrete. If P and Q lie on a common axis of an element of $G \cap F$ of order* 2, *then there are octahedral groups O_P and O_Q such that*

(1) $F \subset O_P$ *and* $G \subset O_Q$,

(2) O_P *and* O_Q *have a common axis of order* 4, *and*

(3) *the group generated by* O_P *and* O_Q, $\langle O_P, O_Q\rangle$, *is discrete.*

Proof. Let g denote the common element of F and G of order 2, and let h be an elliptic element of order 4 with $\mathrm{axis}(h) = \mathrm{axis}(g)$. Then $\langle G, h\rangle = O_P$ and $\langle G, h\rangle = O_Q$ are discrete groups isomorphic to the octahedral group. Since every elliptic of order 3 in O_P lies in G, it is clear that

$$hGh^{-1} = h^{-1}Gh = G, \quad hFh^{-1} = h^{-1}Fh = F, \quad h^2 = g \in G \cap F. \tag{7.6}$$

These equations imply that the group $\langle G, F, h\rangle$ contains $\langle G, F\rangle$ with finite index since one can slide all the h's to one end using the above relations in the obvious fashion. Since the latter group is discrete, so is the former. □

The following is an immediate consequence of the above result and our description of the distances between octahedral points on a common axis of order 4.

COROLLARY 7.7. *Let P and Q be tetrahedral points of a Kleinian group on a common axis of order* 2. *Then*

(1) $\rho(P, Q) = 1.0595$, *or*

(2) $\rho(P, Q) = 1.0612$, *or*

(3) $\rho(P, Q) = 1.12838$, *or*

(4) $\rho(P, Q) > 1.14$.

Each value is uniquely achieved in a two-generator arithmetic Kleinian group.

The extremals for the above result are described in Theorem 7.8. Arithmeticity follows here from the arithmeticity of the extremals in the octahedral cases and our observation that the extremals for the tetrahedral case have finite index in those of the octahedral case.

Octahedral points on a common 4 axis. Suppose that P and Q are octahedral points of a Kleinian group Γ lying on a common axis of order 4. Since the axes of distinct elliptic elements of order 4 meet at right angles at finite points, the common axis of order 4 is actually a common perpendicular between two elliptic axes of order 4, one from Γ_P and the other from Γ_Q. Moreover, at each vertex there are a pair of elliptic axes of order 4 orthogonal to this common perpendicular. Therefore in such a group Γ we must have complex distances between axes of order 4 of the form $\delta + i\theta$ and $\delta + i(\pi/2-\theta)$, $\theta < \pi/4$. The only such entries in Table 6.5 are the complex distances $1.05959 \pm i\ 0.49798$ and 1.06128. Further, for any such pair there will be an elliptic of order 2 at complex distance $\frac{1}{2}(\delta + i\theta)$ and $\frac{1}{2}(\delta + i(\pi/2-\theta))$, either of which together with either elliptic of order 4 perpendicular to the common order 4 axis generates a discrete group, usually containing the first group with index 2. Thus the angle $\tilde{\theta}$ between the elliptics of order 2 and 4 can be chosen with $0 \le \tilde{\theta} \le \pi/8$, and correspondingly there is another angle at $\pi/4 - \tilde{\theta}$. From our improved disk-covering in Theorem 6.7, we immediately deduce the following result.

THEOREM 7.8. *Suppose that P and Q are octahedral points of a Kleinian group on a common axis of order* 4. *Then*

(1) $\rho(P, Q) = 1.0595$ *and the extremal is an arithmetic group generated by elliptics f and g of orders* 2 *and* 3 *with commutator parameter* $\gamma(f, g) = -0.5803 + i\, 0.6062$, *or*

(2) $\rho(P, Q) = 1.0612$ *and the extremal is an arithmetic $\mathbb{Z}_2$-extension of the orientation-preserving subgroup of the tetrahedral reflection group Γ_2 in* [GM05], *or*

(3) $\rho(P, Q) = 1.1283$ *and the extremal is an arithmetic $\mathbb{Z}_2$-extension of the orientation-preserving subgroup of the tetrahedral reflection group Γ_4 in* [GM05], *or*

(4) $\rho(P, Q) > 1.14$.

Octahedral points on a common 3 axis. Suppose that P and Q are octahedral points on a common axis η of order 3. We argue as in the case where P and Q were tetrahedral points on a common axis of order 3. There is an elliptic axis α_4 of order 4 from Q coming as close as possible to P. Choose β_2 of order 2 and β_4 of order 4 from P as before. The angle between β_2 and β_4 at P is $\pi/4$.

If α_4 meets some β_j, then we have formed a (p, q, r)-Kleinian group and the estimates in Table 5.4 imply that

$$\rho(P, O) = 1.3169, \quad \rho(P, Q) = 1.5668 \quad \text{or} \quad \rho(P.Q) > 1.7000. \tag{7.9}$$

