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On Serre's conjecture for 2-dimensional mod p representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

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Abstract

We prove the existence in many cases of *minimally ramified* p-adic lifts of 2-dimensional continuous, odd, absolutely irreducible, mod p representations $\bar{\rho}$ of the absolute Galois group of $\mathbb{Q}$. It is predicted by Serre's conjecture that such representations arise from newforms of optimal level and weight.

Using these minimal lifts, and arguments using compatible systems, we prove some cases of Serre's conjectures in low levels and weights. For instance we prove that there are no irreducible (p, p) type group schemes over $\mathbb{Z}$. We prove that a $\bar{\rho}$ as above of Artin conductor 1 and Serre weight 12 arises from the Ramanujan Delta-function.

In the last part of the paper we present arguments that reduce Serre's conjecture to proving generalisations of modularity lifting theorems of the type pioneered by Wiles.

1. Introduction

Consider an absolutely irreducible, 2-dimensional, odd representation $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ with $\mathbb{F}$ a finite field of characteristic p, and $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ the absolute Galois group of $\mathbb{Q}$. By odd we mean that $\det(\bar{\rho}(c)) = -1$ for a complex conjugation $c \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. We say that such a representation is of *Serre type* or S-type. We abbreviate $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to $G_{\mathbb{Q}}$.

In [43], Serre defines for an S-type representation $\bar{\rho}$ two invariants: the level $N(\bar{\rho})$ which is the (prime to p) Artin conductor of $\bar{\rho}$, and (Serre) weight $k(\bar{\rho})$. Serre has conjectured in [43] that such a $\bar{\rho}$ *arises from* (with respect to some fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$) a newform f of weight $k(\bar{\rho})$ and level $N(\bar{\rho})$. We fix embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ for all primes p hereafter, and when we say (a place above) p, we will mean the place induced by this embedding. By *arises from f* we mean that there is an integral model $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ of

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the p-adic representation ρ_f associated to f, such that $\bar{\rho}$ is isomorphic to the reduction of ρ modulo the maximal ideal of $\mathcal{O}$. For brevity we refer to this as the S-conjecture.

The main technique which is presented in this paper results in the reduction of proving this conjecture to proving a certain *modularity lifting conjecture* that we formulate below.

We say that $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ is *modular* if either it is absolutely irreducible and isomorphic to the reduction of (an integral model of) ρ_f modulo the maximal ideal of $\mathcal{O}$ for some newform f as above, or it is reducible over $\overline{\mathbb{F}_p}$ and odd.

Modularity lifting conjecture (MLC). *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ be a continuous, absolutely irreducible, p-adic representation that is odd* $(\det(\rho(c)) = -1)$, *ramified at finitely many primes, and de Rham at p with Hodge-Tate weights $(k-1, 0)$ with $k \geq 2$. Assume that the reduction of ρ is modular. Then ρ is isomorphic to an integral model of a p-adic representation ρ_f arising from a newform f.* (*The oddness and modularity hypotheses are expected to be superfluous*; *see* [21].)

The first cases of MLC were proved by Wiles, Taylor-Wiles [55], [54]. In their work the conditions at p imposed on ρ were much more stringent than in the conjecture above. There is important work of Kisin [30] that makes serious inroads into allowing more complicated behavior at p of ρ as in the MLC. He proves this conjecture assuming that ρ is potentially crystalline with Hodge-Tate weights $(0, 1)$, $p > 2$, and $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$, the restriction of $\bar{\rho}$ to $G_{\mathbb{Q}(\mu_p)}$, is irreducible.

The ideas of this paper also lead to unconditional proofs of the S-conjecture in low levels and low weights.

The main results of this paper are:

(1) Liftings of $\bar{\rho}$, with $2 \leq k(\bar{\rho}) \leq p+1$, to *minimally ramified representations* ρ (see Theorem 3.3) when $p > 2$ and $k(\bar{\rho}) \neq p$, and $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ absolutely irreducible. These lifts have the property that their conductor $N(\rho) = N(\bar{\rho})$, and they are crystalline at p of Hodge-Tate weight $(0, k(\bar{\rho}) - 1)$. This is predicted by the S-conjecture as if a S-type representation arises from a newform f in $S_{k(\bar{\rho})}(\Gamma_1(N(\bar{\rho})))$, then it has a p-adic lifting $\rho = \rho_f$ that has conductor $N(\rho) = N(\bar{\rho})$ and is crystalline at p of Hodge-Tate weights $(0, k(\bar{\rho}) - 1)$.

(2) Proof of the S-conjectures in low levels and weights (see Theorem 5.2, 5.4 and 5.6). In Theorems 5.2, 5.4 we verify that S-type $\bar{\rho}$ of certain invariants $N(\bar{\rho}), k(\bar{\rho})$ do not exist. This is predicted by the S-conjecture as the corresponding space of cusp forms $S_{k(\bar{\rho})}(\Gamma_1(N(\bar{\rho})))$ is 0.

(3) The reduction of the S-conjecture to the modularity lifting conjecture.

The deduction of the second and third results from the first is based on:

(i) Known cases of the modularity lifting conjecture as in [55], [54], [45], [46], for (1) implies (2).

(ii) The potential version of Serre's conjecture (potential modularity) proved by Taylor in [51], [50].

(iii) A result of Dieulefait [19] that makes the minimal lifting ρ part of a compatible system using (ii) and Brauer's theorem following a method of Taylor (see Theorem 6.6 of [50] and 5.3.3 of [53]) (see also [56]).

(iv) Results of Fontaine, Brumer-Kramer and Schoof [20], [11], [39] which determine semistable abelian varieties over $\mathbb{Q}$ of small conductor.

We say a word about our proof of minimal liftings (see Theorem 3.3 and its proof) and its background. These are deduced from proving that a certain deformation ring $R_{\mathbb{Q}}$ is finite, flat over $\mathbb{Z}_p$.

(a) The finiteness follows from combining Taylor's potential modularity result with modularity lifting results over totally real fields that are proved by Fujiwara [22].

(b) After this the flatness follows from arguments of Böckle that present $R_{\mathbb{Q}}$ in a way so that the number of relations is bounded above by the number of generators [7].

This argument for producing minimal liftings has been suggested in Remark in §5.2 of [27]. The basic principle we exploit here when producing liftings, of proving a finiteness property of a deformation ring, and hence by obstruction theory arguments its flatness, goes back at least to de Jong's paper [16] which is in the setting of function fields. Our proof of Theorem 3.3 proceeds by observing the relevance of the principle in the present context as results stemming from Wiles' breakthrough [55] allow one to prove finiteness of $R_{\mathbb{Q}}$ as in (a) above. The base change arguments used in [16] also influence the work here.

The work of Ramakrishna is an important percursor to this work. In [34] he has produced liftings for odd and even $\bar{\rho}$ to Witt vectors. (For the lifts we produce in Theorem 3.3, their rationality cannot be controlled.) But his lifts are not in general minimally ramified. His ingenious method is purely Galois cohomological, while our methods are more indirect and work only in the odd case.

We end by proposing an inductive approach to the S-conjecture. There are two types of induction involved, one on the number of primes ramified in the residual representation (see Theorem 6.2), and the other on the residual characteristic p of the representation (see Theorem 6.1). For the induction we need a starting point and that is provided by results of Tate and Serre, [47] and [42] page 710. They prove the conjecture for $\bar{\rho}$ of S-type with $N(\bar{\rho}) = 1$ in residue characteristics 2 and 3.

We use at many places in the arguments of this paper ideas or themes that we have learnt from Serre's work. His conjectures in [43] have been a great source of inspiration for people in the field. At a more technical level the work here is influenced by his specification of the weight in [43], and his results on relation between changing weight and p-part of the level, see Théorème 11 of [41]. Further his proof of the level one case of his conjectures for $p = 3$ provides us the toe-hold ("prise d'ongle") in our proposed attack on his conjecture.

Note added in revision. Dieulefait has independently noticed that existence of minimal liftings implies the S-conjecture for $\bar{\rho}$ of weight 2 and small level. He had sought to deduce the existence of minimal lifts from Taylor's potential modularity result. We refer the interested reader to his paper arXiv:math/0412099v1, and its subsequent versions available at `http://lanl.arxiv.org/abs/math/0412099`.

A version of the present paper was circulated in December 2004 (see arXiv:math/0412076v1). (As some of the results of this earlier version are referred to in the literature we indicate when appropriate below the earlier numbering of these results.) At the time of this revision in the summer of 2007, many developments have overtaken some of the work of this paper. These developments grow out of the seeds sown here.

The S-conjecture in the level one case was proven in [25] using the broad strategy outlined here. The main innovation of [25] was to modify the strategy in a way that only known cases of the modularity lifting conjecture sufficed. This was accomplished by using various liftings of a given S-type $\bar{\rho}$, an idea that in a nascent form was introduced in this paper to deduce some higher weight cases from the weight 2 case (see Theorems 5.4, 5.2). The various liftings are produced using the basic method of the proof of Theorem 3.3 which is developed still further in [28], [29] to produce liftings of $\bar{\rho}$ with every possible inertial behavior at primes away from p. The key idea of induction on the prime p introduced in Theorem 6.1 is crucial in [25].

