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On the classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres

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Abstract

In this paper we give a new proof for the classification result in [3]. We show that isoparametric hypersurfaces with four distinct principal curvatures in spheres are of Clifford type provided that the multiplicities m_1, m_2 of the principal curvatures satisfy $m_2 \geq 2m_1 - 1$. This inequality is satisfied for all but five possible pairs (m_1, m_2) with $m_1 \leq m_2$. Our proof implies that for $(m_1, m_2) \neq (1,1)$ the Clifford system may be chosen in such a way that the associated quadratic forms vanish on the higher-dimensional of the two focal manifolds. For the remaining five possible pairs (m_1, m_2) with $m_1 \leq m_2$ (see [13], [1], and [15]) this stronger form of our result is incorrect: for the three pairs $(3,4)$, $(6,9)$, and $(7,8)$ there are examples of Clifford type such that the associated quadratic forms necessarily vanish on the lower-dimensional of the two focal manifolds, and for the two pairs $(2,2)$ and $(4,5)$ there exist homogeneous examples that are not of Clifford type; cf. [5, 4.3, 4.4].

1. Introduction

In this paper we present a new proof for the following classification result in [3].

Theorem 1.1. *An isoparametric hypersurface with four distinct principal curvatures in a sphere is of Clifford type provided that the multiplicities* m_1, m_2 *of the principal curvatures satisfy the inequality* $m_2 \geq 2m_1 - 1$.

An *isoparametric hypersurface* M in a sphere is a (compact, connected) smooth hypersurface in the unit sphere of the Euclidean vector space $V = \mathbb{R}^{\dim V}$ such that the principal curvatures are the same at every point. By [12, Satz 1], the distinct principal curvatures have at most two different multiplicities m_1, m_2. In the following we assume that M has four distinct principal curvatures. Then the only possible pairs (m_1, m_2) with $m_1 = m_2$ are $(1,1)$ and $(2,2)$; see [13], [1]. For the possible pairs (m_1, m_2) with $m_1 < m_2$ we have $(m_1, m_2) = (4,5)$ or $2^{\phi(m_1-1)}$ divides $m_1 + m_2 + 1$, where $\phi : \mathbb{N} \to \mathbb{N}$

is given by

$$\phi(m) = \left|\{i \mid 1 \le i \le m \text{ and } i \equiv 0, 1, 2, 4 \pmod 8\}\right|;$$

see [15]. These results imply that the inequality $m_2 \ge 2m_1 - 1$ in Theorem 1.1 is satisfied for all possible pairs (m_1, m_2) with $m_1 \le m_2$ except for the five pairs $(2, 2)$, $(3, 4)$, $(4, 5)$, $(6, 9)$, and $(7, 8)$.

In [5], Ferus, Karcher, and Münzner introduced (and classified) a class of isoparametric hypersurfaces with four distinct principal curvatures in spheres defined by means of real representations of Clifford algebras or, equivalently, Clifford systems. A *Clifford system* consists of $m + 1$ symmetric matrices $P_0, \ldots, P_m$ with $m \ge 1$ such that $P_i^2 = E$ and $P_iP_j + P_jP_i = 0$ for $i, j = 0, \ldots, m$ with $i \ne j$, where E denotes the identity matrix. Isoparametric hypersurfaces of *Clifford type* in the unit sphere $\mathbb{S}^{2l-1}$ of the Euclidean vector space $\mathbb{R}^{2l}$ have the property that there exists a Clifford system $P_0, \ldots, P_m$ of symmetric $(2l \times 2l)$-matrices with $l - m - 1 > 0$ such that one of their two focal manifolds is given as

$$\{x \in \mathbb{S}^{2l-1} \mid \langle P_i x, x\rangle = 0 \text{ for } i = 0, \ldots, m\},$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the standard scalar product; see [5, Section 4, Satz (ii)]. Families of isoparametric hypersurfaces in spheres are completely determined by one of their focal manifolds; see [12, Section 6], or [11, Proposition 3.2]. Hence the above description of one of the focal manifolds by means of a Clifford system characterizes precisely the isoparametric hypersurfaces of Clifford type. For notions like focal manifolds or families of isoparametric hypersurfaces, see Section 2.

The proof of Theorem 1.1 in Sections 3 and 4 shows that for an isoparametric hypersurface (with four distinct principal curvatures in a sphere) with $m_2 \ge 2m_1 - 1$ and $(m_1, m_2) \ne (1, 1)$ the Clifford system may be chosen in such a way that the *higher-dimensional* of the two focal manifolds is described as above by the quadratic forms associated with the Clifford system. This statement is in general incorrect for the isoparametric hypersurfaces of Clifford type with $(m_1, m_2) = (3, 4)$, $(6, 9)$, or $(7, 8)$; see the remarks at the end of Section 4. Moreover, for the two pairs $(2, 2)$ and $(4, 5)$ there are homogeneous examples that are not of Clifford type. Hence the inequality $m_2 \ge 2m_1 - 1$ is also a necessary condition for this stronger version of Theorem 1.1.

Our proof of Theorem 1.1 makes use of the theory of isoparametric triple systems developed by Dorfmeister and Neher in [4] and later papers. We need, however, only the most elementary parts of this theory. Since our notion of isoparametric triple systems is slightly different from that in [4], we will present a short introduction to this theory in the next section. Based on the triple system structure derived from the isoparametric hypersurface M in the unit sphere of the Euclidean vector space $V = \mathbb{R}^{2l}$, we will introduce in

Section 3 a linear operator defined on the vector space $\mathcal{S}_{2l}(\mathbb{R})$ of real, symmetric $(2l \times 2l)$-matrices. By means of this linear operator we will show that for $m_2 \geq 2m_1 - 1$ with $(m_1, m_2) \neq (1, 1)$ the higher-dimensional of the two focal manifolds may be described by means of quadratic forms as in the Clifford case. These quadratic forms are actually accumulation points of sequences obtained by repeated application of this operator as in a dynamical system. In the last section we will prove that these quadratic forms are in fact derived from a Clifford system. For $(m_1, m_2) = (1, 1)$, even both focal manifolds can be described by means of quadratic forms, but only one of them arises from a Clifford system; see the remarks at the end of this paper.

