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Regularity of flat level sets in phase transitions

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Abstract

We consider local minimizers of the Ginzburg-Landau energy functional

$$\int \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2 dx$$

and prove that, if the 0 level set is included in a flat cylinder then, in the interior, it is included in a flatter cylinder. As a consequence we prove a conjecture of De Giorgi which states that level sets of global solutions of

$$\triangle u = u^3 - u$$

such that

$$|u| \leq 1, \quad \partial_n u > 0, \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$$

are hyperplanes in dimension $n \leq 8$.

1. Introduction

In this paper we establish further properties of phase transitions that are similar to the properties of sets with minimal perimeter.

The Ginzburg-Landau model of phase transitions leads to considerations of local minimizers for the energy functional

$$J(u, \Omega) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2 dx, \qquad |u| \leq 1. \tag{1}$$

If u is a local minimizer then

$$\triangle u = u^3 - u. \tag{2}$$

We explain below some analogies between the theory of phase transitions and the theory of minimal surfaces.

The rescalings $u_\varepsilon(x) = u(\varepsilon^{-1}x)$ are local minimizers for the ε-energy functional

$$J_\varepsilon(u_\varepsilon) = \int \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon}(1-u_\varepsilon^2)^2 dx.$$

In [16] Modica proved that as $\varepsilon \to 0$, u_ε has a subsequence

$$u_{\varepsilon_k} \to \chi_E - \chi_{E^c} \quad \text{in } L^1_{\text{loc}} \tag{3}$$

where E is a set with minimal perimeter.

In [8] Caffarelli and Cordoba proved a uniform density estimate for the level sets of local minimizers u_ε of J_ε. Suppose $u_\varepsilon(0) = 0$; then

$$\frac{|\{u_\varepsilon > 0\} \cap B_\delta|}{|B_\delta|} \geq C$$

for $\varepsilon \leq \delta$, $C > 0$ universal. In particular, this implies that in (3), the level sets $\{u_{\varepsilon_k} = \lambda\}$ converge uniformly on compact sets to ∂E.

In [19] Modica proved a monotonicity formula for the energy functional, i.e. $J(u, B_R)R^{1-n}$ increases with R.

Let us recall some facts about minimal surfaces (see for example Giusti [14]). Suppose that E is a set with minimal perimeter in Ω and $0 \in \partial E$. Then

1) Flatness implies regularity; i.e, if

$$\Omega = \{|x'| < 1\} \times \{|x_n| < 1\}, \quad \partial E \subset \{|x_n| < \varepsilon\},$$

and $\varepsilon \leq \varepsilon_0$, ε_0 small universal, then ∂E is analytic in $\{|x'| < 1/2\}$.

The proof uses an "improvement of flatness" lemma due to De Giorgi (see Chapters 6, 7, 8 from Giusti [14]). More precisely, one can show that, possibly in a different system of coordinates, ∂E can be trapped in a flatter cylinder

$$\{|y'| \leq \eta_2\} \cap \partial E \subset \{|y_n| \leq \varepsilon \eta_1\},$$

with $0 < \eta_1 < \eta_2$ universal. This implies ∂E is $C^{1,\alpha}$, and therefore analytic by the elliptic regularity theory.

2) If $\Omega = \mathbb{R}^n$, and $n \leq 7$ then ∂E is a hyperplane.

3) If $\Omega = \mathbb{R}^n$ and $n = 8$ then there exist nonhyperplane minimal sets, for example Simons cone

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2.$$

If, in addition, we assume that ∂E is a "graph" in some direction, then ∂E is a hyperplane.

4) If $\Omega = \mathbb{R}^n$ and $n \geq 9$ then there exist nonhyperplane minimal graphs (see [6]).

5) If $\Omega = \mathbb{R}^n$ and ∂E is a graph in the e_n direction that has at most linear growth at ∞ then ∂E is a hyperplane.

It is natural to ask if some of these properties hold for level sets of local minimizers of (1), or solutions of (2).

In connection to 3) above De Giorgi made the following conjecture in [11]:

Let $u \in C^2(\mathbb{R}^n)$ be a solution of

$$\triangle u = u^3 - u,$$

such that

$$|u| \leq 1, \quad \partial_n u > 0$$

in whole $\mathbb{R}^n$. Is it true that all the level sets of u are hyperplanes, at least if $n \leq 8$?

The conjecture was proved for $n = 2$ by Ghoussoub and Gui in [12] and for $n = 3$ by Ambrosio and Cabre in [2].

Barlow, Bass and Gui proved in [3] that monotonic solutions in $\mathbb{R}^n$ with Lipschitz level sets are planar in all dimensions.

The main result of this paper is an "improvement of flatness" theorem for 0 level sets of local minimizers (Theorem 2.1). More precisely, if u is a local minimizer of (1) and $\{u = 0\}$ is included in a flat cylinder $\{|x'| < l\} \times \{|x_n| < \theta\}$ with θ, l large and θl^{-1} small then, $\{u = 0\}$ is included, possibly in a different system of coordinates, in a flatter cylinder $\{|x'| < \eta_2 l\} \times \{|x_n| < \eta_1 \theta\}$ with $\eta_2 > \eta_1 > 0$ universal.

If $\{u_{\varepsilon_k} = 0\}$ converges uniformly on compact sets to a hyperplane then, one can apply Theorem 2.1 and conclude that $\{u = 0\}$ is included in flatter and flatter cylinders, therefore it is a hyperplane.

This fact allows us to extend some of the minimal surfaces properties listed above to level sets of local minimizers of (1). In particular, we prove the weak form of De Giorgi's conjecture, i.e. we also assume that

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{4}$$

The approach of Modica to study local minimizers of J_ε uses variational techniques and the notion of $\Gamma-$ convergence. More precisely, by co-area formula, one has

$$\begin{aligned} J_\varepsilon(u_\varepsilon, \Omega) &\geq \frac{1}{\sqrt{2}} \int_\Omega (1 - u_\varepsilon^2)|\nabla u_\varepsilon| dx \\ &= \frac{1}{\sqrt{2}} \int_{-1}^{1} (1 - s^2)\mathcal{H}^{n-1}(\{u_\varepsilon = s\} \cap \Omega) ds. \end{aligned}$$

Heuristically, we minimize $J_\varepsilon(u_\varepsilon, \Omega)$ if, in the interior of Ω, we take the level sets $\{u_\varepsilon = s\}$ to be (almost) minimal and

$$|\nabla u_\varepsilon| = \frac{1}{\sqrt{2}\varepsilon}(1 - u_\varepsilon^2). \tag{5}$$

Notice that, if Γ is a smooth surface then

$$u_\varepsilon(x) = \tanh \frac{d_\Gamma(x)}{\sqrt{2}\varepsilon} \tag{6}$$

satisfies (5), where d_Γ represents the signed distance to the surface Γ.

In this paper we use the method of subsolutions and supersolutions together with the sliding method. Suppose that Γ is a smooth surface and consider the function given by (6). Then

$$\varepsilon^2 \triangle u_\varepsilon = u_\varepsilon^3 - u_\varepsilon - \frac{\varepsilon}{\sqrt{2}}(1-u_\varepsilon^2)\sum \frac{\kappa_i}{1-d_\Gamma \kappa_i}$$

where κ_i represent the principal curvatures of Γ at the point where the distance is realized.

Heuristically, if Γ has positive (negative) mean curvature then we can find a supersolution (subsolution) whose 0 level set is Γ.

In a forthcoming paper we use the same techniques to prove similar results for solutions of

$$F(D^2u) = f(u),$$
$$u \in C^2(\mathbb{R}^n), \quad |u| \le 1, \quad \partial_n u > 0$$

where F is uniformly elliptic, and F, f are such that there exists a one dimensional solution g which solves the equation in all directions; i.e.,

$$F(D^2 g(x\cdot\nu) = f(g(x\cdot\nu)), \quad \forall \nu \in \mathbb{R}^n, \quad |\nu| = 1.$$

2. Main results

Consider the more general energy functional

$$J(u,\Omega) = \int_\Omega \frac{1}{2}|\nabla u|^2 + h_0(u)dx, \quad |u| \le 1, \tag{7}$$

with

$$h_0 \in C^2[-1,1], \quad h_0(-1) = h_0(1) = 0, \quad h_0 > 0 \quad \text{on } (-1,1),$$
$$h_0'(-1) = h_0'(1) = 0, \quad h_0''(-1) > 0, \quad h_0''(1) > 0.$$

We say that u is a local minimizer in Ω if, for every open set $A \subset \Omega$ relatively compact in Ω,

$$J(u,A) \le J(u+v,A), \quad \forall v \in H_0^1(A).$$

A local minimizer of (7) satisfies

$$\triangle u = h_0'(u), \quad |u| \le 1. \tag{8}$$

Our goal is to prove the following theorem for flat level sets of u.

THEOREM 2.1 (Improvement of flatness). *Let u be a local minimizer of (7) in $\{|x'| < l\} \times \{|x_n| < l\}$, and assume that the level set $\{u = 0\}$ stays in the flat cylinder*

$$\{|x'| < l\} \times \{|x_n| < \theta\}$$

and contains the point 0.

Then there exist small constants $0 < \eta_1 < \eta_2 < 1$ depending only on n such that:

Given $\theta_0 > 0$ there exists $\varepsilon_1(\theta_0) > 0$ depending on n, h_0 and θ_0 such that if

$$\frac{\theta}{l} \leq \varepsilon_1(\theta_0), \quad \theta_0 \leq \theta,$$

then

$$\{u = 0\} \cap (\{|\pi_\xi x| < \eta_2 l\} \times \{|x \cdot \xi| < \eta_2 l\})$$

is included in a cylinder

$$\{|\pi_\xi x| < \eta_2 l\} \times \{|x \cdot \xi| < \eta_1 \theta\}$$

for some unit vector ξ (π_ξ denotes the projection along ξ).

We prove Theorem 2.1 by compactness from the following Harnack inequality for flat level sets of minimizers.

THEOREM 2.2. *Let u be a local minimizer of J in the cylinder $\{|x'| < l\} \times \{|x_n| < l\}$ and assume that*

$$\{u = 0\} \subset \{|x_n| < \theta\}, \quad u(0) = 0.$$

There exists a small universal constant $\eta_0 > 0$ depending on n and h_0 such that:

Given $\theta_0 > 0$ there exists $\varepsilon_0(\theta_0) > 0$ depending on n, h_0 and θ_0, such that if

$$\theta l^{-1} \leq \varepsilon_0(\theta_0), \quad \theta_0 \leq \theta,$$

then

$$\{u = 0\} \cap \{|x'| < \eta_0 l\} \subset \{|x_n| < (1 - \eta_0)\theta\}.$$

As a consequence of Theorem 2.1 we prove the following theorems.

THEOREM 2.3. *Suppose that u is a local minimizer of J in $\mathbb{R}^n$, and $n \leq 7$. Then the level sets of u are hyperplanes.*

It is known (see [15]) that monotone solutions of (8) satisfying (4) are local minimizers.

THEOREM 2.4. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of*

$$\triangle u = h_0'(u), \tag{9}$$

such that

$$|u| \leq 1, \quad \partial_n u > 0, \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{10}$$

a) *If $n \leq 8$ then the level sets of u are hyperplanes.*

b) *If the* 0 *level set has at most linear growth at* ∞ *then the level sets of* u *are hyperplanes.*

The paper is organized as follows. In Section 3 we prove Theorems 2.3 and 2.4 assuming Theorem 2.1. In Section 4 we introduce some notation and preliminaries. In Section 5 we show that the 0 level set of u satisfies in some weak viscosity sense a mean curvature equation at large scale. In Section 6 we show that Theorem 2.2 implies Theorem 2.1 by a compactness argument. In the remaining part of the paper we prove Theorem 2.2. The proof uses some ideas of Caffarelli and Cordoba from a paper about regularity of minimal surfaces (see [9]). Next we explain the strategy of its proof.

Let g_0 denote the one dimensional solution of (8), $g_0(0) = 0$, and suppose that at one point $\{u = 0\}$ is close to $x_n = -\theta$. Then, using a family of sliding surfaces (see Section 7), we prove that the graph of u is close in the e_n direction to the graph of $g_0(x_n + \theta)$ at points that project along e_n in sets of positive measure (Section 8). Using an iteration lemma we show that these sets almost fill in measure the strip $\{(x', 0, x_{n+1}) \mid |x_{n+1}| \leq 1/2\}$ (Section 9). From this we obtain a contradiction to the fact that u is a local minimizer and $u(0) = 0$ (Section 10).

3. Proof of Theorems 2.3 and 2.4

In this section we use Theorem 2.1 to prove Theorems 2.3 and 2.4.

Let $\Omega \subset \mathbb{R}^n$ be an open set and E be a measurable set. The perimeter of E in Ω is defined as

$$P_\Omega(E) = \sup \left| \int_E \operatorname{div} g \, dx \right|,$$

where the supremum is taken over all vector fields $g \in C_0^1(\Omega)$ with $\|g\| \leq 1$.

We say that E is a set with minimal perimeter in Ω if, for every open set $A \subset \Omega$, relatively compact in Ω,

$$P_A(E) \leq P_A(F),$$

whenever E and F coincide outside a compact set included in A.