Subsequently in [28] and [29], the authors proved the S-conjecture for (i) $p > 2$, $N(\bar{\rho})$ odd, and (ii) $p = 2, k(\bar{\rho}) = 2$, using the work of [25] and the killing ramification idea in the proof of Theorem 6.2. The general case was reduced to proving the modularity lifting conjecture when $p = 2, k = 2$ and ρ has nonsolvable image. This has been proven by Kisin [31], extending the results of [30] which were for $p > 2$ to the case of $p = 2$, thus finally proving the S-conjecture.

Although the MLC seemed a distant goal at the end of 2004, there have been rapid strides taken towards it by Kisin, Emerton and others using the p-adic Langlands program of Breuil, and important developments in it due to Colmez and others. The approach presented here to the S-conjecture assuming the MLC is very direct, as compared to the winding route taken in the subsequent papers. Because of the progress towards the MLC it might get converted from a blueprint into an actual proof.

1.1. *Notation and terminology.* For F a field, $\mathbb{Q} \subset F \subset \overline{\mathbb{Q}}$, we write G_F for the Galois group of $\overline{\mathbb{Q}}/F$. For λ a prime/place of F, we mean by D_λ (resp., I_λ) a decomposition (resp., inertia) subgroup of G_F at λ. We fix embeddings ι_p, ι_∞ of $\overline{\mathbb{Q}}$ in its completions $\overline{\mathbb{Q}_p}$ and $\mathbb{C}$. Denote by χ_p the p-adic cyclotomic character, and ω_p the Teichmüller lift of the mod p cyclotomic character $\overline{\chi}_p$ (the latter being the reduction mod p of χ_p). By abuse of notation we also denote by ω_p the ℓ-adic character $\iota_\ell \iota_p^{-1}(\omega_p)$ for any prime ℓ: this should not cause confusion as from the context it will be clear where the character is valued. For a number field F we denote the restriction of a character of $G_\mathbb{Q}$ to G_F by the same symbol. We denote by $\mathbb{A}_F$ the adeles of F.

Consider a totally real number field F. Recall that in [48], [49], 2-dimensional p-adic representations ρ_π of G_F are associated to cuspidal automorphic representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ that are discrete series at infinity of weight $(k, \ldots, k)$, $k \geq 2$. For a place v above p we say that the local component π_v at v of π is ordinary if the corresponding eigenvalue of the Hecke operator (T_v or U_v) acting on the representation space of π_v is a unit (with respect to the chosen embedding ι_p). If π_v is ordinary, so is $\rho_\pi|_{D_v}$ in the sense of Definition 3.1 below.

We say that $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$, with $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_p$, is modular if it is isomorphic to (an integral model of) such a ρ_π, and a compatible system of 2-dimensional representations of G_F is modular if one member of the system is modular. We say that $\bar{\rho} : G_F \to \mathrm{GL}_2(\mathbb{F})$, with $\mathbb{F}$ a finite field of characteristic p, is modular if either it is irreducible and isomorphic to the reduction of (an integral model of) such a ρ_π modulo the maximal ideal of $\mathcal{O}$, or it is reducible and totally odd (i.e., $\det(\bar{\rho}(c)) = -1$ for all complex conjugations $c \in G_F$). We denote by $\mathrm{Ad}^0(\bar{\rho})$ the trace zero matrices of $M_2(\mathbb{F})$ and regard it as a G_F-module via the composition of $\bar{\rho}$ with the conjugation action of $\mathrm{GL}_2(\mathbb{F})$ on $M_2(\mathbb{F})$.

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2. Taylor's potential modularity result

The following result of Taylor, proving a potential version of Serre's conjecture, is important for the work of this paper. We just indicate the adjustments needed in the arguments in Taylor's papers [51] and [50] to derive the result in the form we state it. Note that Theorem 2.1 was Proposition 2.5 in an earlier version of the paper.

THEOREM 2.1 (Taylor). *Assume $\bar\rho$ is of S-type in odd residue characteristic, such that $\bar\rho|_{\mathbb{Q}(\mu_p)}$ is irreducible, $2 \le k(\bar\rho) \le p+1$, and $k(\bar\rho) \ne p$. Then there is a totally real field F that is Galois over $\mathbb{Q}$ of even degree, unramified above p, and even split above p if $\bar\rho|_{D_p}$ is irreducible, $\mathrm{im}(\bar\rho) = \mathrm{im}(\bar\rho|_{G_F})$, and $\bar\rho|_{G_{F(\mu_p)}}$ absolutely irreducible such that:*

(i) *$\bar\rho|_{G_F}$ arises from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ that is unramified at all finite places, and is discrete series of weight $k(\bar\rho)$ at the infinite places. If $\bar\rho$ is ordinary at p, then for all places v above p, π_v is ordinary (in the sense defined in* §1.1).

(ii) *$\bar\rho|_{G_F}$ also arises from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ that is unramified at all finite places not above p, and such that π_v, at all places v above p, is of conductor dividing v (and is unramified if $\bar\rho$ is finite flat at v), and is of weight 2 at the infinite places. Further π_v is ordinary at all places v above p in the case when $\bar\rho$ is ordinary at p.*

Proof. This is proved in [25], but the proof there relies on some arguments which appeared in a first version of the present article. We thus present these arguments repeating also parts of the proof in [25] for intelligibility.

The property that $\mathrm{im}(\bar\rho) = \mathrm{im}(\bar\rho|_{G_F})$ is ensured if F is linearly disjoint from the fixed field of kernel of $\bar\rho$. We use the refinement of Moret-Bailly's theorem (see Theorem G of [51]) given in Proposition 2.1 of [23] to ensure that the number fields we consider below (F'', E, F) have this property. In the case when the projective image of $\bar\rho$ is dihedral, we ensure that these fields are split at a prime which splits in the field cut out by the projectivisation of $\bar\rho$, but which is inert in the quadratic subfield of $\mathbb{Q}(\mu_p)$. This ensures that $\bar\rho|_{G_{F(\mu_p)}}$ is irreducible for all the number fields (F'', E, F) considered below.

The supersingular case is covered in [50] explicitly (see Theorem 5.7 of [50]) for $p > 3$ and it is explained in [25] how to extend this to the case $p = 3$.

The ordinary case may be deduced from the arguments of [51] although not explicitly there. Thus we treat below only the case when $\bar\rho$ is ordinary.

Let us borrow for a moment the notations of Taylor [51], even if it contradicts the notations of this paper.

Thus $\bar\rho$ is now a mod ℓ representation. Suppose $\bar\rho|_{I_\ell}$ is of the form

$$\begin{pmatrix} \overline{\chi}_\ell^{k(\bar\rho)-1} & * \\ 0 & 1 \end{pmatrix}.$$

There is a real quadratic extension F'' of $\mathbb{Q}$ (disjoint from the fixed field of the kernel of $\bar\rho$) in which ℓ is unramified and inert such that $\chi \otimes \bar\rho|_{G_{F''}}$ restricted to a decomposition D at the place of F'' above ℓ is of the form

$$\begin{pmatrix} \chi|_D^{-1}\overline{\chi_\ell} & * \\ 0 & \chi|_D \end{pmatrix},$$

for some mod ℓ character χ of $G_{F''}$ such that $\chi^{-2}|_I = \overline{\chi}_\ell{}^{k(\bar\rho)-2}|_I$, with I an inertia subgroup at the place above ℓ of F''.

Applying Moret-Bailly's theorem as in [51] to $\chi\otimes\bar\rho|_{G_{F''}}$, and after a twist, we get an abelian variety A with the following properties:

- A is defined over a totally real field E, unramified above p, that contains F''.
- The abelian variety A is of Hilbert-Blumenthal type with multiplication by the real field M.
- There is a prime λ of M above ℓ such that the restriction of $\bar\rho$ to G_E is isomorphic to the G_E-representation on $A[\lambda]$, the points of A killed by λ.
- The compatible system of G_E-representations attached to A arises from a cuspidal automorphic representation π_A of $\mathrm{GL}_2(\mathbb{A}_E)$ of parallel weight 2.

Let x a place of E above ℓ. We use Lemma 1.5 of [51] to get the needed information for $(\pi_A)_x$, namely we will prove that it is ordinary (with respect to the place λ of M). Let $n = \ell - k(\bar\rho)+1$ if $k(\bar\rho)\neq 2$ and $n=0$ if $k(\bar\rho)=2$. Note that n is as in Lemma 1.5 of [51]. Note that as we are assuming $k(\bar\rho)\neq\ell$, we have $n\neq 1$ and Lemma 1.5 applies. We have $0\leq n<\ell-1$, and we are in the situation of the proof of Lemma 1.5 of [51, p. 137]. For a place x of E above l, the λ-adic representation arising from A when restricted of the decomposition group G_x is of the form:

$$\begin{pmatrix} \varepsilon_\ell\chi_1 & * \\ 0 & \chi_2 \end{pmatrix},$$

with χ_2 unramified and the restriction of χ_1 to the inertia subgroup I_x of G_x is ω_ℓ^{-n}, ε_ℓ being the ℓ-adic cyclotomic character. We know by the proof of Lemma 1.5 that A has mutiplicative reduction over E_x or good reduction over $E_x(\zeta_\ell)$. Furthermore, there is a prime $\wp$ of M above $p\neq\ell$ such that the action of G_x on $A[\wp]$ has the form $\psi_1\oplus\psi_2$, with ψ_2 unramified and the restriction of ψ_1 to I_x is ω_ℓ^{-n} abusing notation as signalled in Section 1.1.

