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# Quasilinear and Hessian equations of Lane-Emden type

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## Abstract

The existence problem is solved, and global pointwise estimates of solutions are obtained for quasilinear and Hessian equations of Lane-Emden type, including the following two model problems:

$$-\Delta_p u = u^q + \mu, \qquad F_k[-u] = u^q + \mu, \qquad u \geq 0,$$

on $\mathbb{R}^n$, or on a bounded domain $\Omega \subset \mathbb{R}^n$. Here $\Delta_p$ is the $p$-Laplacian defined by $\Delta_p u = \operatorname{div}(\nabla u |\nabla u|^{p-2})$, and $F_k[u]$ is the $k$-Hessian defined as the sum of $k \times k$ principal minors of the Hessian matrix $D^2 u$ ($k = 1, 2, \ldots, n$); $\mu$ is a nonnegative measurable function (or measure) on $\Omega$.

The solvability of these classes of equations in the renormalized (entropy) or viscosity sense has been an open problem even for good data $\mu \in L^s(\Omega)$, $s > 1$. Such results are deduced from our existence criteria with the sharp exponents $s = \frac{n(q-p+1)}{pq}$ for the first equation, and $s = \frac{n(q-k)}{2kq}$ for the second one. Furthermore, a complete characterization of removable singularities is given.

Our methods are based on systematic use of Wolff's potentials, dyadic models, and nonlinear trace inequalities. We make use of recent advances in potential theory and PDE due to Kilpeläinen and Malý, Trudinger and Wang, and Labutin. This enables us to treat singular solutions, nonlocal operators, and distributed singularities, and develop the theory simultaneously for quasilinear equations and equations of Monge-Ampère type.


## 1. Introduction

We study a class of quasilinear and fully nonlinear equations and inequalities with nonlinear source terms, which appear in such diverse areas as quasi-regular mappings, non-Newtonian fluids, reaction-diffusion problems, and stochastic control. In particular, the following two model equations are of

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substantial interest:

$$-\Delta_p u = f(x,u), \qquad F_k[-u] = f(x,u), \tag{1.1}$$

on $\mathbb{R}^n$, or on a bounded domain $\Omega \subset \mathbb{R}^n$, where $f(x,u)$ is a nonnegative function, convex and nondecreasing in $u$ for $u \geq 0$. Here $\Delta_p u = \mathrm{div}\,(\nabla u\, |\nabla u|^{p-2})$ is the $p$-Laplacian ($p > 1$), and $F_k[u]$ is the $k$-Hessian ($k = 1, 2, \ldots, n$) defined by

$$F_k[u] = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.2}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix $D^2 u$. In other words, $F_k[u]$ is the sum of the $k \times k$ principal minors of $D^2 u$, which coincides with the Laplacian $F_1[u] = \Delta u$ if $k = 1$, and the Monge–Ampère operator $F_n[u] = \det\,(D^2 u)$ if $k = n$.

The form in which we write the second equation in (1.1) is chosen only for the sake of convenience, in order to emphasize the profound analogy between the quasilinear and Hessian equations. Obviously, it may be stated as $(-1)^k\, F_k[u] = f(x,u)$, $u \geq 0$, or $F_k[u] = f(x,-u)$, $u \leq 0$.

The existence and regularity theory, local and global estimates of sub- and super-solutions, the Wiener criterion, and Harnack inequalities associated with the $p$-Laplacian, as well as more general quasilinear operators, can be found in [HKM], [IM], [KM2], [M1], [MZ], [S1], [S2], [SZ], [TW4] where many fundamental results, and relations to other areas of analysis and geometry are presented.

The theory of fully nonlinear equations of Monge-Ampère type which involve the $k$-Hessian operator $F_k[u]$ was originally developed by Caffarelli, Nirenberg and Spruck, Ivochkina, and Krylov in the classical setting. We refer to [CNS], [GT], [Gu], [Iv], [Kr], [Tru2], [TW1], [Ur] for these and further results. Recent developments concerning the notion of the $k$-Hessian measure, weak continuity, and pointwise potential estimates due to Trudinger and Wang [TW2]–[TW4], and Labutin [L] are used extensively in this paper.

We are specifically interested in quasilinear and fully nonlinear equations of Lane-Emden type:

$$-\Delta_p u = u^q, \quad \text{and} \qquad F_k[-u] = u^q, \qquad u \geq 0 \quad \text{in } \Omega, \tag{1.3}$$

where $p > 1$, $q > 0$, $k = 1, 2, \ldots, n$, and the corresponding nonlinear inequalities:

$$-\Delta_p u \geq u^q, \quad \text{and} \qquad F_k[-u] \geq u^q, \qquad u \geq 0 \quad \text{in } \Omega. \tag{1.4}$$

The latter can be stated in the form of the inhomogeneous equations with measure data,

$$-\Delta_p u = u^q + \mu, \qquad F_k[-u] = u^q + \mu, \qquad u \geq 0 \quad \text{in } \Omega, \tag{1.5}$$

where $\mu$ is a nonnegative Borel measure on $\Omega$.

The difficulties arising in studies of such equations and inequalities with competing nonlinearities are well known. In particular, (1.3) may have singular solutions [SZ]. The existence problem for (1.5) has been open ([BV2, Problems 1 and 2]; see also [BV1], [BV3], [Gre]) even for the quasilinear equation $-\Delta_p u = u^q + f$ with good data $f \in L^s(\Omega)$, $s > 1$. Here solutions are generally understood in the renormalized (entropy) sense for quasilinear equations, and viscosity, or the $k$-convexity sense, for fully nonlinear equations of Hessian type (see [BMMP], [DMOP], [JLM], [TW1]–[TW3], [Ur]). Precise definitions of these classes of *admissible solutions* are given in Sections 3, 6, and 7 below.

In this paper, we present a unified approach to (1.3)–(1.5) which makes it possible to attack a number of open problems. This is based on global pointwise estimates, nonlinear integral inequalities in Sobolev spaces of fractional order, and analysis of dyadic models, along with the Hessian measure and weak continuity results [TW2]–[TW4]. The latter are used to bridge the gap between the dyadic models and partial differential equations. Some of these techniques were developed in the linear case, in the framework of Schrödinger operators and harmonic analysis [ChWW], [Fef], [KS], [NTV], [V1], [V2], and applications to semilinear equations [KV], [VW], [V3].

Our goal is to establish *necessary and sufficient* conditions for the existence of solutions to (1.5), sharp pointwise and integral estimates for solutions to (1.4), and a complete characterization of removable singularities for (1.3). We are mostly concerned with admissible solutions to the corresponding equations and inequalities. However, even for locally bounded solutions, as in [SZ], our results yield new pointwise and integral estimates, and Liouville-type theorems.

In the "linear case" $p = 2$ and $k = 1$, problems (1.3)–(1.5) with nonlinear sources are associated with the names of Lane and Emden, as well as Fowler. Authoritative historical and bibliographical comments can be found in [SZ]. An up-to-date survey of the vast literature on nonlinear elliptic equations with measure data is given in [Ver], including a thorough discussion of related work due to D. Adams and Pierre [AP], Baras and Pierre [BP], Berestycki, Capuzzo-Dolcetta, and Nirenberg [BCDN], Brezis and Cabré [BC], Kalton and Verbitsky [KV].

It is worth mentioning that related equations with absorption,

$$-\Delta u + u^q = \mu, \qquad u \geq 0 \quad \text{in } \Omega, \tag{1.6}$$

were studied in detail by Bénilan and Brezis, Baras and Pierre, and Marcus and Véron analytically for $1 < q < \infty$, and by Le Gall, and Dynkin and Kuznetsov using probabilistic methods when $1 < q \leq 2$ (see [D], [Ver]). For a general class of semilinear equations

$$-\Delta u + g(u) = \mu, \qquad u \geq 0 \quad \text{in } \Omega, \tag{1.7}$$

where $g$ belongs to the class of continuous nondecreasing functions such that $g(0) = 0$, sharp existence results have been obtained quite recently by Brezis, Marcus, and Ponce [BMP]. It is well known that equations with absorption generally require "softer" methods of analysis, and the conditions on $\mu$ which ensure the existence of solutions are less stringent than in the case of equations with source terms.

Quasilinear problems of Lane-Emden type (1.3)–(1.5) have been studied extensively over the past 15 years. Universal estimates for solutions, Liouville-type theorems, and analysis of removable singularities are due to Bidaut-Véron, Mitidieri and Pohozaev [BV1]–[BV3], [BVP], [MP], and Serrin and Zou [SZ]. (See also [BiD], [Gre], [Ver], and the literature cited there.) The profound difficulties in this theory are highlighted by the presence of the two critical exponents,

$$q_* = \tfrac{n(p-1)}{n-p}, \qquad q^* = \tfrac{n(p-1)+p}{n-p}, \tag{1.8}$$

where $1 < p < n$. As was shown in [BVP], [MP], and [SZ], the quasilinear inequality (1.5) does not have nontrivial weak solutions on $\mathbb{R}^n$, or exterior domains, if $q \le q_*$. For $q > q_*$ , there exist $u \in W^{1,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ which obeys (1.4), as well as singular solutions to (1.3) on $\mathbb{R}^n$. However, for the existence of nontrivial solutions $u \in W^{1,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ to (1.3) on $\mathbb{R}^n$, it is necessary and sufficient that $q \ge q^*$ [SZ]. In the "linear case" $p = 2$, this is classical ([GS], [BP], [BCDN]).

The following local estimates of solutions to quasilinear inequalities are used extensively in the studies mentioned above (see, e.g., [SZ, Lemma 2.4]). Let $B_R$ denote a ball of radius $R$ such that $B_{2R} \subset \Omega$. Then, for every solution $u \in W^{1,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ to the inequality $-\Delta_p u \ge u^q$ in $\Omega$,

$$\int_{B_R} u^\gamma \, dx \le C\, R^{n-\frac{\gamma p}{q-p+1}}, \qquad 0 < \gamma < q, \tag{1.9}$$

$$\int_{B_R} |\nabla u|^{\frac{\gamma p}{q+1}} \, dx \le C\, R^{n-\frac{\gamma p}{q-p+1}}, \qquad 0 < \gamma < q, \tag{1.10}$$

where the constants $C$ in (1.9) and (1.10) depend only on $p, q, n, \gamma$. Note that (1.9) holds even for $\gamma = q$ (cf. [MP]), while (1.10) generally fails in this case. In what follows, we will substantially strengthen (1.9) in the end-point case $\gamma = q$, and obtain global pointwise estimates of solutions.

In [PV], we proved that all compact sets $E \subset \Omega$ of zero Hausdorff measure, $H^{n-\frac{pq}{q-p+1}}(E) = 0$, are removable singularities for the equation $-\Delta_p u = u^q$, $q > q_*$. Earlier results of this kind, under a stronger restriction $\mathrm{cap}_{1,\,\frac{pq}{q-p+1}+\varepsilon}(E) = 0$ for some $\varepsilon > 0$, are due to Bidaut-Véron [BV3]. Here $\mathrm{cap}_{1,\,s}(\cdot)$ is the capacity associated with the Sobolev space $W^{1,\,s}$.

In fact, much more is true. We will show below that a compact set $E \subset \Omega$ is a removable singularity for $-\Delta_p u = u^q$ if and only if it has zero fractional capacity: $\mathrm{cap}_{p,\,\frac{q}{q-p+1}}(E) = 0$. Here $\mathrm{cap}_{\alpha,\,s}$ stands for the Bessel capacity associated with the Sobolev space $W^{\alpha,\,s}$ which is defined in Section 2. We observe that the usual $p$-capacity $\mathrm{cap}_{1,\,p}$ used in the studies of the $p$-Laplacian [HKM], [KM2] plays a secondary role in the theory of equations of Lane-Emden type. Relations between these and other capacities used in nonlinear PDE theory are discussed in [AH], [M2], and [V4].

Our characterization of removable singularities is based on the solution of the existence problem for the equation

$$-\Delta_p u = u^q + \mu, \qquad u \geq 0, \tag{1.11}$$

with nonnegative measure $\mu$ obtained in Section 6. Main existence theorems for quasilinear equations are stated below (Theorems 2.3 and 2.10). Here we only mention the following corollary in the case $\Omega = \mathbb{R}^n$: If (1.11) has an admissible solution $u$, then

$$\int_{B_R} d\mu \leq C\, R^{n-\frac{pq}{q-p+1}}, \tag{1.12}$$

for every ball $B_R$ in $\mathbb{R}^n$, where $C = C(p,q,n)$, provided $1 < p < n$ and $q > q_*$; if $p \geq n$ or $q \leq q_*$, then $\mu = 0$.

Conversely, suppose that $1 < p < n$, $q > q_*$, and $d\mu = f\,dx$, $f \geq 0$, where

$$\int_{B_R} f^{1+\varepsilon}\, dx \leq C\, R^{n-\frac{(1+\varepsilon)pq}{q-p+1}}, \tag{1.13}$$

for some $\varepsilon > 0$. Then there exists a constant $C_0(p,q,n)$ such that (1.11) has an admissible solution on $\mathbb{R}^n$ if $C \leq C_0(p,q,n)$.

The preceding inequality is an analogue of the classical Fefferman-Phong condition [Fef] which appeared in applications to Schrödinger operators. In particular, (1.13) holds if $f \in L^{\frac{n(q-p+1)}{pq},\infty}(\mathbf{R}^n)$. Here $L^{s,\,\infty}$ stands for the weak $L^s$ space. This sufficiency result, which to the best of our knowledge is new even in the $L^s$ scale, provides a comprehensive solution to Problem 1 in [BV2]. Notice that the exponent $s = \frac{n(q-p+1)}{pq}$ is sharp. Broader classes of measures $\mu$ (possibly singular with respect to Lebesgue measure) which guarantee the existence of admissible solutions to (1.11) will be discussed in the sequel.

A substantial part of our work is concerned with integral inequalities for nonlinear potential operators, which are at the heart of our approach. We employ the notion of Wolff's potential introduced originally in [HW] in relation to the spectral synthesis problem for Sobolev spaces. For a nonnegative Borel measure $\mu$ on $\mathbb{R}^n$, $s \in (1, +\infty)$, and $\alpha > 0$, the Wolff's potential $\mathbf{W}_{\alpha,\,s}\,\mu$ is defined by

$$\mathbf{W}_{\alpha,\,s}\,\mu(x) = \int_0^\infty \Big[\frac{\mu(B_t(x))}{t^{n-\alpha s}}\Big]^{\frac{1}{s-1}} \frac{dt}{t}, \qquad x \in \mathbb{R}^n. \tag{1.14}$$

We write $\mathbf{W}_{\alpha,\,s}\, f$ in place of $\mathbf{W}_{\alpha,\,s}\,\mu$ if $d\mu = f dx$, where $f \in L^1_{\rm loc}(\mathbb{R}^n)$, $f \geq 0$. When dealing with equations in a bounded domain $\Omega \subset \mathbb{R}^n$, a truncated version is useful:

$$\mathbf{W}^r_{\alpha,\,s}\,\mu(x) = \int_0^r \left[\frac{\mu(B_t(x))}{t^{n-\alpha s}}\right]^{\frac{1}{s-1}} \frac{dt}{t}, \qquad x \in \Omega, \tag{1.15}$$

where $0 < r \leq 2{\rm diam}(\Omega)$. In many instances, it is more convenient to work with the dyadic version, also introduced in [HW]:

$$\mathcal{W}_{\alpha,\,s}\,\mu(x) = \sum_{Q\in\mathcal{D}} \left[\frac{\mu(Q)}{\ell(Q)^{n-\alpha s}}\right]^{\frac{1}{s-1}} \chi_Q(x), \qquad x \in \mathbb{R}^n, \tag{1.16}$$

where $\mathcal{D} = \{Q\}$ is the collection of the dyadic cubes $Q = 2^i(k + [0,\,1)^n)$, $i \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $\ell(Q)$ is the side length of $Q$.

An indispensable source on nonlinear potential theory is provided by [AH], where the fundamental Wolff's inequality and its applications are discussed. Very recently, an analogue of Wolff's inequality for general dyadic and radially decreasing kernels was obtained in [COV]; some of the tools developed there are employed below.

The dyadic Wolff's potentials appear in the following discrete model of (1.5) studied in Section 3:

$$u = \mathcal{W}_{\alpha,\,s}\, u^q + f, \qquad u \geq 0. \tag{1.17}$$

As it turns out, this nonlinear integral equation with $f = \mathcal{W}_{\alpha,\,s}\,\mu$ is intimately connected to the quasilinear differential equation (1.11) in the case $\alpha = 1$, $s = p$, and to its $k$-Hessian counterpart in the case $\alpha = \frac{2k}{k+1}$, $s = k+1$. Similar discrete models are used extensively in harmonic analysis and function spaces (see, e.g., [NTV], [St2], [V1]).

The profound role of Wolff's potentials in the theory of quasilinear equations was discovered by Kilpeläinen and Malý [KM2]. They established local pointwise estimates for nonnegative $p$-superharmonic functions in terms of Wolff's potentials of the associated $p$-Laplacian measure $\mu$. More precisely, if $u \geq 0$ is a $p$-superharmonic function in $B_{3r}(x)$ such that $-\Delta_p u = \mu$, then

$$C_1\,\mathbf{W}^r_{1,\,p}\,\mu(x) \leq u(x) \leq C_2 \inf_{B(x,r)} u + C_3\,\mathbf{W}^{2r}_{1,\,p}\,\mu(x), \tag{1.18}$$

where $C_1, C_2$ and $C_3$ are positive constants which depend only on $n$ and $p$.

In [TW1], [TW2], Trudinger and Wang introduced the notion of the Hessian measure $\mu[u]$ associated with $F_k[u]$ for a $k$-convex function $u$. Very recently, Labutin [L] proved local pointwise estimates for Hessian equations analogous to (1.18), where Wolff's potential $\mathbf{W}^r_{\frac{2k}{k+1},\,k+1}\,\mu$ is used in place of $\mathbf{W}^r_{1,\,p}\,\mu$.

In what follows, we will need *global* pointwise estimates of this type. In the case of a $k$-convex solution to the equation $F_k[u] = \mu$ on $\mathbb{R}^n$ such that

$\inf_{x\in\mathbb{R}^n} (-u(x)) = 0$, one has

$$C_1 \, \mathbf{W}_{\frac{2k}{k+1}, \, k+1} \, \mu(x) \le -u(x) \le C_2 \, \mathbf{W}_{\frac{2k}{k+1}, \, k+1} \, \mu(x), \tag{1.19}$$

where $C_1$ and $C_2$ are positive constants which depend only on $n$ and $k$. Analogous global estimates are obtained below for admissible solutions of the Dirichlet problem for $-\Delta_p u = \mu$ and $F_k[-u] = \mu$ in a bounded domain $\Omega \subset \mathbb{R}^n$ (see §2).

In the special case $\Omega = \mathbb{R}^n$, our criterion for the solvability of (1.11) can be stated in the form of the pointwise condition involving Wolff's potentials:

$$\mathbf{W}_{1,\,p} \, (\mathbf{W}_{1,\,p} \, \mu \,)^q \, (x) \le C \, \mathbf{W}_{1,\,p} \, \mu(x) < +\infty \quad \text{a.e.}, \tag{1.20}$$

which is necessary with $C = C_1(p, q, n)$, and sufficient with another constant $C = C_2(p, q, n)$. Moreover, in the latter case there exists an admissible solution $u$ to (1.11) such that

$$c_1 \, \mathbf{W}_{1,\,p} \, \mu(x) \le u(x) \le c_2 \, \mathbf{W}_{1,\,p} \, \mu(x), \qquad x \in \mathbb{R}^n, \tag{1.21}$$

where $c_1$ and $c_2$ are positive constants which depend only on $p, q, n$, provided $1 < p < n$ and $q > q_*$; if $p \ge n$ or $q \le q_*$ then $u = 0$ and $\mu = 0$.

The iterated Wolff's potential condition (1.20) is crucial in our approach. As we will demonstrate in Section 5, it turns out to be equivalent to the fractional Riesz capacity condition

$$\mu(E) \le C \, \mathrm{Cap}_{p, \frac{q}{q-p+1}} \, (E), \tag{1.22}$$

where $C$ does not depend on a compact set $E \subset \mathbb{R}^n$. Such classes of measures $\mu$ were introduced by V. Maz'ya in the early 60-s in the framework of linear problems.

It follows that every admissible solution $u$ to (1.11) on $\mathbb{R}^n$ obeys the inequality

$$\int_E u^q \, dx \le C \, \mathrm{Cap}_{p, \frac{q}{q-p+1}} \, (E), \tag{1.23}$$

for all compact sets $E \subset \mathbb{R}^n$. We also prove an analogous estimate in a bounded domain $\Omega$ (Section 6). Obviously, this yields (1.9) in the end-point case $\gamma = q$. In the critical case $q = q_*$, we obtain an improved estimate (see Corollary 6.13):

$$\int_{B_r} u^{q_*} \, dx \le C \, \left(\log(\tfrac{2R}{r})\right)^{\frac{1-p}{q-p+1}}, \tag{1.24}$$

for every ball $B_r$ of radius $r$ such that $B_r \subset B_R$, and $B_{2R} \subset \Omega$. Certain Carleson measure inequalities are employed in the proof of (1.24). We observe that these estimates yield Liouville-type theorems for all admissible solutions to (1.11) on $\mathbb{R}^n$, or in exterior domains, provided $q \le q_*$ (cf. [BVP], [SZ]).

