Annals of Mathematics, **169** (2009), 1–39

The "Harder-Narasimhan trace" and unitarity of the KZ/Hitchin connection: genus 0

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Abstract

Let a reductive group G act on a projective variety $\mathcal{X}_+$, and suppose given a lift of the action to an ample line bundle $\hat{\theta}$. By definition, all G-invariant sections of $\hat{\theta}$ vanish on the nonsemistable locus $\mathcal{X}_+^{nss}$. Taking an appropriate normal derivative defines a map $H^0(\mathcal{X}_+, \hat{\theta})^G \to H^0(S_\mu, \mathcal{V}_\mu)^G$, where $\mathcal{V}_\mu$ is a $G-$vector bundle on a $G-$variety S_μ. We call this the Harder-Narasimhan trace. Applying this to the Geometric Invariant Theory construction of the moduli space of parabolic bundles on a curve, we discover generalisations of "Coulomb-gas representations", which map conformal blocks to hypergeometric local systems. In this paper we prove the unitarity of the KZ/Hitchin connection (in the genus zero, rank two, case) by proving that the above map lands in a unitary factor of the hypergeometric system. (An ingredient in the proof is a lower bound on the degree of polynomials vanishing on partial diagonals.) This elucidates the work of K. Gawedzki.

Introduction

The Hitchin connection – the algebro-geometric version of the connection described by the Knizhnik-Zamolodchikov-Bernard equation – is generally believed to be unitary. In a series of remarkable works (related in spirit to the differential-geometric version, due to S. Axelrod, V. Della Pietra and E. Witten), K. Gawedzki has derived expressions for a scalar product ([G] and references therein, in particular, [G1] and [G2]). The starting point is a formal functional integral, and a series of transformations results in a finite-dimensional integral. The latter integral is extremely complicated, the domain of integration noncompact, and convergence has been proved in few cases. Fur-

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thermore, invariance under the KZ connection is only shown ([F-G]) under the assumption of convergence of the integrals involved.

This work is part of a project to understand the resulting expressions and to extract a rigorous proof of unitarity. Here is a summary.

(1) In the next section (§1), we introduce a family of injective maps (Proposition 1.1) from the space of invariant sections of a polarising line bundle on a (projective) G-variety, where G is a complex reductive group. In brief, one fixes a nonsemistable (Harder-Narasimhan) stratum, and the map is defined by attaching to each section an appropriate normal derivative on the stratum. We call this map the *Harder-Narasimhan trace*.

(2) In Section 2 we recall the construction via Geometric Invariant Theory of the moduli space of rank 2 (parabolic) bundles on an n-pointed curve. Invariant sections of a polarising line bundle are sections of a theta bundle on the moduli space. The space of these sections – the *generalised theta functions*, to use Narasimhan's terminology that harks back to Weil – is the fibre of a vector bundle over the space of such curves. The Hitchin connection (in its parabolic version, due to P. Scheinost and M. Schottenloher) describes a natural flat projective connection on this bundle.

(3) In Section 3 we describe the Harder-Narasimhan trace for generalised theta functions in geometric terms.

In the rest of the paper, we specialise to the genus zero case, where the n-pointed curve X is $\mathbb{P}^1$, with n (distinct) points $z_1, \ldots, z_n$ marked on it.

(4) In Section 4 we describe how, locally over the configuration space of n distinct points, the (parabolic) Hitchin connection can be identified with the KZ connection. (Much of this is a rewriting, in algebro-geometric language, of [G-K].)

(5) In Section 5 we introduce "hypergeometric local systems" on $\mathcal{Z}_n \equiv \mathbb{C}^n \backslash \Delta$, where Δ denotes the generalised diagonal. These are direct images of unitary one-dimensional local systems on a complement of hyperplanes in $\mathbb{X} \equiv \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times \mathcal{Z}_n$, i.e., the fibre of the local system is the (top) cohomology with values in a flat unitary line bundle. The number of factors as well as the choice of local system depend on the parabolic data and the choice of Harder-Narasimhan stratum. We introduce (for later use) a certain blow-up of $\mathbb{X}$ (the union of hyperplanes becomes a divisor with normal crossings here), and a cyclic cover $\hat{\mathbb{X}}$ thereof, which has locally toric (and therefore rational) singularities. For $\mathbf{z} \in \mathcal{Z}_n$, let us denote by $\mathbb{X}_{\mathbf{z}}$, $\hat{\mathbb{X}}_{\mathbf{z}}$, etc., the fibre above $\mathbf{z}$.

(6) In Section 6 we compute the Harder-Narasimhan trace explicitly for the *simplest* choice of Harder-Narasimhan stratum and show that it takes

values in the space of top (degree) forms in a certain hyperplane complement in $\mathbb{X}_{\mathbf{z}}$. This gives a map of the KZ local system to a hypergeometric local system, and *the dual map* has been much studied by A. Varchenko, V. Schechtman, and others ([V] and references therein). This is the Coulomb-gas representation of conformal blocks discovered by Dotsenko and Fateev in the eighties.

(7) We have generalisations to other strata, and in fact, other genera. In Section 7 we give an example of a computation involving a "deeper" stratum which yields a new map of local systems.

It is very important for what follows to note that the map to the cohomology actually factors through an ***injective*** *map to the space of top-forms.*

(8) We next (§8) prove a preliminary result which addresses the following question. If a nonzero homogeneous polynomial in b variables is known to vanish on all partial diagonals of dimension $b - a$ (that is, whenever $a+1$ of the variables are equal), what is the minimum degree it can have? The answer turns out to involve a basic result in combinatorics: Turán's theorem.

(9) Returning to our main theme, our second insight is that Gawedzki's scalar product is (up to a "central" term) the natural L^2 norm of the top-forms. As described by him, this L^2-norm is an integral over the complement of hyperplanes in $\mathbb{X}_{\mathbf{z}}$ of the pointwise norm-squared of the form, multiplied by the norm-squared of a (multivalued) weight function. We prove (§9) that this is (up to a constant multiplicative factor) the L^2-norm of a *regular* top form on the smooth locus of $\hat{\mathbb{X}}_{\mathbf{z}}$. (One proves that the form, *a priori* defined outside a divisor, extends across the generic point of each component of this divisor.) For small values of the "level", this is contingent on a certain ("Kac-Kazhdan"-type) condition being met. One then uses the fact that $\hat{\mathbb{X}}_{\mathbf{z}}$ has rational singularities to see that such a form must have bounded L^2 norm. This proves Gawedzki's central conjecture (see §4.3 of [G]).

(10) Invariance under the KZ connection follows with some more work (§10). An important ingredient is the injectivity of the map from the KZ local system to the hypergeometric local system, a result due to Varchenko which we are now able to prove directly using Hodge-theoretic results of Deligne.

One way of investigating convergence of integrals, familiar in Physics, is by "power-counting" in various aymptotic directions. Blowing up to display infinity as a divisor with normal crossings systematises this, and is part of what goes into Step (9) above. But there are novelties, so that it is worth summarising the strategy here:

(1) The integrand is of the form $|\mathcal{R}|^2\phi \wedge \bar{\phi}$, where $\mathcal{R} = \mathcal{T}^{\frac{1}{k+2}}$, with $\mathcal{T}$ an invertible holomorphic function, and ϕ a holomorphic top-form, both defined outside a divisor D in a compact variety $\mathbb{X}_{\mathbf{z}}$, with zeroes and poles along different components of D. (Here, k is the "level".)

(2) One blows up along D to get a normal-crossing divisor D'.

(3) One goes to a generically étale and Galois cover $\hat{\mathbb{X}}_{\mathbf{z}}$, where $\mathcal{R}$ becomes single-valued. The variety $\hat{\mathbb{X}}_{\mathbf{z}}$ is normal and locally toric. One keeps track of the behavior of (the pullbacks of) $\mathcal{R}$ and ϕ near the inverse image $\hat{D}$ of D. Remember that ϕ is a *form*, and pulling back tends to produce zeroes.

(4) One checks that $\mathcal{R}\phi$ extends (that is, without poles) "generically" across $\hat{D}$. By normality of $\hat{\mathbb{X}}_{\mathbf{z}}$, $\mathcal{R}\phi$ is now defined and holomorphic everywhere on the smooth part of $\hat{\mathbb{X}}_{\mathbf{z}}$. We now use the fact that in our case $\hat{\mathbb{X}}_{\mathbf{z}}$ has rational singularities to conclude that $\mathcal{R}\phi$ has finite L^2 norm. (The reason is simple: pulled back to any desingularisation of $\hat{\mathbb{X}}_{\mathbf{z}}$, a regular top-form extends everywhere.)

From a strictly logical point of view, unitarity in genus zero can be proved without reference to the work of Gawedzki, once the map to the hypergeometric local system is known. But it is in his formulae that lies the clue that this map in fact lands in a unitary factor of a local system. Also, the description of the hypergeometric map as a trace on a Harder-Narasimhan stratum emerged out of a close inspection of these formulae, and of the support properties of the formal functional measures that figure in the intermediate steps of his calculation. (Note also that Gawedzki's computation yields the "Virasoro central charge", which we totally ignore.)

I close this Introduction with remarks on work in progress:

(1) Clearly the most urgent task is to extend our results to higher genus, and in particular, to the case "without insertions", i.e., when one considers moduli spaces of bundles without parabolic structure. Section 3 is set up so that the target space of the corresponding trace map is evident. In fact I can now interpret the results of F. Falceto and K. Gawedzki [F-G] and G. Felder and A. Varchenko [F-V] in genus one from this point of view, and a proof of unitarity in this case seems at hand.

(2) The relationship between traces on different strata, and induced inner products on conformal blocks, is not clear yet. In higher rank, the choices of Harder-Narasimhan strata possible are much richer, and the story possibly involves the "strange dualities".

After this work was done, I became aware of the work of T. Yoshida ([Y]) which addresses the unitarity of the Hitchin connection. The approach is via "abelianisation" *à la* Hitchin. I should also mention the important work of A. Kirillov ([K]) where a "formal" hermitian structure – invariant under an action of the modular group, presumably the monodromy of the KZ equation – is defined on conformal blocks.

1. The Harder-Narasimhan trace

In this section, we introduce a map from the space of invariant sections of a polarising line bundle. The map takes values in invariant sections of a certain vector bundle.

We consider varieties over $\mathbb{C}$. Let a reductive group G act on an irreducible projective variety $\mathcal{X}_+$, and suppose given a lift of the action to an ample line bundle $\hat{\theta}$. This is the familiar setup of Geometric Invariant Theory (GIT henceforth). One defines a closed subset, the set of nonsemistable points, by

$$\mathcal{X}_+^{nss} = \cap_{l\ \in\mathbb{N}}\{x \in \mathcal{X}_+ | \sigma(x) = 0\ \forall\ \sigma \in H^0(\mathcal{X}_+, \hat{\theta}^l)^G\ \}\ .$$

We define $\mathcal{X}_+^{ss} = \mathcal{X}_+ \setminus \mathcal{X}_+^{nss}$, the open set of semistable points, which we shall assume to be nonempty.

Suppose now that $\mathcal{X}$ is a nonempty open $G-$invariant subset of $\mathcal{X}_+$. We set $\mathcal{X}^{nss} = \mathcal{X} \cap \mathcal{X}_+^{nss}$. Let $\mathcal{Y}$ be a closed, irreducible $G-$invariant subset of $\mathcal{X}^{nss}$, endowed with the structure of a reduced variety. (*In general* $\mathcal{Y}$ *will be only locally closed in* $\mathcal{X}_+$.) Let $\mathcal{I}_\mathcal{Y}$ denote the corresponding sheaf of ideals on $\mathcal{X}$. This is clearly $G-$invariant. Restriction gives an injective map

$$H^0(\mathcal{X}_+, \hat{\theta})^G \hookrightarrow H^0(\mathcal{X}, \hat{\theta} \otimes \mathcal{I}_\mathcal{Y})^G.$$

Let us suppose in addition that $H^0(\mathcal{X}, \hat{\theta})^G$ is finite-dimensional. (This will be the case, for example, if $\mathcal{X}_+^{ss} \subset \mathcal{X}$.) In that case

$$H^0(\mathcal{X}, \hat{\theta} \otimes \mathcal{I}_\mathcal{Y}^Q)^G = 0$$

for Q a sufficiently large integer, so that for such a Q we have an injection

$$H^0(\mathcal{X}_+, \hat{\theta})^G \hookrightarrow H^0(\mathcal{X}, \hat{\theta} \otimes \mathcal{I}_\mathcal{Y}/\mathcal{I}_\mathcal{Y}^Q)^G\ .$$

For q any positive integer, let $\mathcal{N}^{*(q)}$ denote quotient $\mathcal{I}_\mathcal{Y}^q/\mathcal{I}_\mathcal{Y}^{q+1}$. This is a sheaf on $\mathcal{Y}$. The sheaf $\mathcal{I}_\mathcal{Y}/\mathcal{I}_\mathcal{Y}^Q$ is filtered:

$$0 \subset \cdots \subset \mathcal{I}_\mathcal{Y}^{q+1}/\mathcal{I}_\mathcal{Y}^Q \subset \mathcal{I}_\mathcal{Y}^q/\mathcal{I}_\mathcal{Y}^Q \subset \cdots \subset \mathcal{I}_\mathcal{Y}/\mathcal{I}_\mathcal{Y}^Q\ .$$

Denote the successive quotients:

$$\mathcal{N}^{*(q)} = \mathcal{I}_\mathcal{Y}^q/\mathcal{I}_\mathcal{Y}^{q+1}\ .$$

Let $\mathcal{H}^q \equiv H^0(\mathcal{X}, \hat{\theta} \otimes \mathcal{I}_\mathcal{Y}^q/\mathcal{I}_\mathcal{Y}^Q)^G$. Note the injection

$$\mathcal{H}^q/\mathcal{H}^{q+1} \hookrightarrow H^0(\mathcal{Y}, \hat{\theta} \otimes \mathcal{N}^{*(q)})^G\ .$$

Suppose henceforth that $\mathcal{Y}$ is smooth and $\mathcal{X}$ is smooth along $\mathcal{Y}$. In that case $\mathcal{N}^{*(q)}$ is the q^{th} symmetric power $S^q\mathcal{N}^*$ of the co-normal bundle $\mathcal{N}^*$ of $\mathcal{Y}$.

Finally we make the following hypothesis:

- (H) There exists a unique $q = q^*$ such that $H^0(\mathcal{Y}, \hat{\theta} \otimes S^{q^*}\mathcal{N}^*)^G \neq 0$.

This condition, which appears extremely strong, is satisfied in many cases, and the ensuing theory is very pleasant. In particular, we have an injective map

$$\mathrm{TR} : H^0(\mathcal{X}_+, \hat{\theta})^G \hookrightarrow H^0(\mathcal{Y}, \hat{\theta} \otimes S^{q^*}\mathcal{N}^*)^G \ .$$

If $\mathcal{X}_+$ is normal, the restriction map $H^0(\mathcal{X}_+, \hat{\theta})^G \to H^0(\mathcal{X}_+^{ss}, \hat{\theta})^G$ is an isomorphism ([N-R], Lemma 4.15), so that we can conclude:

Proposition 1.1. *Suppose $\mathcal{X}_+$ is normal,* (H) *holds, and the line bundle $\hat{\theta}$ descends to a line bundle θ on the* GIT *quotient $\mathcal{X}_+^{ss}//G$. There is an* ***injective*** *map*:

$$\mathrm{TR} : H^0(\mathcal{X}_+^{ss}//G, \theta) \hookrightarrow H^0(\mathcal{Y}, \hat{\theta} \otimes S^{q^*}\mathcal{N}^*)^G \ .$$

We call this the Harder-Narasimhan trace.

Remark 1.2. We will need a slightly stronger version which also follows from the same lemma: suppose that $\mathcal{X}_+^{ss}$ is contained in the normal locus of $\mathcal{X}_+$, as is the subset $\mathcal{X}$. In this case we have again an injective map, which factors as follows:

$$H^0(\mathcal{X}_+^{ss}//G, \theta) \hookrightarrow H^0(\mathcal{X}, \theta)^G \hookrightarrow H^0(\mathcal{Y}, \hat{\theta} \otimes S^{q^*}\mathcal{N}^*)^G.$$

Here is an illustrative example, which will be useful later.

Notation 1.3. Fix positive integers μ_i, $i = 1, \ldots, n$, satisfying the condition: $\frac{1}{2}\sum_i \mu_i \equiv J$, an integer. For V_1 a two-dimensional vector space, define $\hat{\mathbb{W}}(V_1)$ to be the tensor product

$$\bigotimes_i S^{\mu_i} V_1 \otimes (\det V_1)^{-J}.$$

This is a $\mathrm{PGL}(V_1)$-module. Let $\mathbb{W}(V_1)$ denote the subspace of invariants.

Remark 1.4. Note that if V_2 is another two-dimensional vector space, the spaces $\mathbb{W}(V_1)$ and $\mathbb{W}(V_2)$ are canonically isomorphic. In particular, to a rank two vector bundle $\mathcal{F}$ (on any base) one can associate a canonically trivialised bundle $\mathbb{W}(\mathcal{F})$.