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2. Isoparametric triple systems

The general reference for the subsequent results on isoparametric hypersurfaces in spheres is Münzner's paper [12], in particular Section 6. For further information on this topic, see [2], [5], [13], [17], or [6], [7]. The theory of isoparametric triple systems was introduced in Dorfmeister's and Neher's paper [4]. They wrote a whole series of papers on this subject. For the relation between this theory and geometric properties of isoparametric hypersurfaces, we refer the reader to [7], [8], [9], and [10]. In this section we only present the parts of the theory of isoparametric triple systems that are relevant for this paper.

Let M denote an isoparametric hypersurface with four distinct principal curvatures in the unit sphere $\mathbb{S}^{2l-1}$ of the Euclidean vector space $V = \mathbb{R}^{2l}$. Then the hypersurfaces parallel to M (in $\mathbb{S}^{2l-1}$) are also isoparametric, and $\mathbb{S}^{2l-1}$ is foliated by this *family of isoparametric hypersurfaces* and the two *focal manifolds* M_+ and M_-. Choose $p \in M_+$ and let $p' \in \mathbb{S}^{2l-1}$ be a vector normal to the tangent space $\mathrm{T}_p M_+$ in $\mathrm{T}_p \mathbb{S}^{2l-1}$ (where tangent spaces are considered as subspaces of $\mathbb{R}^{2l}$). Then the great circle S through p and p' intersects the hypersurfaces parallel to M and the two focal manifolds orthogonally at each intersection point. The points of $S \cap M_+$ are precisely the four points $\pm p, \pm p'$, and $S \cap M_-$ consists of the four points $\pm(1/\sqrt{2})(p \pm p')$. For $q \in M_-$ instead of $p \in M_+$, an analogous statement holds. Such a great circle S

will be called a *normal circle* throughout this paper. For every point $x \in \mathbb{S}^{2l-1}\backslash(M_+ \cup M_-)$ there exists precisely one normal circle through x; see [12, in particular Section 6], for these results.

By [12, Satz 2], there is a homogeneous polynomial function F of degree 4 such that $M = F^{-1}(c) \cap \mathbb{S}^{2l-1}$ for some $c \in (-1,1)$. This *Cartan-Münzner polynomial* F satisfies the two partial differential equations

$$\langle \operatorname{grad} F(x), \operatorname{grad} F(x)\rangle = 16\langle x,x\rangle^3,$$
$$\Delta F(x) = 8(m_2 - m_1)\langle x,x\rangle.$$

By interchanging the multiplicites m_1 and m_2 we see that the polynomial $-F$ is also a Cartan-Münzner polynomial. The polynomial F takes its maximum 1 (minimum -1) on $\mathbb{S}^{2l-1}$ on the two focal manifolds. For a fixed Cartan-Münzner polynomial F, let M_+ always denote the focal manifold on which F takes its maximum 1. Then we have $M_+ = F^{-1}(1) \cap \mathbb{S}^{2l-1}$ and $M_- = F^{-1}(-1) \cap \mathbb{S}^{2l-1}$, where $\dim M_+ = m_1 + 2m_2$ and $\dim M_- = 2m_1 + m_2$; see [12, proof of Satz 4].

Since F is a homogeneous polynomial of degree 4, there exists a symmetric, trilinear map $\{\cdot,\cdot,\cdot\} : V \times V \times V \to V$, satisfying $\langle\{x,y,z\},w\rangle = \langle x,\{y,z,w\}\rangle$ for all $x,y,z,w \in V$, such that $F(x) = (1/3)\langle\{x,x,x\},x\rangle$. We call $(V,\langle\cdot,\cdot\rangle,\{\cdot,\cdot,\cdot\})$ an *isoparametric triple system*. In [4, p. 191], isoparametric triple systems were defined by $F(x) = 3\langle x,x\rangle^2 - (2/3)\langle\{x,x,x\},x\rangle$. This is the only difference between the definition of triple systems in [4] and our definition. Hence the proofs of the following results are completely analogous to the proofs in [4]. The description of the focal manifolds by means of the polynomial F implies that

$$M_+ = \{p \in \mathbb{S}^{2l-1} \mid \{p,p,p\} = 3p\} \text{ and } M_- = \{q \in \mathbb{S}^{2l-1} \mid \{q,q,q\} = -3q\};$$

cf. [4, Lemma 2.1]. For $x,y \in V$ we define self-adjoint linear maps $T(x,y) : V \to V : z \mapsto \{x,y,z\}$ and $T(x) = T(x,x)$. Let μ be an eigenvalue of $T(x)$. Then the eigenspace $V_\mu(x)$ is called a *Peirce space*. For $p \in M_+$, $q \in M_-$ we have orthogonal *Peirce decompositions*

$$V = \operatorname{span}\{p\} \oplus V_{-3}(p) \oplus V_1(p) = \operatorname{span}\{q\} \oplus V_3(q) \oplus V_{-1}(q)$$

with $\dim V_{-3}(p) = m_1 + 1$, $\dim V_1(p) = m_1 + 2m_2$, $\dim V_3(q) = m_2 + 1$, and $\dim V_{-1}(q) = 2m_1 + m_2$; cf. [4, Theorem 2.2]. These Peirce spaces have a geometric meaning that we are now going to explain. By differentiating the map $V \to V : x \mapsto \{x,x,x\} - 3x$, which vanishes identically on M_+, we see that $\mathrm{T}_p M_+ = V_1(p)$ and, dually, $\mathrm{T}_q M_- = V_{-1}(q)$. Thus $V_{-3}(p)$ is the normal space of $\mathrm{T}_p M_+$ in $\mathrm{T}_p \mathbb{S}^{2l-1}$; cf. [7, Corollary 3.3]. Hence for every point $p' \in \mathbb{S}^{2l-1} \cap V_{-3}(p)$ there exists a normal circle through p and p'. In particular, we have $\mathbb{S}^{2l-1} \cap V_{-3}(p) \subseteq M_+$ and, dually, $\mathbb{S}^{2l-1} \cap V_3(q) \subseteq M_-$; cf. [4, Equations 2.6 and 2.13], or [8, Section 2].