Analogous results will be established in Section 7 for equations of Lane-Emden type involving the $k$-Hessian operator $F_k[u]$. We will prove that there exists a constant $C_1(k,q,n)$ such that, if

$$\mathbf{W}_{\frac{2k}{k+1},k+1}(\mathbf{W}_{\frac{2k}{k+1},k+1}\mu)^q(x) \leq C\,\mathbf{W}_{\frac{2k}{k+1},k+1}\mu(x) < +\infty \text{ a.e.,} \tag{1.25}$$

where $0 \leq C \leq C_1(k,q,n)$, then the equation

$$F_k[-u] = u^q + \mu, \qquad u \geq 0, \tag{1.26}$$

has a solution $u$ so that $-u$ is $k$-convex on $\mathbb{R}^n$, and

$$c_1\,\mathbf{W}_{\frac{2k}{k+1},k+1}\,\mu(x) \leq u(x) \leq c_2\,\mathbf{W}_{\frac{2k}{k+1},k+1}\,\mu(x), \qquad x \in \mathbb{R}^n, \tag{1.27}$$

where $c_1$, $c_2$ are positive constants which depend only on $k,q,n$, for $1 \leq k < \frac{n}{2}$. Conversely, (1.25) with $C = C_2(k,q,n)$ is necessary in order that (1.26) has a solution $u$ such that $-u$ is $k$-convex on $\mathbb{R}^n$ provided $1 \leq k < \frac{n}{2}$ and $q > q_* = \frac{nk}{n-2k}$; if $k \geq \frac{n}{2}$ or $q \leq q_*$ then $u = 0$ and $\mu = 0$.

In particular, (1.25) holds if $d\mu = f\,dx$, where $f \geq 0$ and $f \in L^{\frac{n(q-k)}{2kq},\infty}(\mathbf{R}^n)$; the exponent $\frac{n(q-k)}{2kq}$ is sharp.

In Section 7, we will obtain precise existence theorems for equation (1.26) in a bounded domain $\Omega$ with the Dirichlet boundary condition $u = \varphi$, $\varphi \geq 0$, on $\partial\Omega$, for $1 \leq k \leq n$. Furthermore, removable singularities $E \subset \Omega$ for the homogeneous equation $F_k[-u] = u^q$, $u \geq 0$, will be characterized as the sets of zero Bessel capacity $\text{cap}_{2k,\frac{q}{q-k}}(E) = 0$, in the most interesting case $q > k$.

The notion of the *$k$-Hessian capacity* introduced by Trudinger and Wang proved to be very useful in studies of the uniqueness problem for $k$-Hessian equations [TW3], as well as associated $k$-polar sets [L]. Comparison theorems for this capacity and the corresponding Hausdorff measure were obtained by Labutin in [L] where it is proved that the $(n-2k)$-Hausdorff dimension is critical in this respect. We will enhance this result (see Theorem 2.20 below) by showing that the $k$-Hessian capacity is in fact locally equivalent to the fractional Bessel capacity $\text{cap}_{\frac{2k}{k+1},k+1}$.

In conclusion, we remark that our methods provide a promising approach for a wide class of nonlinear problems, including curvature and subelliptic equations, and more general nonlinearities.

## 2. Main results

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. We study the existence problem for the quasilinear equation

$$\begin{cases} -\text{div}\mathcal{A}(x,\nabla u) = u^q + \omega, \\ \quad u \geq 0 \quad \text{in} \quad \Omega, \\ \quad u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $p > 1$, $q > p - 1$ and

$$\mathcal{A}(x, \xi) \cdot \xi \geq \alpha \, |\xi|^p, \qquad |\mathcal{A}(x, \xi)| \leq \beta \, |\xi|^{p-1} \tag{2.2}$$

for some $\alpha, \beta > 0$. The precise structural conditions imposed on $\mathcal{A}(x, \xi)$ are stated in Section 4, formulae (4.1)–(4.5). This includes the principal model problem

$$\begin{cases} -\Delta_p u = u^q + \omega, \\ u \geq 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.3}$$

Here $\Delta_p$ is the $p$-Laplacian defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \, \nabla u)$. We observe that in the well-studied case $q \leq p - 1$, hard analysis techniques are not needed, and many of our results simplify. We refer to [Gre], [SZ] for further comments and references, especially in the classical case $q = p - 1$.

Our approach also applies to the following class of fully nonlinear equations

$$\begin{cases} F_k[-u] = u^q + \omega, \\ u \geq 0 \quad \text{in} \quad \Omega, \\ u = \varphi \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.4}$$

where $k = 1, 2, \ldots, n$, and $F_k$ is the $k$-Hessian operator defined by (1.2). Here $-u$ belongs to the class of $k$-subharmonic (or $k$-convex) functions on $\Omega$ introduced by Trudinger and Wang in [TW1]–[TW2]. Analogues of equations (2.1) and (2.4) on the entire space $\mathbb{R}^n$ are studied as well.

To state our results, let us introduce some definitions and notation. Let $\mathcal{M}^+_B(\Omega)$ (respectively $\mathcal{M}^+(\Omega)$) denote the class of all nonnegative finite (respectively locally finite) Borel measures on $\Omega$. For $\mu \in \mathcal{M}^+(\Omega)$ and a Borel set $E \subset \Omega$, we denote by $\mu_E$ the restriction of $\mu$ to $E$: $d\mu_E = \chi_E d\mu$ where $\chi_E$ is the characteristic function of $E$. We define the Riesz potential $\mathbf{I}_\alpha$ of order $\alpha$, $0 < \alpha < n$, on $\mathbb{R}^n$ by

$$\mathbf{I}_\alpha \mu(x) = c(n, \alpha) \int_{\mathbb{R}^n} |x - y|^{\alpha - n} \, d\mu(y), \qquad x \in \mathbb{R}^n,$$

where $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ and $c(n, \alpha)$ is a normalized constant. For $\alpha > 0$, $p > 1$, such that $\alpha p < n$, the Wolff's potential $\mathbf{W}_{\alpha, p}\mu$ is defined by

$$\mathbf{W}_{\alpha, p}\mu(x) = \int_0^\infty \Big[ \frac{\mu(B_t(x))}{t^{n - \alpha p}} \Big]^{\frac{1}{p-1}} \frac{dt}{t}, \qquad x \in \mathbb{R}^n.$$

When dealing with equations in a bounded domain $\Omega \subset \mathbb{R}^n$, it is convenient to use the truncated versions of Riesz and Wolff's potentials. For $0 < r \leq \infty$, $\alpha > 0$ and $p > 1$, we set

$$\mathbf{I}^r_\alpha \mu(x) = \int_0^r \frac{\mu(B_t(x))}{t^{n - \alpha}} \frac{dt}{t}, \qquad \mathbf{W}^r_{\alpha, p}\mu(x) = \int_0^r \Big[ \frac{\mu(B_t(x))}{t^{n - \alpha p}} \Big]^{\frac{1}{p-1}} \frac{dt}{t}.$$

Here $\mathbf{I}_\alpha^\infty$ and $\mathbf{W}_{\alpha,p}^\infty$ are understood as $\mathbf{I}_\alpha$ and $\mathbf{W}_{\alpha,p}$ respectively. For $\alpha > 0$, we denote by $\mathbf{G}_\alpha$ the Bessel kernel of order $\alpha$ (see [AH, §1.2.4]). The Bessel potential of a measure $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ is defined by

$$\mathbf{G}_\alpha\mu(x) = \int_{\mathbb{R}^n} \mathbf{G}_\alpha(x-y)d\mu(y), \qquad x \in \mathbb{R}^n.$$

Various capacities will be used throughout the paper. Among them are the Riesz and Bessel capacities defined respectively by

$$\mathrm{Cap}_{\mathbf{I}_\alpha,\, s}(E) = \inf\{\|f\|^s_{L^s(\mathbb{R}^n)} : \mathbf{I}_\alpha f \geq \chi_E,\ 0 \leq f \in L^s(\mathbb{R}^n)\},$$

and

$$\mathrm{Cap}_{\mathbf{G}_\alpha,\, s}(E) = \inf\{\|f\|^s_{L^s(\mathbb{R}^n)} : \mathbf{G}_\alpha f \geq \chi_E,\ 0 \leq f \in L^s(\mathbb{R}^n)\}$$

for any $E \subset \mathbb{R}^n$.

Our first two theorems are concerned with *global* pointwise potential estimates for quasilinear and Hessian equations on a bounded domain $\Omega$ in $\mathbb{R}^n$.

THEOREM 2.1. *Suppose that $u$ is a renormalized solution to the equation*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x,\nabla u) = \omega & \text{in} \quad \Omega, \\ \qquad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.5}$$

*with data $\omega \in \mathcal{M}_B^+(\Omega)$. Then there is a constant $K = K(n,p,\alpha,\beta) > 0$ such that, for all $x$ in $\Omega$,*

$$\frac{1}{K}\,\mathbf{W}_{1,\,p}^{\frac{\mathrm{dist}(x,\partial\Omega)}{3}}\omega(x) \leq u(x) \leq K\,\mathbf{W}_{1,\,p}^{2\mathrm{diam}(\Omega)}\omega(x). \tag{2.6}$$

THEOREM 2.2. *Let $\omega \in M_B^+(\Omega)$ be compactly supported in $\Omega$. Suppose that $-u$ is a nonpositive $k$-subharmonic function in $\Omega$ such that $u$ is continuous near $\partial\Omega$ and solves the equation*

$$\begin{cases} F_k[-u] = \omega & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

*Then there is a constant $K = K(n,k) > 0$ such that, for all $x \in \Omega$,*

$$\frac{1}{K}\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}^{\frac{\mathrm{dist}(x,\partial\Omega)}{8}}\omega(x) \leq u(x) \leq K\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}^{2\mathrm{diam}(\Omega)}\omega(x). \tag{2.7}$$

We remark that the upper estimate in (2.6) does not hold in general if $u$ is merely a weak solution of (2.5) in the sense of [KM1]. For a counterexample, see [Kil, §2]. Upper estimates similar to the one in (2.7) hold also for $k$-subharmonic functions with nonhomogeneous boundary condition (see §7). Definitions of renormalized solutions for the problem (2.5) are given in Section 6; for definitions of $k$-subharmonic functions see Section 7.

As was mentioned in the introduction, these global pointwise estimates simplify in the case $\Omega = \mathbb{R}^n$; see Corollary 4.5 and Corollary 7.3 below.

In the next two theorems we give criteria for the solvability of quasilinear and Hessian equations on the entire space $\mathbb{R}^n$.

THEOREM 2.3. *Let $\omega$ be a measure in $\mathcal{M}^+(\mathbb{R}^n)$. Let $1 < p < n$ and $q > p-1$. Then the following statements are equivalent.*

(i) *There exists a nonnegative $\mathcal{A}$-superharmonic solution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ to the equation*

$$\begin{cases} \inf_{x\in\mathbb{R}^n} u(x) = 0, \\ -\mathrm{div}\mathcal{A}(x,\nabla u) = u^q + \varepsilon\,\omega \quad \text{in} \quad \mathbb{R}^n \end{cases} \tag{2.8}$$

*for some $\varepsilon > 0$.*

(ii) *The testing inequality*

$$\int_B \Big[\mathbf{I}_p\omega_B(x)\Big]^{\frac{q}{p-1}} dx \le C\omega(B) \tag{2.9}$$

*holds for all balls $B$ in $\mathbb{R}^n$.*

(iii) *For all compact sets $E \subset \mathbb{R}^n$,*

$$\omega(E) \le C\,\mathrm{Cap}_{\mathbf{I}_p,\,\frac{q}{q-p+1}}(E). \tag{2.10}$$

(iv) *The testing inequality*

$$\int_B \Big[\mathbf{W}_{1,\,p}\omega_B(x)\Big]^{q} dx \le C\,\omega(B) \tag{2.11}$$

*holds for all balls $B$ in $\mathbb{R}^n$ .*

(v) *There exists a constant $C$ such that*

$$\mathbf{W}_{1,\,p}\,(\mathbf{W}_{1,\,p}\omega)^q(x) \le C\,\mathbf{W}_{1,\,p}\omega(x) < \infty \quad a.e. \tag{2.12}$$

*Moreover, there is a constant $C_0 = C_0(n,p,q,\alpha,\beta)$ such that if any one of the conditions* (2.9)–(2.12) *holds with $C \le C_0$, then equation* (2.8) *has a solution $u$ with $\varepsilon = 1$ which satisfies the two-sided estimate*

$$c_1\,\mathbf{W}_{1,\,p}\omega(x) \le u(x) \le c_2\,\mathbf{W}_{1,\,p}\omega(x), \qquad x \in \mathbb{R}^n, \tag{2.13}$$

*where $c_1$ and $c_2$ depend only on $n,p,q,\alpha,\beta$. Conversely, if* (2.8) *has a solution $u$ as in statement* (i) *with $\varepsilon = 1$, then conditions* (2.9)–(2.12) *hold with $C = C_1(n,p,q,\alpha,\beta)$. Here $\alpha$ and $\beta$ are the structural constants of $\mathcal{A}$ defined in* (2.2).

Using condition (2.10) in the above theorem, we can now deduce a simple sufficient condition for the solvability of (2.8) from the known inequality (see, e.g., [AH, p. 39])

$$|E|^{1-\frac{pq}{n(q-p+1)}} \le C\,\mathrm{Cap}_{\mathbf{I}_p,\,\frac{q}{q-p+1}}(E).$$

Corollary 2.4. *Suppose that* $f \in L^{\frac{n(q-p+1)}{pq},\infty}(\mathbb{R}^n)$ *and* $d\omega = f dx$. *If* $q > p-1$ *and* $\frac{pq}{q-p+1} < n$, *then equation* (2.8) *has a nonnegative solution for some* $\varepsilon > 0$.

*Remark* 2.5. The condition $f \in L^{\frac{n(q-p+1)}{pq},\infty}(\mathbb{R}^n)$ in Corollary 2.4 can be relaxed by using the Fefferman-Phong condition [Fef]:

$$\int_{B_R} f^{1+\delta} dx \leq C R^{n-\frac{(1+\delta)pq}{q-p+1}}$$

for some $\delta > 0$, which is known to be sufficient for the validity of (2.9); see, e.g., [KS], [V2].

Theorem 2.6. *Let* $\omega$ *be a measure in* $\mathcal{M}^+(\mathbb{R}^n)$, $1 \leq k < \frac{n}{2}$, *and* $q > k$. *Then the following statements are equivalent.*

(i) *There exists a solution* $u \geq 0$, $-u \in \Phi^k(\Omega) \cap L^q_{\rm loc}(\mathbb{R}^n)$, *to the equation*

$$\begin{cases} \inf_{x\in\mathbb{R}^n} u(x) = 0, \\ F_k[-u] = u^q + \varepsilon\,\omega \quad \text{in} \quad \mathbb{R}^n \end{cases} \tag{2.14}$$

*for some* $\varepsilon > 0$.

(ii) *The testing inequality*

$$\int_B \Big[\mathbf{I}_{2k}\omega_B(x)\Big]^{\frac{q}{k}} dx \leq C\,\omega(B) \tag{2.15}$$

*holds for all balls* $B$ *in* $\mathbb{R}^n$.

(iii) *For all compact sets* $E \subset \mathbb{R}^n$,

$$\omega(E) \leq C\,\mathrm{Cap}_{\mathbf{I}_{2k},\,\frac{q}{q-k}}(E). \tag{2.16}$$

(iv) *The testing inequality*

$$\int_B \Big[\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega_B(x)\Big]^{q} dx \leq C\,\omega(B) \tag{2.17}$$

*holds for all balls* $B$ *in* $\mathbb{R}^n$

(v) *There exists a constant* $C$ *such that*

$$\mathbf{W}_{\frac{2k}{k+1},\,k+1}\,(\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega)^q(x) \leq C\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega(x) < \infty \quad a.e. \tag{2.18}$$

*Moreover, there is a constant* $C_0 = C_0(n,k,q)$ *such that if any one of the conditions* (2.15)–(2.18) *holds with* $C \leq C_0$, *then equation* (2.14) *has a solution* $u$ *with* $\varepsilon = 1$ *which satisfies the two-sided estimate*

$$c_1\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega(x) \leq u(x) \leq c_2\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega(x), \qquad x \in \mathbb{R}^n,$$

*where* $c_1$ *and* $c_2$ *depend only on* $n, k, q$. *Conversely, if there is a solution* $u$ *to* (2.14) *as in statement* (i) *with* $\varepsilon = 1$, *then conditions* (2.15)–(2.18) *hold with* $C = C_1(n,k,q)$.

Corollary 2.7. *Suppose that* $f \in L^{\frac{n(q-k)}{2kq},\infty}(\mathbb{R}^n)$ *and* $d\omega = f dx$. *If* $q > k$ *and* $\frac{2kq}{q-k} < n$ *then* (2.14) *has a nonnegative solution for some* $\varepsilon > 0$.

Since $\mathrm{Cap}_{I_\alpha, s}(E) = 0$ in the case $\alpha\, s \geq n$ for all sets $E \subset \mathbb{R}^n$ (see [AH, §2.6]), we obtain the following Liouville-type theorems for quasilinear and Hessian differential inequalities.

Corollary 2.8. *If* $q \leq \frac{n(p-1)}{n-p}$, *then the inequality* $-\mathrm{div}\mathcal{A}(x, \nabla u) \geq u^q$ *admits no nontrivial nonnegative* $\mathcal{A}$*-superharmonic solutions in* $\mathbb{R}^n$. *Analogously, if* $q \leq \frac{nk}{n-2k}$, *then the inequality* $F_k[-u] \geq u^q$ *admits no nontrivial nonnegative solutions in* $\mathbb{R}^n$.

*Remark* 2.9. When $1 < p < n$ and $q > \frac{n(p-1)}{n-p}$, the function $u(x) = c\,|x|^{\frac{-p}{q-p+1}}$ with

$$c = \Big[\frac{p^{p-1}}{(q-p+1)^p}\Big]^{\frac{1}{q-p+1}} [q(n-p) - n(p-1)]^{\frac{1}{q-p+1}},$$

is a nontrivial admissible (but singular) global solution of $-\Delta_p u = u^q$ (see [SZ]). Similarly, the function $u(x) = c'\,|x|^{\frac{-2k}{q-k}}$ with

$$c' = \Big[\frac{(n-1)!}{k!(n-k)!}\Big]^{\frac{1}{q-k}} \Big[\frac{(2k)^k}{(q-k)^{k+1}}\Big]^{\frac{1}{q-k}} [q(n-2k) - nk]^{\frac{1}{q-k}},$$

where $1 \leq k < \frac{n}{2}$ and $q > \frac{nk}{n-2k}$, is a singular admissible global solution of $F_k[-u] = u^q$ (see [Tso] or [Tru1, formula (3.2)]). Thus, we see that the exponent $\frac{n(p-1)}{n-p}$ (respectively $\frac{nk}{n-2k}$) is critical for the homogeneous equation $-\mathrm{div}\mathcal{A}(x, \nabla u) = u^q$ (respectively $F_k[-u] = u^q$) in $\mathbb{R}^n$. The situation is different when we restrict ourselves only to *locally bounded* solutions in $\mathbb{R}^n$ (see [GS], [SZ]).

Existence results on a bounded domain $\Omega$ analogous to Theorems 2.3 and 2.6 are contained in the following two theorems, where Bessel potentials and the corresponding capacities are used in place of respectively Riesz potentials and Riesz capacities.

Theorem 2.10. *Let* $\omega \in \mathcal{M}_B^+(\Omega)$ *be compactly supported in* $\Omega$. *Let* $p > 1$, $q > p-1$, *and let* $R = \mathrm{diam}(\Omega)$. *Then the following statements are equivalent.*

(i) *There exists a nonnegative renormalized solution* $u \in L^q(\Omega)$ *to the equation*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u) = u^q + \varepsilon\, \omega \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.19}$$

*for some* $\varepsilon > 0$.

(ii) *For all compact sets* $E \subset \Omega$,

$$\omega(E) \le C\, \mathrm{Cap}_{\mathbf{G}_p,\, \frac{q}{q-p+1}}(E). \tag{2.20}$$

(iii) *The testing inequality*

$$\int_B \left[\mathbf{W}^{2R}_{1,\,p}\omega_B(x)\right]^q dx \le C\,\omega(B) \tag{2.21}$$

*holds for all balls* $B$ *such that* $B \cap \operatorname{supp}\omega \neq \emptyset$.

(iv) *There exists a constant* $C$ *such that*

$$\mathbf{W}^{2R}_{1,\,p}\,(\mathbf{W}^{2R}_{1,\,p}\omega)^q(x) \le C\,\mathbf{W}^{2R}_{1,\,p}\omega(x) \quad \text{a.e. on } \Omega. \tag{2.22}$$

*Remark* 2.11. In the case where $\omega$ is not compactly supported in $\Omega$, it can be easily seen from the proof of this theorem, given in Section 6, that any one of the conditions (ii)–(iv) above is still *sufficient* for the solvability of (2.19). Moreover, in the subcritical case $\frac{pq}{q-p+1} > n$, these conditions are redundant since the Bessel capacity $\mathrm{Cap}_{\mathbf{G}_p,\, \frac{q}{q-p+1}}$ of a single point is positive (see [AH], §2.6). This ensures that statement (ii) of Theorem 2.10 holds for some constant $C > 0$ provided $\omega$ is a finite measure.

Corollary 2.12. *Suppose that* $f \in L^{\frac{n(q-p+1)}{pq},\infty}(\Omega)$ *and* $d\omega = f dx$. *If* $q > p-1$ *and* $\frac{pq}{q-p+1} < n$ *then the equation* (2.19) *has a nonnegative renormalized* (*or equivalently, entropy*) *solution for some* $\varepsilon > 0$.