Remark 1.5. Fixing a nonzero element τ of $\det V_1$, we have an isomorphism

$$(\bigotimes_i S^{\mu_i} V_1)^{\mathrm{SL}(V_1)} \to \mathbb{W}(V_1) \ ,$$

given by $w \mapsto w \otimes \tau^{-J}$.

Let V be a two-dimensional vector space, and set $\mathbb{X}_J = \underbrace{\mathbb{P}(V) \times \cdots \times \mathbb{P}(V)}_{n \text{ factors}}$. The latter is a $\mathrm{PGL}(V)$-variety. Take as polarisation

$$\hat{\theta} = \mathcal{O}(\mu_1) \otimes .. \otimes \mathcal{O}(\mu_n) \otimes_{\mathbb{C}} (\det V)^{-J} \ .$$

We have then

$$H^0(\mathbb{X}_J, \hat{\theta})^{\mathrm{PGL}(V)} = \mathbb{W}(V) \ .$$

Take $\mathcal{Y}$ to be the image of the diagonal inclusion $\mathbb{P}(V) \hookrightarrow \mathbb{X}_J$. This is clearly part of the nonsemistable locus. Note that the restriction of $\hat{\theta}$ to $\mathcal{Y}$ is $\mathcal{O}(2J) \otimes_{\mathbb{C}} (\det V)^{-J}$, and $\mathcal{N}$ is defined by the naturally split sequence

$$0 \to K^{-1}_{\mathbb{P}(V)} \underset{\text{diagonal}}{\longrightarrow} (K^{-1}_{\mathbb{P}(V)})^n \to \mathcal{N} \to 0 \ . \tag{1}$$

Therefore condition (H) is satisfied with $q^* = J$. Note that $H^0(\mathcal{Y}, \hat{\theta} \otimes S^J \mathcal{N}^*)^{\mathrm{PGL}(V)} = S^J(\mathbb{C}^n/\mathbb{C})^*$.

Notation 1.6. Fixing a basis $(\mathbf{v}, \mathbf{v}')$ of V, we have the standard basis (e, f, h) of $\mathrm{sl}(V)$, defined by

$$\begin{aligned} e\mathbf{v} &= \mathbf{v}', & e\mathbf{v}' &= 0 \\ f\mathbf{v} &= 0, & f\mathbf{v}' &= \mathbf{v} \\ h\mathbf{v} &= -\mathbf{v}, & h\mathbf{v}' &= \mathbf{v}' \ . \end{aligned} \tag{2}$$

Let e_i, f_i, h_i denote the induced maps on $S^{\mu_i}V$, as well as on the tensor product $\hat{\mathbb{W}}(V)$ (with e_i, for example, acting trivially on all factors except the i^{th}). Set $\tau = \mathbf{v}' \wedge \mathbf{v}$, define vectors: $v_J = \bigotimes_i \mathbf{v}^{\mu_i}$ and $v'_J = \bigotimes_i \mathbf{v}'^{\mu_i}$. For future reference we record: $h_i v_J = -\mu_i v_J$, $h_i v'_J = \mu_i v_J$. Let $(\sigma,\ \sigma')$ denote the basis of sections of $\mathcal{O}(1)$ given by the vectors $(\mathbf{v}, \mathbf{v}')$ respectively (via the quotient map $V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}(1) \to 0$), and x the (meromorphic) co-ordinate on $\mathbb{P}(V)$ determined by $\sigma = x\sigma'$. For any positive integer μ, denote by $\langle , \rangle$ the duality pairing $S^\mu V \otimes S^\mu V \to \mathbb{C}$:

$$\langle x_1 \otimes \ldots x_\mu, y_1 \otimes \ldots y_\mu \rangle \tau^\mu = \frac{1}{\mu!} \sum_\rho (x_1 \wedge y_{\rho(1)}) \ldots (x_\mu \wedge y_{\rho(\mu)})$$

where the sum is over all permutations ρ of $\{1, \ldots, \mu\}$. This induces an $\mathrm{SL}(V)$-invariant bilinear form on tensor products, which we continue to denote by $\langle , \rangle$.

Lemma 1.7. *The map $S^\mu V \to H^0(\mathbb{P}(V), \mathcal{O}(\mu)) = S^\mu V$, given by*

$$u \mapsto \langle u, \exp -xe\ \mathbf{v}^\mu \rangle \sigma'^\mu, \tag{3}$$

is the identity.

We omit the proof.

PROPOSITION 1.8. *The trace map*

$$\mathrm{TR} : \mathbb{W}(V) \hookrightarrow S^J(\mathbb{C}^n/\mathbb{C})^*$$

is given by

$$\mathrm{TR}(w\tau^{-J})\ (\mathbf{u}_1, \ldots, \mathbf{u}_J) = \left\langle w, \prod_{\nu=1}^{J} \left(\sum_i u_\nu^i e_i \right) v_J \right\rangle$$

where $\mathbf{u}_\nu = (u_\nu^1, \ldots, u_\nu^n)$.

Remark 1.9. One can check that the above map is independent of the choice of basis. Further: (a) $\mathrm{TR}(w\tau^{-J})\ (\mathbf{u}_1, \ldots, \mathbf{u}_n) = 0$ if $\mathbf{u}_1 = (u, \ldots, u)$ (since w is an invariant vector), and (b) if we define $\mathrm{TR}_L(w\tau^{-J})$ as above with an arbitrary positive integer L replacing J, then $\mathrm{TR}_L(w\tau^{-J})$ is identically zero unless $L = J$ (consider weights). The above map TR is totally canonical, but a universal choice (of sign) has been made in the isomorphism:

$$K^{-1}_{\mathbb{P}(V)} = \mathcal{O}(2) \otimes (\det V)\ .$$

Proof. Under the map (3) the vector $w\tau^{-J} \in \mathbb{W}(V)$ maps to

$$\sigma_w \equiv \langle w, \exp - \sum_i x_i e_i\ v_J \rangle \bigotimes_i \sigma'^{\mu_i}\ \tau^{-J}.$$

Using the above Remark (b), one sees that σ_w vanishes on the diagonal to order $J-1$, and the J^{th} derivative at the point $P = \{x_1 = \cdots = x_n = 0\}$ is the form

$$D^J \sigma_w = (-1)^J \sum_{|\alpha|=J} \frac{1}{\alpha!} \langle w, e^\alpha v_J \rangle dx^\alpha \bigotimes_i \sigma'^{\mu_i}\ \tau^{-J}\ .$$

Here $\alpha \equiv (\alpha_1, \ldots, \alpha_n)$ is a multi-index, and $e^\alpha \equiv \prod_i e_i^{\alpha_i}$, etc. Using the split sequence (1), we get

$$\mathrm{TR}(w\tau^{-J})(\mathbf{u}, \ldots, \mathbf{u})) = (-1)^J \sum_{|\alpha|=J} \frac{1}{\alpha!} \langle w, e^\alpha \mathbf{u}^\alpha v_J \rangle dx^J \sigma'^2 J\ \tau^{-J}\ .$$

One checks that the tensor product $-dx\sigma'^2\ \tau$ can be canonically (up a sign convention) made equal to one. Polarising the resulting expression yields the proposition. □

2. Rank 2 parabolic bundles on a curve

We summarise some basic results on the theta bundle and the "parabolic" Quot scheme. These are extracted from [N-R]. We have made some minor changes of notation to facilitate comparison with the literature.

Let X be a smooth projective curve of genus g. We will consider rank 2 parabolic bundles of (ordinary, not "parabolic") degree d, with parabolic structures at n distinct points $z_i,\ i = 1, \dots, n$. Fix a further base point z_0 on X. We let I denote the set $\{1, \dots, n\}$, and Z the set $\{z_1, \dots, z_n\}$.

Notation 2.1. Define $\chi = d + 2(1 - g)$. Fix a positive integer k, and integers $\mu_i,\ i = 1, \dots, n$ satisfying the conditions:

(1) $0 < \mu_i < k$, and

(2) $l \equiv \frac{1}{2}(k\chi + \sum_i \mu_i) - kn$ is an integer.

Define the set of real numbers $\{t_i\}$ by $t_i = \mu_i/k$. Set $\kappa = k + 2$.

By definition, a rank 2 quasi-parabolic bundle E comes with a quotient sheaf Q with support Z, and with length one at each of the points z_i; the "parabolic weights" $\{t_i\}$ having been chosen, we have in fact a parabolic structure. The *parabolic degree* of E is

$$\operatorname{par\,deg} E = \deg E + \sum_I t_i \ .$$

Given a line sub-bundle L, this inherits a parabolic structure $L \to Q_L \to 0$, where Q_L is the image of the composite map $L \to E \to Q$. (L is regarded as having a parabolic structure on Z – note that this structure is in fact "nontrivial" at $Z \setminus$ *support* Q_L, which explains the expression for its parabolic degree given below.) The quotient line bundle $E/L \equiv L'$ inherits the parabolic structure $L' \to Q_{L'} \to 0$, where $Q/Q_L \equiv Q_{L'}$. We define complementary subsets R and R^c of I by *support* $Q_{L'} = \{z_i | i \in R\}$. The parabolic degree of L is defined to be

$$\operatorname{par\,deg} L = \deg L + \sum_R t_i$$

and that of L' is defined to be

$$\operatorname{par\,deg} L' = \deg L' + \sum_{R^c} t_i \ .$$

Semistability of E (w.r.to parabolic weights t_i) requires that for every such sub-bundle L we have $\operatorname{par\,deg} L \leq \frac{1}{2}\operatorname{par\,deg} E$, that is to say,

$$2\deg L - \deg E \leq \sum_{R^c} t_i - \sum_R t_i \ .$$

If E is not semistable there exists a line sub-bundle contradicting semistability. This is unique (the rank 2 case is deceptively simple), and the corresponding extension

$$0 \to L \to E \to L' \to 0 \tag{4}$$

with $L_{z_i} \to Q_i$ being an isomorphism if and only if $i \in R^c$, is the parabolic Harder-Narasimhan filtration. In general these extensions do not split. In this paper we wish to consider only those that do split, though we do not rule out having to consider other cases in future. Here is a sufficient condition for that to happen. (In what follows we denote by D the degree of the sub-bundle L.)

Lemma 2.2. *Suppose $2D - d > 2g - 2 + |R^c|$. Then E is isomorphic to the direct sum $L \oplus L'$, with parabolic structure given by $Q = Q_L \oplus Q_{L'}$, where support $Q_L = Z_{R^c} \equiv \{z_i | i \in R^c\}$ and support $Q_{L'} = Z_R \equiv \{z_i | i \in R\}$. (We write, informally, $(E, Q) = (L, Q_L) \oplus (L', Q_{L'})$.)*

Proof. We first need the extension to split as an extension of vector bundles, forgetting parabolic structures. The condition $H^1(\mathrm{Hom}(L', L)) = 0$ ensures this, and this in turn is guaranteed if $2D-d > 2g-2$. To guarantee that the extension of parabolic bundles splits, we need the map $H^0(\mathrm{Hom}(L', L)) \to \oplus_{i \in R^c} \mathrm{Hom}(L', L)_{z_i}$ to be surjective. This in turn is the case if $2D - d > 2g - 2 + |R^c|$. □

Let T be a parameter variety. Suppose given a rank two bundle $\mathcal{E}_T$ on $T \times X$, and rank one quotients $\mathcal{E}_{T,z_i} \to Q_{T,i} \to 0$, with $\mathcal{E}_{T,z_i}$ noting the pull-back to T of $\mathcal{E}_T$ by the map $t \mapsto (t, z_i)$. This is a family of parabolic bundles parametrised by T. One defines the line bundle $\mathcal{L}_T$ on T by

$$\mathcal{L}_T = (\det R(\pi_T)_* \mathcal{E})^k \otimes \{\bigotimes_i Q_{T,i}^{\mu_i} \otimes (\det \mathcal{E}_{T,z_i})^{k-\mu_i}\} \otimes (\det \mathcal{E}_{T,z_0})^l \ . \tag{5}$$

The determinant bundle of cohomology is defined with the convention that the determinant line associated to a bundle E on X is $(\det H^0(X, E))^{-1} \otimes (\det H^1(X, E))$. One verifies that if $\mathcal{E}_T$ is twisted by a line bundle from the parameter space T, the line bundle $\mathcal{L}_T$ "does not change" - the last factor is designed to ensure this.

Let $\mathcal{U} = \mathcal{U}(\{t_i\}_I)$ denote the moduli space of (s-equivalence classes of) semistable parabolic bundles of rank 2 and degree d on X, with quasi-parabolic structures at the $\{z_i\}_I$, semistable with respect to the weights $\{t_i\}_I$. There is a natural ample line bundle $\theta = \theta(k, \{\mu_i\}_I)$ on $\mathcal{U}$, which pulls back to $\mathcal{L}_T$ if the family $\mathcal{E}_T$ consists of semistable parabolic bundles. (If the base point z_0 is changed, θ changes in its algebraic equivalence class; in the case when $g = 0$, it is in fact well-defined.)

We outline the construction of $\mathcal{U}$ and the "parabolic Quot scheme" that intervenes. Let $\mathcal{O}(1)$ denote an ample line bundle on X of degree 1. Let $\mathcal{Q}_+$ denote the Quot scheme parametrising rank 2, degree d quotients of $\mathcal{O}(-m)^M$, with m a large positive integer, wisely chosen, and $M = \chi + 2m$. On $\mathcal{Q}_+ \times X$ there is the tautological sequence:

$$\mathcal{O}(-m)^M \to \mathcal{E}_{\mathcal{Q}_+} \to 0 \ .$$

For each i, consider the sheaf $\mathcal{E}_{\mathcal{Q}_+,z_i}$ on $\mathcal{Q}_+$, and let $\mathbb{P}_i$ denote the scheme representing its rank-one locally free quotients. We let $\hat{\mathcal{Q}}_+$ denote the fibre product $\times_{i\,\mathcal{Q}_+}\mathbb{P}_i$ and denote by $\mathcal{Q}$ the open subset of $\mathcal{Q}_+$ consisting of locally free quotients $\mathcal{O}(-m)^M \to E \to 0$ such that the induced map $\mathbb{C}^M \to H^0(E(m))$ is an isomorphism; we let $\hat{\mathcal{Q}}$ denote its inverse image in $\hat{\mathcal{Q}}_+$. (In particular $H^1(E(m)) = 0$ for such E.) By Grothendieck's criterion ([Le-P, Th. 8.2.1]) $\mathcal{Q}$ is regular, and therefore so is $\hat{\mathcal{Q}}$. We choose m large enough such that

- all semistable parabolic bundles occur as quotients in $\hat{\mathcal{Q}}$, and further,
- all *nonsemistable* bundles belonging to a certain finite number of Harder-Narasimhan types (depending on the situation) also occur in $\hat{\mathcal{Q}}$, and the corresponding strata are smooth.

Note that $\hat{\mathcal{Q}}$ is contained in the smooth locus of $\hat{\mathcal{Q}}_+$, and is irreducible.

Notation 2.3. Consider, briefly, the case $g = 0$, $d = 0$. In this case an open set $\mathcal{Q}_0$ of $\mathcal{Q}$ parametrises quotients which are trivial. For later use, we denote by $\hat{\mathcal{Q}}_0$ the open subset of $\hat{\mathcal{Q}}$ over $\mathcal{Q}_0$.

The data chosen at the beginning of this section determine a polarisation linearising the $\mathrm{PGL}(M)$ action on $\hat{\mathcal{Q}}_+$; the space $\mathcal{U}$ is the quotient of the subset of semistable points for this action. On this open set, the polarising line bundle is (up to a power) isomorphic as a $\mathrm{PGL}(M)$ line bundle to $\hat{\theta} \equiv \mathcal{L}_{\hat{\mathcal{Q}}}$. The line bundle θ is the descent to $\mathcal{U}$ of $\hat{\theta}$.

3. Computation of the Harder-Narasimhan trace

If we fix D and $R \subset I$ satisfying

(1) $2D - d > \sum_{R^c} t_i - \sum_R t_i$,

(2) $2D - d > 2g - 2 + |R^c|$

(the second inequality clearly implying the first if the genus is nonzero), there is an obvious family of nonsemistable parabolic bundles parametrised by $\mathcal{J} \equiv J_X^D \times J_X^{d-D}$, direct sums of parabolic line bundles as in Lemma 2.2.