By [8, Theorem 2.1], we have the following structure theorem for isoparametric triple systems; cf. the main result of [4].

THEOREM 2.1. *Let S be a normal circle that intersects M_+ at the four points $\pm p, \pm p'$ and M_- at the four points $\pm q, \pm q'$. Then V decomposes as an orthogonal sum*

$$V = \operatorname{span}(S) \oplus V'_{-3}(p) \oplus V'_{-3}(p') \oplus V'_3(q) \oplus V'_3(q'),$$

where the subspaces $V'_{-3}(p)$, $V'_{-3}(p')$, $V'_3(q)$, $V'_3(q')$ are defined by $V_{-3}(p) = V'_{-3}(p) \oplus \operatorname{span}\{p'\}$, $V_{-3}(p') = V'_{-3}(p') \oplus \operatorname{span}\{p\}$, $V_3(q) = V'_3(q) \oplus \operatorname{span}\{q'\}$, and $V_3(q') = V'_3(q') \oplus \operatorname{span}\{q\}$.

Let p, q, p', and q' in the theorem above be chosen in such a way that $p = (1/\sqrt{2})(q - q')$ and $p' = (1/\sqrt{2})(q + q')$. The linear map $T(p,p') = (1/2)T(q-q', q+q') = (1/2)\big(T(q) - T(q')\big)$ then acts as $2\,\mathrm{id}_{V'_3(q)}$ on $V'_3(q)$, as $-2\,\mathrm{id}_{V'_3(q')}$ on $V'_3(q')$, and vanishes on $V'_{-3}(p) \oplus V'_{-3}(p')$. Dually, the linear map $T(q,q')$ acts as $2\,\mathrm{id}_{V'_{-3}(p)}$ on $V'_{-3}(p)$, as $-2\,\mathrm{id}_{V'_{-3}(p')}$ on $V'_{-3}(p')$, and vanishes on $V'_3(q) \oplus V'_3(q')$; cf. also [8, proof of Theorem 2.1]. In this paper we need this linear map only in the proof of Theorem 1.1 for the case $m_2 = 2m_1 - 1$; see Section 4.

3. Quadratic forms vanishing on a focal manifold

Let M be an isoparametric hypersurface with four distinct principal curvatures in the unit sphere $\mathbb{S}^{2l-1}$ of the Euclidean vector space $V = \mathbb{R}^{2l}$. Let Φ denote the linear operator on the vector space $\mathcal{S}_{2l}(\mathbb{R})$ of real, symmetric $(2l \times 2l)$-matrices that assigns to each matrix $D \in \mathcal{S}_{2l}(\mathbb{R})$ the symmetric matrix associated with the quadratic form $\mathbb{R}^{2l} \to \mathbb{R} : v \mapsto \operatorname{tr}(T(v)D)$, where $T(v)$ is defined as in the preceding section. For $D \in \mathcal{S}_{2l}(\mathbb{R})$ and a subspace $U \leq V$ we denote by $\operatorname{tr}(D|_U)$ the trace of the restriction of the quadratic form $\mathbb{R}^{2l} \to \mathbb{R} : v \mapsto \langle v, Dv\rangle$ to U, i.e. $\operatorname{tr}(D|_U)$ is the sum of the values of the quadratic form associated with D on an arbitrary orthonormal basis of U.

LEMMA 3.1. *Let $D \in \mathcal{S}_{2l}(\mathbb{R})$, $p \in M_+$, and $q \in M_-$. Then we have*

$$\langle p, \Phi(D)p\rangle = 2\langle p, Dp\rangle - 4\operatorname{tr}(D|_{V_{-3}(p)}) + \operatorname{tr}(D),$$
$$\langle q, \Phi(D)q\rangle = -2\langle q, Dq\rangle + 4\operatorname{tr}(D|_{V_3(q)}) - \operatorname{tr}(D).$$

Proof. For reasons of duality it suffices to prove the first statement. We choose orthonormal bases of $V_{-3}(p)$ and $V_1(p)$. Together with p, the vectors in these bases yield an orthonormal basis of V. With respect to this basis, the linear map $T(p)$ is given by a diagonal matrix; see the preceding section. Hence we get

$$\langle p, \Phi(D)p\rangle = \operatorname{tr}(T(p)D) = 3\langle p, Dp\rangle - 3\operatorname{tr}(D|_{V_{-3}(p)}) + \operatorname{tr}(D|_{V_1(p)}).$$

Then the claim follows because of $\langle p, Dp\rangle + \operatorname{tr}(D|_{V_{-3}(p)}) + \operatorname{tr}(D|_{V_1(p)}) = \operatorname{tr}(D)$.
□

Motivated by the previous lemma we set

$$\Phi_+ : \mathcal{S}_{2l}(\mathbb{R}) \to \mathcal{S}_{2l}(\mathbb{R}) : D \mapsto -\tfrac{1}{4}\big(\Phi(D) - 2D - \operatorname{tr}(D)E\big),$$

where E denotes the identity matrix. Then we have for $p \in M_+$ and $q \in M_-$

$$\begin{aligned} \langle p, \Phi_+(D)p\rangle &= \operatorname{tr}(D|_{V_{-3}(p)}),\\ \langle q, \Phi_+(D)q\rangle &= \langle q, Dq\rangle - \operatorname{tr}(D|_{V_3(q)}) + \tfrac{1}{2}\operatorname{tr}(D). \end{aligned}$$

