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Existence of conformal metrics with constant Q-curvature

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Abstract

Given a compact four dimensional manifold, we prove existence of conformal metrics with constant Q-curvature under generic assumptions. The problem amounts to solving a fourth-order nonlinear elliptic equation with variational structure. Since the corresponding Euler functional is in general unbounded from above and from below, we employ topological methods and min-max schemes, jointly with the compactness result of [35].

1. Introduction

In recent years, much attention has been devoted to the study of partial differential equations on manifolds, in order to understand some connections between analytic and geometric properties of these objects.

A basic example is the Laplace-Beltrami operator on a compact surface (Σ, g). Under the conformal change of metric $\tilde{g} = e^{2w} g$, we have

$$\Delta_{\tilde{g}} = e^{-2w} \Delta_g; \qquad -\Delta_g w + K_g = K_{\tilde{g}} e^{2w}, \tag{1}$$

where Δ_g and K_g (resp. $\Delta_{\tilde{g}}$ and $K_{\tilde{g}}$) are the Laplace-Beltrami operator and the Gauss curvature of (Σ, g) (resp. of $(\Sigma, \tilde{g})$). From the above equations one recovers in particular the conformal invariance of $\int_\Sigma K_g dV_g$, which is related to the topology of Σ through the Gauss-Bonnet formula

$$\int_\Sigma K_g dV_g = 2\pi\chi(\Sigma), \tag{2}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. Of particular interest is the classical *Uniformization Theorem*, which asserts that every compact surface carries a (conformal) metric with constant curvature.

On four-dimensional manifolds there exists a conformally covariant operator, the Paneitz operator, which enjoys analogous properties to the Laplace-Beltrami operator on surfaces, and to which is associated a natural concept of curvature. This operator, introduced by Paneitz, [38], [39], and the corresponding Q-curvature, introduced in [6], are defined in terms of the Ricci tensor Ric_g and the scalar curvature R_g of the manifold (M, g) as

$$P_g(\varphi) = \Delta_g^2 \varphi + \mathrm{div}_g \left(\frac{2}{3} R_g g - 2\mathrm{Ric}_g \right) d\varphi; \tag{3}$$

$$Q_g = -\frac{1}{12} \left(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2 \right), \tag{4}$$

where φ is any smooth function on M. The behavior (and the mutual relation) of P_g and Q_g under a conformal change of metric $\tilde{g} = e^{2w} g$ is given by

$$P_{\tilde{g}} = e^{-4w} P_g; \qquad P_g w + 2Q_g = 2Q_{\tilde{g}} e^{4w}. \tag{5}$$

Apart from the analogy with (1), we have an extension of the Gauss-Bonnet formula which is the following:

$$\int_M \left(Q_g + \frac{|W_g|^2}{8} \right) dV_g = 4\pi^2 \chi(M), \tag{6}$$

where W_g denotes the Weyl tensor of (M, g) and $\chi(M)$ the Euler characteristic. In particular, since $|W_g|^2 \, dV_g$ is a pointwise conformal invariant, it follows that the integral of Q_g over M is also a conformal invariant, which is usually denoted with the symbol

$$k_P = \int_M Q_g dV_g. \tag{7}$$

We refer for example to the survey [18] for more details.

To mention some first geometric properties of P_g and Q_g, we discuss some results of Gursky, [29] (see also [28]). If a manifold of nonnegative Yamabe class $Y(g)$ (this means that there is a conformal metric with nonnegative constant scalar curvature) satisfies $k_P \geq 0$, then the kernel of P_g are only the constants and $P_g \geq 0$, namely P_g is a nonnegative operator. If in addition $Y(g) > 0$, then the first Betti number of M vanishes, unless (M, g) is conformally equivalent to a quotient of $S^3 \times \mathbb{R}$. On the other hand, if $Y(g) \geq 0$, then $k_P \leq 8\pi^2$, with the equality holding if and only if (M, g) is conformally equivalent to the standard sphere.

As for the Uniformization Theorem, one can ask whether every four-manifold (M, g) carries a conformal metric $\tilde{g}$ for which the corresponding Q-curvature $Q_{\tilde{g}}$ is a constant. When $\tilde{g} = e^{2w} g$, by (5) the problem amounts to finding a solution of the equation

$$P_g w + 2Q_g = 2\overline{Q} e^{4w}, \tag{8}$$

where $\overline{Q}$ is a real constant. By the regularity results in [43], critical points of the following functional

$$II(u) = \langle P_g u, u \rangle + 4 \int_M Q_g u dV_g - k_P \log \int_M e^{4u} dV_g; \qquad u \in H^2(M), \tag{9}$$

which are weak solutions of (8), are also strong solutions. Here $H^2(M)$ is the space of functions on M which are of class L^2, together with their first and second derivatives, and the symbol $\langle P_g u, v\rangle$ stands for

$$(10) \quad \langle P_g u, v\rangle = \int_M \left(\Delta_g u \Delta_g v + \frac{2}{3} R_g \nabla_g u \cdot \nabla_g v - 2(\mathrm{Ric}_g \nabla_g u, \nabla_g v) \right) dV_g \qquad \text{for } u, v \in H^2(M).$$

Problem (8) has been solved in [16] for the case in which $\ker P_g = \mathbb{R}$, P_g is a nonnegative operator and $k_P < 8\pi^2$. By the above-mentioned result of Gursky, sufficient conditions for these assumptions to hold are that $Y(g) \geq 0$ and that $k_P \geq 0$ (and (M, g) is not conformal to the standard sphere). Notice that if $Y(g) \geq 0$ and $k_P = 8\pi^2$, then (M, g) is conformally equivalent to the standard sphere and clearly in this situation (8) admits a solution. More general conditions for the above hypotheses to hold have been obtained by Gursky and Viaclovsky in [30]. Under the assumptions in [16], by the Adams inequality

$$\log \int_M e^{4(u - \overline{u})} dV_g \leq \frac{1}{8\pi^2} \langle P_g u, u\rangle + C, \qquad u \in H^2(M),$$

where $\overline{u}$ is the average of u and where C depends only on M, the functional II is bounded from below and coercive, hence solutions can be found as global minima. The result in [16] has also been extended in [10] to higher-dimensional manifolds (regarding higher-order operators and curvatures) using a geometric flow.

The solvability of (8), under the above hypotheses, has been useful in the study of some conformally invariant fully nonlinear equations, as is shown in [13]. Some remarkable geometric consequences of this study, given in [12], [13], are the following. If a manifold of positive Yamabe class satisfies $\int_M Q_g dV_g > 0$, then there exists a conformal metric with positive Ricci tensor, and hence M has finite fundamental group. Furthermore, under the additional quantitative assumption $\int_M Q_g dV_g > \frac{1}{8} \int_M |W_g|^2 dV_g$, M must be diffeomorphic to the standard four-sphere or to the standard projective space. Finally, we also point out that the Paneitz operator and the Q-curvature (together with their higher-dimensional analogues, see [5], [6], [26], [27]) appear in the study of Moser-Trudinger type inequalities, log-determinant formulas and the compactification of locally conformally flat manifolds, [7], [8], [14], [15], [16].

We are interested here in extending the *uniformization* result in [16], namely to find solutions of (8) under more general assumptions. Our result is the following.

THEOREM 1.1. *Suppose* $\ker P_g = \{constants\}$, *and assume that* $k_P \neq 8k\pi^2$ *for* $k = 1, 2, \ldots$. *Then* (M, g) *admits a conformal metric with constant* Q*-curvature.*

Remark 1.2. (a). Our assumptions are conformally invariant and generic, so the result applies to a large class of four manifolds, and in particular to some manifolds of negative curvature or negative Yamabe class. Note that, in view of [29], it is not clear whether or not a manifold of negative Yamabe class satisfies the assumptions of the result in [16]. For example, products of two negatively-curved surfaces might have total Q-curvature greater than $8\pi^2$; see [24].

(b). Under the above, imposing the volume normalization $\int_M e^{4u} dV_g = 1$, the set of solutions (which is nonempty) is bounded in $C^m(M)$ for any integer m, by Theorem 1.3 in [35]; see also [25].

(c). Theorem 1.1 does NOT cover the case of locally conformally flat manifolds with positive and even Euler characteristic, by (6).

Our assumptions include those made in [16] and one (or both) of the following two possibilities

$$k_P \in (8k\pi^2, 8(k+1)\pi^2), \qquad \text{for some } k \in \mathbb{N}; \tag{11}$$

(12) P_g possesses $\overline{k}$ (counted with multiplicity) negative eigenvalues.

In these cases the functional II is unbounded from above and below, and hence it is necessary to find extremals which are possibly saddle points. This is done using a new min-max scheme, which we describe below, depending on k_P and the spectrum of P_g (in particular on the number of negative eigenvalues $\overline{k}$, counted with multiplicity). By classical arguments, the scheme yields a *Palais-Smale sequence*, namely a sequence $(u_l)_l \subseteq H^2(M)$ satisfying the following properties

$$II(u_l) \to c \in \mathbb{R}; \qquad II'(u_l) \to 0 \qquad\qquad \text{as } l \to +\infty. \tag{13}$$

We can also assume that such a sequence $(u_l)_l$ satisfies the volume normalization

$$\int_M e^{4u_l} dV_g = 1 \qquad\qquad \text{for all } l. \tag{14}$$

This is always possible since the functional II is invariant under the transformation $u \mapsto u+a$, where a is any real constant. Then, to achieve existence, one should prove for example that $(u_l)_l$ is bounded, or prove a similar compactness criterion.

In order to do this, we apply a procedure from [40], used in [22], [31], [42]. For ρ in a neighborhood of 1, we define the functional $II_\rho : H^2(M) \to \mathbb{R}$ by

$$II_\rho(u) = \langle P_g u, u \rangle + 4\rho \int_M Q_g dV_g - 4\rho k_P \log \int_M e^{4u} dV_g, \qquad u \in H^2(M),$$

whose critical points give rise to solutions of the equation

$$P_g u + 2\rho Q_g = 2\rho k_P e^{4u} \qquad \text{in } M. \tag{15}$$

One can then define the min-max scheme for different values of ρ and prove boundedness of some Palais-Smale sequence for ρ belonging to a set Λ which is dense in some neighborhood of 1; see Section 5. This implies solvability of (15) for $\rho \in \Lambda$. We then apply the following result from [35], with $Q_l = \rho_l Q_g$, where $(\rho_l)_l \subseteq \Lambda$ and $\rho_l \to 1$.

THEOREM 1.3 ([35]). *Suppose* $\ker P_g = \{constants\}$ *and that* $(u_l)_l$ *is a sequence of solutions of*

$$P_g u_l + 2Q_l = 2k_l e^{4u_l} \qquad \text{in } M, \tag{16}$$

satisfying (14), *where* $k_l = \int_M Q_l dV_g$, *and where* $Q_l \to Q_0$ *in* $C^0(M)$. *Assume also that*

$$k_0 := \int_M Q_0 dV_g \neq 8k\pi^2 \qquad \text{for } k = 1, 2, \dots. \tag{17}$$

Then $(u_l)_l$ *is bounded in* $C^\alpha(M)$ *for any* $\alpha \in (0,1)$.

We give now a brief description of the scheme and a heuristic idea of its construction. We describe it for the functional II only, but the same considerations hold for II_ρ if $|\rho - 1|$ is sufficiently small. It is a standard method in critical point theory to find extrema by looking at the difference of topology between sub- or superlevels of functionals. In our specific case we investigate the structure of the sublevels $\{II \leq -L\}$, where L is a large positive number. Looking at the form of the functional II, see (9), one can find two ways for attaining large negative values.

The first, assuming (11), is by bubbling. In fact, for a given point $x \in M$ and for $\lambda > 0$, consider the following function

$$\varphi_{\lambda,x}(y) = \log\left(\frac{2\lambda}{1 + \lambda^2 \mathrm{dist}(y,x)^2}\right),$$

where $\mathrm{dist}(\cdot,\cdot)$ denotes the metric distance on M. Then for λ large one has $e^{4\varphi_{\lambda,x}} \simeq \delta_x$ (the Dirac mass at x), where $e^{4\varphi_{\lambda,x}}$ represents the volume density of a four sphere attached to M at the point x, and one can show that $II(\varphi_{\lambda,x}) \to -\infty$ as $\lambda \to +\infty$. Similarly, for k as given in (11) and for $x_1, \dots, x_k \in M$, $t_1, \dots, t_k \geq 0$, it is possible to construct an appropriate function φ of the above form (near each x_i) with $e^{4\varphi} \simeq \sum_{i=1}^k t_i \delta_{x_i}$, and on which II still attains large negative values. Precise estimates are given in Section 4 and in the appendix. Since II stays invariant if $e^{4\varphi}$ is multiplied by a constant, we can assume that $\sum_{i=1}^k t_i = 1$. On the other hand, if $e^{4\varphi}$ is concentrated at $k+1$ distinct points of M, it is possible to prove, using an improved Moser-Trudinger inequality from Section 2, that $II(\varphi)$ cannot attain large negative values anymore, see Lemmas 2.2 and 2.4. From this argument we see that one is led naturally to consider the family M_k of elements $\sum_{i=1}^k t_i \delta_{x_i}$ with $(x_i)_i \subseteq M$, and $\sum_{i=1}^k t_i = 1$, known in literature as the *formal set of barycenters of M of order k*, which we are going to discuss in more detail below.