Given a line bundle L of degree D, and another, L' of degree $d-D$, consider the cohomology $H^1(\mathrm{Hom}(L(Z_R), L'))$. (The sets Z_R and Z_{R^c} are defined in the statement of Lemma 2.2.) The exact sequence of sheaves

$$0 \to \mathrm{Hom}(L(Z_R), L') \to \mathrm{Hom}(L, L') \to L^{-1} \otimes L'|_{Z_R} \to 0$$

yields the exact sequence of vector spaces

$$0 \to \sum_{i\in R} L^{-1} \otimes L'|_{z_i} \to H^1(\mathrm{Hom}(L(Z_R), L')) \to H^1(\mathrm{Hom}(L, L')) \to 0\ .$$

An element of $H^1(\mathrm{Hom}(L,L'))$ gives an extension

$$0 \to L' \to E \to L \to 0 \ .$$

Writing the extension class as the pullback of an element $H^1(\mathrm{Hom}(L(Z_R),L'))$ corresponds to a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & E' & \longrightarrow & L(Z_R) & \longrightarrow & 0 \ . \end{array}$$

The bundle E has a natural (quasi)parabolic structure at points of Z_R, given by the map $E \to L'|_{Z_R} \to 0$. At points of Z_{R^c}, endow E with the parabolic structure $E \to L|_{Z_{R^c}} \to 0$.

All this can be done in a family. Choose Poincaré line bundles $\mathcal{L}$ on J_X^D and $\mathcal{L}'$ on J_X^{d-D}. Consider the bundle $\mathcal{N}$ on $\mathcal{J}$ defined to be the first direct image (on $\mathcal{J}$) of $\mathrm{Hom}(\mathcal{L}(Z_R),\mathcal{L}')$. We have a family of parabolic bundles parametrised by the corresponding projective bundle $\mathbb{P}(\mathcal{N}^*)$ (of one-dimensional *sub*spaces of the fibres of $\mathcal{N}$). Note that the generalisation of (4) is the tautological exact sequence on $\mathbb{P}(\mathcal{N}^*) \times X$:

$$0 \to \mathcal{L}' \to \mathcal{E}_{\mathbb{P}(\mathcal{N}^*)} \to \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{N}^*)}(-1) \to 0 \ .$$

We have the following:

PROPOSITION 3.1. *The generic parabolic bundle in this family is semistable. Under the rational map $\phi : \mathbb{P}(\mathcal{N}^*) - - \to \mathcal{U}$, θ pulls back to*

$$\begin{aligned} \mathcal{L}_{\mathbb{P}(\mathcal{N}^*)} = {} & (\det R(\pi_{J_X^D})_*\mathcal{L})^k \otimes \bigotimes_{i\in R^c} \mathcal{L}_{z_i}^{\mu_i} \otimes \bigotimes_{i\in I} \mathcal{L}_{z_i}^{k-\mu_i} \otimes \mathcal{L}_{z_0}^{l} \\ & \otimes (\det R(\pi_{J_X^{d-D}})_*\mathcal{L}')^k \otimes \bigotimes_{i\in R} \mathcal{L}'^{\mu_i}_{z_i} \otimes \bigotimes_{i\in I} \mathcal{L}'^{k-\mu_i}_{z_i} \otimes \mathcal{L}'^{l}_{z_0} \\ & \otimes \mathcal{O}_{\mathbb{P}(\mathcal{N}^*)}\left(k\left[D - \frac{d}{2}\right] + \frac{1}{2}\sum_i \mu_i + \sum_{R^c} \mu_i \right) \ . \end{aligned} \tag{6}$$

Proof. We need to show that a semistable parabolic bundle occurs in the family parametrised by $\mathbb{P}(\mathcal{N}^*)$. In fact by a standard argument (adapted to the parabolic case) one sees that the generic such bundle does indeed occur, thus proving the existence of a rational *dominant* map ϕ as above. The expression (6) follows by a simple computation using the definition of the theta bundle (Equation (5)). □

Remark 3.2. (a) One checks that the R.H.S of (6) is invariant under a change of Poincaré bundles: $\mathcal{L} \mapsto \mathcal{L} \otimes M$, $\mathcal{L}' \mapsto \mathcal{L}' \otimes M'$, where $M,\ M'$ are line bundles on the respective jacobians.

(b) If there are no parabolic points (i.e., $n = 0$), the above expression becomes

$$\mathcal{L}_{\mathbb{P}(\mathcal{N}^*)} = (\det R(\pi_{J_X^D})_*\mathcal{L})^k \otimes \mathcal{L}^l_{z_0} \otimes (\det R(\pi_{J_X^{d-D}})_*\mathcal{L}')^k \otimes \mathcal{L}'^l_{z_0} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{N}^*)}\left(k\left[D - \frac{d}{2}\right]\right) .$$

The interest of the above proposition is the following. The data (D, R) determine a Harder-Narasimhan stratum $S_{(D,R)}$. Let $\mathcal{N}$ denote the normal bundle, and let $\mathcal{N}_{\hat{q}}$ be its fibre at a point $\hat{q} \in \hat{\mathcal{Q}}$ which corresponds to a parabolic bundle $(E, Q)\underset{\eta}{\tilde{\to}}(L, Q_L) \oplus (L', Q_{L'})$. Then we have an isomorphism (depending on η, as the notation emphasises)

$$\mathcal{N}_{\hat{q}} \underset{T_\eta}{=} H^1(\mathrm{Hom}(L(Z_R), L')) .$$

We now apply the considerations of Section 1, taking $\mathcal{X}_+ = \hat{\mathcal{Q}}_+$, $\mathcal{W} = S_{(D,R)}$ and $\mathcal{X}$ to be an open $\mathrm{PGL}(M)$-invariant subset of $\hat{\mathcal{Q}}_+$ containing $S_{(D,R)}$ in which this stratum is closed. We check that (thanks to the hypothesis $2D-d > 2g - 2 + |R^c|$), the whole stratum is a bundle over $J_X^D \times J_X^{d-D}$. The isotropy (in $\mathrm{PGL}(M)$) of a point $\hat{q}$ is $P(\mathbb{C}^* \times \mathbb{C}^*)$, and it acts by a nontrivial character on the fibre of $\hat{\theta} \otimes S^q\mathcal{N}^*$ unless

$$q = q^* \equiv k[D - \frac{d}{2}] + \frac{1}{2}\sum_i \mu_i + \sum_{R^c} \mu_i .$$

The condition (H) of Section 1 is therefore satisfied, and we have the following important result. The proof is a matter of unravelling definitions, and we omit it.

Theorem 3.3. *The map ϕ of Proposition* 3.1 *induces the Harder-Narasimhan trace*

$$\mathrm{TR}_{S(D,R)} : H^0(\mathcal{U}, \theta) \to H^0(\mathcal{J}, \theta_{\mathcal{L}} \otimes \theta_{\mathcal{L}'} \otimes S^{q*}\mathcal{N}^*_{(\mathcal{L},\mathcal{L})})$$

where $\theta_{\mathcal{L}}$ is defined by the first line of the RHS *of* (6), *$\theta_{\mathcal{L}'}$ by the second line, and $\mathcal{N}^* \equiv (\pi_{\mathcal{J}})_*(\mathcal{L} \otimes \mathcal{L}'^{-1} \otimes K_X(Z_R))$.*

Remark 3.4. We shall consider two particular cases:

(a) The genus of X is zero, $D = d = 0$, $R = I$. In this case $\mathcal{N}$ is just the vector space $H^1(\mathcal{O}(-Z))$, and the trace map lands in $S^J H^0(K_X(Z))$. Note, for later reference, that the residue map yields an isomorphism

$$H^0(K_X(Z)) \to (\mathbb{C}^n/\mathbb{C})^* . \tag{7}$$

(b) The genus of X is zero, $d = 0, D = 1$, $R = I$. In this case the trace map is into $\mathbb{L}_{(z_1,\dots,z_n)} \otimes S^{J+k}H^0(\mathcal{O}(Z))$, where $\mathbb{L}_{(z_1,\dots,z_n)}$ is a one-dimensional vector space to be defined in the next section.

The generalisation of the above proposition to higher rank involves the moduli spaces studied by Bradlow, Garcia-Prada and others.

4. Generalised theta functions and conformal blocks, genus zero

Much of the material in this section is not new (cf., [G-K]), but I cannot find a reference in algebro-geometric terms.

We take $d = 0$, so that $\chi = 2$. Noting that $\sum_i \mu_i$ is an even integer, we set, as in Section 1,

$$J = \frac{1}{2}\left(\sum_i \mu_i\right) .$$

We remark that setting $d = 0$ leads to no loss of generality. In the presence of parabolic structures, a suitable Hecke transformation gives an isomorphism between even and odd degree (rank 2) moduli spaces. See Section 2.1 of [R].

Write the genus zero curve as the projective space associated to a two-dimensional vector space: $X = \mathbb{P}(V)$. Consider the tautological exact sequence on $X = \mathbb{P}(V)$:

$$0 \to s \to V \otimes_{\mathbb{C}} \mathcal{O} \underset{p}{\to} q \to 0$$

where we denote by q the hyperplane bundle $\mathcal{O}(1)$.

We pause to define the notation used in the Remark 3.4:

$$\mathbb{L}_{(z_1,\ldots,z_n)} = (\det V)^{-J} \underset{i}{\otimes} s_{z_i}^{\mu_i} .$$

Recall the definition of $\hat{\mathcal{Q}}_0$ (§2): this is the open subset of the parabolic Quot scheme $\hat{\mathcal{Q}}$ where the quotient sheaf is the trivial bundle. Its complement is a divisor $\mathcal{D} = \hat{\mathcal{Q}} \setminus \hat{\mathcal{Q}}_0$. We will need to consider one-parameter families of parabolic bundles which cut this divisor, and we deal with this now.

Let T denote the affine line, and let t be a co-ordinate. Let $\mathcal{E}_T$ be the bundle on $X \times T$ defined by

$$0 \to \mathcal{E}_T \to V \otimes_{\mathbb{C}} \mathcal{O} \oplus q \underset{P}{\to} q \to 0 \ , \tag{8}$$

where the map P is defined by

$$P(v, \beta) = p(v) - t\beta \ .$$

We omit the proof of the following:

Lemma 4.1. *There is an exact sequence sequence (on $T \times X$):*

$$0 \to s \to \mathcal{E}_T \to q \to 0$$

such that the induced map $T \to H^1(X, q^{-1}s) = H^1(K_X) = \mathbb{C}$ is given by the co-ordinate t.

One checks that $\mathcal{E}_T$ is a Hecke transform on $X \times T$:

$$0 \to \mathcal{E}_T \to V \otimes_{\mathbb{C}} \mathcal{O} \to q_0 \to 0$$

where $q_0 = (i_0)_* q$, i_0 being the inclusion $X \to X \times \{0\} \subset X \times T$. In turn, this yields on T the Hecke transform:

$$0 \to \mathcal{E}_{T,z_i} \to V \otimes_{\mathbb{C}} \mathcal{O} \to q_{z_i} \to 0 \ . \tag{9}$$

Turning to the corresponding projective bundles, we see that $\mathbb{P}(\mathcal{E}_{T,z_i})$ is related to the the product bundle $\mathbb{P}_T(V)_i \equiv T \times \mathbb{P}(V)$ by a blow-up and a blow-down:

$$\begin{tikzcd} & \mathcal{P}_i \arrow[dl, "L_i"'] \arrow[dr, "R_i"] & \\ \tilde{z}_i \in \mathbb{P}(\mathcal{E}_{T,z_i}) \arrow[dr] & & \mathbb{P}_T(V)_i \ni (0, z_i) \arrow[dl] \\ & T. & \end{tikzcd}$$

Here R_i is the blow-up of $\mathbb{P}_T(V)_i$ at $(0, z_i)$ - we denote the exceptional fibre by E_{R_i}. The map L_i blows down E_{L_i} (the proper transform of $\{0\} \times \mathbb{P}(V)$) to a point, which we denote $\tilde{z}_i$.

Let $\hat{T} \to T$ denote the fibre product $\times_T \mathbb{P}(\mathcal{E}_{T,z_i})$, and $\hat{T}_0 = T \times \underset{i}{\times} \mathbb{P}(V)$. As above, $\hat{T}$ and $\hat{T}_0$ are related by a blow-up and a blow-down:

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "L"'] \arrow[dr, "R"] & \\ \hat{T} \arrow[dr] & & \hat{T}_0 \arrow[dl] \\ & T. & \end{tikzcd}$$

Let D_0 denote the divisor in $\mathcal{P}$ defined by $\{t = 0\}$. Note the equality (for any i):

$$D_0 = E_{R_i} + E_{L_i} \ . \tag{10}$$

Define $\mathcal{L}_0$ to be the line bundle $(\det V)^{-J} \underset{i}{\otimes} q_i^{\mu_i}$ on $\hat{T}_0$, where q_i denotes the quotient line bundle on $\mathbb{P}_T(V)_i$. The isomorphism – on $\{t | t \neq 0\}$ – $\mathcal{E}_T \to V \otimes_C \mathcal{O}$ induced by (8) yields on $\mathcal{P} \setminus D_0$ an isomorphism between $\mathcal{L}_{\hat{T}}$ and $\mathcal{L}_0$.

The following proposition relates $\mathcal{L}_{\hat{T}}$ and $\mathcal{L}_0$ on $\mathcal{P}$:

Proposition 4.2. *The above isomorphism extends to an isomorphism*

$$\mathcal{L}_0 = \mathcal{L}_{\hat{T}} \otimes \mathcal{O}(\sum_i \mu_i E_{R_i}) \mathcal{O}(-(k+J)D_0) \ . \tag{11}$$

Proof. There are three steps. (a) We recall some notation: q_i denotes the tautological quotient bundle on $\mathbb{P}_T(V)_i$, and $\mathcal{Q}_{T,i}$ the quotient bundle on $\mathbb{P}(\mathcal{E}_{T,z_i})$. Pulled back to $\mathcal{P}_i$, these two line bundles are related by the natural isomorphism $q_i = \mathcal{Q}_{T,i}(E_{R_i})$.

(b) The map $\mathcal{E}_T \to V \otimes_{\mathbb{C}} \mathcal{O}$ induced by the exact sequence (8) yields a map of direct images

$$(\pi_T)_*\mathcal{E}_T \to V \otimes_{\mathbb{C}} \mathcal{O}_T \ .$$

These are rank two vector bundles and the above map is an isomorphism away from $t = 0$, where it vanishes. The first direct images are zero. This yields an isomorphism $\det R(\pi_T)_*\mathcal{E}_T \to \det V \otimes_{\mathbb{C}} \mathcal{O}(2D_0)$.

(c) The sequence (9) yields isomorphisms $\det \mathcal{E}_{T,z_i} \to \det V \otimes_{\mathbb{C}} \mathcal{O}(-D_0)$; we have a similar isomorphism at z_0.

It is straightforward now to verify the equation (10). $\square$

THEOREM 4.3. $H^0(\hat{\mathcal{Q}}_0, \hat{\theta})^{\mathrm{PGL}(M)} = \mathbb{W}(V)$. *For $k \geq J$, all invariant sections of $\hat{\theta}$ extend across $\mathcal{D}$, so that $H^0(\hat{\mathcal{Q}}, \hat{\theta})^{\mathrm{PGL}(M)} = \mathbb{W}(V)$. For $k < J$, $H^0(\hat{\mathcal{Q}}, \hat{\theta})^{\mathrm{PGL}(M)} \subset \mathbb{W}(V)$ is defined by the following condition: An element $\phi \in \mathbb{W}(V)$ belongs to the subspace $H^0(\hat{\mathcal{Q}}_0, \hat{\theta})$ if and only if*

$$\text{(G)} \qquad \phi \textit{ vanishes to order } J - k \textit{ at } (z_1, \ldots, z_n) \ .$$

Proof. Consider the universal quotient bundle $\mathcal{E}_{\mathcal{Q}_0}$ on $\mathcal{Q}_0 \times X$, and let $\mathbb{E} = (\pi_1)_*\mathcal{E}_{\mathcal{Q}_0}$ be its direct image on $\mathcal{Q}_0$. The map $\pi_1^*\mathbb{E} \to \mathcal{E}_{\mathcal{Q}_0}$ is an isomorphism, and therefore $H^0(\hat{\mathcal{Q}}_0, \hat{\theta})^{\mathrm{PGL}(M)} = H^0(\mathcal{Q}_0, \hat{\mathbb{W}}(\mathbb{E}))^{\mathrm{PGL}(M)}$. Now, $\mathcal{Q}_0$ is a homogeneous space for $\mathrm{PGL}(M)$, so that the last space is equal to the fixed subspace for the isotropy group at any point.