We introduce the rescaled energies,

$$J_\varepsilon(v, \Omega) := \int_\Omega \frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{\varepsilon} h_0(v) dx. \tag{11}$$

If u is a local minimizer of $J(u, \Omega)$, then the rescalings

$$u_\varepsilon(x) = u(\frac{x}{\varepsilon}),$$

are local minimizers for $J_\varepsilon(\cdot, \varepsilon\Omega)$, and

$$J_\varepsilon(u_\varepsilon, \varepsilon\Omega) = \varepsilon^{n-1} J(u, \Omega).$$

Now we state two theorems mentioned in the introduction. The first theorem was proved by Modica in [16].

THEOREM 3.1. *Let u_k be a sequence of local minimizers for the energies $J_{\varepsilon_k}(\cdot,\Omega)$ with $\varepsilon_k \to 0$. There exists a subsequence u_{k_m} such that*

$$u_{k_m} \to \chi_E - \chi_{E^c} \qquad \textit{in } L^1_{\text{loc}}(\Omega)$$

where E is a set with minimal perimeter in Ω. Moreover, if A is an open set, relatively compact in Ω, such that

$$\int_{\partial A} |D\chi_E| = 0,$$

then

$$\lim_{m\to\infty} J_{\varepsilon_{k_m}}(u_{k_m}, A) = P_A(E)\int_{-1}^{1}\sqrt{2h_0(s)}ds. \tag{12}$$

The second theorem was proved by Caffarelli and Cordoba in [8].

THEOREM 3.2. *Given $\alpha > -1$, $\beta < 1$, if u is a minimizer of J in B_R and $u(0) \geq \alpha$, then*

$$|\{u > \beta\} \cap B_R| \geq CR^n$$

for $R \geq R_0(\alpha,\beta)$, where C is a constant depending on n and h_0.

Next we use Theorem 2.1 to prove the following lemma.

LEMMA 3.3. *Let u be a local minimizer of J in $\mathbb{R}^n$ with $u(0) = 0$. Suppose that there exist sequences of positive numbers θ_k, l_k and unit vectors ξ_k with $l_k \to \infty$, $\theta_k l_k^{-1} \to 0$ such that*

$$\{u = 0\} \cap (\{|\pi_{\xi_k} x| < l_k\} \times \{|x\cdot\xi_k| < l_k\}) \subset \{|x\cdot\xi_k| < \theta_k\}.$$

Then the 0 level set is a hyperplane.

Proof: Fix $\theta_0 > 0$, and choose k large such that $\theta_k l_k^{-1} \leq \varepsilon \leq \varepsilon_1(\theta_0)$. If $\theta_k \geq \theta_0$ then we apply Theorem 2.1 and obtain the fact that $\{u = 0\}$ is trapped in a flatter cylinder. We apply Theorem 2.1 repeatedly until the height of the cylinder becomes less than θ_0.

In some system of coordinates we obtain

$$\{u = 0\} \cap \left(\{|y'| < l'_k\} \times \{|y_n| < l'_k\}\right) \subset \{|y_n| \leq \theta'_k\},$$

with $\theta_0 \geq \theta'_k \geq \eta_1\theta_0$ and $\theta'_k l_k^{'-1} \leq \theta_k l_k^{-1} \leq \varepsilon$; hence $l'_k \geq \varepsilon^{-1}\eta_1\theta_0$.

We let $\varepsilon \to 0$ and then $\{u = 0\}$ is included in an infinite strip of width θ_0. The lemma is proved since θ_0 is arbitrary. □

Proof of Theorem 2.3. The rescalings $u_\varepsilon(x) = u(\varepsilon^{-1}x)$ are local minimizers for J_ε in $\mathbb{R}^n$. From Theorem 3.1, there exists a sequence $\varepsilon_k \to 0$ such that

$$u_{\varepsilon_k} \to \chi_E - \chi_{E^c} \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n) \tag{13}$$

where E is a set with minimal perimeter.

Claim. $\{u_{\varepsilon_k} = 0\}$ *converges uniformly on compact sets to* ∂E.

Assume not; then there exist $\delta > 0$, $z_0 \in \mathbb{R}^n$, and points

$$x_k \in \{u_{\varepsilon_k} = 0\} \cap B(z_0, \delta)$$

with, say, $B(z_0, 2\delta) \subset E$. By Theorem 3.2, the set $\{u_{\varepsilon_k} < 0\}$ has uniform density in $B(z_0, 2\delta)$ for ε_k small, which contradicts (13).

Since ∂E is a minimal surface in $\mathbb{R}^n$, $n \le 7$, and $0 \in \partial E$, we conclude that ∂E is a hyperplane going through the origin. This implies

$$\{u_{\varepsilon_k} = 0\} \cap B_1 \subset \{|x_n| \le \delta_k\},$$

with $\delta_k \to 0$. Rescaling back we find that u satisfies the hypothesis of Lemma 3.3 and the theorem is proved. $\square$

Proof of Theorem 2.4. First we prove that a function u satisfying (9), (10) is a local minimizer in $\mathbb{R}^n$. For this, it suffices to show that in B_R, u is the unique solution of

$$\triangle v = h_0'(v), \quad |v| < 1, \quad v = u \quad \text{on } \partial B_R.$$

Since

$$\lim_{x_n \to \infty} u(x', x_n) = 1,$$

we conclude that the graph of $u(x', x_n + t)$ is above the graph of v for large t. We slide this graph in the e_n direction until we touch v for the first time. From the Strong Maximum Principle we find that the first touching point occurs on ∂B_R. Since u is strictly increasing in the e_n direction, we can slide the graph of $u(x', x_n + t)$ until it coincides with the graph of u; hence $u \ge v$. Similarly we obtain $u \le v$ which proves that u is a local minimizer in $\mathbb{R}^n$.

Assume $u(0) = 0$ and define $u_\varepsilon(x) = u(\varepsilon^{-1}x)$. Again we find that (13) holds for some sequence $\varepsilon_k \to 0$. Moreover, $u_n > 0$ implies E^c is a subgraph; hence ∂E is a quasi-solution in the e_n direction (see Chapters 16, 17 in Giusti [14]).

In both cases a) and b) one has that ∂E is a hyperplane and the theorem follows from Lemma 3.3. $\square$

4. Preliminaries

First we introduce some notation.

Notation.

$(e_1, \ldots, e_n, e_{n+1})$ is the Euclidean orthonormal basis in $\mathbb{R}^{n+1}$.

$X = (x, x_{n+1}) = (x', x_n, x_{n+1}) = (x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$.

$X \in \mathbb{R}^{n+1}, \quad x' \in \mathbb{R}^{n-1}, \quad x \in \mathbb{R}^n, \quad |x_{n+1}| < 1.$

$B(x, r)$ is the ball of center x and radius r in $\mathbb{R}^n$.

$\mathcal{B}(X, r)$ is the ball of center X and radius r in $\mathbb{R}^{n+1}$.

graph $u = \{(x, u(x)), \quad x \in \mathbb{R}^n\}$.

d_Γ is the signed distance to the surface Γ.

ν is a vector in $\mathbb{R}^{n+1}$, ξ a vector in $\mathbb{R}^n$.

$\angle(\nu_1, \nu_2)$ is the angle between the vectors ν_1 and ν_2.

$\pi_\nu X = X - (X \cdot \nu)\nu$ is the projection along ν.

π_i is the projection along e_i.

P_ν is the hyperplane perpendicular to ν going through the origin.

P_i is the hyperplane perpendicular to e_i going through the origin.

Constants depending on n, h_0 are called universal and we denote them by $\bar{C}_i$, $\bar{c}_i$, C_i, c_i ($\bar{C}_i$, $\bar{c}_i$ are constants used throughout the paper).

Preliminaries. In the proof we find, many times, inequalities involving a strictly increasing function g, and its derivatives g', g''. In this cases we consider $s = g$ as the new variable and we define a new function

$$h(s) = \frac{1}{2}\left(\frac{dg}{dt}\right)^2.$$

Now,

$$g' = \frac{dg}{dt} = \sqrt{2h}, \quad g'' = \frac{d^2g}{dt^2} = \frac{d}{dt}\sqrt{2h} = \frac{dh}{ds} = h'$$

and the inequality involves only h and h'. We can reconstruct g from h (up to a translation) since

$$H(s) := \int_0^s \frac{1}{\sqrt{2h(\zeta)}} d\zeta = g^{-1}(s) - g^{-1}(0).$$

In particular we define

$$H_0(s) := \int_0^s \frac{1}{\sqrt{2h_0(\zeta)}} d\zeta, \quad g_0(t) := H_0^{-1}(t)$$

and we find

$$g_0''(t) = h_0'(g_0(t));$$

thus, g_0 is a one-dimensional solution of (8).

5. The limiting equation

In this section we prove the following:

Proposition 5.1 (The limiting equation). *Let u be a local minimizer of J and assume that $u(0) = 0$. For some $\delta_0 > 0$ small, we consider the surfaces*

$$\Gamma = \left\{ x_n = P(x') := \frac{1}{2} x'^T M x' \right\}, \tag{14}$$

$$M \in \mathcal{M}_{(n-1)\times(n-1)}, \quad \triangle P = \operatorname{tr} M > \delta_0 \|M\|, \quad \|M\| \le \delta_0^{-1}.$$

There exists $\sigma_0(\delta_0) > 0$ small, such that if $\varepsilon \le \sigma_0(\delta_0)$ then Γ cannot touch from below $\{u_\varepsilon = 0\}$ at 0 in a $\delta_0(\triangle P)^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}$ neighborhood.

By "Γ touches from below $\{u_\varepsilon = 0\}$ at 0 in a $\delta_0(\triangle P)^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}$ neighborhood" we understand

$$\{u_\varepsilon = 0\} \cap \{x_n < P(x')\} \cap \{|x| < \delta_0(\triangle P)^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}\} = \emptyset.$$

Proposition 5.1 says that $\{u_\varepsilon = 0\}$ satisfies a mean curvature equation in some weak viscosity sense in which we have to specify the size of the neighborhood around the touching point. The size of the neighborhood depends on the polynomial P and ε.

If P is fixed and $\varepsilon \to 0$ then the radius of the neighborhood converges to 0. In particular, if $\{u_\varepsilon = 0\}$ converges uniformly to a surface, then this surface satisfies in the viscosity sense a mean curvature equation.

One way to interpret the above proposition is the following:
Suppose that P has positive mean curvature and let δ_0 be small such that (14) holds. Consider a spherical neighborhood around 0 such that P separates at just one point at a distance $\delta_0\varepsilon$ from $x_n = 0$. If r denotes the radius of this neighborhood then,

$$\|M\| r^2 \ge 2\delta_0\varepsilon \Rightarrow r^2 \ge \delta_0 \|M\|^{-1}\varepsilon \ge \delta_0^2(\triangle P)^{-1}\varepsilon.$$

Hence, if $\varepsilon < \sigma_0(\delta_0)$ then P cannot touch from below $\{u_\varepsilon = 0\}$ at 0 in the r neighborhood.

We shall prove the following version of Proposition 5.1.

LEMMA 5.2. *Let u be a local minimizer of J in $\{|x'| < l\} \times \{|x_n| < l\}$ and assume that $u(0) = 0$ and $u < 0$ below the surface*

$$\Gamma_1 := \left\{ x_n = P_1(x') = \frac{\theta}{l^2} \frac{1}{2} x'^T M_1 x' + \frac{\theta}{l} \xi \cdot x' \right\},$$

$$\|M_1\| < \delta^{-1}, \quad |\xi| < \delta^{-1}$$

for some small $\delta > 0$. There exists $\sigma(\delta) > 0$ small, such that if

$$\theta l^{-1} \le \sigma(\delta), \quad \theta \ge \delta,$$

then

$$\operatorname{tr} M_1 \le \delta.$$

Next we show that Proposition 5.1 follows from Lemma 5.2. Assume by contradiction that for some

$$\varepsilon \le \sigma_0(\delta_0) := \sigma^2 \left(\frac{\delta_0^2}{4} \right)$$

Γ touches $\{u_\varepsilon = 0\}$ from below at 0 in a $\delta_0(\triangle P)^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}}$ neighborhood. By rescaling we find that

$$\left\{ x_n = \frac{\varepsilon}{2} x'^T M x' \right\}$$

touches from below $\{u = 0\}$ at 0 in a $\delta_0(\triangle P)^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}$ neighborhood.

We apply Lemma 5.2 with

$$l = \frac{\delta_0}{2} (\text{tr } M)^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}, \quad \delta = \theta = \frac{\delta_0^2}{4}, \quad M_1 = (\text{tr } M)^{-1} M, \quad \xi = 0$$

thus,

$$\Gamma_1 = \{x_n = \frac{\varepsilon}{2} x'^T M x'\}.$$

Since

$$\|M_1\| = (\text{tr } M)^{-1} \|M\| < \delta_0^{-1} < \delta^{-1},$$

$$\theta l^{-1} = \frac{\delta_0}{2} (\text{tr } M)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} < \varepsilon^{\frac{1}{2}} \le \sigma \left(\frac{\delta_0^2}{4} \right) = \sigma(\delta),$$

we conclude

$$\delta \ge \operatorname{tr} M_1 = 1$$

which is a contradiction.

Before we prove Lemma 5.2 we need to introduce a comparison function. Using this function and the fact that $\{u < 0\}$ below Γ_1, we are able to bound u by the above.

