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The singular set of 1-1 integral currents

By Tristan Rivière and Gang Tian

Abstract

We prove that 2 dimensional integer multiplicity 2 dimensional rectifiable currents which are almost complex cycles in an almost complex manifold admitting locally a compatible positive symplectic form are smooth surfaces aside from isolated points and therefore are J-holomorphic curves.

I. Introduction

Let (M^{2p}, J) be an almost complex manifold. Let $k \in \mathbb{N}$, $k \leq p$. We shall adopt classical notation from Geometric Measure Theory [Fe]. We say that a $2k$-current C in (M^{2p}, J) is an almost complex integral cycle whenever it fulfills the following three conditions

i) *Rectifiability*: There exists an at most countable union of disjoint oriented C^1 $2k$-submanifolds $\mathcal{C} = \cup_i N_i$ and an integer multiplicity $\theta \in L^1_{\text{loc}}(\mathcal{C})$ such that for any smooth compactly supported in M $2k$-form ψ one has

$$C(\psi) = \sum_i \int_{N_i} \theta\ \psi\ .$$

ii) *Closedness*: C is a cycle,

$$\partial C = 0; \quad \text{i.e., } \forall \alpha \in \mathcal{D}^{2k-1}(M), \quad C(d\alpha) = 0\ .$$

iii) *Almost complex*: For $\mathcal{H}^{2k}$ and almost every point x in $\mathcal{C}$, the approximate tangent plane T_x to the rectifiable set $\mathcal{C}$ is invariant under the almost complex structure J; i.e.,

$$J(T_x) = T_x\ .$$

In this work we address the question of the regularity of such a cycle: Does there exist a *smooth* almost complex manifold (Σ^{2k}, j) without boundary and a *smooth* $j - J$-holomorphic map u ($\forall x \in \Sigma$ and $\forall X \in T_x\Sigma$ $du_x j \cdot X = J \cdot du_x X$) such that u would realize an *embedding* in M^{2p} aside from a locally finite $2k-2$

measure closed subset of M and such that $C = u_*[\Sigma^{2k}]$; i.e., $\forall \psi \in C_0^\infty(\wedge^{2k} M)$

$$C(\psi) = \int_\Sigma u^*\psi \, ?$$

In the very particular case where the almost complex structure J is integrable, this regularity result is optimal (C is the integral over multiples of algebraic subvarieties of M) and was established in [HS] and [Ale]. There are numerous motivations for studying the general case of arbitrary, almost complex structures J. First, as explained in [RT], the above regularity question for rectifiable almost complex cycles is directly connected to the regularity question of J-holomorphic maps into complex projective spaces. It is conjectured, for instance, that the singular set of $W^{1,2}(M^{2p}, N)$ J-holomorphic maps between almost complex manifolds M and N should be of finite $(2p-4)$-Hausdorff measure. The resolution of that question leads, for instance, to the characterization of stable-bundle, almost complex structures over almost Kähler manifolds via Hermite-Einstein Structures and extends Donaldson, Uhlenbeck-Yau characterization in the integrable case (see [Do], [UY]) to the nonintegrable one. Another motivation for studying the regularity of almost complex rectifiable cycles is the following. In [Li] and [Ti] it is explained how the loss of compactness of solutions to geometric PDEs having a given conformal invariant dimension q (a dimension at which the PDE is invariant under conformal transformations - $q = 2$ for harmonic maps, $q = 4$ for Yang-Mills Fields...etc) arises along $m - q$ rectifiable cycles (if m denotes the dimension of the domain). These cycles happen sometimes to be almost complex (see more details in [Ri1]).

By trying to produce in $(\mathbb{R}^{2p}, J)$ an almost complex graph of real dimension $2k$ in a neighborhood of a point $x_0 \in \mathbb{R}^{2p}$ as a perturbation of a complex one (J_{x_0}-holomorphic), one realizes easily that, for generic almost complex structures J, the problem is overdetermined whenever $k > 1$ and well posed for $k = 1$. Therefore the case of 2-dimensional integer rectifiable almost complex cycles is the generic one from the existence point of view. We shall restrict to that important case in the present paper. After complexification of the tangent bundle to M^{2p} a classical result asserts that a 2-plane is invariant under J if and only if it has a $1-1$ tangent 2-vector. Therefore we shall also speak about $1-1$ integral cycles for the almost complex 2-dimensional integral cycles. In the present work we consider the locally symplectic case: We say that (M^{2p}, J) has the *locally symplectic property* if at a neighborhood of each point x_0 in M^{2p} there exist a positive symplectic structure compatible with J and a neighborhood U of x_0 and a smooth closed 2-form ω such that $\omega(\cdot, J\cdot)$ defines a scalar product. It was proved in [RT] that arbitrary, 4-dimensional, almost complex manifolds satisfy the *locally symplectic property.* This is no more the case in larger dimension: one can find an almost complex structure in S^6 which admits no compatible positive symplectic form even locally; see [Br].

Our main result is the following.

THEOREM I.1. *Let (M^{2p}, J) be an almost complex manifold satisfying the locally symplectic property above. Let C be an integral 2 dimensional almost complex cycle. Then, there exist a J-holomorphic curve Σ in M, smooth aside from isolated points, and a smooth integer-valued function θ on Σ such that, for any 2 form $\psi \in C_0^\infty(M)$,*

$$C(\psi) = \int_\Sigma \theta \ \psi \ .$$

In the "locally symplectic case" being an almost-complex 2 cycle is equivalent for a 2-cycle to being calibrated by the local symplectic form ω for the local metric $\omega(\cdot, J\cdot)$. Therefore the regularity question for almost complex cycles is embedded into the problem of calibrated current and hence the theory of area-minimizing rectifiable 2-cycles. Therefore our result appears to be a consequence of the "Big Regularity Paper" of F. Almgren [Alm] combined with the PhD thesis of his student S. Chang [Ch]. Our attempt here is to present an alternative proof independent of Almgren's monumental work and adapted to the case we are interested in. The motivation is to give a proof that could be modified in order to solve the general case (non locally symplectic one) which cannot be "embedded" in the theory of area-minimizing cycles anymore.

A proof for the regularity of almost complex cycle in the locally symplectic, $p = 2$ case, independent of the regularity theory for area-minimizing surfaces, was also one of the results of the work Gr$\Longrightarrow$SW of C. Taubes [Ta] for $p = 2$. In particular, [Ta] presents a proof of Theorem I.1 when $p = 2$. In [RT], we gave an alternative proof for this special case. Theorem I.1 can be seen as the generalization to higher dimension ($p > 2$) of these works.

One of the main difficulties arising in dimension $p > 2$ is the nonnecessary existence of J-holomorphic foliations transverse to our almost complex current C in a neighborhood of a point. This then prevents describing the current as a Q-multivalued graph from D^2 into $\mathbb{C}^{p-1}$, $\{(a_i^k(z))_{k=1^{p-1}}\}_{i=1\cdots Q}$ in a neighborhood of a point of density N solving locally an equation of the form

$$\partial_{\overline{z}} a_i^k = \sum_{l=1}^{p-1} A(z, a_i)_l^k \cdot \nabla a_i^l + \alpha^k(a_i, z) \ , \tag{I.1}$$

where A and α are small in C^2 norm, as for $p = 2$ in [RT]. What we can only ensure instead is to describe the current C, in a neighborhood of a point of multiplicity Q, as an "algebraic Q-valued graph" from D^2 into $\mathbb{C}^{p-1}$; that is, a family of points in $\mathbb{C}^{p-1}$, $\{a_1(z), \ldots, a_P(z), b_1(z), \ldots, b_N(z)\}$ where only $P - N = Q$ is independent on z (neither P nor N is *a priori* independent on z), a_i are the positive intersection points and b_j are the negative ones. This "algebraic Q-valued graph" solves locally a much less attractive equation than (I.1),

$$\partial_{\bar{z}} a_i^k = \sum_{l=1}^{p-1} A_k^l(z, a_i, \nabla a_i) \cdot \nabla a_i^l + \sum_{l=1}^{p-1} B_k^l(z, a_i) \cdot \nabla a_i^l + C^k(z, a_i) \,, \tag{I.2}$$

where $A(z,a,p)$, $B(z,a)$ and $C(z,a)$ are also small in C^2 norm but the dependence on p in $A(z,a,p)$ is linear and therefore as ∇a_i gets bigger, which can happen, the right-hand side of (I.2) cannot be handled as a perturbation of the left-hand one in steps such as the "unique continuation argument". This was used in [RT] for proving that singularities of multiplicity Q cannot have an accumulation point in the carrier $\mathcal{C}$ of C.

The strategy of the proof goes as follows. A classical blow-up analysis tells us that, for an arbitrary point x_0 of the manifold M^{2p}, the limiting density $\theta(x_0) = \lim_{r\to 0} r^{-2}\mathrm{M}(C \llcorner B_r(x_0))$. Here M denotes the mass of a current and $\llcorner$ is the restriction operator which equals π times an integer Q. Since the density function $r \to r^{-2}\mathrm{M}(C \llcorner B_r(x_0))$ at every point is a monotonic increasing function, the complement of the set $\mathcal{C}_Q := \{x \in M \ ; \ \theta(x) \leq Q\}$ is closed in M and this permits us to perform an inductive proof of Theorem I.1 restricting the current to $\mathcal{C}_Q$ and considering increasing integers Q. A point of multiplicity Q is called a singular point of C if it is in the closure of points of nonzero multiplicity strictly less than Q.

The goal of the proof is then to show that singularities of multiplicity less than Q are isolated. We assume this fact for $Q-1$ and the paper is devoted to the proof that this then holds for Q itself. From a now classical result of B. White (see [Wh]), the dilated currents at a point x_0 of density $Q \neq 0$ converge in flat norm to a sum of Q flat J_{x_0}-holomorphic disks. Moreover, for any $\varepsilon > 0$ and r sufficiently small $C \llcorner B_r(x_0)$ is supported in the cones whose axes are the limiting disks and angle ε. For $Q > 1$, if two of these limiting disks are different it is then easy to observe that x_0 cannot be an accumulation point of singularities of multiplicity Q; this is the so called "easy case". If the limiting disks are all identical, equal to D_0, then we are in the "difficult case" and much more work has to be done in order to reach the same statement.

Contrary to the special case of dimension 4 ($p = 2$) considered by the authors in [RT], we could not find nice coordinates that would permit us to write C as a Q-valued graph over the limiting disk D_0. Considering then some J_{x_0}-complex coordinates $(z, w_1, \ldots, w_{p-1})$ in a neighborhood $B^{2p}_{\rho_{x_0}}(x_0)$ such that $\cap_i w_i^{-1}\{0\}$ corresponds to D_0, by the mean of the "lower-epiperimetric inequality" proved by the first author in [Ri2], one can construct a Whitney-Besicovitch covering, $\{B^2_{\rho_i}(z_i)\}_{i\in I}$, of the orthogonal projection on D_0 of the points in $B^{2p}_{\rho_{x_0}}(x_0)$ having a positive density strictly less than Q. This covering is such that for every $i \in I$ there exists $x_i = (z_i, w_i) \in B^{2p}_{\rho_{x_0}}(x_0)$ verifying that the restriction of C to the tube $B^2_{\rho_i}(z_i) \times B^{2p-2}_{\rho_{x_0}}(0)$ is in fact supported in the ball $B^{2p}_{2\rho_i}(x_i)$ of radius $2\rho_i$, two times the width of the tube. Moreover if one looks inside $B^{2p}_{\rho_i}(x_i)$, C is "split": this last word means that C restricted to $B^{2p}_{\rho_i}(x_i)$ is at a flat distance comparable to ρ_i^3 from the Q multiple of any graph over $B^2_{\rho_i}(z_i)$. This comes from the fact that the density ratio $\rho_i^{-2}\mathrm{M}(B_{\rho_i}(x_i))$ is strictly less than πQ minus a constant α depending only on p, Q, J and ω.

We then construct an average curve for C. In the 4-dimensional case since C was a Q-valued graph over D_0 we simply took the average of the Q points over any point in D_0. Here, in arbitrary dimension, the construction of the average curve is more delicate and uses the covering. We first approximate $C \llcorner B^{2p}_{\rho_i}(x_i)$ by a J_{x_i}-holomorphic graph C_i using a technique introduced in [Ri3], and choosing a J_{x_i}-holomorphic disk D_i approximating D_0 we can express C_i as a Q-valued graph over D_i for which we take the average $\tilde{C}_i$ that happens to be Lipschitz with a uniformly bounded Lipschitz constant.

Therefore the J_{x_i}-holomorphic curve $\tilde{C}_i$ can be viewed as a graph $\tilde{a}_i$ over $B^2_{\rho_i}(z_i)$. Patching the $\tilde{a}_i$ together we get a graph $\tilde{a}$ that extends over the whole $B^2_{\rho_{x_0}}(0)$ as a $C^{1,\alpha}$ graph for any $\alpha < 1$ which is almost J-holomorphic and which passes through all the $B^{2p}_{\rho_i}(x_i)$. The fact that the average curve is more regular than the J-holomorphic cycle C from which it is produced is clear in the integrable case (since it is holomorphic); $(z, \pm\sqrt{z})$ is a $C^{0,\frac{1}{2}}$ 2-valued graph whereas its average $(z, 0)$ is smooth. This was extended in the nonintegrable case in the particular case of the 4 dimension in [ST].

The points of multiplicity Q in C are contained in the average curve $\tilde{a}$. We then show, by means of a unique continuation argument in the spirit of the one developed in [Ta] in 4 dimensions, that the points where C gets to coincide with $\tilde{a}$ are either isolated or coincide with the whole curve $\tilde{a}$. We have then shown that any point x_0 of multiplicity Q is either surrounded by points of multiplicity Q only, and in $B^{2p}_{\rho_{x_0}}$, C coincides with Q times a smooth graph over D_0 or x_0 is not an accumulation point of points of multiplicity Q and is surrounded in $B^{2p}_{\rho_{x_0}}$ by points of multiplicity strictly less than Q. It remains at the end to show that it cannot be an accumulation point of singularities of lower density. This is obtained again using an approximation argument by holomorphic curves introduced in [Ri3].

The paper is organized as follows. In Section II we establish preliminaries, introduce notation and give the main statement, assertion $\mathcal{P}_Q$, to be proved by induction in the rest of the paper. In Section III, with the help of the "upper-epiperimetric inequality" of B. White, we establish the uniqueness of the tangent cone and a quantitative version of it; see Lemma III.2. In Section IV we prove the relative Lipschitz estimate together with a tilting control of the tangent cones of density Q points in a neighborhood of a density Q point; see Lemma IV.2. In Section V we proceed to the covering argument, Lemma V.3, which is based on the "splitting before tilting" lemma (see Lemma V.1, proved in [Ri2]). In Section VI we construct the approximated average curve and prove the $C^{1,\alpha}$ estimate for this curve, Lemma VI.3. In Section VII we perform the unique continuation argument showing that singularities of multiplicity Q cannot be accumulation points of singularities of multiplicity Q. In Section VIII we show that singularities of multiplicity Q cannot be accumulation points of singularities of multiplicity less than Q either.

II. Preliminaries

Notation. We shall adopt standard notation from the Geometric Measure Theory [Fe] such as $\mathrm{M}(A)$ for the Mass of a current A, $\mathcal{F}(A)$ for its flat norm, $A \llcorner E$ for its restriction to a measurable subset E...etc; we refer the reader to [Fe].

Preliminaries. Since our result is a local one we shall work in a neighborhood U of a point x_0 and use a symplectic form ω compatible with J. We denote by g the metric generated by J and ω: $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$. We also introduce normal coordinates $(x_1, x_2, \ldots, x_{2p-1}, x_{2p})$ about x_0 in U which can be chosen such that at x_0,

$$J_{x_0} \cdot \frac{\partial}{\partial x_{2i+1}} = \frac{\partial}{\partial x_{2i+2}} \qquad \text{for } i = 0 \ldots p-1 \ . \tag{II.1}$$

Since C is a calibrated current in (U, ω, J), it is an area-minimizing current and its generalized mean curvature vanishes (see [All] or [Si]). One may isometrically embed (U, g) into a euclidian space $\mathbb{R}^{2p+k}$ and the generalised mean curvature of C in $\mathbb{R}^{2p+k}$ coincides with the mean curvature of the embedding of (U, g) and is therefore a bounded function. Combining this fact together with the monotonicity formula (17.3) of [Si] we get that

$$\frac{\mathrm{M}(C \llcorner B_r(x_0))}{r^2} = f(r) + O(r) \ , \tag{II.2}$$

where $f(r)$ is an increasing function, M denotes the Mass of a current and $C \llcorner B_r(x_0)$ is the restriction of C to the geodesic ball of center x_0 and radius r. There exists in fact a constant α depending only on g such that $e^{\alpha r} \frac{\mathrm{M}(C \llcorner B_r(x_0))}{r^2}$ is an increasing function in r (see [Si]). The factor $e^{\alpha r}$ is a perturbation of an order which will have no influence on the analysis below; therefore, by an abuse of notation we will often omit to write it and consider outright that $\frac{\mathrm{M}(C \llcorner B_r(x_0))}{r^2}$ is an increasing function.

By means of the coordinates $(x_1 \ldots x_{2p})$ we shall identify U with a subdomain in $\mathbb{R}^{2p}$ and use the same notation C for the push forward of C in $\mathbb{R}^{2p}$ by this chart. For small radii r we introduce the dilation function $\lambda^{r,x_0}(x) = \frac{x - x_0}{r}$, and we introduce the following dilation of C about x_0 with rate r as being the following current in $\mathbb{R}^{2p}$,

$$C_{r,x_0} := (\lambda^{r,x_0}{}_*C)\llcorner B_1^{2p}(0) \ . \tag{II.3}$$

Observe that $r^2M_0(C_{r,x_0}) = M_0(C\llcorner B_r^{2p}(0))$ where M_0 denotes the mass in the flat metric g_0 in $\mathbb{R}^{2p}$. Since $g = g_0 + O(r^2)$, we deduce from (II.2) that $M_0(C_{r,x_0})$ is uniformly bounded as r tends to zero. Again since g and g_0 coincide up to the second order, it does not hurt in the analysis below if one mixes the notations for the two masses M and M_0 and speaks only about M. Since now C is a cycle in U, $\partial C_{r,x_0}\llcorner B_1^{2p}(0) = 0$ and we can apply the Federer-Fleming compactness theorem to deduce that, from any sequence $r_i \to 0$ one can extract a subsequence $r_{i'}$ such that $C_{r_{i'},x_0}$ converges in Flat norm to a limiting current C_{0,x_0} called a tangent cone of C at x_0. One of the purposes of the next section will be to establish that C_{0,x_0} is independent of the subsequence and that the tangent cone is unique. The lower semi-continuity of the mass under weak convergence implies that

$$\lim_{r\to 0}\frac{\mathrm{M}(C\llcorner B_r(x_0))}{r^2} = \lim_{r\to 0}\mathrm{M}(C_{r,x_0}) \geq \mathrm{M}(C_{0,x_0}) \ . \tag{II.4}$$

Now, from the fact that C is calibrated by ω we deduce that the inequality (II.4) is an equality. Indeed

$$\mathrm{M}(C_{r,x_0}) = r^{-2}C\llcorner B_r^{2p}(0)(\omega) = C_{r,x_0}\left(r^2(\lambda^{r,x_0})^*\omega\right) \ .$$

It is clear that $\lim_{r\to 0}\|r^2(\lambda^{r,x_0})^*\omega - \omega_0\|_\infty$ where $\omega_0 = \sum_{i=1}^p dx_{2i-1}\wedge dx_{2i}$. Therefore $C_{r,x_0}(r^2(\lambda^{r,x_0})^*\omega - \omega_0) \longrightarrow 0$ and we get that

$$\lim_{r\to 0}\mathrm{M}(C_{r,x_0}) = \lim_{i'\to+\infty} C_{r_{i'},x_0}(\omega_0) = C_{0,x_0}(\omega_0) \ . \tag{II.5}$$

Since the comass of ω_0 is equal to 1, $C_{0,x_0}(\omega_0) \leq \mathrm{M}(C_{0,x_0})$. Combining this last fact with (II.4) and (II.5) we have established that

$$\lim_{r\to 0}\mathrm{M}(C_{r,x_0}) = \mathrm{M}(C_{0,x_0}) = C_{0,x_0}(\omega_0) \tag{II.6}$$

which means in particular that C_{0,x_0} is calibrated by the Kähler form ω_0 in $(R^{2p}, J_0) \simeq \mathbb{C}^p$ which is equivalent to the fact that C_{0,x_0} is J_0-holomorphic. Using the explicit form of the monotonicity formula (see [Si] page 202), one observes that for any $s \in \mathbb{R}_+^*$

$$C_{0,x_0} = \lambda_*^s C_{0,x_0}$$

which means that for $\mathcal{H}^2$ almost everywhere on the carrier $\mathcal{C}_{0,x_0}$ of C_{0,x_0}, $\frac{\partial}{\partial r}$ is in the approximate tangent plane to C_{0,x_0}; in other words, C_{0,x_0} is a cone. Since it is J_0-holomorphic, $\mathcal{H}^2$-a.e. x in $\mathcal{C}_{0,x_0}$, the approximate tangent cone is given by

$$T_xC_{0,x_0} = \mathrm{Span}\left\{\frac{\partial}{\partial r}, J_0\frac{\partial}{\partial r}\right\} \ .$$

Integral curves of $J_0\frac{\partial}{\partial r}$ are great-circles, fibers of the Hopf fibration

$$(z_1 = x_1 + ix_2, \ldots, z_p) \longrightarrow [z_1, \ldots, z_p] \ .$$

Therefore we deduce that C_{0,x_0} is the sum of the integrals over radial extensions of such great circles $\Gamma_1 \ldots \Gamma_Q$ in S^{2p-1} which is the integral over a sum of Q flat holomorphic disks. We adopt the following notation (in fact identical to the one used in [Wh]) for the radial extensions in $B_1^{2p}(0)$ of currents supported in $\partial B_1^{2p}(0)$:

$$C_{0,x_0} = \oplus_{i=1}^{Q} 0\sharp\Gamma_i \ .$$

Then we deduce that

$$\lim_{r\rightarrow 0} \mathrm{M}(C_{r,x_0}) = \pi Q \in \pi\mathbb{Z} \ . \tag{II.7}$$

For any $x \in U$ one denotes Q_x the integer such that

$$\lim_{r\rightarrow 0} \frac{\mathrm{M}(C \llcorner B_r(x))}{\pi r^2} = Q_x \ .$$

Using the monotonicity formula, it is straightforward to deduce that for any $Q \in \mathbb{N}$,

$$\mathcal{C}_Q = \{x \in U : \ 0 < Q_x \leq Q\}$$

is an open subset of $\mathcal{C}_* = \{x \in U : \ 0 < Q_x\}$. For $Q > 1$, let us also denote

$$\mathrm{Sing}^Q = \{x \in \mathcal{C}_* : \ Q_x = Q \quad \text{and } x \text{ is an acc. point of } \mathcal{C}_{Q-1}\} \ .$$

Observe that, from Allard's theorem, it is clear that $C \llcorner (U \setminus \cup_Q \mathrm{Sing}^Q)$ is the integral along a smooth surface with a smooth integer multiplicity. Although we won't make use of Allard's theorem this justifies *a priori* our notation. The whole purpose of our paper is to show that $\cup_Q \mathrm{Sing}^Q$ is made of isolated points. As we said, we won't make use of Allard's paper below since the relative Lipschitz estimate we establish in Lemma IV.2 gives Allard's result in our case which is more specific. Because of this nice stratification of C ($\mathcal{C}_Q$ is open in $\mathcal{C}_*$) we can argue by induction on Q . Let $\mathcal{P}_Q$ be the following assertion

$$\mathcal{P}_Q : \ \cup_{q\leq Q} \mathrm{Sing}^q \text{ is made of isolated points .} \tag{II.8}$$

From the beginning of Section IV until Section VII we will assume either $Q = 2$ or that $\mathcal{P}_{Q-1}$ holds and the goal will be to establish $\mathcal{P}_Q$.