If α_4 does not meet some β_j, let ϕ_j be the dihedral angle between the two hyperbolic planes containing η, α_4 and η, β_j. Then as before $\phi_2 + \phi_4 = \pi/3$ is the dihedral angle between the hyperbolic planes containing η, β_2 and η, β_4. Next let $\delta_j + i\theta_j$ be the complex distance between α_4 and β_j. Then

$$\begin{aligned} \cosh(\ell + i\phi_2) &= \frac{\cos(o_{2,3})\cos(o_{3,4}) + \cosh(\delta_2 + i\theta_2)}{\sin(o_{2,3})\sin(o_{3,4})}, \\ \cosh(\ell + i\phi_4) &= \frac{\cos(o_{3,4})\cos(o_{3,4}) + \cosh(\delta_4 + i\theta_4)}{\sin(o_{3,4})\sin(o_{3,4})}, \end{aligned} \tag{7.10}$$

by (4.3), where

$$\cos(o_{2,3}) = \sin(o_{3,4}) = \sqrt{2}/\sqrt{3} \quad \text{and} \quad \sin(o_{2,3}) = \cos(o_{3,4}) = 1/\sqrt{3}.$$

These equations imply that $\ell = 1.60306$ if $\delta_2 = 0.53264$ and $\delta_4 = 1.06528$. Hence, by Lemma 4.12, $\ell \geq 1.60306$ if $\delta_2 \geq 0.53264$ and $\delta_4 \geq 1.06528$.

As before we use equations (7.10) to calculate ℓ for the complex distances $\delta_2 + i\,\theta_2$ and $\delta_4 + i\,\theta_4$ given in Tables 6.3 and 6.4. We then examine the groups $\langle f, g_j, h\rangle$, where f, g_j, h are the elliptics with α, β_j, η as their axes. It turns out that the groups that correspond to complex distances $\delta + i\,\theta_j$ with $\ell \leq 1.6004$ are not discrete.

We thus obtain the following result.

THEOREM 7.11. *Suppose P and Q are octahedral points in a Kleinian group on a common axis of order* 3. *Then*

(1) $\rho(P, Q) = 1.3169$ *and the extremal is the orientation-preserving subgroup of the tetrahedral reflection group* Γ_{25} *in* [GM05], *or*

(2) $\rho(P, Q) = 1.5668$ *and the extremal is the orientation-preserving subgroup of the* (3—4—4—3) *tetrahedral reflection group* Γ_6 *in* [GM05], *or*

(3) $\rho(P, Q) > 1.6004$.

Octahedral points on a common 2 axis. We argue again as above using elliptic axes α_4 of order 4 from Q and β_3 and β_4 of orders 3 and 4 from P. If α_4 meets a β_j, then

$$\rho(P, Q) = 1.7000 \tag{7.12}$$

again by Table 5.4. Otherwise, let the dihedral angles ϕ_j be as defined earlier, and let $\delta_j + i\,\theta_j$ be the complex distance between α_4 and β_j. Then $\phi_3 + \phi_4 = \pi/2$ and we obtain the equations

$$\begin{aligned} \cosh(\ell + i\phi_3) &= \frac{\cos(o_{2,3})\cos(o_{2,4}) + \cosh(\delta_3 + i\theta_3)}{\sin(o_{2,3})\sin(o_{2,4})}, \\ \cosh(\ell + i\phi_4) &= \frac{\cos(o_{2,4})\cos(o_{2,4}) + \cosh(\delta_4 + i\theta_4)}{\sin(o_{2,4})\sin(o_{2,4})}, \end{aligned} \tag{7.13}$$

where

$$\cos(o_{2,3}) = \sqrt{2}/\sqrt{3}, \quad \sin(o_{2,3}) = 1/\sqrt{3}, \quad \cos(o_{2,4}) = \sin(o_{2,4}) = 1/\sqrt{2}.$$

Equations (7.13) imply that $\ell = 1.79021$ if $\delta_3 = .61759$ and $\delta_4 = 1.06528$ and hence that $\ell \geq 1.79021$ if $d_3 \geq .61759$ and $d_4 \geq 1.06528$.

Once more we consider the possible complex distances between the axes α_4 and β_3, β_4 in Tables 6.4 and 6.5 and eliminate those with $\ell < 1.79021$ by means of a discreteness criterion. This together with (7.12) establishes this result:

THEOREM 7.14. *Suppose P and Q are octahedral points in a Kleinian group on a common axis of order* 2. *Then*

(1) $\rho(P,Q) = 1.7000$ *and the extremal is the orientation-preserving subgroup of the* $(3-4-4-3)$ *tetrahedral reflection group* Γ_6 *of* [GM05], *or*

(2) $\rho(P,Q) > 1.7902$.

Icosahedral points on a common 5 axis. Derevnin and Mednykh's geometric results [DM88], mentioned in the introduction, give the sharp results in all situations here when there is a common axis of order 2, 3 or 5. However we must extend one of their bounds slightly to find the next extremal in the case of a common axis of order 5. The methods are entirely as above, and we choose axes α_3 of order 3 and β_2, β_3 of orders 2, 3. The dihedral angle sum is $\phi_2 + \phi_3 = \pi/5$, and proceeding as before we obtain the two equations

$$\cosh(\ell + i\phi_2) = \frac{\cos(i_{2,5})\cos(i_{3,5}) + \cosh(\delta_2 + i\theta_2)}{\sin(i_{2,5})\sin(i_{3,5})},$$
$$\cosh(\ell + i\phi_3) = \frac{\cos(i_{3,5})\cos(i_{3,5}) + \cosh(\delta_3 + i\theta_3)}{\sin(i_{3,5})\sin(i_{3,5})},$$

where $\delta_j + i\,\theta_j$ is again the complex distance between α_3 and β_j. These equations imply that $\ell \geq 1.97047$ whenever $d_2 \geq .28088$ and $d_3 \geq .56177$. For the last time we run through the possible complex distances between α_3 and β_2, β_3 in Tables 6.1 and 6.2 and eliminate all those with $\ell < 1.97047$ using a discreteness criterion. This, together with the lesser values of (p,q,r)-Kleinian groups in Table 5.6, yields the following result.

