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Maharam's problem

By Michel Talagrand

Dedicated to J. W. Roberts

Abstract

We construct an exhaustive submeasure that is not equivalent to a measure. This solves problems of J. von Neumann (1937) and D. Maharam (1947).

Contents

1. Introduction

Consider a Boolean algebra $\mathcal{B}$ of sets. A map $\nu : \mathcal{B} \to \mathbb{R}^+$ is called a *submeasure* if it satisfies the following properties:

$$\nu(\varnothing) = 0, \tag{1.1}$$

$$A \subset B, \quad A, B \in \mathcal{B} \Longrightarrow \nu(A) \le \nu(B), \tag{1.2}$$

$$A, B \in \mathcal{B} \Rightarrow \nu(A \cup B) \le \nu(A) + \nu(B). \tag{1.3}$$

If we have $\nu(A \cup B) = \nu(A) + \nu(B)$ whenever A and B are disjoint, we say that ν is a (finitely additive) *measure*.

We say that a sequence (E_n) of $\mathcal{B}$ is *disjoint* if $E_n \cap E_m = \varnothing$ whenever $n \neq m$. A submeasure is *exhaustive* if $\lim_{n\to\infty} \nu(E_n) = 0$ whenever (E_n) is a disjoint sequence in $\mathcal{B}$. A measure is obviously exhaustive. Given two submeasures ν_1 and ν_2, we say that ν_1 is *absolutely continuous with respect to* ν_2 if

$$\forall\varepsilon > 0,\ \exists\alpha > 0,\ \nu_2(A) \le \alpha \Longrightarrow \nu_1(A) \le \varepsilon. \tag{1.4}$$

If a submeasure is absolutely continuous with respect to a measure, it is exhaustive. One of the many equivalent forms of Maharam's problem is whether the converse is true.

Maharam's problem: If a submeasure is exhaustive, is it absolutely continuous with respect to a measure?

In words, we are asking whether the only way a submeasure can be exhaustive is because it really resembles a measure. This question has been one of the longest standing classical questions of measure theory. It occurs in a variety of forms (some of which will be discussed below).

Several important contributions were made to Maharam's problem. N. Kalton and J. W. Roberts proved [11] that a submeasure is absolutely continuous with respect to a measure if (and, of course, only if) it is *uniformly* exhaustive, i.e.

$$\forall\varepsilon > 0,\quad \exists n,\quad E_1, \dots, E_n \text{ disjoint} \Longrightarrow \inf_{i\le n} \nu(E_i) \le \varepsilon. \tag{1.5}$$

Thus Maharam's problem can be reformulated as to whether an exhaustive submeasure is necessarily uniformly exhaustive. Two other fundamental contributions by J.W. Roberts [15] and I. Farah [6] are used in an essential way in this paper and will be discussed in great detail later.

We prove that Maharam's problem has a negative answer.

THEOREM 1.1. *There exists a nonzero exhaustive submeasure ν on the algebra $\mathcal{B}$ of clopen subsets of the Cantor set that is not uniformly exhaustive* (*and thus is not absolutely continuous with respect to a measure*). *Moreover, no nonzero measure μ on $\mathcal{B}$ is absolutely continuous with respect to ν.*

We now spell out some consequences of Theorem 1.1. It has been known for a while how to deduce these results from Theorem 1.1. For the convenience of the reader these (easy) arguments will be given in a self-contained way in the last section of the paper.

Since Maharam's original question and the von Neumann problem are formulated in terms of general Boolean algebras (i.e., that are not *a priori* represented as algebras of sets) we must briefly mention these. We will denote by $\mathbf{0}$ and $\mathbf{1}$ respectively the smallest and the largest element of a Boolean algebra $\mathcal{B}$, but we will denote the Boolean operations by $\cap, \cup$, etc. as in the case of algebras of sets. A Boolean algebra $\mathcal{B}$ is called *σ-complete* if any countable subset $\mathcal{C}$ of $\mathcal{B}$ has a least upper bound $\cup\mathcal{C}$ (and thus a greatest lower bound $\cap\mathcal{C}$). A submeasure ν on $\mathcal{B}$ is called *continuous* if whenever (A_n)

is a decreasing sequence with $\bigcap_n A_n = \mathbf{0}$ we have $\lim_{n\to\infty} \nu(A_n) = 0$. The submeasure is called *positive* if $\nu(A) = 0 \Longrightarrow A = \mathbf{0}$.

A σ-complete algebra $\mathcal{B}$ on which there is a positive continuous submeasure is called a *submeasure algebra*. If there is a positive continuous measure on $\mathcal{B}$, $\mathcal{B}$ is called a *measure algebra*.

Probably the most important consequence of our construction is that it proves the existence of radically new Boolean algebras.

THEOREM 1.2. *There exists a submeasure algebra $\mathcal{B}$ that is not a measure algebra. In fact, not only there is no positive measure on $\mathcal{B}$, but there is no nonzero continuous measure on it.*

A subset $\mathcal{C}$ of a boolean algebra $\mathcal{B}$ is called *disjoint* if $A \cap B = \mathbf{0}$ (= the smallest element of $\mathcal{B}$) whenever $A, B \in \mathcal{C}$, $A \neq B$. A disjoint set $\mathcal{C}$ is called a *partition* if $\cup\mathcal{C} = \mathbf{1}$ (= the largest element of $\mathcal{B}$). If every disjoint collection of $\mathcal{B}$ is countable, $\mathcal{B}$ is said to satisfy the *countable chain condition.*

If Π is a partition of $\mathcal{B}$ we say that $A \in \mathcal{B}$ is *finitely covered* by Π if there is a finite subset $\{A_1, \ldots, A_n\}$ of Π with $A \subset \bigcup_{i\leq n} A_i$. We say that $\mathcal{B}$ satisfies the *weak distributive law* if whenever (Π_n) is a sequence of partitions of $\mathcal{B}$, there is a single partition Π of $\mathcal{B}$ such that every element of Π is finitely covered by each Π_n. (This terminology is not used by every author; such a σ-algebra is called weakly $(\sigma - \infty)$ distributive in [8].)

THEOREM 1.3 (Negative answer to von Neumann's problem). *There exists a σ-complete algebra that satisfies the countable chain condition and the weak distributive law, but is not a measure algebra.*

The original problem of von Neumann was to characterize measure algebras in the class of complete Boolean algebras. Every measure algebra (and in fact every submeasure algebra) satisfies the countable chain condition and the weak distributive law, and von Neumann asked in the Scottish book ([13, problem 163]) whether these conditions are sufficient. This question was historically important, in that it motivated much further work.

The first major advance on von Neumann's problem is due to Maharam [12]. Her work gives a natural decomposition of von Neumann's problem in the following two parts.

Problem I. Does every weakly distributive complete Boolean algbra $\mathcal{B}$ satifying the countable chain condition support a positive continuous submeasure?

Problem II. Given that $\mathcal{B}$ supports a positive continuous submeasure, does it also support a positive continous measure?

Theorem 1.2 shows that (II) has a negative answer, and this is how Theorem 1.3 is proved.

It is now known that (I) cannot be decided with the usual axioms of set theory. Maharam proved [12] that (I) does not hold if one assumes the negation of Suslin's hypothesis. Recent work ([3], [18]) shows on the other hand that it is consistent with the usual axioms of set theory to assume that (I) holds. One can argue in fact that the reason why (I) does not have a very satisfactory answer is that one does not consider the correct notion of "a countable chain condition". Every submeasure algebra (and hence every measure algebra) $\mathcal{B}$ obviously satisfies the following condition (sometimes called the σ-finite chain condition) that is much stronger than the countable chain condition: $\mathcal{B}$ is the union of sets $\mathcal{B}_n$ such that for each n, every disjoint subset of $\mathcal{B}_n$ is finite. If one replaces in (I) the countable chain condition by the σ-finite chain condition one gets a much more satisfactory answer: S. Todorcevic proved [17] the remarkable fact that a complete Boolean algebra is a submeasure algebra if and only if it satisfies the weak distributive law and the σ-finite chain condition.

The reader interested in the historical developements following von Neumann's problem can find a more detailed account in the introduction of [2].

Consider now a topological vector space X with a metrizable topology, and d a translation invariant distance that defines this topology. If $\mathcal{B}$ is a Boolean algebra of subsets of a set T, an (X-valued) *vector measure* is a map $\theta : \mathcal{B} \to X$ such that $\theta(A \cup B) = \theta(A) + \theta(B)$ whenever $A \cap B = \varnothing$. We say that it is *exhaustive* if $\lim_{n\to\infty} \theta(E_n) = 0$ for each disjoint sequence (E_n) of $\mathcal{B}$. A positive measure μ on $\mathcal{B}$ is called a *control measure* for θ if

$$\forall \varepsilon > 0, \quad \exists \alpha > 0, \ \mu(A) \le \alpha \Longrightarrow d(0, \theta(A)) \le \varepsilon.$$

THEOREM 1.4 (Negative solution to the Control Measure Problem). *There exists an exhaustive vector-valued measure that does not have a control measure.*

We now explain the organization of the paper. The submeasure we will construct is an object of a rather new nature, since it is very far from being a measure. It is unlikely that a very simple example exists at all, and it should not come as a surprise that our construction is somewhat involved. Therefore it seems necessary to explain first the main ingredients on which the construction relies. The fundamental idea is due to J. W. Roberts [15] and is detailed in Section 2. Another crucial part of the construction is a technical device invented by I. Farah [6]. In Section 3, we produce a kind of "miniature version" of Theorem 1.1, to explain Farah's device, as well as some of the other main ideas. The construction of ν itself is given in Section 4, and the technical work of proving that ν is not zero and is exhaustive is done in Sections 5 and 6 respectively. Finally, in Section 7 we give the simple (and known) arguments needed to deduce Theorems 1.2 to 1.4 from Theorem 1.1.

Acknowledgments. My warmest thanks go to I. Farah who explained to me the importance of Roberts's work [15], provided a copy of this hard-to-find paper, rekindled my interest in this problem, and, above all, made an essential technical contribution without which my own efforts could hardly have succeeded.

2. Roberts

Throughout the paper we write

$$T = \prod_{n \geq 1} \{1, \ldots, 2^n\}.$$

For $\boldsymbol{z} \in T$, we thus have $\boldsymbol{z} = (z_n)$, $z_n \in \{1, \ldots, 2^n\}$. We denote by $\mathcal{B}_n$ the algebra generated by the coordinates of rank $\leq n$, and $\mathcal{B} = \bigcup_{n \geq 1} \mathcal{B}_n$ the algebra of the clopen sets of T. It is isomorphic to the algebra of the clopen sets of the Cantor set $\{0,1\}^{\mathbb{N}}$.

We denote by $\mathcal{A}_n$ the set of atoms of $\mathcal{B}_n$. These are sets of the form

$$\{\boldsymbol{z} \in T;\, z_1 = \tau_1, \ldots, z_n = \tau_n\} \tag{2.1}$$

where τ_i is an integer $\leq 2^i$. An element A of $\mathcal{A}_n$ will be called an *atom of rank* n.

Definition 2.1 ([15]). Consider $1 \leq m < n$. We say that a subset X of T is (m,n)-*thin* if

$$\forall A \in \mathcal{A}_m, \quad \exists A' \in \mathcal{A}_n, \quad A' \subset A, \quad A' \cap X = \varnothing.$$

In words, in each atom of rank m, X has a hole big enough to contain an atom of rank n. It is obvious that if X is (m,n)-thin, it is also (m,n')-thin when $n' \geq n$.

Definition 2.2 ([15]). Consider a (finite) subset I of $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We say that $X \subset T$ is I-*thin* if X is (m,n)-thin whenever $m < n$, $m,n \in I$.

We denote by $\operatorname{card} I$ the cardinality of a finite set I. For two finite sets $I, J \subset \mathbb{N}^*$, we write $I \prec J$ if $\max I \leq \min J$.

The following is implicit in [15] and explicit in [6].