Lemma 5.3 (Comparison function). *For $l>0$ large, there exists an increasing function g_l supported inside $(-\infty, l/2)$, $g_l(0)=0$, $g_l(s)$ is constant for $s \le -l/2$, such that the rotation surface*

$$\Psi(y,l) := \{x_{n+1} = g_l(|x-y|-l)\}$$

is, in the viscosity sense, a strict supersolution of (8) *everywhere except on the sphere $\{|x-y|=l\}$. Moreover, if $H_l = g_l^{-1}$ there exist universal constants $\bar{c}_1$ small, $\bar{C}_1$ large, such that H_l is defined on $(-1+e^{-\bar{c}_1 l}, 1)$, and*

$$H_0(s) \le H_l(s) - \frac{\bar{C}_1}{l}\log(1-|s|) \qquad \text{if} \quad |s| < 1 - e^{-\bar{c}_1 l/2}.$$

Proof. Define h_l, H_l (the corresponding functions for g_l) as

(15)
$$h_l(s) = \begin{cases} h_0(s) - h_0(s_l - 1) - \bar{C}_2 l^{-1}\left((1+s)^2 - {s_l}^2\right) & \text{if } s_l - 1 < s < 0 \\ h_0(s) + h_0(1-s_l) + \bar{C}_2 l^{-1}\left((1-s)^2 + s_l(1-s)\right) & \text{if } 0 \le s < 1, \end{cases}$$

(16)
$$H_l(s) = \int_0^s \frac{1}{\sqrt{2h_l(\zeta)}} d\zeta.$$

We choose $\bar{C}_2$ large, universal, such that

(17)
$$4(n-1)\sqrt{h_0(s)} < \bar{C}_2(1-|s|).$$

and $s_l = e^{-\bar{c}_1 l}$ with $\bar{c}_1$ small.

For $s_l - 1 < s \le 0$ and l large we have

$$\frac{1}{2}\left(h_0(s) - h_0(s_l-1)\right) \le h_l(s).$$

Hence

$$\begin{aligned} H_l(s_l - 1) &\ge -\int_{s_l-1}^{0} \left(h_0(\zeta) - h_0(s_l-1)\right)^{-\frac{1}{2}} d\zeta \\ &\ge -C_1 \int_{s_l-1}^{0} (1+\zeta-s_l)^{-\frac{1}{2}}(1+\zeta)^{-\frac{1}{2}} d\zeta \ge C_2 \log s_l \ge -l/2, \end{aligned}$$

if $\bar{c}_1$ is small enough. Moreover, for $0 > g_l(t) > s_l - 1$ we have

$$\begin{aligned} g_l''(t) + 2(n-1)l^{-1}g_l'(t) &= h_l'(s) + 2(n-1)l^{-1}\sqrt{2h_l(s)} \\ &< h_0'(s) - 2\bar{C}_2 l^{-1}(1+s) + 4(n-1)l^{-1}\sqrt{h_0(s)} < h_0'(s). \end{aligned}$$

On the domain where g_l is constant, i.e. $g_l = s_l - 1$, one has

$$\triangle \Psi(0,l) = 0 < h_0'(s_l - 1).$$

We remark that g_l is a $C^{1,1}$ function on $(-\infty, 0)$. Its second derivative has a small jump at $H_l(s_l-1)$ from 0 to $h_l'(s_l-1)$. From the above inequalities we can conclude that $g_l(|x|-l)$ is, in the viscosity sense, a strict supersolution for $|x|<l$.

If $e^{-\bar{c}_1 l/2}-1<s\le 0$, then

$$h_0(s)-h_l(s)\le h_0(s_l-1)+\bar{C}_2 l^{-1}(1+s)^2\le C_1 l^{-1}(1+s)^2.$$

Hence

$$\begin{aligned}H_0(s)-H_l(s)&=\int_s^0 \frac{1}{\sqrt{2h_l(\zeta)}}-\frac{1}{\sqrt{2h_0(\zeta)}}d\zeta\\&\le C_2\int_s^0 \frac{h_0(\zeta)-h_l(\zeta)}{(1+\zeta-s_l)^{\frac12}(1+\zeta)^{\frac52}}d\zeta\le -\bar{C}_1 l^{-1}\log(1+s).\end{aligned}$$

For $0<s<1$ we have

$$h_l(s)\le 2\left(h_0(s)+h_0(1-s_l)\right),$$

and

$$\begin{aligned}H_l(1)&\le\int_0^1 \left(h_0(\zeta)+h_0(1-s_l)\right)^{-\frac12}d\zeta\\&\le C_1\int_0^1\left((1-\zeta)^2+{s_l}^2\right)^{-\frac12}d\zeta\le -C_2\log s_l\le l/2.\end{aligned}$$

Also,

$$\begin{aligned}&g_l''(t)+(n-1)l^{-1}g_l'(t)=h_l'(s)+(n-1)l^{-1}\sqrt{2h_l(s)}\\&\quad<h_0'(s)-\bar{C}_2 l^{-1}(1-s+s_l)+4(n-1)l^{-1}\sqrt{h_0(s)+h_0(1-s_l)}<h_0'(s).\end{aligned}$$

Thus, $g_l(|x|-l)$ is a strict supersolution for $|x|>l$. We also remark that $g_l'(H(1))>0$.

If $0\le s<1-e^{-\bar{c}_1 l/2}$ then

$$h_l(s)-h_0(s)\le h_0(1-s_l)+2\bar{C}_2 l^{-1}(1-s)^2\le C_1 l^{-1}(1-s)^2.$$

Hence,

$$\begin{aligned}H_0(s)-H_l(s)&\le\int_0^s \frac{1}{\sqrt{2h_0(\zeta)}}-\frac{1}{\sqrt{2h_l(\zeta)}}d\zeta\\&\le C_2\int_0^s \frac{h_l(\zeta)-h_0(\zeta)}{(1-\zeta)^3}d\zeta\le -\bar{C}_1 l^{-1}\log(1-s).\end{aligned}$$

With this the lemma is proved. □

Next we construct a strict supersolution which is 0 on a surface Γ with positive mean curvature.

Lemma 5.4. *Let* Γ *be such that*

$$\Gamma = \left\{ x_n = P(x') = \frac{\varepsilon}{2} x'^T M x' + \sigma \xi \cdot x' \right\} \cap \{|x'| < \sigma \varepsilon^{-1}\},$$
$$\operatorname{tr} M \geq \delta, \quad \|M\| \leq \delta^{-1}, \quad |\xi| \leq \delta^{-1}$$

for some small $\delta > 0$. *There exists* $\sigma_1(\delta) > 0$, *such that if* $\varepsilon \leq \sigma \leq \sigma_1(\delta)$, *then there is a function* g_Γ *for which* $g_\Gamma(d_\Gamma)$ *is, in the viscosity sense, a strict supersolution, where* d_Γ *represents the signed distance to* Γ, $d_\Gamma > 0$ *above* Γ. (*We consider only the set where the distance* d_Γ *is realized at a point in the interior of* Γ.)

Proof. Define

$$h_\Gamma(s) = \max\{0, h_0(s) + c_1 \delta \varepsilon \rho(s)\}, \tag{18}$$

where

$$c_1 = 1/2 \min_{-1/2 \leq s \leq 1/2} \sqrt{h_0(s)},$$

$$\rho(s) = \begin{cases} -1 & \text{if } s \leq -1/2 \\ 2s & \text{if } -1/2 \leq s \leq 1/2 \\ 1 & \text{if } 1/2 \leq s. \end{cases}$$

Let $s_{\delta,\varepsilon}$ be the point near -1 for which $h_0(s_{\delta,\varepsilon}) = c_1 \delta \varepsilon$; hence $1 + s_{\delta,\varepsilon} \sim (\delta\varepsilon)^{\frac{1}{2}}$. Now,

$$H_\Gamma(s_{\delta,\varepsilon}) = \int_0^{s_{\delta,\varepsilon}} \frac{1}{\sqrt{2h_\Gamma(\zeta)}} d\zeta \geq \int_{s_{\delta,\varepsilon}}^0 \frac{-C_1}{\sqrt{(\zeta - s_{\delta,\varepsilon})(1+\zeta)}} \geq C_2(\delta) \log \varepsilon,$$
$$H_\Gamma(1) = \int_0^1 \frac{1}{\sqrt{2h_\Gamma(\zeta)}} d\zeta \leq \int_0^1 \frac{C_1}{\sqrt{(1-\zeta)^2 + \delta\varepsilon}} d\zeta \leq -C_2(\delta) \log \varepsilon,$$
$$H_\Gamma(s) \leq H_0(s).$$

Thus, $g_\Gamma(d) = H_\Gamma^{-1}(d)$ is defined for $d \leq H_\Gamma(1)$ and it is constant for $d \leq H_\Gamma(s_{\delta,\varepsilon})$.

Let d be the signed distance function to Γ. In an appropriate system of coordinates

$$D^2 d = \operatorname{diag}\left(\frac{-\kappa_1}{1 - d\kappa_1}, \ldots, \frac{-\kappa_{n-1}}{1 - d\kappa_{n-1}}, 0 \right)$$

where κ_i are the principal curvatures of Γ at the point where the distance is realized.

Notice that $|\kappa_i| \leq C_3(\delta)\varepsilon$; hence, for $|d| \leq C_2(\delta) \log \varepsilon^{-1}$ one has

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{-\kappa_i}{1 - d\kappa_i} &\leq -\sum_{i=1}^{n-1} \kappa_i + C(\delta)\varepsilon^2 \log \varepsilon^{-1} \\ &\leq -\triangle P + C_1 |\nabla P|^2 \|D^2 P\| + C(\delta) \varepsilon^{\frac{3}{2}} \\ &\leq -\varepsilon\delta + C(\delta)(\varepsilon\sigma^2 + \varepsilon^{\frac{3}{2}}) \leq -\varepsilon\delta + C_4(\delta)\varepsilon\sigma^{\frac{1}{2}}. \end{aligned}$$

Thus, for $g_\Gamma(d) > s_{\delta,\varepsilon}$,

$$\begin{aligned}\triangle g_\Gamma(d) &\leq g_\Gamma''(d) - \varepsilon(\delta - C_4(\delta)\sigma^{\frac{1}{2}})g_\Gamma'(d)\\ &= h_\Gamma'(s) - \varepsilon(\delta - C_4(\delta)\sigma^{\frac{1}{2}})\sqrt{2h_\Gamma(s)}\\ &\leq h_0'(s) + \varepsilon\left(2c_1\delta\chi_{[-1/2,1/2]} - (\delta - C_4(\delta)\sigma^{\frac{1}{2}})\sqrt{2h_\Gamma(s)}\right).\end{aligned}$$

If $\sigma_0(\delta)$ is chosen small enough, then

$$\triangle g_\Gamma(d) < h_0'(g_\Gamma(d)).$$

On the domain where g_Γ is constant, i.e. $d \leq H_\Gamma(s_{\delta,\varepsilon})$, we have

$$\triangle g_\Gamma = 0 < h_0'(s_{\delta,\varepsilon}).$$

Since g_Γ is a $C^{1,1}$ function we conclude that $g_\Gamma(d)$ is a strict supersolution in the viscosity sense and the lemma is proved. $\square$

Proof of Lemma 5.2. Assume by contradiction that tr $M_1 > \delta$. We apply Lemma 5.4 to the surface

$$\Gamma_2 = \{x_n = P_1(x') - \frac{\varepsilon\delta}{2}|x'|^2\} \cap \{|x'| \leq l\}$$

with $\varepsilon = \theta l^{-2}$, $\sigma = \theta l^{-1}$ and we find that $g_{\Gamma_2}(d_{\Gamma_2})$ is a strict supersolution if σ is small enough.

On the other hand we claim that

$$u(x) \leq g_{l/4}(d_{\Gamma_1}) \quad \text{if } |x'| \leq l/2,\ |x_n| \leq l/2\ . \tag{19}$$

In order to prove this we use Theorem 3.2, choosing $\alpha < 0$ small such that h_0' is strictly increasing on $[-1, \alpha]$ and $\beta = 0$. Then, there exists C_2 universal such that if $u(x) \geq \alpha$ then

$$B(x, C_2) \cap \{u > 0\} \neq \emptyset.$$

If $l \geq 8C_2$, then

$$u(x) < \alpha \quad \text{for} \quad x \in B((0, -l/2), l/4) \tag{20}$$

Since $\Psi((0, -l/2), l/4)$ is a supersolution of (8) in $B((0, -l/2), l/4)$ and it is supported inside $B((0, -l/2), 3l/8)$ (see Lemma 5.3), we conclude from the maximum principle that u is below $\Psi((0, -l/2), l/4)$.

We slide this surface continuously along vectors ν,with $\nu \cdot e_{n+1} = 0$, $\nu \cdot e_n \geq 0$, till we touch the graph of u. Since $\Psi((0, -l/2), l/4)$ is a strict supersolution everywhere except on the 0 level set, we find that the touching points can occur only on the 0 level set.

The inequality (19) now follows from the fact that, if σ is small enough, at each point of Γ_1 we have a tangent sphere of radius $l/4$ from below which can be obtained from the sphere $|x - (0, -l/2)| = l/4$ by sliding it continuously inside the domain $\{u < 0\}$.