LEMMA 3.2. *Let $p, q \in M_-$ be orthogonal points on a normal circle, $q' \in M_-$, $r \in M_+$, $D \in \mathcal{S}_{2l}(\mathbb{R})$, and $n \in \mathbb{N}$. Then we have*

(i) $\big|\langle r, \Phi_+^n(D)r\rangle\big| \le (m_1+1)^n \max_{x\in M_+}\big|\langle x, Dx\rangle\big|$,

(ii) $\big|\langle p, \Phi_+^n(D)p\rangle + \langle q, \Phi_+^n(D)q\rangle\big| \le 2(m_1+1)^n \max_{x\in M_+}\big|\langle x, Dx\rangle\big|$,

(iii) $\big|\langle p, \Phi_+^n(D)p\rangle - \langle q', \Phi_+^n(D)q'\rangle\big| \le 2(m_2+2)^n \max_{y\in M_-}\big|\langle y, Dy\rangle\big|$,

(iv) $\big|\langle p, \Phi_+^n(D)p\rangle\big| \le (m_1+1)^n \max_{x\in M_+}\big|\langle x, Dx\rangle\big| + (m_2+2)^n \max_{y\in M_-}\big|\langle y, Dy\rangle\big|$.

Proof. Because of $\langle r, \Phi_+(D)r\rangle = \operatorname{tr}(D|_{V_{-3}(r)})$ with $\dim V_{-3}(r) = m_1 + 1$ and $\mathbb{S}^{2l-1} \cap V_{-3}(r) \subseteq M_+$ we get

$$\big|\langle r, \Phi_+(D)r\rangle\big| \le (m_1+1)\max_{x\in M_+}\big|\langle x, Dx\rangle\big|.$$

Then (i) follows by induction. Since p and q are orthogonal points on a normal circle, we have $r_\pm = (1/\sqrt{2})(p \pm q) \in M_+$ (see the beginning of Section 2) and hence

$$\begin{aligned} \big|\langle p, \Phi_+^n(D)p\rangle + \langle q, \Phi_+^n(D)q\rangle\big| &= \big|\operatorname{tr}(\Phi_+^n(D)|_{\operatorname{span}\{p,q\}})\big|\\ &= \big|\langle r_+, \Phi_+^n(D)r_+\rangle + \langle r_-, \Phi_+^n(D)r_-\rangle\big|\\ &\le 2(m_1+1)^n \max_{x\in M_+}\big|\langle x, Dx\rangle\big| \end{aligned}$$

by (i). Because of $\langle p, \Phi_+(D)p\rangle = \langle p, Dp\rangle - \operatorname{tr}(D|_{V_3(p)}) + (1/2)\operatorname{tr}(D)$, the analogous equation with p replaced by q', $\dim V_3(p) = \dim V_3(q') = m_2 + 1$ and $\mathbb{S}^{2l-1} \cap V_3(p)$, $\mathbb{S}^{2l-1} \cap V_3(q') \subseteq M_-$ we get

$$\begin{aligned} &\big|\langle p, \Phi_+(D)p\rangle - \langle q', \Phi_+(D)q'\rangle\big|\\ &\quad\le \big|\langle p, Dp\rangle - \langle q', Dq'\rangle\big| + \big|\operatorname{tr}(D|_{V_3(p)}) - \operatorname{tr}(D|_{V_3(q')})\big|\\ &\quad\le (m_2+2)\max_{y,z\in M_-}\big|\langle y, Dy\rangle - \langle z, Dz\rangle\big|. \end{aligned}$$

By induction we obtain

$$\begin{aligned} \big|\langle p, \Phi_+^n(D)p\rangle - \langle q', \Phi_+^n(D)q'\rangle\big| &\le (m_2+2)^n \max_{y,z\in M_-}\big|\langle y, Dy\rangle - \langle z, Dz\rangle\big|\\ &\le 2(m_2+2)^n \max_{y\in M_-}\big|\langle y, Dy\rangle\big|. \end{aligned}$$

Finally, (ii) and (iii) yield

$$\begin{aligned} \big|\langle p, \Phi_+^n(D)p\rangle\big| &\le \tfrac{1}{2}\big|\langle p, \Phi_+^n(D)p\rangle + \langle q, \Phi_+^n(D)q\rangle\big| + \tfrac{1}{2}\big|\langle p, \Phi_+^n(D)p\rangle - \langle q, \Phi_+^n(D)q\rangle\big|\\ &\le (m_1+1)^n \max_{x\in M_+}\big|\langle x, Dx\rangle\big| + (m_2+2)^n \max_{y\in M_-}\big|\langle y, Dy\rangle\big|. \qquad \square \end{aligned}$$

LEMMA 3.3. *Let $p, q \in M_-$ be orthogonal points on a normal circle, $D \in \mathbb{S}_{2l}(\mathbb{R})$, $d_0 \geq \max_{x \in M_+} \big|\langle x, Dx\rangle\big|$, and let $(d_n)_n$ be the sequence defined by*

$$d_1 = \big|\langle p, \Phi_+(D)p\rangle - \langle q, \Phi_+(D)q\rangle\big|,$$
$$d_{n+1} = (m_2+2)d_n - 4m_2(m_1+1)^n d_0$$

for $n \geq 1$. Then we have

$$\big|\langle p, \Phi_+^n(D)p\rangle - \langle q, \Phi_+^n(D)q\rangle\big| \geq d_n$$

for every $n \geq 1$.