LEMMA 2.3 (Roberts's selection lemma). *Consider two integers s and t, and sets $I_1, \ldots, I_s \subset \mathbb{N}^*$ with $\operatorname{card} I_\ell \geq st$ for $1 \leq \ell \leq s$. Then we can relabel the sets $I_1, \ldots, I_s$ so that there are sets $J_\ell \subset I_\ell$ with $\operatorname{card} J_\ell = t$ and $J_1 \prec J_2 \prec \cdots \prec J_s$.*

Proof. We may assume that $\operatorname{card} I_\ell = st$. Let us enumerate $I_\ell = \{i_{1,\ell}, \ldots$ $\ldots, i_{st,\ell}\}$ where $i_{a,\ell} < i_{b,\ell}$ if $a < b$. We can relabel the sets I_ℓ in order to ensure that

$$\forall k \geq 1, \quad i_{t,1} \leq i_{t,k}, \quad \forall k \geq 2, \quad i_{2t,2} \leq i_{2t,k}$$

and more generally, for any $\ell < s$ that

$$\forall k \geq \ell, \qquad i_{\ell t,\ell} \leq i_{\ell t,k} \tag{2.2}$$

We then define

$$J_\ell = \{i_{(\ell-1)t+1,\ell}, \ldots, i_{\ell t,\ell}\}.$$

To see that for $1 \leq \ell < s$ we have $J_\ell \prec J_{\ell+1}$ we use (2.2) for $k = \ell + 1$, so that $i_{\ell t,\ell} \leq i_{\ell t,\ell+1} < i_{\ell t+1,\ell+1}$. □

The reader might observe that it would in fact suffice to assume that $\mathrm{card} I_\ell \geq s(t-1)+1$; but this refinement yields no benefits for our purposes.

Throughout the paper, given an integer $\tau \leq 2^n$, we write

$$S_{n,\tau} = \{\boldsymbol{z} \in T;\ z_n \neq \tau\} \tag{2.3}$$

so that its complement $S^c_{n,\tau}$ is the set $\{\boldsymbol{z} \in T;\ z_n = \tau\}$. Thus on the set $S_{n,\tau}$ we forbid the n^{th} coordinate of $\boldsymbol{z}$ to be τ while on $S^c_{n,\tau}$ we force it to be τ.

Proposition 2.4. *Consider sets $X_1, \ldots, X_q \subset T$, and assume that for each $\ell \leq q$ the set X_ℓ is I_ℓ-thin, for a certain set I_ℓ with $\mathrm{card} I_\ell \geq 3q$. Then for each n and each integer $\tau \leq 2^n$ we have*

$$S^c_{n,\tau} \not\subset \bigcup_{\ell \leq q} X_\ell. \tag{2.4}$$

Proof. We use Lemma 2.3 for $s = q$ and $t = 3$ to produce sets $J_\ell \subset I_\ell$ with $J_1 \prec J_2 \prec \cdots \prec J_q$ and $\mathrm{card} J_\ell = 3$. Let $J_\ell = (m_\ell, n_\ell, r_\ell)$, and then $r_\ell \leq m_{\ell+1}$ since $J_\ell \prec J_{\ell+1}$.

To explain the idea (on which the paper ultimately relies) let us prove first that $T \not\subset \bigcup_{\ell \leq q} X_\ell$. We make an inductive construction to avoid in turn the sets X_ℓ. We start with any $A_1 \in \mathcal{A}_{m_1}$. Since X_1 is (m_1, n_1)-thin, we can find $C_1 \in \mathcal{A}_{n_1}$ with $C_1 \subset A_1$ and $C_1 \cap X_1 = \varnothing$. Since $n_1 \leq m_2$ we can find $A_2 \in \mathcal{A}_{m_2}$ and $A_2 \subset C_1$, and we continue in this manner. The set C_q does not meet any of the sets X_ℓ.

To prove (2.4), we must ensure that $C_q \cap S^c_{n,\tau} \neq \varnothing$. The fundamental fact is that at each stage we have *two* chances to avoid X_ℓ, using either that X_ℓ is (m_ℓ, n_ℓ)-thin or that it is (n_ℓ, r_ℓ)-thin. The details of the construction depend on the "position" of n with respect to the sets J_ℓ. Rather that enumerating the cases, we explain what happens when $m_1 < n \leq r_1$, and this should make what to do in the other cases obvious.

Case 1. We have $m_1 < n \leq n_1$. Since $S_{n,\tau} \in \mathcal{B}_n \subset \mathcal{B}_{n_1}$, we can choose $A_1 \in \mathcal{A}_{n_1}$ with $A_1 \subset S^c_{n,\tau}$. Since X_1 is (n_1, r_1)-thin, we choose $C_1 \in \mathcal{A}_{r_1}$ with

$C_1 \subset A_1$ and $C_1 \cap X_1 = \varnothing$. We then continue as before, choosing $A_2 \subset C_1$, $A_2 \in \mathcal{A}_{m_2}$, etc.

Case 2. We have $m_1 < n_1 < n \le r_1$. We choose any $A_1 \in \mathcal{A}_{m_1}$. Since X is (m_1, n_1)-thin, we can choose $C_1 \in \mathcal{A}_{n_1}$ with $C_1 \subset A_1$ and $C_1 \cap X_1 = \varnothing$. It is obvious from (2.1) that, since $n_1 < n$, we have $C_1 \cap S^c_{n,\tau} \ne \varnothing$. Since $C_1 \cap S^c_{n,\tau} \in \mathcal{B}_n \subset \mathcal{B}_{r_1} \subset \mathcal{B}_{m_2}$, we can find $A_2 \subset C_1 \cap S^c_{n,\tau}$, $A_2 \in \mathcal{A}_{m_2}$, and we continue as before. □

Definition 2.5. Given $\varepsilon > 0$, a submeasure ν on an algebra $\mathcal{B}$ is called ε-*exhaustive* if for each disjoint sequence (E_n) of $\mathcal{B}$ we have $\limsup_{n\to\infty} \nu(E_n) \le \varepsilon$.

THEOREM 2.6 (Roberts). *For each q there exists a submeasure ν on T such that*

$$\forall n, \forall \tau \le 2^n, \quad \nu(S^c_{n,\tau}) = 1, \tag{2.5}$$

$$\nu \textit{ is } \frac{1}{q+1}\textit{-exhaustive.} \tag{2.6}$$

Of course, (2.5) implies that ν is not uniformly exhaustive. Let us consider the class $\mathcal{C}$ of subsets X of T that are I-thin (for a set I depending on X) with $\mathrm{card} I \ge 3q$. For $B \in \mathcal{B}$ we define

$$\nu(B) = \min\left(1, \inf\left\{\frac{1}{q+1}\mathrm{card}F;\ F \subset \mathcal{C};\ B \subset \cup F\right\}\right), \tag{2.7}$$

where F runs over the finite subsets of $\mathcal{C}$ and $\cup F$ denotes the union of F. It is obvious that ν is a submeasure, and (2.5) is an immediate consequence of Proposition 2.4.

To prove (2.6) it suffices, given a disjoint sequence (E_n) of $\mathcal{B}$, to prove that $\liminf_{n\to\infty} \nu(E_n) \le 1/(1+q)$.

For $X \subset T$, let us define

$$(X)_m = \bigcap\{B \in \mathcal{B}_m;\ B \supset X\} = \bigcup\{A,\ A \in \mathcal{A}_m,\ A \cap X \ne \varnothing\}. \tag{2.8}$$

Since each algebra $\mathcal{B}_m$ is finite, by taking a subsequence we can assume that for some integers $m(n)$ we have $E_n \in \mathcal{B}_{m(n)}$, while

$$\forall k > n, \quad (E_k)_{m(n)} = (E_{n+1})_{m(n)}. \tag{2.9}$$

We claim that for each $k > n+1$, E_k is $(m(n), m(n+1))$-thin. To prove this, consider $A \in \mathcal{A}_{m(n)}$. If $A \cap E_k = \varnothing$, any $A' \in \mathcal{A}_{m(n+1)}$ with $A' \subset A$ satisfies $A' \cap E_k = \varnothing$. Otherwise $A \subset (E_k)_{m(n)} = (E_{n+1})_{m(n)}$ by (2.9). Therefore, $E_{n+1} \cap A \ne \varnothing$. Since $E_{n+1} \in \mathcal{B}_{m(n+1)}$, we can find $A' \in \mathcal{A}_{m(n+1)}$ with $A' \subset A$ and $A' \subset E_{n+1}$. But then $A' \cap E_k = \varnothing$ since E_{n+1} and E_k are disjoint. This proves the claim.

It follows that for $n \ge 3q+1$, E_n is I-thin for $I = (m(1), \ldots, m(3q))$ and thus $E_n \in \mathcal{C}$, so that $\nu(E_n) \le 1/(q+1)$. □

3. Farah

In [6] I. Farah constructs for each ε an ε-exhaustive submeasure ν that is also pathological, in the sense that every measure that is absolutely continuous with respect to ν is zero. In this paper, we learned several crucial technical ideas, that are essential for our approach. The concepts and the techniques required to prove Proposition 3.5 below are essentially all Farah's.

A class $\mathcal{C}$ of *weighted sets* is a subset of $\mathcal{B} \times \mathbb{R}^+$. For a finite subset $F = \{(X_1, w_1), \ldots, (X_n, w_n)\}$ of $\mathcal{C}$, we write throughout the paper

$$w(F) = \sum_{i \le n} w_i; \quad \cup F = \bigcup_{i \le n} X_i, \tag{3.1}$$

and for $B \in \mathcal{B}$ we set

$$\varphi_{\mathcal{C}}(B) = \inf\{w(F);\, B \subset \cup F\}. \tag{3.2}$$

This is well defined provided there exists a finite set $F \subset \mathcal{C}$ for which $T \subset \cup F$. It is immediate to check that $\varphi_{\mathcal{C}}$ is a submeasure. This construction generalizes (2.7). It is generic; for a submeasure ν, we have $\nu = \varphi_{\mathcal{C}}$ where $\mathcal{C} = \{(B, \nu(B));\, B \in \mathcal{B}\}$. Indeed, it is obvious that $\varphi_{\mathcal{C}} \le \nu$, and the reverse inequality follows by subadditivity of ν.

For technical reasons, when dealing with classes of weighted sets, we find it convenient to keep track for each pair (X, w) of a distinguished finite subset I of $\mathbb{N}^*$. For this reason we define a class of *marked weighted sets* as a subset of $\mathcal{B} \times \mathcal{F} \times \mathbb{R}^+$, where $\mathcal{F}$ denotes the collection of finite subsets of $\mathbb{N}^*$.

For typographical convenience we write

$$\alpha(k) = \frac{1}{(k+5)^3} \tag{3.3}$$

and we fix a sequence $(N(k))$ to be specified later. The specific choice is anyway completely irrelevant, what matters is that this sequence increases fast enough. In fact, there is nothing magic about the choice of $\alpha(k)$ either. Any sequence such that $\sum_k k\alpha(k) < \infty$ would do. We like to stress than none of the numerical quantities occurring in our construction plays an essential role. These are all simple choices that are made for convenience. No attempts whatsoever have been made to make optimal or near optimal choices. Let us also point out that for the purpose of the present section it would work just fine to take $\alpha(k) = (k+5)^{-1}$, and that the reasons for taking a smaller value will become clear only in the next section. For $k \ge 1$ we define the class $\mathcal{D}_k$ of marked weighted sets by

$$\mathcal{D}_k = \Big\{ (X, I, w);\, \exists (\tau(n))_{n \in I},\, X = \bigcap_{n \in I} S_{n,\tau(n)};\, \mathrm{card} I \le N(k), \quad w = 2^{-k} \left(\frac{N(k)}{\mathrm{card} I} \right)^{\alpha(k)} \Big\}. \tag{3.4}$$

The most important part of $\mathcal{D}_k$ consists of the triples (X, I, w) where $\mathrm{card}I = N(k)$ and $w = 2^{-k}$. The purpose of the relation

$$w = 2^{-k}\,(N(k)/\mathrm{card}I)^{\alpha(k)}$$

is to allow the crucial Lemma 3.1 below. To understand the relation between the different classes $\mathcal{D}_k$ it might help to observe the following. Whenever X and I are as in (3.4) and whenever $N(k) \geq \mathrm{card}I$ we have $(X, I, w_k) \in \mathcal{D}_k$ for $w_k = 2^{-k}\,(N(k)/\mathrm{card}I)^{\alpha(k)}$. If we assume, as we may, that the sequence $2^{-k}N(k)^{\alpha(k)}$ increases, we see that the sequence (w_k) increases. It is then the smallest possible value of k that gives the smallest possible value of w_k. This is the only value that matters, as will be apparent from the way we use the classes $\mathcal{D}_k$; see the formula (3.7) below. Let us also note that for each k there is a finite subset F of $\mathcal{D}_k$ such that $T \subset \cup F$.