Now it suffices to prove that for $\sigma \le \sigma_1(\delta)$ we have

$$g_{\Gamma_2}(d_{\Gamma_2}) > g_{l/4}(d_{\Gamma_1}) \quad \text{on } \{|x'| = l/2\} \cap \{|d_{\Gamma_1}| \le l/4\}. \tag{21}$$

Then we slide $g_{\Gamma_2}(d_{\Gamma_2})$ from below in the e_n direction in the cylinder

$$\{|x'| \le l/2\} \times \{|x_n| \le l/2\}$$

till we touch u. By (19), (21) this cannot happen on $\{|x'| = l/2\}$ therefore the contact point is an interior point which is a contradiction to the fact that $g_{\Gamma_2}(d_{\Gamma_2})$ is a strict supersolution.

We notice that on $\{|x'| = l/2\} \cap \{|d_{\Gamma_1}| \le l/4\}$ we have

$$d_{\Gamma_2} \ge d_{\Gamma_1} + c_1(\delta);$$

thus, in order to prove (21), it suffices to show that

$$H_{\Gamma_2}(s) < H_{l/4}(s) + c_1(\delta). \tag{22}$$

From (15), (18) we find that for $l = \theta\sigma^{-1} \ge C_1(\delta)$ large,

$$h_{\Gamma_2}(s) \le h_{l/4}(s) \quad \text{if } s \le -1 + c_2(\delta)l^{-\frac{1}{2}}$$
$$h_{\Gamma_2}(s) \ge h_{l/4}(s) \quad \text{if } s \ge 1 - c_2(\delta)l^{-\frac{1}{2}}.$$

This implies that the maximum of $H_{\Gamma_2}(s) - H_{l/4}(s)$ occurs for $1-|s| \ge c_2(\delta)l^{-\frac{1}{2}}$. For these values of s we have

$$H_{\Gamma_2}(s) \le H_0(s) \le H_{l/4}(s) + 4\bar{C}_1 l^{-1} \log \frac{l^{\frac{1}{2}}}{c_2(\delta)} < H_{l/4}(s) + c_1(\delta).$$

With this the lemma is proved. □

6. Theorem 2.2 implies Theorem 2.1

The proof is by compactness. Assume by contradiction that there exists u_k, θ_k, l_k, ξ_k such that u_k is a local minimizer of J, $u_k(0) = 0$, the level set $\{u_k = 0\}$ stays in the flat cylinder

$$\{|x'| < l_k\} \times \{|x_n| < \theta_k\},$$

$\theta \ge \theta_0$, $\theta_k l_k^{-1} \to 0$ as $k \to \infty$ for which the conclusion of Theorem 2.1 doesn't hold.

Let A_k be the rescaling of the 0 level sets given by

$$(x', x_n) \in \{u_k = 0\} \mapsto (y', y_n) \in A_k,$$
$$y' = x' l_k^{-1}, \quad y_n = x_n \theta_k^{-1}.$$

Claim 1. *A_k has a subsequence that converges uniformly on $|y'| \le 1/2$ to a set $A_\infty = \{(y', w(y')), \quad |y'| \le 1/2\}$ where w is a Holder continuous function. In other words, given ε, all but a finite number of the A_k's from the subsequence are in an ε neighborhood of A_∞.*

Proof. Fix y_0', $|y_0'| \le 1/2$ and suppose $(y_0', y_k) \in A_k$. We apply Theorem 2.2 for the function u_k in the cylinder

$$\{|x' - l_k y_0'| < l_k/2\} \times \{|x_n - \theta_k y_k| < 2\theta_k\}$$

in which the set $\{u_k = 0\}$ is trapped. Thus, there exist a universal constant $\eta_0 > 0$ and an increasing function $\varepsilon_0(\theta) > 0$, $\varepsilon_0(\theta) \to 0$ as $\theta \to 0$, such that $\{u_k = 0\}$ is trapped in the cylinder

$$\{|x' - l_k y_0| < \eta_0 l_k/2\} \times \{|x_n - \theta_k y_k| < 2(1-\eta_0)\theta_k\}$$

provided that $4\theta_k l_k^{-1} \le \varepsilon_0(2\theta_k)$. Rescaling back we find that

$$A_k \cap \{|y' - y_0'| \le \eta_0/2\} \subset \{|y_n - y_k| \le 2(1-\eta_0)\}.$$

We apply the Harnack inequality repeatedly and we find that

$$A_k \cap \{|y' - y_0'| \le \eta_0^m/2\} \subset \{|y_n - y_k| \le 2(1-\eta_0)^m\} \tag{23}$$

provided that

$$4\theta_k l_k^{-1} \le \eta_0^{m-1} \varepsilon_0(2(1-\eta_0)^m \theta_k).$$

Since these inequalities are satisfied for all k large we conclude that (23) holds for all but a finite number of k's.

There exist positive constants α, β depending only on η_0, such that if (23) holds for all $m \le m_0$ then A_k is above the graph

$$y_n = y_k - 2(1-\eta_0)^{m_0-1} - \alpha|y' - y_0'|^\beta$$

in the cylinder $|y'| \le 1/2$.

Taking the supremum over these functions as y_0' varies we obtain that A_k is above the graph of a Holder function $y_n = a_k(y')$. Similarly we obtain that A_k is below the graph of a Holder function $y_n = b_k(y')$. Notice that

$$b_k - a_k \le 4(1-\eta_0)^{m_0-1} \tag{24}$$

and that a_k, b_k have a modulus of continuity bounded by the Holder function αt^β. From the Arzela-Ascoli Theorem we find that there exists a subsequence a_{k_p} which converges uniformly to a function w. Using (24) we obtain that b_{k_p}, and therefore A_{k_p}, converge uniformly to w.

Claim 2. *The function w is harmonic* (*in the viscosity sense*).

Proof. The proof is by contradiction. Fix a quadratic polynomial

$$y_n = P(y') = \frac{1}{2} {y'}^T M y' + \xi \cdot y', \qquad \|M\| < \delta^{-1}, \quad |\xi| < \delta^{-1}$$

such that $\triangle P > \delta$, $P(y') + \delta|y'|^2$ touches the graph of w, say, at 0 for simplicity, and stays below w in $|y'| < 2\delta$. Thus, for all k large we find points $({y_k}', y_{kn})$ close to 0 such that $P(y') + \text{const}$ touches A_k from below at $({y_k}', y_{kn})$ and stays below it in $|y' - y_k'| < \delta$. This implies that, eventually, after a translation, there exists a surface

$$\left\{ x_n = \frac{\theta_k}{l_k^2}\frac{1}{2}x'^T M x' + \frac{\theta_k}{l_k}\xi_k \cdot x' \right\}, \quad |\xi_k| < 2\delta^{-1}$$

that touches $\{u_k = 0\}$ at the origin and stays below it in the cylinder $|x'| < \delta l_k$. We write the above surface in the form

$$\left\{ x_n = \frac{\delta^2\theta_k}{2(\delta l_k)^2}x'^T M x' + \frac{\delta^2\theta_k}{\delta l_k}\delta^{-1}\xi_k \cdot x' \right\},$$

and we contradict Lemma 5.2 since $\theta_k \geq \theta_0$, $\theta_k l_k^{-1} \to 0$ and $\triangle P > \delta$.

Since w is harmonic, there exist $0 < \eta_1 < \eta_2$ small (depending only on n) such that

$$|w - \xi \cdot y'| < \eta_1/2 \qquad \text{for } |y'| < 2\eta_2\ .$$

Rescaling back and using the fact that the A_k converge uniformly to the graph of w we conclude that for k large enough

$$\{u_k = 0\} \cap \{|x'| < 3l_k\eta_2/2\} \subset \{|x_n - \theta_k l_k^{-1}\xi \cdot x'| < 3\theta_k\eta_1/4\}.$$

This is a contradiction to the fact that u_k doesn't satisfy the conclusion of Theorem 2.1.

7. Construction of the sliding surfaces $\mathcal{S}(Y,R)$

In this section we introduce a family of rotation surfaces in $\mathbb{R}^{n+1}$ which we denote by $\mathcal{S}(Y,R)$. We say that the point Y is the center of $\mathcal{S}$ and R the radius.

The surfaces $\mathcal{S}$ are defined for centers Y in the strip $\{|y_{n+1}| \leq 1/4\}$ and for radius R large. They have the following property:

Suppose that for fixed R, some surfaces $\mathcal{S}(Y,R)$ are tangent by the above to the graph of u. Then the contact points project along e_n into a set with measure comparable with the measure of the projection of the centers Y along e_n (see Proposition 7.1).

We define $\mathcal{S}(Y,R)$ as

$$\mathcal{S}(Y,R) := \{x_{n+1} = g_{y_{n+1},R}(H_0(y_{n+1}) + |x-y| - R)\}, \quad |y_{n+1}| \leq 1/4, \tag{25}$$

where the function $g_{s_0,R}$, respectively $h_{s_0,R}$, $H_{s_0,R}$ associated with it, are constructed below for $|s_0| \leq 1/4$ and large R. For simplicity of notation we denote them by g, h, H.

Denote

$$\bar{C}_3 = 1 + 8(n-1)\max\sqrt{h_0} \tag{26}$$

and let φ be such that

$$\frac{1}{\sqrt{2\varphi(s)}} = \frac{1}{\sqrt{2h_0(s)}} - \frac{\bar{C}_0}{R}(s - s_0). \tag{27}$$

where $\bar{C}_0$ is large enough such that the following hold:

$$\begin{aligned} \varphi(s) &< h_0(s) - 2\bar{C}_3 R^{-1}, \quad \text{if } s \in [-3/4, -1/2], \\ \varphi(s) &> h_0(s) + 2\bar{C}_3 R^{-1}, \quad \text{if } s \in [1/2, 3/4]. \end{aligned} \tag{28}$$

Let s_R near -1 be such that $h_0(s_R) = R^{-1}$; hence $1 + s_R \sim R^{-\frac{1}{2}}$. We define $h_{s_0,R} : [s_R, 1] \to \mathbb{R}$ as

$$h(s) = \begin{cases} h_0(s) - h_0(s_R) - \bar{C}_3 R^{-1}(s - s_R) & \text{if } s \in [s_R, -\frac{1}{2}] \\ \varphi(s) & \text{if } s \in (-1/2, 1/2) \\ h_0(s) + R^{-1} + \bar{C}_3 R^{-1}(1 - s) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases} \tag{29}$$

For R large, $h(s) \geq c_1(1+s)(s-s_R)$ on $[s_R, 0]$; thus h is positive on $(s_R, 1]$. Define

$$H_{s_0,R}(s) = H_0(s_0) + \int_{s_0}^{s} \frac{1}{\sqrt{2h(\zeta)}} d\zeta \tag{30}$$

and for R large enough

$$\begin{aligned} H(s_R) &\geq H_0(s_0) - \int_{s_R}^{s_0} \frac{1}{\sqrt{c_1(1+\zeta)(\zeta - s_R)}} d\zeta \geq -C_1 \log R, \\ H(1) &\leq H_0(s_0) + \int_{s_0}^{1} \frac{1}{\sqrt{c_2(1-\zeta)^2 + R^{-1}}} d\zeta \leq C_1 \log R. \end{aligned} \tag{31}$$

Finally we define $g_{s_0,R}$ as

$$g_{s_0,R}(t) = \begin{cases} s_R & \text{if } t < H(s_R), \\ H^{-1}(t) & \text{if } H(s_R) \leq t \leq H(1). \end{cases} \tag{32}$$

Next we list some properties of the surfaces $\mathcal{S}(Y, R)$:

1) Notice that

$$\begin{aligned} h(s) &> h_0(s) - 2\bar{C}_3 R^{-1} > \varphi(s) \quad \text{if } s \in [-3/4, -1/2], \\ h(s) &< h_0(s) + 2\bar{C}_3 R^{-1} < \varphi(s) \quad \text{if } s \in [1/2, 3/4]. \end{aligned} \tag{33}$$

From (27), (29), (33) we have

$$\begin{aligned} H(s) &= H_0(s) - \frac{\bar{C}_0}{2R}(s - s_0)^2, \quad \text{if } |s| \leq 1/2, \\ H(s) &> \ H_0(s) - \frac{\bar{C}_0}{2R}(s - s_0)^2 \quad \text{if } 1/2 < |s| < 3/4. \end{aligned} \tag{34}$$

Let $\rho_{s_0,R}$ be the function whose graph is obtained from the graph of g_0 by the transformation

$$(t,s) \mapsto \left(t - \frac{\bar{C}_0}{2R}(s-s_0)^2, s\right) \quad \text{for } |s| < 3/4.$$

From (34) we obtain that $g = \rho$ for $|s| \leq 1/2$, and $g < \rho$ at all other points where ρ is defined.

In other words, if $S(Y,R)$ is the rotation surface,

$$S(Y,R) := \{x_{n+1} = \rho_{y_{n+1},R}(H_0(y_{n+1}) + |x-y| - R)\}, \qquad |y_{n+1}| \leq 1/4, \tag{35}$$

then, $\mathcal{S}(Y,R)$ coincides with $S(Y,R)$ in the set $|x_{n+1}| \leq 1/2$ and stays below it at all the other points where S is defined.

Notice that $S(Y,R) \subset \{|x_{n+1}| \leq 3/4\}$ and is defined only in a neighborhood of the sphere $|x-y| = R$ which is the y_{n+1} level set of $S(Y,R)$.