The second way to attain large negative values, assuming (12), is by considering the negative-definite part of the quadratic form $\langle P_g u, u\rangle$. When $V \subseteq H^2(M)$ denotes the direct sum of the eigenspaces of P_g corresponding to negative eigenvalues, the functional II will tend to $-\infty$ on the boundaries of large balls in V, namely boundaries of sets homeomorphic to the unit ball $B_1^{\overline{k}}$ of $\mathbb{R}^{\overline{k}}$.

Having these considerations in mind, we will use for the min-max scheme a set, denoted by $A_{k,\overline{k}}$, which is constructed using some contraction of the product $M_k \times B_1^{\overline{k}}$; see formula (21) and the figure in Section 2 (when $k_P < 8\pi^2$, we just take the sphere $S^{\overline{k}-1}$ instead of $A_{k,\overline{k}}$). It is possible indeed to map (nontrivially) this set into $H^2(M)$ in such a way that the functional II on the image is close to $-\infty$; see Section 4. On the other hand, it is also possible to do the opposite, namely to map appropriate sublevels of II into $A_{k,\overline{k}}$; see Section 3. The composition of these two maps turns out to be homotopic to the identity on $A_{k,\overline{k}}$ (which is noncontractible by Corollary 3.8) and therefore they are both topologically nontrivial.

Some comments are in order. For the case $k = 1$ and $\overline{k} = 0$, which is presented in [24], the min-max scheme is similar to that used in [22], where the authors study a mean field equation depending on a real parameter λ (and prove existence for $\lambda \in (8\pi, 16\pi)$). Solutions for large values of λ have been obtained recently by Chen and Lin, [19], [20], using blow-up analysis and degree theory. See also the papers [32], [34], [42] and references therein for related results. The construction presented in this paper has been recently used by Djadli in [23] to study this problem as well when $\lambda \neq 8k\pi$ and without any assumption on the topology of the surface. Our method has also been employed by Malchiodi and Ndiaye [36] for the study of the 2×2 Toda system.

The set of barycenters M_k (see subsection 3.1 for more comments or references) has been used crucially in literature for the study of problems with lack of compactness; see [3], [4]. In particular, for Yamabe-type equations (including the Yamabe equation and several other applications), it has been used to understand the structure of the *critical points at infinity* (or asymptotes) of the Euler functional, namely the way compactness is lost through a pseudo-gradient flow. Our use of the set M_k, although the map Φ of Section 4 presents some analogies with the Yamabe case, is of different type since it is employed to reach low energy levels and not to study critical points at infinity. As mentioned above, we consider a projection onto the k-barycenters M_k, but starting only from functions in $\{II \leq -L\}$, whose concentration behavior is not as clear as that of the asymptotes for the Yamabe equation. Here also a technical difficulty arises. The main point is that, while in the Yamabe case all the coefficients t_i are bounded away from zero, in our case they can be arbitrarily small, and hence it is not so clear what the choice of the points x_i and the numbers t_i should be when projecting. Indeed, when $k > 1$ M_k is not a smooth manifold but a *stratified set*, namely union of sets of different dimensions (the maximal one is $5k - 1$, when all the x_i's are distinct and all the t_i's are positive). To construct a continuous global projection takes further work, and this is done in Section 3.

The cases which are not included in Theorem 1.1 should be more delicate, especially when k_P is an integer multiple of $8\pi^2$. In this situation noncompactness is expected, and the problem should require an asymptotic analysis as in [3], or a fine blow-up analysis as in [32], [19], [20]. Some blow-up behavior on open flat domains of $\mathbb{R}^4$ is studied in [2].

A related question in this context arises for the standard sphere ($k_P = 8\pi^2$), where one could ask for the analogue of the *Nirenberg problem*. Precisely, since the Q-curvature of the standard metric is constant, a natural problem is to deform the metric conformally in such a way that the curvature becomes a given function f on S^4. Equation (8) on the sphere admits a noncompact family of solutions (classified in [17]), which all arise from conformal factors of Möbius transformations. In order to tackle this loss of compactness, usually finite-dimensional reductions of the problem are used, jointly with blow-up analysis and Morse theory. Some results in this direction are given in [11], [37] and [44] (see also references therein for results on the Nirenberg problem on S^2).

The structure of the paper is as follows. In Section 2 we collect some notation and preliminary results, based on an improved Moser-Trudinger type inequality. We also introduce the set $A_{k,\overline{k}}$ used to perform the min-max construction. In Section 3, we show how to map the sublevels $\{II \leq -L\}$ into $A_{k,\overline{k}}$. We begin by analyzing some properties of the k-barycenters as a stratified set, in order to understand the component of the projection involving the set M_k, which is the most delicate. Then we turn to the construction of the global map. In Section 4 we show how to embed $A_{k,\overline{k}}$ into the sublevel $\{II \leq -L\}$ for L arbitrarily large. This requires long and delicate estimates, some of which are carried out in the appendix (which also contains other technical proofs). Finally in Section 5 we prove Theorem 1.1, defining a min-max scheme based on the construction of $A_{k,\overline{k}}$, solving the modified problem (15), and applying Theorem 1.3.

An announcement of the present results is given in the preliminary note [24].

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2. Notation and preliminaries

In this section we fix our notation and recall some useful known facts. We state in particular an inequality of Moser-Trudinger type for functions in $H^2(M)$, an improved version of it and some of its consequences.

The symbol $B_r(p)$ denotes the metric ball of radius r and center p, while $\mathrm{dist}(x,y)$ stands for the distance between two points $x, y \in M$. $H^2(M)$ is the Sobolev space of the functions on M which are in $L^2(M)$ together with their first and second derivatives. The symbol $\|\cdot\|$ denotes the norm of $H^2(M)$. If $u \in H^2(M)$, $\overline{u} = \frac{1}{|M|}\int_M u dV_g$ stands for the average of u. For l points $x_1, \dots, x_l \in M$ which all lie in a small metric ball, and for l nonnegative numbers $\alpha_1, \dots, \alpha_l$, we consider *convex combinations* of the form $\sum_{i=1}^l \alpha_i x_i$, $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$. To do this, we consider the embedding of M into some $\mathbb{R}^n$ given by Whitney's theorem, take the convex combination of the images of the points $(x_i)_i$, and project it onto the image of M (which we identify with M itself). If $\mathrm{dist}(x_i, x_j) < \xi$ for ξ sufficiently small, $i, j = 1, \dots, l$, then this operation is well-defined and moreover we have $\mathrm{dist}\left(x_j, \sum_{i=1}^l \alpha_i x_i\right) < 2\xi$ for every $j = 1, \dots, l$. Note that these elements are just points, not to be confused with the formal barycenters $\sum t_i \delta_{x_i}$.

Large positive constants are always denoted by C, and the value of C is allowed to vary from formula to formula and also within the same line. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, as C_δ, etc.. Also constants with subscripts are allowed to vary.

Since we allow P_g to have negative eigenvalues, we denote by $V \subseteq H^2(M)$ the direct sum of the eigenspaces corresponding to negative eigenvalues of P_g. The dimension of V, which is finite, is denoted by $\overline{k}$, and since $\ker P_g = \mathbb{R}$, we can find a basis of eigenfunctions $\hat{v}_1, \dots, \hat{v}_{\overline{k}}$ of V (orthonormal in $L^2(M)$) with the properties

$$P_g \hat{v}_i = \lambda_i \hat{v}_i, \quad i = 1, \dots, \overline{k}; \qquad \int_M \hat{v}_i^2 dV_g = 1; \tag{18}$$
$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{\overline{k}} < 0 < \lambda_{\overline{k}+1} \leq \dots,$$

where the λ_i's are the eigenvalues of P_g counted with multiplicity. From (18), since P_g has a divergence structure, it follows immediately that $\int_M \hat{v}_i dV_g = 0$ for $i = 1, \dots, \overline{k}$. We also introduce the following positive-definite (on the space of functions orthogonal to the constants) pseudo-differential operator P_g^+

$$P_g^+ u = P_g u - 2 \sum_{i=1}^{\overline{k}} \lambda_i \left(\int_M u \hat{v}_i dV_g \right) \hat{v}_i. \tag{19}$$

Basically, we are reversing the sign of the negative eigenvalues of P_g.

Now we define the set $A_{k,\overline{k}}$ to be used in the existence argument, where k is as in (11), and where $\overline{k}$ is as in (18). We let M_k denote the family of formal sums

$$M_k = \sum_{i=1}^{k} t_i \delta_{x_i}; \qquad t_i \geq 0, \sum_{i=1}^{k} t_i = 1; \quad x_i \in M, \tag{20}$$

endowed with the weak topology of distributions. This is known in the literature as the *formal set of barycenters* of M (of order k); see [3], [4], [9]. We stress that this set is NOT the family of convex combinations of points in M which is introduced at the beginning of the section. To carry out some explicit computations, we will use on M_k the metric given by $C^1(M)^*$, which induces the same topology, and which will be denoted by $\mathrm{dist}(\cdot,\cdot)$.

Then, recalling that $\overline{k}$ is the number of negative eigenvalues of P_g, we consider the unit ball $B_1^{\overline{k}}$ in $\mathbb{R}^{\overline{k}}$, and we define the set

$$A_{k,\overline{k}} = \widetilde{M_k \times B_1^{\overline{k}}}, \tag{21}$$

where the notation $\sim$ means that $M_k \times \partial B_1^{\overline{k}}$ is identified with $\partial B_1^{\overline{k}}$; namely $M_k \times \{y\}$, for every fixed $y \in \partial B_1^{\overline{k}}$, is collapsed to a single point. In Figure 1 below we illustrate this collapsing drawing, for simplicity, M_k as a couple of points. When $k_P < 8\pi^2$ and $\overline{k} \geq 1$, we will perform the min-max argument just by using the sphere $S^{\overline{k}-1}$.

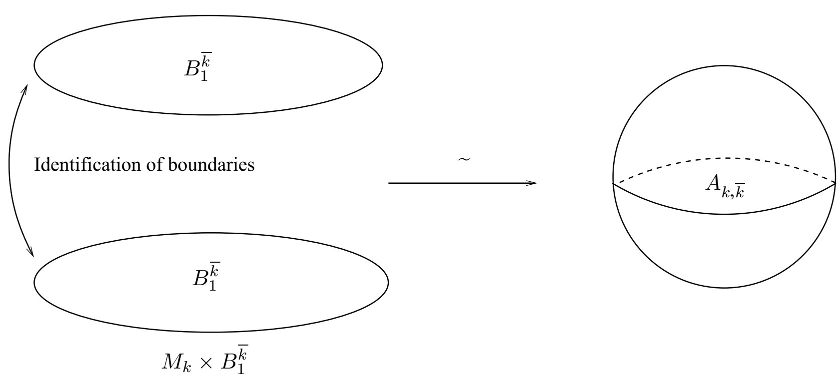

Figure 1. The equivalence relation $\sim$

2.1. *Some improved Adams inequalities.* In this subsection we give some improvements of the Adams inequality (see [1] and [16]) and in particular we consider the possibility of dealing with operators P_g possessing negative eigenvalues. The following lemma is proved in [35] using a modification of the arguments in [16], which in turn extend to the Paneitz operator some previous embeddings due to Adams involving the operator Δ^m in flat domains.

LEMMA 2.1 ([35]). *Suppose* $\ker P_g = \{constants\}$, *let* V *be the direct sum of the eigenspaces corresponding to negative eigenvalues of* P_g, *and let* P_g^+ *be defined as in* (19). *Then there exists a constant* C *such that for all* $u \in H^2(M)$

$$\int_M e^{\frac{32\pi^2 (u-\overline{u})^2}{\langle P_g^+ u, u\rangle}} dV_g \leq C. \tag{22}$$

As a consequence, for all $u \in H^2(M)$,

$$\log \int_M e^{4(u-\overline{u})} dV_g \leq C + \frac{1}{8\pi^2} \langle P_g^+ u, u\rangle. \tag{23}$$

From this result we derive an improved inequality for functions which are concentrated at more than a single point, related to a result in [21]. A consequence of this inequality is that it allows us to give an upper bound (depending on $\int_M Q_g dV_g$) for the number of concentration points of e^{4u}, where u is any function in $H^2(M)$ on which II attains large negative values; see Lemma 2.4.