Given a subset J of $\mathbb{N}^*$ we say that a subset X of T *depends only on the coordinates of rank in* J if whenever $\boldsymbol{z}, \boldsymbol{z}' \in T$ are such that $z_n = z'_n$ for every $n \in J$, we have $\boldsymbol{z} \in T$ if and only if $\boldsymbol{z}' \in T$. Equivalently, we sometimes say that such a set *does not depend on the coordinates of rank in* $J^c = \mathbb{N}^* \setminus J$. One of the key ideas of the definition of $\mathcal{D}_k$ is the following simple fact.

LEMMA 3.1. *Consider* $(X, I, w) \in \mathcal{D}_k$ *and* $J \subset \mathbb{N}^*$. *Then there is* $(X', I', w') \in \mathcal{D}_k$ *such that* $X \subset X'$, X' *depends only on the coordinates in* J *and*

$$w' = w\left(\frac{\mathrm{card}I}{\mathrm{card}I \cap J}\right)^{\alpha(k)}. \tag{3.5}$$

Since $\alpha(k)$ is small, w' is not really larger than w unless $\mathrm{card}I \cap J \ll \mathrm{card}I$. In particular, since $\alpha(k) \leq 1/2$,

$$\mathrm{card}I \cap J \geq \frac{1}{4}\mathrm{card}I \Longrightarrow w' \leq 2w. \tag{3.6}$$

Proof. We define (X', I', w') by (3.5), $I' = I \cap J$, and

$$X' = \bigcap_{n \in I'} S_{n,\tau(n)},$$

where $\tau(n)$ is as in (3.4). $\square$

A class of marked weighted sets is a subset of $\mathcal{B} \times \mathcal{F} \times \mathbb{R}^+$. By projection onto $\mathcal{B} \times \mathbb{R}^+$, to each class $\mathcal{C}$ of marked weighted sets, we can associate a class $\mathcal{C}^*$ of weighted sets. For a class $\mathcal{C}$ of marked weighted sets, we then define $\varphi_{\mathcal{C}}$ as $\varphi_{\mathcal{C}^*}$ using (3.2). As there is no risk of confusion, we will not distinguish between $\mathcal{C}$ and $\mathcal{C}^*$ at the level of notation. We define

$$\mathcal{D} = \bigcup_{k \geq 1} \mathcal{D}_k; \quad \psi = \varphi_{\mathcal{D}}. \tag{3.7}$$

Proposition 3.2. *Let us assume that*

$$N(k) \geq 2^{k+6}(2^{k+5})^{1/\alpha(k)}. \tag{3.8}$$

Then $\psi(T) \geq 2^5$. *Moreover* ψ *is pathological in the sense that if a measure* μ *on* $\mathcal{B}$ *is absolutely continuous with respect to* ψ, *then* $\mu = 0$. *Finally, if* ν *is a submeasure with* $\nu(T) > 0$ *and* $\nu \leq \psi$, ν *is not uniformly exhaustive.*

Pathological submeasures seem to have been constructed first implicitly in [7] and explicitly in [14].

Proof. To prove that $\psi(T) \geq 2^5$, we consider a finite subset F of $\mathcal{D}$, with $w(F) < 2^5$, and we prove that $T \not\subset \cup F$. For $k \geq 1$ consider disjoint sets $F_k \subset F \cap \mathcal{D}_k$ such that $F = \cup_{k\geq 1} F_k$. (We have not proved that the classes $\mathcal{D}_k$ are disjoint.) For $(X, I, w) \in \mathcal{D}_k$, we have $w \geq 2^{-k}$, so that $\mathrm{card} F_k \leq 2^{k+5}$ since $w(F_k) \leq w(F) < 2^5$. Also we have

$$2^{-k}\left(\frac{N(k)}{\mathrm{card} I}\right)^{\alpha(k)} = w \leq w(F) \leq 2^5,$$

so that $\mathrm{card} I \geq (2^{k+5})^{-1/\alpha(k)} N(k) := c(k)$. Under (3.8) we have $c(k) \geq 2^{k+6}$. Let us enumerate F as a sequence $(X_r, I_r, w_r)_{r \leq r_0}$ (where $r_0 = \mathrm{card} F$) in such a way that if $(X_r, I_r, w_r) \in F_{k(r)}$, the sequence $k(r)$ is nondecreasing. Since

$$\sum_{\ell < k} \mathrm{card} F_\ell \leq \sum_{\ell < k} 2^{\ell+5} < 2^{k+5},$$

we see that $r \geq 2^{k+5}$ implies $k(r) \geq k$ and thus $\mathrm{card} I_r \geq c(k)$. Assuming (3.8) we now prove that $\mathrm{card} I_r \geq r + 1$. Indeed this is true if $r < 2^6$ because $\mathrm{card} I \geq c(1) \geq 2^7$, and if $r \geq 2^6$ and if k is the largest integer with $r \geq 2^{k+5}$, then $c(k) \geq 2^{k+6} \geq r + 1$. Since $\mathrm{card} I_r \geq r + 1$, we can then pick inductively integers $i_r \in I_r$ that are all different. If $X_r = \bigcap_{n \in I_r} S_{n,\tau_r(n)}$, any $\boldsymbol{z}$ in T with $z_{i_r} = \tau_r(i_r)$ for $r \leq r_0$ does not belong to any of the sets X_r, and thus $\cup F \neq T$. This proves that $\psi(T) \geq 2^5$.

We prove now that ψ is pathological. Consider a measure μ on $\mathcal{B}$ and $\varepsilon > 0$, and assume that there exists k such that

$$\psi(B) \leq 2^{-k} \Longrightarrow \mu(B) \leq \varepsilon.$$

For each $\boldsymbol{\tau} = (\tau(n))_{n \leq N(k)}$, we consider the set

$$X_{\boldsymbol{\tau}} = \bigcap_{n \leq N(k)} S_{n,\tau(n)}$$

so that if $I = \{1, \ldots, N(k)\}$ we have $(X_{\boldsymbol{\tau}}, I, 2^{-k}) \in \mathcal{D}_k$ and thus $\psi(X_{\boldsymbol{\tau}}) \leq 2^{-k}$, and hence $\mu(X_{\boldsymbol{\tau}}) \leq \varepsilon$.

Let us denote by Av the average over all values of $\boldsymbol{\tau}$, so that

$$\int \mathrm{Av}(1_{X_{\boldsymbol{\tau}}}(\boldsymbol{z})) d\mu(\boldsymbol{z}) = \mathrm{Av} \int 1_{X_{\boldsymbol{\tau}}}(\boldsymbol{z}) d\mu(\boldsymbol{z}) = \mathrm{Av}\mu(X_{\boldsymbol{\tau}}) \leq \varepsilon. \tag{3.9}$$

It should be clear that the quantity $\mathrm{Av}(1_{X_\tau}(\boldsymbol{z}))$ is independent of $\boldsymbol{z}$. Its value a_k satisfies

$$a_k = \int \mathrm{Av} 1_{X_\tau}(\boldsymbol{z}) d\lambda(\boldsymbol{z}) = \mathrm{Av} \int 1_{X_\tau}(\boldsymbol{z}) d\lambda(\boldsymbol{z})$$

where λ denotes the uniform measure on T. Now

$$\int 1_{X_\tau}(\boldsymbol{z}) d\lambda(\boldsymbol{z}) = \lambda(X_{\boldsymbol{\tau}}) = \prod_{n \le N(k)} (1 - 2^{-n})$$

is bounded below independently of k, so that a_k is bounded below independently of k. Finally (3.9) yields

$$\varepsilon \ge \int \mathrm{Av}(1_{X_\tau}(\boldsymbol{z})) d\mu(\boldsymbol{z}) = a_k \mu(T),$$

and since ε is arbitrary this shows that $\mu(T) = 0$.

Consider finally a submeasure $\nu \le \psi$, with $\nu(T) > 0$. We will prove that ν is not uniformly exhaustive, by showing that $\liminf_{n\to\infty} \inf_{\tau \le 2^n} \nu(S^c_{n,\tau}) > 0$. (It is known by general arguments, using in particular the deep Kalton-Roberts theorem [11], that a submeasure that is pathological cannot be uniformly exhaustive. The point of the argument is to show that, in the present setting, there is a very simple reason why this is true.) To see this, consider $I \subset \mathbb{N}^*$, and for $n \in I$ let $\tau(n) \le 2^n$. Then

$$T \subset \bigcup_{n \in I} S^c_{n,\tau(n)} \cup \left(\bigcap_{n \in I} S_{n,\tau(n)} \right)$$

so that by subadditivity we have

$$\begin{aligned} \nu(T) &\le \sum_{n \in I} \nu(S^c_{n,\tau(n)}) + \nu \left(\bigcap_{n \in I} S_{n,\tau(n)} \right) \\ &\le \sum_{n \in I} \nu(S^c_{n,\tau(n)}) + \psi \left(\bigcap_{n \in I} S_{n,\tau(n)} \right). \end{aligned}$$

The definition of $\mathcal{D}$ shows that if k is such that if $2^{-k} \le \nu(T)/2$ and $\mathrm{card} I = N(k)$, the last term is $\le \nu(T)/2$, and thus $\sum_{n \in I} \nu(S^c_{n,\tau(n)}) \ge \nu(T)/2$. This proves that $\limsup_{n\to\infty} \inf_{\tau \le 2^n} \nu(S^c_{n,\tau}) > 0$ and thus that ν is not uniformly exhaustive. $\square$

At the start of the effort that culminates in the present paper, it was not clear whether the correct approach would be, following Roberts, to attempt to directly construct an exhaustive submeasure that is not uniformly exhaustive, or whether it would be, following Farah, to construct an exhaustive measure dominated by a pathological submeasure. The fact, shown in Proposition 3.2, that a submeasure $\nu \le \psi$ is not uniformly exhaustive for "transparent" reasons pointed out that a way to merge these apparently different approaches would be to look for an exhaustive submeasure $\nu \le \psi$. This approach has succeeded, and as a warm up we will prove the following.

THEOREM 3.3. *If the sequence $N(k)$ is chosen as in* (3.8), *for each $\varepsilon > 0$ there is an ε-exhaustive submeasure $\nu \le \psi$.*

This result is of course much weaker than Theorem 1.1. We present its proof for pedagogical reasons. Several of the key ideas required to prove Theorem 1.1 will be needed here, and should be much easier to grasp in this simpler setting.

Given $A \in \mathcal{A}_m$, let us define the map $\pi_A : T \to A$ as follows: If $\tau_1, \ldots, \tau_m$ are such that

$$\boldsymbol{z} \in A \Longleftrightarrow \forall i \le m, \quad z_i = \tau_i$$

then for $\boldsymbol{z} \in T$ we have $\pi_A(\boldsymbol{z}) = \boldsymbol{y}$ where

$$\boldsymbol{y} = (\tau_1, \ldots, \tau_m, z_{m+1}, \ldots).$$

Definition 3.4 (Farah). Given $m < n$, we say that a set $X \subset T$ is (m, n, ψ)-thin if

$$\forall A \in \mathcal{A}_m, \quad \exists C \in \mathcal{B}_n, \quad C \subset A, \quad C \cap X = \varnothing, \quad \psi(\pi_A^{-1}(C)) \ge 1.$$

The idea is now that in each atom of rank m, X has a $\mathcal{B}_n$-measurable hole that is large with respect to ψ. Of course, we cannot require that $\psi(C) \ge 1$ because $\psi(C) \le \psi(A)$ will be small, and one should think of $\psi(\pi_A^{-1}(C))$ as measuring the "size of C with respect to A".

Obviously, if $n' \ge n$ and if X is (m, n, ψ)-thin, it is also (m, n', ψ)-thin. For a subset I of $\mathbb{N}^*$, we say that X is (I, ψ)-thin if it is (m, n, ψ)-thin whenever $m, n \in I$, $m < n$. By the previous observation, it suffices that this should be the case when m and n are consecutive elements of I.

Consider a given integer q and consider an integer b, to be determined later. Consider the class $\mathcal{F}$ of marked weighted sets defined as

$$\mathcal{F} = \{(X, I, w);\ X \text{ is } (I, \psi)\text{-thin, card} I = b,\ w = 2^{-q}\}.$$

We define

$$\nu = \varphi_{\mathcal{F} \cup \mathcal{D}},$$

where $\mathcal{D}$ is the class (2.9). Thus $\nu \le \psi = \varphi_{\mathcal{D}}$, so it is pathological.