Proof. We prove this lemma by induction. For $n = 1$, the statement above is true by definition. Now assume that

$$\big|\langle p, \Phi_+^n(D)p\rangle - \langle q, \Phi_+^n(D)q\rangle\big| \geq d_n$$

for some $n \geq 1$. Let $q' \in \mathbb{S}^{2l-1} \cap V_3(p)$. Then $p, q' \in M_-$ are orthogonal points on a normal circle. Hence we have

$$\langle p, \Phi_+^n(D)p\rangle + \langle q', \Phi_+^n(D)q'\rangle \leq 2(m_1+1)^n d_0$$

by Lemma 3.2(ii). Since $q \in V_3(p)$ with $\dim V_3(p) = m_2 + 1$ we conclude that

$$\operatorname{tr}(\Phi_+^n(D)|_{V_3(p)}) \leq \langle q, \Phi_+^n(D)q\rangle + m_2\big(2(m_1+1)^n d_0 - \langle p, \Phi_+^n(D)p\rangle\big).$$

Hence we obtain

$$\begin{aligned}(3.1)\qquad \langle p, \Phi_+^{n+1}(D)p\rangle &= \langle p, \Phi_+^n(D)p\rangle - \operatorname{tr}(\Phi_+^n(D)|_{V_3(p)}) + \tfrac{1}{2}\operatorname{tr}\big(\Phi_+^n(D)\big)\\ &\geq (m_2+1)\langle p, \Phi_+^n(D)p\rangle - \langle q, \Phi_+^n(D)q\rangle + \tfrac{1}{2}\operatorname{tr}\big(\Phi_+^n(D)\big)\\ &\quad -2m_2(m_1+1)^n d_0.\end{aligned}$$

Analogously, for $p' \in \mathbb{S}^{2l-1} \cap V_3(q)$ we get

$$\langle p', \Phi_+^n(D)p'\rangle + \langle q, \Phi_+^n(D)q\rangle \geq -2(m_1+1)^n d_0$$

by Lemma 3.2(ii) and hence

$$\operatorname{tr}(\Phi_+^n(D)|_{V_3(q)}) \geq \langle p, \Phi_+^n(D)p\rangle - m_2\big(2(m_1+1)^n d_0 + \langle q, \Phi_+^n(D)q\rangle\big).$$

As above, we conclude that

$$\begin{aligned}\langle q, \Phi_+^{n+1}(D)q\rangle &\leq (m_2+1)\langle q, \Phi_+^n(D)q\rangle - \langle p, \Phi_+^n(D)p\rangle + \tfrac{1}{2}\operatorname{tr}\big(\Phi_+^n(D)\big)\\ &\quad +2m_2(m_1+1)^n d_0.\end{aligned}$$

Subtracting this inequality from inequality (3.1) we obtain that

$$\begin{aligned}\big|\langle p, \Phi_+^{n+1}(D)p\rangle - \langle q, \Phi_+^{n+1}(D)q\rangle\big| &\geq (m_2+2)\big(\langle p, \Phi_+^n(D)p\rangle - \langle q, \Phi_+^n(D)q\rangle\big)\\ &\quad -4m_2(m_1+1)^n d_0.\end{aligned}$$

Also the analogous inequality with p and q interchanged is satisfied. Thus we get

$$\begin{aligned}\left|\langle p, \Phi_+^{n+1}(D)p\rangle - \langle q, \Phi_+^{n+1}(D)q\rangle\right| &\geq (m_2+2)\left|\langle p, \Phi_+^{n}(D)p\rangle - \langle q, \Phi_+^{n}(D)q\rangle\right| \\ &\quad -4m_2(m_1+1)^n d_0 \\ &\geq (m_2+2)d_n - 4m_2(m_1+1)^n d_0 \\ &= d_{n+1}. \end{aligned}$$

□

Lemma 3.4. *Let $p, q \in M_-$ be orthogonal points on a normal circle and assume that $m_2 \geq 2m_1 - 1$. Then there exist a symmetric matrix $D \in \mathcal{S}_{2l}(\mathbb{R})$ and a positive constant d such that*

$$\frac{1}{(m_2+2)^n}\left|\langle p, \Phi_+^{n}(D)p\rangle - \langle q, \Phi_+^{n}(D)q\rangle\right| > d$$

for every $n \geq 1$.

Proof. We choose $D \in \mathcal{S}_{2l}(\mathbb{R})$ as the symmetric matrix associated with the self-adjoint linear map on $V = \mathbb{R}^{2l}$ that acts as the identity $\mathrm{id}_{V_3(p)}$ on $V_3(p)$, as $-\mathrm{id}_{V_3(q)}$ on $V_3(q)$, and vanishes on the orthogonal complement of $V_3(p) \oplus V_3(q)$ in V. Let $x \in M_+$ and denote by u, v the orthogonal projections of x onto $V_3(p)$ and $V_3(q)$, respectively. Then we have $\langle x, Dx\rangle = \langle u, u\rangle - \langle v, v\rangle$. By [9, Lemma 3.1], or [11, Proposition 3.2], the scalar product of a point of M_+ and a point of M_- is at most $1/\sqrt{2}$. If $u \neq 0$ then we have $(1/\|u\|)u \in M_-$ and hence

$$\langle u, u\rangle = \langle x, u\rangle = \left\langle x, \tfrac{u}{\|u\|}\right\rangle \|u\| \leq \tfrac{1}{\sqrt{2}}\|u\|.$$