III. The uniqueness of the tangent cone

The uniqueness of the tangent cone means that the limiting cone C_{0,x_0}, obtained in the previous section while dilating at a point following a subsequence of radii $r_{i'}$, is independent of the subsequence and is unique. Since our calibrated two-dimensional rectifiable cycle is area-minimizing, this fact is a consequence of B. White upper-epiperimetric inequality in [Wh] (see also [Ri2] for the justification of the prefix "upper"). We need, however, a more quantitative version of this uniqueness of the tangent cone and express how far we are from the unique tangent cone in terms of the closedness of the density of area $\mathrm{M}(C \llcorner B_r(x_0))/\pi r^2$ to the limiting density Q. Precisely the goal of this section is to prove the following lemma:

Lemma III.1 (Uniqueness of the tangent cone). *For any $\varepsilon > 0$ and $Q \in \mathbb{N}$ there exists $\delta > 0$ and $\rho_\varepsilon \leq 1$ such that, for any compatible pair (J, ω) almost complex structure-symplectic form over $B_1^{2p}(0)$ satisfying $J(0) = J_0(0)$, $\omega(0) = \omega_0(0)$,*

$$\|J - J_0\|_{C^2(B_1)} + \|\omega - \omega_0\|_{C^2(B_1)} \leq \delta \ , \tag{III.1}$$

for any J-holomorphic integral 2-cycle C in $B_1(0)$ such that $Q_0 = Q$, if

$$\mathrm{M}(C \llcorner B_1^{2p}(0)) \leq \pi Q + \delta \ .$$

Then, there exist Q J_0 holomorphic flat discs $D_1 \ldots D_Q$ passing through 0, intersection of holomorphic lines of $\mathbb{C}^p$ with $B_1^{2p}(0)$, such that, for any $\rho \leq \rho_\varepsilon$

$$\mathcal{F}(C_{\rho,0} - \oplus_{i=1}^{Q} D_i) \leq \varepsilon \tag{III.2}$$

and for any $\psi \in C_0^\infty(B^1 \setminus \{x \in B^1 \ ; \ \mathrm{dist}(x, \cup_i D_i) \leq \varepsilon |x|\})$,

$$C_{\rho,0}(\psi) = 0 \ . \tag{III.3}$$

Before proving Lemma III.1, we first establish the following intermediate result:

Lemma III.2. *For any $\varepsilon > 0$ and $Q \in \mathbb{N}$ there exists $\delta > 0$ such that the following is true. If (J, ω) is a compatible pair of almost complex structure-symplectic form over $B_1^{2p}(0)$ satisfying $J(0) = J_0(0)$, $\omega(0) = \omega_0(0)$*

$$\|J - J_0\|_{C^2(B_1)} + \|\omega - \omega_0\|_{C^2(B_1)} \leq \delta \ ,$$

for any J-holomorphic integer rectifiable 2-cycle C such that $Q_x = Q$, if

$$\mathrm{M}(C \llcorner B_1^{2p}(0)) \leq \pi Q + \delta$$

then, there exist Q J_0 holomorphic flat discs $D_1 \ldots D_Q$ passing through 0, intersection of holomorphic lines of $\mathbb{C}^p$ with $B_1^{2p}(0)$, such that,

$$\mathcal{F}(C \llcorner B_1(0) - \oplus_{i=1}^{Q} D_i) \leq \varepsilon \ .$$

Remark III.1. Lemma III.2 give much less information than Lemma III.1. Since *a priori* in Lemma III.2 the disks D_i may vary a lot as one dilates C about 0, whereas Lemma III.1 controls such a tilting as one dilates the current further.

Proof of Lemma III.2. We prove Lemma III.2 by contradiction. Assume there exist $\varepsilon_0 > 0$, $\delta_n \to 0$, compatible J_n and ω_n and C_n such that

(III.4)

$$\begin{aligned} &\text{i)} && \|J_n - J_0\|_{C^2} + \|\omega_n - \omega_0\|_{C^2} \le \delta_n \ , \\ &\text{ii)} && \lim_{r\to 0} \pi^{-1} r^{-2} \mathrm{M}(C_n \llcorner B_r(0)) = Q \ , \\ &\text{iii)} && \mathrm{M}(C_n \llcorner B_1) \le \pi Q + \delta_n \ , \\ &\text{iv)} && \inf \left\{ \mathcal{F}(C_n \llcorner B_1 - \oplus_{i=1}^Q D_i) \ \text{ s.t. } D_i \text{ flat holom discs}, 0 \in D_i \right\} \ge \varepsilon_0 \ . \end{aligned}$$

Since $\partial C_n \llcorner B_1 = 0$ and since the mass of C_n is uniformly bounded, one may assume, modulo extraction of a subsequence if necessary, that C_n converges to a limiting rectifiable cycle C_∞. Exactly as in Section III we have the fact that for any $0 < r \le 1$

$$\lim_{n\to +\infty} \mathrm{M}(C_n \llcorner B_r) = \mathrm{M}(C_\infty \llcorner B_r) = C_\infty \llcorner B_r(\omega_0) \ . \tag{III.5}$$

We deduce then that C_∞ is calibrated by ω_0 and is therefore a J_0-holomorphic cycle. Using ii) we deduce also that

$$\lim_{r\to 0} \pi^{-1} r^{-2} \mathrm{M}(C_\infty \llcorner B_r(0)) = Q$$

and finally, from iii) and the lower semicontinuity of the mass, we have that $\mathrm{M}(C_\infty \llcorner B_1) = \pi Q$ Thus, since $\pi^{-1} r^{-2} \mathrm{M}(C_\infty \llcorner B_r(0))$ is an increasing function, we have established that on $[0, 1]$,

$$\pi^{-1} r^{-2} \mathrm{M}(C_\infty \llcorner B_r(0)) \equiv Q \ . \tag{III.6}$$

Let, for almost every r, $S_\infty^r = \langle C_\infty, \mathrm{dist}(\cdot, 0), r\rangle$ be the slice current obtained by slicing C_∞ with $\partial B_r(0)$ (see [Fe, 4.2.1]). By Fubini, we have that for a.e. $0 < r < 1$

$$\mathrm{M}(\langle C_\infty, \mathrm{dist}(\cdot, 0), r\rangle) \le 2\pi Q r \ .$$

Let $0 \sharp S_\infty^r$ be the radial extension of S_∞^r in $B_r(0)$.

$$\mathrm{M}(0\sharp S_\infty^r) = \frac{r}{2} \mathrm{M}(S_\infty^r) = \pi Q r^2 = \mathrm{M}(C_\infty \llcorner B_r(0)) \ .$$

Since $\partial(C_\infty \llcorner B_r(0) - 0\sharp S_\infty^r) = 0$, and since $C_\infty \llcorner B_r(0)$ is area-minimizing we have that $0\sharp S_\infty^r$ is also area-minimizing. Let α be such that $d\alpha = \omega$ and

$$\mathrm{M}(0\sharp S_\infty^r) = \mathrm{M}(C_\infty \llcorner B_r(0)) = C_\infty \llcorner B_r(0)(\omega_0) = S_\infty^r(\alpha) = (0\sharp S_\infty^r)(\omega_0) \ .$$

Therefore $0\sharp S_\infty^r$ is a holomorphic cone which is a cycle. So we deduce as in Section II that $0\sharp S_\infty^r$ is a sum of flat holomorphic disks for any r. Thus C_∞ is also a sum

$$C_\infty \llcorner B_1(0) = \sum_{i=1}^Q D_i$$

where each D_i is the intersection of a complex straight line in $\mathbb{C}^p$ with B_1^{2p}. From the Federer-Fleming compactness theorem we have the fact that the weak convergence of C_n to C_∞ holds in flat norm

$$\mathcal{F}\left(C_n \llcorner B_1 - \sum_{i=1}^{Q} D_i\right) \longrightarrow 0$$

which contradicts iv), and Lemma III.2 is proved. □

Proof of Lemma. III.1. We first prove assertion (III.2) and recall Brian White's upper-epiperimetric inequality adapted to our present context: White's upper-epiperimetric inequality was proved for area-minimizing surfaces in $\mathbb{R}^{2p}$. Here, in the present situation, we are dealing with area-minimizing currents which are J-holomorphic for a metric $g = \omega(\cdot, J\cdot)$ which gets as close as we want to the standard one because of assumption (III.1). Therefore very minor changes have to be provided to adapt White's theorem to the present context. An adaptation of the epiperimetric inequality for ambient nonflat metric is also given in [Ch, App. A]. So we have the following result.

Given an integer Q, there exists a positive number $\varepsilon_Q > 0$, such that, for any compatible pair ω, J in $B_2^{2p}(0)$ satisfying $\|\omega - \omega_0\|_{C^2(B_2)} + \|J - J_0\|_{C^2(B_2)} \leq \varepsilon_Q$ and for any C J-holomorphic 2-rectifiable integral current in $B_2^{2p}(0)$, satisfying $\partial C \llcorner B_2^{2p} = 0$, assuming there exist Q flat holomorphic disks $D_1 \ldots D_Q$ in $(B_1^{2p}(0), J) \simeq \mathbb{C}^p \cap B_1^{2p}(0)$ passing through the origin such that

$$\mathcal{F}\left(C_{2,0} \llcorner B_1(0) - \sum_{i=1}^{Q} D_i\right) \leq \varepsilon_Q \tag{III.7}$$

(where we used a common notation for the oriented 2-disks D_i and the corresponding 2-currents) we have

$$\mathrm{M}(C \llcorner B_1^{2p}) - \pi Q \leq (1 - \varepsilon_Q) \left(\frac{1}{2} M\left(\partial (C \llcorner B_1^{2p})\right) - \pi Q\right) . \tag{III.8}$$

Remark III.2. Observe that in the statement of the epiperimetric property in Definition 2 of [Wh] the epiperimetric constant ε_Q may *a priori* also depend on the cone $\sum_{i=1}^{Q} D_i$. It is however elementary that this space of cones made of the intersection of Q holomorphic straight lines passing through the origin with $B_1^{2p}(0)$ is compact for the flat distance. Now by using a simple finite covering argument for this space of cones by balls (for the flat distance) one may obtain a constant $\varepsilon_Q > 0$ for which the epiperimetric property holds independently of the cone $\sum_{i=1}^{Q} D_i$.

Once again we shall ignore the factor $e^{\alpha r}$ in front of $r^{-2}\mathrm{M}(C \llcorner B_r)$ which induces lower order perturbations and argue as if $r^{-2}\mathrm{M}(C \llcorner B_r)$ itself would be an increasing function (observe also that α may be taken arbitrarily small because J and ω are chosen as close as we want to J_0 and ω_0).

Then $\varepsilon > 0$, $\varepsilon < \varepsilon_Q$, and $\delta > 0$, given by Lemma III.2 for that ε. Assuming then $\mathrm{M}(C\llcorner B_1) \leq \pi Q + \delta$ implies from the monotonicity formula that for any $r < 1$, $r^{-2}\mathrm{M}(C\llcorner B_r(0)) = \mathrm{M}(C_{r,0}) \leq (\pi Q + \delta)$. Applying Lemma III.2 to $C_{2r,0}$ for $r < 1/2$ we deduce the existence of Q flat disks $D_1 \dots D_Q$ such that

$$\mathcal{F}\left(C_{2r,0} - \sum_{i=1}^{Q} D_i\right) \leq \varepsilon \ . \tag{III.9}$$

We can then apply the epiperimetric inequality to $C_{r,0}$ and get, after rescaling,

$$\mathrm{M}(C\llcorner B_r(0)) - \pi Q r^2 \leq (1-\varepsilon_Q)\left(\frac{r}{2}\mathrm{M}(\partial(C\llcorner B_r(0))) - \pi Q r^2\right) \ . \tag{III.10}$$

Denote $f(r) = \mathrm{M}(C\llcorner B_r(0)) - \pi Q r^2$, $f'(r) \geq \mathrm{M}(\partial(C\llcorner B_r(0))) - 2\pi Q r$. Therefore (III.10) implies

$$\frac{1-\varepsilon_Q}{2} r \ f'(r) \geq f(r) \ .$$

Integrating this differential inequality between s and σ $(1/2 > s > \sigma)$, we see that $f(s) \geq \left(\frac{s}{\sigma}\right)^{\frac{2}{1-\varepsilon_Q}} f(\sigma)$. When $\nu = \frac{2}{1-\varepsilon} - 2 > 0$,

$$\frac{f(s)}{s^2} \geq \left(\frac{s}{\sigma}\right)^{\nu} \ \frac{f(\sigma)}{\sigma^2} \ . \tag{III.11}$$

Let $F(x) = \frac{x}{|x|}$. Then,

$$\mathrm{M}(F_*\left(C\llcorner B_s(0)\setminus B_\sigma(0)\right)) = \int_{B_S(0)\setminus B_\sigma(0)} \frac{1}{|x|^3}\left|\tau \wedge \frac{x}{|x|}\right| \ |\theta| \ d\mathcal{H}^2\llcorner\mathcal{C}$$

where τ denotes the unit 2-vector associated to the oriented approximate tangent plane to C and is defined $\mathcal{H}^2$-a.e. along the carrier $\mathcal{C}$ of the rectifiable current, θ is the $L^1(\mathcal{C})$ integer-valued multiplicity of C (i.e using classical GMT notations:$C = \langle\mathcal{C},\theta,\tau\rangle$) and $d\mathcal{H}^2\llcorner\mathcal{C}$ is the restriction to $\mathcal{C}$ of the 2-dimensional Hausdorff measure. Using Cauchy-Schwarz and 5.4.3 (2) of [Fe] (the explicit formulation of the monotonicity formula) and (III.11), we have

$$\begin{aligned}
&\mathrm{M}(F_*\left(C\llcorner B_s(0)\setminus B_\sigma(0)\right)) \\
&\quad\leq \left[\frac{\mathrm{M}(C\llcorner B_s(0))}{s^2} - \frac{\mathrm{M}(C\llcorner B_\sigma(0))}{\sigma^2}\right]^{\frac{1}{2}} \left[\frac{\mathrm{M}(C\llcorner B_s(0))}{\sigma^2}\right]^{\frac{1}{2}} \\
&\quad\leq \left[\frac{\mathrm{M}(C\llcorner B_s(0))}{s^2} - \pi Q\right]^{\frac{1}{2}} \left[\frac{\mathrm{M}(C\llcorner B_s(0))}{\sigma^2}\right]^{\frac{1}{2}} \\
&\quad\leq \left[\frac{f(s)}{s^2}\right]^{\frac{1}{2}} \left[\frac{s^2}{\sigma^2}\frac{\mathrm{M}(C\llcorner B_s(0))}{s^2}\right]^{\frac{1}{2}} \leq K \ s^{\frac{\nu}{2}} \ \frac{s}{\sigma} \ .
\end{aligned} \tag{III.12}$$

With $r < \rho < 1/2$, applying (III.12) for $s = 2^{-k}\rho$ and $\sigma = 2^{-k-1}\rho$ for $k \leq \log_2 \frac{\rho}{r}$ and summing over k we get

$$\text{(III.13)} \qquad \mathrm{M}(F_*\,(C\llcorner B_\rho(0) \setminus B_r(0))) \leq C\rho^{\frac{\nu}{2}} \ .$$

Observe that $\partial(F_*\,(C\llcorner B_\rho(0) \setminus B_r(0))) = \partial C_{\rho,0} - \partial C_{r,0}$. Therefore we deduce

$$\text{(III.14)} \qquad \mathcal{F}\left((C_{\rho,0} - C_{r,0})\llcorner B_1(0) \setminus B_{\frac{1}{2}}(0)\right) \leq C\rho^{\frac{\nu}{2}} \ .$$

Since

$$\mathcal{F}\left((C_{\rho,0} - C_{r,0})\llcorner B_{\frac{1}{2}}(0)\setminus B_{\frac{1}{4}}(0)\right) \leq \left(\frac{1}{3}\right)^{\frac{1}{3}} \mathcal{F}\left((C_{\frac{\rho}{2},0} - C_{\frac{r}{2},0})\llcorner B_1(0)\setminus B_{\frac{1}{2}}(0)\right) \ ,$$

applying (III.14) for ρ, r replaced by $2^{-k}\rho$, $2^{-k}r$ and summing over $k = 1, \ldots, \infty$ we finally obtain

$$\text{(III.15)} \qquad \mathcal{F}\left((C_{\rho,0} - C_{r,0})\llcorner B_1(0)\right) \leq C\rho^{\frac{\nu}{2}}$$

which is the desired inequality (III.2).

It remains to show (III.3) in order to finish the proof of Lemma III.1. We argue by contradiction. Assume there exists $\varepsilon_0 > 0$, $\rho_n \to 0$ and $\psi_n \in C_0^\infty(\wedge^2 B_1)$ such that

$$\operatorname{supp}\psi_n \subset E_0 = \{x \in B^1 \ ; \ \operatorname{dist}(x, \cup_i D_i) \leq \varepsilon_0|x|\} \ ,$$

where $C_{0,0} = \oplus_{i=1}^Q D_i$, and

$$C_{\rho_n,0}(\psi_n) \neq 0 \ .$$

This latter fact implies in particular that there exists $x^n \in E_0$ such that $\lim_{r\to 0} \mathrm{M}(C_{r,x_n}) \neq 0$. Using the monotonicity formula we deduce then that

$$\mathrm{M}\left(C_{\rho_n|x_n|,0}\llcorner B_{\varepsilon_0/2}\left(\frac{x_n}{|x_n|}\right)\right) \geq \frac{\pi}{4}\varepsilon_0^2 \ .$$

We may then extract a subsequence such that $\frac{x_n}{|x_n|} \to x_\infty$. Thus,

$$\mathrm{M}(C_{\rho_n|x_n|,0}\llcorner B_{3\varepsilon_0/4}(x_\infty)) \geq \frac{\pi}{4}\varepsilon_0^2 \ .$$

Now,

$$\mathrm{M}(C_{\rho_n|x_n|,0}\llcorner B_{3\varepsilon_0/4}(x_\infty)) = C_{\rho_n|x_n|,0}\llcorner B_{3\varepsilon_0/4}(x_\infty)\left(\frac{x}{\rho_n|x_n|}^*\omega_n\right) \ .$$

Since $\|\omega_n - \omega_0\|_{C^2} \to 0$ and since $\omega_n(0) = \omega_0(0)$ we clearly have that

$$\|\frac{x}{\rho_n|x_n|}^*\omega_n - \omega_0\|_\infty \longrightarrow 0 \ .$$

Therefore

$$\left|C_{\rho_n|x_n|,0}\llcorner B_{3\varepsilon_0/4}(x_\infty)\left(\frac{x}{\rho_n|x_n|}^*\omega_n - \omega_0\right)\right| \leq \mathrm{M}(C_{\rho_n|x_n|,0}\llcorner B_{3\varepsilon_0/4}(x_\infty))\ \|\frac{x}{\rho_n|x_n|}^*\omega_n - \omega_0\|_\infty \longrightarrow 0 \ .$$

Thus

$$C_{0,0} \llcorner B_{3\varepsilon_0/4}(x_\infty)(\omega_0) = \lim_{n\rightarrow+\infty} C_{\rho_n|x_n|,0} \llcorner B_{3\varepsilon_0/4}(x_\infty)(\omega_0) \geq \frac{\pi}{4}\varepsilon_0^2$$

which contradicts the fact that $B_{3\varepsilon_0/4}(x_\infty) \subset E_0$. Therefore (III.3) holds and Lemma III.1 is proved. □

IV. Consequences of Lemma III.1. No accumulation of points in Sing^Q in the easy case. The relative Lipschitz estimate in the difficult case

In this section we expose two important consequences of Lemma III.1. Before explaining them we first observe that proving the implication $\mathcal{P}_{Q-1} \Longrightarrow \mathcal{P}_Q$, will require considering two cases separately. The first case (the easy one) is the case where the tangent cone at the point x_0 of multiplicity Q (i.e. $\pi^{-1}r^{-2}\mathrm{M}(B_r(x_0)) \rightarrow Q$) is not made of Q times the same disk. The second case is the case where the tangent case is made of Q times the same disk. In the first case we will deduce almost straight from Lemma III.1 that such an x_0 cannot be an accumulation point of points of multiplicity Q also; see Lemma IV.1 below. In the second case, much more analysis will be needed to reach the same statement and this is the purpose of Sections IV through IX.

We can nevertheless deduce in this section an important consequence of our quantitative version of the uniqueness of the tangent cone (Lemma III.1) for the difficult case: this is the so called "relative Lipshitz estimate" (see Lemma IV.2 below). This property says that, given a point x_0 of multiplicity Q whose tangent cone is Q times a flat disk and given an $\varepsilon > 0$, there exists a radius $r_{\varepsilon,x_0} > 0$ such that given any two points of $\mathcal{C}_* \cap B_{r_{\varepsilon,x_0}}(x_0)$, one of the two being also of multiplicity Q, the slope they realize relative to the tangent cone of x_0 is less than ε.

The condition that one of the two points has multiplicity Q (this could be x_0 itself for instance) is a crucial assumption. It is indeed straightforward to find counterexamples to any Lipschitz estimates of multivalued graphs of holomorphic curves. Take for instance $w^2 = z$ in $\mathbb{C}^2 \simeq \{(z,w)\ z,w \in \mathbb{C}\}$ viewed as a 2-valued graph over the line $\{w = 0\}$, all points having multiplicity 1, $(0,0)$ included of course, but the best possible estimate is a Hölder one $C^{0,\frac{1}{2}}$. We cannot exclude that such a configuration exists as we dilate at a point x_0 of mupltiplicity $Q > 1$.

We first prove the following consequence of Lemma III.1

LEMMA IV.1 (no accumulation — the easy case). *Let $Q \in \mathbb{N}$, $Q \geq 2$. Let x_0 be a point in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ (i.e., $\pi^{-1}r^{-2}\mathrm{M}(B_r(x_0)) \rightarrow Q$ as $r \rightarrow 0$). Assume that the tangent cone at x_0, C_{0,x_0}, contains at least two different flat J_{x_0}-holomorphic disks (i.e., $C_{0,x_0} \neq Q\,D$ for the single flat J_{x_0}-holomorphic* *disk D). Then, there exists $r > 0$ such that*

$$B_r(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}) = \{x_0\} .$$

Proof of Lemma IV.1. Let x_0 be as in the statement of the lemma: $x_0 \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ and $C_{0,x_0} = \oplus_{i=1}^K Q_i\ D_i$ (where $D_i \neq D_j$ for $i \neq j$ and $K > 1$). Let $\varepsilon > 0$ be a positive number smaller than 1/4Q times the maximal angle α between the various disks $D_1, \dots, D_K$ in the tangent cones in such a way that there exists $i \neq j$ such that

$$E_\varepsilon(D_i) \cap E_\varepsilon(D_j) = \{x_0\} ,$$

where we use the following notation

$$E_\varepsilon(D_i)\{x \in \mathbb{R}^{2p}\ ;\ \mathrm{dist}(x, D_i) \leq \varepsilon|x - x_0|\}) .$$

By taking ε as small as above, we even have ensured that $\cup E_\varepsilon(D_i) \setminus x_0$ has at least two connected components whose intersections with $\partial B_1(x_0)$ are at a distance larger than $\alpha/2$. We now prove Lemma IV.1 by contradiction. Assume there exists $x_n \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ such that $x_n \to x_0$ and $x_n \neq x_0$. Let $\delta > 0$ be given by Lemma III.1 for ε chosen as above. Let $\rho > 0$ be such that $\rho^{-2}\mathrm{M}(C \llcorner B_\rho(x_0)) \leq \pi Q + \delta/2$. For any $x \in B_{\rho\frac{\delta}{4\pi Q}}(x_0)$ we have

$$\text{(IV.1)}\quad \mathrm{M}(C \llcorner B_{\rho(1-\frac{\delta}{4\pi Q})}(x)) \leq \mathrm{M}(C \llcorner B_\rho(x_0)) \leq \rho^2 \left(\pi Q + \frac{\delta}{2}\right)$$
$$\leq \left(\rho\left(1 - \frac{\delta}{4\pi Q}\right)\right)^2 \left(1 - \frac{\delta}{4\pi Q}\right)^{-2} \left(\pi Q + \frac{\delta}{2}\right)$$
$$\leq \left(\rho\left(1 - \frac{\delta}{4\pi Q}\right)\right)^2 (\pi Q + \delta) .$$

Choose then $x_n \in B_{\rho\frac{\delta}{8\pi Q}}(x_0)$. Applying (III.3) for x_0 we know that x_n is contained in one of the $E_\varepsilon(D_i)$, say $E_\varepsilon(D_1)$. Denote E_1 the connected component of $\cup E_\varepsilon(D_i) \setminus \{x_0\}$ that contains $E_\varepsilon(D_1)$. We have chosen ε small enough such that $\cup E_\varepsilon(D_i)$ has at least two connected components. Therefore we can chose D_j such that $E_\varepsilon(D_j)$ is disjoint from the component containing $E_\varepsilon(D_1) \setminus \{x_0\}$. Let α be the angular distance, relative to x_0, from $E_\varepsilon(D_j)$ and the component containing $E_\varepsilon(D_1) \setminus \{x_0\}$. Clearly, *al* is bounded from below by a positive number as one chooses ε smaller and smaller. Applying Lemma III.1 this time to x_n, we know that in $B_{4|x_n|}(x_n) \setminus B_{|x_n|}(x_n)$ the support of C is at the $\varepsilon|x_n|$ distance from a union of flat disks passing through x_n (the tangent cone at x_n). This implies that the angular distance between the tangent cone at x_n and D_1 is less than $C_Q\varepsilon$, where C_Q depends on Q only . Therefore

$$\text{(IV.2)}\quad \mathrm{supp}(C \llcorner B_{2|x_n|}(x_n) \setminus B_{|x_n|}(x_n)) \subset \tilde{E}_1 = \{x\ ;\ \mathrm{dist}(x, D_1) \leq C_Q\varepsilon|x_n|\} .$$