LEMMA 2.2. *For a fixed integer* ℓ, *let* $\Omega_1, \dots, \Omega_{\ell+1}$ *be subsets of* M *satisfying* $\mathrm{dist}(\Omega_i, \Omega_j) \geq \delta_0$ *for* $i \neq j$, *where* δ_0 *is a positive real number*, *and let* $\gamma_0 \in \left(0, \frac{1}{\ell+1}\right)$. *Then, for any* $\tilde{\varepsilon} > 0$ *and any* $S > 0$ *there exists a constant* $C = C(\ell, \tilde{\varepsilon}, S, \delta_0, \gamma_0)$ *such that*

$$\log \int_M e^{4(u-\overline{u})} dV_g \leq C + \frac{1}{8(\ell+1)\pi^2 - \tilde{\varepsilon}} \langle P_g u, u\rangle$$

for all the functions $u \in H^2(M)$ *satisfying*

$$\frac{\int_{\Omega_i} e^{4u} dV_g}{\int_M e^{4u} dV_g} \geq \gamma_0, \quad \forall\, i \in \{1, \dots, \ell+1\}; \qquad \sum_{i=1}^{\overline{k}} \alpha_i^2 \leq S. \tag{24}$$

Here $\hat{u} = \sum_{i=1}^{\overline{k}} \alpha_i \hat{v}_i$ *denotes the component of* u *in* V.

Proof. We modify the argument in [21] avoiding the use of truncations, which is not allowed in the H^2 setting. Assuming without loss of generality that $\overline{u} = 0$, we can find $\ell + 1$ functions $g_1, \dots, g_{\ell+1}$ satisfying the following properties

$$
\begin{cases} g_i(x) \in [0,1], & \text{for every } x \in M; \\ g_i(x) = 1, & \text{for every } x \in \Omega_i, i = 1, \dots, \ell+1; \\ g_i(x) = 0, & \text{if } \operatorname{dist}(x, \Omega_i) \geq \frac{\delta_0}{4}; \\ \|g_i\|_{C^4(M)} \leq C_{\delta_0}, & \end{cases} \tag{25}
$$

where C_{δ_0} is a positive constant depending only on δ_0. By interpolation, see [33], since P_g^+ is nonnegative with $\ker P_g^+ = \mathbb{R}$, for any $\varepsilon > 0$ there exists C_{ε,δ_0} (depending only on ε and δ_0) such that, for any $v \in H^2(M)$ and for any $i \in \{1, \dots, \ell+1\}$,

$$
\langle P_g^+ g_i v, g_i v \rangle \leq \int_M g_i^2 (P_g^+ v, v) dV_g + \varepsilon \langle P_g^+ v, v \rangle + C_{\varepsilon,\delta_0} \int_M v^2 dV_g. \tag{26}
$$

If we write u as $u = u_1 + u_2$ with $u_1 \in L^\infty(M)$, then from our assumptions we deduce

$$
\begin{aligned} \int_{\Omega_i} e^{4u_2} dV_g &\geq e^{-4\|u_1\|_{L^\infty(M)}} \int_{\Omega_i} e^{4u} dV_g \\ &\geq e^{-4\|u_1\|_{L^\infty(M)}} \gamma_0 \int_M e^{4u} dV_g; \qquad i = 1, \dots, \ell+1. \end{aligned} \tag{27}
$$

Using (25), (27) and then (23) we obtain

$$
\begin{aligned} \log \int_M e^{4u} dV_g &\leq \log \frac{1}{\gamma_0} + 4\|u_1\|_{L^\infty(M)} + \log \int_M e^{4 g_i u_2} dV_g + C \\ &\leq \log \frac{1}{\gamma_0} + 4\|u_1\|_{L^\infty(M)} + C + \frac{1}{8\pi^2} \langle P_g^+ g_i u_2, g_i u_2 \rangle + 4\overline{g_i u_2}, \end{aligned}
$$

where C depends only on M. We now choose i such that $\langle P_g^+ g_i u_2, g_i u_2 \rangle \leq \langle P_g^+ g_j u_2, g_j u_2 \rangle$ for every $j \in \{1, \dots, \ell+1\}$. Since the functions $g_1, \dots g_{\ell+1}$ have disjoint supports, the last formula and (26) imply

$$
\begin{aligned} \log \int_M e^{4u} dV_g \leq& \log \frac{1}{\gamma_0} + 4\|u_1\|_{L^\infty(M)} \\ &+ C + \left(\frac{1}{8(\ell+1)\pi^2} + \varepsilon \right) \langle P_g^+ u_2, u_2 \rangle + C_{\varepsilon,\delta_0} \int_M u_2^2 dV_g + 4\overline{g_i u_2}. \end{aligned}
$$

Next we choose $\lambda_{\varepsilon,\delta_0}$ to be an eigenvalue of P_g^+ such that $\frac{C_{\varepsilon,\delta_0}}{\lambda_{\varepsilon,\delta_0}} < \varepsilon$, where C_{ε,δ_0} is given in the last formula, and we set

$$
u_1 = P_{V_{\varepsilon,\delta_0}} u; \qquad u_2 = P_{V_{\varepsilon,\delta_0}^\perp} u,
$$

where V_{ε,δ_0} is the direct sum of the eigenspaces of P_g^+ with eigenvalues less than or equal to $\lambda_{\varepsilon,\delta_0}$, and $P_{V_{\varepsilon,\delta_0}}, P_{V_{\varepsilon,\delta_0}^\perp}$ denote the projections onto V_{ε,δ_0} and $V_{\varepsilon,\delta_0}^\perp$ respectively. Since $\overline{u} = 0$, the L^2-norm and the L^∞-norm on V_{ε,δ_0} are equivalent (with a proportionality factor which depends on ε and δ_0), and hence by our choice of u_1 and u_2,

$$\|u_1\|^2_{L^\infty(M)} \leq \hat{C}_{\varepsilon,\delta_0} \langle P_g^+ u_1, u_1 \rangle;$$
$$C_{\varepsilon,\delta_0} \int_M u_2^2 dV_g \leq \frac{C_{\varepsilon,\delta_0}}{\lambda_{\varepsilon,\delta_0}} \langle P_g^+ u_2, u_2 \rangle < \varepsilon \langle P_g^+ u_2, u_2 \rangle,$$

where $\hat{C}_{\varepsilon,\delta_0}$ depends on ε and δ_0. Furthermore, by the positivity of P_g^+ and the Poincaré inequality (recall that $\overline{u} = 0$),

$$\overline{g_i u_2} \leq C \|u_2\|_{L^2(M)} \leq C \|u\|_{L^2(M)} \leq C \langle P_g^+ u, u \rangle^{\frac{1}{2}}.$$

Hence the last formulas imply

$$\begin{aligned} \log \int_M e^{4u} dV_g \leq & \log \frac{1}{\gamma_0} + 4\hat{C}_{\varepsilon,\delta_0} \langle P_g^+ u_1, u_1 \rangle^{\frac{1}{2}} + C \\ & + \left(\frac{1}{8(\ell+1)\pi^2} + \varepsilon \right) \langle P_g^+ u_2, u_2 \rangle + \varepsilon \langle P_g^+ u_2, u_2 \rangle + C \langle P_g^+ u_2, u_2 \rangle^{\frac{1}{2}} \\ \leq & \left(\frac{1}{8(\ell+1)\pi^2} + 3\varepsilon \right) \langle P_g^+ u, u \rangle + \overline{C}_{\varepsilon,\delta_0} + C + \log \frac{1}{\gamma_0}, \end{aligned}$$

where $\overline{C}_{\varepsilon,\delta_0}$ depends only on ε and δ_0 (and ℓ, which is fixed). Now, since by (24) we have uniform bounds on $\hat{u}$, we can replace $\langle P_g^+ u, u \rangle$ by $\langle P_g u, u \rangle$ plus a constant on the right-hand side. This concludes the proof. □

In the next lemma we show a criterion which implies the situation described by the first condition in (24).

LEMMA 2.3. *Let ℓ be a given positive integer, and suppose that ε and r are positive numbers. Suppose that for a nonnegative function $f \in L^1(M)$ with $\|f\|_{L^1(M)} = 1$,*

$$\int_{\cup_{i=1}^{\ell} B_r(p_i)} f dV_g < 1 - \varepsilon \qquad \text{for every } \ell\text{-tuple } p_1, \dots, p_\ell \in M.$$

Then there exist $\overline{\varepsilon} > 0$ and $\overline{r} > 0$, depending only on ε, r, ℓ and M (but not on f), and $\ell + 1$ points $\overline{p}_1, \dots, \overline{p}_{\ell+1} \in M$ (which depend on f) satisfying

$$\int_{B_{\overline{r}}(\overline{p}_1)} f dV_g \geq \overline{\varepsilon}, \ \dots, \ \int_{B_{\overline{r}}(\overline{p}_{\ell+1})} f dV_g \geq \overline{\varepsilon};$$
$$B_{2\overline{r}}(\overline{p}_i) \cap B_{2\overline{r}}(\overline{p}_j) = \emptyset \text{ for } i \neq j.$$

Proof. Suppose by contradiction that for every $\overline{\varepsilon}, \overline{r} > 0$ there is f satisfying the assumptions and such that for every $(\ell+1)$-tuple of points $p_1, \dots, p_{\ell+1}$ in M we have the implication

$$\int_{B_{\overline{r}}(p_1)} f dV_g \geq \overline{\varepsilon}, \ \dots, \ \int_{B_{\overline{r}}(p_{\ell+1})} f dV_g \geq \overline{\varepsilon} \tag{28}$$
$$\Rightarrow \qquad B_{2\overline{r}}(p_i) \cap B_{2\overline{r}}(p_j) \neq \emptyset \text{ for some } i \neq j.$$

We let $\overline{r} = \frac{r}{8}$, where r is as given in the statement. We can find $h \in \mathbb{N}$ and h points $x_1, \dots, x_h \in M$ such that M is covered by $\cup_{i=1}^h B_{\overline{r}}(x_i)$. For ε as given in the statement of the lemma, we also set $\overline{\varepsilon} = \frac{\varepsilon}{2h}$. We point out that the choice of $\overline{r}$ and $\overline{\varepsilon}$ depends on r, ε, ℓ and M only, as required.

Let $\{\tilde{x}_1, \dots, \tilde{x}_j\} \subseteq \{x_1, \dots, x_h\}$ be the points for which $\int_{B_{\overline{r}}(\tilde{x}_i)} f dV_g \geq \overline{\varepsilon}$. We define $\tilde{x}_{j_1} = \tilde{x}_1$, and let A_1 denote the set

$$A_1 = \{\cup_i B_{\overline{r}}(\tilde{x}_i) \ : \ B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_1}) \neq \emptyset\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_1}).$$

If there exists $\tilde{x}_{j_2}$ such that $B_{2\overline{r}}(\tilde{x}_{j_2}) \cap B_{2\overline{r}}(\tilde{x}_{j_1}) = \emptyset$, we define

$$A_2 = \{\cup_i B_{\overline{r}}(\tilde{x}_i) \ : \ B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_2}) \neq \emptyset\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_2}).$$

Proceeding in this way, we define recursively some points $\tilde{x}_{j_3}, \tilde{x}_{j_4}, \dots, \tilde{x}_{j_s}$ satisfying

$$B_{2\overline{r}}(\tilde{x}_{j_s}) \cap B_{2\overline{r}}(\tilde{x}_{j_a}) = \emptyset \qquad \forall\, 1 \leq a < s,$$

and some sets $A_3, \dots, A_s$ by

$$A_s = \{\cup_i B_{\overline{r}}(\tilde{x}_i) \ : \ B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_s}) \neq \emptyset\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_s}).$$

By (28), the process cannot go further than $\tilde{x}_{j_\ell}$, and hence $s \leq \ell$. Using the definition of $\overline{r}$ we obtain

$$\cup_{i=1}^j B_{\overline{r}}(\tilde{x}_i) \subseteq \cup_{i=1}^s A_i \subseteq \cup_{i=1}^s B_{4\overline{r}}(\tilde{x}_{j_i}) \subseteq \cup_{i=1}^s B_r(\tilde{x}_{j_i}). \tag{29}$$

Then by our choice of h, $\overline{\varepsilon}$, $\{\tilde{x}_1, \dots, \tilde{x}_j\}$ and by (29),

$$\begin{aligned}\int_{M \setminus \cup_{i=1}^s B_r(\tilde{x}_{j_i})} f dV_g &\leq \int_{M \setminus \cup_{i=1}^j B_{\overline{r}}(\tilde{x}_i)} f dV_g \\ &\leq \int_{(\cup_{i=1}^h B_{\overline{r}}(x_i)) \setminus \left(\cup_{i=1}^j B_{\overline{r}}(\tilde{x}_i)\right)} f dV_g < (h-j)\overline{\varepsilon} \leq \frac{\varepsilon}{2}.\end{aligned}$$

Finally, if we chose $p_i = \tilde{x}_{j_i}$ for $i = 1, \dots, s$ and $p_i = \tilde{x}_{j_s}$ for $i = s+1, \dots, \ell$, we get a contradiction to the assumptions of the lemma. $\square$

Next we characterize some functions in $H^2(M)$ for which the value of II is large negative. Recall that the number k is given in formula (11) and that $\hat{u}$ is the projection of u on the direct sum of the eigenspaces of P_g corresponding to negative eigenvalues.