Theorem 2.13. *Let* $\Omega$ *be a uniformly* $(k-1)$*-convex domain in* $\mathbb{R}^n$, *and let* $\omega \in \mathcal{M}^+_B(\Omega)$ *be compactly supported in* $\Omega$. *Suppose that* $1 \le k \le n$, $q > k$, $R = \operatorname{diam}(\Omega)$, *and* $\varphi \in C^0(\partial\Omega)$, $\varphi \ge 0$. *Then the following statements are equivalent.*

(i) *There exists a solution* $u \ge 0$, $-u \in \Phi^k(\Omega) \cap L^q(\Omega)$, *continuous near* $\partial\Omega$, *to the equation*

$$\begin{cases} F_k[-u] = u^q + \varepsilon\,\omega & \text{in} \quad \Omega, \\ \qquad u = \varepsilon\,\varphi & \text{on} \quad \partial\Omega \end{cases} \tag{2.23}$$

*for some* $\varepsilon > 0$.

(ii) *For all compact sets* $E \subset \Omega$,

$$\omega(E) \le C\, \mathrm{Cap}_{\mathbf{G}_{2k},\, \frac{q}{q-k}}(E).$$

(iii) *The testing inequality*

$$\int_B \left[\mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}\omega_B(x)\right]^q dx \le C\,\omega(B)$$

*holds for all balls* $B$ *such that* $B \cap \operatorname{supp}\omega \neq \emptyset$ .

(iv) *There exists a constant $C$ such that*

$$\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega)^q(x) \le C\,\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega(x) \quad \text{a.e. on } \Omega.$$

*Remark* 2.14. As in Remark 2.11, any one of the conditions (ii)–(iv) in Theorem 2.13 is still *sufficient* for the solvability of (2.23) if $d\omega = d\mu + f\,dx$, where $\mu \in \mathcal{M}^+_B(\Omega)$ is compactly supported in $\Omega$ and $f \in L^s(\Omega)$, $f \ge 0$ with $s > \frac{n}{2k}$ if $k \le \frac{n}{2}$, and $s = 1$ if $k > \frac{n}{2}$. Moreover, in the subcritical case $\frac{2kq}{q-k} > n$ these conditions are redundant.

Corollary 2.15. *Let $d\omega = (f+g)\,dx$, where $f \ge 0$, $g \ge 0$, $f \in L^{\frac{n(q-k)}{2kq},\infty}(\Omega)$ is compactly supported in $\Omega$, and $g \in L^s(\Omega)$ for some $s > \frac{n}{2k}$. If $q > k$ and $\frac{2kq}{q-k} < n$ then* (2.23) *has a nonnegative solution for some $\varepsilon > 0$.*

Our results on *local integral estimates* for quasilinear and Hessian inequalities are given in the next two theorems. We will need the capacity associated with the space $W^{\alpha,s}$ relative to the domain $\Omega$ defined by

$$\operatorname{cap}_{\alpha,s}(E,\Omega) = \inf\{\|f\|^s_{W^{\alpha,s}(\mathbb{R}^n)} : f \in C^\infty_0(\Omega), f \ge 1 \text{ on } E\}. \tag{2.24}$$

Theorem 2.16. *Let $u$ be a nonnegative $\mathcal{A}$-superharmonic function in $\Omega$ such that $-\mathrm{div}\mathcal{A}(x,\nabla u) \ge u^q$. Suppose that $q > p-1$, $\frac{pq}{q-p+1} < n$, and $\Omega$ is a bounded $C^\infty$-domain. Then*

$$\int_E u^q \le C\,\operatorname{cap}_{p,\,\frac{q}{q-p+1}}(E,\Omega)$$

*for any compact set $E \subset \Omega$, where the constant $C$ may depend only on $p, q, n$, and the structural constants $\alpha$, $\beta$ of $\mathcal{A}$.*

Theorem 2.17. *Let $u \ge 0$ be such that $-u$ is $k$-subharmonic and that $F_k[-u] \ge u^q$ in $\Omega$. Suppose that $q > k$, $\frac{2kq}{q-k} < n$, and $\Omega$ is a bounded $C^\infty$-domain. Then*

$$\int_E u^q \le C\,\operatorname{cap}_{2k,\,\frac{q}{q-k}}(E,\Omega)$$

*for any compact set $E \subset \Omega$, where the constant $C$ may depend only on $k, q$ and $n$.*

As a consequence of Theorems 2.10 and 2.13, we will deduce the following characterization of removable singularities for quasilinear and fully nonlinear equations.

Theorem 2.18. *Let $E$ be a compact subset of $\Omega$. Then any solution $u$ to the problem*

$$\begin{cases} u \text{ is } \mathcal{A}\text{-superharmonic in } \Omega \setminus E, \\ u \in L^q_{\mathrm{loc}}(\Omega \setminus E), \quad u \ge 0, \\ -\mathrm{div}\mathcal{A}(x,\nabla u) = u^q \quad \text{in} \quad \mathcal{D}'(\Omega \setminus E) \end{cases} \tag{2.25}$$

*is also a solution to a similar problem with* $\Omega$ *in place of* $\Omega \setminus E$ *if and only if* $\mathrm{Cap}_{\mathbf{G}_p,\, \frac{q}{q-p+1}}(E) = 0$.

THEOREM 2.19. *Let* $E$ *be a compact subset of* $\Omega$. *Then any solution* $u$ *to the problem*

$$\left\{\begin{array}{c} -u \text{ is } k\text{-subharmonic in } \Omega \setminus E, \\ u \in L^q_{\mathrm{loc}}(\Omega \setminus E), \quad u \geq 0, \\ F_k[-u] = u^q \quad \text{in} \quad \mathcal{D}'(\Omega \setminus E) \end{array}\right. \tag{2.26}$$

*is also a solution to a similar problem with* $\Omega$ *in place of* $\Omega \setminus E$ *if and only if* $\mathrm{Cap}_{\mathbf{G}_{2k},\, \frac{q}{q-k}}(E) = 0$.

In [TW3], Trudinger and Wang introduced the so called *k-Hessian* capacity $\mathrm{cap}_k(\cdot, \Omega)$ defined for a compact set $E$ by

$$\mathrm{cap}_k(E, \Omega) = \sup\, \Big\{ \int_E d\mu_k[u] \Big\}, \tag{2.27}$$

where the supremum is taken over all $k$-subharmonic functions $u$ in $\Omega$ such that $-1 < u < 0$, and $\mu_k[u]$ is the $k$-Hessian measure associated with $u$. Our next theorem asserts that locally the $k$-Hessian capacity is equivalent to the Bessel capacity $\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},\, k+1}$. In what follows, $\mathcal{Q} = \{Q\}$ will stand for a Whitney decomposition of $\Omega$ into a union of disjoint dyadic cubes (see §6).

THEOREM 2.20. *Let* $1 \leq k < \frac{n}{2}$ *be an integer. Then there are constants* $M_1$, $M_2$ *such that*

$$M_1\, \mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},\, k+1}(E) \leq \mathrm{cap}_k(E, \Omega) \leq M_2\, \mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},\, k+1}(E) \tag{2.28}$$

*for any compact set* $E \subset \overline{Q}$ *with* $Q \in \mathcal{Q}$. *Furthermore, if* $\Omega$ *is a bounded* $C^\infty$*-domain then*

$$\mathrm{cap}_k(E, \Omega) \leq C\, \mathrm{cap}_{\frac{2k}{k+1},\, k+1}(E, \Omega) \tag{2.29}$$

*for any compact set* $E \subset \Omega$, *where* $\mathrm{cap}_{\frac{2k}{k+1},\, k+1}(E, \Omega)$ *is defined by* (2.24) *with* $\alpha = \frac{2k}{k+1}$ *and* $s = k + 1$.

## 3. Discrete models of nonlinear equations

In this section we consider certain nonlinear integral equations with discrete kernels which serve as a model for both quasilinear and Hessian equations treated in Section 5–7. Let $\mathcal{D}$ be the family of all dyadic cubes $Q = 2^i(k + [0, 1)^n)$, $i \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, in $\mathbb{R}^n$. For $\omega \in \mathcal{M}^+(\mathbb{R}^n)$, we define the dyadic Riesz and Wolff's potentials respectively by

$$\mathcal{I}_\alpha \omega(x) = \sum_{Q \in \mathcal{D}} \frac{\omega(Q)}{|Q|^{1-\frac{\alpha}{n}}} \chi_Q(x), \tag{3.1}$$

$$\mathcal{W}_{\alpha,\,p}\omega(x) = \sum_{Q\in\mathcal{D}} \left[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\right]^{\frac{1}{p-1}} \chi_Q(x). \tag{3.2}$$

In this section we are concerned with nonlinear inhomogeneous integral equations of the type

$$u = \mathcal{W}_{\alpha,\,p}(u^q) + f, \qquad u \in L^q_{\text{loc}}(\mathbb{R}^n), u \geq 0, \tag{3.3}$$

where $f \in L^q_{\text{loc}}(\mathbb{R}^n)$, $f \geq 0$, $q > p-1$, and $\mathcal{W}_{\alpha,\,p}$ is defined as in (3.2) with $\alpha > 0$ and $p > 1$ such that $0 < \alpha p < n$.

It is convenient to introduce a nonlinear operator $\mathcal{N}$ associated with the equation (3.3) defined by

$$\mathcal{N} f = \mathcal{W}_{\alpha,\,p}(f^q), \qquad f \in L^q_{\text{loc}}(\mathbb{R}^n), f \geq 0, \tag{3.4}$$

so that (3.3) can be rewritten as

$$u = \mathcal{N} u + f, \qquad u \in L^q_{\text{loc}}(\mathbb{R}^n), u \geq 0.$$

Obviously, $\mathcal{N}$ is monotonic, i.e., $\mathcal{N} f \geq \mathcal{N} g$ whenever $f \geq g \geq 0$ a.e., and $\mathcal{N}(\lambda f) = \lambda^{\frac{q}{p-1}} \mathcal{N} f$ for all $\lambda \geq 0$. Since

$$(a+b)^{p'-1} \leq \max\{1, 2^{p'-2}\}(a^{p'-1} + b^{p'-1}) \tag{3.5}$$

for all $a, b \geq 0$, it follows that

$$\Big[\mathcal{N}(f+g)\Big]^{\frac{1}{q}} \leq \max\{1, 2^{p'-2}\}\Big[(\mathcal{N} f)^{\frac{1}{q}} + (\mathcal{N} g)^{\frac{1}{q}}\Big]. \tag{3.6}$$

Proposition 3.1. *Let $\mu \in M^+(\mathbb{R}^n)$, $\alpha > 0$, $p > 1$, and $q > p-1$. Then the following quantities are equivalent*:

(a) $$A_1(P,\mu) = \sum_{Q\subset P} \left[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\right]^{\frac{q}{p-1}} |Q|,$$

(b) $$A_2(P,\mu) = \int_P \Big[\sum_{Q\subset P} \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}}} \chi_Q(x)\Big]^q dx,$$

(c) $$A_3(P,\mu) = \int_P \Big[\sum_{Q\subset P} \frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}} \chi_Q(x)\Big]^{\frac{q}{p-1}} dx,$$

*where $P$ is a dyadic cube in $\mathbb{R}^n$, or $P = \mathbb{R}^n$, and the constants of equivalence do not depend on $P$ and $\mu$.*

*Proof.* The equivalence of $A_1$ and $A_3$ is a localized version of Wolff's inequality (5.3) originally proved in [HW], which follows from Proposition 2.2 in [COV]. Moreover, it was proved in [COV] that

$$A_3(P,\mu) \simeq \int_P \Big[\sup_{x\in Q\subset P} \frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{q}{p-1}} dx, \tag{3.7}$$

where $A \simeq B$ means that there exist constants $c_1$ and $c_2$ which depend only on $\alpha$, $p$, $q$, and $n$ such that $c_1 A \le B \le c_2 A$. Since

$$\Big[\sup_{x\in Q\subset P} \frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}} \le \sum_{Q\subset P} \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}}}\chi_Q(x),$$

from (3.7) we obtain $A_3 \le CA_2$. In addition, for $p \le 2$ we clearly have $A_2 \le A_3 \le CA_1$. Therefore, it remains to check that, in the case $p > 2$, $A_2 \le CA_1$ for some $C > 0$ independent of $P$ and $\mu$. By Proposition 2.2 in [COV] we have (note that $q > p-1 > 1$)

$$\begin{aligned} A_2(P,\mu) &= \int_P \Big[\sum_{Q\subset P} \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}}}\chi_Q(x)\Big]^q dx \\ &\le C\sum_{Q\subset P} \frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}+q-2}}\Big[\sum_{Q'\subset Q}\frac{\mu(Q')^{\frac{1}{p-1}}}{|Q'|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}-1}}\Big]^{q-1}. \end{aligned} \tag{3.8}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} &\sum_{Q'\subset Q}\frac{\mu(Q')^{\frac{1}{p-1}}}{|Q'|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}-1}} \\ &\quad= \sum_{Q'\subset Q}\Big(\mu(Q')^{\frac{1}{p-1}}\,|Q'|^{\varepsilon}\Big)\,|Q'|^{-(1-\frac{\alpha p}{n})\frac{1}{p-1}+1-\varepsilon} \\ &\quad\le \Big(\sum_{Q'\subset Q}\mu(Q')^{\frac{r'}{p-1}}\,|Q'|^{\varepsilon r'}\Big)^{\frac{1}{r'}}\Big(\sum_{Q'\subset Q}|Q'|^{-r(1-\frac{\alpha p}{n})\frac{1}{p-1}+r-r\varepsilon}\Big)^{\frac{1}{r}}, \end{aligned}$$

where $r' = p-1 > 1$, $r = \frac{p-1}{p-2}$ and $\varepsilon > 0$ is chosen so that $-r(1-\frac{\alpha p}{n})\frac{1}{p-1} + r - r\varepsilon > 1$, i.e., $0 < \varepsilon < \frac{\alpha p}{(p-1)n}$. Therefore,

$$\begin{aligned} \sum_{Q'\subset Q}\frac{\mu(Q')^{\frac{1}{p-1}}}{|Q'|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}-1}} &\le C\mu(Q)^{\frac{1}{p-1}}\,|Q|^{\varepsilon}\,|Q|^{-(1-\frac{\alpha p}{n})\frac{1}{p-1}+1-\varepsilon} \\ &= C\frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}-1}}. \end{aligned}$$

Hence, combining this with (3.8) we obtain

$$\begin{aligned} A_2(P,\mu) &\le C\sum_{Q\subset P}\frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}+q-2}}\Big[\frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}-1}}\Big]^{q-1} \\ &= C\sum_{Q\subset P}\frac{\mu(Q)^{\frac{q}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{q}{p-1}-1}} = CA_1(P,\mu). \end{aligned}$$

This completes the proof of the proposition. $\square$

THEOREM 3.2. *Let $\alpha > 0$, $p > 1$ be such that $0 < \alpha p < n$, and let $q > p-1$. Suppose $f \in L^q_{\rm loc}(\mathbb{R}^n)$, $f \geq 0$, and $d\omega = f^q dx$. Then the following statements are equivalent.*

(i) *The equation*

$$u = \mathcal{W}_{\alpha,\,p}(u^q) + \varepsilon f \tag{3.9}$$

*has a solution $u \in L^q_{\rm loc}(\mathbb{R}^n)$, $u \geq 0$, for some $\varepsilon > 0$.*

(ii) *The testing inequality*

$$\int_P \Big[ \sum_{Q \subset P} \frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}} \chi_Q(x) \Big]^{\frac{q}{p-1}} dx \leq C\, \omega(P) \tag{3.10}$$

*holds for all dyadic cubes $P$.*

(iii) *The testing inequality*

$$\int_P \Big[ \sum_{Q \subset P} \frac{\omega(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}}} \chi_Q(x) \Big]^{q} dx \leq C\, \omega(P) \tag{3.11}$$

*holds for all dyadic cubes $P$.*

(iv) *There exists a constant $C$ such that*

$$\mathcal{W}_{\alpha,\,p}[\mathcal{W}_{\alpha,\,p}(f^q)]^q(x) \leq C\mathcal{W}_{\alpha,\,p}(f^q)(x) < \infty \quad \text{a.e.} \tag{3.12}$$

*Proof.* Note that by Proposition 3.1 we have (ii)$\Leftrightarrow$(iii). Therefore, it is enough to prove (iv)$\Rightarrow$(i)$\Rightarrow$(iii)$\Rightarrow$(iv).

*Proof of* (iv)$\Rightarrow$(i). The pointwise condition (3.12) can be rewritten as

$$\mathcal{N}^2 f \leq C\mathcal{N} f < \infty \quad \text{a.e.},$$

where $\mathcal{N}$ is the operator defined by (3.4). The sufficiency of this condition for the solvability of (3.9) can be proved using simple iterations:

$$u_{n+1} = \mathcal{N} u_n + \varepsilon f, \qquad n = 0,\ 1,\ 2, \ldots,$$

starting from $u_0 = 0$. Since $\mathcal{N}$ is monotonic it is easy to see that $u_n$ is increasing and that $\varepsilon^{\frac{q}{p-1}} \mathcal{N} f + \varepsilon f \leq u_n$ for all $n \geq 2$. Let $c(p) = \max\{1, 2^{p'-1}\}$, $c_1 = 0$, $c_2 = [\varepsilon^{\frac{1}{p-1}} c(p)]^q$ and

$$c_n = \Big[\varepsilon^{\frac{1}{p-1}} c(p)(1 + C^{1/q}) c_{n-1}^{p'-1}\Big]^q, \qquad n = 3,\ 4, \ldots,$$

where $C$ is the constant in (3.12). Here we choose $\varepsilon$ so that

$$\varepsilon^{\frac{1}{p-1}} c(p) = \Big(\frac{q-p+1}{q}\Big)^{\frac{q-p+1}{q}} \Big(\frac{p-1}{q}\Big)^{\frac{p-1}{q}} C^{\frac{1-p}{q^2}}.$$

By induction and using (3.6) we have

$$u_n \leq c_n \mathcal{N} f + \varepsilon f, \qquad n = 1,\ 2,\ 3, \ldots\ .$$

Note that

$$x_0 = \Big[\frac{q}{p-1}\varepsilon^{\frac{1}{p-1}}c(p)\,C^{\frac{1}{q}}\Big]^{\frac{q(p-1)}{p-1-q}}$$

is the only root of the equation

$$x = \Big[\varepsilon^{\frac{1}{p-1}}c(p)(1+C^{\frac{1}{q}}x)\Big]^{q}$$

and thus $\lim_{n\to\infty} c_n = x_0$. Hence there exists a solution

$$u(x) = \lim_{n\to\infty} u_n(x)$$

to equation (3.9) (with that choice of $\varepsilon$) such that

$$\varepsilon f + \varepsilon^{\frac{q}{p-1}}\mathcal{W}_{\alpha,\,p}(f^q) \leq u \leq \varepsilon f + x_0\mathcal{W}_{\alpha,\,p}(f^q).$$

*Proof of* (i)⇒(iii). Suppose that $u \in L^q_{\rm loc}(\mathbb{R}^n)$, $u \geq 0$, is a solution of (3.9). Let $P$ be a cube in $\mathcal{D}$ and $d\mu = u^q dx$. Since

$$[u(x)]^q \geq [\mathcal{W}_{\alpha,\,p}(u^q)(x)]^q \quad \text{a.e.},$$

we have

$$\int_P [\mathcal{W}_{\alpha,\,p}(u^q)(x)]^q dx \leq \int_P [u(x)]^q dx.$$

Thus,

$$\int_P \Big[\sum_{Q\subset P}\frac{\mu(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}}}\chi_Q(x)\Big]^q dx \leq C\mu(P), \tag{3.13}$$

for all $P\in\mathcal{D}$. By Proposition 3.1, inequality (3.13) is equivalent to

$$\int_P \Big[\sum_{Q\subset P}\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\chi_Q(x)\Big]^{\frac{q}{p-1}} dx \leq C\mu(P)$$

for all $P\in\mathcal{D}$, which in its turn is equivalent to the weak-type inequality

$$\|\mathcal{I}_{\alpha p}(g)\|_{L^{\frac{q}{q-p+1},\,\infty}(d\mu)} \leq C\,\|g\|_{L^{\frac{q}{q-p+1}}(dx)} \tag{3.14}$$

for all $g\in L^{\frac{q}{q-p+1}}(\mathbb{R}^n)$, $g\geq 0$ (see [NTV], [VW]). Note that by (3.9),

$$d\mu = u^q dx \geq \varepsilon^q f^q\,dx = \varepsilon^q\,d\omega.$$

We now deduce from (3.14),

$$\|\mathcal{I}_{\alpha p}(g)\|_{L^{\frac{q}{q-p+1},\,\infty}(d\omega)} \leq \frac{C}{\varepsilon^{q-p+1}}\,\|g\|_{L^{\frac{q}{q-p+1}}(dx)}\,. \tag{3.15}$$

Similarly, by duality and Proposition 3.1 we see that (3.15) is equivalent to the testing inequality (3.11). The implication (i)⇒ (iii) is proved.