To prove the second statement, it suffices to see which invariant sections of $\hat{\theta}$ on $\hat{\mathcal{Q}}_0$ extend across the divisor $\mathcal{D} = \hat{\mathcal{Q}} \setminus \hat{\mathcal{Q}}_0$. The generic degeneration of the trivial rank two bundle on $\mathbb{P}^1$ is as a direct sum $\mathcal{O}(1) \oplus \mathcal{O}(-1)$. We now use Proposition 4.2 and the isomorphism (11). $\square$

Consider now the configuration space $\hat{\mathcal{M}}_n$ of n (distinct) points $z_1, \ldots, z_n$ on X (we do *not* take the quotient modulo the projective group), and repeat the above constructions relative to $\hat{\mathcal{M}}_n$. Once a further point z_0 is chosen, we get on $\hat{\mathcal{M}}_n$ a vector bundle $\mathcal{V}$ of generalised theta functions, which comes embedded in the (canonically) trivial bundle $\mathcal{W}$ with fibre $\mathbb{W}(V)$. The KZ connection is a connection on $\mathcal{W}$ which leaves the sub- bundle $\mathcal{V}$ invariant. This is an actual connection (rather than just a projective connection), and flat, but depends on the choice of a point ∞ on X, and is defined locally on $\hat{\mathcal{M}}_n$. Let $\hat{\mathcal{M}}'_n$ denote the open set of configurations disjoint from ∞. Given a section of $\mathcal{W}$ over $\hat{\mathcal{M}}'_n$, i.e., a function $w : \hat{\mathcal{M}}'_n \to \mathbb{W}(V)$, the KZ connection is defined by:

$$\nabla w = dw - \frac{1}{\kappa}\sum_i dz_i \sum_{j\neq i} \frac{\Omega_{ij}}{z_i - z_j} w$$

where (recall) $\kappa = k+2$. We have chosen a co-ordinate function on $X = \mathbb{P}(V)$ as in Section 1, and will not distinguish the points z_i and their co-ordinates. For each $j \neq i$, Ω_{ij} is an endomorphism of $\mathbb{W}(V)$ defined by:

$$\Omega_{ij} = e_i f_j + f_j e_j + \frac{1}{2} h_i h_j \ .$$

We have the following theorem, whose proof can be extracted by combining [G-K] and [S-S]. The latter work involves a detour via moduli spaces of bundles on elliptic surfaces. A direct proof will be published elsewhere.

THEOREM 4.4. *The bundle $\mathcal{V} \hookrightarrow \mathcal{W}$ is invariant under the above connection. The induced projective connection on $\mathcal{V}$ agrees with the heat equation connection, defined à la Hitchin.*

5. Interlude: hypergeometric local systems

In this section, as in the previous one, we let $z_i \in \mathbb{C}$ be variables, and we set $\mathcal{Z}_n \equiv \{(z_1,\dots,z_n) | z_i \neq z_j \ \forall i,j\}$. We denote by $\mathbb{X}_{\mathbb{J}}$ the product $\underbrace{X \times .. \times X}_{\mathbb{J} \text{ factors}}$. We set $\mathbb{X}_{\mathbb{J},n} \equiv \mathbb{X}_{\mathbb{J}} \times \mathcal{Z}_n$. We next define a collection $\mathcal{C}$ of hyperplanes $H \hookrightarrow \mathbb{X}_{\mathbb{J},n}$, and associate a positive rational number $a(H)$ to each H:

(1) D^i_ν will denote the divisor $D^i_\nu = \{(t_1,\dots,t_{\mathbb{J}}; z_1,\dots,z_n) | t_\nu = z_i\}$. The corresponding weight is $a(D^i_\nu) = \frac{\mu_i}{\kappa}$.

(2) $D_\nu = \{t_\nu = \infty\}$, $\ a(D_\nu) = -\frac{2(1+J-\mathbb{J})}{\kappa}$.

(3) $D_{(\nu,\nu')} = \{t_\nu = t_{\nu'}\}$, $\ a(D_{(\nu,\nu')}) = -\frac{2}{\kappa}$.

Let D denote the union of all the above divisors. Define the meromorphic function $\mathcal{T}$ by

$$\mathcal{T} = \prod_{i<j} (z_i - z_j)^{\frac{-\mu_i\mu_j}{2}} \prod_{\nu,i} (t_\nu - z_i)^{\mu_i} \prod_{\nu<\nu'} (t_\nu - t_{\nu'})^{-2} \ .$$

Note that the divisor of $\mathcal{T}$ is (the diagonals $\{z_i = z_j\}$ having been removed)

$$(\mathcal{T}) = \sum_{\nu,i} \mu_i D^i_\nu - 2\sum_{\nu<\nu'} D_{(\nu,\nu')} - \sum_\nu 2(J - \mathbb{J} + 1) D_\nu \ .$$

The above data define a *hypergeometric local system* on $\mathcal{Z}_n$, as follows. First, consider the twisted algebraic de Rham complex on $\mathbb{X}_{\mathbb{J},n} \setminus D$:

$$\cdots \to \Omega^p_{\text{alg}} \underset{d_{\mathcal{R}}}{\to} \Omega^{p+1}_{\text{alg}} \to \dots$$

where Ω^p_{alg} is the sheaf of algebraic $p-$forms and the differential $d_{\mathcal{R}}$ is defined by

$$d_{\mathcal{R}} \equiv d + \frac{1}{\kappa}\mathcal{T}^{-1}d\mathcal{T} = \mathcal{R}^{-1} \circ d \circ \mathcal{R}\ . \tag{12}$$

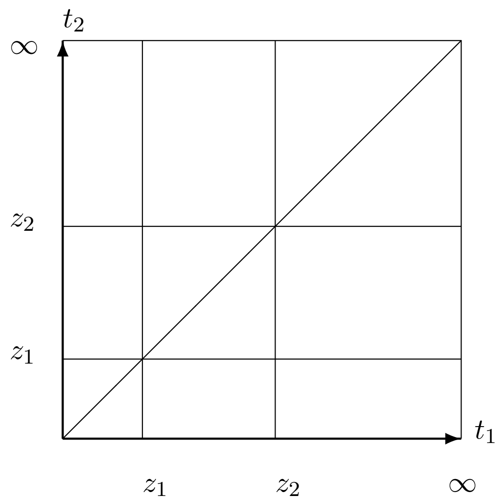
The second equality above is formal, with $\mathcal{R}$ denoting the κ^{th} root of $\mathcal{T}$. By Gothendieck's algebraic de Rham theorem, this complex gives the cohomology of a certain one-dimensional local system on $\mathbb{X}_{\mathbb{J},n}$.

Consider the top direct image of this local system on $\mathcal{Z}_n$. This yields a flat vector bundle $\mathcal{H}^{\mathbb{J}}_{\mathcal{R}}$ on $\mathcal{Z}_n$, which we now describe. Locally, over affine open sets U of the base, a section $\bar{\omega}$ of the vector bundle is represented by a $\mathbb{J}$-form ω along the fibres of the inverse image in $\mathbb{X}_{\mathbb{J},n} \setminus D$, modulo $d_{\mathcal{R}}$-exact forms. The projection $\mathbb{X}_{\mathbb{J},n} \setminus D \to \mathcal{Z}_n$ being affine, ω can be extended to a form ω' on the inverse image of U, and for the same reason, given a vector field v on U one can lift it to a vector field $\tilde{v}$. The covariant derivative of $\bar{\omega}$ is defined by

$$\nabla_v \bar{\omega} = \overline{\mathcal{R}^{-1}\mathcal{L}_{\tilde{v}}\mathcal{R}\omega'}$$

where $\mathcal{L}$ denotes the Lie derivative, and the above equation is to be read in the same spirit as (12). One checks that this is independent of choices.

All this is in the algebraic category. The fibre of the local system $\mathcal{H}^{\mathbb{J}}_{\mathcal{R}}$ at a point $\mathbf{z} \in \mathcal{Z}_n$ is the $\mathbb{J}^{\mathrm{th}}$ cohomology of the local system on the fibre $\mathbb{X}_{\mathbb{J},n} \setminus D_{\mathbf{z}}$.



Here we denote by $D_{\mathbf{z}}$ the union of the divisors $\{t_1, \dots, t_{\mathbb{J}} | t_\nu = z_i\}$, $\{t_1, \dots, t_{\mathbb{J}} | t_\nu = \infty\}$ and $\{t_1, \dots, t_{\mathbb{J}} | t_\nu = t_{\nu'}\}$.

There exists a natural blow-up $\pi' : \mathbb{X}'_{\mathbb{J},n} \to \mathbb{X}_{\mathbb{J},n}$ ([E-S-V], [V]) such that the (reduced) inverse image D' of D is a divisor with normal crossings. A nonempty intersection of elements of $\mathcal{C}$ is called an *edge.* If L is an edge, set $\mathcal{C}_L = \{H \in \mathcal{C} | L \subset H\}$. Note that $L = \cap_{H \in \mathcal{C}_L} H$. The notion of a *dense* edge is due to Varchenko. For a definition, and a proof of the following theorem, see [O-T]. In fact we state a slightly different, relative (to $\mathcal{Z}_n$), version.

THEOREM 5.1. *Let $\mathcal{L}_p$, $p = n, \dots, \mathbb{J}+n-2$, denote the set of dense edges of dimension p, and $\mathbf{L}_p$ the union of the elements of $\mathcal{L}_p$. Set $\mathcal{L}_{\mathbb{J}+n-1} = \mathcal{C}$ and $\mathcal{L} = \coprod_{p=n}^{\mathbb{J}+n-1} \mathcal{L}_p$. Let $\mathbb{X}^n = \mathbb{X}_{\mathbb{J},n}$, and, for $p < \mathbb{J}$ let $\pi_p : \mathbb{X}^{p+n} \to \mathbb{X}^{p-1+n}$ be the blow-up along the proper transform of $\mathbf{L}_{p-1}$ under $\pi_1 \circ \cdots \circ \pi_{p-1}$. Then $\mathbb{X}'_{\mathbb{J},n} = \mathbb{X}^{\mathbb{J}+n-1}$ and $\pi' = \pi_1 \circ \cdots \circ \pi_{\mathbb{J}-1}$. The irreducible components of D' intersect normally, and the set of these divisors is in natural bijection $L \leftrightarrow D'(L)$ with the set of edges $L \in \mathcal{L}$.*

The following is Lemma 10.8.4 of [V]. Here $a(L)$ is the *quasi-classical weight of* an edge L, defined by

$$a(L) = \sum_{L \subset H} a(H) = \sum_{H \in \mathcal{C}_L} a(H)$$

and we have introduced the notation $\alpha(L) = \kappa a(L)$.

LEMMA 5.2. *Let $\mathbf{x}_0$ be a point of D', such that in a neighbourhood D' is given by the equation $x_1 \dots x_\ell = 0$. Then around $\mathbf{x}_0$ one has $\mathcal{T} \circ \pi' = (unit) \times x_1^{\kappa a(L_1)} \dots x_\ell^{\kappa a(L_\ell)}$ where the edge L_i corresponds to the divisor $D'(L_i) = \{x_i = 0\}$. That is, the divisor of $\mathcal{T} \circ \pi'$ in $\mathbb{X}'_{\mathbb{J},n}$ is*

$$(\mathcal{T} \circ \pi') = \sum_L \alpha(L) D'(L) \ .$$

We note also:

LEMMA 5.3. *Let $D'(\mathcal{L}_p)$ denote the sum of divisors corresponding to elements of $\mathcal{L}_p$. The canonical bundles of $\mathbb{X}'_{\mathbb{J},n}$ and $\mathbb{X}_{\mathbb{J},n}$ are related on $\hat{\mathbb{X}}'_{\mathbb{J},n}$ by*

$$K_{\mathbb{X}_{\mathbb{J},n}} = K_{\mathbb{X}'_{\mathbb{J},n}}(-(\mathbb{J}-1)D'(\mathcal{L}_n) - (\mathbb{J}-2)D'(\mathcal{L}_{n+1}) \cdots - D'(\mathcal{L}_{\mathbb{J}+n-2})) \ .$$

We next construct a normal variety $\hat{\mathbb{X}}_{\mathbb{J},n,\mathcal{R}}$, and a generically finite map $\pi : \hat{\mathbb{X}}_{\mathbb{J},n,\mathcal{R}} \to \mathbb{X}_{\mathbb{J},n}$ such that the meromorphic function $\mathcal{T}$, pulled back by $\hat{\pi}$, has a κ^{th} root $\mathcal{R}$. This we do by thinking of $\mathcal{T} \circ \pi'$ as a rational map to $\mathbb{P}^1$, taking the fibre product with the κ^{th}-power map $\mathbb{P}^1 \to \mathbb{P}^1$, and then closing up the result in $\mathbb{X}'_{\mathbb{J},n} \times \mathbb{P}^1$, thus obtaining a (reduced, but in general, *reducible* and nonnormal) variety, denoted $\mathbb{X}''_{\mathbb{J},n,\mathcal{R}}$. Explicitly, if (x, y) are homogeneous co-ordinates on $\mathbb{P}^1$ and $\mathcal{T} \circ \pi'$ is locally given as a quotient F/G of regular functions, $\mathbb{X}''_{\mathbb{J},n,\mathcal{R}}$ is defined by

$$x^\kappa G = y^\kappa F \ .$$

In fact locally F and G can be taken to be monomials (since D' has normal crossings), so that $\mathbb{X}''_{\mathbb{J},n,\mathcal{R}}$ is locally a (nonnormal) toric variety. Its normalisation is therefore locally toric. (See [C, Lecture 1 (7): Third construction: toric ideals].)

Let π'' denote the projection $\mathbb{X}''_{\mathbb{J},n,\mathcal{R}} \to \mathbb{X}'_{\mathbb{J},n}$. We define $\hat{\pi} : \hat{\mathbb{X}}_{\mathbb{J},n,\mathcal{R}} \to \mathbb{X}''_{\mathbb{J},n,\mathcal{R}}$ to be the normalisation map, and set $\pi = \pi' \circ \pi'' \circ \hat{\pi}$. This is the tower that we finally obtain:

$$\begin{array}{ccccccc} \hat{D} & \longrightarrow & D'' & \longrightarrow & D' & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \hat{\mathbb{X}}_{\mathbb{J},n,\mathcal{R}} & \xrightarrow{\hat{\pi}} & \mathbb{X}''_{\mathbb{J},n,\mathcal{R}} & \xrightarrow{\pi''} & \mathbb{X}'_{\mathbb{J},n} & \xrightarrow{\pi'} & \mathbb{X}_{\mathbb{J},n} \ . \end{array}$$

Note that the group μ_κ of κ^{th} roots of unity acts on $\mathbb{X}''_{\mathbb{J},n,\mathcal{R}}$ as well as its normalisation; the map $\hat{\mathbb{X}}_{\mathbb{J},n,\mathcal{R}} \setminus \hat{D} \to \mathbb{X}_{\mathbb{J},n} \setminus D$ is a μ_κ-fibration.

In the rest of this paper we drop, for the most part, the suffixes $\{\}_{\mathbb{J},n}$, *etc.* We have proved:

Proposition 5.4. *The variety* $\hat{\mathbb{X}}$ *is locally toric, and therefore has rational singularities.*

Before going further, we make an important observation:

Remark 5.5. The map

$$\omega \mapsto \pi^*_{\mathcal{R}}\omega \equiv \mathcal{R}\pi^*\omega$$

yields an injective map (of complexes) from the twisted de Rham complex on $\mathbb{X} \setminus D$ to the de Rham complex on $\hat{\mathbb{X}} \setminus \hat{D}$. The image is the subcomplex which transforms under μ_κ by the identity character.

Let D'' denote the (reduced) inverse image of D' in $\mathbb{X}''$. An irreducible component of D'' arises from one of the following three situations:

- D''_1 is above a component $D'_1 = \{x_1 = 0\}$ (corresponding to an edge L_1) of D', such that order $\mathcal{T} \geq 0$ along D'_1. (We abuse notation and write $\mathcal{T}$ when we mean $\mathcal{T} \circ \pi'$.)
- D''_2 is above a component $D'_2 = \{x_2 = 0\}$ (corresponding to an edge L_2) of D', such that order $\mathcal{T} < 0$ along D'_2.
- D''_3 is the inverse image of an intersection $D'_1 \cap D'_2 = \{x_1 = x_2 = 0\}$ of the types 1 and 2 above, with order $\mathcal{T} > 0$ along D'_1.

We will let $\hat{D}$, $\hat{D}_1$, etc., denote the corresponding inverse images in $\hat{\mathbb{X}}$. We will need to have an explicit model of the map π (defined above as the composite morphism $\pi' \circ \pi'' \circ \hat{\pi}$) in the vicinity of a smooth point of $\hat{D}$. We will use the fact that "normalisation commutes with completion" ([EGA-IV], Corollaire 6.14.5).

(i) Consider first a point $\hat{\mathbf{x}}_0$ of $\hat{D}$ above a smooth point $\mathbf{x}'_0$ of a component D'_1 as above. Thus in a neighbourhood of $\mathbf{x}'_0$, $\mathcal{T} = (\text{unit}) \times x_1^{\alpha(L_1)}$, with

$\alpha(L_1) \geq 0$. The variety $\mathbb{X}''$ is defined locally by $\{(u, x_1) | u^{\varepsilon\kappa} = x_1^{\alpha(L_1)}\}$, with ε a set equal to 1 for the present. (The image of $\hat{\mathbf{x}}_0$ under $\hat{\pi}$, which we can denote $\mathbf{x}''_0$, is defined by $u = 0, x_1 = 0$. We are suppressing co-ordinates which do not play a role.) Let $\delta = \gcd(\kappa, \alpha(L_1))$, and write $\kappa = \delta\kappa'$, $\alpha(L_1) = \delta\alpha'$. Then $\hat{\mathbf{x}}_0$ is on one of δ branches of $\hat{\mathbb{X}}$, and locally the map $\hat{\mathbb{X}} \to \mathbb{X}''$ is given by $t \mapsto (u = t^{\varepsilon\alpha'}, x_1 = t^{\kappa'})$. Recall that we have denoted by $\mathcal{R}$ the κ^{th} root of $\mathcal{T}$. Pulled back to $\hat{\mathbb{X}}$, $\mathcal{R} = (unit) \times t^{\alpha'}$ near $\hat{\mathbf{x}}_0$.