PROPOSITION 3.5. *The submeasure ν is 2^{-q}-exhaustive.*

PROPOSITION 3.6. *Assuming*

$$b = 2^{2q+10}, \tag{3.10}$$

we have $\nu(T) \ge 2^4$.

Both these results assume that (3.8) holds. This condition is assumed without further mention in the rest of the paper.

We first prove Proposition 3.5. Again, the arguments are due to I. Farah [6] and are of essential importance.

LEMMA 3.7. *Consider a sequence* $(E_i)_{i\geq 1}$ *of* $\mathcal{B}$ *and assume that*

$$\forall n, \quad \psi\Bigl(\bigcup_{i\leq n} E_i\Bigr) < 1.$$

Assume that for a certain $m \geq 1$, *the sets* E_i *do not depend on the coordinates of rank* $\leq m$. *Then for each* $\alpha > 0$, *there is a set* $C \in \mathcal{B}$, *that does not depend on the coordinates of rank* $\leq m$, *and satisfies that* $\psi(C) \leq 2$ *and*

$$\forall i \geq 1, \quad \psi(E_i \setminus C) \leq \alpha.$$

Proof. By definition of ψ for each n we can find a finite set $F_n \subset \mathcal{D}$ with $w(F_n) < 1$ and $\bigcup_{i\leq n} E_i \subset \cup F_n$. For an integer $r \geq m+2$, let

$$F_n^r = \{(X,I,w) \in F_n;\ \mathrm{card} I \cap \{m+1,\ldots,r-1\} < \mathrm{card} I/2; \qquad \mathrm{card} I \cap \{m+1,\ldots,r\} \geq \mathrm{card} I/2\}, \tag{3.11}$$

so that the sets F_n^r are disjoint as r varies. We use Lemma 3.1 and (3.6) with $J = I \cap \{m+1,\ldots,r\}$ to obtain for each $(X,I,w) \in F_n^r$ an element (X',I',w') of $\mathcal{D}$ such that $X' \supset X$, $w' \leq 2w$, and X' depends only on the coordinates of rank in $\{m+1,\ldots,r\}$ (or, equivalently, $I' \subset \{m+1,\cdots,r\}$). We denote by $F_n'^r$ the collection of the sets (X',I',w') as $(X,I,w) \in F_n^r$. Thus $\cup F_n'^r \supset \cup F_n^r$, and $w(F_n'^r) \leq 2w(F_n^r)$.

Consider an integer i, and j such that $E_i \in \mathcal{B}_j$. We prove that for $n \geq i$ we have $E_i \subset \bigcup_{r\leq j} \cup F_n'^r$. Otherwise, since both these sets depend only on the coordinates of rank in $\{m+1,\ldots,j\}$, we can find a nonempty set A depending only on those coordinates with $A \subset E_i \setminus \bigcup_{r\leq j} \cup F_n'^r$, and thus $A \subset E_i \setminus \bigcup_{r\leq j} \cup F_n^r$. Since $E_i \subset \cup F_n$, we have $A \subset \cup F^\sim$, where $F^\sim = F_n \setminus \bigcup_{r\leq j} F_n^r$. Now, by definition of F_n^r, if $(X,I,w) \in F^\sim$, $\mathrm{card}(I \setminus \{m+1,\ldots,j\}) \geq \mathrm{card} I/2$. Again by Lemma 3.1, now with $J = \{m+1,\ldots,j\}^c$, we can find (X',I',w') in $\mathcal{D}$ with $w' \leq 2w$ and $X' \supset X$, where X' does not depend on the coordinates of rank in $\{m+1,\ldots,j\}$. Let F' be the collection of these triples (X',I',w'), so that $F' \subset \mathcal{D}$ and $w(F') \leq 2w(F_n) \leq 2$. Now $\cup F' \supset \cup F^\sim \supset A$, and since $\cup F'$ does not depend on the coordinates in $\{m+1,\ldots,r\}$, while A is nonempty and determined by these coordinates, we have $\cup F' = T$. But this would imply that $\psi(T) \leq 2$, while we have proved that $\psi(T) \geq 2^5$.

Thus $E_i \subset \bigcup_{r\leq j} \cup F_n'^r$. For (X,I,w) in $F_n'^r$, we have $I \subset \{m+1,\ldots,r\}$. Under (3.8) we have that if $(X,I,w) \in \mathcal{D}_k \cap F_n'^r$ then

$$w = 2^{-k}\left(\frac{N(k)}{\mathrm{card} I}\right)^{\alpha(k)} \geq \frac{2^5}{\mathrm{card} I^{\alpha(k)}} \geq \frac{2^5}{r^{\alpha(k)}}, \tag{3.12}$$

which shows (since $w(F_n'^r) \leq 2$) that k remains bounded independently of n. Since moreover $I \subset \{m+1, \cdots, r\}$ there exists a finite set $\mathcal{D}^r \subset \mathcal{D}$ such that $F_n'^r \subset \mathcal{D}^r$ for all n. Then, by taking a subsequence if necessary, we can assume that for each r the sets $F_n'^r$ are eventually equal to a set F^r. For each triplet (X, I, w) in F^r, the set X depends only on the coordinates of rank in $\{m+1, \ldots, r\}$, and it should be obvious that $\sum_{r \geq m} w(F^r) \leq 2$ and $E_i \subset \bigcup_{r \leq j} \cup F^r$ (whenever j is such that $E_i \in \mathcal{B}_j$).

Consider r_0 such that $\sum_{r > r_0} w(F^r) \leq \alpha$, and let $C = \bigcup_{r \leq r_0} \cup F^r$. Thus $C \in \mathcal{B}$, C does not depend on the coordinates of rank $\leq m$ and $\psi(C) \leq \sum_{r \leq r_0} w(F^r) \leq 2$. Moreover, since $E_i \subset \bigcup_{r \leq j} \cup F^r$ whenever j is large enough that $E_i \in \mathcal{B}_j$, we have

$$E_i \setminus C \subset \bigcup_{r_0 < r \leq j} \cup F^r,$$

so that $\psi(E_i \setminus C) \leq \sum_{r > r_0} w(F^r) \leq \alpha$. □

Lemma 3.8 (Farah). *Consider $\alpha > 0$, $B \in \mathcal{B}_m$, and a disjoint sequence (E_i) of $\mathcal{B}$. Then there exists $n > m$, a set $B' \subset B$, $B' \in \mathcal{B}_n$, so that B' is (m, n, ψ)-thin and $\limsup_{i \to \infty} \psi((B \cap E_i) \setminus B') \leq \alpha$.*

Proof. Consider $\alpha' = \alpha / \mathrm{card}\mathcal{A}_m$. Consider $A \in \mathcal{A}_m$, $A \subset B$.

Case 1. $\exists p$; $\psi\left(\pi_A^{-1}\left(\bigcup_{i \leq p} E_i\right)\right) \geq 1$.

We set $C' = C'(A) = A \setminus \bigcup_{i \leq p} E_i$, so that $\psi(\pi_A^{-1}(A \setminus C')) \geq 1$ and $(A \cap E_i) \setminus C' = \varnothing$ for $i > p$.

Case 2. $\forall p$; $\psi\left(\pi_A^{-1}\left(\bigcup_{i \leq p} E_i\right)\right) < 1$.

The sets $\pi_A^{-1}(E_i)$ do not depend on the coordinates of rank $\leq m$ and so by Lemma 3.7 we can find a set $C \in \mathcal{B}$, that does not depend on the coordinates of rank $\leq m$, with $\psi(C) \leq 2$ and $\limsup_{i \to \infty} \psi\left(\pi_A^{-1}(E_i) \setminus C\right) \leq \alpha'$. Let $C' = C'(A) = \pi_A(C) = A \cap C \subset A$. Since C does not depend on the coordinates of rank $\leq m$, we have $C = \pi_A^{-1}(C')$ so that $\psi\left(\pi_A^{-1}(C')\right) \leq 2$. Since $\pi_A(\boldsymbol{z}) = \boldsymbol{z}$ for $\boldsymbol{z} \in A$, we have

$$(A \cap E_i) \setminus C' \subset \pi_A^{-1}(E_i) \setminus C$$

so that

$$\limsup_{i \to \infty} \psi((A \cap E_i) \setminus C') \leq \limsup_{i \to \infty} \psi\left(\pi_A^{-1}(E_i) \setminus C\right) \leq \alpha'.$$

Let us now define

$$B' = \bigcup \{C' = C'(A);\ A \in \mathcal{A}_m,\ A \subset B\},$$

so that

$$\limsup_{i \to \infty} \psi((B \cap E_i) \setminus B') \leq \sum \limsup_{i \to \infty} \psi((A \cap E_i) \setminus C') \leq \alpha' \mathrm{card}\mathcal{A}_m \leq \alpha, \tag{3.13}$$

where the summation is over $A \subset B$, $A \in \mathcal{A}_m$.

Consider n such that $B' \in \mathcal{B}_n$. To prove that B' is (m,n,ψ)-thin it suffices to prove that $\psi\left(\pi_A^{-1}(A \setminus C')\right) \geq 1$ whenever $A \in \mathcal{A}_m$, $A \subset B$, because $B' \cap A = C'$, and thus $A \setminus B' = A \setminus C'$. This was already done in case 1. In case 2, we observe that

$$\psi\left(\pi_A^{-1}(A \setminus C')\right) = \psi\left(\pi_A^{-1}(C')^c\right)$$

and that

$$2^5 \leq \psi(T) \leq \psi\left(\pi_A^{-1}(C')\right) + \varphi\left(\pi_A^{-1}(C')^c\right) \leq 2 + \psi\left(\pi_A^{-1}(C')^c\right). \qquad \square$$

Proof of Proposition 3.5 (Farah). Consider a disjoint sequence $(E_i)_{i\geq 1}$ of $\mathcal{B}$. Consider $\alpha > 0$. Starting with $B_0 = T$, we use Lemma 3.8 to recursively construct sets $B_\ell \in \mathcal{B}$ and integers $(n_1, n_2, \dots)$ such that B_ℓ is (I_ℓ, ψ)-thin for $I_\ell = \{1, n_1, n_2, \dots, n_\ell\}$ and $B_\ell \subset B_{\ell-1}$,

$$\limsup_{i\to\infty} \psi((E_i \cap B_{\ell-1}) \setminus B_\ell) \leq \alpha. \tag{3.14}$$

We have, since $B_0 = T$,

$$E_i \setminus B_\ell \subset \bigcup_{m\leq \ell} ((E_i \cap B_{m-1}) \setminus B_m),$$

and the subadditivity of ψ then implies that

$$\psi(E_i \setminus B_\ell) \leq \sum_{m\leq \ell} \psi((E_i \cap B_{m-1}) \setminus B_m)$$

and thus

$$\limsup_{i\to\infty} \psi(E_i \setminus B_\ell) \leq \alpha\ell. \tag{3.15}$$

For $\ell = b$ (or even $\ell = b - 1$) (where b is given by (3.10)) the definition of $\mathcal{F}$ shows that $(B_\ell, I_\ell, 2^{-q}) \in \mathcal{F}$, and thus $\nu(B_\ell) \leq 2^{-q}$. Since $\nu \leq \psi$, we have

$$\nu(E_i) \leq \nu(B_\ell) + \psi(E_i \setminus B_\ell) \leq 2^{-q} + \psi(E_i \setminus B_\ell),$$

and (3.15) shows that

$$\limsup_{i\to\infty} \nu(E_i) \leq 2^{-q} + \alpha\ell.$$

Since α is arbitrary, the proof is complete. $\square$

We turn to the proof of Proposition 3.6. Considering $F_1 \subset \mathcal{F}$ and $F_2 \subset \mathcal{D}$, we want to show that

$$w(F_1) + w(F_2) < 2^4 \Longrightarrow T \not\subset (\cup F_1) \cup (\cup F_2).$$

Since $w \geq 2^{-q}$ for $(X, I, w) \in \mathcal{F}$, we have $w(F_1) \geq 2^{-q}\mathrm{card}F_1$, so that $\mathrm{card}F_1 \leq 2^{q+4}$. We appeal to Lemma 2.3 with $s = \mathrm{card}F_1$ and $t = b2^{-q-4}$ (which is an integer by (3.10)) to see that we can enumerate $F_1 = (X_\ell, I_\ell, w_\ell)_{\ell \le s}$ and find sets $J_1 \prec J_2 \prec \cdots \prec J_s$ with $\mathrm{card} J_\ell = t$ and $J_\ell \subset I_\ell$.