*Proof of* (iii)⇒(iv). We first deduce from the testing inequality (3.11) that

$$\omega(P) \le C\,|P|^{1-\frac{\alpha p q}{n(q-p+1)}} \tag{3.16}$$

for all dyadic cubes $P$. In fact, this can be verified by using (3.11) and the obvious estimate

$$\int_P \Big[\frac{\omega(P)}{|P|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{q}{p-1}} dx \le \int_P \Big[\sum_{Q\subset P} \frac{\omega(Q)^{\frac{1}{p-1}}}{|Q|^{(1-\frac{\alpha p}{n})\frac{1}{p-1}}}\chi_Q(x)\Big]^q dx.$$

Following [KV], [V3], we next introduce a certain decomposition of the dyadic Wolff's potential $\mathcal{W}_{\alpha,\,p}\mu$. To each dyadic cube $P\in\mathcal{D}$, we associate the "upper" and "lower" parts of $\mathcal{W}_{\alpha,\,p}\mu$ defined respectively by

$$\mathcal{U}_P\mu(x) = \sum_{Q\subset P} \Big[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_Q(x), \tag{3.17}$$

$$\mathcal{V}_P\mu(x) = \sum_{Q\supset P} \Big[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_Q(x). \tag{3.18}$$

Obviously,

$$\mathcal{U}_P\mu(x) \le \mathcal{W}_{\alpha,\,p}\mu(x), \qquad \mathcal{V}_P\mu(x) \le \mathcal{W}_{\alpha,\,p}\mu(x),$$

and for $x\in P$,

$$\mathcal{W}_{\alpha,\,p}\mu(x) = \mathcal{U}_P\mu(x) + \mathcal{V}_P\mu(x) - \Big[\frac{\mu(P)}{|P|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}.$$

Using the notation just introduced, we can rewrite the testing inequality (3.11) in the form:

$$\int_P [\mathcal{U}_P\omega(x)]^q\,dx \le C\,\omega(P) \tag{3.19}$$

for all dyadic cubes $P$. Recall that $d\omega = f^q\,dx$. The desired pointwise inequality (3.12) can be restated as

$$\sum_{P\in\mathcal{D}} \Big[\frac{\int_P [\mathcal{W}_{\alpha,\,p}\omega(y)]^q\,dy}{|P|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_P(x) \le C\,\mathcal{W}_{\alpha,\,p}\omega(x). \tag{3.20}$$

Obviously, for $y\in P$,

$$\mathcal{W}_{\alpha,\,p}\omega(y) \le \mathcal{U}_P\omega(y) + \mathcal{V}_P\omega(y),$$

and from the testing inequality (3.19) we have

$$\sum_{P\in\mathcal{D}} \Big[\frac{\int_P [\mathcal{U}_P\omega(y)]^q\,dy}{|P|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_P(x) \le C\,\mathcal{W}_{\alpha,\,p}\omega(x).$$

Therefore, to prove (3.20) it enough to prove

$$\sum_{P\in\mathcal{D}} \Big[\frac{\int_P [\mathcal{V}_P\omega(y)]^q\,dy}{|P|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}} \chi_P(x) \le C\,\mathcal{W}_{\alpha,\,p}\omega(x). \tag{3.21}$$

Note that, for $y\in P$,

$$\mathcal{V}_P\omega(y) = \sum_{Q\supset P} \Big[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}} = \text{const}.$$

Using the elementary inequality

$$\Big(\sum_{k=1}^{\infty} a_k\Big)^s \le s\sum_{k=1}^{\infty} a_k\Big(\sum_{j=k}^{\infty} a_j\Big)^{s-1},$$

where $1\le s<\infty$ and $0\le a_k<\infty$, we deduce

$$[\mathcal{V}_P\omega(y)]^{\frac{q}{p-1}} \le C\sum_{Q\supset P}\Big[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\Big\{\sum_{R\supset Q}\Big[\frac{\omega(R)}{|R|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\Big\}^{\frac{q}{p-1}-1}.$$

From this we see that the left-hand side of (3.21) is bounded above by a constant multiple of

$$\sum_{P\in\mathcal{D}} |P|^{\frac{\alpha p}{n(p-1)}} \sum_{Q\supset P}\Big[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\Big\{\sum_{R\supset Q}\Big[\frac{\omega(R)}{|R|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\Big\}^{\frac{q}{p-1}-1}\chi_P(x).$$

Changing the order of summation, we see that it is equal to

$$\sum_{Q\in\mathcal{D}}\Big[\frac{\omega(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}}\chi_Q(x)\Big\{\sum_{P\subset Q}|P|^{\frac{\alpha p}{n(p-1)}}\,\chi_P(x)[\mathcal{V}_Q\omega(x)]^{\frac{q}{p-1}-1}\Big\}.$$

By (3.16), the expression in the curly brackets above is uniformly bounded. Therefore, the proof of estimate (3.21), and hence of (iii) ⇒ (iv), is complete. □

## 4. $\mathcal{A}$-superharmonic functions

In this section, we recall for later use some facts on $\mathcal{A}$-superharmonic functions, most of which can be found in [HKM], [KM1], [KM2], and [TW4]. Let $\Omega$ be an open set in $\mathbb{R}^n$, and $p>1$. We will mainly be interested in the case where $\Omega$ is bounded and $1<p\le n$, or $\Omega=\mathbb{R}^n$ and $1<p<n$. We assume that $\mathcal{A}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is a vector-valued mapping which satisfies the following structural properties:

(4.1) the mapping x $\to\mathcal{A}(x,\xi)$ is measurable for all $\xi\in\mathbb{R}^n$,

(4.2) the mapping $\xi\to\mathcal{A}(x,\xi)$ is continuous for a.e. x $\in\mathbb{R}^n$, and there are constants $0 < \alpha \le \beta < \infty$ such that for a.e. $x$ in $\mathbb{R}^n$, and for all $\xi$ in $\mathbb{R}^n$,

$$\mathcal{A}(x,\xi)\cdot\xi \ge \alpha\,|\xi|^p, \quad |\mathcal{A}(x,\xi)| \le \beta\,|\xi|^{p-1}, \tag{4.3}$$

$$[\mathcal{A}(x,\xi_1) - \mathcal{A}(x,\xi_2)]\cdot(\xi_1-\xi_2) > 0, \quad \text{if } \xi_1 \ne \xi_2, \tag{4.4}$$

$$\mathcal{A}(x,\lambda\xi) = \lambda\,|\lambda|^{p-2}\,\mathcal{A}(x,\xi), \quad \text{if } \lambda \in \mathbb{R}\setminus\{0\}. \tag{4.5}$$

For $u \in W^{1,p}_{\rm loc}(\Omega)$, we define the divergence of $\mathcal{A}(x,\nabla u)$ in the sense of distributions; i.e., if $\varphi \in C^\infty_0(\Omega)$, then

$$\operatorname{div}\mathcal{A}(x,\nabla u)(\varphi) = -\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx.$$

It is well known that every solution $u \in W^{1,p}_{\rm loc}(\Omega)$ to the equation

$$-\operatorname{div}\mathcal{A}(x,\nabla u) = 0 \tag{4.6}$$

has a continuous representative. Such continuous solutions are said to be $\mathcal{A}$-*harmonic* in $\Omega$. If $u \in W^{1,p}_{\rm loc}(\Omega)$ and

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx \ge 0,$$

for all nonnegative $\varphi \in C^\infty_0(\Omega)$, i.e., $-\operatorname{div}\mathcal{A}(x,\nabla u) \ge 0$ in the distributional sense, then $u$ is called a *supersolution* to (4.6) in $\Omega$.

A lower semicontinuous function $u : \Omega \to (-\infty,\infty]$ is called $\mathcal{A}$-*superharmonic* if $u$ is not identically infinite in each component of $\Omega$, and if for all open sets $D$ such that $\overline{D} \subset \Omega$, and all functions $h \in C(\overline{D})$, $\mathcal{A}$-harmonic in $D$, it follows that $h \le u$ on $\partial D$ implies $h \le u$ in $D$.

In the special case $\mathcal{A}(x,\xi) = |\xi|^{p-2}\xi$, $\mathcal{A}$-superharmonicity is often referred to as $p$-superharmonicity. It is worth mentioning that the latter can also be defined equivalently using the language of viscosity solutions (see [JLM]).

We recall here the fundamental connection between supersolutions of (4.6) and $\mathcal{A}$-superharmonic functions [HKM].

PROPOSITION 4.1 ([HKM]). (i) *If $v$ is $\mathcal{A}$-superharmonic on $\Omega$ then*

$$v(x) = \operatorname{ess}\liminf_{y\to x}\, v(y), \qquad x \in \Omega. \tag{4.7}$$

*Moreover, if $v \in W^{1,p}_{\rm loc}(\Omega)$ then*

$$-\operatorname{div}\mathcal{A}(x,\nabla v) \ge 0.$$

(ii) *If $u \in W^{1,p}_{\rm loc}(\Omega)$ is such that*

$$-\operatorname{div}\mathcal{A}(x,\nabla u) \ge 0,$$

*then there is an $\mathcal{A}$-superharmonic function $v$ such that $u = v$ a.e.*

(iii) *If $v$ is $\mathcal{A}$-superharmonic and locally bounded, then $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and*

$$-\mathrm{div}\mathcal{A}(x, \nabla v) \geq 0.$$

A useful consequence of the above proposition is that if $u$ and $v$ are two $\mathcal{A}$-superharmonic functions on $\Omega$ such that $u \leq v$ a.e. on $\Omega$ then $u \leq v$ everywhere on $\Omega$.

Note that an $\mathcal{A}$-superharmonic function $u$ does not necessarily belong to $W^{1,p}_{\mathrm{loc}}(\Omega)$, but its truncation $\min\{u, k\}$ does for every integer $k$ due to Proposition 4.1(iii). Using this, we set

$$Du = \lim_{k\to\infty} \nabla [\min\{u, k\}],$$

defined a.e. If either $u \in L^\infty(\Omega)$ or $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, then $Du$ coincides with the regular distributional gradient of $u$. In general we have the following gradient estimates [KM1] (see also [HKM], [TW4]).

Proposition 4.2 ([KM1]). *Suppose $u$ is $\mathcal{A}$-superharmonic in $\Omega$ and $1 \leq q < \frac{n}{n-1}$. Then both $|Du|^{p-1}$ and $\mathcal{A}(\cdot, Du)$ belong to $L^q_{\mathrm{loc}}(\Omega)$. Moreover, if $p > 2 - \frac{1}{n}$, then $Du$ is the distributional gradient of $u$.*

We can now extend the definition of the divergence of $\mathcal{A}(x, \nabla u)$ to those $u$ which are merely $\mathcal{A}$-superharmonic in $\Omega$. For such $u$ we set

$$-\mathrm{div}\mathcal{A}(x, \nabla u)(\varphi) = \int_\Omega \mathcal{A}(x, Du) \cdot \nabla\varphi \, dx,$$

for all $\varphi \in C^\infty_0(\Omega)$. Note that by Proposition 4.2 and the dominated convergence theorem,

$$-\mathrm{div}\mathcal{A}(x, \nabla u)(\varphi) = \lim_{k\to\infty} \int_\Omega \mathcal{A}(x, \nabla \min\{u, k\}) \cdot \nabla\varphi \, dx \geq 0$$

whenever $\varphi \in C^\infty_0(\Omega)$ and $\varphi \geq 0$.

Since $-\mathrm{div}\mathcal{A}(x, \nabla u)$ is a nonnegative distribution in $\Omega$ for an $\mathcal{A}$-superharmonic $u$, it follows that there is a positive (not necessarily finite) Radon measure denoted by $\mu[u]$ such that

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu[u] \quad \text{in} \quad \Omega.$$

Conversely, given a positive finite measure $\mu$ in a bounded domain $\Omega$, there is an $\mathcal{A}$-superharmonic function $u$ such that $-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu$ in $\Omega$ and $\min\{u, k\} \in W^{1,p}_0(\Omega)$ for all integers $k$.

The following weak continuity result from [TW4] will be used later in Section 5 to prove the existence of $\mathcal{A}$-superharmonic solutions to quasilinear equations.

THEOREM 4.3 ([TW4]). *Suppose that $\{u_n\}$ is a sequence of nonnegative $\mathcal{A}$-superharmonic functions in $\Omega$ that converges a.e. to an $\mathcal{A}$-superharmonic function $u$. Then the sequence of measures $\{\mu[u_n]\}$ converges to $\mu[u]$ weakly; i.e.,*

$$\lim_{n\to\infty}\int_\Omega \varphi\, d\mu[u_n] = \int_\Omega \varphi\, d\mu[u],$$

*for all $\varphi \in C_0^\infty(\Omega)$.*

In [KM2] (see also [Mi, Th. 3.1] and [MZ]) the following pointwise potential estimate for $\mathcal{A}$-superharmonic functions was established, and this serves as a major tool in our study of quasilinear equations of Lane-Emden type.

THEOREM 4.4 ([KM2]). *Suppose $u \geq 0$ is an $\mathcal{A}$-superharmonic function in $B_{3r}(x)$. If $\mu = -\mathrm{div}\mathcal{A}(x, \nabla u)$, then there are positive constants $C_1, C_2$ and $C_3$ which depend only on $n, p$ and the structural constants $\alpha$ and $\beta$ such that* (1.18) *holds.*

A consequence of Theorem 4.4 is the following global version of the above potential pointwise estimate.

COROLLARY 4.5 ([KM2]). *Let $u$ be an $\mathcal{A}$-superharmonic function in $\mathbb{R}^n$ with $\inf_{\mathbb{R}^n} u = 0$. If $\mu = -\mathrm{div}\mathcal{A}(x, \nabla u)$, then*

$$\frac{1}{K}\,\mathbf{W}_{1,\,p}\mu(x) \leq u(x) \leq K\,\mathbf{W}_{1,\,p}\mu(x)$$

*for all $x \in \mathbb{R}^n$, where $K$ is a positive constant depending only on $n, p$ and the structural constants $\alpha$ and $\beta$.*

## 5. Quasilinear equations on $\mathbb{R}^n$

In this section, we study the solvability problem for the quasilinear equation

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = u^q + \omega \tag{5.1}$$

in the class of nonnegative $\mathcal{A}$-superharmonic functions on the entire space $\mathbb{R}^n$, where $\mathcal{A}(x,\xi)\cdot\xi \approx |\xi|^p$ is defined precisely as in Section 4. Here we assume $1 < p < n$, $q > p-1$, and $\omega \in \mathcal{M}^+(\mathbb{R}^n)$. In this setting, all solutions are understood in the "potential-theoretic" sense, i.e., $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, $u \geq 0$, is a solution to (5.1) if $u$ is $\mathcal{A}$-superharmonic, and

$$\int \lim_{k\to\infty} \mathcal{A}(x, \nabla \min\{u,k\})\cdot\nabla\varphi\, dx = \int u^q\varphi\, dx + \int \varphi\, d\omega \tag{5.2}$$

for all test functions $\varphi \in C_0^\infty(\mathbb{R}^n)$.

We first prove a continuous counterpart of Proposition 3.1. Here we use the well-known argument due to Fefferman and Stein [FS] which is based on the averaging over shifts of the dyadic lattice $\mathcal{D}$.

PROPOSITION 5.1. *Let $0 < r \leq \infty$. Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$, $\alpha > 0$, $p > 1$, and $q > p - 1$. Then the following quantities are equivalent.*

$$\text{(a)} \quad \left\|\mathbf{W}^r_{\alpha p, \frac{q}{q-p+1}}\mu\right\|_{L^1(d\mu)} = \int_{\mathbb{R}^n}\int_0^r \left[\frac{\mu(B_t(x))}{t^{n-\frac{\alpha p q}{q-p+1}}}\right]^{\frac{q}{p-1}-1}\frac{dt}{t}d\mu,$$

$$\text{(b)} \quad \left\|\mathbf{W}^r_{\alpha, p}\mu\right\|^q_{L^q(dx)} = \int_{\mathbb{R}^n}\left\{\int_0^r \left[\frac{\mu(B_t(x))}{t^{n-\alpha p}}\right]^{\frac{1}{p-1}}\frac{dt}{t}\right\}^q dx,$$

$$\text{(c)} \quad \left\|\mathbf{I}^r_{\alpha p}\mu\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(dx)} = \int_{\mathbb{R}^n}\left[\int_0^r \frac{\mu(B_t(x))}{t^{n-\alpha p}}\frac{dt}{t}\right]^{\frac{q}{p-1}}dx,$$

*where the constants of equivalence do not depend on $\mu$ and $r$.*

*Remark* 5.2. The equivalence of expressions (a) and (c) in Proposition 5.1 may be regarded as a version of Wolff's inequality [HW] (see also [AH, §4.5]):

$$C_1\int_{\mathbb{R}^n}\mathbf{W}_{\alpha, s}\mu\, d\mu \leq \int_{\mathbb{R}^n}(\mathbf{I}_\alpha\mu)^{\frac{s}{s-1}}dx \leq C_2\int_{\mathbb{R}^n}\mathbf{W}_{\alpha, s}\mu\, d\mu, \tag{5.3}$$

where $1 < s < +\infty$, $0 < \alpha < \frac{n}{s}$, and $C_1$, $C_2$ depend only on $\alpha, s$ and $n$.

Furthermore,

$$\int_{\mathbb{R}^n}(\mathbf{I}_\alpha\mu)^{\frac{s}{s-1}}dx \simeq \int_{\mathbb{R}^n}(\mathcal{I}_\alpha\mu)^{\frac{s}{s-1}}dx \simeq \sum_{Q\in\mathcal{D}}\left[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha}{n}}}\right]^{\frac{s}{s-1}}|Q|\,. \tag{5.4}$$

The second equivalence in (5.4) is a dyadic form of (5.3) proved in [HW] (see also [COV], [V2]).

*Proof of Proposition* 5.1. We will prove only the equivalence of (b) and (c); i.e., there are constants $C_1,\ C_2 > 0$ such that

$$C_1\left\|\mathbf{W}^r_{\alpha, p}\mu\right\|^q_{L^q(dx)} \leq \left\|\mathbf{I}^r_{\alpha p}\mu\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(dx)} \leq C_2\left\|\mathbf{W}^r_{\alpha, p}\mu\right\|^q_{L^q(dx)}. \tag{5.5}$$

The equivalence of (a) and (c), which is actually a consequence of Theorem 3.6.2 in [AH], can also be deduced by a similar argument. We first restrict ourselves to the case $r < \infty$. Observe that there is a constant $C > 0$ such that

$$\left\|\mathbf{I}^{2r}_{\alpha p}\mu\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(dx)} \leq C\left\|\mathbf{I}^r_{\alpha p}\mu\right\|^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(dx)}. \tag{5.6}$$

In fact, since

$$\int_0^{2r}\frac{\mu(B_t(x))}{t^{n-\alpha p}}\frac{dt}{t} \leq C\int_0^r\frac{\mu(B_t(x))}{t^{n-\alpha p}}\frac{dt}{t} + C\frac{\mu(B_{2r}(x))}{r^{n-\alpha p}},$$

(5.6) will follow from the estimate

$$\int_{\mathbb{R}^n}\left[\frac{\mu(B_{2r}(x))}{r^{n-\alpha p}}\right]^{\frac{q}{p-1}}dx \leq C\int_{\mathbb{R}^n}\left[\int_0^r\frac{\mu(B_t(x))}{t^{n-\alpha p}}\frac{dt}{t}\right]^{\frac{q}{p-1}}dx. \tag{5.7}$$

Note that for a partition of $\mathbb{R}^n$ into a union of disjoint cubes $\{Q_j\}$ such that $\mathrm{diam}(Q_j) = \frac{r}{4}$,

$$\begin{aligned}\int_{\mathbb{R}^n} \mu(B_{2r}(x))^{\frac{q}{p-1}} dx &= \sum_j \int_{Q_j} \mu(B_{2r}(x))^{\frac{q}{p-1}} dx \\ &\le C \sum_j \int_{Q_j} \mu(Q_j)^{\frac{q}{p-1}} dx,\end{aligned}$$

where we have used the fact that the ball $B_{2r}(x)$ is contained in the union of at most $N$ cubes in $\{Q_j\}$ for some constant $N$ depending only on $n$. Thus

$$\begin{aligned}\int_{\mathbb{R}^n} \Big[\frac{\mu(B_{2r}(x))}{r^{n-\alpha p}}\Big]^{\frac{q}{p-1}} dx &\le C \sum_j \int_{Q_j} \Big[\frac{\mu(B_{r/2}(x))}{r^{n-\alpha p}}\Big]^{\frac{q}{p-1}} dx \\ &\le C \sum_j \int_{Q_j} \Big[\int_0^r \frac{\mu(B_t(x))}{t^{n-\alpha p}} \frac{dt}{t}\Big]^{\frac{q}{p-1}} dx,\end{aligned}$$

which gives (5.7).