In any case we get $\|u\| \leq 1/\sqrt{2}$ and hence $\langle x, Dx\rangle = \langle u, u\rangle - \langle v, v\rangle \leq 1/2$. Analogously we see that $\langle x, Dx\rangle \geq -1/2$. We set $d_0 = 1/2$. Then we have $d_0 \geq \max_{x\in M_+}\left|\langle x, Dx\rangle\right|$, and we may define a sequence $(d_n)_n$ as in Lemma 3.3. Since $p \in V_3(q)$, $q \in V_3(p)$, and $\dim V_3(p) = \dim V_3(q) = m_2 + 1$ we have

$$d_1 = \left|\langle p, \Phi_+(D)p\rangle - \langle q, \Phi_+(D)q\rangle\right| = 2(m_2+2)$$

and hence

$$\begin{aligned}\frac{1}{m_2+2}d_1 &= 2,\\ \frac{1}{(m_2+2)^2}d_2 &= \frac{1}{m_2+2}d_1 - 2m_2\frac{m_1+1}{(m_2+2)^2},\\ &\;\;\vdots\\ \frac{1}{(m_2+2)^{n+1}}d_{n+1} &= \frac{1}{(m_2+2)^n}d_n - 2m_2\frac{(m_1+1)^n}{(m_2+2)^{n+1}}\end{aligned}$$

for $n \geq 1$. Thus we get

$$\begin{aligned}\frac{1}{(m_2+2)^{n+1}} d_{n+1} &= 2 - 2m_2 \sum_{i=0}^{n-1} \frac{(m_1+1)^{i+1}}{(m_2+2)^{i+2}} \\ &> 2 - 2m_2 \frac{m_1+1}{(m_2+2)^2} \sum_{i=0}^{\infty} \left(\frac{m_1+1}{m_2+2}\right)^i \\ &= 2 - 2m_2 \frac{m_1+1}{(m_2+2)(m_2-m_1+1)}.\end{aligned}$$

We denote the term in the last line by d. Then $d > 0$ is equivalent to

$$(m_2+2)(m_2-m_1+1) > m_2(m_1+1).$$

We put $f : \mathbb{R} \to \mathbb{R} : s \mapsto s^2 - as - a$ with $a = 2(m_1 - 1)$. The latter inequality is equivalent to $f(m_2) > 0$. Since $f(a) = -a \leq 0$ and $f(a+1) = 1$ we see that this inequality is indeed satisfied for $m_2 \geq 2(m_1 - 1) + 1$. By Lemma 3.3, we conclude that for $m_2 \geq 2m_1 - 1$ we have

$$\frac{1}{(m_2+2)^n} \Big| \langle p, \Phi_+^n(D)p\rangle - \langle q, \Phi_+^n(D)q\rangle \Big| \geq \frac{1}{(m_2+2)^n} d_n > d > 0$$

for every $n \geq 1$. □

Lemma 3.5. *Set* $\mathcal{A}(M_+) = \{A \in \mathcal{S}_{2l}(\mathbb{R}) \mid \langle x, Ax\rangle = 0$ *for every* $x \in M_+\}$ *and assume that* $m_2 \geq 2m_1 - 1$. *Then we have*

$$M_+ = \{x \in \mathbb{S}^{2l-1} \mid \langle x, Ax\rangle = 0 \textit{ for every } A \in \mathcal{A}(M_+)\}.$$

Proof. For $B \in \mathcal{S}_{2l}(\mathbb{R})$ we set $\|B\| = \max_{x \in M_+ \cup M_-} \big|\langle x, Bx\rangle\big|$. If $\|B\| = 0$ then the quadratic form $\mathbb{R}^{2l} \to \mathbb{R} : v \mapsto \langle v, Bv\rangle$ vanishes on each normal circle S at the eight points of $S \cap (M_+ \cup M_-)$. Therefore it vanishes entirely on each normal circle and hence on V. This shows that $B = 0$, and hence $\|\cdot\|$ is indeed a norm on $\mathcal{S}_{2l}(\mathbb{R})$.

In the sequel we always assume that $p, q \in M_-$ and $D \in \mathcal{S}_{2l}(\mathbb{R})$ are chosen as in Lemma 3.4. By Lemma 3.2(i) and (iv), the sequence

$$\left(\frac{1}{(m_2+2)^n} \Phi_+^n(D)\right)_n$$

is bounded with respect to the norm defined above. Let $A \in \mathcal{S}_{2l}(\mathbb{R})$ be an accumulation point of this sequence. By Lemma 3.2(i) we have

$$\big|\langle r, Ar\rangle\big| \leq \lim_{n\to\infty} \left(\frac{m_1+1}{m_2+2}\right)^n \max_{x \in M_+} \big|\langle x, Dx\rangle\big| = 0$$

for every $r \in M_+$. Thus the quadratic form $\mathbb{R}^{2l} \to \mathbb{R} : v \mapsto \langle v, Av\rangle$ vanishes entirely on M_+. Since $p, q \in M_-$ are orthogonal points on a normal circle we obtain $\langle p, Ap\rangle + \langle q, Aq\rangle = 0$. Furthermore, by Lemma 3.4 we have $\big|\langle p, Ap\rangle - \langle q, Aq\rangle\big| \geq d > 0$. Hence we get $\langle p, Ap\rangle \neq 0$.

Choose $p' \in \mathbb{S}^{2l-1} \backslash M_+$ arbitrarily. Let S' be a normal circle through p' and let q' be one of the four points of $S' \cap M_-$. The previous arguments show that there exists a matrix $A' \in \mathcal{A}(M_+)$ such that $\langle q', A'q' \rangle \neq 0$. Then the quadratic form associated with A' vanishes on S' precisely at the four points of $S' \cap M_+$. In particular, we have $\langle p', A'p' \rangle \neq 0$. Thus we get

$$\{x \in \mathbb{S}^{2l-1} \mid \langle x, Ax \rangle = 0 \text{ for every } A \in \mathcal{A}(M_+)\} \subseteq M_+.$$

Since the other inclusion is trivial, the claim follows. □

4. End of proof

Based on Lemma 3.5 we complete our proof of Theorem 1.1 by means of the following

LEMMA 4.1. *Let M be an isoparametric hypersurface with four distinct principal curvatures in the unit sphere $\mathbb{S}^{2l-1}$ of the Euclidean vector space $\mathbb{R}^{2l}$ and assume that*

$$M_+ = \{x \in \mathbb{S}^{2l-1} \mid \langle x, Ax \rangle = 0 \textit{ for every } A \in \mathcal{A}(M_+)\},$$

where $\mathcal{A}(M_+)$ is defined as in Lemma 3.5. Then M is an isoparametric hypersurface of Clifford type provided that the multiplicities m_1, m_2 of the principal curvatures satisfy the inequality $m_2 \geq 2m_1 - 1$.