2) We remark that $\mathcal{S}(Y,R)$ is constant s_R when

$$|x-y| \leq R - \frac{1}{2}R^{\frac{1}{3}} \leq R - H_0(y_{n+1}) + H(s_R),$$

and grows from s_R to 1 when

$$-\frac{1}{2}R^{\frac{1}{3}} < |x-y| - R \leq -H_0(y_{n+1}) + H(1) < \frac{1}{2}R^{\frac{1}{3}}.$$

3) The function g is $C^{1,1}$ in $(-\infty, H(-1/2)) \cup (H(1/2), H(1))$ and g'' has a small jump from 0 to $h'(s_R)$ at $H(s_R)$.

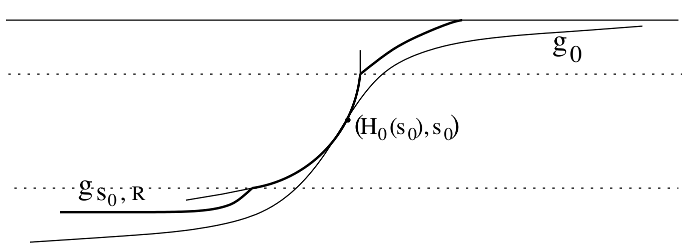
If $s \in (s_R, -1/2) \cup (1/2, 1)$, then on the s level set we have (see (26))

$$\begin{aligned} \triangle\mathcal{S} &\leq h'(s) + 2(n-1)R^{-1}\sqrt{2h(s)} \\ &\leq h_0'(s) - \bar{C}_3 R^{-1} + 4(n-1)\sqrt{h(s)}R^{-1} < h_0'(s). \end{aligned} \tag{36}$$

Moreover, from (29), (33) we have

$$\lim_{s \to -1/2^-} H'(s) < \lim_{s \to -1/2^+} H'(s), \quad \lim_{s \to 1/2^-} H'(s) < \lim_{s \to 1/2^+} H'(s), \quad \lim_{s \to 1^-} H'(s) < \infty \tag{37}$$

which together with (36) implies that $\mathcal{S}(Y,R)$ is, in the viscosity sense, a strict supersolution for $|x_{n+1}| \geq 1/2$. In other words $\mathcal{S}(Y,R)$ cannot touch from above a C^2 subsolution at a point X with $|x_{n+1}| \geq 1/2$.



4) If $|s| < 3/4$, then on the s level set of $S(Y,R)$, defined in (35), one has (see (27))

$$h_0'(s) - C_2R^{-1} \le \varphi'(s) \le \triangle S, \tag{38}$$
$$\triangle S \le \varphi'(s) + \frac{2(n-1)}{R}\sqrt{2\varphi(s)} \le h_0'(s) + C_2R^{-1}.$$

This shows that $S(Y,R)$ is an approximate solution of equation (8) with an R^{-1} error.

5) From (27), (29) we see that if $R_1 \le R_2$, then

$$h_{s_0,R_1}(s) \le h_{s_0,R_2}(s) \quad \text{if } s_{R_1} \le s \le s_0,$$
$$h_{s_0,R_1}(s) \ge h_{s_0,R_2}(s) \quad \text{if } s_0 \le s \le 1;$$

thus,

$$H_{s_0,R_1}(s) \le H_{s_0,R_2}(s) \tag{39}$$

in the domain where H_{s_0,R_1} is defined.

The next proposition is the key tool in proving Theorem 2.2.

Proposition 7.1 (Measure estimate for contact points). *Let u be a C^2 subsolution of* (8); *i.e.*, $\triangle u \ge h_0'(u)$, $|u| \le 1$. *Let ξ be a vector perpendicular to e_{n+1} and A be a closed set in $P_\xi \cap \{|x_{n+1}| \le 1/4\}$. Assume that for each $Y \in A$ the surface $\mathcal{S}(Y + t\xi, R)$, R large, stays above the graph of u when $t \to -\infty$ and, as t increases, it touches the graph from above for the first time at a point* (*contact point*). *If B denotes the projection of the contact points along ξ in P_ξ, then,*

$$\bar{\mu}_0|A| \le |B|$$

where $\bar{\mu}_0 > 0$ is universal, small and $|A|$ represents the n-dimensional Lebesgue measure.

Proof. Assume that $\mathcal{S}(Y,R)$ touches u from above at the point $X = (x, u(x))$. From the discussion above we find $|u(x)| < 1/2$.

Denote by ν the normal to the surface at X; i.e.

$$\nu = (\nu', \nu_{n+1}) = \frac{1}{\sqrt{1+|\nabla u|^2}}(-\nabla u, 1).$$

The center Y is given by

$$Y(X) = \left(x + \frac{\nu'}{|\nu'|}\sigma, x_{n+1} + \omega\right) = F(X,\nu), \tag{40}$$

where

$$\omega = R\bar{C}_0^{-1}(\nu_{n+1}|\nu'|^{-1} - H_0'(x_{n+1})), \tag{41}$$
$$\sigma = -\frac{\bar{C}_0}{2R}\omega^2 + H_0(x_{n+1}) - H_0(x_{n+1}+\omega) + R.$$

The function F is smooth defined on

$$\{X \in \mathbb{R}^{n+1} : |x_{n+1}| < 1/2\} \times \{\nu \in \mathbb{R}^{n+1} : |\nu| = 1, c_1 < \nu_{n+1} < 1 - c_1\}.$$

The differential $D_X Y$ is a linear map defined on T_X, the tangent plane at X, and

$$D_X Y = F_X(X,\nu) + F_\nu(X,\nu) D_X \nu = F_X(X,\nu) - F_\nu(X,\nu) II_u \tag{42}$$

where II_u represents the second fundamental form of u at X. Writing the above formula for the surface $S(Y,R)$ at X, we find

$$0 = F_X(X,\nu) - F_\nu(X,\nu) II_S$$

thus, (42) becomes

$$D_X Y = F_\nu(X,\nu)(II_S - II_u). \tag{43}$$

From (40) and (41), it is easy to check that

$$\|F_\nu(X,\nu)\| \le C_1 R. \tag{44}$$

Since S touches u by above at X, we find that $D^2 S - D^2 u \ge 0$. On the other hand, from (38),

$$\triangle S \le h_0'(x_{n+1}) + C_2 R^{-1} \le \triangle u + C_2 R^{-1}$$

which implies

$$\|D^2 S - D^2 u\| \le C_3 R^{-1}$$

or

$$\|II_S - II_u\| \le C_4 R^{-1}. \tag{45}$$

From (43), (44), (45) we conclude

$$\|D_X Y\| \le C_5.$$

The centers Z for which $X \in S(Z,R)$ describe a rotation surface, around X. Note that if $S(\cdot,R)$ is above u, then its center is above this surface. The normal to the surface at $Y(X)$, which we denote by τ, belongs to the plane spanned by ν and e_{n+1}, and $c_2 < \tau < 1 - c_2$. Thus, if ξ is perpendicular to e_{n+1}, we have

$$|\tau \cdot \xi| \le C_6 |\nu \cdot \xi|.$$

(Notice that the tangent plane to the surface at $Y(X)$ is the range of $F_\nu(X,\nu)$.)

Let $\tilde{B}$ be the set of contact points, $\tilde{A}$ the set of the corresponding centers, $B = \pi_\xi \tilde{B}$ and $A = \pi_\xi \tilde{A}$. Note that π_ξ is injective on $\tilde{A}$ and $\tilde{B}$ by construction. From above, we know that $\tilde{A}$ belongs to a Lipschitz surface. One has

$$\begin{aligned} |A| = \int_{\tilde{A}} |\tau(Y)\cdot\xi| dY &\le \int_{\tilde{B}} |\tau(Y)\cdot\xi||D_X Y| dX \\ &\le C_7 \int_{\tilde{B}} |\nu(X)\cdot\xi| dX = C_7 |B| \end{aligned}$$

and the proposition is proved. $\square$

8. Extension of the contact set

In this section we prove that the contact set from Proposition 7.1 becomes larger and larger when possibly we decrease the radius R.

Denote

$$L = P_n \cap \{|x_{n+1}| \le 1/2\},$$
$$Q_l = \left\{(x', 0, x_{n+1})/\ |x'| \le l, |x_{n+1}| \le 1/2\right\}.$$

Let $\tilde{D}_k$, represent the set of points on the graph of u that have from above a tangent surface $\mathcal{S}(Y, RC^{-k})$, where C is a large universal constant. Suppose that we have some control on the e_n coordinate of these sets and denote by D_k their projections into L.

Recall that $\mathcal{S}(Y, RC^{-k})$ is an approximate solution of equation (8) with a $C^k R^{-1}$ error. If $\mathcal{S}(Y, RC^{-k})$ touches u from above at X_0 then, from Harnack inequality, the two surfaces stay $C^k R^{-1}$ close to each other in a neighborhood of X_0 (see Lemma 8.1). Thus, denoting

$$E_k = \{Z \in L/\ \mathrm{dist}(Z, D_k) \le C_1\},$$

we control the e_n coordinate of a set on the graph of u that projects along e_n into E_k.

We want to prove that, in measure, E_k almost covers Q_l as k becomes larger and larger.

In large scale the interface satisfies a mean curvature equation. In Lemma 8.2 we prove that near (large scale) a point $Z \in D_k$ we can find a set of positive measure in D_{k+1}. Using a covering argument we show that the sets E_k "almost" cover Q_l as k increases.

Next we state and prove two technical lemmas, Lemma 8.1 and Lemma 8.2. At the end of the section we prove a covering lemma which links the two scales.

LEMMA 8.1 (Small scale extension). *Suppose that the surface $\mathcal{S}(Y, R)$ touches a solution u from above at $X_0 = (x_0, u(x_0))$ with*

$$\angle\left(\frac{\nabla u}{|\nabla u|}(x_0), e_n\right) \le \frac{\pi}{8}.$$

Given a constant $a > 1$ large, there exists $C(a) > 0$ depending on universal constants and a such that for each point $Z \in L \cap \mathcal{B}(\pi_n X_0, a)$ there exists x with

1) $\pi_n(x, u(x)) = Z$, $|x - x_0| \le 2a$,
2) $(x - x_0) \cdot \dfrac{\nabla u}{|\nabla u|}(x_0) \le H_0(u(x)) - H_0(u(x_0)) + C(a)R^{-1}$.

LEMMA 8.2 (Large scale extension). *Suppose that the surface $\mathcal{S}(Y_0, R)$ stays above a C^2 subsolution u in the cylinder $\{|x'| < l\} \times \{|x_n| < l\}$, $l > 4R^{\frac{1}{3}}$ and touches the graph of u at $(x_0, u(x_0))$ with*

$$|u(x_0)| < 1/2, \quad |x_{0n}| < l/4, \quad |x_0'| = q, \quad q < l/4,$$
$$\angle\left(\frac{\nabla u}{|\nabla u|}(x_0), e_n\right) \le \frac{\pi}{8}.$$

There exist universal constants $\bar{C}_4$, $\bar{C}_5$, large, $\bar{c}_2$, small, such that if $\bar{C}_4 \le q$, $l \le R\bar{c}_2$ then the set of points $(x, u(x))$ satisfy the following four properties:

1) $|x'| < q/15$, $|u(x)| < 1/2$, $|x - x_0| < 2q$.
2) *There exists a surface $\mathcal{S}(Y, R/\bar{C}_5)$ that stays above u and touches its graph at $(x, u(x))$.*
3) $\angle\left(\frac{\nabla u}{|\nabla u|}(x), \frac{\nabla u}{|\nabla u|}(x_0)\right) \le \bar{C}_4 q R^{-1}$.
4) $(x - x_0) \cdot \frac{\nabla u}{|\nabla u|}(x_0) \le \bar{C}_4 q^2 R^{-1} + H_0(u(x)) - H_0(u(x_0))$ *project along e_n into a set of measure greater than $\bar{c}_2 q^{n-1}$.*

Remark. The term $H_0(u(x)) - H_0(u(x_0))$ that appears in property 2 of Lemma 8.1 and property 4 of Lemma 8.2 represents the distance between the $u(x)$ level surface and the $u(x_0)$ level surface of a one dimensional solution.

Now we state the iteration lemma that links Lemmas 8.2 and 8.1.