Observe that $dist\{E_\varepsilon(D_j) \cap \left(B_{|x_n|}(x_0) \setminus B_{|x_n|/2}(x_0)\right) ; \tilde{E}_1\} \geq \alpha/4$. This later fact combined with (IV.2) contradicts (III.2). Lemma IV.1 is then proved. $\square$

From now on until the beginning of Section X we will be dealing with the difficult case only: the case where the point x_0 of multiplicity Q has a tangent cone which is made of Q times the same disk. As we have been doing since Section II, we will work in a neighborhood of x_0 where a compatible simplectic form ω for J exists, and we shall use normal coordinates for $g(\cdot, cdot) = \omega(\cdot, J\cdot)$ about x_0, compatible with J_{x_0} at x_0, satisfying (II.1); we can also assume that the tangent cone at x_0 is

$$C_{0,x_0} \llcorner B_1(0) = Q[D_0] \tag{IV.3}$$

where D_0 is the flat, oriented disk whose tangent 2-vector is $\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$. From now on we also use the following notation for complex coordinates about x_0:

$$z = x_1 + ix_2 \quad \text{and} \quad w_i = x_{2k+1} + ix_{2k+2} \text{ for } k = 1 \ldots p-1 \ . \tag{IV.4}$$

We will also denote $w = (w_1, \ldots, w_{p-1})$. A second consequence to Lemma III.1 is the following result:

Lemma IV.2 (the relative Lipschitz estimate). *Let x_0 be a point of multiplicity Q (i.e. $x_0 \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$), assume the tangent cone $C_{0,x_0} \llcorner B_1(0)$ at x_0 is Q times a flat disk (i.e. of the form* (IV.3)*). Let $\varepsilon > 0$; then there exists r_{ε,x_0} such that for any $r \leq r_{\varepsilon,x_0}$*

$$\forall x \in B_r(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}), \qquad \mathcal{F}((C_{0,x} - C_{0,x_0}) \llcorner B_1(0)) \leq \varepsilon \ . \tag{IV.5}$$

Also, for all $x = (z, w) \in B_r(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1})$ and $x' = (z', w') \in B_r(x_0) \cap \mathcal{C}_*$ we have

$$|w - w'| \leq \varepsilon |z - z'| \ . \tag{IV.6}$$

Proof of Lemma IV.2. Let $\varepsilon > 0$ and $\delta > 0$ be given by Lemma III.1. Choose r_1 such that

$$\mathrm{M}(C \llcorner B_{r_1}(x_0)) \leq r^2 \ \left(\pi Q + \frac{\delta}{2}\right) \ .$$

This implies in particular that for any $r < r_1$,

$$\mathcal{F}((C_{r,x_0} - C_{0,x_0}) \llcorner B_1(0)) \leq \varepsilon \ . \tag{IV.7}$$

As in the proof of Lemma IV.1, see (IV.1), we have that for any $x \in B_{r_1 \frac{\delta}{4\pi Q}}(x_0)$ and $r < r_1(1 - \frac{\delta}{4\pi Q})$

$$\mathrm{M}(C \llcorner B_r(x)) \leq r^2(\pi Q + \delta) \ .$$

Then, letting $x \in B_{r_1 \frac{\delta}{4\pi Q}}(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1})$ and applying Lemma III.1, we have for $r < r_1(1 - \frac{\delta}{4\pi Q}) = r_2$,

$$\mathcal{F}((C_{r,x} - C_{0,x}) \llcorner B_1(0)) \leq \varepsilon \ . \tag{IV.8}$$

Choose now $x \in B_{\varepsilon r_1}(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1})$. $C_{r_2,x}$ is fast eine ε-translation from C_{r_1,x_0}; therefore, since $\mathrm{M}(C_{r,x_0}) \leq 2\pi Q$,

$$\mathcal{F}((C_{r_2,x_0} - C_{r_2,x}) \llcorner B_1(0)) \leq 2\pi Q \ . \tag{IV.9}$$

Taking $r_{\varepsilon,x_0} = \min\{\varepsilon r_1, \frac{\delta}{4\pi Q} r_1\}$ and combining (IV.7), (IV.8) and (IV.9), we deduce that

(IV.10)

$$\forall x \in B_{r_{\varepsilon,x_0}}(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}), \qquad \mathcal{F}((C_{0,x} - C_{0,x_0}) \llcorner B_1(0)) \leq (2 + 2\pi Q)\varepsilon \ .$$

It remains to check (IV.6) which is in fact an almost direct consequence of (III.3) and (IV.5). Lemma IV.2 is then proved. □

V. The covering argument

Let x_0 be a point of multiplicity $Q > 1$ whose tangent cone C_{0,x_0} is Q times the integral over the flat disk D_0 given by $w_i = 0$ for $i \dots Q-1$ (we use the system of coordinates introduced in the beginning of §IV) . The purpose of this section is to construct a Whitney-Besicovitch covering $B^2_{r_i}(z_i)$ of $\Pi(\mathcal{C}_{Q-1}) \cap B^2_\rho(x_0)$ where Π is the projection on D_0 which gives the first complex coordinate of each point ($\Pi(z, w_1 \dots w_{p-1}) = z$), for some radius ρ, small enough depending on x_0. This covering will be chosen in such a way that the following striking facts hold: first, $C \llcorner \Pi^{-1}(B^2_{r_i}(z_i))$ is in fact supported in a ball of radius $2r_i$, $B^{2p}_{2r_i}(x_i)$, moreover $C \llcorner B^{2p}_{2r_i}(x_i)$ is "split". This last word means that the flat distance between $C \llcorner \Pi^{-1}(B^2_{r_i}(z_i))$ and the Q multiple of any single-valued graph over D_0 is larger than $K\, r_i^3$ where K only depends on p, J and ω. This will come from the fact that r_i may be chosen in such a way that $r_i^{-2}\mathrm{M}(B^{2p}_{r_i}(x_i) \leq \pi - K'$ where again $K' > 0$ only depends on p, Q, J and ω. The existence of such a covering is a consequence of the "splitting before tilting" lemma proved in [Ri2].

Let α be given by Lemma V.1 and let $\varepsilon > 0$ be chosen small enough; compare to α later. Let r_{ε,x_0} be the radius given by Lemma IV.2. We may choose also r_{ε,x_0} small enough in such a way that

$$\forall r \leq r_{\varepsilon,x_0}, \qquad \mathrm{M}(C_{r,x_0} \llcorner B_1(0)) \leq \pi Q + \varepsilon^2 \ . \tag{V.1}$$

Using the proof of Lemma III.1 (from (III.12) until the end of the proof), we deduce that

$$\forall r \leq r_{\varepsilon,x_0}, \qquad \mathcal{F}((C_{r,x_0} - C_{0,x_0}) \llcorner B_1(0)) \leq K\varepsilon \ . \tag{V.2}$$

(In fact $\delta = O(\varepsilon^2)$ works in the statement of Lemma III.1.) On one hand, as in the proof of Lemma IV.1, we have that for any $x \in B_{\varepsilon^2 r_{\varepsilon,x_0}}(x_0)$ and $r \leq r_{\varepsilon,x_0}(1-\varepsilon^2)$,

$$\mathrm{M}(C \llcorner B_r(x)) \leq r^2\ (\pi Q + \varepsilon^2) \ . \tag{V.3}$$

On the other hand, arguing as in the proof of Lemma IV.2, between (IV.8) and (IV.10), we have, using also (V.23),

$$\forall x\in B_{\varepsilon^2}(x_0),\qquad \mathcal{F}(C_{\frac{r_{\varepsilon,x_0}}{2},x_0}\llcorner B_1(0)-\pi Q\ [D_0])\le K\varepsilon\ . \tag{V.4}$$

Having chosen $K\varepsilon<\alpha$ we are in a position to apply the "Splitting before tilting" lemma of [Ri2] which is a key step in our proof of the regularity of 1-1 rectifiable cycles.

LEMMA V.1 (splitting before tilting, [Ri2]). *There exists $\alpha>0$ such that for any $x_0\in\mathcal{C}_{Q-1}$ and for any radius $0<\rho<\alpha$,*

$$\mathcal{F}(C_{2\rho,x_0}\llcorner B_1(0)-Q\ [D_0])\le\alpha \tag{V.5}$$

where D_0 is a flat J_{x_0}-holomorphic disk passing through x_0. Then, for any $r<\rho$ and any J_{x_0}-holomorphic flat disk, D_1, passing through x_0 and satisfying

$$\mathcal{F}([D_0]-[D_1])\ge\frac{1}{4}\ , \tag{V.6}$$

we have

$$\mathcal{F}(C_{r,x_0}\llcorner B_1(0)-Q[D_1])\ge\alpha\ . \tag{V.7}$$

Moreover, there exist $r_0<\rho$ and K_0 a constant depending only on $\|\omega\|_{C^1}$ and $\varepsilon_Q^{\pm}$, the epiperimetric constants, such that

$$\mathrm{M}(C_{r_0,x_0}\llcorner B_1(0))=\pi Q-K_0\alpha\ , \tag{V.8}$$

$$\mathcal{F}(C_{r_0,x_0}\llcorner B_1(0)-Q[D_0])\le K\ \sqrt{\alpha} \tag{V.9}$$

for some constant K depending also only on $\|\omega\|_{C^1}$ and $\varepsilon_Q^{\pm}$. Finally, for any J_{x_0}-holomorphic disk D passing through 0,

$$\forall r\le r_0,\qquad \mathcal{F}(C_{r,x_0}\llcorner B_1(0)-Q\ [D])\ge\alpha\ . \tag{V.10}$$

For any $x\in\mathcal{C}_{p-1}\cap B_{\varepsilon^2 r_{\varepsilon,x_0}}(x_0)$ we denote by r_x the radius r_0 given by the lemma. We then have

$$\mathrm{M}(C_{r_x,x}\llcorner B_1(0))=\pi Q-K_0\alpha\ , \tag{V.11}$$

$$\mathcal{F}(C_{r_x,x}\llcorner B_1(0)-\pi Q[D_0])\le K\ \sqrt{\alpha} \tag{V.12}$$

for some constant K depending only on $\|\omega\|_{C^1}$ and $\varepsilon_Q^{\pm}$. Moreover, for α chosen small enough and ε small enough compared to α, the following lemma holds

LEMMA V.2. *Under the above notation, for any $x\in\mathcal{C}_{p-1}\cap B_{\varepsilon r_{\varepsilon,x_0}}(x_0)$,*

$$\mathrm{supp}\left(C\llcorner\Pi^{-1}(B^2_{r_x}(\Pi(x))\cap B_{r_{\varepsilon,x_0}}(x_0)\right)\subset B^2_{r_x}(\Pi(x))\times B^{2p-2}_{r_x}(0)\ , \tag{V.13}$$

and

$$\mathrm{supp}\left(C\llcorner\Pi^{-1}(B^2_{r_x}(\Pi(x))\cap B_{r_{\varepsilon,x_0}}(x_0)\right)\subset\mathcal{C}_{p-1}\ . \tag{V.14}$$

Proof of Lemma V.2. We claim that for any r between $\frac{r_{\varepsilon,x_0}}{2}$ and r_x one has

$$\text{supp}(C_{r,x} \llcorner B_1(0)) \subset E(\alpha^{\frac{1}{16}}) \tag{V.15}$$

where we use the notation

$$E(\lambda) = \{y = (z, w) \in B_1(0) : \quad |w| \leq \lambda\} \ . \tag{V.16}$$

We show (V.15) arguing by contradiction. First of all from the proof of Lemma V.1 applied to x we have the fact that for any $r \in [r_x, \frac{r_{\varepsilon,x_0}}{2}]$

$$\mathcal{F}(C_{r,x} \llcorner B_1(0) - \pi Q[D_0]) \leq K \ \sqrt{\alpha} \ . \tag{V.17}$$

Let $\omega_\alpha = \chi_\alpha \ \omega = \chi\left(\frac{|w|}{\alpha^{\frac{1}{4}}}\right)\omega$ where χ is a smooth cut-off function on $\mathbb{R}_+$ satisfying $\chi \equiv 1$ on $[0, 1/2]$ and $\chi \equiv 0$ in $[1, +\infty)$. Let S and R be a 3 and a 2-current satisfying $(C_{r,x} \llcorner B_1(0) - \pi Q[D_0]) = \partial S + R$ with $\text{M}(S) + \text{M}(R) \leq 2K\sqrt{\alpha}$. Now,

$$\begin{aligned} |(C_{r,x} \llcorner B_1(0) - \pi Q[D_0])(\omega_\alpha)| &= |S(\omega \wedge d\chi_\alpha) + R(\omega_\alpha)| \\ &\leq \|\nabla\chi_\alpha\|_\infty \ \|\omega_\infty\| \ \text{M}(S) + \|\omega_\alpha\|_\infty \ \text{M}(R) \leq K\alpha^{\frac{1}{4}} \ . \end{aligned}$$

Thus we get in particular

$$\begin{aligned} \pi Q - K\alpha^{\frac{1}{4}} \leq |C_{r,x} \llcorner B_1(0)(\omega_\alpha)| &\leq \text{M}(C_{r,x} \llcorner B_1(0) \cap E(\alpha^{\frac{1}{4}}))\|\omega_\alpha\|_\infty \\ &\leq \text{M}(C_{r,x} \llcorner B_1(0) \cap E(\alpha^{\frac{1}{4}})) \ . \end{aligned} \tag{V.18}$$

Assume now that there exists $y \in (\mathcal{C}_{r,x})_* \cap B_1(0) \cap (\mathbb{R}^{2p} \setminus E(\alpha^{\frac{1}{16}}))$. From the monotonicity formula we deduce that

$$\text{M}(C \llcorner B_{\frac{\alpha^{\frac{1}{16}}}{2} r}(y)) \geq \frac{\pi}{4}\alpha^{\frac{1}{8}} \ r^2 \ . \tag{V.19}$$

Combining (V.18) and (V.19), we obtain

$$\text{M}(C \llcorner B_r(x)) \geq r^2 \left(\pi Q - K\alpha^{\frac{1}{4}} + \frac{\pi}{4}\alpha^{\frac{1}{8}}\right) \ . \tag{V.20}$$

For α small enough (V.20) contradicts (V.11) and (V.15) holds true for any $r \in [r_x, \frac{r_{\varepsilon,x_0}}{2}]$. From this latter fact one deduces (V.13).

It remains to prove (V.14). Again we argue by contradiction. Assume there exists $y \in (\mathcal{C}_p \setminus \mathcal{C}_{p-1}) \cap \Pi^{-1}(B^2_{r_x}(\Pi(x)) \cap B_{r_{\varepsilon,x_0}}(x_0)$. Because of (V.3) and since $y \in B_{r_{x_0,\varepsilon}}(x_0)$ we can apply Lemma IV.2 to y in order to deduce that $\mathcal{C}_\star \cap B_{r_x}(x)$ is included in a cone of center y, axis parallel to D_0 and angle ε. This cone of course contains x and then we can deduce that

$$\text{supp}(C_{r_x,x} \llcorner B_1(0)) \subset E(4\varepsilon) \ . \tag{V.21}$$

(The notation $E(\lambda)$ is introduced in (V.16)). We have $\partial C_{r,x} \llcorner B_1(0) = 0$; moreover, because of (V.17); for α small enough we deduce that the intersection number of $C\llcorner B^2_{r_x}(\Pi(x)) \times B^{2p-2}_{r_x}(0)$ with any vertical current $\Pi^{-1}(z)$ for $z \in B^2_{r_x}$ is Q. Combining this fact with (V.21), using Fubini, one deduces that

$$\mathrm{M}(C_{r_x,x}\llcorner B_1(0)) \geq \pi Q - O(\varepsilon^2) \ . \tag{V.22}$$

For ε small enough compared to α we get a contradiction while comparing (V.22) and (V.11) and (V.14) is proved. This concludes the proof of Lemma V.2. □

In the following second lemma of this section, we show that the covering $(B^2_{r_x}(\Pi(x)))_{x\in\mathcal{C}_{p-1}\cap B^2_{\varepsilon^2 r_{\varepsilon,x_0}}(x_0)}$ of $\Pi(\mathcal{C}_{p-1}\cap B^{2p}_{\varepsilon^2 r_{\varepsilon,x_0}}(x_0)$ has the "Whitney" property: two balls intersecting each-other have comparable size. From now on we adopt the following notation granting the fact that α and ε are fixed small enough for the constraint mentionend above to be fulfilled:

$$\rho_{x_0} := \varepsilon^2 r_{\varepsilon,x_0} \ . \tag{V.23}$$

Precisely we have:

LEMMA V.3 (Whitney property of the covering). *There exists $\gamma > 0$ depending only on Q such that, given $x_0 \in \mathcal{C}_p \setminus \mathcal{C}_{p-1}$ whose tangent cone is $Q[D_0]$ and letting $(B^2_{r_x}(\Pi(x)))$ for $x \in \mathcal{C}_{p-1}\cap B^2_{\rho_{x_0}}(x_0)$ the covering of $\Pi(\mathcal{C}_{p-1}\cap B^{2p}_{\rho_{x_0}}(x_0)$ described above, assuming for some $x, y \in \mathcal{C}_{p-1}\cap B^2_{\rho_{x_0}}(x_0)$*

$$B^2_{r_x}(x)\cap B^2_{r_y}(y) \neq \emptyset \ ,$$

then

$$r_x \geq \alpha^\gamma r_y \ . \tag{V.24}$$

Proof of Lemma V.3. This lemma is again a consequence of the upper and lower-epiperimetric inequalities. Assume for instance that $r_x \leq r_y$. From (V.10) we have

$$\mathcal{F}(C_{r_y,y}\llcorner B_1(0) - Q\ [D_0]) \geq \alpha \ . \tag{V.25}$$

Which implies that for all $r \leq r_y$

$$\mathcal{F}(C\llcorner B^2_r(z_y)\times B^{2p}_{\rho_{x_0}}(0) - Q[B^2_r(z_y)\times\{0\}]) \geq \alpha r^3 \tag{V.26}$$

where $y = (z_y, w_y)$. Since $|z_x - z_y| \leq 2\max\{r_x, r_y\} = 2r_y$ and $B_{r_y}(z_y) \subset B_{3r_y}(z_x)$, (V.26) implies that

$$\mathcal{F}(C\llcorner B^2_{4r_y}(z_x)\times B^{2p}_{\rho_{x_0}}(0) - Q[B^2_{4r_y}(z_x)\times\{w_y\}]) \geq \frac{\alpha}{3} r_y^3 \ . \tag{V.27}$$

This passage from (V.26) to (V.27) is obtained by applying a Fubini type argument. Indeed, let $A = C\llcorner B^2_{4r_y}(z_x)\times B^{2p}_{\rho_{x_0}}(0) - Q[B^2_{4r_y}(z_x)\times\{w_y\}]$ and let

S and R be such that $A = \partial S + R$ and $\mathrm{M}(S) + \mathrm{M}(R)^{\frac{3}{2}} \le 2\mathcal{F}(A)$. For almost every r in $[r_y/2, r_y]$ we have

$$\partial(S \llcorner B^2_r(z_y) \times B^{2p}_{\rho_{x_0}}(0)) = \partial S \llcorner B^2_r(z_y) \times B^{2p}_{\rho_{x_0}}(0)) + \langle S, \mathrm{dist}(\cdot, \{z = z_y\}, r\rangle$$

where $\langle S, \mathrm{dist}(\cdot, \{z = z_y\}), r\rangle$ is the slice current between S and the boundary of the cylinder $B^2_r(z_y) \times B^{2p}_{\rho_{x_0}}(0)$ and $\mathrm{dist}(\cdot, \{z = z_y\})$ denotes the distance function to the axis of this cylinder (see [Fe, 4.2.1, pp. 395...]). Thus

(V.28)
$$\begin{aligned} A \llcorner B^2_r(z_y) \times B^{2p}_{\rho_{x_0}}(0) = & \, p(S \llcorner B^2_r(z_y) \times B^{2p}_{\rho_{x_0}}(0)) \\ & - \langle S, \mathrm{dist}(\cdot, \{z = z_y\}, r\rangle + R \llcorner B^2_r(z_y) \times B^{2p}_{\rho_{x_0}}(0) \ . \end{aligned}$$

We have, see [Fe, 4.2.1, p. 395],

$$\text{(V.29)} \qquad \int_{\frac{r_y}{2}}^{r_y} M\left(\langle S, \mathrm{dist}(\cdot, \{z = z_y\}, r\rangle\right) \le M\left(S \llcorner (B^2_{r_y} \setminus B^2_{\frac{r_y}{2}}) \times B^{2p}_{\rho_{x_0}}(0)\right) \ .$$

Using Fubini's theorem we may then find $r = r_1 \in [r_y/2, r_y]$ such that
(V.30)
$$M\left(\langle S, \mathrm{dist}(\cdot, \{z = z_y\}, r_1\rangle\right) \le \frac{2}{r_y} M\left(S \llcorner (B^2_{r_y} \setminus B^2_{\frac{r_y}{2}}) \times B^{2p}_{\rho_{x_0}}(0)\right) \le \frac{2}{r_y} \mathrm{M}(S) \ .$$

Combining (V.28), (V.29) and (V.30) we deduce that

$$\text{(V.31)} \quad \begin{aligned} & \mathcal{F}(C \llcorner B^2_{r_1}(z_y) \times B^{2p}_{\rho_{x_0}}(0) - Q[B^2_{r_1}(z_y) \times \{0\}]) \\ & \qquad = \mathcal{F}(A \llcorner B^2_r(z_y) \times B^{2p}_{\rho_{x_0}}(0)) \le \mathrm{M}(S) + (\mathrm{M}(R) + \frac{2}{r_y}\mathrm{M}(S))^{\frac{3}{2}} \ . \end{aligned}$$

Thus, combining (V.26) for $r = r_1$ and (V.31), we have

$$\mathrm{M}(S) + (\mathrm{M}(R) + \frac{2}{r_y}\mathrm{M}(S))^{\frac{3}{2}} \ge \alpha r_y^3$$

and since

$$\mathcal{F}(C \llcorner B^2_{4r_y}(z_x) \times B^{2p}_{\rho_{x_0}}(0) - Q[B^2_{4r_y}(z_x) \times \{w_y\}]) \ge \frac{1}{2}\left[\mathrm{M}(S) + \mathrm{M}(R)^{\frac{3}{2}}\right] \ ,$$

we obtain (V.27). Therefore we deduce that

$$\text{(V.32)} \qquad \mathcal{F}(C_{4r_y,x} \llcorner B_1(0) - Q\ [D_0]) \ge \frac{1}{3 \times 4^3}\alpha \ .$$

Let $\frac{\rho_{x_0}}{\varepsilon^2} > s_x > r_x$ be such that

$$\text{(V.33)} \qquad \mathrm{M}(C_{s_x,x} \llcorner B_1(0)) = \pi Q \ .$$

Because of (V.3), arguing as in the proof of Lemma IV.2, between (IV.8) and (IV.10), we have

$$\text{(V.34)} \qquad \forall s_x \le r \le \frac{\rho_{x_0}}{\varepsilon^2}, \qquad \mathcal{F}(C_{r,x} \llcorner B_1(0) - Q\ [D_0]) \le K\varepsilon \ .$$

Assuming, as we did above that $\alpha >> \varepsilon$, comparing (V.32) and (V.26) we deduce that $s_x > 4r_y$. Let $\lambda = 4\frac{r_x}{r_y}$. From the proof of Lemma V.1, in fact from (V.9) precisely, for any $r \in [r_x, s_x]$ we have

$$\mathcal{F}(C_{2r,x} \llcorner B_1(0) - \pi Q[D_0]) \le K\ \sqrt{\alpha} \le \varepsilon_Q^- \ , \tag{V.35}$$

which means in particular that we are in the position to apply the lower-epiperimetric inequality and the differential inequality (III.27) in [Ri2] deduced from it. Integrating then this inequality between r_x and $4r_y$ we have

$$\begin{aligned} \pi Q - \mathrm{M}(C_{4r_y,x} \llcorner B_1(0)) &\le \left(\frac{1}{2^{2\varepsilon_Q^-}}\right)^{\log_2 \lambda} \ [\pi Q - \mathrm{M}(C_{r_x,x} \llcorner B_1(0))] \\ &= \left(\frac{1}{2^{2\varepsilon_Q^-}}\right)^{\log_2 \lambda} \alpha \ . \end{aligned} \tag{V.36}$$