(ii) If the point $\hat{\mathbf{x}}_0$ is above a smooth point $\mathbf{x}'_0$ of a component D'_2, the above paragraph applies, with $\alpha(L_1)$ replaced by $\alpha(L_2)$, which is now *negative* ($\varepsilon = -1$) and finally, with the notation change $x_1 \rightsquigarrow x_2$.

(iii) Finally, suppose $\hat{\mathbf{x}}_0$ is above a smooth point $\mathbf{x}'_0$ of $D'_1 \cap D'_2$, and $\mathbf{x}''_0 \in D''_3$ is the image of $\hat{\mathbf{x}}_0$ under $\hat{\pi}$. We have then, near $\mathbf{x}_0$

$$\mathcal{T} = (\text{unit}) \times x_1^{\alpha(L_1)} x_2^{\alpha(L_2)}$$

(with $\alpha(L_1) > 0,\ \alpha(L_2) < 0$). Near $\mathbf{x}''_0 \in \mathbb{X}' \times \mathbb{P}^1$, $\mathbb{X}''$ is defined by

$$\{(u, x_1, x_2) | u^{\kappa} x_2^{-\alpha(L_2)} = x_1^{\alpha(L_1)}\},$$

with the co-ordinates of $\mathbf{x}''_0$ being $(u_0, 0, 0)$, where $u_0 \neq 0$. (We ignore spectator co-ordinates.) The normalisation map is $(u, t) \mapsto (u, t^{-\alpha'_2} u^{\frac{\kappa}{\alpha_1}}, t^{\alpha'_1})$, where we have written $\alpha(L_i) = \delta\alpha'_i,\ i = 1, 2,\ \delta = \gcd(\alpha(L_1), \alpha(L_2))$. Note that $\mathcal{R}$ is a unit near $\hat{\mathbf{x}}$.

Lemma 5.6. *The canonical bundles of the smooth locus of $\hat{\mathbb{X}}$ and that of $\mathbb{X}'$ are related in the three cases as follows. The behavior of $\mathcal{R}$, derived in the above paragraphs, is also summarised.*

- $K_{\mathbb{X}'} = K_{\hat{\mathbb{X}}}(-(\kappa' - 1)\hat{D}_1)$ *near* $\hat{D}_1$, *and* $(\mathcal{R}) = \alpha'\hat{D}_1$.
- $K_{\mathbb{X}'} = K_{\hat{\mathbb{X}}}(-(\kappa' - 1)\hat{D}_2)$ *near* $\hat{D}_2$, *and* $(\mathcal{R}) = \alpha'\hat{D}_2$.
- $K_{\mathbb{X}'} = K_{\hat{\mathbb{X}}}(-(\alpha'_1 - \alpha'_2 - 1)\hat{D}_3)$ *near* $\hat{D}_3$, *and* $\mathcal{R}$ *is a unit.*

6. The trace map: first case

In this section we calculate the Harder-Narasimhan trace on the stratum $D = 0, R = I$. From now on, we omit the book-keeping factor τ^{-J}. (See Remark 1.5.)

Theorem 6.1.

$$\mathrm{TR}_{(0,I)}(w) = \left\langle w, \prod_{\nu=0}^{J} \left(\sum_i \frac{e_i}{t_\nu - z_i} \right) v_J \right\rangle dt$$

where $dt = dt_1 \dots dt_J$.

Proof. We use the notation of the proof of Theorem 4.3. The Harder-Narasimhan stratum of interest is $\mathcal{Q}_0 \times \mathbb{P}(\mathbb{E})$ diagonally included in $\hat{\mathcal{Q}}_0$, which is isomorphic to $\mathcal{Q} \times \mathbb{P}(\mathbb{E})^n$. The bundle $\hat{\theta}$ restricted to $\hat{\mathcal{Q}}_0$ is the pullback of the corresponding $\hat{\theta}$ on $\mathbb{P}(\mathbb{E})^n$ (defined as in §1). Now use Proposition 1.8, and commutativity of the diagram

$$\begin{array}{ccc} w & \longrightarrow & \mathrm{TR}_{(0,1)}(w) \\ \big\downarrow{=} & & \text{residue}\big\downarrow \\ w & \longrightarrow & \mathrm{TR}(w) \end{array}$$

where TR is the trace calculated in Proposition 1.8 and the residue map is defined as in Remark 3.4. (In particular, we use the isomorphism (7)). □

Remark 6.2. It is instructive to check that the map $\mathrm{TR}_{(0,I)}$ lands in the subspace

$$S^J H^0(K_X(Z)) \subset H^0(\underbrace{K_X(Z) \otimes \cdots \otimes K_X(Z)}_{J \text{ factors}}) = H^0(K_{\mathbb{X}}(Z_1 + \cdots + Z_J))$$

where Z_ν denotes the pullback of Z from the ν^{th} factor. First note that for each ν', the residue at $t_{\nu'} = \infty$ of the form $\mathrm{TR}_{(0,I)}(w)$ is (up to a nonzero constant factor)

$$\left\langle w, \left(\sum_i e_i\right) \prod_{\nu=0}^{J'} \left(\sum_i \frac{e_i}{t_\nu - z_i}\right) v_J \right\rangle dt'$$

which vanishes because w is invariant. (Here the superscript $'$ indicates a product omitting ν'.) Thus the form $\mathrm{TR}_{(0,I)}(w)$ is indeed regular on $\mathbb{X}_J$ except along the divisors D^i_ν where it has a simple pole. The symmetry is clear because the e_i's all commute.

Remark 6.3. Note that the Harder-Narasimhan trace on the stratum $S_{(0,I)}$ is defined on *all of* $\mathbb{W}(V)$ rather than on the subspace satisfying the condition (G) of Theorem 4.3. This will not be the case for other strata.

We now let the points z_i vary over the open set $\mathbb{P}^1 \setminus \infty$. The trace map can then be regarded as a map of vector bundles $\mathcal{W} \to \mathcal{H}^J_{\mathcal{R}}$, where the latter bundle is the local system described in Section 5, with the choice $\mathbb{J} = J$.

Theorem 6.4. *The map* $\mathrm{TR}_{(0,I)}$ *is compatible with connections.*

Proof. I use tricks learnt from [E-F-K] (where the *dual map* is considered), and have adapted some of the following notation from there. I go into some detail because one step of the above reference is not clear to me. Also, we will need to make a similar, more complicated computation in the next section. Set

- $P = \langle w, \prod_{\nu=0}^J (\sum_i \frac{e_i}{t_\nu - z_i}) v_J \rangle$.

- $T_{\nu,i} = \frac{e_i}{t_\nu - z_i}$.
- $Y_\nu = \sum_i \frac{e_i}{t_\nu - z_i} = \sum_i T_{\nu,i}$.
- $H_{\nu,i} = \frac{h_i+\mu_i}{t_\nu - z_i}$.
- $H_\nu = \sum_i \frac{h_i+\mu_i}{t_\nu - z_i} = \sum_i H_{\nu,i}$.

Note the equalities:

$$\frac{\partial Y_\nu}{\partial z_i} = -\frac{\partial T_{\nu,i}}{\partial t_\nu}; \quad \sum_{j\neq i}[\frac{\Omega_{ij}}{z_i - z_j}, Y_\nu] = T_{\nu,i}\sum_j \frac{h_j}{t_\nu - z_j} - Y_\nu \frac{h_i}{t_\nu - z_i} \ .$$

Suppose w satisfies $\nabla w = 0$. Let $\mathcal{R}$ be a multi-valued meromorphic function whose logarithmic differential is well-defined. We have then (as above, J' flags a product omitting ν'):

$$\begin{aligned}
&\mathcal{R}^{-1}\mathcal{L}_{\partial_{z_i}}(\mathcal{R}Pdt)\\
&\quad = \mathcal{R}^{-1}\frac{\partial \mathcal{R}}{\partial z_i}Pdt + \left\langle \frac{\partial w}{\partial z_i}, \prod_{\nu=0}^{J} Y_\nu v_J \right\rangle dt + \sum_{\nu'} \left\langle w, \prod_{\nu=0}^{J'} Y_\nu \frac{\partial Y_{\nu'}}{\partial z_i} v_J \right\rangle dt\\
&\quad = \left(\mathcal{R}^{-1}\frac{\partial \mathcal{R}}{\partial z_i} + \frac{1}{2\kappa}\sum_{j\neq i}\frac{\mu_i\mu_j}{z_i - z_j}\right) Pdt + \sum_{\nu'}\langle w, Y_1 \ldots [\ \ ]^{\nu'} \ldots Y_J v_J\rangle dt
\end{aligned}$$

where in the ν'^{th} place we have

$$\begin{aligned}
[\ \ ]^{\nu'} &\equiv \frac{\partial Y_{\nu'}}{\partial z_i} + \frac{1}{\kappa}\sum_{j\neq i}\left[\frac{\Omega_{ij}}{z_i - z_j}, Y_{\nu'}\right]\\
&= -\mathcal{R}^{-1}\frac{\partial(\mathcal{R}T_{\nu',i})}{\partial t_{\nu'}} + \mathcal{R}^{-1}\frac{\partial \mathcal{R}}{\partial t_{\nu'}}T_{\nu',i} + \frac{1}{\kappa}\Big(T_{\nu',i}\sum_j \frac{h_j}{t_{\nu'} - z_j} - Y_{\nu'}\frac{h_i}{t_{\nu'} - z_i}\Big).
\end{aligned}$$

Re-arranging, and adding and subtracting terms, we get

$$\begin{aligned}
\mathcal{R}^{-1}\mathcal{L}_{\partial_{z_i}}(\mathcal{R}Pdt) = &\Big(\mathcal{R}^{-1}\frac{\partial \mathcal{R}}{\partial z_i} + \frac{1}{2\kappa}\sum_{j\neq i}\frac{\mu_i\mu_j}{z_i - z_j} + \frac{1}{\kappa}\sum_\nu \frac{\mu_i}{t_\nu - z_i}\Big)Pdt\\
&-\mathcal{R}^{-1}\sum_{\nu'}\frac{\partial}{\partial t_{\nu'}}(\mathcal{R}\langle w, Y_1 \ldots [T_{\nu',i}]^{\nu'} \ldots Y_J v_J\rangle)dt\\
&+\sum_{\nu'}\langle w, Y_1 \ldots [\ \ ]^{\nu'} \ldots Y_J v_J\rangle dt
\end{aligned} \tag{13}$$

where in the ν'^{th} place in the third term, we have

$$[\ \ ]^{\nu'} \equiv \frac{1}{\kappa}(T_{\nu',i}H_{\nu'} - Y_{\nu'}H_{\nu',i}) + \underline{\Big(\mathcal{R}^{-1}\frac{\partial \mathcal{R}}{\partial t_{\nu'}} - \frac{1}{\kappa}\sum_j \frac{\mu_j}{t_{\nu'} - z_j}\Big)}T_{\nu',i} \ .$$

We now choose

$$\mathcal{R} = \prod_{i<j}(z_i - z_j)^{\frac{-\mu_i\mu_j}{2\kappa}} \prod_{\nu,i}(t_\nu - z_i)^{\frac{\mu_i}{\kappa}} \prod_{\nu<\nu'}(t_\nu - t_{\nu'})^{-\frac{2}{\kappa}} \ .$$

This has the effect of killing the first line of the R.H.S of (13). As for the last line, after simplifying the underlined term and commuting $H_{\nu'}$ and $H_{\nu',i}$ to the extreme right, one is left with

$$\frac{2}{\kappa}\sum_{\nu'<\nu''}[*]\langle w, Y_1\ldots[1]^{\nu'}\ldots[1]^{\nu''}\ldots Y_J e_i e_j v_J\rangle dt$$

where

$$\begin{aligned}[*] = \sum_j & \frac{1}{(t_{\nu'}-z_i)(t_{\nu'}-z_j)(t_{\nu''}-z_j)} - \frac{1}{(t_{\nu'}-z_i)(t_{\nu'}-z_j)(t_{\nu''}-z_i)} \\ & - \frac{1}{(t_{\nu'}-t_{\nu''})(t_{\nu'}-z_i)(t_{\nu''}-z_j)} + \frac{1}{(t_{\nu'}-t_{\nu''})(t_{\nu'}-z_j)(t_{\nu''}-z_i)} \\ = 0\ . &\end{aligned}$$

(One way of checking that the expression on the L.H.S. above is zero is this: consider it as a function of t'_ν – fixing all other variables – and show that all residues vanish.)

This shows that

$$\mathcal{R}^{-1}\mathcal{L}_{\partial_{z_i}}(\mathcal{R}P dt) = -\mathcal{R}^{-1}\sum_{\nu'}\frac{\partial}{\partial t_{\nu'}}(\mathcal{R}\langle w, Y_1\ldots[T_{\nu',i}]^{\nu'}\ldots Y_J v_J\rangle)dt\ . \tag{14}$$

The R.H.S. is a total derivative (in the t variables), which is what we wanted. □

7. The trace: second case

In this section we calculate the Harder-Narasimhan trace on the stratum $D = 1, R = I$, using the set-up of Section 3. (This section is not used later in the paper, but for Remark 7.1.) Recall that $\tilde{z}_i \in \mathbb{P}(\mathcal{E}_{T,i})$ is the image of E_{L_i}; let $\tilde{\mathbf{z}}$ denote the point $(\tilde{z}_1,\ldots,\tilde{z}_n)$ in $\hat{T}$.

We have a family of parabolic bundles parametrised by $\hat{T}$, such that the point $\tilde{\mathbf{z}}$ corresponds to a nonsemistable bundle of type $(1, I)$. One verifies that the tangent space is $H^1(K_X(-Z))$, and that if we map $\hat{T}$ into $\hat{\mathcal{Q}}$ (by choosing a trivialisation of $(\pi_T)_*\mathcal{E}(m)$), this is an immersion at $\tilde{\mathbf{z}}$, the image cutting $S_{(1,I)}$ transversally. Thus the normal derivative of a section on the stratum can be computed after pulling it back to $\hat{T}$. This is the idea of the following calculation.

We make a couple of preliminary observations:

(1) The space $H^1(K_X(-Z))$ is filtered:

$$0 \to \oplus_i K_{z_i} \to H^1(K_X(-Z)) \to \mathbb{C} \to 0$$

with a splitting being given, once a point ∞ is chosen, by the surjection $H^1(K_X(-Z)) \to H^1(K_X(-Z+\infty))$. (Dually, we have

$$0 \to \mathbb{C} \to H^0(\mathcal{O}(Z)) \to \oplus_i K_{z_i}^{-1} \to 0$$

split by $H^0(\mathcal{O}(Z-\infty)) \hookrightarrow H^0(\mathcal{O}(Z))$.) Note also the isomorphism $H^0(\mathcal{O}(Z-\infty)) = H^0(K_X(Z+\infty))$, given by multiplication by dt, with t a co-ordinate.

(2) From (9) one sees that there are natural isomorphisms (a) $(K_X)_{z_i} = (K_{\hat{T}}^{-1})_{\tilde{z}_i}$ (b) $(\det \mathcal{E}_T)_0 = det\ V$.

The point of the observation (1) is that, with the splitting, we can write:

$$S^{J+k}H^0(\mathcal{O}(Z)) = \oplus_{m=0}^{J+k} S^m H^0(K_X(Z+\infty)) \otimes S^{J+k-m}H^0(K_X(2\infty))$$

The expression for the trace in Theorem 7.3 below is to be read with this in mind.

Step 1. Choose a basis for V and a co-ordinate x on $\mathbb{P}(V)$ as above. As before, we abuse notation and set $x(z_i) = z_i$. Let $(t, x_i = z_i + \zeta_i)$ be co-ordinates on $\mathbb{P}_T(V)_i$. Define co-ordinates $(t, \tilde{\zeta}_i)$ on $\mathbb{P}(\mathcal{E}_{T,i})$ by $t = \zeta_i \tilde{\zeta}_i$.