By arguing as in [COV, §3], we can find constants $a$, $C$ and $c$ depending only on $p$ and $n$ such that

$$\mathbf{W}^r_{\alpha,p}\mu(x) \le C r^{-n} \int_{|t|\le cr} \sum_{\substack{Q\in\mathcal{D}_t\\ \ell(Q)\le 4\frac{r}{a}}} \Big[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}} \chi_Q(x) dt,$$

where $\mathcal{D}_t$, $t \in \mathbb{R}^n$, denotes the lattice $\mathcal{D} + t = \{Q = Q' + t : Q' \in \mathcal{D}\}$ and $\ell(Q)$ is the side length of $Q$. Using Proposition 2.2 in [COV] and arguing as in the proof of Theorem 3.1 we obtain

$$\int_{\mathbb{R}^n} \Big\{ \sum_{\substack{Q\in\mathcal{D}_t\\ \ell(Q)\le 4\frac{r}{a}}} \Big[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}} \chi_Q(x)\Big\}^q dx \simeq \int_{\mathbb{R}^n} \Big[ \sum_{\substack{Q\in\mathcal{D}_t\\ \ell(Q)\le 4\frac{r}{a}}} \frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}} \chi_Q(x)\Big]^{\frac{q}{p-1}} dx,$$

where the constants of equivalence are independent of $\mu$, $r$ and $t$. The last two estimates together with the integral Minkowski inequality then give

$$\begin{aligned}||\mathbf{W}^r_{\alpha,p}\mu||_{L^q(dx)} &\le C r^{-n} \int_{|t|\le cr} \Big\{ \int_{\mathbb{R}^n} \Big( \sum_{\substack{Q\in\mathcal{D}_t\\ \ell(Q)\le 4\frac{r}{a}}} \Big[\frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}}\Big]^{\frac{1}{p-1}} \chi_Q(x)\Big)^q dx\Big\}^{\frac{1}{q}} dt \\ &\le C r^{-n} \int_{|t|\le cr} \Big[ \int_{\mathbb{R}^n} \Big( \sum_{\substack{Q\in\mathcal{D}_t\\ \ell(Q)\le 4\frac{r}{a}}} \frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}} \chi_Q(x)\Big)^{\frac{q}{p-1}} dx\Big]^{\frac{1}{q}} dt.\end{aligned}$$

Note that

$$\begin{aligned}\sum_{\substack{Q\in\mathcal{D}_t\\ \ell(Q)\le 4\frac{r}{a}}} \frac{\mu(Q)}{|Q|^{1-\frac{\alpha p}{n}}} \chi_Q(x) &\le C \sum_{2^k\le 4\frac{r}{a}} \frac{\mu(B(x,\sqrt{n}2^k))}{2^{k(n-\alpha p)}} \\ &\le C \mathbf{I}_{\alpha p}^{\frac{8r\sqrt{n}}{a}} \mu(x)\end{aligned}$$

where $C$ is independent of $t$. Thus, in view of (5.6), we obtain the lower estimate in (5.5).

Now by letting $R \to \infty$ in the inequality

$$||\mathbf{W}^R_{\alpha,\,p}\mu||^q_{L^q(dx)} \le C\,||\mathbf{I}^R_{\alpha p}\mu||^{\frac{q}{p-1}}_{L^{\frac{q}{p-1}}(dx)}, \qquad 0 < R < \infty,$$

we get the lower estimate in (5.5) with $r = \infty$. The upper estimate in (5.5) can be deduced in a similar way. This completes the proof of Proposition 5.1. □

In the next theorem, we give a sufficient condition for the solvability of quasilinear equations in $\mathbb{R}^n$. Later on we will show that it is necessary as well, and give equivalent simpler characterizations.

THEOREM 5.3. *Let $\omega \in \mathcal{M}^+(\mathbb{R}^n)$, $1 < p < n$, and $q > p-1$. Suppose that*

$$\mathbf{W}_{1,\,p}(\mathbf{W}_{1,\,p}\omega)^q \le C\,\mathbf{W}_{1,\,p}\omega < \infty \quad \text{a.e.}, \tag{5.8}$$

*where*

$$C \le \Big(\frac{q-p+1}{qK\max\{1, 2^{p'-2}\}}\Big)^{q(p'-1)}\Big(\frac{p-1}{q-p+1}\Big), \tag{5.9}$$

*and $K$ is the constant used in Theorem* 2.1. *Then there is an $\mathcal{A}$-superharmonic function $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$\begin{cases} \inf_{x\in\mathbb{R}^n} u(x) = 0, \\ -\mathrm{div}\mathcal{A}(x, \nabla u) = u^q + \omega, \end{cases} \tag{5.10}$$

*and*

$$c_1\mathbf{W}_{1,\,p}\omega(x) \le u(x) \le c_2\,\mathbf{W}_{1,\,p}\omega(x)$$

*for all $x$ in $\mathbb{R}^n$, where the constants $c_1, c_2$ depend only $n, p, q$, and the structural constants $\alpha, \beta$.*

*Proof.* For each $m \in \mathbb{N}$, let us construct by an induction argument a nondecreasing sequence $\{u^m_k\}_{k\ge 0}$ of $\mathcal{A}$-superharmonic functions on $B_{m+1}$ such that

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u^m_0) = \omega_{B_m} \quad \text{in} \quad B_{m+1}, \\ u^m_0 = 0 \quad \text{on} \quad \partial B_{m+1}, \end{cases}$$

and

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u^m_k) = (u^m_{k-1})^q + \omega_{B_m} \quad \text{in} \quad B_{m+1}, \\ u^m_k = 0 \quad \text{on} \quad \partial B_{m+1} \end{cases}$$

for each $k \ge 1$, in the renormalized sense (see Lemma 6.9 in §6). Here $B_m$ denotes the ball of radius $m$ and is centered at the origin. The renormalized solutions are needed here only to get the following estimates:

$$u^m_0 \le K\,\mathbf{W}_{1,\,p}\omega \quad \text{and} \quad u^m_k \le K\,\mathbf{W}_{1,\,p}(u^q_k + \omega)$$

for all $k \geq 1$; see Theorem 2.1 whose proof is presented in Section 6. Set $c_0 = K$, where $K$ is the constant in Theorem 2.1. From these estimates and (3.5) we get

$$\begin{aligned} u_1^m &\leq K \max\{1, 2^{p'-2}\} \Big[ \mathbf{W}_{1,p} (u_0^m)^q + \mathbf{W}_{1,p}\omega \Big] \\ &\leq K \max\{1, 2^{p'-2}\} (c_0^{q(p'-1)} C + 1) \mathbf{W}_{1,p}\omega \\ &= c_1 \mathbf{W}_{1,p}\omega, \end{aligned}$$

where $c_1 = K \max\{1, 2^{p'-2}\}(c_0^{q(p'-1)} C+1)$. By induction we can find a sequence $\{c_k\}_{k\geq 0}$ of positive numbers such that $u_k^m \leq c_k \mathbf{W}_{1,p}\omega$, with $c_0 = K$ and $c_{k+1} = K \max\{1, 2^{p'-2}\}(c_k^{q(p'-1)} C + 1)$ for all $k \geq 0$. It is then easy to see that $c_k \leq \frac{K \max\{1, 2^{p'-2}\} q}{q-p+1}$ for all $k \geq 0$ as long as (5.9) is satisfied. Thus

$$u_k^m \leq \frac{K \max\{1, 2^{p'-2}\} q}{q-p+1} \mathbf{W}_{1,p}\omega \quad \text{on} \quad B_{m+1}.$$

Now by weak continuity (Theorem 4.3) or stability results for renormalized solutions in [DMOP] we see that $u_k^m \uparrow u^m$ for an $\mathcal{A}$-superharmonic function $u^m \geq 0$ on $B_{m+1}$ such that

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u^m) = (u^m)^q + \omega_{B_m} \quad \text{in} \quad B_{m+1}, \\ \qquad u^m = 0 \quad \text{on} \quad \partial B_{m+1}, \end{cases} \tag{5.11}$$

and

$$u^m \leq C\, \mathbf{W}_{1,p}\omega \quad \text{on} \quad B_{m+1}. \tag{5.12}$$

By Theorem 1.17 in [KM1] we can find a subsequence $\{u^{m_j}\}_j$ of $\{u^m\}_m$ and an $\mathcal{A}$-superharmonic function $u$ on $\mathbb{R}^n$ such that $u^{m_j} \to u$ a.e. Thus by (5.11) and weak continuity (Theorem 4.3) we see that $u$ is a solution to the equation $-\mathrm{div}\mathcal{A}(x, \nabla u) = u^q + \omega$ in $\mathbb{R}^n$. On the other hand, from (5.12) we have

$$u \leq C\, \mathbf{W}_{1,p}\omega \quad \text{a.e. on} \quad \mathbb{R}^n,$$

which by Corollary 4.5 gives

$$u \leq C(u - \inf_{\mathbb{R}^n} u).$$

Thus $\inf_{\mathbb{R}^n} u = 0$, which completes the proof of the theorem. □

We can now prove Theorem 2.3 stated in Section 2 which gives the existence criteria for quasilinear equations in $\mathbb{R}^n$.

*Proof of Theorem* 2.3. It is well-known that statements (ii) and (iii) in Theorem 2.3 are equivalent (see, e.g., [V2]). Note that (2.9) is also equivalent to the testing inequality (see, e.g., [VW]):

$$\int_{\mathbb{R}^n} \Big[ \mathbf{I}_p \omega_B(x) \Big]^{\frac{q}{p-1}} dx \leq C\, \omega(B).$$

By applying Proposition 5.1 we deduce (ii)⇒(iv). The implication (v)⇒(i) clearly follows from Theorem 5.3. Therefore, it remains to check (i)⇒(ii) and (iv)⇒(v).

*Proof of* (i)⇒(ii). Let $u$ be a nonnegative solution of (2.8) and let $\mu = u^q + \varepsilon\omega$. Then $\mu$ is a nonnegative measure such that $\mu \geq u^q$, $\mu \geq \varepsilon\omega$ and $u(x) \geq C\, \mathbf{W}_{1,\,p}\mu(x)$ by Corollary 4.5. Therefore,

$$\begin{aligned}\int_P d\mu \geq \int_P u^q\, dx &\geq C \int_P (\mathbf{W}_{1,\,p}\mu)^q\, dx\\ &\geq C \int_P \Big[ \sum_{Q\subset P} \Big( \frac{\mu(Q)}{|Q|^{1-\frac{p}{n}}} \Big)^{\frac{1}{p-1}} \chi_Q(x) \Big]^q\, dx\end{aligned}$$

for all dyadic cubes $P$ in $\mathbb{R}^n$. Using this and Proposition 3.1, we get

$$\sum_{Q\subset P} \Big[ \frac{\mu(Q)}{|Q|^{1-\frac{p}{n}}} \Big]^{\frac{q}{p-1}} |Q| \leq C\, \mu(P), \qquad P \in \mathcal{D}.$$

It is known that the preceding condition is equivalent to the inequality (see [V1, §3])

$$\|\mathbf{I}_p(f)\|_{L^{\frac{q}{q-p+1}}(d\mu)} \leq C\, \|f\|_{L^{\frac{q}{q-p+1}}(dx)},$$

where $C$ does not depend on $f \in L^{\frac{q}{q-p+1}}(dx)$. Since $\mu \geq \varepsilon\, \omega$, we have

$$\|\mathbf{I}_p(f)\|_{L^{\frac{q}{q-p+1}}(d\omega)} \leq \varepsilon^{\frac{q-p+1}{-q}} C\, \|f\|_{L^{\frac{q}{q-p+1}}(dx)}.$$

Therefore, by duality we obtain the testing inequality (2.9). This completes the proof of (i)⇒(ii).

*Proof of* (iv)⇒(v). We first claim that (2.11) yields

$$\int_r^\infty \Big[ \frac{\omega(B_t(x))}{t^{n-p}} \Big]^{\frac{1}{p-1}} \frac{dt}{t} \leq C\, r^{\frac{-p}{q-p+1}}, \tag{5.13}$$

where $C$ is independent of $x$ and $r$. Note that for $y \in B_t(x)$ and $\tau \geq 2t$, we have $B_t(x) \subset B_\tau(y)$. Thus,

$$\begin{aligned}\mathbf{W}_{1,\,p}\omega_{B_t(x)}(y) &\geq \int_{2t}^\infty \Big[ \frac{\omega(B_\tau(y)\cap B_t(x))}{\tau^{n-p}} \Big]^{\frac{1}{p-1}} \frac{d\tau}{\tau}\\ &\geq C \Big[ \frac{\omega(B_t(x))}{t^{n-p}} \Big]^{\frac{1}{p-1}}.\end{aligned}$$

Combining this with (2.11) we obtain

$$\omega(B_t(x)) \leq C\, t^{n-\frac{pq}{q-p+1}}, \tag{5.14}$$

which clearly implies (5.13).

Next, we introduce a decomposition of the Wolff potential $\mathbf{W}_{1,\,p}$ into its "upper" and "lower" parts, which are the continuous analogues of the discrete ones given in (3.17) and (3.18) above:

$$\mathbf{U}_r\mu(x) = \int_0^r \Big[\frac{\mu(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dt}{t}, \quad r > 0, \quad x \in \mathbb{R}^n,$$
$$\mathbf{L}_r\mu(x) = \int_r^\infty \Big[\frac{\mu(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dt}{t}, \quad r > 0, \quad x \in \mathbb{R}^n.$$

Let $d\nu = (\mathbf{W}_{1,\,p}\omega)^q dx$. For each $r > 0$ let $d\mu_r = (\mathbf{U}_r\omega)^q dx$ and $d\lambda_r = (\mathbf{L}_r\omega)^q dx$. Then

$$\nu \le C(q)(\mu_r + \lambda_r). \tag{5.15}$$

Let $x \in \mathbb{R}^n$ and $B_r = B_r(x)$. Since $\mathbf{W}_{1,\,p}(\mathbf{W}_{1,\,p}\omega)^q = \mathbf{W}_{1,\,p}\nu$, we have to prove that

$$\mathbf{W}_{1,\,p}\nu(x) = \int_0^\infty \Big[\frac{\nu(B_r)}{r^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dr}{r} \le C\,\mathbf{W}_{1,\,p}\omega(x).$$

For $r > 0$, $t \le r$ and $y \in B_r$ we have $B_t(y) \subset B_{2r}$. Therefore it is easy to see that $\mathbf{U}_r\omega = \mathbf{U}_r\omega_{B_{2r}}$ on $B_r$. Using this together with (2.11), we have

$$\mu_r(B_r) = \int_{B_r} (\mathbf{U}_r\omega)^q dx = \int_{B_r} (\mathbf{U}_r\omega_{B_{2r}})^q dx \le C\omega(B_{2r}).$$

Hence,

$$\begin{aligned}\int_0^\infty \Big[\frac{\mu_r(B_r)}{r^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dr}{r} &\le C \int_0^\infty \Big[\frac{\omega(B_{2r})}{r^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dr}{r} \\ &\le C'\,\mathbf{W}_{1,\,p}\omega(x).\end{aligned} \tag{5.16}$$

On the other hand, for $y \in B_r$ and $t \ge r$, we have $B_t(y) \subset B_{2t}$, and consequently

$$\begin{aligned}\mathbf{L}_r\omega(y) &\le \int_r^\infty \Big[\frac{\omega(B_{2t})}{t^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dt}{t} \\ &\le C \int_{2r}^\infty \Big[\frac{\omega(B_t)}{t^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dt}{t} \\ &\le C\,\mathbf{L}_r\omega(x).\end{aligned} \tag{5.17}$$

Using (5.17), we obtain

$$\lambda_r(B_r) = \int_{B_r} (\mathbf{L}_r\omega(y))^q dy \le C(\mathbf{L}_r\omega(x))^q r^n.$$

Thus,

$$\begin{aligned}\int_0^\infty \Big[\frac{\lambda_r(B_r)}{r^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dr}{r} &\le C' \int_0^\infty (\mathbf{L}_r\omega(x))^{\frac{q}{p-1}} r^{\frac{p}{p-1}} \frac{dr}{r} \\ &= C' \int_0^\infty \Big[\int_r^\infty \Big(\frac{\omega(B_t)}{t^{n-p}}\Big)^{\frac{1}{p-1}} \frac{dt}{t}\Big]^{\frac{q}{p-1}} r^{\frac{p}{p-1}} \frac{dr}{r} \\ &= C'\frac{q}{p} \int_0^\infty r^{\frac{p}{p-1}} [\mathbf{L}_r\omega(x)]^{\frac{q}{p-1}-1} \Big[\frac{\omega(B_r)}{r^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dr}{r},\end{aligned}$$

where we have used integration by parts in the last equality. It then follows from (5.13) that

$$\begin{aligned} (5.18) \qquad \int_0^\infty \Big[\frac{\lambda_r(B_r)}{r^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dr}{r} &\leq C'' \int_0^\infty \Big[\frac{\omega(B_r)}{r^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dr}{r} \\ &= C''\, \mathbf{W}_{1,\,p}\omega(x). \end{aligned}$$

Combining (5.15), (5.16) and (5.18) gives

$$\mathbf{W}_{1,\,p}\nu(x) = \int_0^\infty \Big[\frac{\nu(B_r)}{r^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dr}{r} \leq C\, \mathbf{W}_{1,\,p}\omega(x)$$

for a suitable constant $C$ independent of $\omega$. Thus, (iv) implies (v) as claimed which completes the proof of the theorem. □

## 6. Renormalized solutions of quasilinear Dirichlet problems

Let $\Omega$ be a bounded, open subset of $\mathbb{R}^n$, $n \geq 2$. We denote by $\mathcal{M}_B(\Omega)$ the set of all Radon measures with bounded variation on $\Omega$. Recall that $\mathcal{M}_B^+(\Omega)$ denotes the set of nonnegative finite measures on $\Omega$. Let $\mathcal{A}$ be as in Section 4, and let $1 < p < \infty$. In this section we consider the Dirichlet problem

$$(6.1) \qquad \begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u) = u^q + \omega, \\ \qquad u \geq 0 \quad \text{in} \quad \Omega, \\ \qquad u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\omega \in \mathcal{M}_B^+(\Omega)$ and $q > p-1$.

It is well known that when the data are not regular enough, a solution to nonlinear equations of Leray–Lions type does not necessarily belong to the Sobolev space $\mathrm{W}_0^{1,\,p}(\Omega)$. Therefore, we have to use the framework of renormalized solutions (see, e.g., [DMOP], [BMMP], [Gre], [Kil]).

For a measure $\mu$ in $\mathcal{M}_B(\Omega)$, its positive and negative parts are denoted by $\mu^+$ and $\mu^-$, respectively. We say that a sequence of measures $\{\mu_n\}$ in $\mathcal{M}_B(\Omega)$ converges in the narrow topology to $\mu \in \mathcal{M}_B(\Omega)$ if

$$\lim_{n\to\infty} \int_\Omega \varphi\, d\mu_n = \int_\Omega \varphi\, d\mu$$

for every bounded and continuous function $\varphi$ on $\Omega$.

Denote by $\mathcal{M}_0(\Omega)$ (respectively $\mathcal{M}_s(\Omega)$) the set of all measures in $\mathcal{M}_B(\Omega)$ which are absolutely continuous (respectively singular) with respect to the capacity $\mathrm{cap}_{1,\,p}(\cdot, \Omega)$. Here $\mathrm{cap}_{1,\,p}(\cdot, \Omega)$ is the capacity relative to the domain $\Omega$ defined by

$$(6.2) \qquad \mathrm{cap}_{1,\,p}(E, \Omega) = \inf\Big\{ \int_\Omega |\nabla\phi|^p\, dx : \phi \in C_0^\infty(\Omega), \phi \geq 1 \text{ on } E\Big\}$$

for any compact set $E \subset \Omega$. Recall that, for every measure $\mu$ in $\mathcal{M}_B(\Omega)$, there exists a unique pair of measures $(\mu_0, \mu_s)$ with $\mu_0 \in \mathcal{M}_0(\Omega)$ and $\mu_s \in \mathcal{M}_s(\Omega)$, such that $\mu = \mu_0 + \mu_s$. If $\mu$ is nonnegative, then so are $\mu_0$ and $\mu_s$ (see [FST, Lemma 2.1]).

For $k > 0$ and for $s \in \mathbb{R}$ we denote by $T_k(s)$ the truncation $T_k(s) = \max\{-k, \min\{k, s\}\}$. Recall also from [BBG] that if $u$ is a measurable function on $\Omega$ which is finite almost everywhere and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$, then there exists a measurable function $v : \Omega \to \mathbb{R}^n$ such that

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \quad \text{a.e.} \quad \text{on} \quad \Omega, \quad \text{for all } k > 0.$$

Moreover, $v$ is unique up to almost everywhere equivalence. We define the gradient $Du$ of $u$ as this function $v$, and set $Du = v$.

In [DMOP], several equivalent definitions of renormalized solutions are given. In what follows, we will need the following ones.

*Definition* 6.1. Let $\mu \in \mathcal{M}_B(\Omega)$. Then $u$ is said to be a renormalized solution of

$$\begin{cases} -\text{div}\mathcal{A}(x, \nabla u) = \mu & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{6.3}$$

if the following conditions hold:

(a) The function $u$ is measurable and finite almost everywhere, and $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$.

(b) The gradient $Du$ of $u$ satisfies $|Du|^{p-1} \in L^q(\Omega)$ for all $q < \frac{n}{n-1}$.

(c) If $w$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and if there exist $w^{+\infty}$ and $w^{-\infty}$ in $W^{1,r}(\Omega) \cap L^\infty(\Omega)$, with $r > n$, such that

$$\begin{cases} w = w^{+\infty} & \text{a.e. on the set } \{u > k\}, \\ w = w^{-\infty} & \text{a.e. on the set } \{u < -k\} \end{cases}$$

for some $k > 0$, then

$$\int_\Omega \mathcal{A}(x, Du) \cdot \nabla w dx = \int_\Omega w d\mu_0 + \int_\Omega w^{+\infty} d\mu_s^+ - \int_\Omega w^{-\infty} d\mu_s^-. \tag{6.4}$$

*Definition* 6.2. Let $\mu \in \mathcal{M}_B(\Omega)$. Then $u$ is a *renormalized solution* of (6.3) if $u$ satisfies (a) and (b) in Definition 6.1, and if the following conditions hold:

(c) For every $k > 0$ there exist two nonnegative measures in $\mathcal{M}_0(\Omega)$, $\lambda_k^+$ and $\lambda_k^-$, concentrated on the sets $\{u = k\}$ and $\{u = -k\}$, respectively, such that $\lambda_k^+ \to \mu_s^+$ and $\lambda_k^- \to \mu_s^-$ in the narrow topology of measures.