In the case $n=0$ ($k(\bar\rho)=\ell+1$ or 2), we see by looking at the Tate module $T_\wp(A)$ that A has semistable reduction over E_x. If $k(\bar\rho)=\ell+1$, A has mutiplicative reduction at all x over ℓ and $(\pi_A)_x$ is Steinberg. When $k(\bar\rho)=2$, and $\chi_1\chi_2^{-1}\neq 1$, Taylor finds, for a place v of F above ℓ, an abelian variety A_v over F_v with ordinary good reduction. The theorem of Moret-Bailly [33] produces for us an abelian variety A with good reduction at all primes x of E above ℓ such that the restriction of $\bar\rho$ to G_E is isomorphic to the

G_E-representation on $A[\lambda]$. We see that, if we choose A like this, A has good ordinary reduction at x and $(\pi_A)_x$ is unramified.

If $k(\bar{\rho}) = 2$ and $\chi_1\chi_2^{-1} = 1$, we are in the case $\chi_v^2 = 1$ of the proof of Lemma 1.2 of Taylor. But, as the restriction of $\bar{\rho}$ to G_v comes from a finite flat group scheme over the ring of integers of F_v, we can choose the abelian variety A_v that figures in Lemma 1.2 to have good ordinary reduction, by the same arguments as Taylor uses when $\chi_v^2 \neq 1$. This is because the class in $H^1(G_v, O_M/\lambda(\epsilon))$ of the extension defined by $\bar{\rho}_{|G_v}$ comes from units by Kummer theory. (See also proof of Theorem 6.1 of [29] for more details.) Then, as above, we can choose A with good ordinary reduction at all places x of E above ℓ. Then $(\pi_A)_x$ is unramified at these places.

Suppose now $n \neq 0$. Then, looking at the Tate module $T_\wp(A)$, we see that the abelian variety A has good reduction over $E_x(\zeta_\ell)$. Let $A[\lambda]^0$ and $A[\lambda]^{\text{et}}$ be the connected and étale components of the λ-kernel of the reduction at x of the Néron model of A over $E_x(\zeta_\ell)$. Let $T_\lambda(A)$, $T_\lambda^0(A)$ and $T_\lambda^{\text{et}}(A)$ be the corresponding Tate-modules and let D, D^0 and D^{et} the corresponding Dieudonné modules. We have $D = D^0 \oplus D^{\text{et}}$. Taylor proves in Lemma 1.5 that I_x acts on $\text{Lie}(A[\lambda]^0)$ by multiplication by ω_ℓ^{-n} and trivially on $A[\lambda]^{\text{et}}$. As the action of I_x on D factors through $\text{Gal}(E_x(\zeta_\ell)/E_x)$, it follows that I_x acts on D by multiplication by ω_ℓ^{-n} on D^0 and trivially on D^{et}. By the appendix B in Conrad-Diamond-Taylor [14] it follows that the action of the Weil-Deligne group WD_x on the compatible system of Galois representations attached to A factors through the Weil group. Further this has the form $\eta_1 \oplus \eta_2$, with η_2 unramified and $\eta_2(\text{Frob}_x)$ a λ-adic unit, and the restriction of η_1 to I_x is ω_ℓ^{-n}. It follows that (π_A) is ordinary at x.

We revert now to the notation of the present paper, i.e., ℓ is now p.

To summarise we get that there exists a totally real number field E/F'' (disjoint from the fixed field of the kernel of $\bar{\rho}|_{G_{F''}}$), unramified at the places above the place of E above p, and an abelian variety A over E with endomorphisms by some number field M with $[M : \mathbb{Q}] = \dim(A)$ with the following properties. Firstly, the mod p representation (with respect to the embedding ι_p) that arises from A is isomorphic to $\bar{\rho}|_{G_E}$. Secondly, at places $\wp$ of E above p, the abelian variety A has

- good ordinary reduction over $E_\wp(\zeta_p)$ if $2 < k(\bar{\rho}) < p+1$,
- good ordinary reduction over $E_\wp$ if $k(\bar{\rho}) = 2$,
- semistable reduction over $E_\wp$ if $k(\bar{\rho}) = p+1$.

Further A arises from a cuspidal automorphic representation π_A of $\text{GL}_2(\mathbb{A}_E)$, in the sense that the compatible systems they give rise to are isomorphic, and $\pi_{A,v}$ is ordinary at all places v above p. The conductor of $\pi_{A,v}$ divides v, and $\pi_{A,v}$ is unramified if $\bar{\rho}$ is finite flat at v, for all places v above p. The cuspidal automorphic representation π_A is of weight 2 at the infinite places.

Now using the main theorem of [45], we construct a totally real, solvable extension F/E that is unramified at places above p, Galois over $\mathbb{Q}$ and such that $\bar{\rho}|_{G_F}$ arises (with respect to ι_p) from a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ that is:

- unramified at all finite places not above p,
- π_v is ordinary at all places v of F above p of conductor dividing v (and is unramified if $\bar{\rho}$ is finite flat at v),
- and is of weight 2 at the infinite places.

This proves part (ii) of the theorem when $\bar{\rho}$ is ordinary at p.

Part (i) in the ordinary case follows from this using Corollary 3.5 of [24]. □

3. Lifts of mod p Galois representations

3.1. *Minimal lifts.* Let p an odd prime. Let $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}_p})$ be an S-type representation. We assume that the Serre weight $k(\bar{\rho})$ is such that $2 \leq k(\bar{\rho}) \leq p+1$, $k(\bar{\rho}) \neq p$ and $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is absolutely irreducible. Note that there is always a twist of $\bar{\rho}$ by some power of the mod p cyclotomic character $\overline{\chi_p}$ that has weights between 2 and $p+1$.

We make a definition that helps fix some terminology.

Definition 3.1. Let E, F be finite extensions of $\mathbb{Q}_p$, and $\mathcal{O}$ the ring of integers of E.

1. Suppose V is a 2-dimensional continuous representation with coefficients in E of G_F. We say that V is of weight k if for all embeddings $\iota : E \hookrightarrow \mathbb{C}_p$, $V \otimes_E \mathbb{C}_p = \mathbb{C}_p \oplus \mathbb{C}_p(k-1)$ as G_F-modules.

2. Suppose V is a continuous representation, with V a free rank 2 module over a complete Noetherian local $\mathcal{O}$-algebra R. We say that V is ordinary if there is a free, rank one submodule W of V that is G_F stable, such that V/W is free of rank one over R with trivial action of the inertia I_F of G_F and the action of an open subgroup of I_F on W is by χ_p^a, for a a rational integer ≥ 0. If $\rho : G_F \to \mathrm{GL}_2(E)$ is a continuous p-adic representation, with E a finite extension of $\mathbb{Q}_p$, we say that ρ is ordinary if an integral model of ρ is ordinary.

Let $\mathbb{F} \subset \overline{\mathbb{F}_p}$ be a finite field such that the image of $\overline{\rho}$ is contained in $\mathrm{GL}_2(\mathbb{F})$, and let W be the Witt vectors $W(\mathbb{F})$. By a *lift* of $\overline{\rho}$, we mean a continuous representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a finite extension of the field of fractions of W, such that the reduction of ρ modulo the maximal ideal of $\mathcal{O}$ is isomorphic to $\overline{\rho}$.

Let ρ be such a lift and let ℓ be a prime. One says that ρ is *minimally ramified at* ℓ if it satisfies the following conditions:

– When $\ell \neq p$, it is minimally ramified at ℓ in the terminology of [18]. In particular, if $\overline{\rho}$ is unramified at ℓ, ρ is unramified at ℓ. More generally, when the image of I_ℓ is of order prime to p, $\rho(I_\ell)$ is isomorphic to its reduction $\bar{\rho}(I_\ell)$.
– When $\ell = p$: If $k(\overline{\rho}) \neq p+1$, ρ is minimally ramified at p if ρ is crystalline of weights $(0, k(\overline{\rho})-1)$. If $k(\overline{\rho}) = p+1$, ρ is *minimally ramified of semi-stable type* if ρ is semi-stable non-crystalline of Hodge-Tate weights $(0,1)$; ρ is *minimally ramified of crystalline type* if ρ is crystalline of Hodge-Tate weights $(0,p)$.

We say that a lift ρ of $\bar{\rho}$ is *minimal*, or *minimally ramified*, if it is minimally ramified at all primes ℓ.