Step 2. Consider the section σ_w of $\mathcal{L}_0$ defined as in the proof of Proposition 1.8 . We have

$$\sigma_w(x_1, \ldots, x_n) = \Big\langle w, \exp - \sum_i (\zeta_i + z_i) e_i\ v_J \Big\rangle \bigotimes_i \sigma'^{\mu_i}\ \tau^{-J}\ .$$

Pulled back to $\mathcal{P}$, this section can be written

$$\begin{aligned}\sigma_w &= \Big\langle w, \exp - \sum_i (\zeta_i + z_i) e_i v_J \Big\rangle \bigotimes_i \sigma'^{\mu_i}\ \tau^{-J} \\ &= \Big\langle w, \exp - \sum_i (\frac{t}{\tilde{\zeta}_i} + z_i) e_i\ v_J \Big\rangle \bigotimes_i \sigma'^{\mu_i}\ \tau^{-J}\ .\end{aligned}$$

Step 3. Composing this with the isomorphism given by Proposition 4.2 one gets a section

$$\tilde{\sigma}_w = \Big\langle w, \exp - \sum_i \Big(\frac{t}{\tilde{\zeta}_i} + z_i\Big) e_i\ v_J \Big\rangle \prod_i (\frac{\tilde{\zeta}_i}{t})^{\mu_i} t^{k+J} \tilde{\sigma}^* \tau^{-J}$$

where $\tilde{\sigma}^*$ is a section of $\mathcal{L}_{\hat{T}}$, nonvanishing at $\tilde{\mathbf{z}}$.

Step 4. Set $E_{\mathbf{z}} = \exp \sum_i z_i e_i$. The above expression can be re-written:

$$\tilde{\sigma}_w = \Big\langle w, \exp - \sum_i \frac{t}{\tilde{\zeta}_i} e_i\ E_{\mathbf{z}}^{-1} v_J \Big\rangle \prod_i \tilde{\zeta}_i^{\mu_i} t^{k-J} \tilde{\sigma}^* \tau^{-J}\ .$$

Expanding the exponential, we get

$$\tilde{\sigma}_w = \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!} \langle w, e^\alpha E_{\mathbf{z}}^{-1} v_J \rangle \prod_i \tilde{\zeta}_i^{\mu_i - \alpha_i} t^{k-J+|\alpha|} \tilde{\sigma}^* \tau^{-J}\ .$$

Remark 7.1. Note that for $\tilde{\sigma}_w$ to be regular at $t = 0$, it is necessary and sufficient that

$$\text{(K)} \qquad \langle w, e^\alpha E_{\mathbf{z}}^{-1} v_J \rangle = \langle E_{\mathbf{z}} w, e^\alpha v'_J \rangle = 0 \text{ for } |\alpha| < J - k.$$

We recognise a new *avatar* of the condition (G) of 4.3: once (K) is satisfied, the section $\tilde{\sigma}_w$ vanishes to order $J + k$ at the point $\tilde{\mathbf{z}} \in \hat{T}$.

Step 5. With a little work, one can re-write the above expression for $\tilde{\sigma}_w$ as follows:

$$\tilde{\sigma}_w = \sum_\beta \frac{(-1)^{|\beta|}}{\beta!} \langle E_{\mathbf{z}} w, f^\beta v'_J \rangle \prod_i \tilde{\zeta}_i^{\beta_i} t^{k+J-|\beta|} \tilde{\sigma}^* \tau^{-J}$$

and the condition for regularity becomes:

$$\text{(K}'\text{)} \qquad \langle E_{\mathbf{z}} w, f^\beta v'_J \rangle = 0 \text{ for } |\beta| > J + k.$$

Notation 7.2. We have introduced above the notation: $E_{\mathbf{z}} = \exp \sum_i z_i e_i$.

THEOREM 7.3. *Define*

$$\widehat{\mathrm{TR}}_{(1,I)}(w) \equiv \sum_L \gamma_{|L|} \Big\langle E_{\mathbf{z}} w, \prod_{\nu \in L} \Big(\sum_i \frac{f_i}{t_\nu - z_i} \Big) v'_J \Big\rangle dt$$

where the sum is over all subsets $L = \{\nu_1, \ldots, \nu_{|L|}\} \subset \{1, \ldots, J + k\}$, $dt = dt_1 \ldots dt_{J+k}$. *The map* $\widehat{\mathrm{TR}}_{(1,I)}$ *is compatible with connections provided*

$$\text{(15)} \qquad \gamma_{|L|} = (k - |L| + J) \gamma_{|L|+1} \ .$$

Note that because of (K′) the sum only goes up to $|L| = J + k$, and equation 15 can be solved by taking $\gamma_{J+k-l} = l!$.

Proof. The condition (15) will emerge in the course of the proof. We will use the following notation:

- $\hat{P}_L = \langle E_{\mathbf{z}} w, \prod_{\nu \in L} (\sum_i \frac{f_i}{t_\nu - z_i}) v'_J \rangle$.
- $\mathbb{H} = \sum_j h_j$.
- $\hat{T}_{\nu,i} = \frac{f_i}{t_\nu - z_i}$.
- $\hat{Y}_\nu = \sum_i \frac{f_i}{t_\nu - z_i} = \sum_i \hat{T}_{\nu,i}$.

We begin with some identities, whose proofs are straightforward.

(1) $E_{\mathbf{z}} \Omega_{ij} E_{\mathbf{z}}^{-1} = \Omega_{ij} + (z_i - z_j)(h_i e_j - h_j e_i) - e_i e_j (z_i - z_j)^2$.
(2) $E_{\mathbf{z}} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} w = (\sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} - e_i - \frac{\mathbb{H} e_i}{2}) E_{\mathbf{z}} w$ (using invariance of w).
(3) $\frac{\partial \hat{Y}_\nu}{\partial z_i} = -\frac{\partial \hat{T}_{\nu,i}}{\partial t_\nu}$.

(4) $\sum_{j\neq i}[\frac{\Omega_{ij}}{z_i-z_j},\hat{Y}_\nu]=-\hat{T}_{\nu,i}\sum_j\frac{h_j}{t_\nu-z_j}+\hat{Y}_\nu\frac{h_i}{t_\nu-z_i}$.

We now imitate the calculation of the proof of Theorem 6.4.

$$\begin{aligned}\hat{\mathcal{R}}^{-1}\mathcal{L}_{\partial_{z_i}}\Big(\hat{\mathcal{R}}\hat{P}_L dt\Big)=&\Big(\hat{\mathcal{R}}^{-1}\frac{\partial\hat{\mathcal{R}}}{\partial z_i}+\frac{1}{2\kappa}\sum_{j\neq i}\frac{\mu_i\mu_j}{z_i-z_j}\Big)\hat{P}_L dt\\&+\sum_{l'}\langle E_{\mathbf{z}}w,\hat{Y}_{\nu_1}\dots[\ \ ]^{l'}\dots\hat{Y}_{\nu_{|L|}}v'_J\rangle dt\\&+\Big\langle\Big(1-\frac{1}{\kappa}\Big(1+\frac{\mathbb{H}}{2}\Big)\Big)e_iE_{\mathbf{z}}w,\hat{Y}_{\nu_1}\dots\hat{Y}_{\nu_{|L|}}v'_J\Big\rangle dt\end{aligned}$$

where in the l'^{th} place we have

$$\begin{aligned}[\ \ ]^{l'}=&-\hat{\mathcal{R}}^{-1}\frac{\partial(\hat{\mathcal{R}}\hat{T}_{\nu_{l'},i})}{\partial t_{\nu_{l'}}}+\hat{\mathcal{R}}^{-1}\frac{\partial\hat{\mathcal{R}}}{\partial t_{\nu_{l'}}}\hat{T}_{\nu_{l'},i}\\&-\frac{1}{\kappa}\Big(\hat{T}_{\nu_{l'},i}\sum_j\frac{h_j}{t_{\nu_{l'}}-z_j}-\hat{Y}_{\nu_{l'}}\frac{h_i}{t_{\nu_{l'}}-z_i}\Big)\ .\end{aligned}$$

As before we re-arrange, and add and subtract terms to find

(16) $\hat{\mathcal{R}}^{-1}\mathcal{L}_{\partial_{z_i}}(\hat{\mathcal{R}}\hat{\mathcal{P}}_L dt)$

$$\begin{aligned}&=\Big(\hat{\mathcal{R}}^{-1}\frac{\partial\hat{\mathcal{R}}}{\partial z_i}+\frac{1}{2\kappa}\Big(\sum_{j\neq i}\frac{\mu_i\mu_j}{z_i-z_j}+2\sum_{l\in L}\frac{\mu_i}{t_l-z_i}\Big)\Big)\hat{P}_L dt\\&-\hat{\mathcal{R}}^{-1}\frac{\partial}{\partial t_{\nu_{l'}}}(\hat{\mathcal{R}}\sum_{\nu_{l'}}<E_{\mathbf{z}}w,\hat{Y}_{\nu_1}\dots[\hat{T}_{\nu_{l'},i}]^{l'}\dots\hat{Y}_{\nu_{|L|}}v'_J>)dt\\&+\sum_{\nu_{l'}}\langle E_{\mathbf{z}}w,\hat{Y}_{\nu_1}\dots[\ \ ]^{l'}\dots\hat{Y}_{\nu_{|L|}}v'_J\rangle dt\\&+\Big\langle\Big(1-\frac{1}{\kappa}\Big(1+\frac{\mathbb{H}}{2}\Big)\Big)e_iE_{\mathbf{z}}w,\hat{Y}_{\nu_1}\dots\hat{Y}_{\nu_{|L|}}v'_J\Big\rangle dt\end{aligned}$$

where in the l'^{th} place we now have

$$[\ \ ]^{l'}\equiv-\frac{1}{\kappa}(\hat{T}_{\nu_{l'},i}\hat{H}_{\nu_{l'}}-\hat{Y}_{\nu_{l'}}\hat{H}_{\nu_{l'},i})+(\hat{\mathcal{R}}^{-1}\frac{\partial\hat{\mathcal{R}}}{\partial t_{\nu_{l'}}}-\frac{1}{\kappa}\sum_j\frac{\mu_j}{t_{\nu_{l'}}-z_j})\frac{f_i}{t_{\nu_{l'}}-z_i}\ .$$

The last term in (16) can be simplified to yield

(17) $\Big\langle\Big(1-\frac{1}{\kappa}\Big(1+\frac{\mathbb{H}}{2}\Big)\Big)e_iE_{\mathbf{z}}w,\hat{Y}_{\nu_1}\dots\hat{Y}_{\nu_{|L|}}v'_J\Big\rangle dt$

$$\begin{aligned}&=-\frac{1}{\kappa}(\kappa-1-|L|+J)\langle E_{\mathbf{z}}w,e_i\hat{Y}_{\nu_1}\dots\hat{Y}_{\nu_{|L|}}v'_J\rangle dt\\&=-\frac{1}{\kappa}(\kappa-1-|L|+J)\sum_{l'}\frac{\mu_i}{t_{\nu_{l'}}-z_i}\langle E_{\mathbf{z}}w,\hat{Y}_{\nu_1}\dots[1]^{l'}\dots\hat{Y}_{\nu_{|L|}}v'_J\rangle dt\\&-\frac{2}{\kappa}(\kappa-1-|L|+J)\end{aligned}$$

$$\times \sum_{l''>l'} \frac{1}{t_{\nu_{l''}} - t_{\nu_{l'}}} \Big(\frac{1}{t_{\nu_{l''}} - z_i} - \frac{1}{t_{\nu_{l'}} - z_i} \Big)$$
$$\times \langle E_{\mathbf{z}} w, \hat{Y}_{\nu_1} \ldots [1]^{l'} \ldots [1]^{l''} \ldots \hat{Y}_{\nu_{|L|}} f_i v'_J \rangle dt$$
$$= -\frac{1}{\kappa}(\kappa - 1 - |L| + J) \sum_{l'} \frac{\mu_i}{t_{\nu_{l'}} - z_i} \langle E_{\mathbf{z}} w, \hat{Y}_{\nu_1} \ldots [1]^{l'} \ldots \hat{Y}_{\nu_{|L|}} v'_J \rangle dt$$
$$-\frac{2}{\kappa}(\kappa - 1 - |L| + J)$$
$$\times \sum_{l'',l'} \frac{1}{t_{\nu_{l''}} - t_{\nu_{l'}}} \frac{1}{t_{\nu_{l''}} - z_i} \langle E_{\mathbf{z}} w, \hat{Y}_{\nu_1} \ldots [1]^{l'} \ldots [1]^{l''} \ldots \hat{Y}_{\nu_{|L|}} f_i v'_J \rangle dt \ .$$

We now choose

$$\hat{\mathcal{R}} = \prod_{i<j} (z_i - z_j)^{\frac{-\mu_i \mu_j}{2\kappa}} \prod_{l,i} (t_l - z_i)^{\frac{\mu_i}{\kappa}} \prod_{l<l'} (t_l - t_{l'})^{\frac{-2}{\kappa}} \ .$$

This kills most of the terms as before, and we are left with

$$\hat{\mathcal{R}}^{-1} \mathcal{L}_{\partial_{z_i}} (\hat{\mathcal{R}} \hat{\mathcal{P}}_L dt) = \hat{\mathcal{R}}^{-1} \frac{\partial}{\partial t_{\nu_{l'}}} \Big(\hat{\mathcal{R}} \sum_{\nu_{l'}} \langle E_{\mathbf{z}} w, \hat{Y}_1 \ldots [\hat{T}_{\nu_{l'},i}]^{l'} \ldots \hat{Y}_L v_J \rangle \Big) dt \tag{18}$$
$$-\frac{1}{\kappa} \Big(\sum_{\nu_l \notin L} \frac{\mu_i}{t_{\nu_l} - z_i} \Big) \hat{P}_L dt$$
$$+\frac{2}{\kappa} \sum_{\nu_{l'}} \sum_{\nu_l \notin L} \frac{1}{t_{\nu_{l'}} - t_{\nu_l}} \frac{1}{t_{\nu_{l'}} - z_i} \langle E_{\mathbf{z}} w, \hat{Y}_{\nu_1} \ldots [1]^{l'} \ldots \hat{Y}_{\nu_{|L|}} f_i v_J \rangle dt$$
$$+ \text{(RHS of (17))} \ .$$

Consider now the variation of the sum $\widehat{\mathrm{TR}}_{(1,I)}(w) = \sum_L \gamma_{|L|} \hat{P}_L dt$. We have (up to exact terms)

$$\hat{\mathcal{R}}^{-1} \mathcal{L}_{\partial_{z_i}} (\hat{\mathcal{R}} \widehat{\mathrm{TR}}_{(1,I)}(w)) = -\sum_L \gamma_{|L|} \Big(\sum_{\nu_l \notin L} \frac{\mu_i}{t_{\nu_l} - z_i} \Big) \hat{P}_L dt$$
$$+ 2 \sum_L \gamma_{|L|} \sum_{\nu_{l'}} \sum_{\nu_l \notin L} \frac{1}{t_{\nu_{l'}} - t_{\nu_l}} \frac{1}{t_{\nu_{l'} - z_i}} \langle E_{\mathbf{z}} w, \hat{Y}_{\nu_1} \ldots [1]^{l'} \ldots \hat{Y}_{\nu_{|L|}} f_i v_J rangle dt$$
$$- \sum_L \gamma_{|L|} (k + 1 - |L| + J) \sum_{l'} \frac{\mu_i}{t_{\nu_{l'}} - z_i} \langle E_{\mathbf{z}} w, \hat{Y}_{\nu_1} \ldots [1]^{l'} \ldots \hat{Y}_{\nu_{|L|}} v'_J \rangle dt$$
$$- 2 \sum_L \gamma_{|L|} (k + 1 - |L| + J)$$
$$\times \sum_{l'',l'} \frac{1}{t_{\nu_{l''}} - t_{\nu_{l'}}} \frac{1}{t_{\nu_{l''}} - z_i} \langle E_{\mathbf{z}} w, \hat{Y}_{\nu_1} \ldots [1]^{l'} \ldots [1]^{l''} \ldots \hat{Y}_{\nu_{|L|}} f_i v'_J \rangle dt$$
$$= -\sum_L \gamma_{|L|} \Big(\sum_{\nu_l \notin L} \frac{\mu_i}{t_{\nu_l} - z_i} \Big) \hat{P}_L dt$$
$$- \sum_L \gamma_{|M|+1} (k - |M| + J) \sum_{\nu_{l'} \notin M} \frac{\mu_i}{t_{\nu_{l'}} - z_i} \hat{P}_M dt$$

$$
\begin{aligned}
&+ 2\sum_{M} \gamma_{M+1} \sum_{\nu_{l'},\nu_l \notin L} \frac{1}{t_{\nu_{l'}} - t_{\nu_l}} \frac{1}{t_{\nu_{l'}-z_i}} \langle E_{\mathbf{z}} w, \prod_{\nu_m \in M} \hat{Y}_{\nu_m} f_i v_J \rangle dt \\
&- 2\sum_{N} \gamma_{|N|+2}(k-1-|N|+J) \\
&\times \sum_{\nu_{l''},\nu_{l'}} \frac{1}{t_{\nu_{l''}} - t_{\nu_{l'}}} \frac{1}{t_{\nu_{l''}} - z_i} \langle E_{\mathbf{z}} w, \prod_{\nu_m \in n} \hat{Y}_{\nu_N} f_i v_J \rangle dt \\
= 0
\end{aligned}
$$

provided the coefficients $\gamma_{|L|}$ are chosen as in the statement of the theorem. □

8. Polynomials vanishing on partial diagonals

Let a, b be positive integers satisfying $a < b$. Set $\mathbf{B} = \{1, \ldots, b\}$. Consider *homogeneous* polynomials P in b complex variables $\mathbf{t} = (t_1, \ldots, t_b)$. We denote by $C_{(a,b)}$ the condition:

- P vanishes on all the partial diagonals of dimension $b - a$. That is, $P(t_1, \ldots, t_b) = 0$ if there is a subset $\mathbf{i} = (i_1 < \cdots < i_{a+1})$ of $\mathbf{B}$ such that $t_{i_1} = \cdots = t_{i_{a+1}}$. (We shall denote by $\Delta_{\mathbf{i}}$ the above partial diagonal.)