Lemma 8.3 (Covering lemma). *Let D_k be closed sets, $D_k \subset L$, with the following properties*:

1) $D_0 \cap Q_l \ne \emptyset$, $\quad D_0 \subset D_1 \subset D_2 \ldots$;
2) *If $Z_0 \in D_k \cap Q_{2l}$, $Z_1 \in L$, $|Z_1 - Z_0| = q$ and $2l \ge q \ge a$ then,*
$$|D_{k+1} \cap \mathcal{B}(Z_1, q/10)| \ge \mu_1 |\mathcal{B}(Z_1, q) \cap L|$$
where $a > 1$ (large), μ_1 (small) are given positive constants and $l > 2a$. Denote by E_k the set
$$E_k := \{Z \in L / \operatorname{dist}(Z, D_k) \le a\}.$$
Then there exists $\mu > 0$ depending on n, μ_1 such that
$$|Q_l \setminus E_k| \le (1 - \mu)^k |Q_l|.$$

Proof of Lemma 8.1. Let $S(Y, R)$ be the surface defined in (35). Notice that $S(Y, R)$ touches u from above at X_0. The restrictions
$$\pi_{n|S} : S(Y_0, R) \to P_n, \quad \pi_{n+1|S} : S(Y_0, R) \to P_{n+1}$$
are diffeomorphisms in a $3a$ neighborhood of X_0 for R large. Denote by T the map
$$T := \pi_{n+1|S} \circ \pi_{n|S}^{-1} : P_n \cap \{|x_{n+1}| < 3/4\} \to P_{n+1}.$$

In the set

$$O_1 := T\left(P_n \cap \{|x_{n+1}| < 3/4\} \cap \mathcal{B}(\pi_n X_0, a+2)\right)$$

we have $0 \le S - u$, $0 = S(x_0) - u(x_0)$. From (38) and the fact that h_0' is Lipschitz we find

$$C_1(S-u) + C_1 R^{-1} \ge |\triangle(S-u)|.$$

The open set

$$O_2 := T\left(P_n \cap \{|x_{n+1}| < 5/8\} \cap \mathcal{B}(\pi_n X_0, a+1)\right)$$

satisfies $O_2 \subset O_1$, $\mathrm{dist}(O_2, \partial O_1) \ge c_1$, with $c_1 > 0$, universal. From Harnack inequality, one obtains

$$\sup_{x \in O_2} (S-u) \le C'(a) R^{-1}. \tag{46}$$

For each $Z \in L \cap \mathcal{B}(\pi_n X_0, a)$ we consider the line $Z + te_n$ and denote by X_1 its intersection with $S(Y,R)$.

Notice that in O_1 we have $\partial_n S \ge c_2$, $c_2 > 0$ universal. From this, (46), and the continuity of u we find that $Z + te_n$ intersects the graph of u at a point $X_2 = (x_2, u(x_2))$ with

$$|X_2 - X_1| \le C''(a) R^{-1}.$$

Since

$$(x_1 - x_0) \cdot \frac{\nabla u}{|\nabla u|}(x_0) \le H_0(z_{n+1}) - H_0(u(x_0)) + C_2 R^{-1}$$

we conclude that

$$(x_2 - x_0) \cdot \frac{\nabla u}{|\nabla u|}(x_0) \le H_0(u(x_2)) - H_0(u(x_0)) + C(a) R^{-1}$$

and the lemma is proved. □

Proof of Lemma 8.2. The proof consists of two steps. In Step 1 we find a point that satisfies properties 2–4 and property 1 with $q/40$ instead of $q/15$. In Step 2 we use Proposition 7.1 to extend properties 2–4 from that point to a set of positive measure.

Before we start, we introduce some notation. For a surface $\mathcal{S}(Y,R)$ we associate its 0 level surface, the $n-1$ dimensional sphere

$$\Sigma(y,r) = \left\{ |x-y| = r := R - H_0(y_{n+1}) - \frac{\bar{C}_0}{2R} y_{n+1}^2 \right\}.$$

We remark that the s level surface of $\mathcal{S}$, $|s| < 1/2$, is a concentric sphere at a (signed) distance

$$H_0(s) + O(1)\bar{C}_0 R^{-1}, \qquad |O(1)| < 1/2 \tag{47}$$

from $\Sigma(y,r)$. Also for a point $X=(x,x_{n+1})\in\mathcal{S}(Y,R)$, $|x_{n+1}|<1/2$ we associate the point

$$\tilde{x}=[y,x)\cap\Sigma(y,r)$$

where $[y,x)$ represents the half line from y going through x.

First we prove the lemma in the following situation (this is a rotation of the above configuration):

The surface $\mathcal{S}(Y_0,R_0)$ stays above the graph of u in the cylinder $\{|x'|\le 2q\}\times\{|x_n|\le l/2\}$ and touches it at $X_0=(x_0,u(x_0))$, $|u(x_0)|<1/2$. Assume

$$\tilde{x}_0\in\{|x'|=q\}\cap\{x_n=0\},\quad y_0=-e_n\sqrt{r_0^2-q^2},$$

$q\ge c_1^{-1}$ large, and $q/R_0\le c_1$, c_1 small, universal.

Step 1. We prove the existence of a surface $\mathcal{S}(Y_*,R_*)$ that stays above u in the cylinder $|x'|\le 2q$ and touches it at $(x_*,u(x_*))$ such that

$$Y_*=Y_0+t_*e_n,\quad R_*>R_0/C_3,\quad \tilde{x}_*\in\left\{x_n<C_4\frac{q^2}{R_0}\right\}\cap\left\{|x'|<\frac{q}{100}\right\}$$

where C_3, C_4 are large universal constants.

From (26), (27), (29) we obtain the existence of C_1, $\bar{C}_3$ universal such that

$$\begin{gathered}(2h_{s_0,R}(s))^{-1/2}|h'_{s_0,R}(s)-h'_0(s)|\le C_1R^{-1}\quad\text{if } |s|<1/2,\\ h'_{s_0,R}(s)=h'_0(s)-\bar{C}_3R^{-1}\quad\text{if } s\in(s_R,-1/2)\cup(1/2,1)\ ,\\ \bar{C}_3=1+8(n-1)\max\sqrt{h_0}.\end{gathered}\tag{48}$$

We consider the function $\psi:\mathbb{R}^{n-1}\to\mathbb{R}$:

$$\psi(z')=\frac{1}{\gamma}(|z'|^{-\gamma}-1),\qquad z'\in\mathbb{R}^{n-1},$$

where γ is such that

$$\gamma=4(C_1+6(n-2)).\tag{49}$$

Finally, we choose $\omega<1$, universal, such that $\omega^{-\gamma-2}=2$. The graph

$$x_n=\frac{q^2}{\sqrt{r_0^2-q^2}}\psi\left(\frac{x'}{q}\right)\tag{50}$$

has by below the tangent sphere $\Sigma(y_0,r_0)$ when $|x'|=q$, and a tangent sphere of radius r_ω and center y_ω when $|x'|=\omega q$, where

$$r_\omega=\omega^{\gamma+2}\sqrt{r_0^2+q^2(\omega^{-2\gamma-2}-1)}\ge r_0/2.$$

Let Γ_1 denote the graph of $\Sigma(y_0,r_0)$ for $|x'|>q$ below $x_n=0$, Γ_2 the graph of the above function for $\omega q\le|x'|\le q$ and Γ_3 the graph of $|x-y_\omega|=r_\omega$

when $|x'| < \omega q$, $x_n > 0$. We notice that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is a $C^{1,1}$ surface in $\mathbb{R}^n$. We define the following surface in $\mathbb{R}^{n+1}$

$$\Psi = \left\{ x_{n+1} = g_{y_{0_{n+1}}, R_0} \left(d_\Gamma + H_{y_{0_{n+1}}, R_0}(0) \right) \right\},$$

where d_Γ represents the signed distance to the surface Γ (d_Γ positive in the exterior of Γ). Note that Ψ coincides with $\mathcal{S}(Y_0, R_0)$ if d_Γ is realized on Γ_1.

Claim. *The surface* Ψ *is a supersolution of* (8) *everywhere except the set where* $|x_{n+1}| < 1/2$ *and* $d_\Gamma(x)$ *is realized on* $\Gamma_1 \cup \Gamma_3$.

Proof. Let $h_{y_{0_{n+1}}, R}$ be the corresponding function for $g_{y_{0_{n+1}}, R}$ denoted by h and g for simplicity. At distance d from Γ we have in an appropriate system of coordinates

$$D^2 g = \operatorname{diag}\left[\frac{-\kappa_1}{1 - \kappa_1 d} g', \dots, g'' \right] = \operatorname{diag}\left[\frac{-\kappa_1}{1 - \kappa_1 d} \sqrt{2h(s)}, \dots, h'(s) \right]$$

where κ_i represent the principal curvatures of Γ (upwards) at the point where d is realized.

Case 1. If d is realized at a point on Γ_1, then the result follows from the construction of $\mathcal{S}(Y, R)$.

Case 2. If d is realized at a point on Γ_2, then

$$\begin{aligned} 0 \geq \kappa_i \geq -r_\omega^{-1} \geq -3R_0^{-1}, \qquad & i = 1, \dots, n-2, \\ \kappa_{n-1} \geq \frac{\gamma + 1}{2} R_0^{-1} & \end{aligned}$$

provided that q/r_0 is small. Without loss of generality we assume $|d| \leq R_0^{\frac{1}{3}}$ since otherwise, g is constant. On the $-1/2$, respectively $1/2$, level sets $g(d)$ is a supersolution from (37). On the other level sets one has

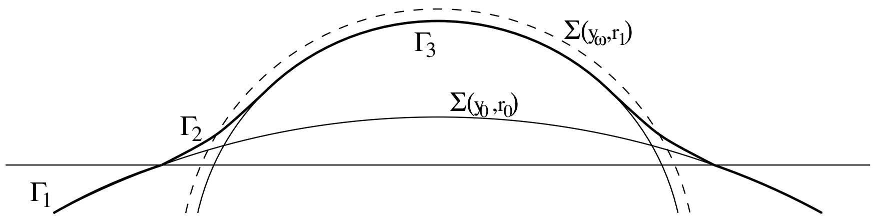
$$\begin{aligned} h' + \sum_{i=1}^{n-1} \frac{-\kappa_i}{1 - \kappa_i d} \sqrt{2h} &\leq h' + \left(-2 \sum_{i=1}^{n-2} \kappa_i - \frac{\kappa_{n-1}}{2} \right) \sqrt{2h} \\ &< h_0' + C_1 R_0^{-1} \sqrt{2h} + \left(6(n-2) - \frac{\gamma+1}{4} \right) R_0^{-1} \sqrt{2h} < h_0'(s) \end{aligned}$$

(we used (48) and (49)).

Case 3. If d is realized at a point on Γ_3 and $|s| > 1/2$, then

$$h' + \sum_{i=1}^{n-1} \frac{-\kappa_i}{1 - \kappa_i d} \sqrt{2h} \leq h_0' - \bar{C}_3 R_0^{-1} + 4(n-1) R_0^{-1} \sqrt{2h} < h_0'(s),$$

(by (48)) and the claim is proved.



We remark that Ψ and $\mathcal{S}(Y_0, R_0)$ coincide outside the cylinder $|x'| < 2q$. Next we consider $\mathcal{S}(Y_\omega, R_1)$ with

$$R_1 = r_\omega + H_0(y_{0n+1}) + 5\frac{\bar{C}_0}{R_0}, \qquad Y_\omega = (y_\omega, y_{0n+1}).$$

The sphere $\Sigma(y_\omega, r_1)$ stays at a distance greater than $3\bar{C}_0 R_0^{-1}$ above Γ_3 and stays at a distance greater than $3\bar{C}_0 R_0^{-1}$ below Γ if $|x'| > q(1+\omega)/2 > \omega q + 8\bar{C}_0^{1/2}$. This implies (see (47)):

1) The region of Ψ where $|x_{n+1}| < 1/2$ and the distance to Γ is realized on Γ_3 is above $\mathcal{S}(Y_\omega, R_1)$.
2) The region of $\mathcal{S}(Y_\omega, R_1)$ where $|x_{n+1}| < 1/2$ and the distance to $\Sigma(y_\omega, r_1)$ is realized at a point outside $\{|x'| < q(1+\omega)/2\}$ is above Ψ.
3) $\mathcal{S}(Y_\omega, R_1)$ is above Ψ outside $\{|x'| < 2q\}$.

We slide from below Ψ in the e_n direction till we touch u for the first time. This cannot happen at $(x_0, u(x_0))$ since Ψ is a strict supersolution in the viscosity sense at x_0 and $u \in C^2$ is a subsolution. We conclude that there exists $\beta > 0$ such that the surface $\Psi - \beta e_n = \{X - \beta e_n, X \in \Psi\}$ touches u at a point $(z, u(z))$ with $|u(z)| < 1/2$ and the distance from $z + \beta e_n$ to Γ is realized on Γ_3.

Now we consider the surfaces $\mathcal{S}(Y_0 + te_n, R_1)$ and increase t till we touch for the first time the graph of u. We notice that when $Y_0 + te_n = Y_\omega - \beta e_n$ then the point $(z, u(z))$ is above the surface $\mathcal{S}(Y_0 + te_n, R_1)$. Thus we can find $0 < t_1 < |Y_0 - Y_\omega| - \beta$ such that $\mathcal{S}(Y_1, R_1)$, $Y_1 = Y_0 + t_1 e_n$ touches u from above at a point $(x_1, u(x_1))$, $|u(x_1)| < 1/2$ in the cylinder $|x'| < 2q$. Moreover from the above remarks

$$\tilde{x}_1 \in \{|x'| < q(1+\omega)/2\} \cap \{x_n < C_2 q^2 R_0^{-1}\}, \qquad R_1 > R_0/3.$$

We apply the above argument with $(x_1, u(x_1))$ and $\mathcal{S}(Y_1, R_1)$ instead of $(x_0, u(x_0))$ and $\mathcal{S}(Y_0, R_0)$ and continue inductively at most a finite number of times till we find a point $(x_*, u(x_*))$ with the required properties.