LEMMA 2.4. *Under the assumptions of Theorem 1.1, and for $k_P \in (8k\pi^2, 8(k+1)\pi^2)$ with $k \geq 1$, the following property holds. For any $S > 0$, any $\varepsilon > 0$ and any $r > 0$ there exists a large positive $L = L(S, \varepsilon, r)$ such that for every $u \in H^2(M)$ with $II(u) \leq -L$ and $\|\hat{u}\| \leq S$ there exist k points $p_{1,u}, \dots, p_{k,u} \in M$ such that*

$$\int_{M \setminus \cup_{i=1}^k B_r(p_{i,u})} e^{4u} dV_g < \varepsilon. \tag{30}$$

Proof. Suppose by contradiction that the statement is not true, namely that there exist $S, \varepsilon, r > 0$ and $(u_n)_n \subseteq H^2(M)$ with $\|\hat{u}_n\| \leq S$, $II(u_n) \to -\infty$ and such that for every k-tuple $p_1, \dots, p_k$ in M there holds

$$\int_{\cup_{i=1}^k B_r(p_i)} e^{4u_n} dV_g < 1 - \varepsilon.$$

Recall that without loss of generality, since II is invariant under translation by constants in the argument, we can assume that for every n $\int_M e^{4u_n} dV_g = 1$. Then we can apply Lemma 2.3 with $\ell = k$, $f = e^{4u_n}$, and in turn Lemma 2.2 with $\delta_0 = 2\overline{r}$, $\Omega_1 = B_{\overline{r}}(\overline{p}_1)$, $\dots$, $\Omega_{k+1} = B_{\overline{r}}(\overline{p}_{k+1})$ and $\gamma_0 = \overline{\varepsilon}$, where $\overline{\varepsilon}$, $\overline{r}$ and $(\overline{p}_i)_i$ are as given by Lemma 2.3. This implies that for any given $\tilde{\varepsilon} > 0$ there exists $C > 0$ depending only on $S, \varepsilon, \tilde{\varepsilon}$ and r such that

$$\begin{aligned} II(u_n) \geq \; & \langle P_g u_n, u_n \rangle \\ & + 4 \int_M Q_g u_n dV_g - C k_P - \frac{k_P}{8(k+1)\pi^2 - \tilde{\varepsilon}} \langle P_g u_n, u_n \rangle - 4 k_P \overline{u}_n, \end{aligned}$$

where C is independent of n. Since $k_P < 8(k+1)\pi^2$, we can choose $\tilde{\varepsilon} > 0$ so small that $1 - \frac{k_P}{8(k+1)\pi^2 - \tilde{\varepsilon}} := \delta > 0$. Hence, using also the Poincaré inequality, we deduce

$$\begin{aligned} II(u_n) & \geq \delta \langle P_g u_n, u_n \rangle + 4 \int_M Q_g (u_n - \overline{u}_n) dV_g - C k_P \\ & \geq \delta \langle P_g u_n, u_n \rangle - 4C \langle P_g u_n, u_n \rangle^{\frac{1}{2}} - C k_P \geq -C. \end{aligned} \tag{31}$$

This violates our contradiction assumption, and concludes the proof. □

3. Mapping sublevels of II into $A_{k,\overline{k}}$

In this section we show how to map nontrivially some sublevels of the functional II into the set $A_{k,\overline{k}}$. Since adding a constant to the argument of II does not affect its value, we can always assume that the functions $u \in H^2(M)$ we are dealing with satisfy the normalization (14) (with u instead of u_l). Our goal is to prove the following result.

PROPOSITION 3.1. *For $k \geq 1$ (see (11)) there exists a large $L > 0$ and a continuous map Ψ from the sublevel $\{II < -L\}$ into $A_{k,\overline{k}}$ which is topologically nontrivial. For $k_P < 8\pi^2$ and $\overline{k} \geq 1$ the same is true with $A_{k,\overline{k}}$ replaced by $S^{\overline{k}-1}$.*

We divide the section into two parts. First we derive some properties of the set M_k for $k \geq 1$. Then we turn to the construction of the map Ψ. Its nontriviality will follow from Proposition 4.1 below, where we show that there is another map Φ from $A_{k,\overline{k}}$ into $H^2(M)$ such that $\Psi \circ \Phi$ is homotopic to the identity on $A_{k,\overline{k}}$, which is not contractible by Corollary 3.8.

3.1. *Some properties of the set M_k.* In this subsection we collect some useful properties of the set M_k, beginning with some local ones near the singularities, namely the subsets $M_j \subseteq M_k$ with $j < k$. Although the topological structure of the barycenters is well-known, we need some estimates of quantitative type concerning the metric distance. The reason, as mentioned in the introduction, is that the amount of concentration of e^{4u} (where $u \in \{II \leq -L\}$, see Lemma 2.4) near a single point can be arbitrarily small. In this way we are forced to define a projection which depends on *all* the distances from the M_j's; see subsection 3.2, which requires some preliminary considerations. We recall that on M_k we are adopting the metric induced by $C^1(M)^*$, see Section 2, and for $j < k$ we set $d_j(\sigma) = \mathrm{dist}(\sigma, M_j)$, $\sigma \in M_k$. Then for $\varepsilon > 0$ and $2 \leq j \leq k$, we define

$$M_j(\varepsilon) = \{\sigma \in M_j \ : \ d_{j-1}(\sigma) > \varepsilon\} .$$

For convenience, we extend the definition also to the case $j = 1$, setting

$$M_1(\varepsilon) := M_1.$$

We give a first quantitative description of the set $M_j(\varepsilon)$, which leads immediately to (the well known) Corollary 3.3.

LEMMA 3.2. *Let $j \in \{2, \dots, k\}$. Then there exists ε sufficiently small with the following property. If $\sigma \in M_j(\varepsilon)$, $\sigma = \sum_{i=1}^{j} t_i \delta_{x_i}$, then*

$$t_i \geq \frac{\varepsilon}{2}; \qquad \mathrm{dist}(x_i, x_l) \geq \frac{\varepsilon}{2}; \qquad\qquad i, l = 1, \dots, j, i \neq l. \tag{32}$$

Proof. Let $\sigma = \sum_{i=1}^{j} t_i \delta_{x_i} \in M_j(\varepsilon)$. Assuming by contradiction that the first inequality in (32) is not satisfied, we see that there would exists $\bar\imath \in \{1, \dots, j\}$ such that $t_{\bar\imath} < \frac{\varepsilon}{2}$. Then, for $\tilde\imath \in \{1, \dots, j\}$, $\tilde\imath \neq \bar\imath$, we consider the following element

$$\hat\sigma = (t_{\bar\imath} + t_{\tilde\imath})\delta_{x_{\tilde\imath}} + \sum_{i=1,\dots,j,\ i \neq \bar\imath, \tilde\imath} t_i \delta_{x_i} \in M_{j-1}.$$

For any function $f \in C^1(M)$ with $\|f\|_{C^1(M)} \leq 1$ one has clearly

$$|(\sigma, f) - (\hat\sigma, f)| \leq t_{\bar\imath} \left(|f(x_{\bar\imath})| + |f(x_{\tilde\imath})|\right) \leq 2 t_{\bar\imath}.$$

Taking the supremum with respect to f we deduce

$$\varepsilon < \mathrm{dist}(\sigma, M_{j-1}) \leq \mathrm{dist}(\sigma, \hat\sigma) = \sup_f |(\sigma, f) - (\hat\sigma, f)| \leq 2 t_{\bar\imath}.$$

This gives us a contradiction. Let us prove now the second inequality. Assuming that there are $x_i, x_l \in M$ with $x_i \neq x_l$ and $\mathrm{dist}(x_i, x_l) < \frac{\varepsilon}{2}$ (for ε sufficiently small), we define the element

$$\hat\sigma = (t_i + t_l)\delta_{\frac{1}{2}x_i + \frac{1}{2}x_l} + \sum_{s=1,\dots,j,\ s \neq i,l} t_s \delta_{x_s} \in M_{j-1}.$$

See the notation introduced in Section 2 for the convex combination of the points x_i and x_l. Similarly, as before, for $\|f\|_{C^1(M)} \leq 1$ we obtain

$$
\begin{aligned}
|(\sigma, f) - (\hat{\sigma}, f)| &\leq t_i \left| f(x_i) - f\left(\frac{x_i + x_l}{2}\right)\right| + t_l \left| f(x_l) - f\left(\frac{x_i + x_l}{2}\right)\right| \\
&\leq \left| f(x_i) - f\left(\frac{x_i + x_l}{2}\right)\right| + \left| f(x_l) - f\left(\frac{x_i + x_l}{2}\right)\right|.
\end{aligned}
$$

Taking the supremum with respect to such functions f, since they all have Lipschitz constant less than or equal to 1, we deduce

$$
\varepsilon < \operatorname{dist}(\sigma, M_{j-1}) \leq \operatorname{dist}(\sigma, \hat{\sigma}) = \sup_f |(\sigma, f) - (\hat{\sigma}, f)| \leq 2\operatorname{dist}(x_i, x_l).
$$

This gives us a contradiction and concludes the proof. □

COROLLARY 3.3 (well-known). *The set M_1 is a smooth manifold in $C^1(M)^*$. Furthermore, for any $\varepsilon > 0$ and for $j \geq 2$, the set $M_j(\varepsilon)$ is also a smooth* (*open*) *manifold of dimension* $5j - 1$.

Proof. The first assertion is obvious. As to the second one, the previous lemma guarantees that all the numbers t_i are uniformly bounded away from zero and that the mutual distance between the points x_i is also uniformly bounded from below. Therefore, recalling that the t_i's satisfy the constraint $\sum_i t_i = 1$, each element of $M_j(\varepsilon)$ can be smoothly parametrized by $4j$ coordinates locating the points x_i and by $j - 1$ coordinates identifying the numbers t_i. □

We show next that it is possible to define a continuous homotopy which brings points in M_k, which are close to $M_j(\varepsilon)$, onto $M_j\left(\frac{\varepsilon}{2}\right)$. We also provide some quantitative estimates on the deformation. Our goal is to patch together projections onto sets of different dimensions (of the form $M_j(\varepsilon_j)$ for suitable ε_j's), as shown in Figure 2 below. The proof of Lemma 3.4 is postponed to the appendix.

LEMMA 3.4. *Let $j \in \{1, \dots, k-1\}$, and let $\varepsilon > 0$. Then there exist $\hat{\varepsilon}$* ($\ll \varepsilon^2$), *depending only on ε and k, and a map T_j^t, $t \in [0, 1]$, from the set*

$$
\hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon} := \{\sigma \in M_k \; : \; \operatorname{dist}(\sigma, M_j(\varepsilon)) < \hat{\varepsilon}\}
$$

into M_k such that the following five properties hold true:

(i) $T_j^0 = \mathrm{Id}$ *and* $T_j^t|_{M_j} = Id|_{M_j}$ *for every* $t \in [0, 1]$;

(ii) $T_j^1(\sigma) \in M_j\left(\frac{\varepsilon}{2}\right)$ *for every* $\sigma \in \hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon}$;

(iii) $\operatorname{dist}(T_j^0(\sigma), T_j^t(\sigma)) \leq C_{k,\varepsilon}\sqrt{\hat{\varepsilon}}$ *for every* $\sigma \in \hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon}$ *and for every* $t \in [0, 1]$;

(iv) *If* $\sigma \in \hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon} \cap M_l$ *for some* $l \in \{j, \dots, k-1\}$, *then* $T_j^t(\sigma) \in M_l$ *for every* $t \in [0,1]$;

(v) *If* $\sigma \in \hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon} \cap M_j$, *then* $T_j^t(\sigma) = \sigma$ *for every* $t \in [0,1]$.