The determinant of $\bar{\rho}$ is $\overline{\chi_p}^{k(\bar{\rho})-1}\epsilon$ where ϵ is a character of conductor prime to p ([43]). For $\ell \neq p$, the restriction to I_ℓ of the determinant of a minimal lift of $\bar{\rho}$ is the Teichmuller lift ([18]). A semi-stable representation of I_p of Hodge-Tate weights $(0, k-1)$ has determinant χ_p^{k-1}. So we see that a minimal lift of $\bar{\rho}$ (of crystalline type if $k(\bar{\rho}) = p+1$) has determinant $\chi_p^{k(\bar{\rho})-1}\widehat{\epsilon}$, where $\widehat{\epsilon}$ is the Teichmüller lift of ϵ. If $k(\bar{\rho}) = p+1$, a minimal lift of semi-stable type has determinant $\chi_p\widehat{\epsilon}$.

Let us make a few comments on the condition for $\ell = p$, $k(\bar{\rho}) = p+1$. Let $\chi_p : G_{\mathbb{Q}} \to \mathbb{Z}_p^*$ be the p-adic cyclotomic character and $\overline{\chi_p}$ its reduction modulo p. If $k(\bar{\rho}) = p+1$, the restriction of $\bar{\rho}$ to the decomposition group D_p is of the form:

$$\begin{pmatrix} \overline{\chi_p}\epsilon & \eta \\ 0 & \epsilon \end{pmatrix},$$

where ϵ is an unramified character, and η is a “très ramifié” 1-cocycle, which corresponds via Kummer theory to an element of $\mathbb{Q}_p^* \otimes \mathbb{F}$ whose image by the map defined by the valuation of $\mathbb{Q}_p$ is a nonzero element of $\mathbb{F}$.

The lifting ρ is minimally ramified of semi-stable type if the restriction of ρ to I_p is of the form:

$$\begin{pmatrix} \chi_p & * \\ 0 & 1 \end{pmatrix}.$$

As Kummer theory easily shows, this implies that the restriction of ρ to the decomposition group D_p is of the form:

$$\begin{pmatrix} \chi_p\widehat{\epsilon} & * \\ 0 & \widehat{\epsilon} \end{pmatrix},$$

where $\widehat{\epsilon}$ is an unramified character lifiting ϵ. This is Proposition 6.1 of [17].

The lifting ρ is minimally ramified of crystalline type if the restriction of ρ to I_p is of the form:

$$\begin{pmatrix} \chi_p^p & * \\ 0 & 1 \end{pmatrix}.$$

Indeed, by Bloch and Kato (3.9 of [6]), we know that such p-adic representations are exactly the crystalline reducible representations of D_p of Hodge-Tate weights $(0,p)$.

We record a result of Berger-Li-Zhu that we need several times later.

PROPOSITION 3.2 (Berger-Li-Zhu). *Let E be a finite extension of $\mathbb{Q}_p$.*

(i) *A reduction of an irreducible crystalline representation $\rho : D_p \to \mathrm{GL}_2(E)$ of Hodge-Tate weights $(0,p)$ is isomorphic to an unramified twist of $\mathrm{ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p}(\omega_2)$, where $\mathbb{Q}_{p^2}$ is the quadratic unramified extension of $\mathbb{Q}_p$ and ω_2 is the fundamental character of level 2. In particular, it is not isomorphic to a très ramifiée representation.*

(ii) *A reducible crystalline representation $\rho : D_p \to \mathrm{GL}_2(E)$ of D_p of Hodge-Tate weights $(0,p)$ is ordinary and the semisimplification of $\rho|_{I_p}$ is $\chi_p^p \oplus 1$. The reduction of (an integral model of) ρ is ordinary and has Serre weight either 2 or $p+1$.*

Proof. This follows from Corollary 4.1.3. and Proposition 4.1.4. of [4]. □

One of the main results of the paper is:

THEOREM 3.3. *Let $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ be of S-type, of residue characteristic $p > 2$, and such that $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is absolutely irreducible. We suppose that $2 \le k(\bar{\rho}) \le p+1$ and $k(\bar{\rho}) \neq p$. Then $\bar{\rho}$ has a lift ρ which is minimally ramified at every ℓ, and if the Serre weight is $k(\overline{\rho}) = p+1$, one can impose that ρ be either of crystalline type (of weight $p+1$) or of semi-stable type (of weight 2).*

We deduce this at the end of this section from the flatness of a certain deformation ring that we first define.

For the proof of the theorem, we have to consider minimally ramified deformations $G_{\mathbb{Q}} \to \mathrm{GL}_2(R)$ of $\overline{\rho}$. Here R is a complete Noetherian local W-algebra (CNL$_W$-algebra), with an isomorphism of $R/\mathcal{M}_R$ with $\mathbb{F}$ with $\mathcal{M}_R$ the maximal ideal of R (W is as above the Witt ring $W(\mathbb{F})$). A deformation of $\bar{\rho}$ is a continuous representation $\gamma : G_{\mathbb{Q}} \to \mathrm{GL}_2(R)$ such that γ mod $\mathcal{M}_R$ is $\bar{\rho}$, where we take γ up to conjugation by matrices that are 1 mod $\mathcal{M}_R$. We say that the deformation is minimally ramified, if:

– for $\ell \neq p$, γ is minimal in the sense of [18];

– if $k(\bar{\rho}) < p$, the restriction of γ to D_p comes from a Fontaine-Laffaille module (for the precise definition, see Section 2 of [35]);

– if $k(\bar{\rho}) = p+1$, the restriction of γ to I_p is of the form:

$$\begin{pmatrix} \chi_p^{k-1} & * \\ 0 & 1 \end{pmatrix},$$

with $k = p+1$ if we are in the crystalline type, and $k = 2$ if we are in the semi-stable type.

The condition of being minimally ramified is a deformation condition in the sense of [32], and hence the minimally ramified deformation problem has a universal object. More precisely, if $k(\bar{\rho}) \neq p+1$, there exists a universal minimally ramified deformation $\rho_{\rm univ} : G_{\mathbb{Q}} \to {\rm GL}_2(R_{\rm univ})$; if $k(\bar{\rho}) = p+1$, we have two universal rings $R_{\rm univ,ss}$ and $R_{\rm univ,crys}$. The determinant of $\rho_{\rm univ}$ is $\chi_p^{k-1}\widehat{\epsilon}$, with $k = k(\bar{\rho})$ except in the case $k(\bar{\rho}) = p+1$ and we are in the case of semi-stable type, and then $k = 2$.

Define for each ℓ, the W-algebra R_ℓ of versal deformations of $\overline{\rho}_{|D_\ell}$ which are minimally ramified (if $k(\bar{\rho}) = p+1$, we have to consider the two W-algebras $R_{p,\rm crys}$ and $R_{p,\rm ss}$) and such that the determinant is the restriction to D_ℓ of $\chi_p^{k-1}\widehat{\epsilon}$, with $k = k(\bar{\rho})$ except in the case $k(\bar{\rho}) = p+1$ and we are in the case of semi-stable type, and then $k = 2$.

We have the following result of Böckle that is very important for our needs:

PROPOSITION 3.4. *The W-algebra $R_{\rm univ}$ (resp. $R_{\rm univ,crys}, R_{\rm univ,ss}$) has a presentation as a* CNL *W-algebra as*

$$W[[X_1, \ldots, X_r]]/(f_1, \ldots, f_s)$$

with $r \geq s$.

Proof. We may deduce this from the method of Böckle (see for instance Proposition 1 in appendix to [26]) if we know that the W-algebras R_ℓ for all primes ℓ are flat, complete intersections of relative dimension

$$\dim_\kappa(H^0(D_\ell, {\rm ad}^0(\overline{\rho})) + \epsilon_\ell$$

with $\epsilon_\ell = 0$ if $\ell \neq p$ and $\epsilon_p = 1$. Except in the case of $k(\bar{\rho}) = p+1$ and R_ℓ is $R_{p,\rm crys}$, this property of R_ℓ follows from the work of Ramakrishna ([34]) and Taylor ([52]). (The previous sentences are justified in greater detail in §3 of [25].) For $R_\ell = R_{p,\rm crys}$ in the case when $k(\bar{\rho}) = p+1$, it is proved by Böckle (see also Proposition 3.5) that $R_{p,\rm crys}$ is a relative complete intersection of relative dimension 1 (Remark 7.5 (iii) of [8]). □

For the convenience of the reader we give a proof of this result of Böckle (Remark 7.5 (iii) of [8]) used above. The following proposition is Proposition 2.3 of an earlier version.

PROPOSITION 3.5. *In the case $k(\bar{\rho}) = p+1$, the W-algebra $R_{p,\rm crys}$ is formally smooth of dimension* 1.

Proof. We owe the following succinct proof to the referee. By a standard calculation, the tangent space of $R_{p,\rm crys}/(p)$ is of dimension 1 over $\mathbb{F}$. Thus

$R_{p,\mathrm{crys}}$ is a quotient of $W[[T]]$, and hence it suffices to show that for each très ramifié $\eta \in H^1(D_p, \mathbb{F}(\overline{\chi}_p))$ there are infinitely many unramified $\mu : D_p \to 1 + pW$ such that η is in the image of the map from $H^1(D_p, W(\chi_p^p \mu))$ induced by reduction mod p. This image is the same as that of $H^1(D_p, W/p^2(\chi_p^p \mu))$ (in particular it depends only on μ mod p^2). By Tate duality it is the orthogonal complement of the image of the connecting homomorphism $H^0(D_p, \mathbb{F}) \to H^1(D_p, \mathbb{F})$ defined by the extension $W/p^2(\chi_p^{1-p}\mu^{-1})$. The different choices of μ mod p give precisely the ramified lines in $H^1(D_p, \mathbb{F})$, and their orthogonal complements are precisely the très ramifié lines in $H^1(D_p, \mathbb{F}(\overline{\chi}_p))$. □

We will prove below that R_{univ} is finite as a $\mathbb{Z}_p$-module, or equivalently that $R_{\mathrm{univ}}/(p)$ is finite. The following criterion for this finiteness is very useful and is inspired by Lemma 3.15 of [16]. Note that this was Lemma 2.4 in an earlier version of the paper.