Let us enumerate

$$J_\ell = \{i_{1,\ell}, \dots, i_{t,\ell}\}. \tag{3.16}$$

An essential idea is that each of the pairs $\{i_{u,\ell}, i_{u+1,\ell}\}$ for $1 \le u \le t-1$ gives us a chance to avoid X_ℓ. We are going for each ℓ to choose one of these chances using a counting argument. For

$$\boldsymbol{u} = (u(\ell))_{\ell \le s} \in \{1, \dots, t-1\}^s, \tag{3.17}$$

we define the set

$$W(\boldsymbol{u}) = \bigcup_{\ell \le s}]i_{u(\ell),\ell}, i_{u(\ell)+1,\ell}],$$

where for integers $m < n$ we define $]m, n] = \{m+1, \dots, n\}$.

We consider the quantity

$$S(\boldsymbol{u}) = \sum \{w;\ (X, I, w) \in F_2,\ \mathrm{card}(I \cap W(\boldsymbol{u})) \ge \mathrm{card} I/2\}.$$

We will choose $\boldsymbol{u}$ so that $S(\boldsymbol{u})$ is small. Let us denote by Av the average over all possible choices of $\boldsymbol{u}$. Then, for any set I, by linearity of Av, we have

$$\begin{aligned} \mathrm{Av}(\mathrm{card}(I \cap W(\boldsymbol{u}))) &= \sum_{\ell \le s} \mathrm{Av}(\mathrm{card}(I \cap]i_{u(\ell),\ell}, i_{u(\ell)+1,\ell}])) \\ &= \sum_{\ell \le s} \frac{1}{t-1} \mathrm{card}(I \cap]i_{1,\ell}, i_{t,\ell}]) \le \frac{1}{t-1} \mathrm{card} I. \end{aligned}$$

Thus, by Markov's inequality,

$$\mathrm{Av}(1_{\{\mathrm{card}(I \cap W(\boldsymbol{u})) \ge \mathrm{card} I/2\}}) \le \frac{2}{t-1}$$

and, using linearity of average, we get

$$\mathrm{Av}(S(\boldsymbol{u})) \le \frac{2}{t-1} w(F_2) \le \frac{2^5}{t-1} \le \frac{2^{q+10}}{b}.$$

Thus, we can find $\boldsymbol{u}$ such that $S(\boldsymbol{u}) \le 2^{q+10}/b$. We fix this value of $\boldsymbol{u}$ once and for all. To lighten notation we set

$$W = W(\boldsymbol{u}); \quad m_\ell = i_{u(\ell),\ell}, \quad n_\ell = i_{u(\ell)+1,\ell}, \quad W_\ell =]m_\ell, n_\ell] \tag{3.18}$$

so that $W = \bigcup_{\ell \le s} W_\ell$, and $n_\ell \le m_{\ell+1}$ since $n_\ell \in J_\ell$, $m_{\ell+1} \in J_{\ell+1}$, $J_\ell \prec J_{\ell+1}$.

Let us define

$$F_3 = \{(X, I, w) \in F_2;\ \mathrm{card}(I \cap W) \ge \mathrm{card} I/2\}, \tag{3.19}$$

$$F_4 = \{(X, I, w) \in F_2;\ \mathrm{card}(I \cap W) < \mathrm{card} I/2\}, \tag{3.20}$$

so that $F_2 = F_3 \cup F_4$, and the condition $S(\boldsymbol{u}) \le 2^{q+10}/b$ means that

$$w(F_3) \le \frac{2^{q+10}}{b}.$$

In particular if $(X, I, w) \in F_3$ we have $w \leq 2^{q+10}/b$. Since $w \geq 2^{-k}$ for $(X, I, w) \in \mathcal{D}_k$ we see that under (3.10) we have

$$(X, I, w) \in \mathcal{D}_k \cap F_3 \Longrightarrow k \geq q. \tag{3.21}$$

Since $s = \mathrm{card} F_1 \leq 2^{q+4}$ and $W = \bigcup_{\ell \leq s} W_\ell$, if $\mathrm{card}(I \cap W) \geq \mathrm{card} I/2$, there must exist $\ell \leq s$ with $\mathrm{card}(I \cap W_\ell) \geq 2^{-q-5} \mathrm{card} I$. This shows that if we define

$$F_3^\ell = \{(X, I, w) \in F_3; \mathrm{card}(I \cap W_\ell) \geq 2^{-q-5} \mathrm{card} I\}, \tag{3.22}$$

then we have $F_3 = \bigcup_{\ell \leq s} F_3^\ell$.

We appeal to Lemma 3.1 with $J = W_\ell$, using the fact that if $k \geq q$ we have

$$(2^{q+5})^{\alpha(k)} \leq 2$$

(with huge room to spare!), to find for each $(X, I, w) \in F_3^\ell$ a triplet $(X', I', w') \in \mathcal{D}$ with $X \subset X'$, $w' \leq 2w$, such that X' depends only on the coordinates of rank in W_ℓ. Let $F_3'^\ell$ be the collection of these triples, so that under (3.10) we have

$$w(F_3'^\ell) \leq 2w(F_3^\ell) \leq 2w(F_3) \leq \frac{2^{q+11}}{b} \leq \frac{1}{2}.$$

We use again Lemma 3.1, this time for J the complement of W, so that $\mathrm{card}(I \cap J) \geq \mathrm{card} I/2$ for $(X, I, w) \in F_4$, and we can find $(X', I', w') \in \mathcal{D}$ with $w' \leq 2w$, X' contains X and depends only on coordinates whose rank is not in W. Let F_4' be the collection of these triples, so that $w(F_4') \leq 2w(F_4) < 2^5$.

Since $\psi(T) \geq 2^5$, we have $T \not\subset \cup F_4'$, so that we can find $\boldsymbol{z} \in T \setminus \cup F_4'$. Since $\cup F_4'$ depends only on the coordinates whose rank is not in W, if $\boldsymbol{z}' \in T$ is such that $z_i = z_i'$ for $i \notin W$, then $\boldsymbol{z}' \notin \cup F_4'$. To conclude the proof, we are going to construct such a $\boldsymbol{z}'$ that does not belong to any of the sets X_ℓ or $\cup F_3'^\ell$. (Thus $\boldsymbol{z}'$ will not belong to $(\cup F_1) \cup (\cup F_2)$.) First, let $A_1 \in \mathcal{A}_{m_1}$ such that $\boldsymbol{z} \in A_1$. Since X_1 is (m_1, n_1, ψ)-thin, there exists $C \in \mathcal{B}_{n_1}$, $C \cap X_1 = \varnothing$, $\psi\left(\pi_{A_1}^{-1}(C)\right) \geq 1$. Since $w(F_3'^1) \leq 1/2$, we therefore have $\pi_{A_1}^{-1}(C) \setminus C' \neq \varnothing$, where $C' = \cup F_3'^1$. Since C' does not depend on the coordinates of rank $\leq m_1$ we have $C' = \pi_{A_1}^{-1}(C')$, so that $\pi_{A_1}^{-1}(C) \setminus \pi_{A_1}^{-1}(C') \neq \varnothing$, and hence $C \setminus C' \neq \varnothing$. Since C' depends only on the coordinates of rank in W_1, we have $C' \in \mathcal{B}_{n_1}$, and since $C \in \mathcal{B}_{n_1}$, we can find $A' \in \mathcal{A}_{n_1}$ with $A' \subset C \setminus C'$, so that $A' \cap X_1 = \varnothing$ and $A' \cap \cup F_3'^1 = \varnothing$. Next, we find $A_2 \in \mathcal{A}_{m_2}$ with $A_2 \subset A'$ such that if $\boldsymbol{y} \in A_2$ then

$$\forall i, \quad n_1 < i \leq m_2 \Longrightarrow y_i = z_i,$$

and we continue the construction in this manner. $\square$

4. The construction

Given an integer p, we will make a construction "with p levels", and we will then take a kind of limit as $p \to \infty$. We consider the sequence $\alpha(k)$ as in (3.3), and we fix a sequence $(M(k))$ to be specified later. The only requirement is that this sequence increases fast enough. We recall the class $\mathcal{D}$ constructed in the previous section.

We construct classes $(\mathcal{E}_{k,p})_{k \le p}$, $(\mathcal{C}_{k,p})_{k \le p}$ of marked weighted sets, and submeasures $(\varphi_{k,p})_{k \le p}$ as follows. First, we set

$$\mathcal{C}_{p,p} = \mathcal{E}_{p,p} = \mathcal{D}, \qquad \varphi_{p,p} = \varphi_{\mathcal{D}} = \psi.$$

Having defined $\varphi_{k+1,p}$, $\mathcal{E}_{k+1,p}$, $\mathcal{C}_{k+1,p}$, we then set

$$\begin{aligned}\mathcal{E}_{k,p} = \Bigg\{ &(X, I, w);\ X \in \mathcal{B},\ X \text{ is } (I, \varphi_{k+1,p})\text{-thin}, \\ &\operatorname{card} I \le M(k),\ w = 2^{-k} \left(\frac{M(k)}{\operatorname{card} I} \right)^{\alpha(k)} \Bigg\} \\ \mathcal{C}_{k,p} = &\, \mathcal{C}_{k+1,p} \cup \mathcal{E}_{k,p}, \qquad \varphi_{k,p} = \varphi_{\mathcal{C}_{k,p}}.\end{aligned}$$

To take limits, we fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}^*$ and we define the class $\mathcal{E}_k$ of marked weighted sets by

$$(X, I, w) \in \mathcal{E}_k \Longleftrightarrow \{p;\quad (X, I, w) \in \mathcal{E}_{k,p}\} \in \mathcal{U}. \tag{4.1}$$

Of course, one can also work with subsequences if one so wishes. It seems plausible that with further effort one might prove that $(X, I, w) \in \mathcal{E}_k$ if and only if $(X, I, w) \in \mathcal{E}_{k,p}$ for all p large enough, but this fact, if true, is not really relevant for our main purpose.

We define

$$\mathcal{C}_k = \mathcal{D} \cup \bigcup_{\ell \ge k} \mathcal{E}_\ell = \mathcal{C}_{k+1} \cup \mathcal{E}_k; \quad \nu_k = \varphi_{\mathcal{C}_k}; \quad \nu = \nu_1.$$

Let us assume that

$$M(k) \ge 2^{(k+5)/\alpha(k)}. \tag{4.2}$$

Then if $w < 2^5$ and $(X, I, w) \in \mathcal{E}_{r,p}$, since

$$w = 2^{-r} \left(\frac{M(r)}{\operatorname{card} I} \right)^{\alpha(r)} \ge \frac{2^5}{\operatorname{card} I^{\alpha(r)}}, \tag{4.3}$$

r remains bounded independently of p. It then follows from (4.1) that if $w < 2^5$ we have

$$(X, I, w) \in \mathcal{C}_k \Longleftrightarrow \{p; (X, I, w) \in \mathcal{C}_{k,p}\} \in \mathcal{U}. \tag{4.4}$$

THEOREM 4.1. *We have $\nu(T) > 0$, ν is exhaustive, ν is pathological, and ν is not uniformly exhaustive.*

The hard work will of course be to show that $\nu(T) > 0$ and that ν is exhaustive, but the other two claims are consequences of Proposition 3.2, since $\nu \le \psi$.

It could be of interest to observe that the submeasure ν has nice invariant properties. For each n it is invariant under any permutation of the elements of T_n. It was observed by Roberts [15] that if there exists an exhaustive submeasure that is not uniformly exhaustive, this submeasure can be found with the above invariance property. This observation was very helpful to the author, as it pointed to the rather canonical construction of ψ.

5. The main estimate

Before we can say anything at all about ν, we must of course control the submeasures $\varphi_{k,p}$. Let us define

$$c_1 = 2^4; \quad c_{k+1} = c_k 2^{2\alpha(k)}$$

so that since $\sum_{k\ge1} \alpha(k) \le 1/2$ we have

$$c_k \le 2^5. \tag{5.1}$$

THEOREM 5.1. *If the sequence $M(k)$ satisfies*

$$M(k) \ge 2^{2k+10} 2^{(k+5)/\alpha(k)} (2^3 + N(k-1)), \tag{5.2}$$

then

$$\forall p, \forall k \le p, \quad \varphi_{k,p}(T) \ge c_k. \tag{5.3}$$

Of course (5.2) implies (4.2). It is the only requirement we need on the sequence $(M(k))$.