THEOREM 7.15. *Suppose P and Q are icosahedral points in a Kleinian group on a common axis of order* 5. *Then*

(1) $\rho(P,Q) = 1.3825$ *and the extremal is an orientation-preserving subgroup of the* $(3-5-3)$ *tetrahedral reflection group* Γ_1 *of* [GM05, Tab. 9], *or*

(2) $\rho(P,Q) = 1.6169$ *and the extremal is the orientation-preserving subgroup the tetrahedral reflection group* Γ_2 *of* [GM05, Tab. 9], *or*

(3) $\rho(P,Q) > 1.9704$.

8. Spherical points not on a common axis

We derive here bounds on the distance between spherical points not on a common axis.

Tetrahedral points not on a common axis. In this section we shall identify the smallest possible distance between tetrahedral points P and Q not on a common axis. Let

$$\ell_t = \inf_\Gamma \rho(P,Q),$$

where the infimum is over all Kleinian groups Γ containing distinct tetrahedral points P and Q not on a common axis. Examples show that $\ell_t < \infty$, and since the space of discrete nonelementary groups is closed, we can choose a Kleinian group Γ that contains tetrahedral points P and Q with $\rho(P, Q) = \ell_t$. The bound $\ell_t \geq 1.059$ will be enough for our purposes, and so we may assume that $\ell_t < 1.059$ and see where this leads us. Let $s_t = 1.059$, and note that this is the minimal distance between tetrahedral points on a common axis of order two; see Corollary 7.7. Let γ be the hyperbolic segment joining P and Q, set $G = \Gamma_Q$ and $F = \Gamma_P$, and consider small hyperbolic spheres Σ_F and Σ_G about P and Q. Then Σ_F is stabilized by F and Σ_G by G. Since F tessellates Σ_F by $(2, 3, 3)$ spherical triangles, γ must pass through one such triangle but, by assumption, not a vertex. Thus there is an elliptic element of order 3 in F whose axis α_3 comes as close as possible to Q and forms the angle $0 < \psi_P < t_{2,3}$ with γ. There is a similar elliptic element of order 3 in G with axis β that forms an angle $0 < \psi_Q < t_{2,3}$ with γ.

Notice that if ψ_P or ψ_Q were to equal $t_{2,3}$, then $\overline{PQ}$ would have to be the axis of an elliptic of order 2 contrary to assumption. We shall use this observation to improve our bounds on ψ_P and ψ_Q a little. Let ϕ be the angle formed between γ and the axis α_2 of the elliptic g of order 2 at P that comes closest to Q. See Figure 4.

Now Q and $g(Q)$ are tetrahedral points. If they are on a common axis then by Theorem 7.4 and Corollary 7.7, we may assume $\rho(\alpha_2, Q) > t_0 = 0.7209/2$, unless we are in a configuration arrived at in one of the two extremals for tetrahedral points on a common order 3 axis. Let us set aside these cases (where the extremal will occur) for the moment. Then, using hyperbolic trigonometry on the right triangle formed by P, Q and the point on α_2 nearest Q, we get

$$\sin(\phi) = \sinh(t_0)/\sinh(\ell_t), \quad \text{and so } \phi > 0.2946. \tag{8.1}$$

If Q and $g(Q)$ are not on a common axis, then

$$\sin(\phi) \geq \sinh(\ell_t/2)/\sinh(\ell_t) \geq \sinh(s_t/2)/\sinh(s_t) = 0.4371. \tag{8.2}$$

Thus $\phi > 0.2865$ in both cases.

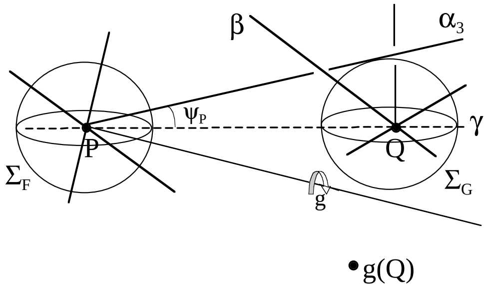

Figure 4. Tetrahedral points with no common axis.

Spherical trigonometry then shows that if γ lies outside the angular disk $\{\phi > 0.2865\}$ about the right-angled vertex in the spherical (2,3,3) triangle it passes through, then

$$\psi_P < 0.7714 = \psi_0. \tag{8.3}$$

This value is achieved at the midpoint of the angular arc on the boundary of the disk $\{\phi > 0.2865\}$ and is maximal since α_3 was chosen closest to Q, and therefore the angle ψ_P between γ and α_3 is smaller than the angle between γ and any other elliptic of order 3 emanating from P, and in particular less than the angle to the other vertex of the (2,3,3) triangle. Similarly $\psi_Q < \psi_0$ by symmetry.

Remark. At this point we have to make an observation we will use later in a special case. If we know *apriori* that $\rho(P, Q) > 0.815$, then we find, following the arguments we used to get (8.2) and (8.3), that $\phi > 0.333$ and

$$\psi_P, \psi_Q < 0.7442. \tag{8.4}$$

We also remark that clearly (8.2) continues to hold for $\ell < 1.0709$, the bound we shall obtain later.