LEMMA 3.6. *Let κ be a finite field of characteristic p, G a profinite group satisfying the p-finiteness condition* (*Chapter* 1 *of Mazur* [32]) *and $\eta : G \to \mathrm{GL}_N(\kappa)$ be an absolutely irreducible continuous representation. Let $\mathcal{F}_N(\kappa)$ be a subcategory of deformations of η in κ-algebras which satisfy the conditions of* 23 *of* [32]. *Let $\eta_{\mathcal{F}} : G \to \mathrm{GL}_N(R_{\mathcal{F}})$ be the universal deformation of η in $\mathcal{F}_N(\kappa)$. Then $R_{\mathcal{F}}$ is finite if and only if $\eta_{\mathcal{F}}(G)$ is finite.*

Proof. It is clear that if $R_{\mathcal{F}}$ is finite, $\eta_{\mathcal{F}}(G)$ is finite. Let us suppose that $\eta_{\mathcal{F}}(G)$ is finite. As η is absolutely irreducible, a theorem of Carayol says that $R_{\mathcal{F}}$ is generated by the traces of the $\eta_{\mathcal{F}}(g)$, $g \in G$ ([13]). As $\eta_{\mathcal{F}}(G)$ is finite, for each prime ideal $\wp$ of $R_{\mathcal{F}}$, the images of these traces in the quotient $R_{\mathcal{F}}/\wp$ are sums of roots of unity, and there is a finite number of them. We see that $R_{\mathcal{F}}/\wp$ is a finite extension of κ. It follows that the noetherian ring $R_{\mathcal{F}}$ is of dimension 0, and so is finite. □

3.2. *Flatness of minimal deformation ring.* We deduce Theorem 3.3 from the following theorem as we explain at the end of this section.

THEOREM 3.7. *Let $\bar{\rho}$ as in Theorem* 3.3, *i.e.*, *of S-type*, *of residue characteristic $p > 2$, and such that $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is absolutely irreducible, with $2 \le k(\bar{\rho}) \le p+1$ and $k(\bar{\rho}) \neq p$. Then R_{univ}* (*if $k(\bar{\rho}) \neq p+1$*) *and $R_{\mathrm{univ,crys}}$ and $R_{\mathrm{univ,ss}}$* (*if $k(\bar{\rho}) = p+1$*) *are finite*, *flat W-modules*, *and complete intersections over W.*

Proof. From now on we denote by R_{univ} the deformation ring we consider. We prove in Proposition 3.8 below that R_{univ} is a finitely generated W-module. From this we deduce from Proposition 3.4, using its notation, that $s = r$ and the sequence $f_1, \ldots, f_s, p$ is regular. Thus R_{univ} is a finite flat complete intersection over W. □

PROPOSITION 3.8. *$R_{\mathrm{univ}}/pR_{\mathrm{univ}}$ is of finite cardinality.*

Proof. By Lemma 3.6 it will suffice to prove that ρ_{univ} mod p, which we denote by $\overline{\rho_{\mathrm{univ}}}$, has finite image.

We have need for the following lemma (see also Lemma 2.12 of [16]):

LEMMA 3.9. *For each $\ell \neq p$, $\overline{\rho_{\mathrm{univ}}}$ is finitely ramified at ℓ. In fact, the order of $\overline{\rho_{\mathrm{univ}}}(I_\ell)$ is the same as that of $\bar\rho(I_\ell)$.*

Proof. The only case that needs argument is when the restriction of $\bar\rho$ to I_ℓ is of type:

$$\xi \otimes \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix},$$

with ϕ a ramified character. The minimality condition implies that the restriction of ρ_{univ} to I_ℓ is of the form:

$$\tilde\xi \otimes \begin{pmatrix} 1 & \tilde\phi \\ 0 & 1 \end{pmatrix},$$

with $\tilde\xi$ being the Teichmüller lift of ξ. The morphism $\tilde\phi$ is tamely ramified, so its image is cyclic. As $R_{\mathrm{univ}}/pR_{\mathrm{univ}}$ is a $\mathbb{F}_p$-algebra, $p\tilde\phi = 0$ and $\tilde\phi$ has image of order p. □

We return to the proof of Proposition 3.8, and show that $\overline{\rho_{\mathrm{univ}}}$ has finite image.

Choose F as in Theorem 2.1. We show that $\overline{\rho_{\mathrm{univ}}}|_{G_F}$ has finite image for this choice of F, which clearly implies that $\overline{\rho_{\mathrm{univ}}}$ has finite image.

Let $\rho_{\mathrm{univ},F} : G_F \to \mathrm{GL}_2(R_{\mathrm{univ},F})$ be the universal, minimally ramified W-deformation of the restriction of $\overline{\rho}$ to G_F: this is unramified at every prime of F of residual characteristic $\neq p$, and for primes above p, we take the same conditions as we have taken to define ρ_{univ} (and their variants for the 2 deformation rings when $k(\bar\rho) = p+1$). We demand that determinant of this deformation is the restriction to G_F of $\hat\epsilon\chi_p^{k(\bar\rho)-1}$ in the case when we consider crystalline lifts above p and otherwise $\hat\epsilon\chi_p$. Because of Lemma 3.9, there is a morphism of CNL_W algebras $\phi : R_{\mathrm{univ},F}/(p) \to R_{\mathrm{univ}}/(p)$ such that $\overline{\rho_{\mathrm{univ}}}|_{G_F}$ is $\phi \circ \overline{\rho_{\mathrm{univ},F}}$. Thus it will be enough to prove that $\overline{\rho_{\mathrm{univ},F}}$ has finite image.

From Fujiwara's generalisation in Theorem 0.2 of [22] of $R = T$ theorems of [55] to the case of totally real fields in the ordinary case, and Theorem 3.2 of [50] in the supersingular case we deduce from Theorem 2.1 that $R_{\mathrm{univ},F}$ can be identified with a certain Hecke algebra $\mathbb{T}_F$, known to be finite, flat as a $\mathbb{Z}_p$-module. We may apply the results of [22] and [50] as we are excluding weight p, in weight $p+1$ all our lifts are ordinary, $\bar\rho|_{F(\mu_p)}$ is absolutely irreducible, and in the supersingular case we assume that p splits in F. Note that we are allowing $p = 3$ as it is explained in [25] how to extend the results of [50] to this case. From this it follows that $\overline{\rho_{\mathrm{univ},F}}$ has finite image. □

3.3. *Proof of Theorem* 3.3. We denote the deformation rings in Theorem 3.7 by R_{univ}. We deduce that there is a (minimal) prime ideal $\wp$ of R_{univ} that does not contain p, and $R_{\text{univ}}/\wp$ is a CNL_W-algebra that can be embedded in $\mathcal{O}$ for the ring of integers of a finite extension of $\mathbb{Q}_p$ that contains W. The corresponding CNL_W-algebra morphism $R_{\text{univ}} \to \mathcal{O}$ gives a minimal lifting $\rho : G_{\mathbb{Q}} \to \text{GL}_2(\mathcal{O})$ of the desired kind of $\bar{\rho}$. □

4. Compatible system lifts of mod p Galois representations

Let $F \subset \overline{\mathbb{Q}}$ be a number field and let $\rho : G_F \to \text{GL}_d(\overline{\mathbb{Q}_\ell})$ be a (continuous) Galois representation. We consider only ρ that are unramified outside a finite set of primes of F. Such a representation defines for every prime $q \neq \ell$ of F a representation of the Weil-Deligne group WD_q with values in $\text{GL}_d(\overline{\mathbb{Q}_\ell})$, well defined up to conjugacy.

For a number field E, we call an E-rational, 2-dimensional *compatible system* of Galois representations (ρ_ι) of G_F the data of:

(i) for each rational prime ℓ and each embedding $\iota : E \hookrightarrow \overline{\mathbb{Q}_\ell}$, a continuous, semisimple representation $\rho_\iota : G_F \to \text{GL}_2(\overline{\mathbb{Q}_\ell})$:

(ii) for each prime q of F, a F-semisimple (Frobenius semisimple) representation r_q of the Weil-Deligne group WD_q with values in $\text{GL}_2(E)$ such that:

a) r_q is unramified for all q outside a finite set,

b) for each rational prime ℓ, each prime q of F of characteristic different from ℓ and each $\iota : E \hookrightarrow \overline{\mathbb{Q}_\ell}$, the Frobenius-semisimple Weil-Deligne parameter $\text{WD}_q \to \text{GL}_2(\overline{\mathbb{Q}_\ell})$ associated to $\rho_\iota|_{D_q}$ is conjugate to r_q (via the embedding $E \hookrightarrow \overline{\mathbb{Q}_\ell}$)).