Step 2. Using the result from Step 1, we prove that the set of contact points $(x, u(x))$ such that

1) $|x'| < q/40$, $|u(x)| < 1/2$, $|x - x_0| < 4q/3$,
2) in the cylinder $\{|x'| < 2q\}$, u is touched by the above at $(x, u(x))$ by $\mathcal{S}(Y, R_0/C_5)$, and $\mathcal{S}(Y, R_0/C_5)$ is above $\mathcal{S}(Y_0, R_0)$ outside this cylinder,
3) $\angle\left(\frac{\nabla u}{|\nabla u|}(x), \frac{\nabla u}{|\nabla u|}(x_0)\right) < C_9 \frac{q}{R_0}$ and the contact points belong in each level set to a Lipschitz graph with Lipschitz constant less than $C_9 q R_0^{-1}$,
4) $$(x - x_0) \cdot \frac{\nabla u}{|\nabla u|}(x_0) \leq H_0(u(x)) - H_0(u(x_0)) + C_9 \frac{q^2}{R_0}$$

project along e_n in a set of measure greater than $c_2 q^{n-1}$, where C_5, C_9, c_2 are appropriate universal constants.

We slide from below, in the e_n direction, the surfaces $\mathcal{S}(Y, R)$ with

$$|y' - \tilde{x}'_*| \leq \frac{q}{500}, \quad |y_{n+1}| \leq \frac{1}{4}, \quad R = \frac{R_0}{C_5}, \quad C_5 = 4C_4(400)^2 \tag{51}$$

till they touch u.

First we show that $(\tilde{x}'_*, 2C_4 q^2 R_0^{-1})$ is in the exterior of $\Sigma(y, r)$. Assume not; then $\Sigma(y, r)$ is above $x_n = 3C_4 q^2 (2R_0)^{-1}$ in the cylinder $|x' - \tilde{x}'_*| \leq q(100)^{-2}$. Now,

$$x_* = \tilde{x}_* + \frac{\nabla u}{|\nabla u|}(x_*)(H_0(u(x_*)) + O(1)\bar{C}_0 C_3 R_0^{-1}),$$
$$\angle\left(\frac{\nabla u}{|\nabla u|}(x_*), e_n\right) \leq q C_3 R_0^{-1};$$

hence

$$x_* \cdot e_n \leq \tilde{x}_* \cdot e_n + H_0(u(x_*)) + C_6(q^2 R_0^{-2} + R_0^{-1}).$$

Thus, if q is greater than a large universal constant, x_* is at a signed distance less than

$$H_0(u(x_*)) + C_6(q^2 R_0^{-2} + R^{-1}) - C_4 q^2 (2R_0)^{-1} < H_0(u(x_*)) - \bar{C}_0 C_5 R_0^{-1}$$

from $\Sigma(y, r)$. This implies that x_* is in the interior of the $u(x_*)$ level surface of $\mathcal{S}(Y, R)$ which is a contradiction.

Since $(\tilde{x}'_*, 2C_4 q^2 R_0^{-1})$ is in the exterior of $\Sigma(y, r)$, we find from (51) that $\Sigma(y, r)$ is below $x_n = 4C_4 q^2 R_0^{-1}$ and below $x_n = 0$ outside $|x'| < q/50$. Thus, $\Sigma(y, r)$ is at a distance greater than $q^2 (4R_0)^{-1}$ in the interior of $\Sigma(y_0, r_0)$ outside $\{x_n > 0\} \times \{|x'| < q/50\}$.

The s level surface of $\mathcal{S}(Y_0, R_0)$ is at distance greater than (see (39))

$$\begin{aligned} &H_{y_{0_{n+1}}, R_0}(s) - H_{y_{0_{n+1}}, R_0}(0) \\ &\qquad \geq H_{y_{0_{n+1}}, R}(s) - H_{y_{0_{n+1}}, R_0}(0) \geq H_{y_{0_{n+1}}, R}(s) - \frac{\bar{C}_0}{2R_0} \end{aligned}$$

from $\Sigma(y_0, r_0)$.

The s level surface of $\mathcal{S}(Y,R)$ is at a distance less than

$$H_{y_{n+1},R}(s) - H_{y_{n+1},R}(0) \le H_{y_{0_{n+1}},R}(s) + \frac{\bar{C}_0 C_5}{R_0}$$

from $\Sigma(y,r)$.

Hence, at the points x for which

$$d_{\Sigma(y,r)}(x) - d_{\Sigma(y_0,r_0)}(x) \ge 2\bar{C}_0 C_5 R_0^{-1},$$

$\mathcal{S}(Y,R)$ is above $\mathcal{S}(Y_0,R_0)$. Since $\mathcal{S}(Y_0,R_0)$ is constant outside a $R_0^{\frac{1}{3}}$ neighborhood of $\Sigma(y_0,r_0)$, we can conclude that, for q greater than a large universal constant, $\mathcal{S}(Y,R)$ is above $\mathcal{S}(Y_0,R_0)$ outside $|x'| < q/40$. This implies that the contact points $(x,u(x))$ have the properties $|u(x)| < 1/2$,

$$\angle\left(\frac{\nabla u}{|\nabla u|}(x), e_n\right) < C_7\frac{q}{R_0}, \quad \tilde{x}_n < 4C_4\frac{q^2}{R_0}, \quad |\tilde{x}'| < \frac{q}{40}$$

and, from Proposition 7.1 they project along e_n in a set of measure greater than $c_2 q^{n-1}$. We notice that on each level set the contact points belong to a Lipschitz graph with Lipschitz constant less than $2C_7 q R_0^{-1}$. Also, one has

$$|x - x_0| < \frac{4}{3}q,$$

$$x = \tilde{x} + \frac{\nabla u}{|\nabla u|}(x)\left(H_0(u(x)) + O(1)\bar{C}_0 C_5 R_0^{-1}\right),$$

$$x_n = \tilde{x}_n + H_0(u(x)) + C_8(q^2 R_0^{-2} + R_0^{-1}).$$

Thus,

$$(x - x_0)\cdot e_n \le 5C_4\frac{q^2}{R_0} + H_0(u(x)) - H_0(u(x_0)),$$

$$(x - x_0)\cdot\frac{\nabla u}{|\nabla u|}(x_0) \le C_9\frac{q^2}{R_0} + H_0(u(x)) - H_0(u(x_0)),$$

which proves Step 2.

End of proof of Lemma 8.2. In the general case we denote by $X_1 \in \mathcal{S}(Y_0,R_0)$ the point such that $\pi_n X_1 = 0$ and let

$$\xi = \frac{x_1 - y_0}{|x_1 - y_0|}.$$

The cylinder

$$\{|(x - x_1)\cdot\xi| < l/2\} \times \{|\pi_\xi(x - x_1)| < 2|\pi_\xi(\tilde{x}_0 - x_1)|\}$$

is included in $\{|x'| < l\} \times \{|x_n| < l\}$. Also, $|x_0'|/2 < |\pi_\xi(\tilde{x}_0 - x_1)| < |x_0'|3/2$, hence we are in the situation above. The contact points obtained in Step 2 belong in each level set to a Lipschitz graph (in the e_n direction) with Lipschitz constant less than 1. The result follows now by projecting these points along the e_n direction. With this the lemma is proved. □

Proof of Lemma 8.3. Denote by $F_k \subset E_k$ the closed set

$$F_k = \{Z \in L / \operatorname{dist}(X, D_k \cap Q_{l+a}) \leq a\}.$$

We prove that there exists $\mu(n, \mu_1) > 0$ small, such that

$$|Q_l \setminus F_k| \leq (1-\mu)^k |Q_l|. \tag{52}$$

Let $Z \in Q_l \setminus F_k$, $Z_1 \in F_k$ be such that $|Z - Z_1| = \operatorname{dist}(Z, F_k) = r$. We claim that for some $\mu_2(n, \mu_1) > 0$

$$|F_{k+1} \cap Q_l \cap \mathcal{B}(Z, r)| \geq \mu_2 |Q_l \cap \mathcal{B}(Z, r)|. \tag{53}$$

Let $Z_0 \in D_k \cap Q_{l+a}$ be the point for which $|Z - Z_0| = r + a$ and Z_1 belongs to the segment $[Z, Z_0]$.

If $2r \geq a$, let Z_2 be such that

$$|Z - Z_2| = \frac{r}{2}, \qquad \mathcal{B}\left(Z_2, \frac{r}{2}\right) \cap L \subset Q_l.$$

From property 2 and $a + r/2 \leq |Z_2 - Z_0| \leq 5r$ we obtain

$$\begin{aligned} |F_{k+1} \cap Q_l \cap \mathcal{B}(Z, r)| &\geq \left| D_{k+1} \cap \mathcal{B}\left(Z_2, \frac{r}{2}\right)\right| \\ &\geq |D_{k+1} \cap \mathcal{B}(Z_2, |Z_2 - Z_0|/10)| \\ &\geq \mu_1 \left| \mathcal{B}\left(Z_2, \frac{r}{2}\right) \cap L \right| \geq \mu_2 |\mathcal{B}(Z, r) \cap Q_l|. \end{aligned}$$

If $2r < a$ then, from property 2, there exists a point

$$Z_3 \in D_{k+1} \cap \mathcal{B}\left(Z, \frac{r+a}{10}\right) \subset Q_{l+a}.$$

Thus,

$$Q_l \cap \mathcal{B}(Z, r) \subset Q_l \cap \mathcal{B}(Z_3, a) \subset F_{k+1},$$

which proves (53).

We take a finite overlapping cover of $Q_l \setminus F_k$ with balls $\mathcal{B}(Z, r)$. Using (53) we find a constant $\mu(\mu_2, n) > 0$ such that

$$|F_{k+1} \cap (Q_l \setminus F_k)| \geq \mu |Q_l \setminus F_k|;$$

hence,

$$|Q_l \setminus F_{k+1}| \leq (1-\mu) |Q_l \setminus F_k|,$$

and (52) is proved. □

9. Estimate for the projection of the contact set

In this section we use the results of the previous section and prove the following:

LEMMA 9.1. *Let u be a local minimizer of J in $\{|x'| < 32l\} \times \{|x_n| < 32l\}$, and assume that $u(0) = 0$, $u < 0$ if $x_n < -\theta$. There exists universal constants $\bar{C}_*$, $\bar{\mu}$, $\bar{c}_4$ such that*:

Given $\theta_0 > 0$, there exists $\varepsilon_0(\theta_0) > 0$ such that if

$$\theta l^{-1} = \varepsilon \le \varepsilon_0(\theta_0), \quad \theta \ge \theta_0, \quad \bar{C}_*^k \varepsilon \le \bar{c}_4,$$

then the set of points

$$(x, u(x)) \in \{|x'| \le l\} \times \{|x_{n+1}| \le 1/2\}$$

that satisfy

$$x_n \le \bar{C}_*^k \theta + H_0(u(x))$$

project along e_n into a set of measure greater than $(1 - (1 - \bar{\mu})^k)|Q_l|$.

Before we prove Lemma 9.1 we need another lemma that gives us a first surface $\mathcal{S}(Y, R)$ that touches u from above.

LEMMA 9.2 (The first touching surfaces). *Let u be a local minimizer of J in $\{|x'| < 32l\} \times \{|x_n| < 32l\}$, and assume that $u(0) = 0$, $u < 0$ if $x_n < -\theta$. Given $\theta_0 > 0$, there exists $\varepsilon_1(\theta_0) > 0$ such that if*

$$\theta l^{-1} \le \varepsilon_1(\theta_0), \quad \theta \ge \theta_0,$$

then the points $(x, u(x))$ have the following properties:

1) $|x'| < l$, $|u(x)| < 1/2$.
2) *There exists a surface $\mathcal{S}(Y, R_0)$ that stays above u in the cylinder $\{|x'| < 16l\} \times \{|x_n| < 16l\}$ and touches its graph at $(x, u(x))$, where*

$$R_0 = \frac{l^2}{32\theta}, \quad l > R_0^{\frac{1}{3}}.$$

3) $\angle \left(\dfrac{\nabla u}{|\nabla u|}(x), e_n \right) \le l R_0^{-1}$,
4) $x_n \le \dfrac{\theta}{4} + H_0(u(x))$ *project along e_n into a set of measure greater than $\bar{c}_3 l^{n-1}$, where $\bar{c}_3 > 0$ is small, universal.*

Proof of Lemma 9.2. We slide from below surfaces $\Psi(y, l)$ and as in the proof of (19) we obtain

$$g_l(x_n + \theta) \ge u(x) \quad \text{if } |x'| < 16l,\ |x_n| < 16l \tag{54}$$

where g_l (respectively $\Psi(y,l)$) is the comparison function (surface) constructed in Lemma 5.3.

Let $R_0 = l^2(32\theta)^{-1}$ and notice that lR_0^{-1} is small and $l > R_0^{\frac{1}{3}}$ if $\varepsilon_1(\theta_0)$ is small. Consider the surfaces $\mathcal{S}(Y,R_0)$ that contain the point $(0,0)$ with

$$|y'| \le l/16, \quad |y_{n+1}| \le 1/4.$$

Claim. *The surfaces $\mathcal{S}(Y,R_0)$ are above $g_l(x_n+\theta)$ (and therefore above u) in the region $l < |x'| < 16l$.*

Proof. The 0 level surface of $\mathcal{S}(Y,R_0)$ is a sphere $|x-y| = r$, which is below the hyperplane $x_n = \theta/8$. Let d_1, d_2, denote the signed distance to the sphere $|x-y| = r$, respectively to the hyperplane $x_n = -\theta$.