Suppose now that α_3 and β intersect at the point R. If R is not a finite point, then α and β meet on the Riemann sphere at angle 0,

$$\cosh(\rho(P, Q)) = \frac{\cos(\psi_P)\cos(\psi_Q) + 1}{\sin(\psi_P)\sin(\psi_Q)} > \frac{\cos(\psi_0)\cos(\psi_0) + 1}{\sin(\psi_0)\sin(\psi_0)} = 3.1777,$$

and hence $\rho(P, Q) > 1.827175$. If R is a finite point, then R is fixed by a spherical triangle group H. Now G and H and also F and H are spherical triangle groups on a common axis. Thus the distances $\rho(P, R)$ and $\rho(Q, R)$ are bounded below by those bounds given in the previous section. The worst case where the distances are smallest is easily seen to be when $H \equiv A_4$. The angle formed between the order 3 axes α and β at R is $t_{3,3}$ or $\pi - t_{3,3}$. Again we set aside the two extremal cases. Thus $\ell_{P,R} = \rho(P, R)$ and $\ell_{Q,R} = \rho(Q, R)$ can be both assumed to exceed 0.7209. We consider the two angles separately. If the angle of intersection of the two axes is $t_{3,3}$, then we apply the second cosine law of plane hyperbolic trigonometry to get

$$\begin{aligned}\cosh(\ell_t) &= (\cos(\psi_P)\cos(\psi_Q) + \cos(t_{3,3}))/(\sin(\psi_P)\sin(\psi_Q)) \\ &\geq (\cos(\psi_0)\cos(\psi_0) + 1/3)/(\sin(\psi_0)\sin(\psi_0)) = 1.7851.\end{aligned}$$

Otherwise the first cosine law yields

$$\begin{aligned}\cosh(\ell_t) &= \cosh(\ell_{P,R})\cosh(\ell_{Q,R}) + \sinh(\ell_{P,R})\sinh(\ell_{Q,R})\cos(t_{3,3}) \\ &\geq \cosh^2(t_0) + \sinh^2(t_0)/3 = 1.8542.\end{aligned}$$

We deduce in either case that $\ell_t > s_t$.

Suppose next that α and β do not meet. Then

$$\sin(\psi_P)\sin(\psi_Q)\cosh(\ell_t+i\phi)-\cos(\psi_P)\cos(\psi_Q)=\cosh(\delta+i\theta) \tag{8.5}$$

by (4.3), where $\delta+i\theta$ is the complex distance between α and β coming from Table 6.2. Next Lemma 4.4 implies that we get a lower bound for ℓ_t if we consider the case where $\psi_P=\psi_Q=\psi_0$ while keeping $\delta+i\theta$ fixed (or one can see this directly from the geometry). We check through the various possibilities for $\delta+i\theta$ for small values of δ. This time we obtain $\ell_t>1.3$ except in the case of the 3−5−3 reflection group, which we also set aside. Thus there is no need to prove certain groups are not discrete, and we may assume that $\delta>0.561$ by the tables of complex distances in Section 6, specifically Table 6.2. Hence

$$\sinh(\ell)\geq\frac{\sinh(\delta)}{\sin^2(\psi_0)}=1.2158 \tag{8.6}$$

by Lemma 4.4, and we have proved that if P and Q are tetrahedral points of a Kleinian group that do not lie on a common axis, then $\rho(P,Q)>1.026$ unless P and Q together generate a subgroup of either of the two extremal groups for tetrahedral points on a common axis of order 3 or the 3−5−3 reflection group.

We now complete our analysis of the situation of tetrahedral points not on a common axis by examining the cases we have left aside. In particular the two reflection groups are eliminated simply by constructing the tessellation about any one of the vertices and looking at the nearest points. Using a little geometry we find in the 3−5−3 reflection group tetrahedral subgroups of the icosahedral points on a common axis of order 5. This distance is 1.3825. While for the 3−3−6 group, if we consider the closest tetrahedral points on a common axis, say P and Q, and take the elliptic g of order 2 coming from Q as close as possible to P, this distance is $\operatorname{arcsinh}(\sqrt{2}\sinh(0.6931)/\sqrt{3})=0.57936$. Therefore $\rho(P,g(P))=1.1587$ is a distance between tetrahedral points not on a common axis. A little further consideration of the tessellation about the vertices shows us that this is best possible in this group. We are left with the extremal for tetrahedral points on a common axis, an axis of order 3, and again we take the elliptic of order 2 with axis η coming from P as close as possible to Q. The distance between Q and η is

$$\begin{aligned}\rho(Q,\eta)&=\operatorname{arcsinh}(\sinh(\rho(P,Q))\sin(t_{2,3}))\\&=\operatorname{arcsinh}(0.68736\sqrt{2}/\sqrt{3})=0.5353.\end{aligned}$$

As before we get tetrahedral points not on a common axis at distance 1.0707.

THEOREM 8.7. *If P and Q are tetrahedral points of a Kleinian group that do not lie on a common axis, then $\rho(P,Q)>1.026$.*

Remark. Again, if we know *a priori* that $\rho(P, Q) > 0.815$, so that

$$\psi_P, \psi_Q < 0.7442 \quad \text{and} \quad \sinh(\ell) > 1.2877,$$

we obtain the bound

$$\rho(P, Q) > 1.0709, \tag{8.8}$$

which exceeds the value of 1.0707 attained in the extremal case above.