(d) For every $k > 0$

$$\int_{\{|u|<k\}} \mathcal{A}(x, Du) \cdot \nabla \varphi dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_\Omega \varphi d\lambda_k^+ - \int_\Omega \varphi d\lambda_k^- \tag{6.5}$$

for every $\varphi$ in $\mathrm{W}_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

*Remark* 6.3. By [DMOP, Remark 2.18], if $u$ is a renormalized solution of (6.3) then (the $\mathrm{cap}_{1,p}$-quasi continuous representative of) $u$ is finite $\mathrm{cap}_{1,p}$-quasieverywhere. Therefore, $u$ is finite $\mu_0$-almost everywhere.

*Remark* 6.4. By (6.5), if $u$ is a renormalized solution of (6.3) then

$$-\mathrm{div}\mathcal{A}(x, \nabla T_k(u)) = \mu_k \quad \text{in} \quad \Omega, \tag{6.6}$$

where

$$\mu_k = \chi_{\{|u|<k\}}\mu_0 + \lambda_k^+ - \lambda_k^-.$$

Since $T_k(u) \in W_0^{1,p}(\Omega)$, by (4.3) we see that $-\mathrm{div}\mathcal{A}(x, \nabla T_k(u))$ and hence $\mu_k$ belong to the dual space $W^{-1,p'}(\Omega)$ of $W_0^{1,p}(\Omega)$. Moreover, by Remark 6.3, $|u| < \infty$ $\mu_0$-almost everywhere and hence $\chi_{\{|u|<k\}} \to \chi_\Omega$ $\mu_0$-almost everywhere as $k \to \infty$. Therefore, by the monotone convergence theorem, $\mu_k$ converges to $\mu$ in the narrow topology of measures.

*Remark* 6.5. If $\mu \geq 0$, i.e., $\mu \in \mathcal{M}_B^+(\Omega)$, and $u$ is a renormalized solution of (6.3) then $u$ is nonnegative. To see this, for each $k > 0$ we "test" (6.4) with $w = T_k(\min\{u, 0\})$, $w^{+\infty} = 0$ and $w^{-\infty} = -k$:

$$\int_\Omega \mathcal{A}(x, Du) \cdot \nabla w dx = \int_\Omega w d\mu_0 + \int_\Omega k d\mu_s^- = \int_\Omega w d\mu_0 \leq 0,$$

since $\mu_s^- = 0$ and $w \leq 0$. Thus using (4.3) we get

$$\int_\Omega |\nabla T_k(\min\{u, 0\})|^p dx \leq 0$$

for every $k > 0$. Therefore $\min\{u, 0\} = 0$ a.e., i.e., $u$ is nonnegative.

*Remark* 6.6. Let $\mu \in \mathcal{M}_B^+(\Omega)$ and let $u$ be a renormalized solution of (6.3). Since $u^- = 0$ a.e. (by Remark 6.5) and hence $u^- = 0$ $\mathrm{cap}_{1,p}$-quasi everywhere (see [HKM, Th. 4.12]), in Remark 6.4 we may take $\lambda_k^- = 0$, and thus $\mu_k$ is nonnegative. Hence by (6.6) and Proposition 4.1, the functions $v_k$ defined by $v_k(x) = \mathrm{ess}\liminf_{y\to x} T_k(u)(y)$ are $\mathcal{A}$-superharmonic and increasing. Using Lemma 7.3 in [HKM], it is then easy to see that $v_k \to v$ as $k \to \infty$ everywhere in $\Omega$ for some $\mathcal{A}$-superharmonic function $v$ on $\Omega$ such that $v = u$ a.e. In other words, $v$ is an $\mathcal{A}$-superharmonic representative of $u$.

*Remark* 6.7. When dealing with pointwise values of a renormalized solution u to (6.3) with measure data $\mu \geq 0$, we always identify $u$ with its $\mathcal{A}$-superharmonic representative mentioned in Remark 6.6.

We now establish a comparison principle for renormalized solutions.

LEMMA 6.8. *Let $\mu$, $\nu \in \mathcal{M}_B^+(\Omega)$ be such that $\mu \le \nu$. Suppose that $u$ and $v$ are renormalized solutions of*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u) = \mu \quad \text{in} \quad \Omega, \\ \quad u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

*and*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla v) = \nu \quad \text{in} \quad \Omega, \\ \quad v = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

*respectively. If $u$ is uniformly bounded then $u \le v$.*

*Proof.* Let $w = \min\{(u-v)^+, k\}$. Then $w = 0$ on the set $\{v > k + M\}$ and $w = k$ on the set $\{v < -k - M\}$, where $M = \sup_\Omega u$. Moreover, $w \in W_0^{1,p} \cap L^\infty(\Omega)$ as $w = \min\{(u - T_{k+M}(v))^+, k\}$. Thus by Definition 6.1 we have

$$\int_\Omega \mathcal{A}(x, Dv) \cdot \nabla w dx = \int_\Omega w d\nu_0. \tag{6.7}$$

On the other hand, since $u$ is bounded (hence belongs to $W_0^{1,p}(\Omega)$) we have

$$\int_\Omega \mathcal{A}(x, Du) \cdot \nabla w dx = \int_\Omega w d\mu. \tag{6.8}$$

From (6.7) and (6.8) we get

$$\int_\Omega [\mathcal{A}(x, Du) - \mathcal{A}(x, Dv)] \cdot \nabla w dx \le 0.$$

Consequently,

$$\int_{0<u-v<k} [\mathcal{A}(x, Du) - \mathcal{A}(x, Dv)] \cdot (Du - Dv) dx \le 0,$$

since $\nabla w = \nabla \max\{T_k(u-v), 0\} = D(u-v)\chi_{\{0<u-v<k\}}$. Thus by (4.4) we have $\nabla w = 0$ and hence $w = 0$ a.e. for every $k > 0$, which gives $u \le v$. □

In the following lemma we drop the assumption that $u$ is uniformly bounded in Lemma 6.8, but claim only the existence of $v$ such that $v \ge u$. This lemma was referred to in the proof of Theorem 5.3 given in Section 5 above.

LEMMA 6.9. *Let $\mu$, $\nu \in \mathcal{M}_B^+(\Omega)$ be such that $\nu \ge \mu$. Suppose that $u$ is a renormalized solution of*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u) = \mu \quad \text{in} \quad \Omega, \\ \quad u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

*Then there exists $v \ge u$ such that*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla v) = \nu \quad \text{in} \quad \Omega, \\ \quad v = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

*in the renormalized sense.*

*Proof.* Let $u_k = \min\{u, k\}$ for each $k \in \mathbb{N}$. From Definition 6.2 of renormalized solutions we have

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla u_k) = \mu_{0\{u<k\}} + \lambda_k^+ \quad \text{in} \quad \Omega, \\ u_k = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

in the renormalized sense for a sequence of nonnegative measures $\{\lambda_k^+\}$ that converges to $\mu_s^+$ in the narrow topology of measures. Thus, by Lemma 6.8 we have $u_k \le v_k$, where $v_k$ are renormalized solutions of

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla v_k) = \mu_0 + \lambda_k^+ + \nu - \mu \quad \text{in} \quad \Omega, \\ v_k = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Finally, it follows from the stability results in [DMOP] that we can find a subsequence of $\{v_k\}$ that converges a.e. to a required function $v$. □

We will also need the following version of Lemma 6.9 which will be used in the proof of Theorem 2.1 on global potential estimates for renormalized solutions stated in Section 2.

Lemma 6.10. *Suppose that $u$ is a renormalized solution to* (6.3) *with data $\mu \in \mathcal{M}_B^+(\Omega)$. Let $B$ be a ball that contains $\Omega$. Then there exists a renormalized solution $w$ on $B$ to*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla w) = \mu \quad \text{in} \quad B, \\ w = 0 \quad \text{on} \quad \partial B \end{cases} \tag{6.9}$$

*such that $u \le w$ on $\Omega$, and*

$$||w||_{L^{p-1}(B)} \le C R^{\frac{p}{p-1}} \mu(\Omega)^{\frac{1}{p-1}}. \tag{6.10}$$

*Proof.* Let $u_k = \min\{u, k\}$, $k > 0$, and let $\mu_k = \chi_{\{u<k\}}\mu_0 + \lambda_k^+$ be as in Remark 6.4 (note that $\lambda_k^- = 0$ by Remark 6.6). We see that $u_k \in W_0^{1,p}(\Omega)$ is the unique solution of problem (6.3) with data $\mu_k$. We next extend $u_k$ by zero outside $\Omega$, and set

$$\Psi = \min\{w_k - u_k, 0\} = \min\{\min\{w_k, k\} - u_k, 0\},$$

where $w_k$, $k > 0$, is a renormalized solution to the problem

$$\begin{cases} -\mathrm{div}\mathcal{A}(x, \nabla w_k) = \mu_0 + \lambda_k^+ \quad \text{in} \quad B, \\ w_k = 0 \quad \text{on} \quad \partial B. \end{cases}$$

Note that $\Psi \in W_0^{1,p}(\Omega) \cap W_0^{1,p}(B) \cap L^\infty(B)$ since $|\Psi| \le u_k$. Then using $\Psi$ as a test function we have

$$\begin{aligned}
0 &\geq \int_B \mathcal{A}(x, \nabla w_k) \cdot \nabla \Psi dx - \int_\Omega \mathcal{A}(x, \nabla u_k) \cdot \nabla \Psi dx \\
&= \int_{B \cap \{w_k < u_k\}} \mathcal{A}(x, \nabla w_k) \cdot \nabla \Psi dx - \int_{B \cap \{w_k < u_k\}} \mathcal{A}(x, \nabla u_k) \cdot \nabla \Psi dx \\
&= \int_{B \cap \{w_k < u_k\}} [\mathcal{A}(x, \nabla w_k) - \mathcal{A}(x, \nabla u_k)] \cdot (\nabla w_k - \nabla u_k) dx.
\end{aligned}$$

Thus $\nabla w_k = \nabla u_k$ a.e. on the set $B \cap \{w_k < u_k\}$ by hypothesis (4.4) on $\mathcal{A}$. Hence $\Psi = 0$ a.e.; i.e.,

$$u_k \le w_k \quad \text{a.e.} \tag{6.11}$$

Now by the stability results for renormalized solutions established in [DMOP] we can find a subsequence $\{w_{k_j}\}$ of $\{w_k\}$ such that $w_{k_j} \to w$ a.e., where $w$ is a renormalized solution to (6.9). By (6.11) we have $u \le w$ a.e. on $\Omega$, and hence $u \le w$ everywhere on $\Omega$ due to Remark 6.7 and Proposition 4.1.

Finally, note that for $p < n$ we have

$$||w||_{L^{\frac{n(p-1)}{n-p}, \infty}(B)} \le C\, \mu(\Omega)^{\frac{1}{p-1}},$$

for a constant $C$ independent of $R$ (see [DMOP, Th. 4.1] or [BBG, Lemma 4.1]). Thus we obtain (6.10). Inequality (6.10) also holds for $p \ge n$; see for example [Gre, Lemma 2.1]. This completes the proof of the lemma. □

*Proof of Theorem* 2.1. The lower estimate in (2.6) is just a restatement of the local estimate given in Theorem 4.4. To prove the upper estimate we let $B = B_{2R}(a)$, where $R = \mathrm{diam}(\Omega)$, $a \in \Omega$ so that $\Omega \subset B$. Also, let $w$ be as in Lemma 6.10 with respect to that choice of $B$. For $x \in \Omega$ we denote by $d(x)$ the distance from $x$ to the boundary $\partial B$ of $B$. By Theorem 4.4, Lemma 6.10, and the fact that $d(x) \ge R$,

$$\begin{aligned}
u(x) \le w(x) &\le C\, \mathbf{W}_{1,p}^{\frac{2}{3} d(x)} \mu(x) + C \inf_{B_{\frac{1}{3} d(x)}(x)} w \\
&\le C\, \mathbf{W}_{1,p}^{2R} \mu(x) + C d(x)^{\frac{-n}{p-1}} ||w||_{L^{p-1}(B)} \\
&\le C\, \mathbf{W}_{1,p}^{2R} \mu(x) + C R^{\frac{-n}{p-1}} ||w||_{L^{p-1}(B)}.
\end{aligned}$$

Thus from (6.10) we get the desired upper estimate in (2.6). □

We next give a sufficient condition for the existence of renormalized solutions to quasilinear equations in a bounded domain $\Omega$, which is an analogue of Theorem 5.3 related to the case $\Omega = \mathbb{R}^n$. Its proof is based on stability results for renormalized solutions in place of the weak continuity of measures generated by $\mathcal{A}$-superharmonic functions used in the proof of Theorem 5.3.

Theorem 6.11. *Let $\omega \in \mathcal{M}_B^+(\Omega)$. Let $p>1$ and $q>p-1$. Suppose that $R=\mathrm{diam}(\Omega)$, and*

$$\mathbf{W}_{1,p}^{2R}\,(\mathbf{W}_{1,p}^{2R}\omega)^q \le C\,\mathbf{W}_{1,p}^{2R}\omega \quad \text{a.e.},$$

*where*

$$C \le \Big(\frac{q-p+1}{qK\max\{1,2^{p'-2}\}}\Big)^{q(p'-1)}\Big(\frac{p-1}{q-p+1}\Big),$$

*and $K$ is the constant in Theorem 2.1. Then there is a renormalized solution $u\in L^q(\Omega)$ to the Dirichlet problem*

$$\begin{cases} -\mathrm{div}\mathcal{A}(x,\nabla u) = u^q+\omega & \text{in} \quad \Omega, \\ u=0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{6.12}$$

*such that*

$$u(x) \le M\,\mathbf{W}_{1,p}^{2R}\omega(x)$$

*for all $x$ in $\Omega$, where the constant $M$ depends only on $p,q,n$, and the structural constants $\alpha$ and $\beta$.*

*Proof.* By Lemma 6.9 we can find a nondecreasing sequence $\{u_k\}_{k\ge0}$ of renormalized solutions to the following Dirichlet problems:

$$\begin{cases} -\mathrm{div}\mathcal{A}(x,\nabla u_0) = \omega & \text{in} \quad \Omega, \\ u_0=0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{6.13}$$

and

$$\begin{cases} -\mathrm{div}\mathcal{A}(x,\nabla u_k) = u_{k-1}^q+\omega & \text{in} \quad \Omega, \\ u_k=0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{6.14}$$

for $k\ge1$. By Theorem 2.1 we have

$$u_0 \le K\,\mathrm{W}_{1,p}^{2R}\omega, \qquad u_k \le K\,\mathrm{W}_{1,p}^{2R}(u_{k-1}^q+\omega).$$

Thus arguing as in the proof of Theorem 5.3, we obtain a constant $M>0$ such that

$$u_k \le M\,\mathrm{W}_{1,p}^{2R}\omega < \infty \quad \text{a.e.}$$

for all $k\ge0$. Therefore, $\{u_k\}$ converges pointwise to a nonnegative function $u$ for which

$$u \le M\,\mathrm{W}_{1,p}^{2R}\omega < \infty \quad \text{a.e.},$$

and $u_k^q \to u^q$ in $\mathrm{L}^1(\Omega)$. Finally, in view of (6.14), the stability result in [DMOP, Th. 3.4] asserts that $u$ is a renormalized solution of (6.12), which proves the theorem. □

Let $\mathcal{Q} = \{Q\}$ be a Whitney decomposition of $\Omega$, i.e., $\mathcal{Q}$ is a disjoint subfamily of the family of dyadic cubes in $\mathbb{R}^n$ such that $\Omega = \cup_{Q\in\mathcal{Q}} Q$, where we can assume that $2^5\text{diam}(Q) \le \text{dist}(Q, \partial\Omega) \le 2^7\text{diam}(Q)$. Let $\{\phi_Q\}_{Q\in\mathcal{Q}}$ be a partition of unity associated with the Whitney decomposition of $\Omega$ above: $0 \le \phi_Q \in C_0^\infty(Q^*)$, $\phi_Q \ge 1/C(n)$ on $\overline{Q}$, $\sum_Q \phi_Q = 1$ and $|D^\gamma \phi_Q| \le A_\gamma(\text{diam}(Q))^{-|\gamma|}$ for all multi-indices $\gamma$. Here $Q^* = (1+\varepsilon)Q$, $0 < \varepsilon < \frac{1}{4}$ and $C(n)$ is a positive constant depending only on $n$ such that each point in $\Omega$ is contained in at most $C(n)$ of the cubes $Q^*$ (see [St1]).

The following theorem is an extension of Theorem 2.16 on local estimates for solutions of quasilinear equations.

THEOREM 6.12. *Let $\omega$ be a locally finite, nonnegative measure on an open (not necessarily bounded) set $\Omega$. Let $p > 1$ and $q > p - 1$. Suppose that there exists a nonnegative $\mathcal{A}$-superharmonic function $u$ in $\Omega$ such that*

$$-\text{div}\mathcal{A}(\text{x}, \nabla \text{u}) = u^q + \omega \quad \text{in} \quad \Omega.$$

*Then, for each cube $P \in \mathcal{Q}$ and compact set $E \subset \Omega$,*

$$\mu_P(E) \le C\, \text{Cap}_{\mathbf{I}_p, \frac{q}{q-p+1}}(E) \tag{6.15}$$

*if $\frac{pq}{q-p+1} < n$, and*

$$\mu_P(E) \le C(P)\, \text{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}(E) \tag{6.16}$$

*if $\frac{pq}{q-p+1} \ge n$. Here $d\mu = u^q dx + d\omega$, and the constant $C$ in* (6.15) *is independent of $P \in \mathcal{Q}$ and $E \subset \Omega$, but the constant $C(P)$ in* (6.16) *may depend on the side length of $P$.*

*Moreover, if $\frac{pq}{q-p+1} < n$ and $\Omega$ is a bounded $C^\infty$-domain, then*

$$\mu(E) \le C\, \text{cap}_{p, \frac{q}{q-p+1}}(E, \Omega)$$

*for all compact sets $E \subset \Omega$, where $\text{cap}_{p, \frac{q}{q-p+1}}(E, \Omega)$ is defined by* (2.24).

*Proof.* Let $P$ be a fixed dyadic cube in $\mathcal{Q}$. For a dyadic cube $P' \subset P$ we have

$$\text{dist}(P', \partial\Omega) \ge \text{dist}(P, \partial\Omega) \ge 2^5\text{diam}(P) \ge 2^5\text{diam}(P').$$

The lower estimate in Theorem 4.4 then yields

$$\begin{aligned} u(x) &\ge C\, \mathbf{W}_{1,p}^{2^3\text{diam}(P')}\mu(x) \\ &\ge C \sum_{k=0}^{\infty} \int_{2^{-k+2}\text{diam}(P')}^{2^{-k+3}\text{diam}(P')} \Big[\frac{\mu(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dt}{t} \\ &\ge C \sum_{Q\subset P'} \Big[\frac{\mu(Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{1}{p-1}} \chi_Q(x) \end{aligned}$$

for all $x \in P'$. Thus it follows from Proposition 3.1 that

$$\sum_{Q\subset P'} \Big[\frac{\mu(Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}} |Q| \leq C\int_{P'} u^q dx \leq C\mu(P'), \qquad P' \subset P. \tag{6.17}$$

Hence

$$\mu(P') \leq C\left|P'\right|^{1-\frac{pq}{n(q-p+1)}}, \qquad P' \subset P. \tag{6.18}$$

To get a better estimate for $\mu(P')$ in the case $\frac{pq}{q-p+1} = n$, we observe that (6.17) is a dyadic Carleson condition. Thus by the dyadic Carleson imbedding theorem (see, e.g., [NTV], [V1]) we obtain, for $\frac{pq}{q-p+1} = n$,

$$\sum_{Q\subset P} \mu(Q)^{\frac{q}{p-1}} \Big[\frac{1}{\mu(Q)}\int_Q f d\mu\Big]^{\frac{q}{p-1}} \leq C\int_P f^{\frac{q}{p-1}} d\mu, \tag{6.19}$$

where $f \in L^{\frac{q}{p-1}}(d\mu_P)$, $f \geq 0$. From (6.19) with $f = \chi_{P'}$, one gets

$$\mu(P') \leq C\Big(\log\frac{2^n|P|}{|P'|}\Big)^{\frac{1-p}{q-p+1}}, \quad P' \subset P, \tag{6.20}$$

if $\frac{pq}{q-p+1} = n$. Now let $P'$ be a dyadic cube in $\mathbb{R}^n$. From Wolff's inequality (5.4) we have

$$\begin{aligned} &\int_{\mathbb{R}^n} (\mathbf{I}_p\mu_{P'\cap P})^{\frac{q}{p-1}} dx \\ &\quad\leq C\sum_{Q\in\mathcal{D}} \Big[\frac{\mu_P(P'\cap Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}} |Q| \\ &\quad= C\sum_{Q\subset P'} \Big[\frac{\mu_P(Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}} |Q| + C\sum_{P'\subsetneq Q} \Big[\frac{\mu_P(P')}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}} |Q|. \end{aligned} \tag{6.21}$$

Thus, for $\frac{pq}{q-p+1} < n$, by combining (6.17) and (6.21) we deduce

$$\int_{\mathbb{R}^n} (\mathbf{I}_p\mu_{P'\cap P})^{\frac{q}{p-1}} dx \leq C\,\mu_P(P'). \tag{6.22}$$

In the case $\frac{pq}{q-p+1} \geq n$, a similar argument using (6.17), (6.18), (6.20) and Wolff's inequality for Bessel potentials:

$$\int_{\mathbb{R}^n} (\mathbf{G}_p\mu_{P'\cap P})^{\frac{q}{p-1}} dx \leq C(P) \sum_{Q\in\mathcal{D},\, Q\subset P} \Big[\frac{\mu_P(P'\cap Q)}{|Q|^{1-\frac{p}{n}}}\Big]^{\frac{q}{p-1}} |Q|,$$

(see [AH, §4.5]), also gives

$$\int_{\mathbb{R}^n} (\mathbf{G}_p\mu_{P'\cap P})^{\frac{q}{p-1}} dx \leq C(P)\mu_P(P'), \tag{6.23}$$

where the constant $C(P)$ may depend on the side length of $P$. Note that (6.22) which holds for all dyadic cubes $P'$ in $\mathbb{R}^n$ is the well-known Kerman-Sawyer condition. Therefore by the results of [KS],

$$\|\mathbf{I}_p f\|_{L^{\frac{q}{q-p+1}}(d\mu_P)} \leq C\,\|f\|_{L^{\frac{q}{q-p+1}}(dx)}$$

for all $f \in L^{\frac{q}{q-p+1}}(\mathbb{R}^n)$ which is equivalent to the capacitary condition

$$\mu_P(E) \le C \operatorname{Cap}_{\mathbf{I}_p, \frac{q}{q-p+1}}(E)$$

for all compact sets $E \subset \mathbb{R}^n$. Thus we obtain (6.15). The inequality (6.16) is proved in the same way using (6.23). From (6.15) and the definition of $\operatorname{cap}_{p, \frac{q}{q-p+1}}(\cdot, \Omega)$, we see that, for each cube $P \in \mathcal{Q}$,

$$\mu_P(E) \le C \operatorname{cap}_{p, \frac{q}{q-p+1}}(E \cap P, \Omega)$$

for all compact sets $E \subset \Omega$. Thus

$$\begin{aligned}\mu(E) &\le \sum_{P \in \mathcal{Q}} \mu_P(E) \\ &\le C \sum_{P \in \mathcal{Q}} \operatorname{cap}_{p, \frac{q}{q-p+1}}(E \cap P, \Omega) \\ &\le C \operatorname{cap}_{p, \frac{q}{q-p+1}}(E, \Omega),\end{aligned}$$

where the last inequality follows from the quasi-additivity of the capacity $\operatorname{cap}_{p, \frac{q}{q-p+1}}(\cdot, \Omega)$ which is considered in the next theorem. □

Let $B_R$ be a ball such that $B_{2R} \subset \Omega$. It is easy to see that there exists a constant $c > 0$ such that $\ell(P) \ge cR$ for any Whitney cube P that intersects $B_R$. On the other hand, if $B_r$ is a ball in $B_R$ then we can find at most $N$ dyadic cubes $P_i$ with $c\,\frac{r}{4} \le \ell(P_i) < c\,\frac{r}{2}$ that cover $B_r$, where $N$ depends only on $n$. Thus we can deduce from (6.20) the following corollary which gives an improved estimate in the critical case $q = \frac{n(p-1)}{n-p}$, $1 < p < n$.