The constant $C_{k,\varepsilon}$ *in* (iii) *depends only on* k *and* ε, *and not on* t *and* $\hat{\varepsilon}$ (*provided the latter is sufficiently small*).

Remark 3.5. We notice that, by the property (iv) in the statement of Lemma 3.4, the above homotopy is well defined also from each M_l into itself, for $l \in \{1, \dots, k-1\}$, and extends continuously to a neighborhood of M_l in M_k.

Since $M_j\left(\frac{\varepsilon}{4}\right)$ is a smooth finite-dimensional manifold in $C^1(M)^*$ by Corollary 3.3, we can define a continuous projection P_j (see the comments at the beginning of the proof of Lemma 3.4 in the appendix) from $\hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon}$ into $M_j\left(\frac{\varepsilon}{2}\right)$, which is compactly contained in $M_j\left(\frac{\varepsilon}{4}\right)$. We have then an immediate consequence of the previous lemma.

Corollary 3.6. *Let* T_j^t *denote the map constructed in Lemma* 3.4 *above. Then for* $\hat{\varepsilon}$ *sufficiently small there exists a homotopy* H_j^t, $t \in [0,1]$, *between* $T_j^1(\sigma)$ *and* $P_j(\sigma)$ *within* $M_j\left(\frac{\varepsilon}{2}\right)$, *namely a map satisfying the following properties*

$$\begin{cases} H_j^t(\sigma) \in M_j\left(\frac{\varepsilon}{2}\right) & \text{for every } t \in [0,1] \text{ and every } \sigma \in \hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon}; \\ H_j^0(\sigma) = T_j^1(\sigma) & \text{for every } \sigma \in \hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon}; \\ H_j^1(\sigma) = P_j(\sigma) & \text{for every } \sigma \in \hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon}. \end{cases} \tag{33}$$

In view of Corollary 3.6, we can also modify the map T_j^t by composing it with the above homotopy H_j^t; namely, we set

$$\hat{T}_j^t(\sigma) = \begin{cases} T_j^{2t}(\sigma), & \text{for } t \in \left[0, \frac{1}{2}\right]; \\ H_j^{2t-1} \circ T_j^1(\sigma), & \text{for } t \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{34}$$

In this way, if for some element of $C^1(M)^*$ both the projections P_j and P_l are defined, for $1 \leq j < l \leq k$, composing $\hat{T}_j^t$ with P_l we obtain a homotopy between P_l and P_j within $M_l \cap M_j\left(\frac{\varepsilon}{2}\right)$; see Remark 3.5. This fact will be used crucially in the proof of Lemma 3.10 below.

Next we recall the following result, which is necessary in order to carry out the topological argument below. For completeness, we give a brief idea of the proof.

Lemma 3.7 (well-known). *For any* $k \geq 1$, *the set* M_k *is noncontractible.*

Proof. For $k = 1$ the statement is obvious, so we consider the case $k \geq 2$. The set $M_k \setminus M_{k-1}$, see Corollary 3.3, is an open manifold of dimension $5k-1$. It is possible to prove, even if M_{k-1} is not a smooth manifold (for $k \geq 3$), that it is anyway a Euclidean neighborhood retract; namely it is a contraction of some of its neighborhoods which has a smooth boundary (of dimension $5k-2$); see [4], [9]. Therefore M_k has an orientation (mod 2) with respect to M_{k-1}; namely the relative homology class $H_{5k-1}(M_k, M_{k-1}; \mathbb{Z}_2)$ is nontrivial. Consider now this part of the exact homology sequence of the pair (M_k, M_{k-1}):

$$\cdots \to H_{5k-1}(M_{k-1}; \mathbb{Z}_2) \to H_{5k-1}(M_k; \mathbb{Z}_2) \to H_{5k-1}(M_k, M_{k-1}; \mathbb{Z}_2) \to H_{5k-2}(M_{k-1}; \mathbb{Z}_2) \to \cdots .$$

Since the dimension of (the stratified set) M_{k-1} is less than or equal to $5(k-1)-1 < 5k-2$, both the homology groups $H_{5k-1}(M_{k-1}; \mathbb{Z}_2)$ and $H_{5k-2}(M_{k-1}; \mathbb{Z}_2)$ vanish, and therefore $H_{5k-1}(M_k; \mathbb{Z}_2) \simeq H_{5k-1}(M_k, M_{k-1}; \mathbb{Z}_2) \neq 0$. The proof is concluded. □

From the preceding lemma and from a standard application of the Majer-Vietoris theorem is it easy to deduce the following result.

Corollary 3.8. *For any (relative) integers $k \geq 1$ and $\overline{k} \geq 0$, the set $A_{k,\overline{k}}$ is noncontractible.*

3.2. *Construction of* Ψ. In this subsection we finally construct the map Ψ, using the preceding results about the set M_k. First we show how to construct some partial projections on the sets $M_j(\varepsilon)$ for $\varepsilon > 0$. When referring to the distance of a function in $L^1(M)$ from a set M_j, we always adopt the metric induced by $C^1(M)^*$. The comments before Corollary 3.6 yield the following result.

Lemma 3.9. *Suppose that $f \in L^1(M)$, $f \geq 0$ and that $\int_M f dV_g = 1$. Then, given any $\varepsilon > 0$ and any $j \in \{1, \dots, k\}$, there exists $\hat{\varepsilon} > 0$, depending on j and ε with the following property. If $\operatorname{dist}(f, M_j(\varepsilon)) \leq \hat{\varepsilon}$, then there is a continuous projection P_j mapping f onto $M_j\left(\frac{\varepsilon}{2}\right)$.*

Next we define an auxiliary map $\hat{\Psi}$ from a suitable sublevel of II into M_k.

Lemma 3.10. *For $k \geq 1$ there exist a large $\hat{L} > 0$ and a continuous map $\hat{\Psi}$ from $\{II \leq -\hat{L}\} \cap \{\|\hat{u}\| \leq 1\}$ into M_k. Here, as before, $\hat{u}$ denotes the component of u belonging to V, the direct sum of the negative eigenspaces of P_g (if any, otherwise we impose no restriction on u except for $II(u) \leq -\hat{L}$).*

Proof. First we define some numbers

$$\varepsilon_k \ll \varepsilon_{k-1} \ll \cdots \ll \varepsilon_2 \ll \varepsilon_1 \ll 1$$

in the following way. We choose ε_1 so small that there is a projection P_1 from the nonnegative $L^1(M)$ functions in an ε_1-neighborhood of M_1 onto M_1 (by Lemma 3.9). We now can apply again Lemma 3.9 with $j = 2$, $\varepsilon = 4\varepsilon_1$ and,

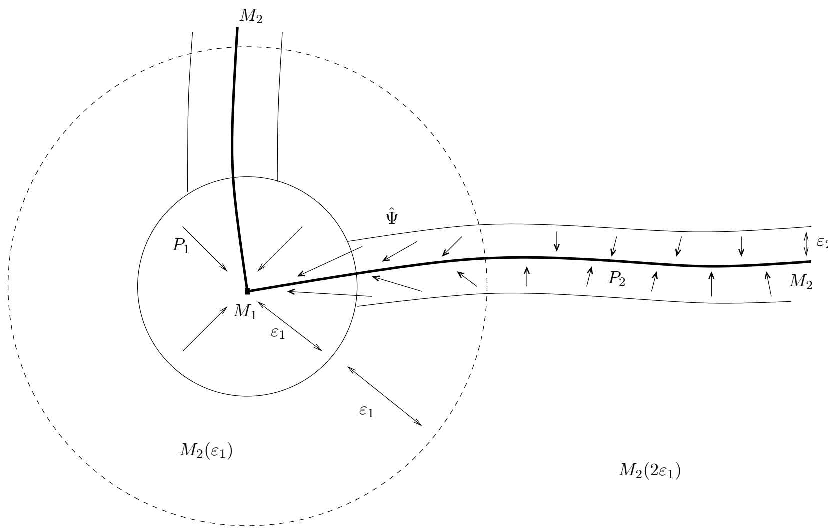

Figure 2. The map $\hat{\Psi}$ for $k = 2$

obtaining the corresponding (sufficiently small) $\hat{\varepsilon}$, we define $\varepsilon_2 = \frac{\hat{\varepsilon}}{4}$. Then we choose the numbers $\varepsilon_3, \dots, \varepsilon_k$ iteratively in the same way.

For any $i = 1, \dots, k$, let f_i be a smooth nonincreasing cutoff function which satisfies the following properties

$$\begin{cases} f_i(t) = 1 & \text{for } t \leq \varepsilon_i; \\ f_i(t) = 0 & \text{for } t \geq 2\varepsilon_i. \end{cases} \tag{35}$$

Next we choose suitably the large number $\hat{L}$. In order to do this, we apply Lemma 2.4 with $S = 1$ and some small ε. It is easy to see that if ε is chosen first sufficiently small, and then $\hat{L} = L$ sufficiently large, then for any $u \in H^2(M)$ with $II(u) \leq -\hat{L}$ (and $\int_M e^{4u} dV_g = 1$), $\operatorname{dist}(e^{4u}, M_k) < \varepsilon_k$.

Now, given $u \in H^2(M)$ with $II(u) \leq -\hat{L}$, we let j (depending on u) denote the first integer such that $f_j(\operatorname{dist}(e^{4u}, M_j)) = 1$. We notice that for $j > 1$, since $f_{j-1}(\operatorname{dist}(e^{4u}, M_{j-1})) < 1$, $\operatorname{dist}(e^{4u}, M_{j-1}) > \varepsilon_{j-1}$ and $\operatorname{dist}(e^{4u}, M_{j-1}) < \varepsilon_j$. Therefore, by Lemma 3.9 and our choice of the ε_i's, the projection $P_j(e^{4u})$ is well-defined. Then we set

$$\hat{\Psi}(u) = \hat{T}_1^{f_1(\operatorname{dist}(e^{4u}, M_1))} \circ \hat{T}_2^{f_2(\operatorname{dist}(e^{4u}, M_2)} \circ \cdots \circ \hat{T}_{j-1}^{f_{j-1}(\operatorname{dist}(e^{4u}, M_{j-1}))} \circ P_j(e^{4u}).$$

Some comments are in order. This definition depends in principle on the index j, which is a function of u. Nevertheless, since all the distance functions from the M_l's are continuous, and since $\hat{T}_l^1 = P_l$, see (34), the above map $\hat{\Psi}$ is indeed well defined and continuous in u, see Remark 3.5 and the comments after Corollary 3.6. □

In Figure 2, we sketch the construction of the map $\hat{\Psi}$ for the case $k = 2$, which is the simplest among the nontrivial ones. M_1 is depicted as a single point, while M_2 is depicted as a couple of curves. The region between the two circles, which represents $\{\varepsilon_1 \leq \mathrm{dist}(e^{4u}, M_1) \leq 2\varepsilon_1\}$, is where the homotopy $\hat{T}_1^t$ (and hence the construction of Lemma 3.4) is used.

We are finally in position to introduce the global map Ψ. If $\hat{v}_1, \dots, \hat{v}_{\overline{k}}$ form an orthonormal basis (in $L^2(M)$) of V, V being the direct sum of the eigenspaces of P_g corresponding to negative eigenvalues, see Section 2, we define the $\overline{k}$-vector

$$s(u) = \left((\hat{v}_1, u)_{L^2(M)}, \dots, (\hat{v}_{\overline{k}}, u)_{L^2(M)}\right) \in \mathbb{R}^{\overline{k}}.$$

Then, if $\hat{L}$ is as in Lemma 3.10 and if $\overline{\sigma}$ is any fixed element of M_k, in the case $k \geq 1$ we let $\Psi : \{II \leq -\hat{L}\} \to A_{k,\overline{k}}$ be defined by

$$\Psi(u) = \begin{cases} (\hat{\Psi}(u), s(u)) & \text{for } |s(u)| \leq 1; \\ \left(\overline{\sigma}, \frac{s(u)}{|s(u)|}\right) & \text{for } |s(u)| > 1. \end{cases} \tag{36}$$

Since for $|s|$ tending to 1 the set M_k is collapsing to a single point in $A_{k,\overline{k}}$, see (21), the map Ψ is continuous.