Octahedral points not on a common axis. Let

$$\ell_o = \inf\nolimits_\Gamma \rho(P, Q), \tag{8.9}$$

where the infimum is over all Kleinian groups Γ containing distinct octahedral points P and Q not on a common axis. Examples show that $\ell_o < \infty$, and hence this infimum is attained for octahedral points P and Q in a Kleinian group Γ. We shall show that

$$\ell_o > \operatorname{arccosh}(\cosh^2(1.05959)) \approx 1.6140.$$

We may assume $\ell_o \le 1.6140$, and we note that this number is the minimal distance between octahedral points on a common order 4 axis. Arguing as in the case for tetrahedral points, we choose axes β_1 and β_2, both of order 4, which form angles ψ_P and ψ_Q satisfying $0 < \psi_P, \psi_Q < o_{3,4}$ with the hyperbolic segment γ joining P and Q. Note that if ψ_P or ψ_Q were equal to $o_{3,4}$, then P and Q would lie in a common axis of order 3 contrary to our supposition. Let ϕ be the angle formed by γ and the axis α' of the elliptic g of order 3 at P that comes closest to Q. Now Q, $g(Q)$ and $g^2(Q)$ are octahedral points. If a pair were to lie on a common axis, then so would both other pairs, the axis being translated by g. If the order of the common axis was either 3 or 4, we would obtain a hyperbolic triangle all of whose sides are axes of the same order, either 3 or 4, and the sum of whose interior angles would be three times the angle between pairs of elliptics of order 3 or of order 4 in an octahedral group and thus exceed π. Hence either there is no common axis, or if there is a common axis, it has order 2. In either case we would have from Theorem 7.14 that $\rho(Q, g(Q)) \ge \ell_o$, since we have assumed that $\ell_o \le 1.6140 < 1.700$, the last number here being the minimal distance between octahedral points on a common 2 axis.

We consider the triangle with vertices Q, $g(Q)$ and $g^2(Q)$. Each side length is at least ℓ_o, and hence the distance to the center, the axis of g, is at least $\operatorname{arcsinh}(\sinh(\ell/2)/\sin(\pi/3))$ by the hyperbolic law of sines. Thus the angle ϕ between the axis of g and γ satisfies

$$\sin(\phi) \ge \sinh(\delta)/\sinh(\ell) = 2\sinh(\ell/2)/(\sqrt{3}\sinh(\ell)) \ge 0.417295,$$

where $\delta + i\theta$ is the complex distance between α and β. Thus $\phi > 0.43046$. Spherical trigonometry, as in the tetrahedral case, shows that if γ lies outside this angular disk about the axis of order 3 in the spherical triangle it passes through, then $\psi_P < 0.804786 = \psi_0$. Again $\psi_Q < \psi_0$ holds by symmetry.

Suppose that α and β intersect at the point R. If R is not a finite point, then one obtains good bounds as in the argument for tetrahedral points. If R is finite, then these axes meet at angle $\pi/2$ and R is stabilized by the group S_4. Thus

$$\cosh(\ell_o) \geq \cosh^2(1.05959), \quad \text{whence} \quad \ell_o \geq 1.614055.$$

This bound occurs in the extremal group for octahedral points on a common axis of order 4.

Suppose next that α and β do not meet. Then

$$\sin(\psi_P)\sin(\psi_Q)\cosh(\ell_t + i\,\phi) - \cos(\psi_P)\cos(\psi_Q) = \cosh(\delta + i\,\theta)$$

by (4.3). Next Lemma 4.4 implies that we get a lower bound for ℓ_o if we consider the case where $\psi_P = \psi_Q = \psi_0$ while keeping $d + i\,\theta$ fixed. Again we check through the possible values of $\delta + i\,\theta$ for small values of δ. For all values except $0.881374 + i\,1.5708$ and $0.8314 + i\,1.19606$ we find that $\ell_o > 1.68$. These two cases are eliminated by directly examining the arithmetic Kleinian group generated by the two elliptics of order 4 in question. Hence we may assume that $\delta > 1.06$ and we obtain

$$\sinh(\ell_o) \geq \sinh(\delta)/\sin^2(\psi_0) = 2.44513 \quad \text{and} \quad \ell_o > 1.6266.$$

THEOREM 8.10. *If P and Q are octahedral points of a Kleinian group that do not lie on a common axis, then $\rho(P, Q) > 1.6140$.*

9. **Summary**

We summarize here the results of Sections 7 and 8 as well as recall the results of Derevnin and Mednykh for the A_5 case.

THEOREM 9.1. *Let P and Q be spherical points of the same type in a Kleinian group.*

(1) *P, Q tetrahedral points: If $\rho(P, Q) < 1.026$, then P and Q lie on a common axis of order 3 and*

$$\rho(P, Q) = 0.64244, \quad \rho(P, Q) = 0.6931 \quad \textit{or} \quad \rho(P, Q) > 0.7209.$$

(2) *P, Q octahedral points: If $\rho(P, Q) < 1.6140$, then P and Q lie on a common axis of order 4 and*

$$\rho(P, Q) = 1.0595, \qquad \rho(P, Q) = 1.0612,$$
$$\rho(P, Q) = 1.1283, \quad \textit{or} \quad \rho(P, Q) > 1.14.$$

(3) P, Q *icosahedral points*: *If* $\rho(P, Q) \leq \operatorname{arccosh}(2+\sqrt{5}) = 2.1225$, *then* P *and* Q *lie on a common axis. If this axis has order* 5, *then*

$$\rho(P, Q) = 1.3825, \quad \rho(P, Q) = 1.6169, \quad \textit{or} \quad \rho(P, Q) > 1.9704.$$

If this axis has order 3, *then* $\rho(P, Q) \geq \operatorname{arccosh}((5+3\sqrt{5})/4) = 1.7365$. *If this axis has order* 2, *then* $\rho(P, Q) = \operatorname{arccosh}(2+\sqrt{5})$.