We treat the cases $m_2 \geq 2m_1$ and $m_2 = 2m_1 - 1$ separately because of the essentially different proofs for these two cases. Whereas the proof in the first case is based on results of [8], the proof in the second case involves, in addition, representation theory of Clifford algebras. For more information on the special case $(m_1, m_2) = (1, 1)$, see the remarks at the end of this section.

Proof of Lemma 4.1 (case $m_2 \geq 2m_1$). For every matrix $A \in \mathcal{A}(M_+)$ we have a well-defined linear map $\varphi_A : \mathcal{A}(M_+) \to \mathcal{A}(M_+) : B \mapsto ABA$; see [8, Proposition 3.1 (i)]. We first want to show that φ_A is injective for every $A \in \mathcal{A}(M_+) \backslash \{0\}$. Without loss of generality we may assume that $A = (a_{ij})_{i,j}$ is a diagonal matrix with $a_{ii} = 0$ for $i > t$, where t denotes the rank of A. Assume that there exists a matrix $B = (b_{ij})_{i,j} \in \ker(\varphi_A) \backslash \{0\}$. Then we have $t < 2l$ and $b_{ij} = 0$ for $1 \leq i, j \leq t$. Hence the nonzero entries of B lie in the two blocks given by $t+1 \leq i \leq 2l$ and $1 \leq i \leq t$, $t+1 \leq j \leq 2l$. By [8, Proposition 3.1 (ii)], we have $t \geq 2(m_2+1)$ and hence the rank of both blocks is bounded by $2l - t \leq 2(m_1 + m_2 + 1) - 2(m_2 + 1) = 2m_1$. Thus the rank of B is at most $4m_1$ and, again by [8, Proposition 3.1 (ii)], at least $2(m_2+1)$. We conclude that $2m_1 \geq m_2 + 1$ in contradiction to $m_2 \geq 2m_1$. Hence φ_A is a bijection.

The only nonzero entries of a matrix $C = (c_{ij})_{i,j}$ in the image of φ_A lie in the block given by $1 \leq i, j \leq t$. Thus every matrix in $\mathcal{A}(M_+)$, considered as a self-adjoint linear map on $\mathbb{R}^{2l}$, vanishes on the kernel of A. We want to show that $\ker(A) = \{0\}$. Otherwise there exists a point $q \in \mathbb{S}^{2l-1} \cap \ker(A)$. For every $C \in \mathcal{A}(M_+)$ we have $\langle q, Cq\rangle = 0$. Hence we get $q \in M_+$. Let S be a normal circle through q and choose $p \in S$ with $\langle p, q\rangle = 0$. Then we have $p \in M_+$ and $\langle p, Cp\rangle = \langle p, Cq\rangle = \langle q, Cq\rangle = 0$ for every $C \in \mathcal{A}(M_+)$. This implies that $S \subseteq M_+$, a contradiction. Since $A \in \mathcal{A}(M_+)\backslash\{0\}$ was chosen arbitrarily we conclude that every matrix in $\mathcal{A}(M_+)\backslash\{0\}$ is regular. Hence M is an isoparametric hypersurface of Clifford type; see [8, Theorem 4.1]. Note that the inequality $l - m - 1 > 0$ is satisfied by [5, Section 4, Satz (i)]. □

Proof of Lemma 4.1 (case $m_2 = 2m_1 - 1$). We use the notation of the proof above. If the linear map φ_A is injective for every $A \in \mathcal{A}(M_+)\backslash\{0\}$, then we see precisely as in the preceding proof that M is an isoparametric hypersurface of Clifford type. Thus we may assume that there exists a matrix $A \in \mathcal{A}(M_+)\backslash\{0\}$ such that φ_A is not injective. The arguments used above to prove that φ_A is always injective for $A \in \mathcal{A}(M_+)\backslash\{0\}$ for $m_2 \geq 2m_1$ then show that for $m_2 = 2m_1 - 1$ the rank t of A must be equal to $2(m_2 + 1)$.

Without loss of generality we may assume that the quadratic form $\mathbb{R}^{2l} \to \mathbb{R} : v \mapsto \langle v, Av\rangle$ takes the maximum 1 on $\mathbb{S}^{2l-1}$. For $p \in \mathbb{S}^{2l-1}$ with $\langle p, Ap\rangle = 1$ we then have $Ap = p$ and $p \in M_-$ by [8, Proposition 3.1 (ii)]. By the same result the minimum of this quadratic form on $\mathbb{S}^{2l-1}$ is equal to -1. For an arbitrary point $q \in \mathbb{S}^{2l-1} \cap V_3(p)$ we get $\langle q, Aq\rangle = -\langle p, Ap\rangle = -1$ since $p, q \in M_-$ are orthogonal points on a normal circle S and the quadratic form associated with A vanishes at the four points of $S \cap M_+$. This shows that the matrix A acts as $-\mathrm{id}_{V_3(p)}$ on $V_3(p)$ and, by an analogous argument, as the identity $\mathrm{id}_{V_3(q)}$ on $V_3(q)$. Since $t = 2(m_2 + 1)$ and $\dim V_3(p) = \dim V_3(q) = m_2 + 1$ we conclude that A vanishes on the orthogonal complement W of $V_3(p) \oplus V_3(q)$ in $\mathbb{R}^{2l}$.