Corollary 6.13. *Let $\omega$, $\Omega$, $p$, $q$ and $u$ be as in Theorem 6.12. Then in the case $\frac{pq}{q-p+1} = n$,*

$$\int_{B_r} u^q \, dx + \omega(B_r) \le C (\log \tfrac{2R}{r})^{\frac{1-p}{q-p+1}}$$

*for all balls $B_r \subset B_R$ such that $B_{2R} \subset \Omega$.*

Theorem 6.14. *Suppose that $\Omega$ is a $C^\infty$-domain in $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that*

$$\sum_{Q \in \mathcal{Q}} \operatorname{cap}_{p, \frac{q}{q-p+1}}(E \cap Q, \Omega) \le C \operatorname{cap}_{p, \frac{q}{q-p+1}}(E, \Omega)$$

*for all compact sets $E \subset \Omega$.*

*Proof.* Obviously, we may assume that $\operatorname{cap}_{p, \frac{q}{q-p+1}}(E, \Omega) > 0$. Then by definition there exists $f \in C_0^\infty(\Omega)$, $f \ge 1$, on $E$ such that

$$2 \operatorname{cap}_{p, \frac{q}{q-p+1}}(E, \Omega) \ge \|f\|^{\frac{q}{q-p+1}}_{W^{p, \frac{q}{q-p+1}}(\mathbb{R}^n)}.$$

By the refined localization principle on the smooth domain $\Omega$ for the function space $W^{p,\frac{q}{q-p+1}}$ (see, e.g., [Tri, Th. 5.14]) we have

$$\|f\|^{\frac{q}{q-p+1}}_{W^{p,\frac{q}{q-p+1}}(\mathbb{R}^n)} \geq C \sum_{Q\in\mathcal{Q}} \|f\phi_Q\|^{\frac{q}{q-p+1}}_{W^{p,\frac{q}{q-p+1}}(\mathbb{R}^n)}.$$

Thus

$$\sum_{Q\in\mathcal{Q}} \|f\phi_Q\|^{\frac{q}{q-p+1}}_{W^{p,\frac{q}{q-p+1}}(\mathbb{R}^n)} \leq C\mathrm{cap}_{p,\frac{q}{q-p+1}}(E,\Omega). \tag{6.24}$$

Note that for $x \in E\cap\overline{Q}$,

$$f\phi_Q \geq \phi_Q \geq 1/C(n).$$

Hence by definition we have

$$\mathrm{cap}_{p,\frac{q}{q-p+1}}(E\cap\overline{Q},\Omega) \leq C\,\|f\phi_Q\|^{\frac{q}{q-p+1}}_{W^{p,\frac{q}{q-p+1}}(\mathbb{R}^n)}.$$

From this and (6.24) we deduce the desired inequality. □

We now prove Theorem 2.10 stated in Section 2. which gives existence criteria for quasilinear equations in a bounded domain.

*Proof of Theorem* 2.10. Since $\omega$ is compactly supported in $\Omega$, we have (i)⇒(ii) by Theorem 6.12. Thus we need to show (ii)⇒(iii)⇒(iv) ⇒(i). Note that the capacitary inequality (2.20) is equivalent to the Kerman–Sawyer condition

$$\int_{\mathbb{R}^n}\Big[\mathbf{G}_p\omega_B(x)\Big]^{\frac{q}{p-1}}dx \leq C\,\omega(B), \tag{6.25}$$

(see [KS], [V2]). Note also that

$$\int_{\mathbb{R}^n}\Big[\mathbf{G}_p\mu(x)\Big]^{\frac{q}{p-1}}dx \simeq \int_{\mathbb{R}^n}\Big[\int_0^{2R}\frac{\mu(B_t(x))}{t^{n-p}}\Big]^{\frac{q}{p-1}}dx, \tag{6.26}$$

where the constants of equivalence are independent of the measure $\mu$, (see [HW], [AH]). Thus from (6.25), (6.26), and Proposition 5.1 we deduce the implication (ii)⇒(iii). By Theorem 6.11 we have (iv)⇒(i). It remains to show that (iii)⇒(iv). In fact, the proof of this implication is similar to the proof of (iv)⇒(v) in Theorem 2.3. We will only sketch some crucial steps here. We define the "lower" and "upper" parts of the truncated Wolff's potential $\mathbf{W}^{2R}_{1,p}$ respectively by

$$\mathbf{L}^{2R}_r\mu(x) = \int_r^{2R}\Big[\frac{\mu(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}}\frac{dt}{t}, \qquad 0<r<2R,\ x\in\mathbb{R}^n,$$

and

$$\mathbf{U}^{2R}_r\mu(x) = \int_0^{r}\Big[\frac{\mu(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}}\frac{dt}{t}, \qquad 0<r<2R,\ x\in\mathbb{R}^n.$$

Since $R = \mathrm{diam}(\Omega)$ and $\omega \in M_B^+(\Omega)$, to prove (2.22), it is enough to verify that, for $x \in \Omega$,

$$\int_0^{2R} \Big[\frac{\mu_r(B_r(x))}{r^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dr}{r} \le C\, \mathbf{W}_{1,p}^{2R}\omega(x), \tag{6.27}$$

and

$$\int_0^{2R} \Big[\frac{\lambda_r(B_r(x))}{r^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dr}{r} \le C\, \mathbf{W}_{1,p}^{2R}\omega(x), \tag{6.28}$$

where $d\mu_r = (\mathbf{U}_r^{2R}\omega)^q dx$, $d\lambda_r = (\mathbf{L}_r^{2R}\omega)^q dx$ and $0 < r < 2R$. The proof of (6.27) is the same as before. For the proof of (6.28), we need an estimate similar to (5.13) in the proof of Theorem 2.3, namely,

$$\int_r^{4R} \Big[\frac{\omega(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dt}{t} \le C(R, \omega(\Omega))\, r^{\frac{-p}{q-p+1}} \tag{6.29}$$

for all $0 < r \le 4R$ and $x \in \Omega$. In fact, note that for $0 < t < \frac{R}{2}$ and $y \in B_t(x)$,

$$\begin{aligned}\mathbf{W}_{1,p}^{2R}\omega_{B_t(x)}(y) &\ge \int_{2t}^{2R} \Big[\frac{\omega(B_\tau(y)\cap B_t(x))}{\tau^{n-p}}\Big]^{\frac{1}{p-1}} \frac{d\tau}{\tau}\\ &\ge C(n,p)\Big[\frac{\omega(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}}.\end{aligned}$$

As before, from this inequality and (2.21) we get

$$\omega(B_t(x)) \le C t^{n-\frac{pq}{q-p+1}}, \qquad 0 < t < \tfrac{R}{2}. \tag{6.30}$$

To prove (6.29), we can assume that $0 < r < \frac{R}{2}$ and write the left-hand side of (6.29) in the form

$$\int_r^{\frac{R}{2}} \Big[\frac{\omega(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dt}{t} + \int_{\frac{R}{2}}^{4R} \Big[\frac{\omega(B_t(x))}{t^{n-p}}\Big]^{\frac{1}{p-1}} \frac{dt}{t}. \tag{6.31}$$

Applying (6.30) to the first term of (6.31) and using the fact that $\omega \in \mathcal{M}_B^+(\Omega)$ in the second term of (6.31), we finally obtain (6.29). This completes the proof of (iii)⇒(iv), and hence Theorem 2.10 is proved. □

We are now in a position to obtain the characterization of removable singularities for homogeneous quasilinear equations in Theorem 2.18 above.

*Proof of Theorem* 2.18. Let us first prove the "only if" part of the theorem. Suppose that $\mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}(E) = 0$, and $u$ is a solution of (2.25). We have $\mathrm{cap}_{1,p}(E, \Omega) = 0$, where the capacity $\mathrm{cap}_{1,p}(\cdot, \Omega)$ is defined by (6.2). Thus $u$ can be extended so that it is a nonnegative $\mathcal{A}$-superharmonic function in $\Omega$ (see [HKM]). Let $\mu[u]$ be the Radon measure on $\Omega$ associated with $u$, and let $\varphi$ be an arbitrary nonnegative function in $C_0^\infty(\Omega)$. As in [BP, Lemme 2.2], we can find a sequence $\{\varphi_n\}$ of nonnegative functions in $C_0^\infty(\Omega \setminus E)$ such that

$$0 \le \varphi_n \le \varphi; \qquad \varphi_n \to \varphi \quad \mathrm{Cap}_{\mathbf{G}_p, \frac{q}{q-p+1}}\text{-quasi everywhere.} \tag{6.32}$$

By Fatou's lemma we have

$$\begin{aligned}\int_\Omega u^q\,\varphi\,dx &\le \liminf_{n\to\infty}\int_\Omega u^q\,\varphi_n\,dx\\ &= \liminf_{n\to\infty}\int_\Omega \varphi_n\,d\mu[u]\\ &\le \int_\Omega \varphi\,d\mu[u] < \infty.\end{aligned}$$

Therefore $u \in L^q_{\rm loc}(\Omega)$, and $\mu[u] \ge u^q$ in $\mathcal{D}'(\Omega)$. It is then easy to see that

$$-{\rm div}\mathcal{A}(x,\nabla u) = u^q + \mu^E \quad \text{in} \quad \mathcal{D}'(\Omega)$$

for some nonnegative measure $\mu^E$ such that $\mu^E(A) = 0$ for any Borel set $A \subset \Omega\setminus E$. Moreover, by Theorem 6.12 we have

$$\mu^E(E) \le C(E)\,{\rm Cap}_{\mathbf{G}_p,\,\frac{q}{q-p+1}}(E) = 0.$$

Thus $\mu^E = 0$ and $u$ solves (2.25) with $\Omega$ in place of $\Omega\setminus E$.

The "if" part of the theorem is proved in the same way as in the linear case $p = 2$ using the existence results obtained in Theorem 2.10. We refer to [AP] for details. □

## 7. Hessian equations

In this section, we study a fully nonlinear counterpart of the theory presented in the previous sections. Here the notion of $k$-subharmonic ($k$-convex) functions associated with the fully nonlinear $k$-Hessian operator $F_k$, $k = 1,\dots,n$, introduced by Trudinger and Wang in [TW1]–[TW3] will play a role similar to that of $\mathcal{A}$-superharmonic functions in the quasilinear theory.

Let $\Omega$ be an open set in $\mathbb{R}^n$, $n \ge 2$. For $k = 1,\dots,n$ and $u \in C^2(\Omega)$, the $k$-Hessian operator $F_k$ is defined by

$$F_k[u] = S_k(\lambda(D^2u)),$$

where $\lambda(D^2u) = (\lambda_1,\dots,\lambda_n)$ denotes the eigenvalues of the Hessian matrix of second partial derivatives $D^2u$, and $S_k$ is the $k^{th}$ symmetric function on $\mathbb{R}^n$ given by

$$S_k(\lambda) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\cdots\lambda_{i_k}.$$

Thus $F_1[u] = \Delta u$ and $F_n[u] = \det D^2u$. Alternatively, we may also write

$$F_k[u] = [D^2u]_k,$$

where for an $n\times n$ matrix $A$, $[A]_k$ is the $k$-trace of $A$, i.e., the sum of its $k\times k$ principal minors. Several equivalent definitions of $k$-subharmonicity were given in [TW2], one of which involves the language of viscosity solutions: An upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is said to be $k$-subharmonic in $\Omega$, $1 \le k \le n$, if $F_k[q] \ge 0$ for any quadratic polynomial $q$ such that $u-q$ has a local finite maximum in $\Omega$. Equivalently, an upper-semicontinuous function $u : \Omega \to [-\infty, \infty)$ is $k$-subharmonic in $\Omega$ if, for every open set $\Omega' \Subset \Omega$ and for every function $v \in C^2_{\mathrm{loc}}(\Omega') \cap C^0(\overline{\Omega'})$ satisfying $F_k[v] \ge 0$ in $\Omega'$, the following implication holds:

$$u \le v \text{ on } \partial\Omega' \Longrightarrow u \le v \text{ in } \Omega',$$

(see [TW2, Lemma 2.1]). Note that a function $u \in C^2_{\mathrm{loc}}(\Omega)$ is $k$-subharmonic if and only if

$$F_j[u] \ge 0 \text{ in } \Omega \text{ for all } j = 1, \dots, k.$$

We denote by $\Phi^k(\Omega)$ the class of all $k$-subharmonic functions in $\Omega$ which are not identically equal to $-\infty$ in each component of $\Omega$. It was proved in [TW2] that $\Phi^n(\Omega) \subset \Phi^{n-1}(\Omega) \cdots \subset \Phi^1(\Omega)$ where $\Phi^1(\Omega)$ coincides with the set of all proper classical subharmonic functions in $\Omega$, and $\Phi^n(\Omega)$ is the set of functions convex on each component of $\Omega$.

The following weak convergence result proved in [TW2] is fundamental to potential theory associated with $k$-Hessian operators.

Theorem 7.1 ([TW2]). *For each $u \in \Phi^k(\Omega)$, there exists a nonnegative Borel measure $\mu_k[u]$ in $\Omega$ such that*

(i) *$\mu_k[u] = F_k[u]$ for $u \in C^2(\Omega)$, and*

(ii) *if $\{u_m\}$ is a sequence in $\Phi^k(\Omega)$ converging in $L^1_{\mathrm{loc}}(\Omega)$ to a function $u \in \Phi^k(\Omega)$, then the sequence of the corresponding measures $\{\mu_k[u_m]\}$ converges weakly to $\mu_k[u]$.*

The measure $\mu_k[u]$ in the theorem above is called the $k$-Hessian measure associated with $u$. Due to (i) in Theorem 7.1 we sometimes write $F_k[u]$ in place of $\mu_k[u]$ even in the case where $u \in \Phi^k(\Omega)$ does not belong to $C^2(\Omega)$. The $k$-Hessian measure is an important tool in potential theory for $\Phi^k(\Omega)$. It was used by D. A. Labutin to derive pointwise estimates for functions in $\Phi^k(\Omega)$ in terms of Wolff's potential, which is an analogue of Wolff's potential estimates for $\mathcal{A}$-superharmonic functions considered in Theorem 4.4.

Theorem 7.2 ([L]). *Let $u \ge 0$ be such that $-u \in \Phi^k(B_{3r}(x))$, where $1 \le k \le n$. If $\mu = \mu_k[-u]$ then*

$$C_1 \mathbf{W}^{\frac{r}{8}}_{\frac{2k}{k+1}, k+1}\mu(x) \le u(x) \le C_2 \inf_{B(x,r)} u + C_3 \mathbf{W}^{2r}_{\frac{2k}{k+1}, k+1}\mu(x),$$

*where the constants $C_1$, $C_2$ and $C_3$ depend only on $n$ and $k$.*

The following global estimate is deduced from the preceding theorem as in the quasilinear case.

Corollary 7.3. *Let* $u \geq 0$ *be such that* $-u \in \Phi^k(\mathbb{R}^n)$, *where* $1 \leq k < \frac{n}{2}$. *If* $\mu = \mu_k[-u]$ *and* $\inf_{\mathbb{R}^n} u = 0$ *then for all* $x \in \mathbb{R}^n$,

$$\frac{1}{K}\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}\mu(x) \leq u(x) \leq K\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}\mu(x),$$

*for a constant* $K$ *depending only on* $n$ *and* $k$.

Let $\Omega$ be a bounded uniformly $(k-1)$-convex domain in $\mathbb{R}^n$, that is, $\partial\Omega \in C^2$ and $H_j(\partial\Omega) > 0$, $j = 1, ..., k-1$, where $H_j(\partial\Omega)$ denotes the $j$-mean curvature of the boundary $\partial\Omega$. We consider the following fully nonlinear problem:

$$\left\{\begin{array}{lll} F_k[-u] = u^q + \omega & \text{in} & \Omega, \\ u \geq 0 & \text{in} & \Omega, \\ u = \varphi & \text{on} & \partial\Omega \end{array}\right. \tag{7.1}$$

in the class of functions $u$ such that $-u$ is $k$-subharmonic in $\Omega$. Here $\omega$ is a nonnegative finite Borel measure which is regular enough near $\partial\Omega$ so that the boundary condition in (7.1) can be understood in the classical sense (see [TW1], [TW2]). Characterizations of the existence of $u \geq 0$, $-u \in \Phi^k(\Omega)$, continuous near $\partial\Omega$, which solves (7.1), can be obtained using the iteration scheme employed in the proof of Theorem 6.11 along with weak continuity of Hessian measures and testing inequalities analogous to those used in the proof of Theorem 2.10. For this purpose, we first prove an extension of Theorem 2.2 concerning the global potential estimates on a bounded domain, which is an analogue of Theorem 2.1 established for quasilinear operators.

Theorem 7.4. *Suppose that* $\varphi \geq 0$, $\varphi \in C^0(\partial\Omega)$ *and* $\nu = \mu + f$ *where* $\mu \in \mathcal{M}^+(\Omega)$ *has compact support in* $\Omega$ *and* $f \geq 0$, $f \in L^s(\Omega)$ *with* $s > \frac{n}{2k}$ *if* $1 \leq k \leq \frac{n}{2}$, *and* $s = 1$ *if* $\frac{n}{2} < k \leq n$. *Let* $u \geq 0$, $-u \in \Phi^k(\Omega)$ *be such that* $u$ *is continuous near* $\partial\Omega$ *and solves*

$$\left\{\begin{array}{lll} \mu_k[-u] = \nu & \text{in} & \Omega, \\ u = \varphi & \text{on} & \partial\Omega. \end{array}\right.$$

*Then for all* $x \in \Omega$,

$$u(x) \leq K\left[\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\nu(x) + \max_{\partial\Omega}\varphi\right],$$

*where* $R = \operatorname{diam}(\Omega)$ *and* $K$ *is a constant depending only on* $n$ *and* $k$.