10. **Volume estimates**

We conclude by showing how the above estimates can be used to provide co-volume bounds for hyperbolic 3-orbifolds. An interesting, and important point, is that all our extremals are two generator arithmetic Kleinian groups.

We assume in what follows that G is a finite subgroup of a Kleinian group Γ. We will study how the nature of G affects the volume of $\mathbb{H}^3/\Gamma$. We begin with the case where G is the icosahedral group since this is the easiest for us to deal with and since it also provides the essential ideas that we shall have to refine in the other two cases.

Suppose that G stabilizes the point $P \in \mathbb{H}^3$. If $f \in \Gamma \setminus G$, then $f(P)$ is an icosahedral point, stabilized by fGf^{-1}. From Theorem 9.1 we see that $\rho(P, f(P)) > 1.7365$, unless Γ contains the orientation-preserving subgroup of the 3–5–3 reflection group or the orientation-preserving subgroup of the 4–3–5 reflection group, both of which are arithmetic. Hence we may assume $\rho(P, f(P)) > 1.7365 = 2r_0$ for all $f \in \Gamma \setminus G$. Let $B = B(P, r_0)$. For such f we see $B \cap f(B) = \varnothing$, that is, the ball B is *precisely invariant*: it is stabilized precisely by G and every other element moves it off itself. Therefore a fundamental set for the action of G on B is moved off itself by every nontrivial element of Γ, and

$$\begin{aligned}\operatorname{Vol}(\mathbb{H}^3/\Gamma) &\geq \operatorname{Vol}(B/G) = \operatorname{Vol}(B)/60 \\ &= \pi(\sinh(2r_0) - 2r_0)/60 = 0.0531 \\ &> \operatorname{Vol}(\mathbb{H}^3/\Gamma_1) = 0.0408\end{aligned}$$

since the hyperbolic volume of a ball of radius r is $\pi(\sinh(2r) - 2r)$). In fact a little more is true. The orbit of B under the action of Γ gives a packing of $\mathbb{H}^3$ by congruent hyperbolic balls. There is an optimal density for a packing by such balls, depending on the radius [Bör78]. This observation for improving volume estimates was first used by Meyerhoff [Mey86]. The density function $d(r)$ is strictly increasing. For the radius r_0, this constant is less than 0.81. Thus we may increase the volume estimate $\operatorname{Vol}(B/G) = \operatorname{Vol}(B)/|G|$ by this packing density to get the lower bound 0.06555 on volume. Estimates for this density can be found in [Mar91]. We have proved the following result.

THEOREM 10.1. *If* Γ *is a Kleinian group that contains a finite subgroup isomorphic to the icosahedral group, then* Γ *is either a subgroup of the* 3–5–3 *reflection group or the* 4–3–5 *reflection group, or* $\mathrm{Vol}(\mathbb{H}^3/\Gamma) > 0.06555$.

In fact the next volume in this case is 0.093326 from a $\mathbb{Z}_2$ extension of the 5–3–5 reflection group when icosahedral vertices are on a common axis of order 3. We could have achieved this bound if we had been willing to extend the known spectrum of distances for icosahedral points on a common axis of order 3 a little, from 1.7365 to 1.93, and identified all the attained values (and maximal groups) between.

We treat next the case of a Kleinian group containing octahedral points. We obtain from Theorem 9.1 a precisely invariant ball of radius $r_0 = 1.14/2$ unless one of the three exceptions occurs. This ball of radius r_0 together with the packing density estimate is sufficient to give a co-volume bound of 0.0425 slightly exceeding that of Γ_1. Thus we need only discuss what happens in the three exceptional cases where we have octahedral points on a common axis of order 4 at distances $\rho(P, Q) = 1.0595$, $\rho(P, Q) = 1.0612$ and $\rho(P, Q) = 1.1283$.

The first value 1.0595 corresponds to the arithmetic Kleinian group $G_{4,7}$; see [GMMR97, Tab. 2]. In Table 7 of that paper we computed the minimal co-volume of any arithmetic lattice in which this group embeds as 0.0661. The second exceptional value $\rho(P, Q) = 1.0612$ has the order 4 axes at each vertex parallel or perpendicular to the axes at the other vertex. Then two elliptic axes of order 2 meet at angle $\pi/5$. The group is actually a subgroup of the arithmetic 4–3–5 tetrahedral reflection group. In any case, there is an icosahedral point that gives sufficient volume by Theorem 10.1. The third exceptional value example $\rho(P, Q) = 1.1283$ again arises in an arithmetic tetrahedral reflection group.

THEOREM 10.2. *Suppose* Γ *is a Kleinian group containing a finite subgroup isomorphic to the octahedral group. Then*

$$\mathrm{Vol}(\mathbb{H}^3/\Gamma) > \mathrm{Vol}(\mathbb{H}^3/\Gamma_1) > \mathrm{Vol}(\mathbb{H}^3/\Gamma_0).$$

Finally we turn to the case that there is a tetrahedral subgroup in our Kleinian group. We may assume that there are no icosahedral or octahedral subgroups, for these have already been dealt with.