We wish to bound from below the degree of a nonzero P satisfying $C_{(a,b)}$. In fact, for our purpose we will need to consider polynomials invariant under the symmetric group S_b; so we introduce the condition $C^*_{(a,b)}$:

- P is invariant under permutations of its arguments and vanishes on all the partial diagonals of dimension $b - a$. That is, $P(t_{\tau(1)}, \ldots, t_{\tau(b)}) = P(t_1, \ldots, t_b)$ for any permutation $\tau \in S_b$, and $P(t_1, \ldots, t_b) = 0$ if there is a subset $\mathbf{i} = (i_1 < \cdots < i_{a+1})$ of $\mathbf{B}$ such that $t_{i_1} = \cdots = t_{i_{a+1}}$.

We start with an *ansatz* which gives a family of polynomials satisfying $C_{(a,b)}$, as products of linear factors.

Consider sets

$$\mathbb{B} \subset \{(i,j) | 1 \le i < j \le b\}$$

such that every subset $\mathbf{A}$ of $\mathbf{B}$ with $a+1$ elements contains at least two elements i, j such that $(i,j) \in \mathbb{B}$. The theorem of Turán ([A-S, p. 72]) produces such a $\mathbb{B}$ of least possible size: Write $b = qa + r$, with $0 \le r < a$, and divide $\mathbf{B}$ into disjoint sets $\mathbf{B}_1, \ldots \mathbf{B}_l, \ldots, \mathbf{B}_a$ such that $\mathbf{B}_1, \ldots, \mathbf{B}_r$ are of size $q+1$ and $\mathbf{B}_{r+1}, \ldots, \mathbf{B}_a$ are of size q; denote by $\hat{\mathbb{B}}$ the resulting partition and let $\mathbb{B}$ be the set of all pairs of elements which belong to the same "box" $\mathbf{B}_l$. Note that

$$|\mathbb{B}| = rC_2^{q+1} + (a-r)C_2^q \equiv d(a,b) \ .$$

For any partition of $\mathbf{B}$ as above, the polynomial

$$P_{\hat{\mathbb{B}}} = \prod_{l=1,\ldots a} \left(\prod_{i,j \in \mathbf{B}_l, i<j} (t_i - t_j) \right)$$

has degree $d(a,b)$ and satisfies $C_{(a,b)}$. Set $P^* = \sum_{\hat{\mathbb{B}}} P_{\hat{\mathbb{B}}}^2$, where the sum $\hat{\mathbb{B}}$ runs through all partitions of the above type. This (nonzero) polynomial has degree $2d(a,b)$, and satisfies $C^*_{(a,b)}$.

THEOREM 8.1. *If a nonzero polynomial satisfies $C_{(a,b)}$, its degree is at least $d(a,b)$, and if it satisfies $C^*_{(a,b)}$, its degree is at least $2d(a,b)$.*

Proof. Clearly, we can assume that the polynomial is homogeneous. We consider the case without permutation symmetry and then indicate where the argument has to be modified to cover this. The proof is by induction, and so we begin by labelling particular cases of the statement of the theorem.

- $T(a,b)$: The polynomials $P_{\hat{\mathbb{B}}}$ have minimum degree among those that satisfy the condition $C_{(a,b)}$.
- $T^*(a,b)$: The polynomial P^* has minimum degree among those satisfying $C^*_{(a,b)}$.

Step 1. Note that the result for $a = 1$ follows from unique factorisation. This proves $T(1,b)$.

Step 2a. Consider now the case when a is arbitrary, but $b \leq 2a$. Let P be a nonzero polynomial satisfying $C_{(a,b)}$, and for any a-subset $\mathbf{j} = (j_1 < \cdots < j_a)$ of $\mathbf{B}$, consider its restriction $P_{\mathbf{j}}$ to the $(b-a)$-dimensional subspace $H_{\mathbf{j}}$ defined by

$$H_{\mathbf{j}} = \{\mathbf{t} | t_{j_1} = \cdots = t_{j_a} = 0\}.$$

The restriction $P_{\mathbf{j}}$ vanishes along the hyperplanes $t_i = 0,\ i \notin \mathbf{j}$, and is therefore divisible by $\prod_{i \notin \mathbf{j}} t_i$. Thus either

(1) the degree of P is at least $b - a$ (this is the desired bound), or

(2) all the restrictions $P_{\mathbf{j}}$ are zero.

In the second case, a simple induction (Step 2b below) shows that the degree of P is at least $b - a + 1$, so that we have the desired bound in any case. Thus we have proved $T(a,b)$ for $b \leq 2a$.

Step 2b. Let $b > c > 0$. We prove that if a nonzero homogeneous polynomial in b variables vanishes whenever $b - c$ of the variables are zero, the degree is at least $c + 1$. We do induction on b, keeping c fixed. First, this is clear if $b = c + 1$. For the inductive step, consider the restrictions of the polynomial to the b co-ordinate hyperplanes. If all these are zero, the degree is at least b. If not, consider a nonzero restriction. This is a polynomial in $b - 1$ variables, vanishing whenever $b - c - 1$ variables are put equal to zero.

Step 3. Let now $b > 2a$. Consider a polynomial P satisfying $C_{(a,b)}$ and a restriction $P_{\mathbf{j}}$. Write

$$P_{\mathbf{j}} = \left(\prod_{i \notin \mathbf{j}} t_i \right) P'_{\mathbf{j}} \ . \tag{19}$$

Note that $P'_{\mathbf{j}}$ satisfies $C_{(a,b-a)}$. Suppose $T(a, b-a)$ holds. Then either

(1) the degree of P is at least $b - a + d(a, b-a)$ - this is the desired bound $d(a, b)$ - or
(2) all the restrictions $P_{\mathbf{j}}$ are zero.

Step 4. We need to deal with the second possibility. For $\lambda \in \mathbb{C}$, consider the map $D_\lambda : \mathbb{C}^b \to \mathbb{C}^b$ defined by

$$D_\lambda(t_1, \ldots, t_b) = \left(t_1 + \lambda \sum_i t_i, \ldots, t_b + \lambda \sum_i t_i \right) \ .$$

This is invertible as long as $b\lambda \neq -1$. Consider the polynomial $P_\lambda = P \circ D_\lambda$; one sees easily that P_λ satisfies $C(a, b)$ as well. Clearly its degree is the same as that of P. Thus if P were to have a degree smaller than $d(a, b)$, P_λ must also vanish on all the subspaces $H_{\mathbf{j}}$. In terms of P, this translates to $P(t_1, \ldots, t_b) = 0$ whenever

$$t_{j_1} = \cdots = t_{j_a} = -\lambda \sum_i t_i \ .$$

As λ runs over all possible complex numbers, this covers a dense set of the $(a-1)$-dimensional diagonal $\Delta_{\mathbf{j}}$:

$$t_{j_1} = \cdots = t_{j_a}$$

so that P is seen to satisfy $C(a-1, b)$. Suppose now that $T(a-1, b)$ holds. Then $\deg P \geq d(a-1, b) \geq d(a, b)$, where this last inequality follows from Turán's Theorem, though no doubt it can be proved directly from the definition of $d(a, b)$.

Step 5. Summarising the last two steps, we have, if $b > 2a$,

$$T(a-1, b) \text{ and } T(a, b-a) \Rightarrow T(a, b) \ .$$

By induction, we are done.

To incorporate symmetry under S_b and thus prove $T^*(a, b)$ note the following: if a polynomial P is invariant under permutation of its arguments and vanishes along a diagonal $\Delta_{\mathbf{i}}$, then *its derivatives* $\partial P/\partial t_{i_p}, i_p \in \mathbf{i}$ *vanish there as well.* This follows from the equalities

- $\partial P/\partial t_{i_p} = \partial P/\partial t_{i_q}, \ p, q \in \mathbf{i}$ on $\Delta_{\mathbf{i}}$.
- $\sum_{i \in \mathbf{i}} \partial P/\partial t_i = 0$.

As a consequence, in Step 3 above, equation (19) gets replaced by

$$P_{\mathbf{j}} = (\prod_{i \notin \mathbf{j}} t_i^2) P'_{\mathbf{j}} \ . \tag{20}$$

Once this is noted, the proof goes through with obvious changes. □

9. Finiteness

We return to the situation in Section 6.

THEOREM 9.1. *Suppose that the level k satisfies the inequality $k \geq J$. Then the (multi-valued) form $\mathcal{R}\ \mathrm{TR}_{(0,I)}(w)$, pulled back to $\hat{\mathbb{X}} \setminus \hat{D}$ (where it becomes single-valued), extends across the smooth locus of $\hat{D}$.*

Proof. Much of the work for the following proof has already been done in Sections 5 and 8. We retain the notation from there, and in addition we set $\mathbf{J} = \{1, \ldots, J\}$, and for any nonempty $\mathbf{J}' \subset \mathbf{J}$, set $\wedge^2 \mathbf{J}' = \{(\nu, \nu') | \nu < \nu' \ \in \mathbf{J}'\}$. Such a subset determines dense edges $L_{\mathbf{J}'}$ and $L_{(\mathbf{J}',r)}$ $r = 1, \ldots, n+1$, defined as intersections:

(1) $L_{(\mathbf{J}',r)} = \bigcap_{\nu \in \mathbf{J}'} D_\nu^r \cap \bigcap_{(\nu,\nu') \in \wedge^2 \mathbf{J}'} D_{(\nu,\nu')} \quad (r = 1, \ldots, n).$

$\alpha(L_{(\mathbf{J}',r)}) = \mu_r |\mathbf{J}'| - |\mathbf{J}'|(|\mathbf{J}'| - 1), \quad C = |\mathbf{J}'|.$

(2) $L_{(\mathbf{J}',n+1)} = \bigcap_{\nu \in \mathbf{J}'} D_\nu \cap \bigcap_{(\nu,\nu') \in \wedge^2 \mathbf{J}'} D_{(\nu,\nu')}.$

$\alpha(L_{\mathbf{J}',n+1}) = -2|\mathbf{J}'| - |\mathbf{J}'|(|\mathbf{J}'| - 1), \quad C = |\mathbf{J}'|.$

(3) $L_{\mathbf{J}'} = \bigcap_{(\nu,\nu') \in \wedge^2 \mathbf{J}'} D_{(\nu,\nu')}.$

$\alpha(L_{\mathbf{J}'}) = -|\mathbf{J}'|(|\mathbf{J}'| - 1), \quad C = |\mathbf{J}'| - 1.$

(In each case we have also written the corresponding α, as well as the codimension C.) It is easy to check (and in fact a very special case of [S-T-V, Prop. 12]) that these are all the dense edges.

Consider the form $\mathcal{R}\ \mathrm{TR}_{(0,I)}(w)$ on $\mathbb{X}$. Locally, this has the shape $\mathcal{R} f \tau$, where τ is a regular nonvanishing top-form and f a function with at most simple poles along the divisors D_ν^i. We introduce – for later use – a nonnegative integer l associated to each dense edge. In the last two cases l is the order of vanishing of f along the edge; in particular, $l > 0$. In the first case $l + |\mathbf{J}'|$ is the order of vanishing (along $L_{\mathbf{J}'}$) of the function $f' = (\prod_{\nu = \in \mathbf{J}'} (t_\nu - z_r)) f$; we have $l + |\mathbf{J}'| \geq 0$.

We now consider in turn each of the three situations described in Section 5.

(i) This is the simplest case, for $\alpha(L_1) > 0$. From the list above, we see that L_1 must be of the form $L_{(\mathbf{J}',r))}$. Pulled back to $\mathbb{X}'$, $\mathcal{R}\ \mathrm{TR}_{(0,I)}(w)$ becomes

$$x_1^{\frac{\alpha(L_1)}{\kappa}+C-1+l}\tau'$$

where τ' is a regular top-form. On further pulling back to $\hat{\mathbb{X}}$, we get an expression

$$t^{\alpha'+\kappa'(C-1+l)+\kappa'-1}\hat{\tau}$$

with $\hat{\tau}$ a regular form. For this expression to be a regular form, we need

$$\alpha' + \kappa'(C+l) \geq 1$$

or equivalently (multiplying by δ)

$$\alpha(L_1) + \kappa(C+l) \geq \delta\ .$$

Since $\delta = \gcd(\kappa, \alpha(L_1))$, it clearly suffices to ensure (since $C = |\mathbf{J}'|$)

$$\alpha(L_1) + \kappa(|\mathbf{J}'| + l) > 0\ .$$

This is clearly the case if $\alpha(L_1) > 0$. If $\alpha(L_1) = 0$, $|\mathbf{J}'| - 1 = \mu_r$, and Lemma 9.2 below yields $|\mathbf{J}'| + l \geq |\mathbf{J}'| - \mu_r = 1$. For later reference, we write out the L.H.S., and find we have proved

$$|\mathbf{J}'|\mu_r - |\mathbf{J}'|(|\mathbf{J}'|-1) + \kappa(|\mathbf{J}'|+l) = |\mathbf{J}'|\left\{\mu_r - |\mathbf{J}'| + 1 + \kappa\left(1 + \frac{l}{|\mathbf{J}'|}\right)\right\} > 0\ .$$

(ii) When the quasi-classical weight is negative, the situation is subtler. We argue as before, and find that $\mathcal{R}\ \mathrm{TR}_{(0,I)}(w)$ pulls back to $\hat{\mathbb{X}}$ to yield a form of the shape

$$t^{\alpha'+\kappa'(C-1+l)+\kappa'-1}\hat{\tau}$$

except, of course, that now $\alpha' < 0$. As before, we need to ensure that

$$\alpha(L_2) + \kappa(C+l) > 0\ . \tag{21}$$

Substituting the expressions for α and the codimension C, the L.H.S. of the above inequality becomes (respectively)

(1) $\mu_r|\mathbf{J}'| - |\mathbf{J}'|(|\mathbf{J}'|-1) + \kappa(|\mathbf{J}'|+l) = |\mathbf{J}'|\{\mu_r - |\mathbf{J}'| + 1 + \kappa(1+\frac{l}{|\mathbf{J}'|})\}$.

(2) $-2|\mathbf{J}'| - |\mathbf{J}'|(|\mathbf{J}'|-1) + \kappa(|\mathbf{J}'|+l) = |\mathbf{J}'|\{-|\mathbf{J}'| - 1 + \kappa(1+\frac{l}{|\mathbf{J}'|})\}$.

(3) $-|\mathbf{J}'|(|\mathbf{J}'|-1) + \kappa(|\mathbf{J}'|-1+l) = (|\mathbf{J}'|-1)\{-|\mathbf{J}'| + \kappa + \frac{\kappa l}{|\mathbf{J}'|-1})\}$.

If the condition $k \geq J$ is met, we have in the respective cases

(1) $\mu_r - |\mathbf{J}'| + 1 + \kappa(1+\frac{l}{|\mathbf{J}'|}) > 0$ by Lemma 9.2 below.

(2) $\kappa - 1 - |\mathbf{J}'| > 0$, $l \geq 0$.

(3) $\kappa - |\mathbf{J}'| > 0$, $|\mathbf{J}'| - 1 > 0$, $l \geq 0$.

This proves (21).