Using now (III.28) from [Ri2], we deduce from (V.36) that

$$\begin{aligned} \mathcal{F}((C_{s_x,x} - C_{4r_y,x}) \llcorner B_1(0)) &\le \sqrt{\pi Q - \mathrm{M}(C_{4r_y,x} \llcorner B_1(0))} \\ &= \left(\frac{1}{2^{\varepsilon_Q^-}}\right)^{\log_2 \lambda} \alpha^{\frac{1}{2}} \ . \end{aligned} \tag{V.37}$$

Combining (V.34) and (V.37) one gets that

$$\mathcal{F}(C_{4r_y,x} \llcorner B_1(0) - Q[D_0]) \le \left(\frac{1}{2^{\varepsilon_Q^-}}\right)^{\log_2 \lambda} \alpha^{\frac{1}{2}} + K\varepsilon \ . \tag{V.38}$$

Comparing (V.32) and (V.38) we obtain

$$\frac{1}{3 \times 4^3}\alpha \le \left(\frac{1}{2^{\varepsilon_Q^-}}\right)^{\log_2 \lambda} \alpha^{\frac{1}{2}} \ . \tag{V.39}$$

Since $K\varepsilon \le \frac{1}{6\times 4^3}\alpha$ we have $\frac{1}{6\times 4^3}\alpha \le \left(\frac{1}{2^{\varepsilon_Q^-}}\right)^{\log_2 \lambda} \alpha^{\frac{1}{2}}$. Taking the log of this last inequality we obtain

$$\frac{\log \lambda}{\log 2}\varepsilon_Q^- \log \frac{1}{2} + \frac{1}{2} \log \alpha \ge \log \alpha - \log(6 \times 4^3) \ .$$

Thus

$$\frac{1}{2} \log \frac{1}{\alpha} + \log(6 \times 4^3) \ge \varepsilon_Q^- \log \lambda \ .$$

By taking α small enough, we may always assume that $\log \frac{1}{\alpha} \ge 4 \times \log(6 \times 4^3)$ and we finally get that

$$\frac{1}{\varepsilon_Q^-} \log \frac{1}{\alpha} \ge \log \lambda \ ,$$

which leads to the desired inequality (V.24) and, now, Lemma V.3 is proved.□

Constructing a partition of unity adapted to the covering. From the covering $(B^2_{r_x}(x))$ for $x \in \mathcal{C}_{p-1} \cap B_{\rho_{x_0}}(x_0)$ of $\Pi(\mathcal{C}_{p-1} \cap B_{\rho_{x_0}}(x_0))$ we extract a Besicovitch covering $(B^2_{r_{x_i}}(x_i))$ for $i \in I$ (I is a countable set) of $\Pi(\mathcal{C}_{p-1} \cap B_{\rho_{x_0}}(x_0))$ that is a covering such that

$$\text{(V.40)} \qquad \forall z \in B^2_{\rho_{x_0}}(x_0), \qquad \text{Card}\left\{i \in I : z \in B^2_{r_{x_i}}(x_i)\right\} \leq n \ ,$$

where N is some universal number (see [Fe]). To simplify the notation we will simply write r_i for r_{x_i}. Note that since balls intersecting each other have comparable size (see Lemma V.3), each ball $B^2_{r_i}(z_i)$ intersects a uniformly bounded number of other balls: there exists $N_{Q,\alpha}$ such that

$$\text{(V.41)} \qquad \forall i \in I, \qquad \text{Card}\left\{j \in I : B^2_{r_j}(z_j) \cap B^2_{r_i}(z_i) \neq \emptyset\right\} \leq N_{Q,\alpha} \ .$$

We now construct a partition of unity adapted to a slightly modified covering. Considering the covering $(B^2_{r_{z_i}}(z_i))$ for $i \in I$ (I is a countable set) of $\Pi(\mathcal{C}_{p-1} \cap B^{2p}_{\rho_{x_0}}(x_0))$, we can apply Lemma A.1 and obtain δ depending on α and Q such that (A.3) holds true for some $P \in \mathbb{N}$. Letting $i \in I$, we can deduce from (A.3) and (V.24) that the radii r_j of balls $B^2_{r_j}(z_j)$ intersecting $B^2_{r_i(1+\delta)}(z_i)$ satisfy $\alpha^{\gamma P} r_i \leq r_j \leq \alpha^{-\gamma P} r_i$. From this latter fact we deduce that there exists a number $M \in \mathbb{N}$ depending only on α and Q such that

$$\text{(V.42)} \qquad \text{Card}\left\{j \in I : B_{r_i}(z_i) \cap B_{(1+\delta)r_j}(z_j) \neq \emptyset\right\} \leq M \ .$$

Indeed, assuming $B_{r_i}(z_i) \cap B_{r_j}(z_j) = \emptyset$, if $B_{r_i}(z_i) \cap B_{(1+\delta)r_j}(z_j)$ we have just seen that $\alpha^{\gamma P} r_j \leq r_i \leq \alpha^{-\gamma P} r_j$: the two radii have comparable size which is of course also comparable with the distance $|z_i - z_j|$. From (V.40) it is then clear that the number of such ball $B_{r_j}(z_j)$ is bounded by a constant depending only on the variables α and Q. It is now not difficult to deduce that $(B^2_{r_i(1+\frac{\delta}{2})})_{i\in I}$ realizes a locally finite covering of $\Pi(\mathcal{C}_{p-1} \cap B^{2p}_{\rho_{x_0}}(x_0))$ satisfying

$$\text{(V.43)} \quad \forall i \in I, \qquad \text{Card}\left\{j \in I : B_{(1+\frac{\delta}{2})r_i}(z_i) \cap B_{(1+\frac{\delta}{2})r_j}(z_j) \neq \emptyset\right\} \leq M + P \ .$$

Indeed, when we assume for instance that $r_i \geq r_j$, then

$$B_{(1+\frac{\delta}{2})r_i}(z_i) \cap B_{(1+\frac{\delta}{2})r_j}(z_j) \neq \emptyset$$

implies clearly that $B_{(1+\delta)r_i}(z_i) \cap B_{r_j}(z_j) \neq \emptyset$ and the number of such a j is controlled by P (see A.3), whereas if $r_i \leq r_j$, $B_{(1+\frac{\delta}{2})r_i}(z_i) \cap B_{(1+\frac{\delta}{2})r_j}(z_j) \neq \emptyset$ implies clearly that $B_{r_i}(z_i) \cap B_{r_j(1+\delta)}(z_j) \neq \emptyset$ and the number of such a j is controlled by M (see V.42). Thus (V.43) holds true. And we shall use from now on the notation

$$\text{(V.44)} \qquad \forall i \in I, \qquad \rho_i := r_i \left(1 + \frac{\delta}{2}\right) \ .$$

For any $i \in I$ we define χ_i to be a smooth nonnegative function satisfying

$$\text{(V.45)} \qquad \text{i)} \qquad \chi_i \equiv 1 \quad \text{in } B^2_{r_i}(z_i)\ .$$

$$\text{(V.46)} \qquad \text{ii)} \qquad \chi_i \equiv 0 \quad \text{in } \mathbb{R}^2 \setminus B^2_{(1+\frac{\delta}{2}r_i)}(z_i)$$

$$\text{(V.47)} \qquad \text{iii)} \qquad \forall k \in \mathbb{N}, \quad \|\nabla^k \chi_i\|_\infty \le \frac{K_k}{r_i^k}\ ,$$

where K_k depends only on k and Q.

We define now

$$\text{(V.48)} \qquad \varphi_i := \frac{\chi_i}{\sum_{i\in I}\chi_i}\ .$$

It is clear that (φ_i) defines a partition of unity adapted to $B^2_{(1+\frac{\delta}{2}r_i)}(z_i)$ and satisfying the following estimates

$$\text{(V.49)} \qquad \forall k \in \mathbb{N}, \qquad \|\nabla^k \varphi_i\|_\infty \le \frac{K_k}{r_i^k}\ ,$$

where K_k depends only on k and Q.

VI. The approximated average curve

This section is another step towards the proof that $\mathcal{P}_{Q-1} \Longrightarrow \mathcal{P}_Q$ which continues until Section VIII. We thus assume that $\mathcal{P}_{Q-1}$ holds (or that $Q = 1$). Again in this part we consider the difficult case which is the case where we are blowing-up the current at a point x_0 of multiplicity $Q > 1$ whose tangent cone C_{0,x_0} is Q times the integral over the flat disk D_0 given by $w_i = 0$ for $i \dots Q-1$ (we use the system of coordinates introduced at the beginning of §II) and where x_0 belongs to the closure of $\mathcal{C}_{Q-1}$. The purpose of this section is to approximate first our current over each ball of the covering introduced in the previous section $C \llcorner \Pi^{-1}(B^2_{r_i}(z_i))$ by a Q-valued graph $\{a_i^k\}_{k=1\dots Q}$ over $B^2_{r_i}(z_i)$ which is almost J-holomorphic (J_{x_i}-holomorphic in fact where $x_i \in \mathcal{C}_*$ and $\Pi(x_i) = z_i$). Gluing the average curves $\tilde{a}_i = \frac{1}{Q}\sum_{k=1}^{Q} a_i^k$ of each of these J_{x_i}-holomorphic Q-valued graphs together we shall produce a single-valued graph $\tilde{a}$ over $B^2_{\rho_{x_0}}(x_0)$ which approximates C and for which we will study regularity properties that will be used in the following Section VII devoted to the unique continuation argument . Finally in the second subsection of this section we construct new coordinates adapted to the average curve.

VI.1. *Constructing the average curve.* Let ρ_{x_0} be given by (V.23) and let $(B^2_{\rho_i}(z_i))_{i\in I}$ be the Besicovitch-Whitney covering of $\Pi(\mathcal{C}_{Q-1} \cap B_{\rho_{x_0}}(x_0))$ obtained at the end of the previous section. As we have seen above, for any

$i \in I$, $\mathcal{C}_* \cap \Pi^{-1}(B^2_{\rho_i}(z_i)) \subset \mathcal{C}_{Q-1} \cap B^2_{\rho_i}(z_i) \times B^{2p-2}_{2\rho_i}(w_i)$ where $x_i = (z_i, w_i)$ is in $\mathcal{C}_*$ (see Lemma V.2). For convenience we shall adopt the following notation:

$$N^i_r := B^2_r(z_i) \times B^{2p-2}_{2\rho_i}(w_i) \ . \tag{VI.1}$$

Assuming $\mathcal{P}_{Q-1}$, $C \llcorner N_{2\rho_i}$ is a J-holomorphic curve: there exist a smooth Riemannian surface and a smooth J-holomorphic map

$$\begin{aligned} \Psi_i \ : \ \Sigma_{2,i} &\longrightarrow N^i_{2\rho_i} \\ \xi &\longrightarrow \Psi_i(\xi) \end{aligned} \tag{VI.2}$$

such that $\Psi_*[\Sigma_{2,i}] = C \llcorner N_{2\rho_i}$. Let $H^0_{\pm}(\Sigma_{2,i})$ be the sets respectively of holomorphic and antiholomorphic functions on $\Sigma_{2,i}$. We introduce now η_i the map from $\Sigma_{2,i}$ into $\mathbb{R}^{2p-2}$ chosen such that the perturbation $\Psi_i + \eta_i$ is J_{x_i}-holomorphic; precisely, η_i is given by

$$\left\{ \begin{array}{ll} \dfrac{\partial}{\partial \xi_1}(\Psi_i + \eta_i) + J_{x_i} \dfrac{\partial}{\partial \xi_2}(\Psi_i + \eta_i) = 0 & \text{in } \Sigma_{2,i} \\ \forall h \in H(\Sigma_{2,i}) \qquad \displaystyle\int_{\partial \Sigma_{2,i}} h \ d\eta_i = 0 & \end{array} \right. \tag{VI.3}$$

where (ξ_1, ξ_2) are local coordinates on $\Sigma_{2,i}$ compatible with the complex structure. The existence of η_i is justified in a few lines below. To this aim we shall make use of the following notation. Since J is smooth the inverse function theorem gives the existence of a smooth map $\Lambda \ : \ B^{2p}_{\rho_{x_0}}(x_0) \times \mathbb{R}^{2p} \longrightarrow \mathbb{R}^{2p}$ - for ρ_{x_0} chosen small enough such that

$$\text{i)} \qquad \Lambda_x := \Lambda(x, .) \qquad \text{is a linear isomorphism of } \mathbb{R}^{2p} \ , \tag{VI.4}$$

$$\text{ii)} \qquad \Lambda_{x_0} = \text{id} \ , \tag{VI.5}$$

$$\text{iii)} \qquad \forall x \in B^{2p}_{\rho_{x_0}}(x_0) \qquad J_{x_0} = \begin{pmatrix} 0 & -1 & 0 \ldots & 0 & 0 \\ 1 & 0 & 0 \ldots & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & \ldots . & 0 & -1 \\ 0 & 0 & \ldots . & 1 & 0 \end{pmatrix} = \Lambda_x \ J_x \ \Lambda_x^{-1} \ . \tag{VI.6}$$

We shall denote by $(z^i = x_1^i + ix_2^i, w_1^i = x_3^i + ix_4^i, \dots, w_p^i = x_{2p-1}^i + ix_{2p}^i)$ the following complex coordinates in $N_{2\rho_i}$

$$\text{(VI.7)} \qquad \begin{pmatrix} x_1^i \\ x_2^i \\ . \\ . \\ . \\ x_{2p-1}^i \\ x_{2p}^i \end{pmatrix} = \Lambda_{x_i} \cdot \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_{2p-1} \\ x_{2p} \end{pmatrix} - x_i \ .$$

We will also denote by Π_i the map that assign to any point x in $N^i_{\rho_i}$ the complex coordinate z^i and by D^i we denote the J_{x_i}-holomorphic 2-disk

$$\text{(VI.8)} \qquad D^i := \left\{x \ ; \ \forall k = 1 \dots p-1 \quad w_k^i = 0\right\} = \Lambda_{x_i}^{-1} D_0 \ .$$

Using these complex coordinates in $N^i_{2\rho_i}$, we see that (VI.3) means

$$\text{(VI.9)} \qquad \begin{cases} \overline{\partial}\eta_i = -\overline{\partial}\Psi_i \quad \text{ in } \Sigma_{2,i} \\ \\ \forall h \in H(\Sigma_{2,i}), \qquad \displaystyle\int_{\partial\Sigma_{2,i}} h \ d\eta_i = 0 \ . \end{cases}$$

The existence and uniqueness of η_i are given by Proposition A.3 of [Ri3]. Since Ψ_i is J-holomorphic we have $\partial_{\xi_1}\Psi_i + J(\Psi_i(\xi))\partial_{\xi_2}\Psi_i = 0$; thus

$$|\partial_{\xi_1}\Psi_i + J(x_i)\partial_{\xi_2}\Psi_i| \le |J(\Psi_i(\xi)) - J(x_i)| \ |\nabla\psi| \ .$$

Combining this fact with the second part of Proposition A.3 (i.e. estimate (A.13) of [Ri3]) we obtain

$$\text{(VI.10)} \qquad \int_{\Sigma_{2,i}} |\nabla\eta_i|^2 \le K r_i^2 \int_{\Sigma_{2,i}} |\nabla\Psi_i|^2 \le K r_i^4 \ .$$

For $\lambda \le 2$, we denote by $\Sigma_{\lambda,i}$ the surface $\Sigma_{\lambda,i} = \Sigma_{2,i} \cap \Psi_i^{-1}(N^i_{\lambda\rho_i})$ such that

$$\text{(VI.11)} \qquad \Psi_{i*}[\Sigma_i] = C \llcorner B^2_{\rho_i}(z_i) \times B^{n-2}_{\rho_{x_0}}(0) \ .$$

We then prove in [Ri3] the following lemma:

Lemma VI.1. *Under the notation above*,

$$\text{(VI.12)} \qquad \|\eta_i\|_{L^\infty(\Sigma_{\frac{3}{2},i})} \le K r_i^2$$

where K is a constant depending only on $\|\nabla J\|_\infty$ and the choice of α made in the previous section.

Consider now the J_{x_i}-holomorphic curve C^i given by the image by $\Psi_i+\eta_i$ of $\Sigma_{\frac{3}{2},i}$. Since $\partial\Psi_{i*}[\Sigma_{\frac{3}{2},i}]$ is supported in $\Pi^{-1}(\partial B^2_{\frac{3}{2}\rho_i})$, we know from Lemma VI.1 that $|\eta_i|_\infty \leq C r_i^2$. Therefore $\partial\Psi_i + \eta_{i*}[\Sigma_{\frac{3}{2},i}]$ is supported in an r_i^2 neighborhood of $\Pi^{-1}(\partial B^2_{\frac{3}{2}\rho_i}(0))$ (for r_i small enough : that holds if ε has been chosen small enough in Section VI). Thus we have that $\Psi_{i*}[\Sigma_{\frac{3}{2},i}])$ is a cycle in $\Pi_i^{-1}(B^2_{\frac{5}{4}\rho_i}(0))$ and the part of the image of $\Sigma_{\frac{3}{2},i}$ included in $\Pi_i^{-1}(B^2_{\frac{5}{4}\rho_i}(0))$ by $\Psi_i+\eta_i$ is a J_{x_i}-holomorphic cycle and therefore it is a Q-valued graph over D^i for the complex coordinates given by (z^i,w^i). We denote by $\{a^i_k\}_{k=1\ldots Q}$ this Q-valued graph (i.e. $a^i_k(z^i_0)$ are the w^i coordinates, in the chart (z^i,w^i), of the Q intersection points between the J_{x_i}-holomorphic curve $\Psi_i+\eta_i(\Sigma_{\frac{3}{2},i})$ and the J_{x_i}-holomorphic submanifold given by $z^i=z^i_0$). We now define $\tilde{C}^i$ to be the J_{x_i} holomorphic curve in $\Pi_i^{-1}(B^2_{\rho_i})$ given by

(VI.13)

$$\tilde{C}^i := \left\{ x = \Lambda^{-1}_{x_i}\left((z^i, \tilde{a}^i_i(z_i) = \frac{1}{Q}\sum_{k=1}^{Q} a^i_k(z^i)) + x_i \right) \quad \forall z^i \in B^2_{\frac{5}{4}\rho_i}(0) \right\} .$$

Observe that

(VI.14) $$\frac{\partial}{\partial z^i}\tilde{a}^i_i = 0 \qquad \text{in } \mathcal{D}'(B^2_{\frac{5}{4}\rho_i}(0)) .$$

Moreover, the conformal invariance of the Dirichlet energy gives

(VI.15) $$\int_{B^2_{\frac{5}{4}\rho_i}(0)} \sum_{k=1}^{Q} |\nabla a^i_k|^2(z^i)\ dz^i\wedge d\overline{z}^i = \int_{(\Psi_i+\eta_i)^{-1}(C_i\cap\Pi_i^{-1}(B^2_{\frac{5}{4}\rho_i}(0)))} |\nabla(\Psi_i+\eta_i)|^2(\xi)\ d\xi \leq K r_i^2 .$$

We then deduce that

(VI.16) $$\int_{B^2_{\frac{5}{4}\rho_i}(0)} |\nabla \tilde{a}^i_i|^2(z^i)\ dz^i \leq K r_i^2 .$$

Combining (VI.15) and (VI.16) and using standard elliptic estimates we get that for any $l\in\mathbb{N}$

(VI.17) $$\|\nabla^l \tilde{a}^i_i\|_{L^\infty(B^2_{\frac{6}{5}\rho_i}(0))} \leq K_l r_i^{-l+1} .$$

The subscript i in the notation $\tilde{a}^i_i$ is here to recall that we express $\tilde{C}^i$ as a graph in the (z^i,w^i) coordinates. The same J_{x_i}-holomorphic curve $\tilde{C}^i$ can also, due to (VI.17), be expressed as a graph in a neighboring system of coordinate (z^j,w^j)

where $B_{\rho_i}(z_i) \cap B_{\rho_j}(z_j) \neq \emptyset$ (indeed the passage from (z^i, w^i) to (z^j, w^j) is given by a transformation matrix in $\mathbb{R}^{2p}$ close to the identity at a distance of the order r_i). In such system of coordinates (z^j, w^j), we shall denote $\tilde{a}^i_j(z^j)$ the graph corresponding to $\tilde{C}^i$.

Since $\tilde{C}^i$ is a graph over $w^i = 0$ given by $(z^i, \tilde{a}^i_i(z^i))$ whose gradient is bounded (see (VI.17)), and since the passage from the (z, w) coordinates to (z^i, w^i) coordinates is given by a transformation Λ_{x_i} whose distance to the identity is bounded by $|x_i|$ that tends to zero, $\tilde{C}^i$ is then also realized by a graph over $B^2_{\frac{7}{6}\rho_i}(\Pi(x_i))$ that we shall now denote $(z, \tilde{a}_i(z))$:

$$\tilde{C}^i \llcorner \Pi^{-1}(B^2_{\frac{7}{6}\rho_i}(\Pi(x_i))) = (z, \tilde{a}_i(z))_*[B^2_{\frac{7}{6}\rho_i}(\Pi(x_i))] \ . \tag{VI.18}$$

Consider now i and j such that $B_{\rho_i}(z_i) \cap B_{\rho_j}(z_j) \neq \emptyset$. We shall compare $\tilde{a}_i$ and $\tilde{a}_j$ in $\Pi^{-1}(B_{\rho_i}(z_i) \cap B_{\rho_j}(z_j))$. Precisely, we have the following lemma:

Lemma VI.2. *Under the notation above*,

$$\forall l \in \mathbb{N}, \qquad \|\nabla^l(\tilde{a}_i - \tilde{a}_j)\|_{L^\infty(B^2_{\rho_i}(z_i)\cap B^2_{\rho_j}(z_j))} \leq K_l \rho_i^{2-l} \ . \tag{VI.19}$$

Proof of Lemma VI.2. First of all we compare C_i and C_j in $\Pi^{-1}(B_{\rho_i}(z_i) \cap B_{\rho_j}(z_j))$. We can always assume that $\Sigma_{2,i}$ and $\Sigma_{2,j}$ are part of a same Riemannian surface Σ with a joint parametrization $\Psi = \Psi_i$ on $\Sigma_{2,i}$ and $\Psi = \Psi_j$ on $\Sigma_{2,j}$ and such that $\Psi_*[\Sigma] = C \llcorner N^i_{2\rho_i} \cup N^j_{2\rho_j}$. Let $\Sigma^{ij} := \Psi^{-1}(\mathrm{supp}(C)) \cap N^i_{2\rho_i} \cap N^j_{2\rho_j}$. We consider the following mapping

$$\begin{aligned} \Xi^{ij} \Sigma^{ij} \times [0,1] &\longrightarrow N^i_{3\rho_i} \cap N^j_{3\rho_j} \\ (\xi, t) &\longrightarrow \Psi(\xi) + t\eta_j(\xi) + (1-t)\eta_i(\xi) \ . \end{aligned} \tag{VI.20}$$

Clearly for any $\lambda < \frac{3}{2}$

$$\partial \Xi^{ij}{}_*[\Sigma^{ij}] \times [0,1] \llcorner N^i_{\lambda\rho_i} \cap N^j_{\lambda\rho_j} = C^j - C^i \llcorner N^i \lambda\rho_i \cap N^j_{\lambda\rho_j} \ . \tag{VI.21}$$

We have

$$\mathrm{M}(\Xi^{ij}{}_*[\Sigma^{ij}] \times [0,1]) = \int_0^1 \int_{\Sigma^{ij}} J_3 \Xi^{ij} \ , \tag{VI.22}$$

where $(J_3\Xi^{ij})^2$ is the sum of the squares of the determinants of the 3×3 submatrices of $D\Xi^{ij}$. Clearly

$$|J_3\Xi^{ij}|(\xi, t) \leq K \ \left[\|\eta_i\|_\infty + \|\eta_j\|_\infty\right] \ \left[|\nabla\Psi|^2(\xi) + |\nabla\eta_i|^2(\xi) + |\nabla\eta_j|^2(\xi)\right] \ . \tag{VI.23}$$

Combining Lemma V.3, Lemma VI.1, (VI.22) and (VI.23), we get that for any $\lambda < \frac{3}{2}$

$$\mathrm{M}(\Xi^{ij}{}_*[\Sigma^{ij}] \times [0,1] \llcorner N^i \lambda\rho_i \cap N^j_{\lambda\rho_j}) \leq K \ r_i^4 \ . \tag{VI.24}$$

Therefore, combining (VI.21) and (VI.24), using a standard slicing and Fubini type argument, we may find $\lambda \in (\frac{5}{4}, \frac{3}{2})$ such that

$$\mathcal{F}((C^i - C^j) \llcorner N^i_{\lambda\rho_i} \cap N^j_{\lambda\rho_j}) \le K r_i^4 \ . \tag{VI.25}$$