Let P be a tetrahedral point and Q its closest translate. If P and Q do not lie on a common axis of order 3, then $\rho(P, Q) \geq 1.026$, and so the volume contribution of the precisely invariant ball around P of this radius, together with the sphere packing estimate, gives a volume contribution, as described above, greater than 0.06. In fact as soon as $\rho(P, Q) \geq 0.9$ we obtain a sufficient volume contribution from the precisely invariant ball about P.

Therefore we suppose P and Q do lie on a common axis of order three. If the distance between them is no more than 0.7209, then this distance is either

$\rho(P, Q) = 0.64244$ or $\rho(P, Q) = 0.6931$, by Theorem 9.1. We have also identified the groups that uniquely achieve these values in Theorem 7.4 as arithmetic. In fact the first distance occurs in the maximal arithmetic lattice Γ_1 and the second distance in an arithmetic tetrahedral reflection group. Again we may therefore leave aside these cases so as to be able to assume $\rho(P, Q) > 0.7209$.

A precisely invariant ball about P of radius $0.7209/2$ gives a volume contribution of at least 0.021. Thus, if $\mathrm{Vol}(\mathbb{H}^3/\Gamma) < 0.042$, there cannot be more than one conjugacy class of tetrahedral subgroup since each conjugacy class yields disjoint balls. We may suppose that $\mathbb{H}^3/\Gamma$ is compact, the minimal co-volume noncompact lattice having been identified in [Mey86].

Along the elliptic axis shared by P and Q are other tetrahedral points periodically spaced out. Let Q' be the tetrahedral point on this axis closest to P but on the other side from Q. The axis of Γ_P that comes closest to Q contains a tetrahedral point Q'' at distance $\rho(P, Q)$ from P.

It is not hard to show that Q' and Q'' do not share a common axis: If they did share a common axis, by assumption it must be order 2 or 3. We would then obtain a hyperbolic triangle with all edges of order 2 or 3, and the further supposition that there are no octahedral or icosahedral points severely constrains the angles at these vertices. So much in fact that Lemma 3.1 implies the interior angle sum would exceed π. This is not possible. Since Q' and Q'' may not share an axis, $\rho(Q', Q'') > 1.026$.

Then, using hyperbolic trigonometry on the $PQ'Q''$ triangle, we get

$$\begin{aligned}
&\cosh(\rho(Q', Q'')) \\
&\quad = \cosh(\rho(P, Q'))\cosh(\rho(P, Q'')) - \sinh(\rho(P, Q'))\sinh(\rho(P, Q''))\cos(t_{3,3}) \\
&\quad = \cosh(\rho(P, Q'))\cosh(\rho(P, Q)) - \sinh(\rho(P, Q'))\sinh(\rho(P, Q))/3.
\end{aligned}$$

If $\rho(P, Q) < 0.815$, then this formula gives $\rho(P, Q') > 0.8437$. If

$$0.815 < \rho(P, Q) < 0.85,$$

then we know from (8.8) that $\rho(Q', Q'') > 1.0709$ so that $\rho(P, Q') > 0.87$. Hence in all cases $\rho(P, Q')$ is larger than 0.843. Further, note that any tetrahedral point R with $\rho(P, R) < 1.026$ must share an axis with P.

We already know bounds on how close the tetrahedral points on the axes of order 2 can be. Then consideration of the vertices around P and their orbit under the stabilizer Γ_P and the possible involution interchanging P and Q identifies a partial fundamental domain obtained as the convex hull of the tetrahedral and dihedral vertices. The volume of this region is sufficient for our purposes. Unfortunately, this volume is exceedingly difficult to estimate. We prefer a rather simpler approach

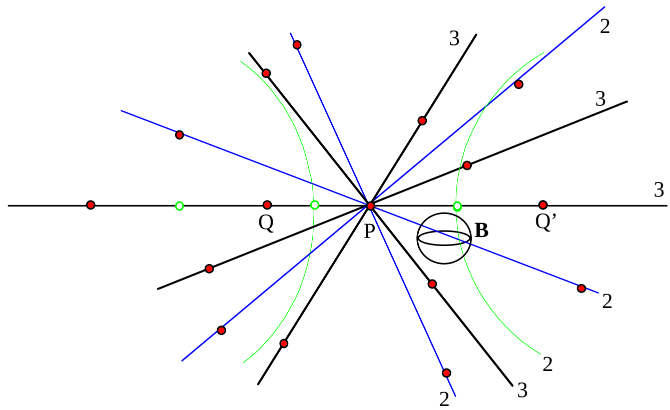

Figure 5. A precisely invariant ball.

by again considering precisely invariant balls, this time not centered on a vertex. See Figure 5.

Let ℓ be the line passing through the tetrahedral point P making the angle $\pi/4$ with the elliptics of order 2 and the angle $o_{3,4}$ with the order 3 elements. Thus ℓ would be an axis of an elliptic of order 2 should our tetrahedral point in fact be an octahedral point. Let z_0 be a point on ℓ at distance 0.28102 from P. Let $B = B(z_0, 0.2)$. Then B is actually tangent to an elliptic axis of order 2, while $\rho(z_0, \beta) \geq 0.2304$ for any axis β of an elliptic of order 3. This implies the orbit of B around any elliptic of order 3 consists of 3 disjoint balls. Thus if $g \in \Gamma_P$ and $g(B) \cap B \neq \varnothing$, then g is the identity.