If $|b| \le R_0^{\frac{1}{3}}$, the sphere $|x-y| = r+b$ is below $x_n = -2\theta + b$ outside $|x'| < l/2$; thus

$$d_1 \ge d_2 + \theta, \quad \text{in } \{|d_1| \le R_0^{\frac{1}{3}}\} \cap \{l < |x'| < 16l\}. \tag{55}$$

Now it suffices to show

$$H_{y_{n+1},R_0}(s) - H_{y_{n+1},R_0}(0) \le H_l(s) + \theta \tag{56}$$

which implies

$$g_{y_{n+1},R_0}(d + H_{y_{n+1},R_0}(0)) \ge g_l(d-\theta).$$

Hence,

$$g_{y_{n+1},R_0}(d_1 + H_{y_{n+1},R_0}(0)) \ge g_l(d_1-\theta) \ge g_l(d_2),$$

or $\mathcal{S}(Y,R_0)$ is above $g_l(x_n+\theta)$ in the region $l < |x'| < 16l$.

The proof of (56) is similar to the proof of (22). Notice that

$$H_{y_{n+1},R_0}(s) - H_{y_{n+1},R_0}(0) = \int_0^s \frac{1}{\sqrt{2h_{y_{n+1},R_0}(\zeta)}} d\zeta, \tag{57}$$

$$\begin{aligned} H_{y_{n+1},R_0}(s) - H_{y_{n+1},R_0}(0) &\le H_0(s) + \frac{\bar{C}_0}{2R_0} \\ &\le H_0(s) + C_1\theta l^{-2} \le H_0(s) + \theta/2 \end{aligned} \tag{58}$$

for l large.

From (15), (29) we find that

$$\begin{aligned} h_{y_{n+1},R_0}(s) &\le h_l(s), \quad \text{if } s \le -1 + c_1(\theta_0)l^{-\frac{1}{2}}, \\ h_{y_{n+1},R_0}(s) &\ge h_l(s), \quad \text{if } s \ge 1 - c_1(\theta_0)l^{-\frac{1}{2}}, \end{aligned}$$

provided that $l \ge \theta_0\varepsilon_1(\theta_0)^{-1} \ge C_1(\theta_0)$ large. This implies that the maximum of $H_{y_{n+1},R_0}(s) - H_l(s)$ occurs for $1 - |s| \ge c_1(\theta_0)l^{-\frac{1}{2}}$. For these values of s we have (see Lemma 5.3)

$$H_0(s) \le H_l(s) + \bar{C}_1 l^{-1} \log \frac{l^{\frac{1}{2}}}{c_1(\theta_0)} < H_l(s) + \theta_0/2,$$

which together with (58) proves (56).

In conclusion, we slide from below surfaces $\mathcal{S}(Y, R_0)$ in the cylinder $\{|x'| \le 16l\} \times \{|x_n| \le 16l\}$ with

$$|y'| \le l/16, \quad |y_{n+1}| \le 1/4,$$

and we touch u for the first time at points $(x, u(x))$ that satisfy properties 1, 2, 3 of the lemma and

$$\begin{aligned} x_n &\le H_{y_{n+1},R_0}(u(x)) - H_{y_{n+1},R_0}(0) + \theta/8 \\ &\le H_0(u(x)) + \frac{32\bar{C}_0\theta}{2l^2} + \theta/8 \le H_0(u(x)) + \theta/4. \end{aligned}$$

Now the lemma follows from Proposition 7.1. $\square$

Proof of Lemma 9.1. Let $R_0 = l^2(32\theta)^{-1}$ and define $\tilde{D}_k$ as the set of points $(x, u(x))$ with the following properties:

1) $|x'| \le 16l$, $|u(x)| < 1/2$.
2) The graph of u is touched from above in $\{|x'| \le 16l\} \times \{|x_n| \le 16l\}$ at $(x, u(x))$ by $\mathcal{S}(Y, R_k)$ with $R_k \ge R_0\bar{C}_6^{-k}$.
3) $\angle\left(\frac{\nabla u}{|\nabla u|}(x), e_n\right) \le \bar{C}_6^k l R_0^{-1}$.
4) $x \cdot e_n \le \bar{C}_6^k\theta + H_0(u(x))$ where

$$\bar{C}_6 = \max\{\bar{C}_5, 200\bar{C}_4, C(\bar{C}_4)\}.$$

Also, we define $D_k = \pi_n(\tilde{D}_k)$. From Lemma 9.2 we find that if $\varepsilon \le \varepsilon_1(\theta_0)$, then $D_0 \cap Q_l \ne \emptyset$.

Claim. *As long as*

$$8\bar{C}_6^k l R_0^{-1} \le \min\{\bar{c}_2, \pi\}$$

D_k satisfies property 2 *of Lemma* 8.3 *with $a = \bar{C}_4$.*

Proof. Let $Z_k = \pi_n(x_k, u(x_k)) \in Q_{2l} \cap D_k$ and let $\tilde{Z} \in L$, $|x_k' - \tilde{z}'| = q$, $2l \ge q \ge \bar{C}_4$. We apply Lemma 8.2 in the cylinder

$$\{|x' - \tilde{z}'| \le 8l\} \times \{|x_n| \le 8l\}$$

and obtain that the points $(x, u(x))$ with the following four properties project along e_n in a set of measure greater than $\bar{c}_2 q^{n-1}$.

1) $|x' - \tilde{x}'| \le q/15$, $|u(x)| < 1/2$, $|x - x_k| \le 4l$.
2) The graph of u is touched from above in $\{|x'| \le 16l \times \{|x_n| \le 16l\}$ at $(x, u(x))$ by $\mathcal{S}(Y, R_{k+1})$ with

$$R_{k+1} \ge R_k\bar{C}_5^{-1} \ge R_0\bar{C}_6^{-k-1}.$$

3) $\angle\left(\frac{\nabla u}{|\nabla u|}(x), \frac{\nabla u}{|\nabla u|}(x_k)\right) \le 2\bar{C}_4\bar{C}_6^k l R_0^{-1}$; hence,

$$\angle\left(\frac{\nabla u}{|\nabla u|}(x), e_n\right) \le 2\bar{C}_4\bar{C}_6^k l R_0^{-1} + \bar{C}_6^k l R_0^{-1} \le \bar{C}_6^{k+1} l R_0^{-1}.$$

4) $(x - x_k) \cdot \frac{\nabla u}{|\nabla u|}(x_k) \le 4\bar{C}_4 l^2 \bar{C}_6^k R_0^{-1} + H_0(u(x)) - H_0(u(x_k))$

$$(x - x_k) \cdot e_n \le 4\bar{C}_6^k l^2 R_0^{-1} + 4\bar{C}_4 l^2 \bar{C}_6^k R_0^{-1} + H_0(u(x)) - H_0(u(x_k)).$$

Thus,

$$x_n \le \bar{C}_6^{k+1}\theta + H_0(u(x)).$$

All these points are in $\tilde{D}_{k+1}$ which proves the claim. □

Let E_k be the sets defined in Lemma 8.3. From Lemma 8.1 we know that each point in E_k is the projection of a point $(x, u(x))$ with $|x - x_k| \le 2\bar{C}_4$ and

$$(x - x_k) \cdot \frac{\nabla u}{|\nabla u|}(x_k) \le H_0(u(x)) - H_0(u(x_k)) + C(\bar{C}_4) R_k^{-1},$$

for some point $(x_k, u(x_k)) \in \tilde{D}_k$. Thus,

$$(x - x_k) \cdot e_n \le C(\bar{C}_4)\bar{C}_6^k R_0^{-1} + 2\bar{C}_4\bar{C}_6^k l R_0^{-1} + H_0(u(x)) - H_0(u(x_k))$$

or

$$x_n \le \bar{C}_6^{k+1}\theta + H_0(u(x)).$$

We apply Lemma 8.3 and obtain that there exist positive universal constants $\bar{c}_4$, $\bar{\mu}$, small, $\bar{C}_* := \bar{C}_6^2$ such that if

$$\bar{C}_*^k \varepsilon \le \bar{c}_4 := 300^{-1} \min\{\bar{c}_2, \pi\}, \quad \varepsilon \le \varepsilon_0(\theta_0),$$

then

$$|E_k \cap Q_l| \ge (1 - (1 - \bar{\mu})^k)|Q_l|.$$

With this the lemma is proved. □

10. Proof of Theorem 2.2

We assume that u is a local minimizer of J in the cylinder

$$\{|x'| < 32l\} \times \{|x_n| < 32l\},$$

and

$$u > 0 \quad \text{if } x_n > \theta, \quad u < 0 \quad \text{if } x_n < -\theta, \quad u(0) = 0.$$

We show that if the 0 level set is close to $x_n = -\theta$ at a point in $|x'| < l/4$ then the energy of u is large and we obtain a contradiction.

As in (54) (or (19)) we can prove

$$\rho_l(x_n - \theta) \le u(x) \le g_l(x_n + \theta) \quad \text{in } \{|x'| < 16l\} \times \{|x_n| < 16l\}, \tag{59}$$

where ρ_l is the function similar to g_l which rotated gives a subsolution (it is supported in $(-l/2, \infty)$, $\rho_l(0) = 0$ and is constant for $s \ge l/2$).

Next we apply Lemma 9.2 upside-down and obtain that there exists a small universal constant c_1 such that the points $(x, u(x))$ with

$$x_n \ge -\frac{\theta}{4} + H_0(u(x)), \qquad |x'| \le \frac{l}{2}, \quad |u(x)| \le \frac{1}{2} \tag{60}$$

project along e_n in a set of measure greater than $c_1 l^{n-1}$, provided that $\theta l^{-1} \le \varepsilon_2(\theta_0)$ is small.

On the other hand, from Lemma 9.1 we find universal constants $\bar{C}_*$, $\bar{\mu}$ such that if

$$\begin{gathered} \{u = 0\} \cap \{|x'| < l/4\} \cap \{x_n < (-1 + \bar{C}_*^{-k_0}/4)\theta\} \ne \emptyset, \\ \theta l^{-1} \le \varepsilon_3(\theta_0, k_0), \end{gathered} \tag{61}$$

then the set of points $(x, u(x))$ with

$$x_n \le -\frac{\theta}{2} + H_0(u(x)), \qquad |x'| \le \frac{l}{2}, \quad |u(x)| \le \frac{1}{2} \tag{62}$$

project along e_n in a set of measure greater than $(1 - (1 - \bar{\mu})^{k_0})|Q_{l/2}|$. We show that if we choose k_0 large, universal such that

$$c_1 l^{n-1} > 2(1 - \bar{\mu})^{k_0} |Q_{l/2}| \tag{63}$$

and ε small enough, then we obtain a contradiction.

Now,

$$\begin{aligned} \left(\frac{1}{2}|\nabla u|^2 + h_0(u)\right) dx' dx_n &\ge \sqrt{2h_0(u)}|u_n| dx' dx_n \\ &= \sqrt{2h_0(x_{n+1})} dx' dx_{n+1}. \end{aligned}$$

Denote

$$A_l := \{|x'| < l\} \times \{|x_n| < l\}.$$

Project along e_n the points $(x, u(x))$, with $|x_n| \le l$. From (60), (62), (63) we find that there exists a set of measure $c_1 l^{n-1}/2$ included in $Q_{l/2}$ where these points project twice. Using also (59) we can find two small universal constants

c_2, $c_3 > 0$ such that

$$J(u, A_l) = \int_{A_l} \frac{1}{2}|\nabla u|^2 + h_0(u)dx \tag{64}$$
$$\geq \omega_{n-1} l^{n-1} \int_{c_2-1}^{1-c_2} \sqrt{2h_0(x_{n+1})}dx_{n+1}$$
$$+\frac{c_1}{2}\left(\min_{|s|\leq 1/2} \sqrt{2h_0(s)}\right) l^{n-1}$$
$$\geq \omega_{n-1} l^{n-1} \int_{-1}^{1} \sqrt{2h_0(s)}ds + c_3 l^{n-1},$$

where ω_{n-1} represents the volume of the $n-1$ dimensional unit sphere.

Assume by contradiction that there exist numbers l_k, θ_k with

$$\theta_k l_k^{-1} \to 0, \quad \theta_k \geq \theta_0,$$

and local minimizers u_k in A_{32l_k} satisfying the hypothesis of Theorem 2.2 and property (61).

Denote by $\varepsilon_k := l_k^{-1}$ and $v_k(x) := u_k(\varepsilon_k^{-1}x)$. From (64) we obtain

$$J_{\varepsilon_k}(v_k, A_1) = \varepsilon_k^{n-1} J(u_k, A_{l_k}) \geq \omega_{n-1} \int_{-1}^{1} \sqrt{2h_0(s)}ds + c_3. \tag{65}$$

On the other hand, as $k \to \infty$ we have

$$v_k \to \chi_E - \chi_{E^c} \quad \text{ in } L^1_{\text{loc}}(A_1),$$

where $E = A_1 \cap \{x_n > 0\}$. By Theorem 3.1 one has

$$\lim_{k\to\infty} J_{\varepsilon_k}(v_k, A_1) = P_{A_1} E \int_{-1}^{1} \sqrt{2h_0(s)}ds = \omega_{n-1} \int_{-1}^{1} \sqrt{2h_0(s)}ds$$

which contradicts (65).

With this, Theorem 2.2 is proved.

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