The proof of Theorem 5.1 resembles that of Proposition 3.6. The key fact is that the class $\mathcal{E}_{k,p}$ has to a certain extent the property of $\mathcal{D}_k$ stressed in Lemma 3.1, at least when the set J is not too complicated.

The following lemma expresses such a property when J is an interval. We recall the notation $(X)_n$ of (2.8).

LEMMA 5.2. *Consider $(X, I, w) \in \mathcal{E}_{k,p}$, $k < p$, and $m_0 < n_0$. Let $I' = I \cap]m_0, n_0]$ and $A \in \mathcal{A}_{m_0}$. Then if $X' = \left(\pi_A^{-1}(X)\right)_{n_0}$ we have $(X', I', w') \in \mathcal{E}_{k,p}$ where $w' = w(\mathrm{card} I/\mathrm{card} I')^{\alpha(k)}$.*

Proof. It suffices to prove that X' is $(I', \varphi_{k+1,p})$-thin. Consider $m, n \in I'$, $m < n$, so that $m_0 < m < n \le n_0$. Consider $A_1 \in \mathcal{A}_m$, and set $A_2 = \pi_A(A_1) \subset A$, so that $A_2 \in \mathcal{A}_m$. Since X is $(m, n, \varphi_{k+1,p})$-thin, there exists $C \subset A_2$, $C \in \mathcal{B}_n$, with $C \cap X = \varnothing$, $\varphi_{k+1,p}(\pi_{A_2}^{-1}(C)) \geq 1$. Let $C' = A_1 \cap \pi_{A_2}^{-1}(C)$, so that $C' \in \mathcal{B}_n$.

We observe that if a set B does not depend on the coordinates of rank $\leq m$, we have

$$\pi_{A_1}^{-1}(B) = B = \pi_{A_1}^{-1}(B \cap A_1).$$

Using this for $B = \pi_{A_2}^{-1}(C)$, we get that $\pi_{A_1}^{-1}(C') = \pi_{A_2}^{-1}(C)$, and consequently $\varphi_{k+1,p}\left(\pi_{A_1}^{-1}(C')\right) \geq 1$.

It remains only to prove that $C' \cap X' = \varnothing$. This is because on A_1 the maps π_A and π_{A_2} coincide, so that, since $C' \subset A_1$, we have $\pi_A(C') = \pi_{A_2}(C') \subset C$ and hence $\pi_A(C') \cap X = \varnothing$. Thus $C' \cap \pi_A^{-1}(X) = \varnothing$ and since $C' \in \mathcal{B}_n$ we have $C' \cap X' = \varnothing$. □

Given p, the proof of Theorem 5.1 will go by decreasing induction over k. For $k = p$, the result is true since by Proposition 3.2 we have $\varphi_{p,p}(T) = \psi(T) \geq 2^5 \geq c_k$.

Now we proceed to the induction step from $q+1$ to q. Considering $F \subset \mathcal{C}_{q,p}$, with $w(F) < c_q$, our goal is to show that $\cup F \neq T$. Since $\mathcal{C}_{q,p} = \mathcal{C}_{q+1,p} \cup \mathcal{E}_{q,p}$ we have $F = F_1 \cup F_2$, $F_1 \subset \mathcal{E}_{q,p}$, $F_2 \subset \mathcal{C}_{q+1,p}$.

Let $F_2' = F_2 \cap \bigcup_{k<q} \mathcal{D}_k$. When $(X, I, w) \in \mathcal{D}_k$ we have $w \geq 2^{-k} \geq 2^{-q}$, and thus

$$2^{-q} \mathrm{card} F_2' \leq w(F_2') \leq w(F) \leq c_q \leq 2^5$$

so that $\mathrm{card} F_2' \leq 2^{q+5}$. Also, for $(X, I, w) \in \mathcal{D}_k$ we have $\mathrm{card} I \leq N(k)$, so that if

$$I^* = \bigcup \{I; (X, I, w) \in F_2'\} \tag{5.4}$$

then

$$\mathrm{card} I^* \leq t' := 2^{q+5} N(q-1). \tag{5.5}$$

When $(X, I, w) \in \mathcal{E}_{q,p}$ we have $w \geq 2^{-q}$. Thus

$$2^{-q} \mathrm{card} F_1 \leq w(F_1) \leq w(F) \leq c_q \leq 2^5$$

and thus $s := \mathrm{card} F_1 \leq 2^{q+5}$. Also, when $(X, I, w) \in \mathcal{E}_{q,p}$,

$$2^{-q} \left(\frac{M(q)}{\mathrm{card} I} \right)^{\alpha(q)} = w \leq 2^5$$

so that

$$\mathrm{card} I \geq M(q) 2^{-(q+5)/\alpha(q)} \tag{5.6}$$

and hence, if

$$t = 2^{q+8} + t' \tag{5.7}$$

under (5.2) then $\mathrm{card} I \geq st$ where $s = \mathrm{card} F_1$. Now following the proof of Proposition 3.6, we appeal to Roberts' selection lemma to enumerate F_1 as $(X_\ell, I_\ell, w_\ell)_{\ell \leq s}$ and find sets $J_1 \prec J_2 \prec \cdots \prec J_s$ with $\mathrm{card} J_\ell = t$ and $J_\ell \subset I_\ell$. Then appealing to the counting argument of Proposition 3.6, but instead of allowing in (3.17) all the values of $u(\ell) \leq t-1$, we now restrict the choice of $u(\ell)$ by

$$u(\ell) \in U_\ell = \{u; 1 \leq u \leq t-1, I^* \cap]i_{u,\ell}, i_{u+1,\ell}] = \varnothing\}.$$

We observe that by (5.5) and (5.7), $\mathrm{card} U_\ell \geq 2^{q+8} - 1$.

The counting argument then allows us to find $\boldsymbol{u}$ such that (since $w(F_2) \leq 2^5$)

$$S(\boldsymbol{u}) \leq \frac{2}{2^{q+8}-1} w(F_2) \leq 2^{-q-1}.$$

Using the notation (3.18) we have thus constructed intervals $W_\ell =]m_\ell, n_\ell]$, $\ell \leq s$, with $n_\ell \leq m_{\ell+1}$, in such a manner that X_ℓ is $(m_\ell, n_\ell, \varphi_{q+1,p})$-thin and that if F_3 is defined by (3.19),

$$w(F_3) \leq 2^{-q-1} \leq \frac{1}{4}. \tag{5.8}$$

Moreover, if $W = \bigcup_{\ell \leq s}]m_\ell, n_\ell]$ we have ensured that

$$(X, I, w) \in F_2' \Longrightarrow W \cap I = \varnothing,$$

so that in particular if we define F_4 by (3.20) then

$$(X, I, w) \in F_4\,,\ (X, I, w) \in \bigcup_{k<q} \mathcal{D}_k \Longrightarrow W \cap I = \varnothing. \tag{5.9}$$

As before, (5.8) implies that if $(X, I, w) \in \mathcal{D}_k \cap F_3$, then $k \geq q$. Let us define the classes F_3^ℓ, $\ell \leq s$ by

$$F_3^\ell = \{(X, I, w) \in F_3; \mathrm{card}(I \cap W_\ell) \geq 2^{-q-6} \mathrm{card} I\},$$

so that $F_3 = \bigcup_{\ell \leq s} F_3^\ell$, since $s \leq 2^{q+5}$.

LEMMA 5.3. *Consider $(X, I, w) \in F_3^\ell$ and $A \in \mathcal{A}_{m_\ell}$. Then there is (X', I', w') in $\mathcal{C}_{q+1,p}$ with $X' \supset \pi_A^{-1}(X)$, $X' \in \mathcal{B}_{n_\ell}$, $w' \leq 2w$.*

Proof. If $(X, I, w) \in \mathcal{D}$ we have already proved this statement in the course of the proof of Proposition 3.6, and so, since $\mathcal{C}_{q+1,p} = \mathcal{D} \cup \bigcup_{q+1 \leq r \leq p} \mathcal{E}_{r,p}$, it suffices to consider the case where $(X, I, w) \in \mathcal{E}_{r,p}$, $r \geq q+1$. In that case, if $I' = I \cap W_\ell$,

$$\left(\frac{\mathrm{card} I}{\mathrm{card} I'}\right)^{\alpha(r)} \leq (2^{q+6})^{\alpha(r)} \leq 2$$

and the result follows from Lemma 5.2. □

Corollary 5.4. *Consider $A \in \mathcal{A}_{m_\ell}$. Then there is $A' \in \mathcal{A}_{n_\ell}$ such that $A' \subset A$, $A' \cap X_\ell = \varnothing$ and $A' \cap \cup F_3^\ell = \varnothing$.*

Proof. Lemma 5.3 shows that $\pi_A^{-1}(\cup F_3^\ell) \subset C'$, where $C' \in \mathcal{B}_{n_\ell}$ and $\varphi_{q+1,p}(C') \le 2w(F_3^\ell) \le 1/2$. Since X_ℓ is $(m_\ell, n_\ell, \varphi_{q+1,p})$-thin, there is $C \in \mathcal{B}_{n_\ell}$, $C \subset A$, $C \cap X = \varnothing$ with $\varphi_{q+1,p}\left(\pi_A^{-1}(C)\right) \ge 1$. Thus we cannot have $\pi_A^{-1}(C) \subset C'$ and hence since both these sets belong to $\mathcal{B}_{n_\ell}$ we can find $A_1 \in \mathcal{A}_{n_\ell}$ with

$$A_1 \subset \pi_A^{-1}(C) \setminus C' \subset \pi_A^{-1}(C) \setminus \pi_A^{-1}(\cup F_3^\ell).$$

Now, $A' = \pi_A(A_1) \in \mathcal{A}_{n_\ell}$, $A' \cap \cup F_3^\ell = \varnothing$, $A' \subset C$, so that $A' \cap X_\ell = \varnothing$. □

We now construct a map $\Xi : T \to T$ with the following properties. For $\boldsymbol{y} \in T$, $\boldsymbol{z} = \Xi(\boldsymbol{y})$ is such that $\boldsymbol{z}_i = \boldsymbol{y}_i$ whenever $i \notin W = \bigcup_{\ell \le s}]m_\ell, n_\ell]$. Moreover, for each ℓ, and each $A \in \mathcal{A}_{m_\ell}$, there exists $A' \in \mathcal{A}_{n_\ell}$ with

$$\boldsymbol{y} \in A \Longrightarrow \Xi(\boldsymbol{y}) \in A',$$

and A' satisfies $A' \cap X_\ell = \varnothing$ and $A' \cap \cup F_3^\ell = \varnothing$.

The existence of this map is obvious from Corollary 5.4. It satisfies

$$\ell \le s \Longrightarrow \Xi(T) \cap X_\ell = \varnothing, \quad \Xi(T) \cap \cup F_3^\ell = \varnothing. \tag{5.10}$$

It has the further property that for each integer j the first j coordinates of $\Xi(\boldsymbol{y})$ depend only on the first j coordinates of $\boldsymbol{y}$.

We recall that F_4 is as in (3.20).

Lemma 5.5. *We have $\varphi_{q+1,p}\left(\Xi^{-1}(\cup F_4)\right) < c_{q+1}$.*

Proof of Theorem 5.1. Using the induction hypothesis $\varphi_{q+1,p}(T) \ge c_{q+1}$ we see that there is $\boldsymbol{y}$ in $T \setminus \Xi^{-1}(\cup F_4)$, so that $\Xi(\boldsymbol{y}) \notin \cup F_4$. Combining with (5.10) we see that $\Xi(\boldsymbol{y}) \notin \bigcup_{\ell \le s} X_\ell = \cup F_1$, $\Xi(\boldsymbol{y}) \notin \cup F_3$, so that $\Xi(\boldsymbol{y}) \notin \cup F$. □

Proof of Lemma 5.5. We prove that if $(X, I, w) \in F_4$, then $\varphi_{q+1,p}\left(\Xi^{-1}(X)\right) \le w 2^{2\alpha(q)}$. This suffices since $w(F_4) < c_q$.

Case 1. $(X, I, w) \in \mathcal{D}_k$, $k < q$. In that case, by (5.9), $I \cap W = \varnothing$, so that $\Xi^{-1}(X) = X$ and thus $\varphi_{q+1,p}\left(\Xi^{-1}(X)\right) = \varphi_{q+1,p}(X) \le w$.