We shall now compare $\tilde{C}^i$ and $\tilde{C}^j$. Denote $(x_1^t, x_2^t \dots x_{2p}^t)$ the coordinates given by $(x_1^t, x_2^t \dots x_{2p}^t)^T = \Lambda_{x_t} \cdot [(x_1, x_2 \dots x_{2p})^T - x_i]$ where we keep denoting $(x_1, x_2 \dots x_{2p})$ our original normal coordinates vanishing at the center x_0 introduced in (II.1) and Λ_{x_t} is the transformation matrix introduced in (VI.4). Observe that with this notation $(x_1^0, x_2^0 \dots x_{2p}^0) = (x_1^i, x_2^i \dots x_{2p}^i)$, that $(x_1^1, x_2^1 \dots x_{2p}^1) = (x_1^j, x_2^j \dots x_{2p}^j) + \Lambda_{x^j} \cdot (x_i - x_j)$ and that $(x_1^t, x_2^t \dots x_{2p}^t)$ has been chosen in order to vanish at a fixed point x_i. Observe also that

$$\left|\frac{d}{dt}\Lambda_{x^t}\right| \le K r_i \ , \quad \left\|\frac{d}{dt}x^t\right\|_{L^\infty(B^{2p}_{4r_i}(x_i))} + \left\|\frac{d}{dt}y^t\right\|_{L^\infty(B^{2p}_{4r_i}(x_i))} \le K r_i^2 \ . \tag{VI.26}$$

We also adopt the notation $z^t := x_1^t + i x_2^t$ and for $k = 1 \dots p-1$ $w^t := x_{2k+1}^t + i x_{2k+2}^t$. Observe then that $z^t =$ constant or $w_k^t =$ constant are J_{x^t}-holomorphic $2p-2$ submanifolds, or simply complex varieties in $(\mathbb{R}^{2p}, J_{x^t})$. In order to compare $\tilde{C}^i$ and $\tilde{C}^j$ we shall perturb Ξ^{ij} in the following way: denote first Ψ^t, η_i^t and η_j^t the maps Ψ,η_i and η_j expressed in the coordinates (z^t, w^t), and consider the map $s^t \ : \ (\Sigma')^{ij} \to \mathbb{C}^p$ solving

$$\begin{aligned} &\partial_{\overline{\xi}} s^t = \partial_{\overline{\xi}}(\Psi^t + t\eta_j^t + (1-t)\eta_i^t) \quad \text{ in } (\Sigma')^{ij} \\ &\forall h \in H(\Sigma^{ij}), \qquad \int_{\partial\Sigma^{ij}} s^t \, dh = 0 \ , \end{aligned} \tag{VI.27}$$

where $\Sigma^{ij} := \Psi^{-1}(\text{supp}(C)) \cap N^i_{\frac{3}{2}\rho_i} \cap N^j_{\frac{3}{2}\rho_j}$. The existence and uniqueness of s^t are given by Proposition A.3 of [Ri3]. We shall now replace the map Ξ^{ij} on $(\Sigma')^{ij}$ by the map

(VI.28)

$$\begin{aligned} (\Xi')^{ij} \ : \ (\Sigma')^{ij} \times [0,1] &\longrightarrow N^i_{3\rho_i} \cap N^j_{3\rho_j} \\ (\xi, t) &\longrightarrow \Lambda^{-1}_{x^t} \cdot [\Psi^t(\xi) + t\eta_j^t(\xi) + (1-t)\eta_i^t(\xi) - s^t] + x_i \ . \end{aligned}$$

Observe that for each $t \in [0,1]$ the map $\Xi'^{ij}(\cdot, t)$ is a J_{x^t}-holomorphic curve. Observe also that, for $t = 0$, $s^t = 0$ and for $t = 1$, $\partial_{\overline{\xi}}(\Psi^1 + \eta_j^1) = 0$, since $\Psi + \eta_j$ is J_{x^j}-holomorphic and (z^1, w^1) are J_{x^j} coordinates, thus we have also $s^1 = 0$. One can easily verify, as in proving (VI.10) that for all $t \in [0,1]$

$$\int_{(\Sigma')^{ij}} |\partial_{\overline{\xi}}(\Psi^t + t\eta_j^t + (1-t)\eta_i^t)|^2 \le K r_i^4 \ .$$

Now, using Lemma VI.1 we have $\|s^t\|_\infty((\Sigma'')^{ij}) \le K r_i^2$ where

$$(\Sigma'')^{ij} := \Psi^{-1}(\text{supp}(C)) \cap N^i_{\frac{5}{4}\rho_i} \cap N^j_{\frac{5}{4}\rho_j} \ .$$

Therefore for any $\lambda < \frac{6}{5}$ we have

$$\partial (\Xi')^{ij}{}_*[(\Sigma')^{ij}]\times[0,1]\llcorner N^i\lambda\rho_i\cap N^j_{\lambda\rho_j} = C^j - C^i\llcorner N^i\lambda\rho_i\cap N^j_{\lambda\rho_j}\ . \tag{VI.29}$$

We consider now the following interpolation between $\tilde{C}^i$ and $\tilde{C}^j$: let $\tilde{\Xi}^{ij}$ be the following map

$$\begin{aligned}\tilde{\Xi}^{ij}\Pi(\Psi((\Sigma'')^{ij}))\times[0,1] &\longrightarrow N^i_{3\rho_i}\cap N^j_{3\rho_j}\\ (z,t) &\longrightarrow \Lambda^{-1}_{x^t}\cdot[(z,\tilde{a}^t(z)]+x_i\ ,\end{aligned} \tag{VI.30}$$

where the $p-1$ complex components are given by the slices of

$$C^t := (\Xi')^{ij}{}_*[(\Sigma')^{ij}]\times\{t\}$$

by $z^t=\xi$ evaluated on the functions w^t_k. Precisely, using the notation of [Fe, 4.3], we have

$$\tilde{a}^t_k(z) := \left\langle C^t, z^t, z\right\rangle(w^t_k)\ . \tag{VI.31}$$

It is clear that for any $\lambda < \frac{6}{5}$

$$\tilde{\Xi}^{ij}_*[\Pi_i(\Psi((\Sigma')^{ij}))]\times[0,1]\llcorner N^i\lambda\rho_i\cap N^j_{\lambda\rho_j} = \tilde{C}^j - \tilde{C}^i\llcorner N^i\lambda\rho_i\cap N^j_{\lambda\rho_j}\ . \tag{VI.32}$$

In order to get a bound for $\mathcal{F}(\tilde{C}^j - \tilde{C}^i\llcorner N^i\lambda\rho_i\cap N^j_{\lambda\rho_j})$, it remains to evaluate the mass of $\tilde{\Xi}^{ij}_*[\Pi_i(\Psi((\Sigma')^{ij}))]\times[0,1]\llcorner N^i\lambda\rho_i\cap N^j_{\lambda\rho_j}$ for $\lambda=\frac{6}{5}$ for instance. We have

$$|J_3\tilde{\Xi}^{ij}|(z,t)\le\left|\frac{\partial}{\partial t}\Lambda^{-1}_{x^t}\cdot[(z,\tilde{a}^t(z)]\right|(z,t)\ \left[1+|\nabla_z\tilde{a}^t(z)|^2\right]\ . \tag{VI.33}$$

Because of the same arguments developed to prove (VI.17), since $\tilde{a}^t(\xi)$ is holomorphic, we have

$$\|\nabla_z\tilde{a}^t(z)\|_{L^\infty((\Sigma'')^{ij})}\le K\ . \tag{VI.34}$$

Thus

$$\begin{aligned}\mathrm{M}(\tilde{\Xi}^{ij}_*[\Pi\Psi((\Sigma')^{ij})]\times[0,1]\llcorner N^i\lambda\rho_i\cap N^j_{\lambda\rho_j}) &\le \int_0^1\int_{\Pi(\Psi((\Sigma'')^{ij}))}|J_3\tilde{\Xi}^{ij}|\\ &\le K\int_0^1\int_{\Pi(\Psi((\Sigma'')^{ij}))}\left|\frac{\partial}{\partial t}\Lambda^{-1}_{x^t}\cdot[(z,\tilde{a}^t(z)]\right|\ dz\wedge d\overline{z}\wedge dt\\ &\le K\int_0^1\int_{\Pi(\Psi((\Sigma'')^{ij}))}\left[r_i|(z,\tilde{a}^t(z))|+\left|\frac{\partial\tilde{a}^t}{\partial t}\right|\right]\ .\end{aligned} \tag{VI.35}$$

On the one hand, since $\Pi_i(\Psi((\Sigma'')^{ij}))\ |(z,\tilde{a}^t(z))|\le K\ r_i$, we have

$$\int_0^1\int_{\Pi(\Psi((\Sigma'')^{ij}))} r_i|(z,\tilde{a}^t(z))|\le K r_i^4\ . \tag{VI.36}$$

On the other hand

(VI.37)

$$\int_0^1\int_{\Pi(\Psi((\Sigma'')^{ij}))}\left|\frac{\partial\tilde a^t}{\partial t}\right| = \lim_{N\to+\infty}\frac{1}{N}\sum_{l=1}^{N-1}\int_{\Pi(\Psi((\Sigma'')^{ij}))} N\ |\tilde a^{\frac{l}{N}}(z)-\tilde a^{\frac{l+1}{N}}(z)|\ dz\wedge d\overline{z}\ .$$

We have

$$\begin{aligned}(\text{VI.38})\quad |\tilde a_k^{\frac{l}{N}}(z)-\tilde a_k^{\frac{l+1}{N}}(z)| &= |\langle C^{\frac{l}{N}},z^{\frac{l}{N}},z\rangle(w_k^{\frac{l}{N}})-\langle C^{\frac{l+1}{N}},z^{\frac{l+1}{N}},z\rangle(w_k^{\frac{l+1}{N}})|\\ &\le |\langle C^{\frac{l}{N}}-C^{\frac{l+1}{N}},z^{\frac{l}{N}},z\rangle(w_k^{\frac{l}{N}})|\\ &\quad +|\langle C^{\frac{l+1}{N}},z^{\frac{l}{N}},z\rangle(w_k^{\frac{l}{N}})-\langle C^{\frac{l+1}{N}},z^{\frac{l+1}{N}},z\rangle(w_k^{\frac{l}{N}})|\\ &\quad +|\langle C^{\frac{l+1}{N}},z^{\frac{l+1}{N}},z\rangle(w_k^{\frac{l}{N}}-w_k^{\frac{l+1}{N}})|\ .\end{aligned}$$

We have to control the sum over l of the integral over $\Pi(\Psi((\Sigma'')^{ij}))$ of the three absolute values on the right-hand side of (VI.38) one by one. For the first term of the right-hand side of (VI.38) we have, using [Fe, 4.3.1], since $\|w_k^{\frac{l}{N}}\|_\infty+\|dw_k^{\frac{l}{N}}\|_\infty\le 1$,

$$\begin{aligned}(\text{VI.39})\quad &\int_{\Pi(\Psi((\Sigma'')^{ij}))}|\langle C^{\frac{l}{N}}-C^{\frac{l+1}{N}},z^{\frac{l}{N}},z\rangle(w_k^{\frac{l}{N}})|\ dz\wedge d\overline{z}\\ &\le \text{Lip}(z^{\frac{k}{N}})\ \mathcal{F}_{N^i_{\lambda\rho_i}\cap N^j_{\lambda\rho_j}}(C^{\frac{k}{N}}-C^{\frac{k+1}{N}})\le K\ \mathcal{F}_{N^i_{\lambda\rho_i}\cap N^j_{\lambda\rho_j}}(C^{\frac{k}{N}}-C^{\frac{k+1}{N}})\ .\end{aligned}$$

Similarly, as we established estimate (VI.25) we can show that

$$(\text{VI.40})\qquad \mathcal{F}_{N^i_{\lambda\rho_i}\cap N^j_{\lambda\rho_j}}(C^{\frac{k}{N}}-C^{\frac{k+1}{N}})\le K\ \frac{1}{N}\ r_i^4\ .$$

Thus

$$(\text{VI.41})\qquad \lim_{N\to+\infty}\sum_{l=1}^{N-1}\int_{\Pi(\Psi((\Sigma'')^{ij}))}|\langle C^{\frac{l}{N}}-C^{\frac{l+1}{N}},z^{\frac{l}{N}},z\rangle(w_k^{\frac{l}{N}})|\le K\ r_i^4\ .$$

For the second term on the right-hand side of (VI.38) we use 4.3.9 (3) of [Fe] and get

$$\begin{aligned}(\text{VI.42})\quad &\int_{\Pi(\Psi((\Sigma'')^{ij}))}|\langle C^{\frac{l+1}{N}},z^{\frac{l}{N}},z\rangle(w_k^{\frac{l}{N}})-\langle C^{\frac{l+1}{N}},z^{\frac{l+1}{N}},z\rangle(w_k^{\frac{l}{N}})|\\ &K\int_{\frac{l}{N}}^{\frac{l+1}{N}}dt\int_{(z^t)^{-1}(\Pi(\Psi((\Sigma'')^{ij})))}|z^{\frac{l}{N}}-z^{\frac{l+1}{N}}|\ d\|C^{\frac{l+1}{N}}\|\\ &K\ \frac{r_i^2}{N}\text{M}(C^{\frac{l+1}{N}}\llcorner N^i\lambda\rho_i\cap N^j_{\lambda\rho_j})\\ &\le K\frac{r_i^4}{N}\end{aligned}$$

where we have used (VI.26). Therefore we obtain

(VI.43)
$$\lim_{l\to+\infty}\sum_{l}^{N-1}\int_{\Pi(\Psi((\Sigma'')^{ij}))}|\langle C^{\frac{l+1}{N}},z^{\frac{l}{N}},z\rangle(w_k^{\frac{l}{N}})-\langle C^{\frac{l+1}{N}},z^{\frac{l+1}{N}},z\rangle(w_k^{\frac{l}{N}})|\le Kr_i^4\ .$$

Finally for the second term of the right-hand side of (VI.38), we use again (VI.26) and 4.3.2 (2) of [Fe] to obtain that

(VI.44)
$$\int_{\Pi(\Psi((\Sigma'')^{ij}))}|\langle C^{\frac{l+1}{N}},z^{\frac{l+1}{N}},z\rangle(w_k^{\frac{l}{N}}-w_k^{\frac{l+1}{N}})|$$
$$\le \mathrm{M}(C^{\frac{l+1}{N}}\llcorner N^i\lambda\rho_i\cap N^j_{\lambda\rho_j})\ \frac{r_i^2}{N}\quad\le K\ \frac{r_i^4}{N}\ .$$

Combining (VI.35), (VI.36), (VI.37), (VI.38), (VI.41), (VI.43) and (VI.44), we obtain that

(VI.45)
$$\mathrm{M}(\tilde{\Xi}_*^{ij}[\Pi\Psi((\Sigma')^{ij})]\times[0,1]\llcorner N^i_{\lambda\rho_i}\cap N^j_{\lambda\rho_j})\le r_i^4\ .$$

Combining this last inequality with (VI.32) and a Fubini type argument we obtain that there exists $\lambda\in[\frac{7}{6},\frac{6}{5}]$ such that

(VI.46)
$$\mathcal{F}((\tilde{C}_i-\tilde{C}_j)\llcorner N^i\lambda\rho_i\cap N^j_{\lambda\rho_j})\le r_i^4\ .$$

From this fact we then deduce, since $\tilde{C}_i$ and $\tilde{C}_j$ are single valued graphs with uniformly bounded gradients

(VI.47)
$$\int_{\Pi_i(N^i\frac{7}{6}\rho_i\cap N^j_{\frac{7}{6}\rho_j})}|\tilde{a}_j^i(z^i)-\tilde{a}_j^j|\le Kr_i^4\ .$$

Using the notation introduced in (VI.4) and (VI.13), we have that for any z there exists ξ such that

(VI.48)
$$(z-z_i,\tilde{a}_i(z)-w_i)=\Lambda_{x_i}^{-1}(\xi,\tilde{a}_i^i(\xi))\ ,$$

where $|\Lambda_{x_i}-\mathrm{id}|\le K\ |x_i|\le K\ \rho_{x_0}$. Let $z':=\rho_i^{-1}(z-z_i)$ and $\hat{a}_i(z'):=\rho_i^{-1}(\tilde{a}_i(z)-w_i)$. Let also $\xi':=\rho_i^{-1}\xi$ and $\hat{a}_i^i(\xi'):=\tilde{a}_i^i(\xi)$. Since $\tilde{a}_i^i$ is holomorphic (see (VI.14), $\hat{a}_i^i$ is also clearly holomorphic and since $\|\hat{a}_i^i\|_{L^\infty(B_{\frac{3}{2}}(0))}\le K$, we have that for any $l\in\mathbb{N}$

(VI.49)
$$\|\nabla^l\hat{a}_i^i\|_{L^\infty(B_{\frac{5}{4}}(0))}\le K_l\ .$$

Using the above notation we have

(VI.50)
$$(z',\hat{a}_i(z'))=\Lambda_{x_i}^{-1}(\xi',\hat{a}_i^i(\xi'))\ .$$

From the inverse function theorem, since $|\Lambda_{x_i}-id|\le K\ |x_i|\le K$, ρ_{x_0} can be taken as small as we want by taking ρ_{x_0} small enough, we have that for all $l\in\mathbb{N}$ there exists K_l such that

(VI.51)
$$\|\nabla^l_{z'}\xi'\|_\infty\le K_l\ .$$

Therefore, combining (VI.49), (VI.50 and (VI.51, we obtain

$$\|\nabla^l_{z'}\hat{a}_i(z')\|_\infty \le K_l\ . \tag{VI.52}$$

From that estimate we then deduce

$$\|\nabla^l_z\tilde{a}_i\|_{L^\infty(B^2_{\frac{7}{6}\rho_i}(z_i))} \le K_l\ r_i^{l-1}\ . \tag{VI.53}$$

Since $\tilde{C}^i$ is J_{x_i}-holomorphic, we have the existence of $\lambda^i_1, \mu^i_1, \lambda^i_2, \mu^i_2$ such that

$$\left\{\begin{array}{l} J_{x_i}\cdot\begin{pmatrix}1\\0\\\dfrac{\partial\tilde{a}_i}{\partial x}\end{pmatrix} = \lambda^i_1\begin{pmatrix}1\\0\\\dfrac{\partial\tilde{a}_i}{\partial x}\end{pmatrix} + \mu^i_1\begin{pmatrix}0\\1\\\dfrac{\partial\tilde{a}_i}{\partial y}\end{pmatrix}\\[2ex] J_{x_i}\cdot\begin{pmatrix}0\\1\\\dfrac{\partial\tilde{a}_i}{\partial y}\end{pmatrix} = \lambda^i_2\begin{pmatrix}1\\0\\\dfrac{\partial\tilde{a}_i}{\partial x}\end{pmatrix} + \mu^i_2\begin{pmatrix}0\\1\\\dfrac{\partial\tilde{a}_i}{\partial y}\end{pmatrix}\ . \end{array}\right. \tag{VI.54}$$

Writing $J_{x_i} = J_0 + \delta(x_i)$, we first observe that

$$|\delta(x_i)| \le \|J\|_{C^1}|x_i| \le K\rho_{x_0}\ . \tag{VI.55}$$

From this notation we deduce using (VI.54),

$$\left\{\begin{array}{l} \lambda^i_1 = \delta_{1,1}(x_i) + \displaystyle\sum_{l=3}^{2p}\delta_{1,l}(x_i)\frac{\partial\tilde{a}^l_i}{\partial x}\\ \mu^i_1 = 1 + \delta_{2,1}(x_i) + \displaystyle\sum_{l=3}^{2p}\delta_{2,l}(x_i)\frac{\partial\tilde{a}^l_i}{\partial x}\\ \lambda^i_2 = -1 + \delta_{1,2}(x_i) + \displaystyle\sum_{l=3}^{2p}\delta_{1,l}(x_i)\frac{\partial\tilde{a}^l_i}{\partial y}\\ \mu^i_2 = \delta_{2,2}(x_i) + \displaystyle\sum_{l=3}^{2p}\delta_{2,l}(x_i)\frac{\partial\tilde{a}^l_i}{\partial y}\ . \end{array}\right. \tag{VI.56}$$

Therefore the equation solved by $\tilde{a}_i$ is for any $k = 1 \dots p-1$,

$$\text{(VI.57)} \quad \left\{ \begin{aligned} \frac{\partial \tilde{a}_i^{2k+1}}{\partial x} - \frac{\partial \tilde{a}_i^{2k+2}}{\partial y} &= \left[\delta_{1,1}(x_i) + \sum_{l=3}^{2p} \delta_{1,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial x} \right] \frac{\partial \tilde{a}_i^{2k+2}}{\partial x} \\ &\quad + \left[\delta_{2,1}(x_i) + \sum_{l=3}^{2p} \delta_{2,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial x} \right] \frac{\partial \tilde{a}_i^{2k+2}}{\partial y} \\ &\quad -\delta_{2k+2,1}(x_i) - \sum_{l=3}^{2p} \delta_{2k+2,l} \frac{\partial \tilde{a}_i^l}{\partial x} , \\ \frac{\partial \tilde{a}_i^{2k+1}}{\partial y} + \frac{\partial \tilde{a}_i^{2k+2}}{\partial x} &= \left[\delta_{1,2}(x_i) + \sum_{l=3}^{2p} \delta_{1,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial y} \right] \frac{\partial \tilde{a}_i^{2k+2}}{\partial x} \\ &\quad + \left[\delta_{2,2}(x_i) + \sum_{l=3}^{2p} \delta_{2,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial y} \right] \frac{\partial \tilde{a}_i^{2k+2}}{\partial y} \\ &\quad -\delta_{2k+2,1}(x_i) - \sum_{l=3}^{2p} \delta_{2k+2,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial y} . \end{aligned} \right.$$

Then we deduce that there exist a linear map

$$A(x_i, \cdot) \ : \ \mathbb{R}^2 \otimes \mathbb{R}^{2p-2} \longrightarrow \mathbb{C}^{p-1} \otimes_{\mathbb{R}} \left(\mathbb{R}^2 \otimes \mathbb{R}^{2p-2} \right)^* ,$$

and an element

$$B(x_i, \cdot) \in \mathbb{C}^{p-1} \otimes_{\mathbb{R}} \left(\mathbb{R}^2 \otimes \mathbb{R}^{2p-2} \right)^* ,$$

such that $\tilde{a}_i$ solves

$$\text{(VI.58)} \qquad \frac{\partial \tilde{a}_i}{\partial \overline{z}} = A(x_i, \nabla \tilde{a}_i) \cdot \nabla \tilde{a}_i + B(x_i, \nabla \tilde{a}_i) + D(x_i, \tilde{a}_i) .$$

Observe also that the dependence of A and B in $B_{\rho_{x_0}}(x_0)$ is smooth and that $A(x_0, \cdot) = 0$, $B(x_0, \cdot) = 0$, $D(x_0) = 0$ and because of (VI.55) we have an estimate of the sort

$$\text{(VI.59)} \qquad \forall p \in \mathbb{R}^2 \otimes \mathbb{R}^{2p} \qquad |A(x_i, p)| + |B(x_i, p)| \leq K \ |x_i| (1 + |p|) .$$

Consider now i and j such that $B^2_{\rho_i}(z_i) \cap B^2_{\rho_j}(z_j) \neq \emptyset$. On $B^2_{\frac{7}{6}\rho_i}(z_i) \cap B^2_{\frac{7}{6}\rho_j}(z_j)$ $\tilde{a}_i - \tilde{a}_j$ solves the following equation

$$\text{(VI.60)} \qquad \begin{aligned} \partial_{\overline{z}}(\tilde{a}_i - \tilde{a}_j) &= A(x_i, \nabla \tilde{a}_i) \cdot \nabla \tilde{a}_i + B(x_i, \nabla \tilde{a}_i) \\ &\quad - A(x_j, \nabla \tilde{a}_j) \cdot \nabla \tilde{a}_j - B(x_j, \nabla \tilde{a}_j) \\ &= C(x_i, \nabla \tilde{a}_i, \nabla \tilde{a}_j) \cdot \nabla (\tilde{a}_i - \tilde{a}_j) \\ &\quad + E(x_i, \nabla \tilde{a}_j) - E(x_j, \nabla \tilde{a}_j) , \end{aligned}$$

where

$$\text{(VI.61)} \quad C(x_i, \nabla \tilde{a}_i, \nabla \tilde{a}_j) \cdot \nabla (\tilde{a}_i - \tilde{a}_j) := A(x_i, \nabla \tilde{a}_i) \cdot \nabla \tilde{a}_i - A(x_i, \nabla \tilde{a}_j) \cdot \nabla \tilde{a}_j$$

$$+B(x_i,\nabla\tilde a_i)-B(x_i,\nabla\tilde a_j)\ ,$$

(where we have used the linear dependence in p of $A(x_i,p)$ and $B(x_i,p)$), and where

$$E(x,p):=A(x,p)\cdot p+B(x,p)+D(x)\ . \tag{VI.62}$$

Observe, on the one hand, that $C(x,p,q)$ has a linear dependence in p and q in $\mathbb{R}^2\otimes\mathbb{R}^{2p-2}$, that

$$|C(x,p,q)|\le K\ |x|\ (1+|p|+|q|)\ , \tag{VI.63}$$

and that the following estimates hold for $D(x,p)$, for all $l\in\mathbb{N}$:

$$|\nabla^l_x E(x,p)|\le K_l\ (1+|p|^2)\ . \tag{VI.64}$$

On $B^2_{\frac{7}{6}}(\rho_i^{-1}z_i)\cap B^2_{\frac{7}{6}}(\rho_i^{-1}z_j)$ the function $f(z'):=a_i(\rho_i z')-a_j(\rho_i z')$ solves

$$\partial_{\overline{z'}}f-C(z')\cdot\nabla f=g(z')\ , \tag{VI.65}$$

where

$$C(z'):=C(x_i,\nabla\tilde a_i,\nabla\tilde a_j)(\rho_i z')\ ,$$

and

$$g(z'):=\rho_i\left[D(x_i,\nabla\tilde a_j(\rho_i z'))-D(x_j,\nabla\tilde a_j(\rho_i z'))\right]\ .$$