(iii) for each prime λ of F above a rational prime $\ell > 2$, at which r_λ is unramified, and for every embedding $\iota : E \hookrightarrow \overline{\mathbb{Q}_\ell}$, the restriction of ρ_ι to the decomposition group D_λ is crystalline and there are two integers a, b independent of λ, $a \leq b$, such that ρ_ι has Hodge-Tate (HT) weights (a, b).

If λ is the place of E associated with ι, we also denote a compatible system by (ρ_λ).

The primes of F such that r_q is unramified are called the unramified primes of the compatible system. The restriction to $I_q \times \mathbb{G}_{\text{a}}$ of r_q is called the inertial WD parameter at q. The parameter is said to be unramified if this restriction is trivial. We refer to a, b, as the weights of the compatible system and when $b \geq 0, a = 0$ we say that ρ_ι and the compatible system (ρ_λ), is of weight $b + 1$.

When we say that for some number field E, an E-rational compatible system (ρ_ι) of 2-dimensional representations of $G_{\mathbb{Q}}$ lifts $\bar{\rho}$ we mean that the residual representation arising from ρ_{ι_p} is isomorphic to $\bar{\rho}$. We say that a compatible system (ρ_ι) is odd if ρ_ι is odd for every ι. For a prime ℓ we abuse notation and denote by ρ_ℓ the ℓ-adic representation ρ_ι for ι the chosen embedding above ℓ, and by $\bar{\rho}_\ell$ a reduction of ρ_ℓ. We say that a compatible system (ρ_ι) is *irreducible* if all the ρ_ι are (absolutely) irreducible.

We recall a standard definition:

Definition 4.1. We say an S-type $\bar{\rho}$ is semi-stable if $\bar{\rho}(I_\ell)$ is of p-power order for $\ell \neq p$ and $k(\bar{\rho}) = 2$ or $p+1$.

The following theorem is proved using a method of Taylor, Theorem 6.6 of [50] and 5.3.3 of [53], and a refinement of Dieulefait [19] (see also [56]).

THEOREM 4.2. *Let $\bar{\rho}$ be a S-type representation in odd residue characteristic p, such that $2 \leq k(\bar{\rho}) \leq p+1$, $k(\bar{\rho}) \neq p$ and such that $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is absolutely irreducible.*

(i) *There is an irreducible compatible system (ρ_λ) that lifts $\bar{\rho}$ such that ρ_p is a minimal lift of $\bar{\rho}$ which is crystalline of weight $k(\bar{\rho})$ at p. The WD parameter of the compatible system (ρ_λ) is unramified at p, and for all primes $\ell > 2$ not ramified in ρ, ρ_λ for λ above ℓ is crystalline at ℓ of weight $k(\bar{\rho})$, i.e., of* HT *weights $(0, k(\bar{\rho}) - 1)$.*

(ii) *Assume further that $\bar{\rho}$ is semistable, and that either $N(\bar{\rho}) \neq 1$ or $k(\bar{\rho}) = p+1$. Then there is an irreducible compatible system (ρ_λ) that lifts $\bar{\rho}$ such that ρ_p is a minimal lift of $\bar{\rho}$ which at p is crystalline of weight* 2 *when $k(\bar{\rho}) = 2$, and semistable of weight* 2 *if $k(\bar{\rho}) = p+1$. Further there is a number field E and an abelian variety A over $\mathbb{Q}$ of dimension $[E : \mathbb{Q}]$ and an embedding $\mathcal{O}_E \hookrightarrow \mathrm{End}(A/\mathbb{Q})$ such that (ρ_λ) arises from A. The abelian variety A has multiplicative reduction exactly at the primes dividing the prime to p part of the Artin conductor of $\bar{\rho}$, and also p when $k(\bar{\rho}) = p+1$, and at all other places has good reduction.*

Proof. We start by remarking that much of the proof below can now also be found in the proof of Theorem 5.1 of [25].

The fact that there is a p-adic lift ρ_p as asserted follows from Corollary 3.3. Consider $\bar{\rho}|_{G_F}$ with F as in Taylor's Theorem 2.1.

The existence of a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbb{A}_F)$ that gives rise to $\bar{\rho}|F$ as in Theorem 2.1, and the modularity lifting results in Theorem 5.1 of [46] and Therorem 3.3 of [50], yield that $\rho|_{G_F}$ arises from a holomorphic, cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ with respect to the embedding ι_p. The cuspidal automorphic representation π gives rise by [48] to an irreducible compatible system $(\rho_{\pi,\iota})$: see the arguments below for justification of property (iii) of our definition of compatibility, and Proposition 3.1 of [49] for the irreducibility.

Let $G = \mathrm{Gal}(F/\mathbb{Q})$. Using Brauer's theorem we get subextensions F_i of F such that $G_i = \mathrm{Gal}(F/F_i)$ is solvable, characters χ_i of G_i (that we may also regard as characters of G_{F_i}) with values in $\overline{\mathbb{Q}}$ (that we embed in $\overline{\mathbb{Q}}_p$ using ι_p), and $n_i \in \mathbb{Z}$ such that $1_G = \sum_{G_i} n_i \mathrm{Ind}_{G_i}^{G} \chi_i$. Using the base change results of Arthur-Clozel in [2], we get holomorphic cuspidal automorphic representations π_i of $\mathrm{GL}_2(\mathbb{A}_{F_i})$ such that if ρ_{π_i,ι_p} is the representation of G_{F_i} corresponding to π_i w.r.t. ι_p, then $\rho_{\pi_i,\iota_p} = \rho|_{G_{F_i}}$. Thus $\rho = \sum_{G_i} n_i \mathrm{Ind}_{G_{F_i}}^{G_\mathbb{Q}} \chi_i \otimes \rho_{\pi_i,\iota_p}$.

Now for any prime ℓ and any embedding $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$, we define the virtual representation $\rho_\iota = \sum_{G_i} n_i \mathrm{Ind}_{G_{F_i}}^{G_\mathbb{Q}} \chi_i \otimes \rho_{\pi_i,\iota}$ of $G_\mathbb{Q}$ with the χ_i's now regarded as ℓ-adic characters via the embedding ι. We check that ρ_ι is a true representation by computing its inner product in the Grothendieck group of $\overline{\mathbb{Q}}_\ell$-valued (continuous, linear) representations of $G_\mathbb{Q}$. We claim that this is independent of ι. This is because, as the $\mathrm{Ind}_{G_{F_i}}^{G_\mathbb{Q}} \chi_i \otimes \rho_{\pi_i,\iota}$ are semisimple, the value of the inner product is the dimension of $\mathrm{End}(\rho_\iota)$ as a $\overline{\mathbb{Q}}_\ell$-vector space. Using Mackey's formula, and the fact that the compatible system is irreducible on restriction to G_F, we see that this dimension is independent of ι. As for $\iota = \iota_p$ this dimension is 1 we see that $(\pm 1)\rho_\iota$ is a true, irreducible representation. As the dimension of ρ_ι is independent of ι i.e. is 2, we see that ρ_ι is a true representation.

The representations ρ_ι together constitute the compatible system (ρ_λ) we seek as we proceed to justify. As $(\rho_{\pi_i,\iota})$ satisfies property (ii) of our definition of compatibility, so does $(\rho_\iota = \sum_{G_i} n_i \mathrm{Ind}_{G_{F_i}}^{G_\mathbb{Q}} \chi_i \otimes \rho_{\pi_i,\iota})$ (see proof of Theorem 6.6 of [50]). As $(\rho_\lambda|_{G_F})$ is irreducible as remarked above, so is (ρ_λ). The properties at p of the compatible system follow from the construction.

We now prove (ρ_λ) satisfies property (iii) of our definition of compatibility. By Arthur-Clozel solvable base change ([2]), we know that for each $F' \subset F$ such that F/F' has solvable Galois group, the restriction of ρ to $G_{F'}$ comes from an automorphic representation $\pi_{F'}$ of $GL_2(\mathbb{A}_{F'})$. One uses following [19], that for $F' \subset F$ such that F/F' has solvable Galois group, the system (ρ_λ) restricted to $G_{F'}$ comes from $\pi_{F'}$. Let q be a prime number. Let $\mathcal{Q}$ be a prime of F above q and let $F(\mathcal{Q})$ be the subfield of F fixed by the decomposition group $\subset \mathrm{Gal}(F/\mathbb{Q})$ at $\mathcal{Q}$. We know that the restriction of (ρ_λ) to $G_{F(\mathcal{Q})}$ comes from $\pi_{F(\mathcal{Q})}$. We deduce the finer properties required by applying to $\pi_{F(\mathcal{Q})}$:

– if we are in the case (i) and λ is above $q \neq 2$, the theorems of Breuil ([9]) and Berger ([3]) to get that ρ is crystalline at q;

– if we are in the case (ii), the theorem of Saito ([38]) because we know that $\pi_{F(\mathcal{Q})}$ is Steinberg at one prime of $F(\mathcal{Q})$ (note that if ρ is unramified outside p, ρ is semistable not crystalline of weight 2 at p, and it follows from [9] and [3] that $\pi_{F(\mathcal{Q})}$ is not unramified at primes above p).