Case 2. We have $(X, I, w) \in \mathcal{D}_k$, $k \ge q$. We use Lemma 3.1 with $J = \mathbb{N}^* \setminus W$ and the fact that $\alpha(k) \le \alpha(q) \le (q+5)^{-3}$. This has already been done in the previous section.

Case 3. $(X, I, w) \in \mathcal{E}_{r,p}$ for some $q+1 \le r < p$. In a first stage we prove the following. Whenever $m, n \in I$ are such that $m < n$, and $]m, n] \cap W = \varnothing$, then $\Xi^{-1}(X)$ is $(m, n, \varphi_{r+1,p})$-thin. Since for each integer j the first j coordinates of $\Xi(\boldsymbol{y})$ depend only on the first j coordinates of $\boldsymbol{y}$, whenever

$A \in \mathcal{A}_m$ there is $A' \in \mathcal{A}_m$ with $\Xi(A) \subset A'$. Since X is $(m, n, \varphi_{r+1,p})$-thin we can find $C' \in \mathcal{B}_n$ with $C' \cap X = \varnothing$, $C' \subset A'$, and $\varphi_{r+1,p}\left(\pi_{A'}^{-1}(C')\right) \geq 1$. We first prove that

$$\Xi\left(\pi_A\left(\pi_{A'}^{-1}(C')\right)\right) \subset C'. \tag{5.11}$$

Consider $\tau_1, \ldots, \tau_m$ and $\tau'_1, \ldots, \tau'_m$ such that

$$\begin{aligned} A &= \{\boldsymbol{z} \in T;\ \ \forall i \leq m, \ \ z_i = \tau_i\}, \\ A' &= \{\boldsymbol{z} \in T;\ \ \forall i \leq m, \ \ z_i = \tau'_i\}. \end{aligned}$$

Consider $\boldsymbol{y} \in \pi_{A'}^{-1}(C')$. Then there exists $\boldsymbol{y}' \in C'$ with $y_i = y'_i$ for $i > m$. Thus $\boldsymbol{y}'' = \pi_A(\boldsymbol{y})$ is such that $y''_i = \tau_i$ for $i \leq m$, and $y''_i = y'_i$ for $i > m$, so that $\boldsymbol{z} = \Xi(\boldsymbol{y}'')$ is such that $z_i = \tau'_i$ for $i < m$. Moreover $z_i = y''_i$ for $i \notin W$, and since $]m, n] \cap W = \varnothing$, we have $z_i = y''_i = y'_i$ for $m < i \leq n$. Since $C' \subset A'$, we have $y'_i = \tau'_i$ for $i < m$, so that $z_i = y'_i$ for all $i \leq n$, and thus $\boldsymbol{z} \in C'$ because $\boldsymbol{y}' \in C' \in \mathcal{B}_n$. Since $\boldsymbol{y}$ is arbitrary this proves (5.11).

Let $C = \Xi^{-1}(C') \cap A \in \mathcal{B}_n$, so that (5.11) implies that

$$\pi_{A'}^{-1}(C') \subset \pi_A^{-1}\left(\Xi^{-1}(C')\right) = \pi_A^{-1}(C),$$

so that $\varphi_{r+1,p}\left(\pi_A^{-1}(C)\right) \geq 1$ and since $C \cap \Xi^{-1}(X) = \varnothing$ we have proved that $\Xi^{-1}(X)$ is $(m, n, \varphi_{r+1,p})$-thin.

For each $\ell \leq 1$, consider the largest element $i(\ell)$ of I that is $\leq m_\ell$. (Trivial modifications of the argument take care of the case where I has no elements $\leq m_\ell$). Let

$$I' = I \setminus (W \cup \{i(1), \ldots, i(s)\}),$$

so that, since $\operatorname{card}(I \setminus W) \geq \operatorname{card}I/2$, we have

$$\operatorname{card}I' \geq \frac{\operatorname{card}I}{2} - s \geq \frac{\operatorname{card}I}{2} - 2^{q+5} \geq \frac{\operatorname{card}I}{4},$$

using (5.6) and (5.2). We claim that $\Xi^{-1}(X)$ is $(m, n, \varphi_{r+1,p})$-thin whenever $m < n$, $m, n \in I'$. To see this, consider the smallest element n' of I such that $m < n'$. Then $n' \leq n$, so it suffices to show that $\Xi^{-1}(X)$ is $(m, n', \varphi_{r+1,p})$-thin. By the first part of the proof, it suffices to show that $W \cap]m, n'] = \varnothing$. Assuming $W_\ell \cap]m, n'] \neq \varnothing$, we see that $m_\ell < n'$. Since $m \notin W_\ell$ we have $m \leq i(\ell)$ and since $m \neq i(\ell)$, we have $m < i(\ell) \leq m_\ell$, contradicting the choice of n'.

Let $w' = w(\operatorname{card}I/\operatorname{card}I')^{\alpha(q)} \leq w2^{2\alpha(q)}$. Then, obviously, $(\Xi^{-1}(X), I', w') \in \mathcal{E}_{r,q}$, so that $\varphi_{q+1,p}\left(\Xi^{-1}(X)\right) \leq w2^{2\alpha(q)}$. □

6. Exhaustivity

Lemma 6.1. *Consider $B \in \mathcal{B}$ and $a > 0$. If $\nu_k(B) < a$ then*

$$\{p;\ \varphi_{k,p}(B) < a\} \in \mathcal{U}.$$

Proof. By definition of $\nu_k = \varphi_{\mathcal{C}_k}$, there exists a finite set $F \subset \mathcal{C}_k = \mathcal{D} \cup \bigcup_{r \geq k} \mathcal{E}_r$ with $w(F) < a$ and $\cup F \supset B$. By definition of $\mathcal{E}_r$, for $(X, I, w) \in \mathcal{E}_r$,

$$\{p;\ (X, I, w) \in \mathcal{E}_{r,p}\} \in \mathcal{U},$$

so that since $\mathcal{C}_{k,p} = \mathcal{D} \cup \bigcup_{k \leq r < p} \mathcal{E}_{r,p}$ we have $\{p;\ F \subset \mathcal{C}_{k,p}\} \in \mathcal{U}$ and thus $\varphi_{k,p}(B) \leq w(F) < a$ for these p. □

COROLLARY 6.2. *We have* $\nu(T) \geq 16$.

Proof. By Lemma 6.1, and since $\varphi_{1,p}(T) \geq c_1 = 16$, by Theorem 5.1. □

The next lemma is a kind of converse to Lemma 6.1, and lies much deeper.

LEMMA 6.3. *Let* $B \in \mathcal{B}$ *with* $\nu_k(B) \geq 4$. *Then*

$$\{p;\ \varphi_{k,p}(B) \geq 1\} \in \mathcal{U}.$$

Proof. Consider n such that $B \in \mathcal{B}_n$, and assume for contradiction that

$$U = \{p;\ \varphi_{k,p}(B) < 1\} \in \mathcal{U}.$$

Thus, for $p \in U$, we can find $F_p \subset \mathcal{C}_{k,p}$ with $B \subset \cup F_p$ and $w(F_p) \leq 1$. Let

$$\begin{aligned} F_p^1 &= \{(X, I, w) \in F_p;\ \operatorname{card}(I \cap \{1, \ldots, n\}) \geq \operatorname{card} I/2\}, \\ F_p^2 &= F_p \setminus F_p^1 = \{(X, I, w) \in F_p;\ \operatorname{card}(I \cap \{1, \ldots, n\}) < \operatorname{card} I/2\}. \end{aligned}$$

Using Lemmas 3.1 and 5.2 we find a family $F_p^\sim$ of triples (X', I', w') in $\mathcal{C}_{k,p}$ with $\cup F_p^\sim \supset \cup F_p^1$, $w(F_p^\sim) \leq 2$ and $I' \subset \{1, \ldots, n\}$, $X' \in \mathcal{B}_n$, so that $\cup F_p^\sim \in \mathcal{B}_n$.

We claim that $B \subset \cup F_p^\sim$. For, otherwise, since B and $\cup F_p^\sim$ both belong to $\mathcal{B}_n$, we can find $A \in \mathcal{A}_n$ with $A \subset B \setminus \cup F_p^\sim$, so that $A \subset \cup F_p^2$. By Lemma 5.2 again (or, to be exact, its obvious extension to the case $n_0 = \infty$) and Lemma 3.1 we get

$$\varphi_{k,p}(T) = \varphi_{k,p}\left(\pi_A^{-1}(\cup F_p^2)\right) \leq 2w(F_p^2) \leq 2,$$

which is impossible because $\varphi_{k,p}(T) \geq 16$.

Using (3.12) and (4.3) we see that for $(X', I', w') \in \mathcal{C}_{k,p}$ the value of $\operatorname{card} I'$ determines w'. Since $X' \in \mathcal{B}_n$, it follows that there exists a finite collection $\mathcal{G}$ of triples (X, I, w) such that $F_p^\sim \subset \mathcal{G}$ for all p. Thus there exists a set F such that $\{p \in U; F_p^\sim = F\} \in \mathcal{U}$. If follows from (4.4) that $F \subset \mathcal{C}_k$ and it is obvious that $B \subset \cup F$ and $w(F) \leq 2$, so that $\nu_k(B) \leq 2$, a contradiction. □

COROLLARY 6.4. *Consider a triplet* (X, I, w) *and* k *with* $\operatorname{card} I \leq M(k)$ *and*

$$w = 2^{-k}\Big(\frac{M(k)}{\operatorname{card} I}\Big)^{\alpha(k)}.$$

Assume that X is $(I, \nu_{k+1}/4)$-thin, i.e.

(6.1)

$$\forall m, n \in I,\ m < n,\quad \forall A \in \mathcal{A}_m,\ \exists C \in \mathcal{B}_n,\ C \cap X = \varnothing,\ \nu_{k+1}\left(\pi_A^{-1}(C)\right) \geq 4.$$

Then $(X, I, w) \in \mathcal{E}_k$.

Proof. If $\nu_{k+1}(\pi_A^{-1}(C)) \geq 4$ then by Lemma 6.3 we have $\{p;\ \varphi_{k+1,p}(\pi_A^{-1}(C)) \geq 1\} \in \mathcal{U}$ and

$$\{p;\ (X, I, w) \in \mathcal{E}_{k,p}\} \supset \bigcap \left\{p;\ \varphi_{k+1,p}\left(\pi_A^{-1}(C)\right) \geq 1\right\} \in \mathcal{U},$$

where the intersection is over all sets A, C as in (6.1). □

Lemma 6.5. *Consider a sequence (E_i) of $\mathcal{B}$, and assume that these sets do not depend on the coordinates of rank $\leq m$ for a certain m. Assume that*

$$\forall n,\quad \nu_k\left(\bigcup_{i \leq n} E_i\right) < 4.$$

Then for each $\alpha > 0$ there is $C \in \mathcal{B}$, which does not depend on the coordinates of rank $\leq m$, and such that $\nu_k(C) \leq 8$ and $\nu_k(E_i \setminus C) \leq \alpha$ for each i.