For every $x \in \mathbb{S}^{2l-1} \cap V_3(p)$ we see as above that A acts as the identity $\mathrm{id}_{V_3(x)}$ on $V_3(x)$. Hence we get $V_3(x) = V_3(q)$ for every $x \in \mathbb{S}^{2l-1} \cap V_3(p)$. Thus the self-adjoint map $T(p, x)$ leaves the subspace W invariant, and $T(p, x)|_W$ has the eigenvalues ± 2; see the end of Section 2. Denote by $\mathcal{S}(W)$ the vector space of self-adjoint linear maps on W. Then we have a well-defined linear map

$$\psi : V_3(p) \to \mathcal{S}(W) : x \mapsto \tfrac{1}{2} T(p, x)|_W$$

with the property that $\psi(x)^2 = \mathrm{id}_W$ for every $x \in \mathbb{S}^{2l-1} \cap V_3(p)$. In particular, the linear map ψ is injective, and if we identify the Euclidean vector space W with $\mathbb{R}^{2m_1}$ we see as in [8, proof of Theorem 4.1], that the image of ψ is generated by a Clifford system $Q_0, \ldots, Q_{m_2}$ of $(2m_1 \times 2m_1)$-matrices. Since $m_2 = 2m_1 - 1$, this yields a contradiction to the representation theory of Clifford algebras except for the case $(m_1, m_2) = (1, 1)$; see [5, Section 3.5]. For this special case there exists up to isometry precisely one family of isoparametric hypersurfaces; see [16]. This family is (homogeneous and) of Clifford type. □

Remarks. (i) In the proof above we referred the reader for the case $(m_1, m_2) = (1, 1)$ to [16]. Let us now have a closer look at this particular case. By Lemma 3.5, both focal manifolds may be described by means of quadratic forms. In order to see this, it suffices to interchange the focal manifolds M_+ and M_-. Note that this argument does not work for $(m_1, m_2) \neq (1, 1)$. If we interchange M_+ and M_- we also have to interchange the multiplicities m_1 and m_2 since we required in Section 2 that M_+ and M_- be given by $F^{-1}(1) \cap \mathbb{S}^{2l-1}$ and $F^{-1}(-1) \cap \mathbb{S}^{2l-1}$, respectively, where F denotes a Cartan-Münzner polynomial. Hence both of the inequalities $m_2 \geq 2m_1 - 1$ and $m_1 \geq 2m_2 - 1$ must be satisfied in order to conclude from Lemma 3.5 that both focal manifolds may be described by means of the vanishing of quadratic forms. This is only possible for $(m_1, m_2) = (1, 1)$.

Based on this observation, the proof of Lemma 4.1 can also be completed independently of [16] for this case. It turns out that one of the focal manifolds, say M_+, can be described by means of a Clifford system as in the introduction, but there does not exist any quadratic form associated with a regular symmetric matrix that vanishes entirely on the other focal manifold M_-. Nevertheless, for every point $p \in M_+$ there exists a symmetric matrix of rank 4 such that the associated quadratic form takes its maximum at p and vanishes identically on M_-. These statements may be proved by means of calculations based on orthonormal bases in accordance with Theorem 2.1.

(ii) As we have seen in the introduction, the inequality $m_2 \geq 2m_1 - 1$ is satisfied for all but five possible pairs (m_1, m_2) with $m_1 \leq m_2$. For $(m_1, m_2) = (2, 2)$ or $(4, 5)$ the only known examples are two homogeneous families of isoparametric hypersurfaces; cf. [5, Section 4.4]. In the first case, the example is unique; see [14]. Note that it is an immediate consequence of the representation theory of Clifford algebras that there does not exist any example of Clifford type with these multiplicities; see [5, Section 3.5]. For an overview of isoparametric hypersurfaces of Clifford type with small multiplicities, we refer the reader to [5, Section 4.3]. In the sequel we want to give some information on the three remaining cases $(3, 4)$, $(6, 9)$ and $(7, 8)$.

By [5, Sections 5.2, 5.8, 6.1, and, in particular, 6.5], there are (up to isometry) precisely two isoparametric families of Clifford type with $(m_1, m_2) = (3, 4)$. One of these families is inhomogeneous and has the property that even both focal manifolds can be described by means of a Clifford system as in the introduction. The other family is homogeneous, and only the lower-dimensional of the two focal manifolds may be described in this way.

Also for $(m_1, m_2) = (6, 9)$ there are up to isometry precisely one inhomogeneous and one homogeneous isoparametric family of Clifford type; see

[5, Sections 5.4 and 6.3]. For the inhomogeneous family, only the higher-dimensional of the two focal manifolds may be described by means of a Clifford system as in the proof above. In contrast to that, for the homogeneous family only the lower-dimensional of the two focal manifolds may be described by means of the vanishing of quadratic forms associated with a Clifford system.

For $(m_1, m_2) = (7, 8)$ there are even three nonisometric isoparametric families of Clifford type, all of which are inhomogeneous; see [5, Sections 5.4, 5.5, and, in particular, 6.6]. For one of these examples, only the higher-dimensional of the two focal manifolds may be described by means of the vanishing of the quadratic forms associated with a Clifford system. In the other two cases, only the lower-dimensional of the two focal manifolds may be described in this way. For one of these two families, both focal manifolds (and not only the isoparametric hypersurfaces) are inhomogeneous, while for the other family only the higher-dimensional focal manifold is inhomogeneous.

(iii) In (ii) we have seen that for $(m_1, m_2) = (3, 4)$ there exists an isoparametric family of Clifford type such that both focal manifolds can be described by means of a Clifford system as in the proof above. The same property also occurs for the three pairs $(1, 2)$, $(1, 6)$, and $(2, 5)$ (and does not occur for any other pair (m_1, m_2) with $m_1 \leq m_2$); see [5, Section 4.3]. Moreover, for each of these three pairs there exists (up to isometry) precisely one isoparametric family of Clifford type. These three examples are homogeneous; see [5, Section 6.1].

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