We now go about proving that B is precisely invariant. So suppose that $g \in \Gamma \setminus \Gamma_P$ with $g(B) \cap B \neq \varnothing$. Then

$$\begin{aligned}\rho(g(P), P) &\leq \rho(P, z_0) + \rho(z_0, g(z_0)) + \rho(g(z_0), g(P)) \\ &= 2 \times 0.28 + 2 \times 0.2 = 0.96 < 1.02.\end{aligned}$$

Thus the tetrahedral points P and $g(P)$ lie on a common axis, and of course $g(z_0)$ must be a distance 0.28 from the tetrahedral point $g(P)$. Note that $\rho(z_0, g(z_0)) < 0.4$. Thus the image of P must be a tetrahedral point at distance at most 0.68 from z_0. One easily calculates that the distance from z_0 to the orbit of Q' under Γ_P, given $\rho(P, Q') > 0.843$, is at least 0.726, since

$$\operatorname{arccosh}\big(\cosh(0.281)\cosh(0.843) - \sinh(0.281)\sinh(0.843)\cos(o_{3,4})\big)$$

is equal to 0.7268. Thus $g(P)$ lies in the orbit of Q under Γ_P, all other tetrahedral points being too far away. Indeed, if $\rho(P, Q) > 0.8$, we calculate as above that the distance to the orbit of Q exceeds 0.68. Then $g(P) = P$, and so as above $g \in \Gamma_P$.

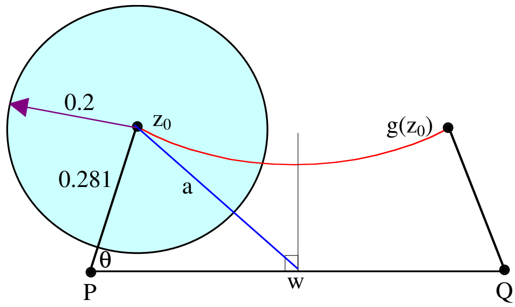

Figure 6. Distance from z_0 to $g(z_0)$.

So we are now supposing $\rho(P,Q) < 0.8$ and $g(P) = Q$, the closest pair of tetrahedral points. In particular, we have already eliminated the case that $\rho(P,Q) = \rho(P,Q')$. Of course $g(B)$ sits in the same relative position to Q as B does to P.

Consider the figure with edges $\overline{z_0 P}$, $\overline{PQ}$, $\overline{Qg(z_0)}$ and $\overline{z_0 g(z_0)}$, as shown in Figure 6. We want to use this figure to bound $\rho(z_0, g(z_0))$ from below. Keeping the angles at P and Q fixed, we decrease this distance if we rotate one edge, say $\overline{Qg(z_0)}$, until the figure is planar. Let w be the bisector of $\overline{PQ}$, let $r = \rho(P,Q)$, let $a = \rho(z_0, w)$ and let $\theta = \angle(\overline{z_0 w}, \overline{wP})$. We calculate

$$\cosh(a) = \cosh(0.28)\cosh(r/2) - \sinh(0.28)\sinh(r/2)\cos(o_{3,4}),$$

$$\sin(\theta) = \sinh(0.28)\sin(o_{3,4})/\sinh(a),$$

$$\sinh(\rho(z_0, g(z_0))/2) = \sinh(a)\cos(\theta).$$

From this we deduce that $\rho(z_0, g(z_0)) > 0.4$ if $r > 0.7209$. In particular we conclude that $g(B)$ cannot meet B.

Hence the ball B is precisely invariant, and its stabilizer is the identity. Thus we find $\mathrm{Vol}(\mathbb{H}^3/\Gamma) > 0.0422$ after we apply the packing density estimate. We define a *maximal* finite subgroup of a Kleinian group Γ to be a finite subgroup not contained in a larger finite subgroup.

THEOREM 10.3. *Let Γ be a Kleinian group containing a maximal finite subgroup isomorphic to the tetrahedral group. Then we have*

$$\mathrm{Vol}(\mathbb{H}^3/\Gamma) \geq \mathrm{Vol}(\mathbb{H}^3/\Gamma_1) = 0.0408.$$

This estimate is sharp and uniquely achieved in Γ_1.

Remark. This remark concerns arithmeticity and how we have used this in our co-volume estimates. The reader should be well aware that we have from time to time used our knowledge of the minimal co-volume arithmetic lattice, as identified by Chinburg and Friedman [CF86], to eliminate various small configurations. In fact in each case we have identified the commutator parameter of the arithmetic group in question. From this single complex number, all the relevant arithmetic data such as trace fields and quaternion algebras can be obtained [GMMR97], given the order of the two generators. In fact the commutator determines the group uniquely up to conjugacy in this situation. From this arithmetic data we may easily identify the minimal co-volume in the commensurability class using a formula of Borel [Bor81]. Thus our results do not really depend in any significant way on the results of [CF86].

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E-mail address: fgehring@umich.edu
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, UNITED STATES

E-mail address: g.j.martin@massey.ac.nz
INSTITUTE OF INFORMATION AND MATHEMATICAL SCIENCES, MASSEY UNIVERSITY, ALBANY CAMPUS, PRIVATE BAG 102-904, NORTH SHORE MAIL CENTRE, AUCKLAND, NEW ZEALAND