The existence of the abelian variety in part (ii) follows from the arguments used for Corollary E, or Corollary 2.4, of [51] which use results of Carayol and

Blasius-Rogawski ([12], [5]). By [12], we know that there exists an abelian variety B over F, a number field E with $[E:\mathbb{Q}]=\dim(B)$, an embedding $E\hookrightarrow \mathrm{End}(B)_{\mathbb{Q}}$, and an embedding $\tau : E\hookrightarrow \overline{\mathbb{Q}_p}$ such that the restriction of ρ to G_F is isomorphic to the Galois representation on the factor of $\overline{\mathbb{Q}_p}\otimes_{\mathbb{Q}_p} V_p(B)$ defined by τ. Let A' be the abelian variety over $\mathbb{Q}$ obtained from B by Weil-restriction. The embedding $E\hookrightarrow \mathrm{End}(B)_{\mathbb{Q}}$ defines an embedding $E\hookrightarrow \mathrm{End}(A')_{\mathbb{Q}}$. The Tate-module $V_p(A')$ is the Galois module obtained from $V_p(B)$ by induction from G_F to $G_{\mathbb{Q}}$. The Frobenius reciprocity formula implies that ρ appears with multiplicity 1 in the factor of $\overline{\mathbb{Q}_p}\otimes_{\mathbb{Q}_p} V_p(A')$ defined by τ. As in [51], it follows from Faltings' proof of the Tate conjecture for abelian varieties that there exists an abelian subvariety $A\subset A'$, defined over $\mathbb{Q}$, stable by the action of E, a finite extension E' of E with $[E':\mathbb{Q}]=\dim(A)$ and an embedding $E'\hookrightarrow \mathrm{End}(A)_{\mathbb{Q}}$ extending the embedding $E\hookrightarrow \mathrm{End}(A)_{\mathbb{Q}}$, an embedding τ' of E' in $\overline{\mathbb{Q}_p}$, such that ρ is isomorphic to the Galois representation on the factor of $\overline{\mathbb{Q}_p}\otimes_{\mathbb{Q}_p} V_p(A)$ defined by τ'. We have $\mathrm{End}(A)_{\mathbb{Q}}=E'$. The Galois module $\overline{\mathbb{Q}_p}\otimes_{\mathbb{Q}_p} V_p(A)$ is the direct sum of the Galois submodules of dimension 2 over $\overline{\mathbb{Q}_p}$ defined by the different embeddings of E' in $\overline{\mathbb{Q}_p}$. It follows from compatibility that A has semistable reduction at all primes and has multiplicative reduction exactly at those primes $\neq p$ which are ramified in ρ and at p if and only if $k(\bar\rho)=p+1$. □

5. Low levels and weights

Theorem 4.2 when combined with modularity lifting results in [55], [45] and [46], and the theorems of Fontaine, [20], together with their generalisations due to Brumer and Kramer, and Schoof, [11], [39], has a number of corollaries.

We state a special case of the results of [45], [46] that we need repeatedly in this section.

THEOREM 5.1 (Skinner-Wiles). *Let $\rho : G_{\mathbb{Q}}\to \mathrm{GL}_2(E)$ be a continuous, absolutely irreducible, p-adic representation with $p>2$ that is ramified at finitely many primes, and ordinary at p in the sense of Definition* 3.1. *Assume that the* HT *weights of ρ are $(0,k-1)$ with $k\geq 2$. Assume that a reduction $\bar\rho$ of (an integral model of) ρ is modular and is such that* $\det(\bar\rho)|_{I_p}=\overline{\chi}_p^{a}$ *for some odd integer a. Then ρ arises from a newform f of weight k.*

Proof. We need only to remark that the D_p-distinguished hypothesis of [45], [46] on ρ follows from our assumption on the determinant of $\bar\rho$. □

Remark. The hypothesis on the determinant of $\bar\rho$ of Theorem 5.1 is satisfied if: (i) $\det(\bar\rho)$ is unramified outside p, and (ii) $\bar\rho$ is odd. The residual representations we consider in this section thus satisfy the hypothesis of Theorem 5.1. This will be used without further comment.

Part (i) of the following theorem follows immediately from Theorem 3.3 and the result of [19].

THEOREM 5.2. (i) *There is no S-type $\bar{\rho}$ with $N(\bar{\rho}) = 1, k(\bar{\rho}) = 2$.*

(ii) *Assume $p > 2$. There is no semistable $\bar{\rho}$ with (prime to p) Artin conductor $N(\bar{\rho}) = q = 2, 3, 5, 7$, or 13 and $k(\bar{\rho}) = 2$.*

(iii) *For $p = 3, 5, 7, 13$ there is no Serre type $\bar{\rho}$ that is unramified outside p and such that $k(\bar{\rho}) = p + 1$.*

Proof. We assume throughout for $p > 2$ that $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is irreducible as otherwise we are done by known cases of the S-conjecture. Namely we know in this case that $\bar{\rho}$ has projectively dihedral image, and thus by a result of Hecke is modular (see remark in Section 5.1 of [43]). Then results of [36], [17] prove that it arises from $S_{k(\bar{\rho})}(\Gamma_1(N(\bar{\rho})))$.

Let us prove (i). The case $p = 2$ is taken care of by [47] which proves that there is no S-type $\bar{\rho}$ with $N(\bar{\rho}) = 1$ and the characteristic of $\bar{\rho}$ is 2. For $p \geq 3$, consider a S-type $\bar{\rho}$ with $N(\bar{\rho}) = 1, k(\bar{\rho}) = 2$. We use Theorem 4.2 (i) to get an irreducible compatible system lift (ρ_λ) of $\bar{\rho}$. This contradicts the main theorem of [19] or the arXiv version (see arXiv:math/0406576v1) of [56]. For instance, the latter considers the 7-adic representation ρ_7, and uses arguments of [20] to show that ρ_7 is reducible. Thus no such $\bar{\rho}$ exists.

Consider a S-type $\bar{\rho}$ as in the statement of (ii). We use Theorem 4.2 (ii) to get a compatible lift (ρ_λ) of $\bar{\rho}$ (of weight 2) which arises from an abelian variety A with good reduction outside q and semistable reduction at q. The results Brumer-Kramer and Schoof, [11] and [39], yield that A is zero which is a contradiction.

Consider a S-type $\bar{\rho}$ as in the statement of (iii). We use Theorem 4.2 (ii) to get a compatible lift (ρ_λ) of $\bar{\rho}$ (of weight 2) which arises from an abelian variety A with good reduction outside p and semistable reduction at p. The results of Brumer-Kramer and Schoof, [11] and [39], yield that $A = 0$. □

COROLLARY 5.3. *If p is odd then the only (p, p)-type finite flat group schemes over $\mathbb{Z}$ are $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z} \oplus \mu_p$ or $\mu_p \oplus \mu_p$.*

For $p = 2$, see Abrashkin [1], or Remarque (3) in Section 4.5 of [43].

Proof. After Theorem 5.2 (i) this follows from Serre's arguments in Section 4.5 of [43]. □

In fact using our methods we can also rule out the existence of some higher weight $\bar{\rho}$ in accordance with the predictions of Serre.

THEOREM 5.4. *There is no S-type $\bar{\rho}$ in residue characteristic p with $N(\bar{\rho}) = 1$ such that $2 \leq k(\bar{\rho}) \leq 8$, or $k(\bar{\rho}) = 14$ where in the latter case we assume $p \neq 11$.*

Proof. The case $k(\bar{\rho}) = 2$ is done in Theorem 5.2. We may assume that $p > 2$ because of [47]. We may assume as before that $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is irreducible as otherwise we are done by known cases of the S-conjecture.

- $k(\bar{\rho}) = 4$: Let p be a prime > 2, and $\bar{\rho}$ be an S-type representation in characteristic p with $N(\bar{\rho}) = 1, k(\bar{\rho}) = 4$. We use Theorem 4.2 (i) to get a compatible system (ρ_λ), of weight 4. Consider ρ_3, and the residual representation $\bar{\rho}_3$. If $\bar{\rho}_3$ is irreducible, then as by Proposition 3.2 we get that $k(\bar{\rho}_3) = 2$ or 4, this contradicts Theorem 5.2. Hence $\bar{\rho}_3$ is reducible, and thus by Proposition 3.2, ρ_3 is ordinary. Then by Theorem 5.1, and known properties of p-adic representations ρ_f associated to newforms f due to [12] and [37], we get that (ρ_λ) arises from $S_4(\mathrm{SL}_2(\mathbb{Z})) = 0$, a contradiction. Thus there is no S-type $\bar{\rho}$ with $N(\bar{\rho}) = 1, k(\bar{\rho}) = 4$.
- $k(\bar{\rho}) = 6$: We may assume now that $p > 3$. This is because up to twist a S-type $\bar{\rho}$ in characteristic 3 has weight ≤ 4. After this the argument is identical to the case $k(\bar{\rho}) = 4$, but using $\bar{\rho}_5$ instead of $\bar{\rho}_3$.
- $k(\bar{\rho}) = 8$: We may assume now by an identical argument that $p > 5$, and the rest of the argument is identical.
- $k(\bar{\rho}) = 14$: We may assume now that $p > 7$ by an identical argument, and as we are assuming $p \neq 11$ in this case, we may in fact assume that $p \geq 13$. After this the argument is identical. □

Remark. As noted in the introduction the technique used implicitly in the proof above of considering two different compatible systems that lift a given residual representation has been developed and used in [25] in the proof of the level one case of Serre's conjecture.