Using (VI.53), observe that for any $l\in\mathbb{N}$,

$$\begin{aligned}&\|\nabla^l_{z'}(C(x_i,(\nabla_z\tilde a_i)(\rho_i z'),(\nabla_z\tilde a_j)(\rho_i z')))\|_\infty\\&\qquad\le K\ |x_i|\ \rho_i^l\left[\|\nabla_z^{l+1}\tilde a_i\|_\infty+\|\nabla_z^{l+1}\tilde a_j\|_\infty\right]\le\ K_l\ \rho_{x_0}\ .\end{aligned} \tag{VI.66}$$

Therefore, for ρ_{x_0} small enough, $L:=\partial_{\overline{z'}}-C(z')\cdot\nabla_{z'}$ is an elliptic coercive first order operator with smooth coefficients whose derivatives are uniformly bounded. Observe also that, using again (VI.53),

$$\begin{aligned}&\|\nabla^l_{z'}\rho_i\left[D(x_i,\nabla\tilde a_j(\rho_i z'))-D(x_j,\nabla\tilde a_j(\rho_i z'))\right]\|_\infty\\&\qquad\le K\rho_i^2\ \left[\rho_i^l\sum_{s=0}^{[l/2]}\|\nabla_z^{s+1}\tilde a_j\|_\infty\ \|\nabla_z^{l-s+1}\tilde a_j\|_\infty+\rho_i^l\ \|\nabla_{z'}^{l+1}\tilde a_j\|_\infty\right]\ .\end{aligned} \tag{VI.67}$$

Then we have

$$\|\nabla^l_{z'}g\|_\infty\le K_l\ \rho_i^2\ . \tag{VI.68}$$

From (VI.46) we deduce that

$$\int_{B^2_{\frac{7}{6}}(\rho_i^{-1}z_i)\cap B^2_{\frac{7}{6}}(\rho_i^{-1}z_j)}|f|\le K\ \rho_i^2\ . \tag{VI.69}$$

Thus combining (VI.65)...(VI.69) and using standard elliptic estimates we obtain that for any $l \in \mathbb{N}$

$$\|\nabla^l f\|_{L^\infty(B_1^2(\rho_i^{-1}z_i)\cap B_1^2(\rho_i^{-1}z_j))} \le K_l\ \rho_i\ , \tag{VI.70}$$

which yields, going back to the original scale, the estimate (VI.19) and Lemma VI.2 is proved. □

Definition of the approximated average curve. On $B^2_{\rho_{x_0}}(0) = \Pi(B_{\rho_{x_0}}(x_0))$ we define the approximated average curve as follows. Let φ be the partition of unity of $\Pi(\mathcal{C}_{Q-1} \cap B_{\rho_{x_0}}(x_0))$ defined in (V.48). We denote

$$\begin{cases} \tilde{a}(z_0) := \displaystyle\sum_{i\in I} \varphi(z_0)\tilde{a}_i(z_0) & \forall z_0 \in \Pi(\mathcal{C}_{Q-1} \cap B_{\rho_{x_0}}(x_0))\ , \\ \tilde{a}(z_0) := \langle C, z, z_0\rangle(w) & \forall z_0 \in \Pi((\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}) \cap B_{\rho_{x_0}}(x_0))\ . \end{cases} \tag{VI.71}$$

Observe that, because of Lemma IV.2, for any $z_0 \in \Pi((\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}) \cap B_{\rho_{x_0}}(x_0))$ the slice $\langle C, z, z_0\rangle$ consists of exactly one point and $\tilde{a}(z_0)$ is simply the w coordinates of that point. The following estimates for $\tilde{a}$ holds :

Lemma VI.3. *Under the above notation, for any $q < +\infty$ there exists a constant K_q independent of ρ_{x_0} such that*

$$\int_{B^2_{\frac{\rho_{x_0}}{2}}(0)} |\nabla^2 \tilde{a}|^q \le K_q\ \rho^2_{x_0}\ . \tag{VI.72}$$

Proof of Lemma VI.3. We claim first that $\tilde{a}$ is a Lipschitz map over $B^2_{\rho_{x_0}}(0)$. In $\Pi(\mathcal{C}_{Q-1}\cap B_{\rho_{x_0}}(x_0))$, by the assumptions of the inductive procedure, we know that $\tilde{a}$ is smooth and using both (VI.53) and (VI.19), because also of (V.49), we have

$$\|\nabla \tilde{a}\|_{L^\infty(\Pi(\mathcal{C}_{Q-1}\cap B_{\rho_{x_0}}(x_0)))} \le K\ . \tag{VI.73}$$

Consider now two arbitrary points x and y of $B^2_{\rho_{x_0}}(0)$. Either the segment $[x, y]$ in $B^2_{\rho_{x_0}}(0)$ is included in $\Pi(\mathcal{C}_{Q-1} \cap B_{\rho_{x_0}}(x_0))$ and then we can integrate (VI.73) all along that segment to get

$$|\tilde{a}(x) - \tilde{a}(y)| \le K|x - y|\ , \tag{VI.74}$$

or there exists $z \in [x, y]\cap\Pi((\mathcal{C}_Q \setminus \mathcal{C}_{Q-1})\cap B_{\rho_{x_0}}(x_0))$. Using this time (IV.6) we also get (VI.74), which proves the desired claim. Using the equation (VI.58) solved by the $\tilde{a}_i$s we obtain that, in $\Pi(\mathcal{C}_{Q-1} \cap B_{\rho_{x_0}}(x_0))$, $\tilde{a}$ is a solution of

$$\partial_{\overline{z}}\tilde{a} - A((z, \tilde{a}(z)), \nabla\tilde{a}) \cdot \nabla\tilde{a} - B((z, \tilde{a}(z)), \nabla\tilde{a}) - D(z, \tilde{a}(z)) = \zeta(z)\ , \tag{VI.75}$$

where

$$\text{(VI.76)}\qquad \zeta(z) := \sum_{i\in I}\varphi_i\ \left[A(x_i,\nabla\tilde{a}_i)\cdot\nabla\tilde{a}_i - A((z,\tilde{a}(z)),\nabla\tilde{a})\cdot\nabla\tilde{a}\right] + \sum_{i\in I}\ \left[B(x_i,\nabla\tilde{a}_i) - B((z,\tilde{a}(z)),\nabla\tilde{a})\right] + \sum_{i\in I}\partial_{\overline{z}}\varphi_i\ \left[\tilde{a}_i - \tilde{a}\right]\ .$$

Observe that $\sum_{i\in I}\partial_{\overline{z}}\varphi_i\tilde{a}$ was added at the end because this quantity vanishes. Since for any x_i and r_i, (V.10) holds, since also for any $z\in\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1})\cap B_{\rho_{x_0}}(x_0))$ we have for an ε as small as we want (recall α was fixed independently of ε), the relative Lipschitz estimate (IV.6) holds and granting the fact that $\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1})\cap B_{\rho_{x_0}}(x_0))$ is a compact subset of $B_{\rho_{x_0}}(x_0)$, it is clear that

$$\text{(VI.77)}\qquad \forall\eta>0\quad \exists\delta>0:\forall i\in I \qquad \operatorname{dist}(B_{\rho_i}(z_i),\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1})))\le\delta \quad\Longrightarrow\quad |\rho_i|\le\eta\ .$$

Observe that

$$\text{(VI.78)}\qquad |\zeta(z)|\le K\sum_{i,j\in I}\varphi_i(z)\varphi_j(z)\ |\tilde{a}_j - w_i| \quad \sum_{i,j\in I}\ \varphi_i(z)\varphi_j(z)\ |\nabla(\tilde{a}_i-\tilde{a}_j)| \quad \sum_{i,j\in I}\ |\nabla\varphi_i(z)|\varphi_j(z)\ |\tilde{a}_i(z)-\tilde{a}_j(z)|\ .$$

Using (VI.19), we have that

$$\text{(VI.79)}\quad |\zeta(z)|\le\ K\max\left\{r_i\ i\in I;\ z\in B^2_{\rho_i}(z_i)\right\}\le K\ \operatorname{dist}(z,\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1})))\ .$$

Combining (VI.77) and (VI.79), we have that $\zeta(z)$ converges uniformly to 0 as z tends to $\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))$. We then extend ζ by 0 in $\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))$. Now ζ is a continuous function in $B^2_{\rho_{x_0}}(x_0)$ and we claim that

$$\text{(VI.80)}\qquad \partial_{\overline{z}}\tilde{a} - A((z,\tilde{a}(z)),\nabla\tilde{a})\cdot\nabla\tilde{a} - B((z,\tilde{a}(z)),\nabla\tilde{a}) = \zeta(z)\quad\text{ in }\ \mathcal{D}'(B^2_{\rho_{x_0}}(x_0))\ .$$

Since $\tilde{a}$ is a Lipschitz function in $B^2_{\rho_{x_0}}(x_0)$, $\partial_{\overline{z}}\tilde{a} - A((z,\tilde{a}(z)),\nabla\tilde{a})\cdot\nabla\tilde{a} - B((z,\tilde{a}(z)),\nabla\tilde{a})$ is a bounded function in $B^2_{\rho_{x_0}}(x_0)$ and therefore, in order to prove (VI.80), it suffices to prove that for $\mathcal{H}^2$ almost every z in $\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))\cap B^2_{\rho_{x_0}}(x_0)$,

$$\text{(VI.81)}\qquad \partial_{\overline{z}}\tilde{a} - A((z,\tilde{a}(z)),\nabla\tilde{a})\cdot\nabla\tilde{a} - B((z,\tilde{a}(z)),\nabla\tilde{a}) - D(z,\tilde{a}(z)) = 0\ .$$

This latter equality, because of the computations between (VI.54)…(VI.58), is just equivalent to the fact that the tangent plane of the graph $(z,\tilde{a}(z))$ at that point is J-holomorphic, which is the case at every point of $\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))\cap$

$B^2_{\rho_{x_0}}(x_0)$ due to Lemma IV.2. Thus (VI.80) is proved. Using again (VI.19) we observe that

(VI.82)

$$
\begin{aligned}
|\nabla\zeta(z)| \leq K \sum_{i,j\in I} & \varphi_i(z)\varphi_j(z) \ \ \left[|\nabla\tilde{a}_j| + |\nabla\tilde{a}_j|^3\right] \\
& \sum_{i,j\in I} \ \varphi_i(z)\varphi_j(z) \ |\nabla\tilde{a}_i|(z) \ |\nabla^2(\tilde{a}_i - \tilde{a}_j)|(z) \\
& \sum_{i,j\in I} \ \left[|\nabla^2\varphi_i(z)|\varphi_j(z) + |\nabla\varphi_i|(z)|\nabla\varphi_j(z)|\right] |\tilde{a}_i(z) - \tilde{a}_j(z)| + |\tilde{a}_j - w_i| \\
& \leq K \ .
\end{aligned}
$$

We claim now that ζ is Lipschitz in $B^2_{\rho_{x_0}}(x_0)$. Indeed, arguing as for $\tilde{a}$, given x and y in $B^2_{\rho_{x_0}}(x_0)$, if the segment $[x,y]$ has no intersection with $\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))$, then, integrating (VI.78) on that segment gives

$$
|\zeta(x) - \zeta(y)| \leq K \ |x-y| \ . \tag{VI.83}
$$

Otherwise, if there exists $z \in [x,y]\cap\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))$, then, (VI.79) gives

$$
|\zeta(x)| + |\zeta(y)| \leq K \ |x-z| + |y-z| \ ,
$$

which gives (VI.83) and the claim is proved. We claim now that $\tilde{a} \in W^{2,q}(B^2_{\rho_{x_0}/2}(0))$ for any $q < +\infty$. Let e be a unit vector in $B^2_{\rho_{x_0}/2}(0)$. For small h we denote $\tilde{a}_h(z) := \tilde{a}(z+h)$ and $\zeta_h(z) := \zeta(z+h)$. We have, using the linear dependencies of

(VI.84)

$$
\begin{aligned}
\partial_{\overline{z}}(\tilde{a} - \tilde{a}_h) & - A((z,\tilde{a}),\nabla\tilde{a})\cdot\nabla(\tilde{a}-\tilde{a}_h) - A((z,\tilde{a}),\nabla(\tilde{a}-\tilde{a}_h))\cdot\nabla\tilde{a}_h \\
& - B((z,\tilde{a}),\nabla(\tilde{a}-\tilde{a}_h)) \\
& = \zeta - \zeta_h + A((z,\tilde{a}),\nabla\tilde{a}_h)\cdot\nabla\tilde{a}_h - A((z+h,\tilde{a}_h),\nabla\tilde{a}_h)\cdot\nabla\tilde{a}_h \\
& \quad B((z,\tilde{a}),\nabla\tilde{a}_h) - B((z+h,\tilde{a}_h),\nabla\tilde{a}_h) + D(z,\tilde{a}) - D(z+h,\tilde{a}_h) \ .
\end{aligned}
$$

Denote Δ_h the right-hand side of (VI.84) and observe that there exists a constant K such that

$$
|\Delta_h| \leq K \ h \ . \tag{VI.85}
$$

Let $\chi_{\rho_{x_0}}$ be a cut-off function such that $\chi_{\rho_{x_0}} \equiv 1$ in $B^2_{\frac{\rho_{x_0}}{2}}(0)$ and $\chi_{\rho_{x_0}} \equiv 0$ in $\mathbb{R}^2\setminus B^2_{\rho_{x_0}}(0)$. Let $f_h := \chi_{\rho_{x_0}}\ (\tilde{a}-\tilde{a}_h)$, and let L_h be the operator such that

$$
L_h f := \partial_{\overline{z}} f - A((z,\tilde{a}),\nabla\tilde{a})\cdot\nabla f - A((z,\tilde{a}),\nabla f)\cdot\nabla\tilde{a}_h - B((z,\tilde{a}),\nabla f) \ . \tag{VI.86}
$$

Now,

$$\text{(VI.87)}\quad L_h f_h = \chi_{\rho_{x_0}} \Delta_h + (\tilde{a}-\tilde{a}_h)\partial_{\overline{z}}\chi_{\rho_{x_0}} - A((z,\tilde{a}),\nabla\tilde{a})\cdot(\tilde{a}-\tilde{a}_h)\nabla\chi_{\rho_{x_0}}$$
$$-A((z,\tilde{a}),(\tilde{a}-\tilde{a}_h)\nabla\chi_{\rho_{x_0}})\cdot\nabla\tilde{a}_h - B((z,\tilde{a}),(\tilde{a}-\tilde{a}_h)\nabla\chi_{\rho_{x_0}})\ .$$

Since $\tilde{a}$ is Lipschitz, using (VI.85) and (VI.59), we have that

$$\text{(VI.88)}\qquad |L_h f_h| \le K\ h\ .$$

Observe that $|L_h f_h| \ge |\partial_{\overline{z}} f_h| - K\rho_{x_0}|\nabla f_h|$. Since $f_h = 0$ on $\partial B^2_{\rho_{x_0}}(0)$, for any $p < +\infty$ we have that

(VI.89)

$$\int_{B^2_{\rho_{x_0}}(0)} |\nabla f_h|^q \le K_q \int_{B^2_{\rho_{x_0}}(0)} |\partial_{\overline{z}} f_h|^q \le K_q \int_{B^2_{\rho_{x_0}}(0)} |L_h f_h|^q + K\rho^q_{x_0}|\nabla f_h|^q\ .$$

Dividing by h^q and making h tend to zero, we get that for ρ_{x_0} small enough inequality (VI.72) holds and Lemma VI.3 is proved. □

VI.2. *Constructing adapted coordinates to C in a neighborhood of $x_0 \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$.* We consider a point x_0 in the support of our J-holomorphic current C and we assume, as above, that the multiplicity at x_0 is Q and that the tangent cone C_{0,x_0} is Q times a J_{x_0}-holomorphic disk D. We start with the coordinates $(z, w_1, \ldots, w_{p-1})$ chosen in (II.1) such that $C_{0,x_0} = Q[D]$ is Q times the "horizontal" disk given by $w_i = 0$ for $i = 1, \ldots, p-1$ and we work in the ball $B^{2p}_{\rho_{x_0}}(x_0)$ whose radius ρ_{x_0} is given by (V.23). The purpose of this subsection is to construct new coordinates $(\xi, \lambda_1, \ldots, \lambda_{p-1})$ in $B^{2p}_{\rho_{x_0}}(x_0)$ such that the set $\lambda_i = 0$ for $i = 1, \ldots, p-1$ coincides with the graph of the average map $\tilde{a}$ constructed in the previous subsection.

On the graph $\tilde{A}(z) := (z, \tilde{a}(z))$ we consider the complex structure j given by the metric induced by $g := \omega(J\cdot,\cdot)$. Let X be a vector tangent to $\tilde{A}$ at $(z, \tilde{a}(z))$. We compare jX and JX in $\mathbb{R}^{2p}$. Let $n(z, \tilde{a}(z))$ be the $2p-2$-unit vector normal to $T_{(z,\tilde{a}(z))}\tilde{A}$, making the identification between $2p-1$-vectors and the vector given by the ambient metric,

$$jX := n \wedge X\ .$$

We have seen that

$$|Jn - n|(z, \tilde{a}(z)) \le r_{(z,\tilde{a}(z))}\ .$$

Therefore

$$|jJX - JjX| \le |n\wedge JX - J(n\wedge X)| \le |n\wedge JX - Jn\wedge JX| \le r_{(z,\tilde{a}(z))}|X|\ .$$

Thus, $|(J-j)(J+j)X| \le r_{(z,\tilde{a}(z))}|X|$ where we extend j to the normal bundle to $\tilde{A}$ again by means of the induced metric. Since $|(J+j)X|$ and $|X|$ are comparable independent of X, we have

$$\text{(VI.90)}\qquad \forall X \in T_{(z,\tilde{a}(z))}\tilde{A}\qquad |(J-j)X| \le K r_{(z,\tilde{a}(z))}|X|\ .$$

We choose now coordinates $\xi = (\xi_1, \xi_2)$ on $\tilde{A}$ compatible with j (i.e. $j\frac{\partial}{\partial \xi_1} = \frac{\partial}{\partial \xi_2}$). Let $(z', \hat{a}(z')) := (\rho_{x_0}^{-1}, \rho_{x_0}^{-1}\tilde{a}(\rho_{x_0} z'))$ and in $B_1^{2p}(0)$ consider the metric $\hat{g}(z', w') := \rho_{x_0}^{-2}(\rho_{x_0} z', \rho_{x_0} w')^* g$ where g in $B_{\rho_{x_0}}^{2p}(0)$ is the original metric $g(\cdot, \cdot) = \omega(J\cdot, \cdot)$. After this scaling we have, using (VI.72),

$$\text{(VI.91)} \qquad \int_{B_1^2} |\nabla^2 \hat{a}|^p \, dz' \leq K_q \rho_{x_0}^q$$

and

$$\text{(VI.92)} \qquad \hat{g}^{ij} = \delta^{ij} + h^{ij} \quad \text{where } h^{ij}(0,0) = 0 \quad \text{and } \|\nabla h^{ij}\|_\infty \leq K\rho_{x_0} \, .$$

We look for isothermal coordinates (ξ_1', ξ_2') in $\hat{A} = \{(z', \hat{a}(z')), \ z' \in B_1^2(0)\}$ of the form $\xi' = z' + \delta(z')$ where δ will be small in $W^{2,p}$. On $B_1^2(0)$ we consider the metric $\hat{k} = (z', \hat{a}(z'))^* \hat{g} = (1 + k_{11})(dx_1')^2 + 2k_{12} dx_1' dx_2' + (1 + k_{22})(dx_2')^2$. From the estimates above we have for any $q > 0$, (since $\nabla \hat{a}(0,0) = 0$ and $\|\nabla^2 \hat{a}\|_q \leq \rho_{x_0}$ that for $\|\nabla \hat{a}\|_\infty \leq \rho_{x_0}$),

$$\text{(VI.93)} \qquad \int_{B_1^2} |\nabla k|^q \leq K_q \rho_{x_0}^{2q} \, .$$

Following [DNF, pp. 110–111], it suffices to find δ_1 solving

(VI.94)

$$-\frac{\partial}{\partial x_1'}\left[\frac{(1+k_{11})\frac{\partial \delta_1}{\partial x_1'} - k_{12}\frac{\partial \delta_1}{\partial x_2'}}{\sqrt{(1+k_{11})(1+k_{22}) - k_{12}^2}}\right] - \frac{\partial}{\partial x_2'}\left[\frac{(1+k_{22})\frac{\partial \delta_1}{\partial x_2'} - k_{12}\frac{\partial \delta_1}{\partial x_1'}}{\sqrt{(1+k_{11})(1+k_{22}) - k_{12}^2}}\right] = 0 \, .$$

Taking $\delta_1 = 0$ on $\partial B_1^2(0)$ we get a well-posed elliptic problem and obtain the existence of δ_1 satisfying

$$\sum_{i=1}^{2} a_{ij} \frac{\partial^2 \delta_1}{\partial x_i' \partial x_j'} = F \cdot \nabla \delta_1 \, ,$$

where a_{ij} are Hölder continuous, $\|a_{ij} - \delta_{ij}\|_{C^{0,\alpha}(B_1^2)} \leq K_\alpha \rho^2$ and $F \in L^q$ with $\int |F|^q \leq K_p \rho_{x_0}^{2q}$. Standard elliptic estimates give then

$$\text{(VI.95)} \qquad \|\delta_1\|_{W^{2,q}(B_1^2)} \leq K\rho_{x_0}^2 \, .$$

Therefore, going back to the original scale, we have found coordinates $\xi_i = x_i + \rho_{x_0}\delta_i(\rho_{x_0}^{-1} z) = x_i + \alpha_i(z)$ such that

$$\text{(VI.96)} \qquad \|\nabla \alpha\|_\infty \leq K\rho_{x_0}^2 \quad \text{and} \quad j\frac{\partial}{\partial \xi_1} = \frac{\partial}{\partial \xi_2} \, .$$

We translate these coordinates in such a way that $\alpha(0,0) = (0,0)$.