*Proof.* Suppose that the support of $\mu$ is contained in $\Omega'$ for some open set $\Omega' \Subset \Omega$. Let $M = \sup_{\overline{\Omega}\setminus\Omega'} u$ and $u_m = \min\{u, m\}$ for $m > M$. Then $-u_m \in \Phi^k(\Omega)$, continuous near $\partial\Omega$, solves

$$\left\{\begin{array}{lll} \mu_k[-u_m] = \nu_m & \text{in} & \Omega, \\ u_m = \varphi & \text{on} & \partial\Omega \end{array}\right.$$

for certain nonnegative Borel measures $\nu_m$ in $\Omega$. Since $u_m \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, by Theorem 7.1 we have

$$\mu_m \to \nu = \mu + f \text{ weakly as measures in } \Omega. \tag{7.2}$$

Note that $u_m = u$ in $\overline{\Omega} \setminus \Omega'$ since $m > M$. Thus $\nu_m = \mu_k[u] = f$ in $\Omega \setminus \overline{\Omega'}$ for all $m > M$. Using this and (7.2) it is easy to see that

$$\int_\Omega \phi d\mu_m \to \int_\Omega \phi d\mu + \int_\Omega \phi f dx$$

as $m \to \infty$ for all $\phi \in C^0(\overline{\Omega})$, i.e.,

$$\mu_m \to \nu = \mu + f \text{ in the narrow topology of measures.}$$

We now take a ball $B = B_{2R}(a)$ where $a \in \Omega$ so that $\Omega \subset B$, and consider solutions $w_m \geq 0$, $-w_m \in \Phi^k(B)$, continuous near $\partial B$, of

$$\begin{cases} \mu_k[-w_m] = \nu_m & \text{in} \quad B, \\ w_m = \max_{\partial\Omega} \varphi & \text{on} \quad \partial B, \end{cases}$$

where $m > M$. Since $u_m$ is bounded in $\Omega$ the measure $\nu_m$ is absolutely continuous with respect to the capacity $\mathrm{cap}_k(\cdot, \Omega)$, and hence with respect to the capacity $\mathrm{cap}_k(\cdot, B)$ (see [TW3]). Here $\mathrm{cap}_k(\cdot, \Omega)$ is the $k$-Hessian capacity defined by (2.27). By a comparison principle (see [TW3, Th. 4.1]), we have $w_m \geq \max_{\partial\Omega} \varphi$ in $B$, and hence $w_m \geq u_m$ on $\partial\Omega$. Thus, applying the comparison principle again, we have

$$w_m \geq u_m \quad \text{in} \quad \Omega. \tag{7.3}$$

Since $\nu_m \to \nu$ in the narrow topology of measures in $\Omega$, we see that $\nu_m \to \nu$ weakly as measures in $B$. Therefore, arguing as in [TW2, §6] we can find a subsequence $\{w_{m_j}\}$ such that $w_{m_j} \to w$ a.e. for some $w \geq 0$, $-w \in \Phi^k(B)$, such that $w$ is continuous near $\partial B$ and

$$\begin{cases} \mu_k[-w] = \nu & \text{in} \quad B, \\ w = \max_{\partial\Omega} \varphi & \text{on} \quad \partial B. \end{cases}$$

Note that from (7.3), $w \geq u$ a.e. on $\Omega$ and hence $w \geq u$ everywhere on $\Omega$. Using this and Theorem 7.2 applied to the function $w$ on $B_{d(x)}(x)$, where $d(x) = \mathrm{dist}(x, \partial B)$ we have, for $x \in \Omega$,

$$\begin{aligned} u(x) &\leq C\, \mathbf{W}^{2R}_{\frac{2k}{k+1}, k+1} \nu(x) + C \inf_{B_{\frac{1}{3}d(x)}(x)} w \\ &\leq C\, \mathbf{W}^{2R}_{\frac{2k}{k+1}, k+1} \nu(x) + C\, d(x)^{-n} \int_{B_{\frac{1}{3}d(x)}(x)} w\, dy \\ &\leq C \Big( \mathbf{W}^{2R}_{\frac{2k}{k+1}, k+1} \nu(x) + \max_{\partial\Omega} \varphi + R^{2-\frac{n}{k}} \nu(\Omega)^{\frac{1}{k}} \Big), \end{aligned} \tag{7.4}$$

where the last inequality in (7.4) follows from the estimate (6.3) in [TW2]. The proof of Theorem 7.4 is then completed by noting that

$$\int_R^{2R} \Big[\frac{\nu(B_t(x))}{t^{n-2k}}\Big]^{\frac{1}{k}} \frac{dt}{t} \geq C R^{2-\frac{n}{k}} \nu(\Omega)^{\frac{1}{k}}$$

for all $x \in \Omega$. □

The following lemma is an analogue of Lemma 6.9. It is needed in the proof of Theorem 7.6 below to construct a solution to Hessian equations.

Lemma 7.5. *Let $\Omega$, $\nu$, $\varphi$ and $u$ be as in Theorem 7.4. Let $\nu'$ be a measure which belongs to the same class as $\nu$, i.e., $\nu' = \mu' + f'$, where $\mu' \in \mathcal{M}^+(\Omega)$ has compact support in $\Omega$ and $f' \geq 0$, $f' \in L^s(\Omega)$ with $s > \frac{n}{2k}$ if $1 \leq k \leq \frac{n}{2}$, and $s = 1$ if $\frac{n}{2} < k \leq n$. Then there exists $w \geq u$ such that $-w \in \Phi^k(\Omega)$ and*

$$\begin{cases} \mu_k[-w] = \nu + \nu' & \text{in} \quad \Omega, \\ w = \varphi & \text{on} \quad \partial\Omega. \end{cases}$$

*Proof.* By approximation we may assume that $\mu'$ is absolutely continuous with respect to the capacity $\mathrm{cap}_k(\cdot, \Omega)$. Let $u_m$ and $\nu_m$ be as in the proof of Theorem 7.4. Then by the comparison principle in [TW3, Th. 4.1], we have $u_m \leq w_m$ where $w_m$ is the solution of

$$\begin{cases} \mu_k[-w_m] = \nu_m + \nu' & \text{in} \quad \Omega, \\ w_m = \varphi & \text{on} \quad \partial\Omega. \end{cases}$$

Thus arguing as in [TW2, §6] we obtain a subsequence $\{w_{m_j}\}$ that converges a.e. to a required function $w$. □

From Lemma 7.5 and Theorem 2.2, along with the weak continuity of Hessian measures (Theorem 7.1), we deduce the following existence theorem for fully nonlinear equations whose proof, which we will omit, is analogous to that of Theorem 5.3 in the quasilinear case.

Theorem 7.6. *Let $\omega \in \mathcal{M}^+(\mathbb{R}^n)$, $1 \leq k < \frac{n}{2}$, and $q > k$. Suppose that*

$$\mathbf{W}_{\frac{2k}{k+1}, k+1}(\mathbf{W}_{\frac{2k}{k+1}, k+1}\omega)^q \leq C\, \mathbf{W}_{\frac{2k}{k+1}, k+1}\omega < \infty \quad \text{a.e.},$$

*where*

$$C \leq \Big(\frac{q-k}{qK}\Big)^{q/k} \frac{k}{q-k},$$

*and $K$ is the constant in Theorem 2.2. Then there exists $u \geq 0$, $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, such that $-u \in \Phi^k(\mathbb{R}^n)$ and*

$$\begin{cases} \inf_{x\in\mathbb{R}^n} u(x) = 0, \\ F_k[-u] = u^q + \omega. \end{cases}$$

*Moreover, $u$ satisfies the two-sided estimate*

$$c_1\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega(x) \le u(x) \le c_2\,\mathbf{W}_{\frac{2k}{k+1},\,k+1}\omega(x)$$

*for all $x$ in $\mathbb{R}^n$, where the constants $c_1, c_2$ depend only on $n, k, q$.*

We are now in a position to establish the main results of this section.

*Proof of Theorem* 2.6. The proof of Theorem 2.6 is completely analogous to that of Theorem 2.3 in the quasilinear case using $\mathbf{W}_{\frac{2k}{k+1},\,k+1}$ in place of $\mathbf{W}_{1,\,p}$ and Theorem 7.6 in place of Theorem 5.3. □

The proof of our next theorem on the existence of solutions for Hessian equations with nonhomogeneous boundary condition is similar to that of Theorem 6.11. However, due to the inhomogeneity we will need to take care of the boundary term. Moreover, the weak continuity of Hessian measures is used in place of the stability result for renormalized solutions in the quasilinear case.

THEOREM 7.7. *Let $\Omega$ be a bounded uniformly $(k-1)$-convex domain in $\mathbb{R}^n$. Suppose that $\omega \in \mathcal{M}_B^+(\Omega)$ such that $\omega = \mu + f$, where $\mu \in \mathcal{M}^+(\Omega)$ has compact support in $\Omega$, and $0 \le f \in L^s(\Omega)$ with $s > \frac{n}{2k}$ if $1 \le k \le \frac{n}{2}$ and $s = 1$ if $\frac{n}{2} < k \le n$. Let $q > k$, $R = \mathrm{diam}(\Omega)$ and $0 \le \varphi \in C^0(\partial\Omega)$. Suppose that*

$$\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\,(\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega)^q \le A\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega, \tag{7.5}$$

*and*

$$(\max_{\partial\Omega}\varphi)^{\frac{q}{k}-1} \le \frac{B^{\frac{k}{q}}}{2R^2|B_1(0)|^{\frac{1}{k}}}, \tag{7.6}$$

*where $A$, $B$ are positive constants such that*

$$A \le \Big(\frac{q-k}{3^{\frac{q-1}{q}}qK}\Big)^{\frac{q}{k}}\Big(\frac{k}{q-k}\Big), \quad \text{and} \quad B \le \Big(\frac{q-k}{3^{\frac{q-1}{q}}qK^{\frac{q}{k}}}\Big)^{\frac{q}{k}}\Big(\frac{k}{q-k}\Big). \tag{7.7}$$

*Here $K$ is the constant in Theorem 7.4. Then there exists a function $u \ge 0$, $-u \in \Phi^k(\Omega) \cap L^q(\Omega)$, continuous near $\partial\Omega$ such that*

$$\begin{cases} F_k[-u] = u^q + \omega \quad \text{in} \quad \Omega, \\ \qquad u = \varphi \quad \text{on} \quad \partial\Omega. \end{cases} \tag{7.8}$$

*Moreover, there is a constant $C = C(n,k,q)$ such that*

$$u \le C\Big\{\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega + \mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q + \max_{\partial\Omega}\varphi\Big\}.$$

*Proof.* First observe by direct calculations that condition (7.6) is equivalent to

$$\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\Big[\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q\Big]^q \le B\,\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q. \tag{7.9}$$

From Lemma 7.5 it follows that we can choose inductively a nondecreasing sequence $\{u_m\}$ of nonnegative functions on $\Omega$ such that

$$\begin{cases} F_k[-u_0] = \omega \quad \text{in} \quad \Omega, \\ \quad u_0 = \varphi \quad \text{on} \quad \partial\Omega, \end{cases}$$

and

$$\begin{cases} F_k[-u_m] = u_{m-1}^q + \omega \quad \text{in} \quad \Omega, \\ \qquad u_m = \varphi \quad \text{on} \quad \partial\Omega \end{cases} \tag{7.10}$$

for $m \geq 1$. Here for each $m \geq 0$, $-u_m$ is $k$-subharmonic and is continuous near $\partial\Omega$. By Theorem 7.4 we have

$$\begin{aligned} u_0 &\leq K\, \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}\omega + K \max_{\partial\Omega} \varphi \\ &= a_0 \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}\omega + b_0 \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}(\max_{\partial\Omega}\varphi)^q + K \max_{\partial\Omega}\varphi, \end{aligned}$$

where $a_0 = K$ and $b_0 = 0$. Thus

$$\begin{aligned} u_1 &\leq K\, \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}(u_0^q + \omega) + K \max_{\partial\Omega}\varphi \\ &\leq K \Big\{ (3^{q-1}a_0^q)^{\frac{1}{k}} \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1} (\mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}\omega)^q \\ &\quad + (3^{q-1}b_0^q)^{\frac{1}{k}} \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1} \Big[\mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}(\max_{\partial\Omega}\varphi)^q\Big]^q \\ &\quad + K^{\frac{q}{k}} \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}(\max_{\partial\Omega}\varphi)^q + \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}\omega \Big\} + K \max_{\partial\Omega}\varphi. \end{aligned}$$

Then by (7.5) and (7.9),

$$\begin{aligned} u_1 &\leq K[(3^{q-1}a_0^q)^{\frac{1}{k}} A + 1] \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}\omega \\ &\quad + K[(3^{q-1}b_0^q)^{\frac{1}{k}} B + K^{\frac{q}{k}}] \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}(\max_{\partial\Omega}\varphi)^q + K \max_{\partial\Omega}\varphi \\ &= a_1 \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}\omega + b_1 \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}(\max_{\partial\Omega}\varphi)^q + K \max_{\partial\Omega}\varphi, \end{aligned}$$

where

$$a_1 = K[(3^{q-1}a_0^q)^{\frac{1}{k}} A + 1], \qquad b_1 = K[(3^{q-1}b_0^q)^{\frac{1}{k}} B + K^{\frac{q}{k}}].$$

By induction we have

$$u_m \leq a_m \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}\omega + b_m \mathbf{W}^{2R}_{\frac{2k}{k+1},\, k+1}(\max_{\partial\Omega}\varphi)^q + K \max_{\partial\Omega}\varphi,$$

where

$$a_{m+1} = K[(3^{q-1}a_m^q)^{\frac{1}{k}} A + 1], \qquad b_{m+1} = K[(3^{q-1}b_m^q)^{\frac{1}{k}} B + K^{\frac{q}{k}}]$$

for all $m \geq 0$. It is then easy to see that

$$a_m \leq \frac{Kq}{q-k}, \quad \text{and} \quad b_m \leq \frac{K^{\frac{q}{k}+1} q}{q-k},$$

provided (7.7) is satisfied. Thus

$$u_m \le \frac{Kq}{q-k}\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}\omega + \frac{K^{\frac{q}{k}+1}q}{q-k}\mathbf{W}^{2R}_{\frac{2k}{k+1},\,k+1}(\max_{\partial\Omega}\varphi)^q + K\max_{\partial\Omega}\varphi. \tag{7.11}$$

Using (7.5) and (7.11) we see that $u_m \uparrow u$ for a function $u \ge 0$ such that $-u$ is $k$-subharmonic and $u_m^q \to u^q$ in $L^1(\Omega)$. Thus in view of (7.10) and Theorem 7.1 we see that $u$ is a desired solution of (7.8). □

We will omit the proof of the next theorem, which contains Theorem 2.17 in Section 2, as it is completely analogous to the proof of Theorem 6.12 in the quasilinear case.

THEOREM 7.8. *Let $\omega$ be a locally finite nonnegative measure on an open (not necessarily bounded) set $\Omega$. Let $1 \le k \le n$ and $q > k$. Suppose that $u \ge 0$, $-u \in \Phi^k(\Omega)$, such that $u$ is a solution to*

$$F_k[-u] = u^q + \omega \qquad \text{in} \quad \Omega.$$

*Then for each cube $P \in \mathcal{Q}$, where $\mathcal{Q} = \{Q\}$ is a Whitney decomposition of $\Omega$ (see §6),*

$$\mu_P(E) \le C\,\mathrm{Cap}_{\mathbf{I}_{2k},\,\frac{q}{q-k}}(E) \tag{7.12}$$

*if $\frac{2kq}{q-k} < n$, and*

$$\mu_P(E) \le C(P)\,\mathrm{Cap}_{\mathbf{G}_{2k},\,\frac{q}{q-k}}(E) \tag{7.13}$$

*if $\frac{2kq}{q-k} \ge n$, for all compact sets $E \subset \Omega$. Here $d\mu = u^q dx + d\omega$, and the constant $C$ in (7.12) does not depend on $P \in \mathcal{Q}$ and $E \subset \Omega$; however, the constant $C(P)$ in (7.13) may depend on the side length of $P$.*

*Moreover, if $\frac{2kq}{q-k} < n$, and $\Omega$ is a bounded $C^\infty$-domain then*

$$\mu(E) \le C\,\mathrm{cap}_{2k,\,\frac{q}{q-k}}(E,\Omega)$$

*for all compact sets $E \subset \Omega$, where $\mathrm{cap}_{2k,\,\frac{q}{q-k}}(E,\Omega)$ is defined by (2.24).*

*Remark* 7.9. Let $B_R$ be a ball such that $B_{2R} \subset \Omega$. Then in the critical case $q = \frac{nk}{n-2k}$, $(k < \frac{n}{2})$, as in Corollary 6.13 we have

$$\mu(B_r) \le C(\log\tfrac{2R}{r})^{\frac{-k}{q-k}}$$

for all balls $B_r \subset B_R$.

We are now in a position to deduce Theorem 2.13 concerning the characterizations of solvability for Hessian equations in a bounded domain.

*Proof of Theorem* 2.13. The proof of this theorem is analogous to that of Theorem 2.6 in the quasilinear case. One only has to use Theorems 7.7 and 7.8 in place of Theorems 6.11 and 6.12 respectively. □

We next prove Theorem 2.19 on removable singularities for Hessian equations.

*Proof of Theorem* 2.19. To prove this theorem, we will proceed as in the proof of Theorem 2.18. For the "only if" part, we may assume that $k < \frac{n}{2}$, since otherwise $\frac{2kq}{q-k} > n$, and so $E = \emptyset$. Note that if $\mathrm{Cap}_{\mathbf{G}_{2k},\,\frac{q}{q-k}}(E) = 0$ then $\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},\,k+1}(E) = 0$ (see [AH, §5.5]), which implies that $\mathrm{cap}_k(E, \Omega) = 0$ due to Theorem 2.20, whose proof is given below. Here $\mathrm{cap}_k(\cdot, \Omega)$ is the (relative) $k$-Hessian capacity associated with the domain $\Omega$ defined by (2.27). Thus by [L, Th. 4.2], $E$ is a $k$-polar set, i.e., it is contained in the $(-\infty)$-set of a $k$-subharmonic function in $\mathbb{R}^n$. Suppose that $u$ is a solution of (2.26). It is easy to see that the function $\tilde{u}$ defined by

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega \setminus E, \\ \liminf\limits_{y \to x,\, y \notin E} u(y), & x \in E \end{cases} \tag{7.14}$$

is an extension of $u$ to $\Omega$ such that $-\tilde{u} \in \Phi^k(\Omega)$. The rest of the proof is then the same as in the quasilinear case. □

Finally, we prove Theorem 2.20 on the local equivalence of the $k$-Hessian capacity and an appropriate Bessel capacity.

*Proof of Theorem* 2.20. Let $R$ be the diameter of $\Omega$. From Wolff's inequality (5.3) it follows that $\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},\,k+1}(E)$ is equivalent to

$$\sup \{\mu(E) : \ \mu \in \mathcal{M}^+(E), \quad \mathbf{W}^{4R}_{\frac{2k}{k+1},\,k+1}\mu \le 1 \text{ on } \operatorname{supp} \mu\}$$

for any compact set $E \subset \Omega$ (see [HW, Prop. 5]). To prove the left-hand inequality in (2.28), let $\mu \in \mathcal{M}^+(E)$ such that $\mathbf{W}^{4R}_{\frac{2k}{k+1},\,k+1}\mu \le 1$ on $\operatorname{supp} \mu$, and let $u \in \Phi^k(B)$ be a nonpositive solution of

$$\begin{cases} F_k[u] = \mu & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B, \end{cases}$$

where $B$ is a ball of radius $R$ containing $\Omega$. By Theorem 7.4 and the boundedness principle for nonlinear potentials (see [AH, §2.6]), we have

$$|u(x)| \le C\, \mathbf{W}^{4R}_{\frac{2k}{k+1},\,k+1}\mu(x) \le C, \qquad x \in B.$$

Thus

$$\mu(E) = \mu_k[u](E) \le C\, \mathrm{cap}_k(E, \Omega),$$

which shows that

$$\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},\,k+1}(E) \le C\,\mathrm{cap}_k(E,\Omega).$$

To prove the upper estimate in (2.28), we let $Q \in \mathcal{Q}$, and fix a compact set $E \subset \overline{Q}$. Note that for $\mu \in \mathcal{M}^+(E)$ and $x \in E$,

$$\mathbf{W}^{4R}_{\frac{2k}{k+1},\,k+1}\mu(x) = \mathbf{W}^{2\mathrm{diam}(Q)}_{\frac{2k}{k+1},\,k+1}\mu(x) + \int_{2\mathrm{diam}(Q)}^{4R} \left[\frac{\mu(E)}{t^{n-2k}}\right]^{\frac{1}{k}} \frac{dt}{t}.$$

Thus, for $k < \frac{n}{2}$,

$$\mathbf{W}^{4R}_{\frac{2k}{k+1},\,k+1}\mu(x) \le C\,\mathbf{W}^{2\mathrm{diam}(Q)}_{\frac{2k}{k+1},\,k+1}\mu(x), \qquad x \in E. \tag{7.15}$$

Now for $u \in \Phi^k(\Omega)$ such that $-1 < u < 0$ by Theorem 7.2 we obtain

$$\mathbf{W}^{2\mathrm{diam}(Q)}_{\frac{2k}{k+1},\,k+1}\mu_E(x) \le \mathbf{W}^{2\mathrm{diam}(Q)}_{\frac{2k}{k+1},\,k+1}\mu(x) \le C\,|u(x)| \le C$$

for all $x \in E$, where $\mu = \mu_k[u]$. Thus, we deduce from (7.15) that

$$\mathbf{W}^{4R}_{\frac{2k}{k+1},\,k+1}\mu_E(x) \le C, \qquad x \in E,$$

which implies

$$\mathrm{cap}_k(E,\Omega) \le C\mathrm{Cap}_{\mathbf{G}_{\frac{2k}{k+1}},\,k+1}(E). \tag{7.16}$$

Finally, if $\Omega$ is a $C^\infty$-domain in $\mathbb{R}^n$, and $1 \le k < \frac{n}{2}$, then by (7.16) and the quasi-additivity of the capacity $\mathrm{cap}_{\frac{2k}{k+1},\,k+1}(\cdot,\Omega)$ (see Theorem 6.14) we obtain the global upper estimate (2.29) for the $k$-Hessian capacity. $\square$

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