On the other hand, if $k_P < 8\pi^2$ and if $\overline{k} \geq 1$ we just set

$$\Psi(u) = \frac{s(u)}{|s(u)|}. \tag{37}$$

Proof of Proposition 3.1. It remains only to prove the nontriviality of the map Ψ. This follows from Corollary 3.8 and from (b) in Proposition 4.1 below. □

4. Mapping $A_{k,\overline{k}}$ into low sublevels of II

The next step consists in finding a map Φ from $A_{k,\overline{k}}$ (resp. from $S^{\overline{k}-1}$) into $H^2(M)$ on which image the functional II attains large negative values.

PROPOSITION 4.1. *Let Ψ be the map defined in the previous section. Then, if $k \geq 1$ (resp. $k_P < 8\pi^2$ and $\overline{k} \geq 1$), for any $L > 0$ sufficiently large (such that Proposition* 3.1 *applies) there exists a map $\Phi_{\overline{S},\overline{\lambda}} : A_{k,\overline{k}} \to H^2(M)$ (resp. $\Phi_{\overline{S}} : S^{\overline{k}-1} \to H^2(M)$) with the following properties*

(a) *$II(\Phi_{\overline{S},\overline{\lambda}}(z)) \leq -L$ for any $z \in A_{k,\overline{k}}$ (resp. $II(\Phi_{\overline{S}}(z)) \leq -L$ for any $z \in S^{\overline{k}-1}$);*

(b) *$\Psi \circ \Phi_{\overline{S},\overline{\lambda}}$ is homotopic to the identity on $A_{k,\overline{k}}$ (resp. $\Psi \circ \Phi_{\overline{S}}$ is homotopic to the identity on $S^{\overline{k}-1}$).*

In order to prove this proposition we need some preliminary notation and lemmas. For $\delta > 0$ small, consider a smooth nondecreasing cut-off function

$\chi_\delta : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following properties

$$\begin{cases} \chi_\delta(t) = t & \text{for } t \in [0,\delta]; \\ \chi_\delta(t) = 2\delta & \text{for } t \geq 2\delta; \\ \chi_\delta(t) \in [\delta, 2\delta] & \text{for } t \in [\delta, 2\delta]. \end{cases} \tag{38}$$

Then, given $\sigma \in M_k$ $\left(\sigma = \sum_{i=1}^k t_i \delta_{x_i}\right)$ and $\lambda > 0$, we define the function $\varphi_{\lambda,\sigma} : M \to \mathbb{R}$ as

$$\varphi_{\lambda,\sigma}(y) = \frac{1}{4} \log \sum_{i=1}^k t_i \left(\frac{2\lambda}{1 + \lambda^2 \chi_\delta^2 \left(d_i(y)\right)} \right)^4, \qquad y \in M, \tag{39}$$

where we have set

$$d_i(y) = \mathrm{dist}(y, x_i), \qquad y \in M,$$

with $\mathrm{dist}(\cdot,\cdot)$ denoting the distance function on M. We are now in position to define the function $\Phi_{\overline{S},\overline{\lambda}} : A_{k,\overline{k}} \to H^2(M)$. For large $\overline{S}$ and $\overline{\lambda}$ we let

$$\Phi_{\overline{S},\overline{\lambda}}(\sigma, s) = \begin{cases} \varphi_s + \varphi_{\overline{\lambda},\sigma} & \text{for } |s| \leq \frac{1}{4}; \\ \varphi_s + \varphi_{2\overline{\lambda}-1+4(1-\overline{\lambda})|s|,\sigma} & \text{for } \frac{1}{4} \leq |s| \leq \frac{1}{2}; \\ \varphi_s + 2(1 - \varphi_{1,\sigma})|s| + 2\varphi_{1,\sigma} - 1 & \text{for } |s| \geq \frac{1}{2}, \end{cases}$$

where

$$s = (s_1, \dots, s_{\overline{k}}); \qquad \varphi_s(y) = \overline{S} \sum_{i=1}^{\overline{k}} s_i \hat{v}_i(y).$$

For $k_P < 8\pi^2$ and for $\overline{k} \geq 1$ we just set

$$\Phi_{\overline{S}}(s) = \varphi_s, \qquad |s| = 1.$$

Notice that the map is well defined on $A_{k,\overline{k}}$.

We state now two preliminary lemmas, postponing the proof of the first to the appendix.

LEMMA 4.2. *Suppose $\varphi_{\lambda,\sigma}$ is as in (39). Then as $\lambda \to +\infty$ one has*

$$\langle P_g \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \rangle \leq \left(32k\pi^2 + o_\delta(1)\right) \log \lambda + C_\delta \qquad (\textit{uniformly in } \sigma \in M_k),$$

where $o_\delta(1) \to 0$ as $\delta \to 0$, and where C_δ is a constant independent of λ and $(x_i)_i$.

LEMMA 4.3. *For $k \geq 1$ (resp. for $k_P < 8\pi^2$ and for $\overline{k} \geq 1$), given any $L > 0$, there exist a small δ, some large $\overline{S}$ and $\overline{\lambda}$ such that $II(\Phi_{\overline{S},\overline{\lambda}}(\sigma, s)) \leq -L$ for every $(\sigma, s) \in A_{k,\overline{k}}$ (resp. $II(\Phi_{\overline{S}}(s)) \leq -L$ for every $s \in S^{\overline{k}-1}$).*

Proof. We begin with the case $k \geq 1$, and prove first the following three estimates (recall that $\lambda_{\overline{k}}$ is the biggest negative eigenvalue of P_g):

(40)
$$\int_M Q_g(\varphi_s + \varphi_{\lambda,\sigma}) dV_g = -k_P \log \lambda + O(\delta^4 \log \lambda) + O(|\log \delta|) + \overline{S} O(|s|) + O(1);$$

(41)
$$\log \int_M \exp\left(4(\varphi_s + \varphi_{\lambda,\sigma})\right) dV_g = O(1) + O(\overline{S}|s|);$$

(42)
$$\langle P_g(\varphi_s + \varphi_{\lambda,\sigma}), (\varphi_s + \varphi_{\lambda,\sigma})\rangle \leq -|\lambda_{\overline{k}}| |s|^2 \overline{S}^2 + 32 k \pi^2 (1 + o_\delta(1)) \log \lambda + C_\delta + O(\delta^4 |s| \overline{S}).$$

Proof of (40). We have

$$\varphi_{\lambda,\sigma}(y) = \log \frac{2\lambda}{1 + 4\lambda^2 \delta^2}, \qquad \text{for } y \in M \setminus \cup_{i=1}^k B_{2\delta}(x_i),$$

and

$$\log \frac{2\lambda}{1 + 4\lambda^2 \delta^2} \leq \varphi_{\lambda,\sigma}(y) \leq \log 2\lambda, \qquad \text{for } y \in \cup_{i=1}^k B_{2\delta}(x_i).$$

Writing

$$\int_M Q_g(y) \varphi_{\lambda,\sigma}(y) dV_g(y) = \log \frac{2\lambda}{1 + 4\lambda^2\delta^2} \int_M Q_g(y) dV_g(y) + \int_M Q_g(y) \left(\varphi_{\lambda,\sigma}(y) - \log \frac{2\lambda}{1 + 4\lambda^2\delta^2}\right) dV_g(y),$$

we have from the last three formulas

(43)
$$\int_M Q_g(y) \varphi_{\lambda,\sigma}(y) dV_g(y) = k_P \log \frac{2\lambda}{1 + 4\lambda^2\delta^2} + O\left(\delta^4 \log(1 + 4\lambda^2\delta^2)\right).$$

Furthermore recalling that the average of φ_s is zero (since all the $\hat{v}_i$'s have zero average, see Section 2), we deduce that

(44)
$$\int_M Q_g(y) \varphi_s(y) dV_g(y) = \overline{S} \sum_{i=1}^{\overline{k}} s_i \int_M Q_g(y) \hat{v}_i(y) dV_g(y) = \overline{S} O(|s|).$$

Hence (43) and (44) yield

$$\int_M Q_g(y)(\varphi_s + \varphi_{\lambda,\sigma}(y)) dV_g(y) = k_P \log \frac{2\lambda}{1 + 4\lambda^2\delta^2} + O\left(\delta^4 \log(1 + 4\lambda^2\delta^2)\right) + \overline{S} O(|s|),$$

which immediately implies (40).

Proof of (41). We recall that in V the L^2-norm and the L^∞ norm are equivalent. Therefore, noticing that

(45)
$$\exp\left(4(\varphi_s(y))\right) \in \left[\exp(4\inf_M \varphi_s), \exp(4\sup_M \varphi_s)\right] \subseteq \left[\exp(-4C\overline{S}|s|), \exp(4C\overline{S}|s|)\right],$$

we obtain

$$\log \int_M \exp\left(4(\varphi_s + \varphi_{\lambda,\sigma})\right) dV_g = \log \int_M \exp\left(4\varphi_s\right) dV_g + \log \int_M \exp\left(4\varphi_{\lambda,\sigma}\right) dV_g$$

$$= \log \int_M \exp\left(4\varphi_{\lambda,\sigma}\right) dV_g + O(\overline{S}|s|). \tag{46}$$

By the definition of $\varphi_{\lambda,\sigma}$,

$$\int_M \exp\left(4\varphi_{\lambda,\sigma}(y)\right) dV_g(y) = \sum_{i=1}^k t_i \int_M \left(\frac{2\lambda}{1+\lambda^2\chi_\delta^2\left(\operatorname{dist}(y,x_i)\right)}\right)^4 dV_g(y).$$

We divide each of the above integrals into the metric ball $B_\delta(x_i)$ and its complement. By construction of χ_δ, working in normal coordinates centered at x_i, we have (for δ sufficiently small)

$$\int_{B_\delta(x_i)} \left(\frac{2\lambda}{1+\lambda^2\chi_\delta^2\left(\operatorname{dist}(y,x_i)\right)}\right)^4 dV_g(y) = \int_{B_\delta^{\mathbb{R}^4}(0)} (1+O(\delta)) \left(\frac{2\lambda}{1+\lambda^2|y|^2}\right)^4 dy$$
$$= \int_{B_{\lambda\delta}^{\mathbb{R}^4}(0)} (1+O(\delta)) \left(\frac{2}{1+|y|^2}\right)^4 dy = (1+O(\delta)) \left(\frac{8}{3}\pi^2 + O\left(\frac{1}{\lambda^4\delta^4}\right)\right).$$

On the other hand, for $\operatorname{dist}(y,x_i) \geq \delta$, $\left(\frac{2\lambda}{1+\lambda^2\chi_\delta^2(\operatorname{dist}(y,x_i))}\right)^4 \leq \left(\frac{2\lambda}{1+\lambda^2\delta^2}\right)^4$. Hence, from these two formulas we deduce

$$\int_M \exp\left(4\varphi_{\lambda,x}(y)\right) dV_g(y) = \frac{8}{3}\pi^2 + O(\delta) + O\left(\frac{1}{\lambda^4\delta^4}\right) + O\left(\frac{2\lambda}{1+\lambda^2\delta^2}\right)^4. \tag{47}$$

It follows from (46) and (47) that

$$\int_M \exp\left(4\varphi_s + 4\varphi_{\lambda,\sigma}\right) dV_g = O(\overline{S}|s|) + O(1). \tag{48}$$

This concludes the proof of (41).