(iii) The form $\mathcal{R}\ \mathrm{TR}_{(0,I)}(w)$ pulls back to $\hat{\mathbb{X}}$ as a form of the shape

$$t^{\alpha_1'(C_2+l_2-1)-\alpha_2'(C_1+l_1-1)+\alpha_1'-\alpha_2'-1}\hat{\tau}\ .$$

We have used subscripts to distinguish the codimensions of the two edges, as well as the corresponding values of l. Regularity requires

$$\alpha_1(l_2+|\mathbf{J}_2|)-\alpha_2(l_1+|\mathbf{J}_1|)>0\ .$$

Since $\alpha_2 < 0$, $l_2 + |\mathbf{J}_2| > 1$, the L.H.S. above is greater than or equal to $\alpha_1 > 0$. □

Lemma 9.2. *Consider an edge of type $L_{(\mathbf{J}',r)}$. Now, $l \geq \sup(-\mu_r, -|\mathbf{J}'|)$.*

Proof. Consider the function $P = \langle w, \prod_{\nu=0}^{J}(\sum_i \frac{e_i}{t_\nu - z_i})v_J\rangle$ around the edge. We write

$$P = \left(\prod_{\nu=\in\mathbf{J}'}\frac{1}{t_\nu - z_r}\right)P'$$

where

$$P' = \left\langle w, \prod_{\nu\notin\mathbf{J}'}\left(\sum_i \frac{e_i}{t_\nu - z_i}\right)\prod_{\nu\in\mathbf{J}'}\left(e_r + \sum_{i\neq r}\frac{(t_\nu - z_r)e_i}{t_\nu - z_i}\right)v_J\right\rangle\ .$$

We need to show that P' vanishes on L to order $|\mathbf{J}'| - \mu_i$, once $|\mathbf{J}'| > \mu_i$. For $\nu \in \mathbf{J}'$, set $y_\nu = t_\nu - z_r$, and rewrite the above expression:

$$P' = \left\langle w, \prod_{\nu\notin\mathbf{J}'}\left(\sum_i \frac{e_i}{t_\nu - z_i}\right)\prod_{\nu\in\mathbf{J}'}\left(e_r + \sum_{i\neq r}\frac{y_\nu e_i}{z_r - z_i + y_\nu}\right)v_J\right\rangle\ .$$

Note that $e_r^m v_J = 0$ if $m > \mu_r$. The lemma now follows. □

Theorem 9.3. *Suppose that $k < J$. Then the form $\mathcal{R}\ \mathrm{TR}_{(0,I)}(w)$, pulled back to $\hat{\mathbb{X}} \setminus \hat{D}$, extends across the smooth locus of $\hat{D}$, provided the condition* (K) *holds.*

An examination of the previous proof reveals that it is the part (ii) which calls for more work. The key fact is the following consequence of the condition (K): the functions $P(t_1, \dots t_J) = \langle w, \prod_{\nu=0}^{J}(\sum_i \frac{e_i}{t_\nu - z_i})v_J\rangle$ vanish along the partial diagonals of codimension k or more (Proposition 9.5 below).

It is useful to introduce some notation at this point. Set, for $\mathbf{u} = (u_1, u_2, \dots, u_n)$ a sequence of points in $\mathbb{C}$:

- $E(t, \mathbf{u}) = \sum_i e_i(t - u_i)^{-1}$,
- $\hat{E}(t, \mathbf{u}) = \sum_i e_i(t - u_i)$, and
- $\hat{F}(t, \mathbf{u}) = \sum_i f_i(t - u_i)$.

LEMMA 9.4. *The condition* (K) *is equivalent to either one of the following equalities* (*of functions of the variables* $\mathbf{u}$).

(1) $\langle w, \hat{E}(t', \mathbf{u})^{J-q} \hat{E}(t, \mathbf{z})^q v_J \rangle = 0,\ k < q$.
(2) $\langle w, \hat{F}(t', \mathbf{u})^{J-q} \hat{F}(t, \mathbf{z})^q v'_J \rangle = 0,\ k < q$.

Proof. The condition

$$\langle E_{\mathbf{z}} w, e^\alpha v_J \rangle = 0 \tag{22}$$

(for fixed α) is equivalent to $\langle w, e^\alpha (\sum_i z_i e_i)^{J-|\alpha|} v_J \rangle = 0$ (consider the weights). Demanding that (22) be satisfied for all $|\alpha| < J - k$ is equivalent to demanding that $\sum_{|\alpha|=p} \langle w, \mathbf{u}^\alpha e^\alpha (\sum_i z_i e_i)^{J-p} v_J \rangle = 0,\ p < J - k$. This in turn, is true if and only if

$$\left\langle w, \left(\sum_i u_i e_i \right)^p \left(\sum_i z_i e_i \right)^{J-p} v_J \right\rangle = 0,\ p < J - k\ . \tag{23}$$

This is easily seen to translate to the first condition of the lemma, because w is an invariant vector.

Set $\mathbb{F} = \sum_j f_j$. It is easy to check that $v_J = \mathbb{F}^{2J} v'_J$, up to a nonzero constant. So (1) implies

$$\langle w, \hat{E}(t', \mathbf{u})^{J-q} \hat{E}(t, \mathbf{z})^q \mathbb{F}^{2J} v'_J \rangle = 0,\ k < q\ .$$

On the other hand, one computes (for B a positive integer)

$$\hat{E}(t, \mathbf{u}) \mathbb{F}^B v'_J = -B(B-1) \mathbb{F}^{B-2} \hat{F}(t, \mathbf{u}) v'_J + B \mathbb{F}^{B-1} \left(\sum_i \mu_i (t - u_i) \right) v'_J\ .$$

If we therefore commute all the operators $\hat{E}_1$ across until they kill v'_J, the pairing with the invariant vector w results in the vanishing of all terms which contain a factor $\mathbb{F}$. The second relation (2) of the lemma is the result. □

PROPOSITION 9.5. *Suppose $k < J$, and that condition* (K) *holds. Then* $\langle w, \prod_{\nu \geq l} E(t_\nu, \mathbf{z}) E(t, \mathbf{z})^l v_J \rangle = 0$ *for all integers* $J \geq l > k$.

Proof. It suffices to prove $\langle w, e^\beta E(t, \mathbf{z})^{J-|\beta|} v_J \rangle = 0$ for all multi-indices β with $J - |\beta| > k$. One checks that $v_J = \hat{F}^{2J}(t, \mathbf{z}) v'_J$, up to a nonzero factor. So the above condition translates (using the relation $[E(t, \mathbf{z}), \hat{F}(t, \mathbf{z})] = \mathbb{H} = \sum_i h_i$) to $\langle w, e^\beta \hat{F}(t, \mathbf{z})^{J+|\beta|} v'_J \rangle = 0$. Commuting e^β across, one obtains a sum of terms proportional to $\langle w, \hat{F}(t, \mathbf{z})^{J-|\beta'|} f^{\beta'} v'_J \rangle$ where $\beta' \leq \beta$ (the inequality of multi-indices has an obvious meaning). This in turn follows easily from part (2) of Lemma 9.4. □

We can now give the proof of Theorem 9.3:

Proof. We have to deal with the cases in part (ii) of the proof of Theorem 9.1 remembering that now $k < |\mathbf{J}|$. Combining Proposition 9.5 and Theorem 8.1, we obtain the following lower bound on the degrees l: write $|\mathbf{J}'| = qk + \hat{r},\ 0 \leq \hat{r} < k$; then

$$l \geq 2\{\hat{r}(q+1)q + (k-\hat{r})q(q-1)\} \ .$$

The required inequalities now follow. □

We have proved the conjecture of Gawedzki (§4.3 of [G])). In our notation:

Corollary 9.6. *The integral*

$$\int_{\mathbb{X}\setminus D} |\mathcal{R}|^2 \mathrm{TR}_{(0,I)}(w)\overline{\mathrm{TR}_{(0,I)}(w)}$$

is finite if w satisfies the condition (K) (*or equivalently*, (G)).

Proof. It suffices to prove that the integral, pulled back to the μ_κ-cover $\hat{\mathbb{X}} \setminus \hat{D}$, is finite. We have proved that the form $\mathcal{R}\ \mathrm{TR}_{(0,I)}(w)$ extends across the smooth part of $\hat{D}$, and thus to the smooth locus of $\hat{\mathbb{X}}$. On the other hand, $\hat{\mathbb{X}}$ has rational singularities, which implies that pulled back to any desingularisation $\hat{\mathbb{X}}_s$, $\mathcal{R}\ \mathrm{TR}_{(0,I)}(w)$ extends to a globally regular form. □

The *necessity* of the condition (K) for the convergence to hold has already been noted by Gawedzki.

10. Unitarity

Theorem 10.1. *The* (*parabolic*) *Hitchin connection in genus zero*, *or equivalently the* KZ *connection on the bundle of conformal blocks* (*the sub-bundle $\mathcal{W}$ defined in* §4) *is unitary in the rank two case.*

Proof. Consider, over $\mathcal{Z}_n$, a μ_κ-equivariant resolution of singularities $\pi_s : \hat{\mathbb{X}}_s \to \hat{\mathbb{X}}$. This is an isomorphism outside $\hat{D}$, and so we have the diagram:

$$\begin{array}{ccccccc}
\mathbb{X}\setminus D & \xleftarrow{\pi} & \hat{\mathbb{X}}\setminus\hat{D} & \xrightarrow[\hookrightarrow]{i} & \hat{\mathbb{X}}_s & \xrightarrow{\pi_s} & \hat{\mathbb{X}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{Z}_n & = & \mathcal{Z}_n & = & \mathcal{Z}_n & = & \mathcal{Z}_n \ .
\end{array}$$

The map π above is a μ_κ-fibration. Consider the corresponding bundles of degree J cohomology (i.e., the J^{th} direct images of the constant sheaf $\mathbb{C}$):

$$\begin{tikzcd}
\mathcal{H}^J_{\mathcal{R}} \arrow[r, hook, "\pi^*_{\mathcal{R}}"] \arrow[d] & \mathcal{H}^J(\hat{\mathbb{X}} \setminus \hat{D}) \arrow[d] & \mathcal{H}^J(\hat{\mathbb{X}}_s) \arrow[l, "i^*"'] \arrow[d] \\
\mathcal{Z}_n \arrow[r, equal] & \mathcal{Z}_n \arrow[r, equal] & \mathcal{Z}_n .
\end{tikzcd}$$

The map i^* is covariant between flat μ_κ-bundles, and $\pi^*_{\mathcal{R}}$ (defined in Remark 5.5) an isomorphism onto an isotypical component (corresponding to the identity character of μ_κ) of $\mathcal{H}^J(\hat{\mathbb{X}} \setminus \hat{D})$. Let us denote by χ_0 this character, and by $\mathcal{H}^J_{\chi_0}(\mathbb{X} \setminus \hat{D})$, $\mathcal{H}^J_{\chi_0}(\hat{\mathbb{X}}_s)$ the corresponding isotypical components. Our work so far has yielded the following:

(1) The map of bundles $\mathcal{V} \to \mathcal{H}^J_{\mathcal{R}}$ factors as in the following diagram:

$$\mathcal{V} \hookrightarrow \Omega^J_{\chi_0} \hookrightarrow \mathcal{H}^J_{\chi_0}(\hat{\mathbb{X}}_s) \underset{i^*}{\to} \mathcal{H}^J_{\chi_0}(\mathbb{X} \setminus \hat{D}) = \mathcal{H}^J_{\mathcal{R}}$$

where Ω^J is the direct image of a relative canonical bundle of $\hat{\mathbb{X}}_s \to \mathcal{Z}_n$. The injectivity of the first arrow follows from the injectivity of the trace map; see Proposition 1.1.

(2) The map $\mathcal{V} \to \mathcal{H}^J_{\mathcal{R}}$ is flat.

The bundle $\mathcal{H}^J(\hat{\mathbb{X}}_s)$ carries a skew-hermitian pairing invariant under the flat connection:

$$(\alpha, \beta) \mapsto i^J \int \alpha \wedge \bar{\beta}$$

which induces a definite scalar product on the sub-bundle Ω^J. In general the sub-bundle Ω^J is not invariant under the connection, the second fundamental form being given by cupping with the Kodaira-Spencer map. By the results of Deligne [D] the composite map

$$\Omega^J(\hookrightarrow \mathcal{H}^J(\hat{\mathbb{X}}_s)) \underset{i^*}{\to} \mathcal{H}^J(\hat{\mathbb{X}} \setminus \hat{D}) \tag{24}$$

is an injection, which yields that the map of isotypical components

$$\Omega^J_{\chi_0}(\hookrightarrow \mathcal{H}^J_{\chi_0}(\hat{\mathbb{X}}_s)) \underset{i^*}{\to} \mathcal{H}^J_{\chi_0}(\mathbb{X} \setminus \hat{D}) = \mathcal{H}^J_{\mathcal{R}}$$

is an injection as well. This proves that the image of $\mathcal{V} \to \Omega^J_{\chi_0}$ is a flat sub-bundle of the latter. Thus the induced scalar product on $\mathcal{V}$ is invariant. $\square$

Note that in the course of the above proof we have obtained

COROLLARY 10.2. *The map $\mathcal{V} \to \mathcal{H}^J_{\mathcal{R}}$ is injective; therefore the map of dual local systems is surjective.*

This is in fact a result of A. Varchenko (Theorem 14.6.4 of [V]). In an earlier version of this paper, I had used this result to reach the conclusion of Theorem 10.1. M. Nori and N. Fakhruddin pointed out the injectivity of (24), which therefore yields an independent proof of the fact that the Coulomb-gas construction yields a complete set of solutions to the KZ equation.

It remains to add that everything seems to go through as above for the other trace that we have studied. I have no idea at present why this should be so, and close this paper with the question: Are the two metrics induced on the bundle of conformal blocks the same (modulo of course, scalars)?

Acknowledgments. It is a pleasure to thank A. Bruguières, P. Deligne, N. Fakhruddin, N. J. Hitchin, M. S. Narasimhan, M. Perling, J. Radhakrishnan, N. Ressayre, K. Smith, and V. Srinivas for discussions, and A. Varchenko for a correspondence. Conversations with M. Nori regarding mixed Hodge structures were crucial. Also, though I had been considering branched covers for a while, it was while talking to him that I saw clearly how this would work. I gratefully acknowledge his help.

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References

[A-S] N. Alon and J. Spencer, *The Probabilistic Method*, Second edition, *Wiley-Interscience Series in Discrete Mathematics and Optimization*, Wiley-Interscience, New York, 2000.

[C] D. A. Cox, Mini-course on toric varieties, University of Buenos Aires, July 2001, at `http://www.amherst.edu/~dacox/`.

[D] P. Deligne, Théorie de Hodge. II, *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5–57.

[E-F-K] P. I. Etingof, I. B. Frenkel, and A. A. Kirillov, Jr., *Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations*, *Mathematical Surveys and Monographs* **58**, Amer. Math. Soc., Providence, RI, 1998.

[EGA-IV] A. Grothendieck, in collaboration with J. Dieudonné, Éléments de Géométrie Algébrique, Chapitre IV (Seconde Partie), *Publ. Math.* **24**, 1965.

[E-S-V] H. Esnault, V. Schechtman, and E. Viehweg, Cohomology of local systems on the complement of hyperplanes, *Invent. Math.* **109** (1992), 557–561.

[F-G] F. Falceto and K. Gawedzki, Unitarity of the Knizhnik-Zamolodchikov-Bernard connection and the Bethe Ansatz for the elliptic Hitchin systems, *Comm. Math. Phys.* **183** (1997), 267–290.

[F-V] G. Felder and A. Varchenko, Integral representation of solutions of the elliptic Knizhnik-Zamolodchikov-Bernard equations, *Internat. Math. Res. Notices* **1995**, no. 5, 221–233.

[G] K. Gawedzki, Lectures on conformal field theory, in *Quantum Fields and Strings: a Course for Mathematicians*, Vol. 1, 2 (Princeton, NJ, 1996/1997), 727–805, Amer. Math. Soc., Providence, RI, 1999.

[G1] ———, Quadrature of conformal field theories, *Nuclear Physics* **328** (1989), 733–752.

[G2] ———, Constructive conformal field theory, in (Z. Haba and J. Sobczyk, eds.), *Functional Integration*, *Geometry and Strings*, 277–302, Birkhäuser, Boston, 1989.

[G-K] K. Gawedzki and A. Kupiainen, SU(2) Chern-Simons theory at genus zero, *Comm. Math. Phys.* **135** (1991), 531–546.

[K] A. Kirillov, On inner products in modular tensor categories II. Inner product on conformal blocks and affine inner product identities, *Adv. Theor. Math. Phys.* **2** (1998), 155–180.

[Le-P] J. Le Potier, *Lectures on Vector Bundles*, Translated by A. Maciocia, *Cambridge Studies in Advanced Mathematics* **54**, Cambridge Univ. Press, Cambridge, 1997.

[N-R] M. S. Narasimhan and T. R. Ramadas, Factorisation of generalised theta functions I, *Invent. Math.* **114** (1993), 565–623.

[O-T] P. Orlik and H. Terao, *Arrangements and Hypergeometric Arrangements*, MSJ *Memoirs* **9**, Mathematical Society of Japan, Tokyo, 2001.

[R] T. R. Ramadas, Factorisation of generalised theta functions II: the Verlinde formula, *Topology* **35** (1996), 641–654.

[S-S] P. Scheinost and M. Schottenloher, Metaplectic quantization of the moduli spaces of flat and parabolic bundles, *J. Reine Angew. Math.* **466** (1995), 145–219.

[S-T-V] V. Schechtman, H. Terao, and A. Varchenko, Local systems over complements of hyperplanes and the Kac-Kazhdan condition for singular vectors, *J. Pure Appl. Algebra* **100** (1995), 93–102.

[V] A. Varchenko, *Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups*, *Advanced Series in Mathematical Physics* **21**, World Sci. Publ., Inc., River Edge, NJ, 1995.

[Y] T. Yoshida, An abelianization of SU(2) WZW model, *Ann. of Math.* **164** (2006), 1–49.

(Received February 17, 2003)
(Revised February 22, 2005)