Inside $Gl(\mathbb{R}^{2p})$, the space of invertible $2p \times 2p$ matrices with real coefficients, we denote $U(p)$ the subspace of matrices M which commute with J_0. $U(p)$ is a compact submanifold of $Gl(\mathbb{R}^{2p})$ and for some metric in $Gl(\mathbb{R}^{2p})$ we denote $\pi_{U(p)}$ the orthogonal projection from a neighborhood of $U(p)$ onto $U(p)$. We consider $\mathrm{M}(z)$ the matrix which is given by

$$\mathrm{M}(z) := \Lambda^{-1}_{(z,\tilde{a}(z))}\pi_{U(p)}(\Lambda_{(z,\tilde{a}(z))}) \ , \tag{VI.97}$$

where we recall that Λ_x is given by (VI.6). We have clearly, since $\|\nabla\tilde{a}\| \le K$ and $\int_{B^2_{\rho_{x_0}}(0)} |\nabla^2\tilde{a}|^q \le K_q\rho^2_{x_0}$ for any $p < +\infty$,

$$\|\nabla\mathrm{M}(z)\|_{L^\infty(B^2_{\rho_{x_0}}(0))} \le K \quad \text{ and } \quad \|\nabla^2\mathrm{M}(z)\|_{L^q(B^2_{\rho_{x_0}}(0))} \le K_q\rho_{x_0}^{\frac{2}{q}} \ . \tag{VI.98}$$

We continue denoting $e_1, e_2, \ldots, e_{2p}$ the canonical basis of $\mathbb{R}^{2p}$. Let

$$\varepsilon_k(z) := \mathrm{M}(z)\cdot e_i \ . \tag{VI.99}$$

We have for all $i = 1\ldots p$ that

$$\begin{aligned} J((z,\tilde{a}(z))\cdot\varepsilon_{2i-1}(z) &= J((z,\tilde{a}(z))\cdot\Lambda^{-1}_{(z,\tilde{a}(z))}\pi_{U(p)}(\Lambda_{(z,\tilde{a}(z))})\cdot e_{2i-1} \\ &= \Lambda^{-1}_{(z,\tilde{a}(z))}J_0\pi_{U(p)}(\Lambda_{(z,\tilde{a}(z))})\cdot e_{2i-1} \\ &= \Lambda^{-1}_{(z,\tilde{a}(z))}\pi_{U(p)}(\Lambda_{(z,\tilde{a}(z))})J_0\cdot e_{2i-1} \\ &= \Lambda^{-1}_{(z,\tilde{a}(z))}\pi_{U(p)}(\Lambda_{(z,\tilde{a}(z))})\cdot e_{2i} \\ &= \varepsilon_{2i}(z) \ . \end{aligned} \tag{VI.100}$$

In $B^{2p}_{\rho_{x_0}}(x_0)$ we consider the new coordinates (ξ,λ) given by

$$\Psi \ : (\xi,\lambda) \longrightarrow \Psi(\xi,\lambda) := (z(\xi),\tilde{a}(z(\xi))) + \sum_{l=1}^{2p}\lambda_l\varepsilon_{l+2}(z(\xi)) \ . \tag{VI.101}$$

Letting $\tilde{J}$ be the expression of the almost complex structure in these coordinates (i.e. $\tilde{J}_{(\xi,\lambda)}\cdot X := d\Psi^{-1}J_{\Psi(\xi,\lambda)}\cdot d\Psi\cdot X$), we shall now estimate $|\tilde{J}-J_0|$ for points satisfying $|\lambda| \le r_{\Psi(\xi,\lambda)}$; recall that r_x was defined in the beginning of Section VI (see (V.11) and (V.12)), which corresponds in $B^{2p}_{\rho_{x_0}}(x_0)$ to a neighborhood of $\tilde{A}(z) = (z,\tilde{a}(z))$ containing the support of C. We have first for $i = 1, 2$, using (VI.90),

$$\begin{aligned} d\Psi\tilde{J}_{(\xi,0)}e_i = J_{\Psi(\xi,0)}\cdot\frac{\partial}{\partial\xi_i} &= j_{\Psi(\xi,0)}\cdot\frac{\partial}{\partial\xi_i} + (J_{\Psi(\xi,0)} - j_{\Psi(\xi,0)})\cdot\frac{\partial}{\partial\xi_i} \\ &= (-1)^{i+1}\frac{\partial}{\partial\xi_{i+1}} + O(r_\Psi(\xi,0)) \ , \end{aligned} \tag{VI.102}$$

where we are using the convention $\frac{\partial}{\partial\xi_{i+1}} = \frac{\partial}{\partial\xi_{i-1}}$. We have then, for $1 < l \le p$,

$$d\Psi\tilde{J}_{(\xi,0)}e_{2l} = J_{\Psi(\xi,0)}\frac{\partial}{\partial\lambda_{2l}} = J_{\Psi(\xi,0)}\varepsilon_{2l} = -\varepsilon_{2l-1} = \frac{\partial}{\partial\xi_{2l+1}} \ . \tag{VI.103}$$

For $|\lambda|\le r_{\Psi(\xi,\lambda)}$ we have for $i=1,2$,

$$d\Psi\tilde{J}_{(\xi,\lambda)}e_i = J_{\Psi(\xi,\lambda)}\cdot d\Psi_{(\xi,\lambda)}e_i = J_{\Psi(\xi,0)}\cdot d\Psi_{(\xi,0)}e_i + J_{\Psi(\xi,0)}\cdot\left[d\Psi_{(\xi,\lambda)}e_i - d\Psi_{(\xi,0)}e_i\right] + \left[J_{\Psi(\xi,\lambda)} - J_{\Psi(\xi,0)}\right]\cdot d\Psi_{(\xi,\lambda)}e_i\ . \tag{VI.104}$$

Using the facts that $|J_{\Psi(\xi,\lambda)} - J_{\Psi(\xi,0)}| \le \|J\|_{C^1}\ |\Psi(\xi,\lambda)-\Psi(\xi,0)| \le Kr_{\Psi(\xi,\lambda)}$ and that $d\Psi_{(\xi,\lambda)}e_i - d\Psi_{(\xi,0)}e_i = \frac{\partial\Psi}{\partial\xi_i}(\xi,\lambda) - \frac{\partial\Psi}{\partial\xi_i}(\xi,0) = \sum_{l=1}\lambda_l\partial_{\xi_i}\varepsilon_{l+2}$, we have then

$$|d\Psi\tilde{J}_{(\xi,\lambda)}e_1 - J_{\Psi(\xi,0)}\cdot d\Psi_{(\xi,0)}e_1| \le O(r_{\Psi(\xi,\lambda)})\ . \tag{VI.105}$$

Using (VI.90) again, we have $|[J_{\Psi(\xi,0)} - j(\Psi(\xi,0))]\cdot d\Psi_{(\xi,0)}e_i| \le O(r_{\Psi(\xi,0)})$, thus, since Ψ is Lipschitz, $r_{\Psi(\xi,\lambda)}$ and $r_{\Psi(\xi,0)}$ from Lemma V.3 are comparable and (VI.105) implies

$$|d\Psi\tilde{J}_{(\xi,\lambda)}e_1 - d\Psi_{(\xi,\lambda)}\cdot e_2| \le |d\Psi\tilde{J}_{(\xi,\lambda)}e_1 - d\Psi_{(\xi,0)}\cdot e_2| + |d\Psi_{(\xi,0)}\cdot e_2 - d\Psi_{(\xi,\lambda)}\cdot e_2|\ . \tag{VI.106}$$

Using again the fact that $|d\Psi_{(\xi,0)}\cdot e_2 - d\Psi_{(\xi,\lambda)}\cdot e_2| = |\frac{\partial\Psi}{\partial\xi_2}(\xi,\lambda) - \frac{\partial\Psi}{\partial\xi_2}(\xi,0)| = |\sum_{l=1}\lambda_l\partial_{\xi_2}\varepsilon_{l+2}| \le O(r_{\Psi(\xi,\lambda)})$, we have finally that

$$|d\Psi\tilde{J}_{(\xi,\lambda)}e_1 - d\Psi_{(\xi,\lambda)}\cdot e_2| \le O(r_{\Psi(\xi,\lambda)})\ . \tag{VI.107}$$

Finally, for $1<l\le p$,

$$d\Psi\tilde{J}_{(\xi,\lambda)}e_{2l} = J_{\Psi(\xi,\lambda)}\cdot d\Psi_{(\xi,\lambda)}e_{2l} = -d\Psi_{(\xi,\lambda)}e_{2l-1} + [d\Psi_{(\xi,\lambda)} - d\Psi_{(\xi,0)}]e_{2l-1} + J_{\Psi(\xi,0)}\cdot[d\Psi_{(\xi,\lambda)} - d\Psi_{(\xi,0)}e_{2l}] + [J_{\Psi(\xi,\lambda)} - J_{\Psi(\xi,0)}]\cdot d\Psi_{(\xi,\lambda)}\ . \tag{VI.108}$$

Using the estimates from the above lines, (VI.108) becomes for $1<l\le p$,

$$|d\Psi\tilde{J}_{(\xi,\lambda)}e_{2l} + d\Psi_{(\xi,\lambda)}e_{2l-1}| \le O(r_{\Psi(\xi,\lambda)})\ . \tag{VI.109}$$

Thus combining (VI.107) and (VI.109), we obtain

$$\forall(\xi,\lambda),\ \text{such that } |\lambda|\le r_{\Psi(\xi,\lambda)},\quad |\tilde{J}_{(\xi,\lambda)} - J_0| \le K\ r_{\Psi(\xi,\lambda)}\ . \tag{VI.110}$$

VII. The unique continuation argument

In this part we show that, assuming $\mathcal{P}_{Q-1}$ (recall that the definition is given by (II.8)), a point x_0 in $\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}$ is isolated in $\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}$ unless all points in $\mathcal{C}_*$ in a neighborhood from x_0 are of multiplicity Q. This fact has already been proved in Section IV in the case where C_{0,x_0} was not Q times the same flat holomorphic disk (the easy case). Here we assume that we are in the difficult case $C_{0,x_0} = Q[D_0]$ where D_0 is the horizontal unit disk as before.

We adopt the coordinate system about x_0 constructed in Section VIII.2. We denote by Π the map that assigns the first complex coordinate $\xi = \xi_1 + i\xi_2$. Assuming there exists a sequence of points $x_n \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ different from x_0 and converging to x_0, the goal of this section is to show that C in a neighborhood is a Q times the same graph. The strategy is inspired by [Ta, Ch. 1]; in our coordinates, the points in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ are contained in the disk $\lambda_i = 0$ for $i = 1, \ldots, 2p-2$ and we shall use a unique continuation argument based on the proof of a Carleman estimate to show that our assumption implies that the whole cycle in the neighborhood of x_0 is included in that disk. Let $(\xi_n, 0)$ be the coordinates of $x_n \to x_0$. We can always extract a subsequence such that $|\xi_{n+1}| \leq |\xi_n|^2$. We then introduce the function $g_N(\xi) := \prod_{j=1}^{N} (\xi - \xi_n)$. Because of the speed of convergence of our sequence ξ_n to zero it is not difficult to check that there exists a constant K independent of N such that for any $\xi \in B^2_{\rho_{x_0}}$ the following holds:

$$\text{(VII.1)} \quad \frac{K^{-1}}{|\xi|^{N-N_\xi}} \frac{1}{|\xi - \xi_{N_\xi}|} \prod_{j=1}^{N_\xi - 1} \frac{1}{|\xi_j|} \leq |g_N^{-1}|(\xi) \leq \frac{K}{|\xi|^{N-N_\xi}} \frac{1}{|\xi - \xi_{N_\xi}|} \prod_{j=1}^{N_\xi - 1} \frac{1}{|\xi_j|}$$

where N_ξ is the index less than N such that $|\xi - \xi_{N_\xi}|$ is minimal among the $|\xi - \xi_n|$. It is also straightforward to check that

$$\text{(VII.2)} \quad |\nabla g_N^{-1}|(\xi) \leq \frac{K(N - N_\xi)}{|\xi|^{N-N_\xi+1}} \frac{1}{|\xi - \xi_{N_\xi}|} \prod_{j=1}^{N_\xi - 1} \frac{1}{|\xi_j|} + \frac{K}{|\xi|^{N-N_\xi}} \frac{1}{|\xi - \xi_{N_\xi}|^2} \prod_{j=1}^{N_\xi - 1} \frac{1}{|\xi_j|} + \frac{K}{|\xi|^{N-N_\xi}} \frac{1}{|\xi - \xi_{N_\xi}|} \sum_{l=1}^{N_\xi - 1} \frac{1}{|\xi_l|} \prod_{j=1}^{N_\xi - 1} \frac{1}{|\xi_j|}$$

and we have a corresponding estimate for $|\nabla^k g_N^{-1}|(\xi)$ for arbitrary k in general. Let $\xi \in \Pi(\mathcal{C}_{Q-1})$ with ξ belonging to some $B^2_{\rho_i}(\xi_i)$ of the covering constructed in Section VI. To every such i we assign k_i, an index such that $|\xi_{k_i}| + \rho_{k_i} \leq |\xi_i| - \rho_i$ and such that $|\xi_{k_i} - \xi_i| \leq K\rho_i$ and such that ρ_i and ρ_{k_i} are comparable:

$$\text{(VII.3)} \qquad K^{-1}\rho_i \leq \rho_{k_i} \leq K\rho_i \ .$$

This is always possible due to the Whitney-Besicovitch nature of our covering; moreover for every k there exists a uniformly bounded number of i such that $k_i = k$. Observe also, because of the relative Lipschitz estimate (IV.6) with constant ε and because of the "splitting stage" of C_{ξ_i,ρ_i} characterized by (V.10) we have that for any $\delta > 0$ one may choose ε small enough compared to α defined in Section V such that for any $\xi \in \Pi(\mathcal{C}_{Q-1})$

$$\text{(VII.4)} \qquad \operatorname{dist}(\xi, \Pi(\mathcal{C}_Q \setminus \mathcal{C}_{Q-1})) \geq \delta^{-1}\rho_i \ ,$$

where $\xi \in B^2_{\rho_i}(\xi_i)$. Combining (VII.3) and (VII.4) we get that $\forall i \in I$, $\forall \xi \in B^2_{\rho_i}(\xi_i)$, $\forall \zeta \in B^2_{\rho_{k_i}}(\xi_{k_i})$,

(VII.5)
$$\frac{1}{2}\text{dist}\,(\xi, \Pi(\mathcal{C}_Q \setminus \mathcal{C}_{Q-1})) \leq \text{dist}\,(\zeta, \Pi(\mathcal{C}_Q \setminus \mathcal{C}_{Q-1})) \leq 2\text{dist}\,(\xi, \Pi(\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}))\ .$$

From (VII.1), (VII.2) and (VII.5) we get that for any $N \in \mathbb{N}$, for any $\xi \in B^2_{\rho_i}(\xi_i)$ and for any $\zeta \in B^2_{\rho_{k_i}}(\xi_{k_i})$

$$|g_N^{-1}|(\xi) \leq K|g_N^{-1}|(\zeta) \tag{VII.6}$$

and

$$\rho_i\frac{|\nabla g_N|}{|g_N|^2}(\xi) + \rho_i^2\frac{|\nabla^2 g_N|}{|g_N|^2}(\xi) \leq K\frac{1}{|g_N|}(\zeta)\ . \tag{VII.7}$$

Let $\chi_{\rho_{x_0}}$ be a cut-off function identically equal to 1 in $B^2_{\rho_{x_0}/2}(0)$ and equal to 0 outside $B^2_{\rho_{x_0}}$. In $B^2_{\rho_{x_0}}(z_0) \times \mathbb{R}^{2p-2}$ we introduce the cycle C^{g_N} which is given by

$$\forall \xi \in B^2_{\rho_{x_0}} \quad \langle C^{g_N}, \Pi, \xi\rangle := g_N^{-1}(\xi)_*\,\langle C, \Pi, \xi\rangle\ . \tag{VII.8}$$

In other words if $\Psi\ :\ \Sigma \to B^2_{\rho_{x_0}}(0) \times \mathbb{R}^{2p-2}$ is a parametrization of a piece of C, a parametrization of the corresponding piece in C^{g_N} is given by $(\Psi_\xi, g_N^{-1} \circ \Pi \circ \Psi\ \Psi_\lambda)$ where (Ψ_ξ, Ψ_λ) are the coordinates of Ψ. Since C^{g_N} is a cycle in $B^2_{\rho_{x_0}}(z_0) \times \mathbb{R}^{2p-2}$, we have, denoting $\Omega := \sum_{l=1}^{p-1} d\lambda_{2l-1} \wedge d\lambda_{2l}$),

$$C^{g_N}(\chi_{\rho_{x_0}} \circ \Pi\ \Omega) = -C^{g_N}\left(d\chi_{\rho_{x_0}} \circ \Pi \wedge \sum_{l=1}^{p-1} \lambda_{2l-1} d\lambda_{2l}\right)\ . \tag{VII.9}$$

Splitting $C^{g_N}(\chi_{\rho_{x_0}} \circ \Pi\ \Omega) = C^{g_N} \llcorner B^2_{\rho_{x_0}/2} \times \mathbb{R}^{2p-2}(\Omega) + C^{g_N} \llcorner (B^2_{\rho_{x_0}} \setminus B^2_{\rho_{x_0}/2}) \times \mathbb{R}^{2p-2}(\chi_{\rho_{x_0}} \circ \Pi\ \Omega)$, we have

(VII.10)
$$C^{g_N} \llcorner B^2_{\rho_{x_0}/2} \times \mathbb{R}^{2p-2}(\Omega) = \sum_{i\in I} C^{g_N} \llcorner B^2_{\rho_{x_0}/2} \times \mathbb{R}^{2p-2}\left(\varphi_i \circ \Pi \sum_{l=1}^{p-1} \Omega\right)\ ,$$

where we recall that the partition of unity was constructed in (V.48) adapted to the covering $B^2_{\rho_i}(\xi_i)$. Let $\Psi_i\ :\ \Sigma_i \to B^2_{2\rho_i}(\xi_i) \times B^{2p-2}_{2\rho_i}(0)$ be a smooth parametrization of $C \llcorner B^2_{2\rho_i}(\xi_i) \times B^{2p-2}_{\rho_{x_0}}(0)$ and denote by η_i the map from Σ_i into $\mathbb{R}^{2p}$ given by [Ri3, Prop. A.3], such that $\Psi_i + \eta_i$ is J_0-holomorphic. Since J in $B^{2p}_{2\rho_i}((\xi_i, 0))$ is closed to J_0 at a distance comparable to ρ_i — see (VI.110) — we have

$$\|\nabla \eta_i\|_{L^2(\Sigma_{\frac{3}{2},i})} \leq \rho_i^2 \tag{VII.11}$$

where we recall that $\Sigma_{\frac{3}{2},i} = \Sigma_i \cap \Psi_i^{-1}(B^2_{\frac{3\rho_i}{2}}(\xi_i) \times \mathbb{R}^2)$. Using now Lemma II.2 of [Ri3], which does not require J to be C^1 in these coordinates but just the metric g to be close to the flat one, one has

$$\|\eta_i\|_{L^2(\Sigma_{\frac{4}{3},i})} \leq \rho_i^3 \ . \tag{VII.12}$$

From the parametrization $\Psi_i = (\Psi_{i,\xi}, \Psi_{i,\lambda})$,

$$\begin{aligned} C^{g_N} \llcorner B^2_{\rho_{x_0}/2} \times \mathbb{R}^{2p-2} (\varphi_i \circ \Pi\ \Omega) \\ = \int_{\Sigma_i} \varphi(\Psi_{i,\xi}) \sum_{l=1}^{p-1} d\left[\frac{\Psi_{i,\lambda}^{2l-1}}{g(\Psi_{i,\xi})}\right] \wedge d\left[\frac{\Psi_{i,\lambda}^{2l}}{g(\Psi_{i,\xi})}\right] \ . \end{aligned} \tag{VII.13}$$

We compare this quantity with

$$\begin{aligned} C_i^{g_N} \llcorner (\varphi_i \circ \Pi\ \Omega) \\ := \int_{\Sigma_i} \varphi(\Psi_{i,\xi}) \sum_{l=1}^{p-1} d\left[\frac{\Psi_{i,\lambda}^{2l-1} + \eta_{i,\lambda}^{2l-1}}{g(\Psi_{i,\xi} + \eta_{i,\xi})}\right] \wedge d\left[\frac{\Psi_{i,\lambda}^{2l} + \eta_{i,\lambda}^{2l}}{g(\Psi_{i,\xi} + \eta_{i,\xi})}\right] \ . \end{aligned} \tag{VII.14}$$

Now,

$$\begin{aligned} &\left|(C_i^{g_N} - C^{g_N})\left(\varphi_i \circ \Pi \sum_{l=1}^{p-1} d\lambda_{2l-1} \wedge d\lambda_{2l}\right)\right| \\ &\leq K \int_{\Sigma_i} \left|\nabla\left[\frac{\Psi_{i,\lambda} + \eta_{i,\lambda}}{g(\Psi_{i,\xi} + \eta_{i,\xi})} - \frac{\Psi_{i,\lambda}}{g(\Psi_{i,\xi})}\right]\right| \ \left|\nabla\left[\frac{\Psi_{i,\lambda} + \eta_{i,\lambda}}{g(\Psi_{i,\xi} + \eta_{i,\xi})}\right]\right| \\ &\leq \delta K \int_{\Sigma_i} \varphi_{k_i} \left|\nabla\left[\frac{\Psi_{i,\lambda} + \eta_{i,\lambda}}{g(\Psi_{i,\xi} + \eta_{i,\xi})}\right]\right|^2 \\ &\quad + \frac{K}{\delta} \int_{\Sigma_i} \left|\nabla\left[\frac{\Psi_{i,\lambda} + \eta_{i,\lambda}}{g(\Psi_{i,\xi} + \eta_{i,\xi})} - \frac{\Psi_{i,\lambda}}{g(\Psi_{i,\xi})}\right]\right|^2 \ . \end{aligned} \tag{VII.15}$$

Next,

$$\begin{aligned} &\int_{\Sigma_i} \left|\nabla\left[\frac{\Psi_{i,\lambda} + \eta_{i,\lambda}}{g(\Psi_{i,\xi} + \eta_{i,\xi})} - \frac{\Psi_{i,\lambda}}{g(\Psi_{i,\xi})}\right]\right|^2 \\ &\leq \int_{\Sigma_i} \left|\nabla\left[\Psi_i \left(\frac{1}{g(\Psi_{i,\xi} + \eta_{i,\xi})} - \frac{1}{g(\Psi_{i,\xi})}\right)\right]\right|^2 \\ &\quad + \int_{\Sigma_i} |\nabla \eta_i| \sup_{\xi \in B^2_{\rho_i}(\xi_i)} \frac{1}{|g(\xi)|^2} + \int_{\Sigma_i} |\eta_i|^2 \ \sup_{\xi \in B^2_{\rho_i}(\xi_i)} \frac{|\nabla g|^2}{|g|^4}(\xi) \ . \end{aligned} \tag{VII.16}$$

Let $f_0(\xi)$ be the flat norm of the slice of C^{g_N} minus the average curve, $\Psi_\lambda = 0$, by $\Pi^{-1}(\xi)$,

$$f_N(\xi) = \mathcal{F}(\langle C^{g_N}, \Pi, \xi\rangle - Q\delta_0) \ .$$

Using (V.8) (observe that the difference of the densities for the metrics $\omega(\cdot, J\cdot)$ and $\omega_0(\cdot, J_0\cdot)$ is as small as we want for ρ_{x_0} chosen small enough) and using also (VII.11), (VII.12), (VII.6) and (VII.7), we have

$$\text{(VII.17)}\quad \int_{\Sigma_i} |\nabla\eta_i|^2 \sup_{\xi\in B^2_{\rho_i}(\xi_i)} \frac{1}{|g(\xi)|^2} + \int_{\Sigma_i} |\eta|^2 \sup_{\xi\in B^2_{\rho_i}(\xi_i)} \frac{|\nabla g_N|^2}{|g_N|^4}(\xi) \le K \int_{B^2_{\rho_{k_i}}(\xi_{k_i})} |f_N|^2$$

where we have also used the fact that $\int_{\Sigma_{k_i}} \varphi_{k_i} |\Psi_{k_i,\lambda}|^2 \ge K\rho_i^4$. This lower bound, a crucial point in our paper, comes from the fact that C restricted to $B_{\rho_{k_i}}(x_{k_i})$ is split; $\rho_{k_i}^{-2}\mathrm{M}(C \llcorner B_{\rho_{k_i}}(x_{k_i}))$ is less that $\pi Q - K_0\alpha$ (see (V.11)) where K_0 and α only depend on p, Q, J and ω. If $\Psi_{k_i,\lambda}$ had been too close to 0 in the L^2 norm, since the intersection number between $(\Psi_{k_i})_*[\Sigma_{k_i}]$ and the $2p-2$-planes $\Pi^{-1}(\xi)$ for $\xi \in B^2_{\rho_{k_i}}(\xi_{k_i})$ is Q, $\rho_{k_i}^{-2}\mathrm{M}(C \llcorner B_{\rho_{k_i}}(x_{k_i}))$ would have been too large which contradicts the upper bound (V.11). The first term on the right-hand side of (VII.16) can be bounded as follows

$$\begin{aligned}
\text{(VII.18)}\quad & \int_{\Sigma_i} \left|\nabla\left[\Psi_i\left(\frac{1}{g_N(\Psi_{i,\xi}+\eta_{i,\xi})} - \frac{1}{g_N(\Psi_{i,\xi})}\right)\right]\right|^2 \\
&\le \int_{\Sigma_i} \left|\frac{1}{g_N(\Psi_{i,\xi}+\eta_{i,\xi})} - \frac{1}{g_N(\Psi_{i,\xi})}\right|^2 \\
&\quad + \int_{\Sigma_i} \rho_i^2 \left|\nabla\left(\frac{1}{g_N(\Psi_{i,\xi}+\eta_{i,\xi})} - \frac{1}{g_N(\Psi_{i,\xi})}\right)\right|^2 \\
&\le K \int_{\Sigma_i} \rho_i^4 \left|\sup_{\xi\in B^2_{\rho_i}(\xi_i)} \frac{|\nabla g_N|}{|g_N|^2}\right|^2 (\xi) \\
&\quad + K \int_{\Sigma_i} \rho_i^2 |\nabla\eta_i|^2 \left|\sup_{\xi\in B^2_{\rho_i}(\xi_i)} \frac{|\nabla g_N|}{|g_N|^2}\right|^2 (\xi) \\
&\quad + K \int_{\Sigma_i} \rho_i^2 |\eta_i|^2 \sup_{\xi\in B^2_{\rho_i}(\xi_i)} \left|\frac{|\nabla g_N|^2}{|g_N|^3}\right|^2 (\xi) \\
&\quad + K \int_{\Sigma_i} \rho_i^2 |\eta_i|^2 \sup_{\xi\in B^2_{\rho_i}(\xi_i)} \left|\frac{|\nabla^2 g_N|}{|g_N|^2}\right|^2 (\xi)\ .
\end{aligned}$$