Proof of (42). We have trivially

$$\langle P_g(\varphi_s + \varphi_{\lambda,\sigma}), (\varphi_s + \varphi_{\lambda,\sigma})\rangle$$
$$= \int_M (P_g\varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma}) dV_g + 2\int_M (P_g\varphi_s, \varphi_{\lambda,\sigma}) dV_g + \int_M (P_g\varphi_s, \varphi_s) dV_g.$$

By Lemma 4.2 it is sufficient to estimate the last two quantities. Since P_g is negative-definite on V (and since the largest negative eigenvalue is $\lambda_{\overline{k}}$), we have clearly

$$\int_M (P_g\varphi_s, \varphi_s) dV_g \leq -|\lambda_{\overline{k}}||s|^2\overline{S}^2. \tag{49}$$

To evaluate the second term we write

$$2\int_M (P_g \varphi_s, \varphi_{\lambda,\sigma}) dV_g = 2\overline{S} \sum_{i=1}^{\overline{k}} s_i \lambda_i \int_M \hat{v}_i \varphi_{\lambda,\sigma} dV_g.$$

Hence it is sufficient to study each of the terms $\int_M \hat{v}_i \varphi_{\lambda,\sigma} dV_g$. We claim that for each $i \in \{1, \dots, k\}$

$$\int_M \hat{v}_i \varphi_{\lambda,\sigma} dV_g = O(\delta^4). \tag{50}$$

In order to prove this claim, we notice first that the following inequality holds (recall that we have chosen χ_δ nondecreasing):

$$\log\left(\frac{2\lambda}{1+4\lambda^2\delta^2}\right) \leq \varphi_{\lambda,\sigma} \leq \log\left(\frac{2\lambda}{1+\lambda^2\chi_\delta^2\left(d_{min}(y)\right)}\right),$$

where $d_{min}(y) = \mathrm{dist}(y, \{x_1\} \cup \cdots \cup \{x_k\})$. Recalling also that $\int_M \hat{v}_i dV_g = 0$, we write

$$\int_M \hat{v}_i(y) \varphi_{\lambda,\sigma}(y) dV_g(y) = \int_M \hat{v}_i \left(\varphi_{\lambda,\sigma}(y) - \log\frac{2\lambda}{1+4\lambda^2\delta^2}\right) dV_g.$$

Therefore we deduce that

$$\left|\int_M \hat{v}_i \varphi_{\lambda,\sigma} dV_g\right| \leq \|\hat{v}_i\|_{L^\infty(M)} \int_M \left(\varphi_{\lambda,\sigma}(y) - \log\frac{2\lambda}{1+4\lambda^2\delta^2}\right) dV_g(y)$$

$$\leq \|\hat{v}_i\|_{L^\infty(M)} \sum_{j=1}^{k} \int_{B_{2\delta}(x_j)} \left(\log\left(\frac{2\lambda}{1+\lambda^2\chi_\delta^2\left(d_j(y)\right)}\right) - \log\frac{2\lambda}{1+4\lambda^2\delta^2}\right) dV_g(y).$$

Working in geodesic coordinates around the point x_j we get

$$\int_{B_{2\delta}(x_j)} \left(\varphi_{\lambda,\sigma}(y) - \log\frac{2\lambda}{1+4\lambda^2\delta^2}\right) dV_g(y)$$
$$\leq C\int_0^\delta s^3 \left(\log\frac{2\lambda}{1+\lambda^2 s^2} - \log\frac{2\lambda}{1+4\lambda^2\delta^2}\right) ds + C\int_\delta^{2\delta} s^3 \log\frac{1+4\lambda^2\delta^2}{1+\lambda^2\chi_\delta^2(s)} ds.$$

Using elementary computations we then find

$$\left|\int_M \hat{v}_i(y)\varphi_{\lambda,\sigma}(y) dV_g(y)\right| \leq C\frac{1}{\lambda^4}\left[\lambda^4\delta^4 \log\frac{1+4\lambda^2\delta^2}{1+\lambda^2\delta^2} + \frac{1}{8}\lambda^4\delta^4\right] + C\delta^4 \leq C\delta^4,$$

which proves our claim (50). Notice that this expression is independent of λ: this will also be used at the end of the section. From the above formulas we obtain

$$\langle P_g(\varphi_s + \varphi_{\lambda,\sigma}), (\varphi_s + \varphi_{\lambda,\sigma})\rangle$$
$$\leq -|\lambda_{\overline{k}}||s|^2\overline{S}^2 + 32k\pi^2(1+o_\delta(1))\log\lambda + C_\delta + O(\delta^4|s|\overline{S}),$$

which concludes the proof of (42).

From the three estimates (40), (41) and (42) we deduce that

(51)
$$II(\varphi_{\lambda,\sigma}) \leq \big(32k\pi^2 - 4k_P + o_\delta(1)\big)\log\lambda - |\lambda_{\overline{k}}||s|^2\overline{S}^2 + O(|s|\overline{S}) + C_\delta + O(1).$$

Since $k_P > 8k\pi^2$, with δ sufficiently small, the coefficient of $\log\lambda$ is negative. In order to show the upper bound on $II \circ \Phi_{\overline{S},\overline{\lambda}}$, we fix $L > 0$. It is easy to see that for $\overline{S}$ sufficiently large one has

$$\begin{cases} II(\varphi_s + 2(1-\varphi_{1,\sigma})|s| + 2\varphi_{1,\sigma} - 1) \leq -L & \forall\sigma \in M_k, \forall|s| \geq \frac{1}{2}; \\ \\ II(\varphi_s + \varphi_{\overline{\lambda},\sigma}) \leq -L & \forall\sigma \in M_k, \forall|s| \in \left[\frac{1}{4},\frac{1}{2}\right], \forall\lambda \geq 1. \end{cases}$$

After this choice of $\overline{S}$, we can also take $\overline{\lambda}$ so large that

$$II(\varphi_s + \varphi_{\overline{\lambda},\sigma}) \leq -L, \qquad \forall|s| \leq \frac{1}{4}.$$

This concludes the proof of the lemma for $k \geq 1$. In the case $k_P < 8\pi^2$ and $\overline{k} \geq 1$, it is sufficient to use the estimates (44), (45) and (49) to obtain

$$II(\varphi_s) \leq -|\lambda_{\overline{k}}|\overline{S}^2 + O(\overline{S}|s|).$$

The proof is thereby complete. □

Proof of Proposition 4.1. The statement (a) follows from Lemma 4.3. Let us prove property (b), starting from the case $k \geq 1$. From the expression $e^{4\varphi_{\lambda,\sigma}}$ it is easy to see that $\Psi \circ \Phi_{0,\overline{\lambda}}$ is homotopic to the identity on M_k (to prove this, it is sufficient to consider $\Psi \circ \Phi_{0,\lambda}$ for λ varying from $\overline{\lambda}$ to $+\infty$). Furthermore, by continuity and by the estimate (50) one can check that for $|s| \leq \frac{1}{8\overline{S}}$ (if $\overline{S} > 1$ and if δ is chosen sufficiently small),

$$\Psi(\varphi_s + \varphi_{\overline{\lambda},\sigma}) = \Big(\hat{\Psi}(\varphi_s + \varphi_{\overline{\lambda},\sigma}), s\overline{S} + O(|s|\overline{S}\delta^4)\Big), \tag{52}$$

where $\hat{\Psi}$ is as defined in Lemma 3.10, and therefore $\Psi \circ \Phi_{\overline{S},\overline{\lambda}}$ is homotopic (in $A_{k,\overline{k}}$) to the identity on $M_k \times B^{\overline{k}}_{\frac{1}{8\overline{S}}} \subseteq A_{k,\overline{k}}$.

On the other hand, by (52), for $|s| \geq \frac{1}{8\overline{S}}$, the $\overline{k}$-vector $s\overline{S} + O(|s|\overline{S}\delta^4)$ is almost parallel to s (and nonzero), and therefore on this set $\Psi \circ \Phi_{\overline{S},\overline{\lambda}}$ can be easily contracted to the boundary of $B^{\overline{k}}_1$ (recall the definition of $A_{k,\overline{k}}$), as for the identity map. This concludes the proof in the case $k \geq 1$. The proof for $k_P < 8\pi^2$ and under the assumption (12) is analogous. □

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1 employing a min-max scheme based on the construction of the above set $A_{k,\overline{k}}$; see Lemma 5.1. As anticipated in the introduction, we then define a modified functional II_ρ for which we can prove existence of solutions in a dense set of the values of ρ. Following an idea of Struwe, this is done proving the a.e. differentiability of the map $\rho \mapsto \overline{\Pi}_\rho$, where $\overline{\Pi}_\rho$ is the min-max value for the functional II_ρ.

We now introduce the scheme which provides existence of solutions for (8), beginning with the case $k \geq 1$. Let $\widehat{A_{k,\overline{k}}}$ denote the (contractible) cone over $A_{k,\overline{k}}$, which can be represented as $\widehat{A_{k,\overline{k}}} = A_{k,\overline{k}} \times [0,1]$ with $A_{k,\overline{k}} \times \{0\}$ collapsed to a single point. Let L be so large that Proposition 3.1 applies with $\frac{L}{4}$, and then let $\overline{S}, \overline{\lambda}$ be so large (and δ so small) that Proposition 4.1 applies for this value of L. Fixing these numbers $\overline{S}$ and $\overline{\lambda}$, we define the following class:

$$\Pi_{\overline{S},\overline{\lambda}} = \Big\{ \pi : \widehat{A_{k,\overline{k}}} \to H^2(M) \ : \ \pi \text{ is continuous and } \pi(\cdot \times \{1\}) = \Phi_{\overline{S},\overline{\lambda}}(\cdot) \text{ on } A_{k,\overline{k}} \Big\}. \tag{53}$$

In the case $k_P < 8\pi^2$ and $\overline{k} \geq 1$ we simply use the closed unit $\overline{k}$-dimensional ball $\overline{B}_1^{\overline{k}}$ and we set (still for large values of L)

$$\Pi_{\overline{S}} = \left\{ \pi : \overline{B}_1^{\overline{k}} \to H^2(M) \ : \ \pi \text{ is continuous and } \pi(\cdot) = \Phi_{\overline{S}}(\cdot) \text{ on } S^{\overline{k}-1} \right\}.$$

Then we have the following properties.

Lemma 5.1. *The set* $\Pi_{\overline{S},\overline{\lambda}}$ (*resp.* $\Pi_{\overline{S}}$) *is nonempty and moreover*, *when*

$$\overline{\Pi}_{\overline{S},\overline{\lambda}} = \inf_{\pi \in \Pi_{\overline{S},\overline{\lambda}}} \sup_{m \in \widehat{A_{k,\overline{k}}}} II(\pi(m)), \qquad \overline{\Pi}_{\overline{S},\overline{\lambda}} > -\frac{L}{2},$$

$$\left(\text{resp. } \quad \overline{\Pi}_{\overline{S}} = \inf_{\pi \in \Pi_{\overline{S}}} \sup_{m \in \overline{B}_1^{\overline{k}}} II(\pi(m)), \qquad \overline{\Pi}_{\overline{S}} > -\frac{L}{2} \right).$$

Proof. To prove that $\Pi_{\overline{S},\overline{\lambda}} \neq \emptyset$, we just notice that the following map

$$\overline{\pi}(z,t) = t\Phi_{\overline{S},\overline{\lambda}}(z), \qquad (z,t) \in \widehat{A_{k,\overline{k}}}, \tag{54}$$

belongs to $\Pi_{\overline{S},\overline{\lambda}}$. Assuming by contradiction that $\overline{\Pi}_{\overline{S},\overline{\lambda}} \leq -\frac{L}{2}$, we see that there would exist a map $\pi \in \Pi_{\overline{S},\overline{\lambda}}$ with $\sup_{m \in \widehat{A_{k,\overline{k}}}} II(\pi(m)) \leq -\frac{3}{8}L$. Then, since Proposition 3.1 applies with $\frac{L}{4}$, with $m = (z,t)$, and $z \in A_{k,\overline{k}}$, the map

$$t \mapsto \Psi \circ \pi(\cdot, t)$$

would be a homotopy in $A_{k,\overline{k}}$ between $\Psi \circ \Phi_{\overline{S},\overline{\lambda}}$ and a constant map. But this is impossible since $A_{k,\overline{k}}$ is noncontractible (see Corollary 3.8) and since $\Psi \circ \Phi_{\overline{S},\overline{\lambda}}$

is homotopic to the identity on $A_{k,\overline{k}}$, by Proposition 4.1. Therefore we deduce $\overline{\Pi}_{\overline{S},\overline{\lambda}} > -\frac{L}{2}$.

In the case $k_P < 8\pi^2$ and $\overline{k} \geq 1$ it is sufficient to take $\overline{\pi}(z,t) = t\Phi_{\overline{S}}(z)$ and to proceed in the same way. □

Next we introduce a variant of the above min-max scheme, following [40] and [22]. When $k_P < 8\pi^2$, we define for convenience $A_{k,\overline{k}} = S^{\overline{k}}$, $\widehat{A_{k,\overline{k}}} = \overline{B}_1^{\overline{k}}$, $\Phi_{\overline{S},\overline{\lambda}} = \Phi_{\overline{S}}$, etc. For ρ in a small neighborhood of 1, $[1-\rho_0, 1+\rho_0]$, we define the modified functional $II_\rho : H^2(M) \to \mathbb{R}$ as

$$II_\rho(u) = \langle P_g u, u\rangle + 4\rho \int_M Q_g u - 4\rho k_P \log \int_M e^{4u} dV_g. \tag{55}$$

Following the estimates of the previous sections, one easily checks that the above min-max scheme applies uniformly for $\rho \in [1-\rho_0, 1+\rho_0]$ and for $\overline{S}$, $\overline{\lambda}$ sufficiently large. More precisely, given any large number $L > 0$, there exist ρ_0 sufficiently small and $\overline{S}$, $\overline{\lambda}$ sufficiently large so that

$$\sup_{\pi \in \Pi_{\overline{S},\overline{\lambda}}} \sup_{m \in \partial \widehat{A_{k,\overline{k}}}} II_\rho(\pi(m)) < -2L; \tag{56}$$

$$\overline{\Pi}_\rho := \inf_{\pi \in \Pi_{\overline{S},\overline{\lambda}}} \sup_{m \in \widehat{A_{k,\overline{k}}}} II(\pi(m)) > -\frac{L}{2}; \qquad \rho \in [1-\rho_0, 1+\rho_0],$$

where $\Pi_{\overline{S},\overline{\lambda}}$ is as defined in (53). Moreover, using for example the test map (54), one shows that for ρ_0 sufficiently small there exists a large constant $\overline{L}$ such that

$$\overline{\Pi}_\rho \leq \overline{L} \qquad \text{for every } \rho \in [1-\rho_0, 1+\rho_0]. \tag{57}$$

We have the following result, regarding the dependence in ρ of the min-max value $\overline{\Pi}_\rho$. A similar statement has been proved in [22], but here we allow the presence of negative eigenvalues for the elliptic operator, so that the proof is more involved. Since this is rather technical, we give it in the appendix.