Proof. For each n, let

$$U_n = \left\{p;\ \varphi_{k,p}\left(\bigcup_{i \leq n} E_i\right) < 4\right\}$$

so that $U_n \in \mathcal{U}$ by Lemma 6.1. For $p \in U_n$ we can find $F_{n,p} \subset \mathcal{C}_{k,p}$ with $\bigcup_{i \leq n} E_i \subset \cup F_{n,p}$ and $w(F_{n,p}) \leq 4$. For $r \geq m+1$ we define

$$F_{n,p}^r = \left\{(X, I, w) \in F_{n,p};\ \operatorname{card}(I \cap \{m+1, \ldots, r-1\}) \leq \frac{1}{2}\operatorname{card} I;\right.$$
$$\left.\operatorname{card}(I \cap \{m+1, \ldots, r\}) \geq \frac{1}{2}\operatorname{card} I\right\},$$

and

$$F'_{n,p} = \left\{(X, I, w) \in F_{n,p};\ \operatorname{card}(I \cap \{1, \ldots, m\}) \geq \frac{1}{4}\operatorname{card} I\right\}.$$

We use Lemmas 3.1 and 5.2 to find a set $B \in \mathcal{B}_m$ with $\varphi_{k,p}(B) \leq 8$ and $B \supset \cup F'_{n,p}$ so that since $\varphi_{k,p}(T) \geq 16$, $B \neq T$ and thus there exists $A_{n,p} \in \mathcal{A}_m$ with $A_{n,p} \cap \cup F'_{n,p} = \varnothing$. We use again Lemmas 3.1 and 5.2 to see that for $(X, I, w) \in F_{n,p}^r$ we can find $w' \leq 2w$ such that $(X', I', w') = ((\pi_{A_{n,p}}^{-1}(X))_r,\ I \cap \{m+1, \ldots, r\}, w') \in \mathcal{C}_{k,p}$. We observe that X' does not depend on the coordinates of rank $\leq m$. Let $F'^r_{n,p}$ be the collection of the sets (X', I', w') for $(X, I, w) \in F_{n,p}^r$ so that $w(F'^r_{n,p}) \leq 2w(F_{n,p}^r)$. We claim that if $E_i \in \mathcal{B}_j$ then

$$E_i \subset \bigcup_{r \leq j} \cup F'^r_{n,p}. \tag{6.2}$$

Otherwise, since both sets depend only on the coordinates of rank $\geq m$ and $\leq j$, and since $A_{n,p} \in \mathcal{A}_m$, we would find $A \in \mathcal{A}_j$ with $A \subset A_{n,p}$ and $A \subset E_i \setminus \bigcup_{r \leq j} \cup F'^r_{n,p}$. Since $\pi^{-1}_{A_{n,p}}(X) \cap A_{n,p} \supset X \cap A_{n,p}$, this shows that $A \subset E_i \setminus \bigcup_{r \leq j} \cup F^r_{n,p}$. Since $A_{n,p} \cap \cup F'_{n,p} = \varnothing$, we would have $A \cap \cup F'_{n,p} = \varnothing$. Also, since $E_i \subset \cup F_{n,p}$, we would have $A \subset \cup F''_{n,p}$, where

$$
\begin{aligned}
F''_{n,p} &= F_{n,p} \setminus \left(F'_{n,p} \cup \bigcup_{r \leq j} F^r_{n,p} \right) \\
&\subset \left\{ (X, I, w) \in F_{n,p};\, \mathrm{card} I \cap \{j+1, \dots\} \geq \frac{1}{4} \mathrm{card} I \right\}.
\end{aligned}
$$

A new application of Lemmas 3.1 and 5.2 then shows that $T = \pi^{-1}_A(A)$ would satisfy $\varphi_{k,p}(T) \leq 8$, and this is impossible. So we have proved (6.2).

Given r, we prove, using (3.12) and (4.3), that $F'^r_{n,p} \subset \mathcal{G}^r$ where $\mathcal{G}^r$ is finite and does not depend on n or p. It should then be clear by (4.1) how to take limits as $p \to \infty$, $n \to \infty$ to define for $r \geq m+1$ sets $F^r \subset \mathcal{C}_k$ with $\sum_{r \geq m+1} w(F^r) \leq 8$ such that $E_i \subset \bigcup_{r \leq j} \cup F^r$ provided $E_i \in B_j$. The elements of F^r are of the type (X, I, w) where X does not depend on the coordinates of rank $\leq m$, and $X \in \mathcal{B}_r$.

Consider r_0 such that $\sum_{r > r_0} w(F^r) < \alpha$ and let $C = \bigcup_{r \leq r_0} \cup F^r$. Then $\nu_k(C) \leq \sum_{r \leq r_0} w(F^r) \leq 8$ and

$$
E_i \setminus C \subset \bigcup_{r_0 \leq r \leq j} \cup F^r
$$

so that $\nu_k(E_i \setminus C) \leq \alpha$. □

LEMMA 6.6. *Consider $k > 0$, $\alpha > 0$, $B \in \mathcal{B}_m$, a disjoint sequence (E_i) of $\mathcal{B}$. Then we can find $n > m$, a set $B' \in \mathcal{B}_n$, $B' \subset B$ such that B' is $(m, n, \nu_k/4)$-thin and*

$$
\limsup_{i \to \infty} \nu_k((B \cap E_i) \setminus B') \leq \alpha.
$$

Proof. This is nearly identical to that of Lemma 3.8, by Lemma 6.5, and since $\nu_k(T) \geq 16$. □

Proof that ν is exhaustive. For each k we show that ν is 2^{-k} exhaustive following the method of Proposition 3.5, and using the fact that by Corollary 6.4, if X is $(I, \nu_{k+1}/4)$-thin where $\mathrm{card} I = M(k)$, then $(X, I, 2^{-k}) \in \mathcal{E}_k$, so that $\nu(X) \leq \nu_k(X) \leq 2^{-k}$. □

7. Proofs of Theorems 1.2 to 1.4

The simple arguments we present here are essentially copied from the paper of Roberts [15], and are provided for the convenience of the reader.

To prove Theorem 1.4, we simply consider the space $\mathcal{L}_0$ of real-valued functions, defined on the Cantor set, that are $\mathcal{B}$-measurable, provided with the topology induced by the distance d such that

$$d(f,0) = \sup\{\varepsilon;\ \nu(\{|f| \geq \varepsilon\}) \geq \varepsilon\}, \tag{7.1}$$

where ν is the submeasure of Theorem 1.1. We consider the $\mathcal{L}_0$-valued vector measure θ given by $\theta(A) = 1_A$. Thus $d(0, \theta(A)) = \nu(A)$, which makes it obvious that θ is exhaustive and does not have a control measure. Let us also note that d satisfies the nice formula

$$d(f+g,0) \leq d(f,0) + d(g,0),$$

as follows from the relation $\{|f+g| \geq \varepsilon_1 + \varepsilon_2\} \subset \{|f| \geq \varepsilon_1\} \cup \{|g| \geq \varepsilon_2\}$.

We start the proof of Theorem 1.2 first observing that the submeasure ν of Theorem 1.1 is strictly positive, i.e., $\nu(A) > 0$ if $A \neq \varnothing$. This follows from subadditivity and the fact that by construction we have $\nu(A) = \nu(A')$ for $A, A' \in \mathcal{A}_n$ and any n.

Since ν is strictly positive we can define a distance d on $\mathcal{B}$ by

$$d(A,B) = \nu(A \vartriangle B),$$

where $\vartriangle$ denotes the symmetric difference. It is simple to see that the completion $\widehat{\mathcal{B}}$ of $\mathcal{B}$ with respect to this distance is still a Boolean algebra, the operations being defined by continuity, and that ν extends to $\widehat{\mathcal{B}}$ in a positive submeasure, still denoted by ν. We claim that ν is exhaustive. To see this, consider a disjoint sequence (E_n) in $\widehat{\mathcal{B}}$. Consider $\varepsilon > 0$, and for each n find A_n in $\mathcal{B}$ with $\nu(A_n \vartriangle E_n) \leq \varepsilon 2^{-n}$. Let $B_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$, so that, since $E_n = E_n \setminus (E_1 \cup \cdots \cup E_{n-1})$ we have

$$\nu(B_n \vartriangle E_n) \leq \sum_{m \leq n} \nu(E_m \vartriangle A_m) \leq \sum_{m \leq n} \varepsilon 2^{-m} \leq \varepsilon. \tag{7.2}$$

Since the sequence (B_n) is disjoint in $\mathcal{B}$, we have $\lim_{n\to\infty} \nu(B_n) = 0$, and by (7.2) we have $\limsup_{n\to\infty} \nu(E_n) \leq \varepsilon$. As ε is arbitrary, this proves the result.

Consider now a decreasing sequence (A_n) of $\widehat{\mathcal{B}}$. The fundamental observation is that it is a Cauchy sequence for d. Otherwise, we could find $\varepsilon > 0$ and numbers $m(k) < n(k) \leq m(k+1) < n(k+1) \cdots$ with $\nu(A_{n(k)} \setminus A_{m(k)}) \geq \varepsilon$, and this contradicts exhaustivity.

The limit of a decreasing sequence (A_n) in $\widehat{\mathcal{B}}$ is clearly the infimum of this sequence. This shows that $\widehat{\mathcal{B}}$ is σ-complete and that ν is continuous.

It follows that ν is countably subadditive, i.e.

$$\nu\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n\geq 1} \nu(A_n). \tag{7.3}$$

This is because for each m, if $B_m = \bigcup_{n\geq 1} A_n \setminus \left(\bigcup_{1\leq n\leq m} A_n\right)$ then

$$\nu\left(\bigcup_{n\geq 1} A_n\right) \leq \nu\left(B_m \cup \bigcup_{1\leq n\leq m} A_n\right) \leq \nu(B_m) + \sum_{1\leq n\leq m} \nu(A_n)$$

and $\nu(B_m) \to 0$ since ν is continuous. Now, the sequence (B_n) decreases because $\bigcap_{m\geq 1} B_m = \mathbf{0}$ (the smallest element of $\widehat{\mathcal{B}}$).

Lemma 7.1. *Consider $A \in \mathcal{B}$, and countable collections $\mathcal{C}_n$, $n \geq 1$ such that $A \subset \cup\mathcal{C}_n$ for each n. Then for each $\eta > 0$ there is $A' \subset A$ with $\nu(A\backslash A') \leq \eta$ such that for each n, A is covered by a finite subset of $\mathcal{C}_n$.*

Proof. Enumerate $\mathcal{C}_n$ as $(C_{n,m})_{m\geq 1}$. Since $A \subset \cup\mathcal{C}_n$, we have

$$\bigcap_k \left(A \setminus \bigcup_{m\leq k} C_{n,m}\right) = \mathbf{0},$$

so that by continuity of ν there exists $k(n)$ with $\nu\left(A \setminus \bigcup_{m\leq k(n)} C_{n,k}\right) \leq \eta 2^{-n}$. The set $A' = \bigcap_n \bigcup_{m\leq k(n)} C_{n,k}$ is for each n covered by a finite subset of $\mathcal{C}_n$ and it satisfies $\nu(A \setminus A') \leq \eta$ by (7.3). □

Consider a measure μ on $\widehat{\mathcal{B}}$. Then μ is not absolutely continuous with respect to ν on $\mathcal{B}$, so that we can find $\varepsilon > 0$ and for each n a set $B_n \in \mathcal{B}$ with $\nu(B_n) \leq 2^{-n}$ and $\mu(B_n) \geq \varepsilon$. Let $A_n = \bigcup_{m\geq n} B_m$. By (7.3) we have $\nu(A_n) \leq \sum_{m\geq n} 2^{-m} \leq 2^{-n+1}$ so that if $A = \bigcap_{n\geq 1} A_n$ we have $\nu(A) = 0$ and thus $A = \mathbf{0}$. But by monotonicity we have $\mu(A_n) \geq \varepsilon$, so that μ is not continuous.

On the other hand, ν is not absolutely continuous with respect to μ on $\mathcal{B}$, so for some $\varepsilon > 0$ and each n we can find $B_n \in \mathcal{B}$ with $\nu(B_n) \geq \varepsilon$ and $\mu(B_n) \leq 2^{-n}$. Let $A_n = \bigcup_{m\geq n} B_m$ and $A = \bigcap_{n\geq 1} A_n$, so that $\nu(A_n) \geq \varepsilon$ and $\nu(A) \geq \varepsilon$ by continuity of ν. We use Lemma 7.1 with $\eta = \varepsilon/2$, $\mathcal{C}_n = \{B_m;\, m \geq n\}$, $A = \bigcap_{n\geq 1} A_n$. Now, $\nu(A') \geq \varepsilon/2$ since $\nu(A) \geq \varepsilon$, so that $A' \neq \mathbf{0}$. For each n, since μ is subadditive, and since A' can be covered by a finite subset of $\mathcal{C}_n$ we have $\mu(A') \leq \sum_{m\geq n} 2^{-m} = 2^{-n-1}$. Thus $\mu(A') = 0$, and hence μ is not positive. This concludes the proof of Theorem 1.2.

To prove Theorem 1.3, we first observe that $\widehat{\mathcal{B}}$ satisfies the countable chain condition, since ν is positive and exhaustive. We prove that it also satisfies the weak distributive law. Given a sequence (Π_n) of partitions of $\widehat{\mathcal{B}}$ and $m \in \mathbb{N}^*$, Lemma 7.1 produces a set C_m with $\nu(C_m^c) \leq 2^{-m}$ such that C_m is finitely covered by every partition Π_n. And $C_1,\ C_2 \setminus C_1,\ C_3 \setminus (C_1 \cup C_2), \cdots$ is the required partition. This concludes the proof of Theorem 1.3.

Equipe d'analyse de l'Institut de Mathématiques, Université Paris VI, Paris, France
E-mail address: spinglass@talagrand.net

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