Using (VII.6) and (VII.7) as above and combining (VII.15)–(VII.18), we have finally for every $\delta > 0$

$$\begin{aligned}
\text{(VII.19)}\quad & \left|(C_i^{g_N} - C^{g_N})\left(\varphi_i \circ \Pi \sum_{l=1}^{p-1} d\lambda_{2l-1}\wedge d\lambda_{2l}\right)\right| \\
&\le \delta K \int_{\Sigma_i} \varphi_i \left|\nabla\left[\frac{\Psi_{i,\lambda}+\eta_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}\right]\right|^2 + \frac{K}{\delta} K \int_{B^2_{\rho_{k_i}}(\xi_{k_i})} |f_N|^2\ .
\end{aligned}$$

Since $\Psi_i + \eta_i/g \circ \Pi \circ \Psi_i + \eta_i$ is a holomorphic map into $\mathbb{C}^p$, the λ-coordinate of it, $\Psi_{i,\lambda} + \eta_{i,\lambda}/g \circ \Pi \circ \Psi_i + \eta_i$, is also a holomorphic map but into $\mathbb{C}^{p-1}$. Now,

(VII.20)

$$\frac{\Psi_{i,\lambda}+\eta_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}^{*}\left(\sum_{l=1}^{p-1} d\lambda_{2l-1}\wedge d\lambda_{2l}\right)=\frac{1}{2}(\Psi_{i,\lambda}+\eta_{i,\lambda})^{*}\left(\sum_{l=1}^{p-1} d\Lambda_l\wedge d\overline{\Lambda_l}\right)$$
$$=\left|\nabla\frac{\Psi_{i,\lambda}+\eta_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}\right|^2\ d\zeta\wedge d\overline{\zeta}\ ,$$

where ζ denotes local complex coordinates on Σ_i, and Λ_l is the complex coordinate $\Lambda_l=\lambda_{2l-1}+i\lambda_{2l}$. Therefore, combining (VII.19) and (VII.20) we have for δ chosen such that $\delta K<\frac{1}{2}$

(VII.21)
$$C^{g_N}(\varphi_i\circ\Pi\sum_{l=1}^{p-1} d\lambda_{2l-1}\wedge d\lambda_{2l})$$
$$\geq\frac{1}{2}\int_{\Sigma_i}\varphi_i\ \left|\nabla\frac{\Psi_{i,\lambda}+\eta_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}\right|^2-\frac{K}{\delta}\int_{B^2_{\rho_{k_i}}(\xi_{k_i})}|f_N|^2\ .$$

Letting $\{a_i^l(\xi)\}_{l=1\dots Q}$ be the holomorphic Q-valued graph realized by

$$(\Psi_{i,\xi}+\eta_{i,\xi},g_N^{-1}(\Psi_{i,\xi}+\eta_{i,\xi})\Psi_{i,\lambda}+\eta_{i,\lambda}),$$

we have

(VII.22)
$$\int_{\Sigma_i}\varphi_i\ \left|\nabla\frac{\Psi_{i,\lambda}+\eta_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}\right|^2=\int_{B^2_{\rho_i}}\varphi_i\sum_{l=1}^{p-1}|\nabla a_i^l|^2(\xi)\ d\xi\wedge d\overline{\xi}\ .$$

Clearly this quantity is larger than $\int_{B^2_{\rho_i}}\varphi_i|\nabla\tilde{f}_N|^2$ where $\tilde{f}_N(\xi)$ is the Flat norm of the slice by $\Pi^{-1}(\xi)$ of the difference between$C_i^{g_N}$ and the average curve. Replacing $\tilde{f}_N$ by f_N itself, we see the slice by $\Pi^{-1}(\xi)$, of C^{g_N} minus the average curve, in the integral $\int_{B^2_{\rho_i}}\varphi_i|\nabla\tilde{f}_N|^2$, induces error terms which can be controlled by $\int_{B^2_{\rho_{k_i}}(\xi_{k_i})}|f_N|^2$ as in the computation of the error between C^{g_N} and $C_i^{g_N}$ above. Therefore we have

(VII.23)
$$C^{g_N}(\varphi_i\circ\Pi\ \Omega)\geq\frac{1}{2}\int_{\Sigma_i}\varphi_i\ |\nabla f_N|^2-\frac{K}{\delta}\int_{B^2_{\rho_{k_i}}(\xi_{k_i})}|f_N|^2\ .$$

Because of the relative Lipschitz estimate, f_N extends as a $W^{1,2}$ function on all of $B^2_{\rho_{x_0}}(0)$. Standard Poincaré estimates yield

(VII.24)
$$\int_{B^2_{\rho_{x_0}}}|\chi_{\rho_{x_0}}f_N|^2\leq K\rho_{x_0}^2\int_{B^2_{\rho_{x_0}}}|\nabla(\chi_{\rho_{x_0}}f_N)|^2\ .$$

Taking ρ_{x_0} small enough we can ensure that $K\rho_{x_0}^2 > O(\delta)$ and combining (VII.9), (VII.23) and (VII.24) we finally get that

$$\text{(VII.25)} \qquad \int_{B^2_{\rho_{x_0}/2}} |f_N|^2 \leq K_1 \int_{B^2_{\rho_{x_0}} \setminus B^2_{\rho_{x_0}/2}} |f_N|^2 + |\nabla f_N|^2 + C^{g_N}\left(d\chi_{\rho_{x_0}} \circ \Pi \wedge \sum_{l=1}^{p-1} \lambda_{2l-1} d\lambda_{2l}\right)$$

where K_1 is a constant independent of N. By taking the sequence ξ_n such that the largest $|\xi_1|$ satisfies $|\xi_1| \leq \frac{\rho_{x_0}}{4}$, if C does not coincide with the average curve (f_N is not identically zero) near the origin we would have $\left(\frac{\rho_{x_0}}{4}\right)^{2N} \int_{B^2_{\rho_{x_0}/2}} |f_N|^2$ tending to infinity, whereas, it is not difficult to check that the right-hand side of (VII.25) which involves quantities supported in $B^2_{\rho_{x_0}} \setminus B^2_{\rho_{x_0}/2}$ is bounded by $K_{\rho_{x_0}} N^2 \left(\frac{\rho_{x_0}}{2}\right)^{-2N}$. The multiplication of it by $\left(\frac{\rho_{x_0}}{4}\right)^{2N}$ tends clearly to zero as N tend to infinity. We have then obtained a contradiction and have proved that any point inside $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ is surrounded in $\mathcal{C}_*$ by points which are all in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ or by points which are all in $\mathcal{C}_{Q-1}$. It remains to show that a point in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ is not an accumulation point of $\cup_{q \leq Q-1} \mathrm{Sing}^q$. This is the purpose of the next section.

VIII. Points in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ are not accumulation points of $\cup_{q \leq Q-1} \mathrm{Sing}^q$

In this section we prove, assuming $\mathcal{P}_{Q-1}$, that points in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ are not accumulation points of $\cup_{q \leq Q-1} \mathrm{Sing}^q$ and combining this fact with the result in the previous section we will have proved $\mathcal{P}_Q$.

Let then $x_0 \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$, and assume that x_0 is an accumulation point of $\mathcal{C}_{Q-1}$, which means, by the monotonicity formula, Lemma IV.1 together with the result obtained in the previous section, that there exists a radius ρ such that $\mathcal{C}_* \cap B_\rho(x_0) \subset \mathcal{C}_Q$ and that $(\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}) \cap B_\rho(x_0) = \{x_0\}$. From the assumed hypothesis $\mathcal{P}_{Q-1}$, we have then that there exists a Riemann surface Σ and a smooth J-holomorphic map Ψ such that $C \llcorner B_{r_0}(x_0) = \Psi_*[\Sigma]$. The goal is to show that Σ has a finite topology and that it is a closed Riemann surface. The idea is to perturb Ψ by finding $\eta \in L^\infty(\Sigma)$ such that $\Psi + \eta$ is J_0-holomorphic and $(\Psi + \eta)_*[\Sigma]$ is a cycle.

For any $r < r_0$, we denote Σ_r the finite Riemann surface obtained by taking $\Sigma \cap \Psi^{-1}(B_\rho(x_0) \setminus B_r(x_0))$ and we shall denote Γ_r the part of the boundary of Σ_r which is disjoint from $\partial\Sigma \subset (|\Psi - x_0|)^{-1}(r_0)$. On Σ_r we consider η_r the map which is given by Proposition A.3 in [Ri3]. It satisfies in particular, when we use the complex coordinates induced by J_0,

$$\text{(VIII.1)} \qquad \overline{\partial}(\Psi+\eta_r)=0 \qquad \text{in } \Sigma_r$$

$$\forall r\le r_0, \qquad \int_{\Sigma_r}|\nabla\eta_r|^2\le\int_{\Sigma_r}|J(\Psi)-J_0|^2|\nabla\Psi|^2\le Kr_0^4\ ,$$

where for the induced metric by Ψ on Σ (Ψ is an isometry), we have $\int_\Sigma|\nabla\Psi|^2 = \mathrm{M}(C\llcorner B_{r_0}(x_0))\le Kr_0^2$. Using local ξ_1, ξ_2 coordinates in Σ_r, we have for all $k=1\dots 2p$,

$$\frac{\partial\Psi_i^k}{\partial\xi_1}=-\sum_{l=1}^{2p}J_l^k(\Psi_i)\frac{\partial\Psi^l}{\partial\xi_2} \qquad \text{and} \qquad \frac{\partial\Psi_i^k}{\partial\xi_2}=\sum_{l=1}^{2p}J_l^k(\Psi_i)\frac{\partial\Psi^l}{\partial\xi_1}\ .$$

Taking respectively the ξ_1 derivative and the ξ_2 derivative of these two equations we obtain

$$\text{(VIII.2)} \qquad \forall k=1\dots 2p, \quad \Delta_{\Sigma_r}\Psi_i^k=*\left(\sum_{l=1}^{2p}d(J_l^k(\Psi_i))\wedge d\Psi_i^l\right)\ .$$

From (VIII.1) we deduce that $\Delta_{\Sigma_r}(\Psi+\eta_r)=0$; therefore this yields

$$\text{(VIII.3)} \qquad \forall k=1\dots 2p, \quad \Delta_{\Sigma_r}\eta_r^k=-*\left(\sum_{l=1}^{2p}d(J_l^k(\Psi_i))\wedge d\Psi_i^l\right)\ .$$

Let δ_r^k be given by

$$\text{(VIII.4)} \qquad \begin{cases}\Delta_{\Sigma_r}\delta_r^k &= *\left(\sum_{l=1}^{2p}d(J_l^k(\Psi_i))\wedge d\Psi_i^l\right) \qquad \text{in } \Sigma_r\\ \delta_r^k &=0 \qquad \text{on } \partial\Sigma_r\ .\end{cases}$$

From [Ge] and [To] there exists a universal constant K such that

$$\text{(VIII.5)} \qquad \|\delta_r\|_{L^\infty(\Sigma_r)}+\|\nabla\delta\|_{L^2(\Sigma_r)}\le K\|J\|_{C^1}\int_{\Sigma_r}|\nabla\Psi|^2\le Kr_0^2\ .$$

Because of the above estimates, taking some sequence $r_n\to 0$, one can always extract a subsequence $r_{n'}$ such that $\eta_{r_{n'}}$ and $\delta_{r_{n'}}$ converge to limits η_0 and δ_0 that satisfy in particular

$$\text{(VIII.6)} \qquad \begin{aligned}\overline{\partial}(\Psi+\eta_0)&=0 \qquad \text{in } \Sigma\ ,\\ \Delta_\Sigma(\eta_0+\delta_0)&=0 \qquad \text{in } \Sigma\ ,\\ \|\nabla\delta_0\|_{L^2(\Sigma)}+\|\delta_0\|_{L^\infty(\Sigma)}&\le Kr_0^2\ .\end{aligned}$$

For any $k=1,\dots,2p$ we consider the harmonic function $u^k:=\eta^k+\delta^k$. Using the coarea formula we have, for any $r<r_0$,

$$\text{(VIII.7)} \qquad \int_0^r ds\int_{\Gamma_s}|\nabla u^k|=\int_{\Sigma\setminus\Sigma_r}|\nabla u^k|\ |\nabla|\Psi||\le r\left(\int_{\Sigma\setminus\Sigma_r}|\nabla u^k|^2\right)^{\frac{1}{2}}\ .$$

Therefore, by use of a mean formula, for any $\varepsilon > 0$, there exists $s > 0$ such that

(VIII.8)
$$\int_{\Gamma_s} |\nabla u^k| \leq \varepsilon \ .$$

Now,

(VIII.9)
$$0 = \int_{\Sigma_s} \Delta_\Sigma u^k = \int_{\Gamma_s} \frac{\partial u^k}{\partial \nu} + \int_{\partial\Sigma} \frac{\partial u^k}{\partial \nu} \ .$$

By choosing ε smaller and smaller and taking the corresponding s given by (VIII.8), one gets

(VIII.10)
$$\int_{\partial\Sigma} \frac{\partial u^k}{\partial \nu} \ .$$

Let $m < M$ be two values such that $\sup_{\partial\Sigma} u^k < m$ and consider the truncation $T_m^M u^k$ equal to m if $u^k \leq m$ equal to M if $u^k \geq M$ and equal to u^k otherwise. We have

(VIII.11)
$$0 = \int_{\Sigma_s} T_m^M u^k \ \Delta_\Sigma u^k = -\int_{\Sigma_s} |\nabla T_m^M u^k|^2 + \int_{\Gamma_s} T_m^M u^k \frac{\partial u^k}{\partial \nu} + m \int_{\partial\Sigma} \frac{\partial u^k}{\partial \nu} \ .$$

Therefore

(VIII.12)
$$\int_{\Sigma_s} |\nabla T_m^M u^k|^2 \leq M \int_{\Gamma_s} |\nabla u^k| \ ,$$

and by choosing again s tending to zero according to (VIII.8), one gets that $T_m^M u^k$ is identically equal to m and we deduce that $u^k \leq m$ in Σ. Similarly one gets that u^k is bounded from below and then we have proved that $\|u\|_{L^\infty(\Sigma)} < +\infty$. Combining this fact with (VIII.6) we have that

(VIII.13)
$$\|\eta_0\|_{L^\infty(\Sigma)} < +\infty \ .$$

Being more careful above by taking eventually Σ_r instead of Σ for some $r \in [r_0/2, r_0]$, and using [Ri3] we could have shown that $\|\eta_0\|_{L^\infty(\Sigma)} < K \ r_0^2$. We claim now that $\partial(\Psi + \eta_0)_*[\Sigma] = (\Psi + \eta_0)_*[\partial\Sigma]$; that is, for any smooth 1-form ϕ equal to zero in a neighborhood of $(\Psi + \eta_0)(\partial\Sigma)$, one has

(VIII.14)
$$\int_\Sigma (\Psi + \eta_0)^* d\phi = 0 \ .$$

Now,

(VIII.15)
$$\left|\int_{\Sigma_s} (\Psi + \eta_0)^* d\phi\right| = \left|\int_{\Gamma_s} (\Psi + \eta_0)^* \phi\right| \leq K_\phi \int_{\Gamma_s} |\nabla \Psi + \eta_0| \ .$$

Arguing as in the proof of (VIII.8), for any ε, we can find s such that $\int_{\Gamma_s} |\nabla\Psi + \eta_0| \leq \varepsilon$ and we then deduce (VIII.14). Thus in $B_{r_0}^{2p}(x_0) \setminus \Psi + \eta_0(\partial\Sigma)$, $(\Psi + \eta_0)_*[\Sigma]$ is an integer-multiplicity, rectifiable, holomorphic cycle. Using the results of Harvey-Shiffman and King ([HS] and [Ki]) we have that there exists a compact Riemann surface with boundary Σ' and a holomorphic map Ψ' such that $(\Psi+\eta_0)_*[\Sigma] = \Psi'_*[\Sigma']$. Therefore $(\phi+\eta)(\Sigma)$ is a holomorphic curve — with boundary — in $\mathbb{C}^{2p}$. We claim that $\Psi+\eta_0$ is a holomorphic simple covering of Σ'. Indeed letting ω_Σ be the pull-back by Ψ of the symplectic form ω in $\mathbb{R}^{2p}$ we have $\int_\Sigma \omega_\Sigma = \int_\Sigma \Psi^*\omega = \int_\Sigma |\nabla\Psi|^2 \geq \pi Q r_0^2$, because of the monotonicity formula ($x_0 \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$). Let $\omega_{\Sigma'}$ be the restriction of ω_0 to Σ'. We have $\int_\Sigma (\Psi+\eta_0)^*\omega_{\Sigma'} = \int_\Sigma(\Psi+\eta_0)^*\omega_0 = \int_\Sigma |\nabla(\Psi+\eta_0)|^2$. Because of (VIII.1), the holomorphic covering $\Psi+\eta_0$ from Σ onto Σ' satisfies

$$\left|\int_\Sigma \omega_\Sigma - \int_\Sigma (\Psi+\eta_0)\omega_{\Sigma'}\right| = o_{r_0}\left(\int_\Sigma \omega_\Sigma\right) . \tag{VIII.16}$$

Therefore, for r_0 small enough this covering has to be a simple one and Σ is a compact Riemann surface. Now Ψ is a J-holomorphic map from a compact Riemann surface Σ into (B^{2p}, J); it is then smooth and $C \llcorner B^{2p}_{r_0}$ is a J-holomorphic curve.

A. Appendix

LEMMA A.1. *Let U be an open subset of $\mathbb{R}^2$, let $0<\lambda<1$ and let $(B^2_{r_i}(z_i))_{i\in I}$ a covering of U which is locally finite. There exists $n \in \mathbb{N}$ such that*

$$\forall z \in U, \qquad \mathrm{Card}\left\{i \in I : z \in B^2_{r_{z_i}}(z_i)\right\} \leq N . \tag{A.1}$$

Moreover one assumes that

$$\forall i,j \in I, \qquad B_{r_i}(z_i) \cap B_{r_j}(z_j) \neq \emptyset \quad \Longrightarrow \quad r_i \geq \lambda r_j . \tag{A.2}$$

Then there exist δ and $P \in \mathbb{N}$ depending on λ only such that

$$\forall i \in I, \qquad \mathrm{Card}\left\{j \in I : B_{r_j}(z_j) \cap B_{(1+\delta)r_i}(z_i) \neq \emptyset\right\} \leq P . \tag{A.3}$$

Proof of Lemma A.1. We argue by contradiction. Assume there exist $\delta_n \to 0$, a sequences of coverings of U, $(B^2_{r_{n,i}}(z_{n,i}))$ for $i \in I$ satisfying (A.1) and (A.2) and a sequence of indices i_n such that

$$\mathrm{Card}\left\{j \in I : B_{r_{j,n}}(z_{j,n}) \cap B_{(1+\delta_n)r_{i_n,n}}(z_{i_n,n}) \neq \emptyset\right\} \to +\infty \quad \text{as } n \to +\infty . \tag{A.4}$$

After a possible rescaling of the whole covering and a translation we can assume that $r_{i_n,n} = 1$ and $z_{i_n,n} = 0$. Also, after extraction of a subsequence, we can ensure that there exists $A \in \partial B_1(0)$ such that for any $r > 0$

$$\mathrm{Card}\left\{j \in I : B_{r_{j,n}}(z_{j,n}) \cap B_r(A) \neq \emptyset\right\} \to +\infty \quad \text{as } n \to +\infty . \tag{A.5}$$

For a given r and n we take the longest sequence of distinct balls of our covering $B_{r_{j_p,n}}(z_{j_p,n})$ for $p = 0 \ldots P_n$ satisfying

i) $\qquad B_{r_{j_0,n}}(z_{j_0,n}) = B_1(0)$,

$$\text{ii)} \qquad \forall p \le P_n - 1, \quad B_{r_{j_p,n}}(z_{j_p,n}) \cap B_{r_{j_{p+1},n}}(z_{j_{p+1},n}) \neq \emptyset \ ,$$

$$\text{iii)} \qquad \forall p \le P_n, \qquad B_{r_{j_p,n}}(z_{j_p,n}) \cap B_r(A) \neq \emptyset \ .$$

It is clear that for a given r, $P_n \to +\infty$; indeed if it were not the case, i.e. $P_n \le P_* < +\infty$, this would imply that the minimal radius for the balls of the covering intersecting $B_r(A)$ is λ^{P_*}. Combining this fact with (A.5) would contradict (A.1). Therefore we can find $r_m \to +\infty$ and $n_m \to +\infty$ as $m \to +\infty$ and sequences $(B_{r_{j_p,n_m}})$ for $1 \le p \le Q_m$ and for $m = 0 \cdots + \infty$ such that

$$\text{i)} \qquad B_{r_{j_0,n_m}}(z_{j_0,n_m}) = B_1(0) \ ,$$

$$\text{ii)} \qquad \forall p \le Q_{m-1}, \quad B_{r_{j_p,n_m}}(z_{j_p,n_m}) \cap B_{r_{j_{p+1},n_m}}(z_{j_{p+1},n_m}) \neq \emptyset \ , \tag{A.6}$$

$$\text{iii)} \qquad \forall p \le Q_m, \qquad B_{r_{j_p,n_m}}(z_{j_p,n_m}) \cap B_{r_m}(A) \neq \emptyset \ , \tag{A.7}$$

$$\text{iv)} \qquad Q_m \to +\infty \ .$$

Since $\lambda \le r_{j_1,n_m} \le \lambda^{-1}$ and since the distance $|z_{j_1,n_m}|$ is bounded, we can extract from n_m a subsequence that we still denote n_m such that $B_{r_{j_1,n_m}}(z_{j_1,n_m})$ converges to a limiting ball $B_{r_{1,\infty}}(z_{1,\infty})$ with $\lambda \le r_{1,\infty} \le \lambda^{-1}$, $z_{1,\infty} \le 2$ and $A \in \overline{B_{r_{1,\infty}}(z_{1,\infty})}$. This procedure can be iterated and using a diagonal argument we can assume that

$$\forall p \in \mathbb{N}, \quad r_{j_p,n_m} \to r_{p,\infty}, \quad z_{j_p,n_m} \to z_{p,\infty}$$

such that

$$\forall p \in \mathbb{N}, \quad \lambda^p \le r_{p,\infty} \le \lambda^{-p}, \quad |z_{p,\infty}| \le 2 \quad \text{ and } A \in \overline{B_{r_{p,\infty}}(z_{p,\infty})} \ .$$

Moreover because of (A.1) we have that

$$\forall z \in \mathbb{R}^2, \qquad \text{Card}\left\{p \in \mathbb{N} : z \in B^2_{r_{p,\infty}}(z_{p,\infty})\right\} \le N \ . \tag{A.8}$$

Because of this latter fact, since $A \in \overline{B_{r_{p,\infty}}(z_{p,\infty})}$ for all p, it is clear that $r_{p,\infty} \to +\infty$ as $p \to +\infty$. Because of (A.8) again, the number of open balls $B_{r_{p,\infty}}(z_{p,\infty})$ containing A is bounded by N and we can therefore forget them while considering the sequence and assume that

$$\forall p, \qquad A \in \partial B^2_{r_{p,\infty}}(z_{p,\infty}) \ .$$

Let $\vec{t_p}(A) \in S^1$ be the unit exterior normal to $\partial B^2_{r_{p,\infty}}(z_{p,\infty})$ at A. Let $\vec{t_\infty}$ be an accumulation unit vector of the sequence $\vec{t_p}(A)$. Given a direction $\vec{t}$ and an open disk containing A in its boundary and whose exterior normal at A is given by $\vec{t}$ any other open disk containing A in its boundary and whose exterior unit at A is not $-\vec{t}$ as a nonempty intersection with that disk. When $B^2_{r_{p_0,\infty}}(z_{p_0,\infty})$, such that $\vec{t_{p_0}}(A) \neq -\vec{t_\infty}$, there exist infinitely many disks $B^2_{r_p,\infty}(z_{p,\infty})$ having nonempty intersections with $B^2_{r_{p_0,\infty}}(z_{p_0,\infty})$. But then, because of (A.2), that passes to the limit and all these infinitely many disks have radii which are bounded from below by positive numbers and this contradicts the fact that $r_{p,\infty} \to +\infty$ as implied by (A.8). Thus Lemma A.1 is proved. □

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ETH Zentrum, Zürich, Switzerland
E-mail address: riviere@math.ethz.ch

Princeton University, Princeton NJ
E-mail address: tian@math.princeton.edu

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