The following corollary is immediate. The cases $p = 2, 3$ are due to Tate and Serre, [47] and [42], and the case $p = 5$ is done in [10] under the GRH. Note that this was Corollary 4.4 in an earlier version of the paper.

Corollary 5.5. *For the primes $p = 2, 3, 5, 7$ there are no S-type $\bar{\rho}$ in characteristic p with $N(\bar{\rho}) = 1$.*

We now prove a case of the S-conjecture for $\bar{\rho}$ with given invariants $N(\bar{\rho}), k(\bar{\rho})$ in a case when a S-type $\bar{\rho}$ of these invariants is known to exist.

Theorem 5.6. *Let $\bar{\rho}$ be of S- type, with $N(\bar{\rho}) = 1, k(\bar{\rho}) = 12$. Then $\bar{\rho}$ arises from the Ramanujan Δ function.*

Proof. Using Corollary 5.5 we may assume that $p \geq 11$. We may assume as before that $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is irreducible as otherwise we are done by known cases of the S-conjecture. Using Theorem 4.2, we get a compatible system (ρ_λ) that lifts $\bar{\rho}$ and such that ρ_p is crystalline of Hodge-Tate weight $(0, 11)$. Consider ρ_{11}, which is unramified outside 11 and crystalline at 11 of weight 12, and a residual representation $\bar{\rho}_{11}$.

Firstly ρ_{11} is ordinary at 11 as otherwise by Proposition 3.2, $\bar{\rho}_{11}$ is of S-type (as $\bar{\rho}_{11}|_{D_{11}}$ is irreducible) and $k(\bar{\rho}_{11}) = 2$. This contradicts Theorem 5.2. If $\bar{\rho}_{11}$ is reducible, as ρ_{11} is ordinary at 11, by Theorem 5.1, (ρ_λ), and hence $\bar{\rho}$ arises from the unique newform Δ of $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$. Otherwise $\bar{\rho}_{11}$ is of S-type, $N(\bar{\rho}) = 1$, $k(\bar{\rho}_{11}) = 2$ or 12 and hence by Theorem 5.2, $k(\bar{\rho}_{11}) = 12$ and is très ramifiée at 11. Note that $\bar{\rho}_{11}|_{\mathbb{Q}(\mu_{11})}$ is irreducible as otherwise some twist of $\bar{\rho}_{11}$ would have weight 6 which contradicts Theorem 5.4 for instance.

We may apply Theorem 3.3 to $\bar{\rho}_{11}$ and get another lift ρ'_{11} of $\bar{\rho}_{11}$, which is unramified outside 11 and semistable at 11 of weight 2. By Theorem 4.2 ρ'_{11} arises from an abelian variety A defined over $\mathbb{Q}$ with good reduction outside 11 and multiplicative reduction at 11. By [39] such an abelian variety A is isogenous to a power of $J_0(11)$. The Galois representation on points of order 11 of the elliptic curve $J_0(11)$ is absolutely irreducible, is ordinary at 11, and is isomorphic to the representation modulo 11 associated to Δ (see Section 3.5 of [40]), and consequently $\bar{\rho}_{11}$ itself arises from Δ. Thus by Theorem 5.1, ρ_{11} arises from $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ and hence $\bar{\rho}$ arises from the Δ function. □

Remark. For finitely many primes p there may be no $\bar{\rho}$ of S-type in characteristic p with $N(\bar{\rho}) = 1$, $k(\bar{\rho}) = 12$. These primes are $p = 2, 3, 5, 7, 691$, as follows from Theorem 5.6 and [40].

6. MLC implies the S-conjecture

6.1. *Level one case of the S-conjecture by induction on the prime p.* By the level one case of the S-conjecture we mean the S-conjecture for S-type $\bar{\rho}$ with $N(\bar{\rho}) = 1$. We reduce this to restricted versions of the MLC that are a little beyond the modularity lifting theorems that are known.

Consider a S-type $\bar{\rho}$ in residue characteristic p and such that $N(\bar{\rho}) = 1$. By the result of Tate we may assume $p > 2$. By twisting we may assume that the Serre weight $k(\bar{\rho}) \leq p + 1$. Note that $k(\bar{\rho}) \neq p$. Using Theorem 4.2 we get a compatible system (ρ_λ) that lifts $\bar{\rho}$. Applying Theorem 5.2 to a residual representation $\bar{\rho}_3$ arising from this system, we get that it is reducible. Thus if we assume the MLC, we conclude that ρ_3 is modular and hence so is $\bar{\rho}$. We refine this argument a little.

THEOREM 6.1. *Assume the* MLC *in the restricted version that we consider only ρ in it that are crystalline at p of* HT *weights $(0, k)$ with $k < 2p$. Then the level one case of the S-conjecture is true.*

Proof. We prove the theorem by induction on the prime p, the case $p = 2, 3$ being known. We may assume as before that $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is irreducible as otherwise we are done by known cases of the S-conjecture.

Suppose the level one S-conjecture conjecture is proven for a prime $p_n > 2$. We want to prove it for the next prime p_{n+1}. Thus assume we have a S-type mod p_{n+1} representation $\bar{\rho}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which by twisting we can assume has Serre weight $k(\bar{\rho}) \leq p_{n+1}+1$, and such that $N(\bar{\rho}) = 1$.

Use Theorem 4.2 to get a compatible system (ρ_λ) that lifts $\bar{\rho}$ such that $\rho_{p_{n+1}}$ is unramified outside p_{n+1} and crystalline at p_{n+1} of weight $k(\bar{\rho})$. We also get that ρ_{p_n} is unramified outside p_n and crystalline at p_n of Hodge-Tate weights$(0, k(\bar{\rho})-1)$. By Bertrand's postulate, $p_{n+1} \leq 2p_n - 1$, and by the induction hypothesis a residual representation $\bar{\rho}_{p_n}$ is modular. Thus by the restricted version of the MLC in the statement of the theorem, ρ_{p_n} arises from $S_{k(\bar{\rho})}(\mathrm{SL}_2(\mathbb{Z}))$, and hence so does $\bar{\rho}$. □

Remark. A modification of this strategy is used to prove the level one case of the S-conjecture in [25].

6.2. *Killing ramification.* The process of killing ramification is the following. Suppose you wish to prove that a compatible system (ρ_λ) is modular. Let λ_0 be above a prime of ramification of (ρ_λ). One applies the theorem 4.2 to a cyclotomic twist of $\bar{\rho}_{\lambda_0}$ to get a compatible system $(\rho'_{\lambda'})$ whose set of ramification primes is smaller than the set of ramification primes of (ρ_λ). If one knows by induction modularity of $(\rho'_{\lambda'})$, one gets modularity of $\bar{\rho}_{\lambda_0}$, hence modularity of (ρ_λ) if one has the needed modularity lifting theorem. We give an example of a more precise statement:

THEOREM 6.2. *Assume the* MLC *and assume Theorem* 3.3 *holds also if* $k(\bar{\rho}) = p$. *Also assume that the compatible systems* (ρ_λ) *of Theorem* 4.2 (i) *are such that* ρ_λ *is de Rham of weight* $k(\bar{\rho})$ *for all* λ. *Then the* S*-conjecture is true for* $\bar{\rho}$ *in characteristic* $p > 2$, *and with* $N(\bar{\rho})$ *odd.*

Proof. We may assume as before that $\bar{\rho}|_{\mathbb{Q}(\mu_p)}$ is irreducible as otherwise we are done by known cases of the S-conjecture.

Consider a S-type $\bar{\rho}$ as in the statement, and assume as we may that $2 \leq k(\bar{\rho}) \leq p+1$. The proof is by induction on the cardinality of the set of prime divisors of $N(\bar{\rho})$ for the type of $\bar{\rho}$ in the statement. The case of $N(\bar{\rho}) = 1$ is dealt with in Theorem 6.1.

Let q be a prime divisor of $N(\bar{\rho})$. Use Theorem 4.2 (i) to get a compatible system (ρ_λ) that lifts $\bar{\rho}$. Consider a residual representation $\bar{\rho}_q$ arising from this system. Observe that $N(\bar{\rho}_q)$ is divisible by at least one prime fewer than $N(\bar{\rho})$. Thus by the inductive hypothesis, we deduce that $\bar{\rho}_q$ is modular, and then by the MLC we see that ρ_q, and hence (ρ_λ) and $\bar{\rho}$, arises from a newform. □

Remark. The idea of the proof above is a key component of the proof of the odd conductor case of the S-conjecture in [28], [29]. The superfluous restriction to $p > 2$ and $N(\bar{\rho})$ odd is because Theorem 3.3 is stated only for $p > 2$.

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