LEMMA 5.2. *Let $\overline{S}$, $\overline{\lambda}$ be so large and ρ_0 be so small that* (56) *holds. Then, taking ρ_0 possibly smaller, there exists a fixed constant C* (*depending only on M and ρ_0*) *such that the function*

$$\rho \mapsto \frac{\overline{\Pi}_\rho}{\rho} - C\rho \qquad \textit{is nonincreasing in } [1-\rho_0, 1+\rho_0].$$

From Lemma 5.2 we deduce that the function $\rho \mapsto \frac{\overline{\Pi}_\rho}{\rho}$ is differentiable a.e., and we obtain the following corollary.

Corollary 5.3. *Let* $\overline{S}$, $\overline{\lambda}$ *and* ρ_0 *be as in Lemma* 5.2, *and let* $\Lambda \subset [1-\rho_0, 1+\rho_0]$ *be the (dense) set of* ρ *for which the function* $\frac{\overline{\Pi}_\rho}{\rho}$ *is differentiable. Then for* $\rho \in \Lambda$ *the functional* II_ρ *possesses a bounded Palais-Smale sequence* $(u_l)_l$ *at level* $\overline{\Pi}_\rho$.

Proof. The existence of a Palais-Smale sequence $(u_l)_l$ follows from Lemma 5.1, and the boundedness is proved exactly as in [22], Lemma 3.2. □

Remark 5.4. When $k_P < 8\pi^2$ one can use a direct approach to prove boundedness of Palais-Smale sequences (satisfying (14)). We test the relation $II'(u_l) \to 0$ (in $H^{-2}(M)$) on $\hat{u}_l$ and $\tilde{u}_l$, where $\hat{u}_l$ is the component of u_l in V and $\tilde{u}_l$ is the component perpendicular to V.

Testing on $\hat{u}_l$ we obtain

$$\langle P_g \hat{u}_l, \hat{u}_l \rangle + 4 \int_M Q_g \hat{u}_l dV_g - 4k_P \int_M e^{4u_l} \hat{u}_l dV_g = o_l(1) \|\hat{u}_l\|_{L^\infty(M)}. \tag{58}$$

Since $\|e^{4u_l}\|_{L^1(M)} = 1$ by (14) and since on V the L^∞-norm is equivalent to the H^2-norm, the last formula implies $-\langle P_g \hat{u}_l, \hat{u}_l \rangle = O(1) \|\hat{u}_l\|_{H^2(M)}$. Therefore, being P_g negative-definite on V, we get uniform bounds on $\|\hat{u}_l\|$.

On the other hand, testing the equation on $\tilde{u}_l$ we find

$$2\langle P_g \tilde{u}_l, \tilde{u}_l \rangle - 4k_P \int_M e^{4u_l} (\tilde{u}_l - \overline{u}_l) dV_g = O(\|\tilde{u}_l - \overline{u}_l\|_{H^2(M)}).$$

This implies, for any $\alpha > 1$ (by (23) and (58)),

$$\begin{aligned} 2\langle P_g \tilde{u}_l, \tilde{u}_l \rangle &\leq C e^{4\overline{u}_l} \int_M e^{4(u_l - \overline{u}_l)} (\tilde{u}_l - \overline{u}_l) dV_g + O(\|\tilde{u}_l - \overline{u}_l\|_{H^2(M)}) \\ &\leq C_\alpha e^{4\overline{u}_l} \int_M e^{4\alpha(u_l - \overline{u}_l)} dV_g + O(\|\tilde{u}_l - \overline{u}_l\|_{H^2(M)}) \\ &\leq C_\alpha e^{4\overline{u}_l} e^{\alpha^2 \frac{\langle P_g \tilde{u}_l, \tilde{u}_l \rangle}{8\pi^2}} + O(\|\tilde{u}_l - \overline{u}_l\|_{H^2(M)}). \end{aligned}$$

Moreover, since we are assuming $II(u_l) \to c \in \mathbb{R}$, for any small ε we get

$$C \geq II(\tilde{u}_l) = \langle P_g \tilde{u}_l, \tilde{u}_l \rangle + 4 \int_M Q_g \tilde{u}_l = (1 + O(\varepsilon)) \langle P_g \tilde{u}_l, \tilde{u}_l \rangle + 4k_P \overline{u}_l + C_\varepsilon,$$

provided l is sufficiently large. Hence from the last two formulas we deduce

$$\langle P_g \tilde{u}_l, \tilde{u}_l \rangle \leq C_{\alpha,\varepsilon} e^{\langle P_g \tilde{u}_l, \tilde{u}_l \rangle \left(\frac{\alpha^2}{8\pi^2} - \frac{1+O(\varepsilon)}{k_P} \right)} + O(\|\tilde{u}_l - \overline{u}_l\|_{H^2(M)}).$$

Now, choosing α close to 1 and ε so small that the exponential factor has a negative coefficient (this is always possible since $k_P < 8\pi^2$), we obtain a uniform bound for $\|\tilde{u}_l - \overline{u}_l\|$. The bound on $\overline{u}_l$ now follows easily from (14).

Now the proof of Theorem 1.1 is an easy consequence of the following proposition and of Theorem 1.3.

Proposition 5.5. *Suppose $(u_l)_l \subseteq H^2(M)$ is a sequence for which (as $l \to +\infty$)*

$$II_\rho(u_l) \to c \in \mathbb{R}; \qquad II'_\rho(u_l) \to 0; \qquad \|u_l\|_{H^2(M)} \leq C,$$

where C is independent of l. Then $(u_l)_l$ has a weak limit u_0 which satisfies (15).

Proof. The existence of a weak limit $u_0 \in H^2(M)$ follows from Corollary 5.3. Now, we show that u_0 satisfies $II'_\rho(u_0) = 0$. For any function $v \in H^2(M)$,

$$II'_\rho(u_0)[v] = II'_\rho(u_l)[v] + 2\langle P_g v, (u_0 - u_l)\rangle \\ +4\rho k_P \left(\frac{\int_M e^{4u_l} v dV_g}{\int_M e^{4u_l} dV_g} - \frac{\int_M e^{4u_0} v dV_g}{\int_M e^{4u_0} dV_g} \right).$$

Since the first two terms on the right-hand side tend to zero by our assumptions, it is sufficient to check that $\int_M e^{4u_l} v dV_g = \int_M e^{4u_0} v dV_g + o(1)\|v\|_{H^2(M)}$ (to deal with the denominators just take $v \equiv 1$). In order to do this, consider $p, p', p'' > 1$ satisfying $\frac{1}{p} + \frac{1}{p'} + \frac{1}{p''} = 1$. Using Lagrange's formula we obtain, for some function θ_l with range in $[0,1]$, $e^{4u_l} - e^{4u_0} = e^{4\theta_l u_l + 4(1-\theta_l)u_0}(u_l - u_0)$ a.e. in x. Then from some elementary inequalities we find

$$\begin{aligned}
&\int_M \left(e^{4u_l} - e^{4u_0}\right) v \, dV_g \\
&\quad \leq C \int_M \left(e^{4u_l} + e^{4u_0}\right) |u_l - u_0||v| dV_g \\
&\quad \leq C \left[\|e^{4u_l}\|_{L^p(M)} + \|e^{4u_0}\|_{L^p(M)}\right] \|u_l - u_0\|_{L^{p'}(M)} \|v\|_{L^{p''}(M)} \\
&\quad \leq o(1)\|v\|_{L^{p''}(M)} = o(1)\|v\|_{H^2(M)},
\end{aligned}$$

by (23), the boundedness of $(u_l)_l$ and the fact that $u_l \rightharpoonup u_0$ weakly in $H^2(M)$. □

6. Appendix

In this section we collect the most technical proofs of the paper, namely those of Lemmas 3.4, 4.2 and 5.2.

Proof of Lemma 3.4. By Corollary 3.3, we know that $M_j\left(\frac{\varepsilon}{4}\right)$ is a smooth finite-dimensional manifold. Therefore, if $\hat{\varepsilon}$ is sufficiently small, there exists a continuous projection P_j from $\hat{M}^{\hat{\varepsilon},\varepsilon}_{k,j}$ onto $M_j\left(\frac{\varepsilon}{2}\right)$ (whose closure lies in $M_j\left(\frac{\varepsilon}{4}\right)$). Since we regard M_k as a subset of $C^1(M)^*$, a Banach space, we cannot in general project elements in a neighborhood of $M_j\left(\frac{\varepsilon}{2}\right)$ onto their closest point in $M_j\left(\frac{\varepsilon}{2}\right)$ (this might not be unique). Nevertheless, using the Implicit Function Theorem and a partition of unity it is possible to define the projection in such a way that

(59) $$\mathrm{dist}(\sigma, P_j(\sigma)) \leq C_{k,\varepsilon}\mathrm{dist}(\sigma, M_j\,(\varepsilon)), \qquad \sigma \in \hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon},$$

where $C_{k,\varepsilon}$ is a constant depending only on k and ε (we are taking $1 \leq j \leq k-1$).

To fix some notation, we use the following convention:

$$\sigma = \sum_{i=1}^{k} t_i \delta_{x_i}; \qquad P_j(\sigma) = \sum_{i=1}^{j} s_i \delta_{y_i}.$$

By Lemma 3.2, since we are assuming that $P_j(\sigma)$ belongs to $M_j(\frac{\varepsilon}{2})$, we have the following estimates

$$s_i \geq \frac{\varepsilon}{4}, \qquad \mathrm{dist}(y_i, y_l) \geq \frac{\varepsilon}{4}; \qquad \forall\ i, l = 1, \dots, s, i \neq l.$$

Moreover the points y_i and the numbers s_i depend continuously on σ.

We define first an auxiliary map $\tilde{T}_j^t$, $\tilde{T}_j^t(\sigma) = \sum \tilde{t}_i \delta_{\tilde{x}_i}$, which misses the normalization condition $\sum_{i=1}^k \tilde{t}_i = 1$, but only up to a small error. This map will then be corrected to the real T_j^t. The idea to construct $\tilde{T}_j^t$ is the following. If a point x_i is far from each y_l, we keep this point fixed and let its coefficient vanish to zero as t varies from 0 to 1. On the other hand, if x_i is close to some of the y_l's, then we translate it to a *weighted convex combination* of the points x_i which are close to the same y_l.

To make this construction rigorous (and the map $\tilde{T}_j^t$ continuous), we consider a small number $\eta \ll \varepsilon$ (this will be chosen later of order $C_{k,\varepsilon}\sqrt{\hat{\varepsilon}}$) (where $C_{k,\varepsilon}$ depends only on k and ε), and define a smooth cutoff function ρ_η satisfying the following properties

(60) $$\begin{cases} \rho_\eta(t) = 1, & \text{for } t \leq \frac{\eta}{16}; \\ \rho_\eta(t) = 0, & \text{for } t \geq \frac{\eta}{8}; \\ \rho_\eta(t) \in [0,1], & \text{for every } t \geq 0. \end{cases}$$

Then we set

(61) $$\rho_{l,\eta}(x) = \rho_\eta(\mathrm{dist}(x, y_l)); \qquad \text{for } l = 1, \dots, j.$$

We define also the following quantities

$$\mathcal{T}_l(\sigma) = \sum_{x_i \in B_{\frac{\eta}{8}}(y_l)} \rho_{l,\eta}(x_i) t_i;$$

$$\mathcal{X}_l(\sigma) = \frac{1}{\mathcal{T}_l(\sigma)} \sum_{x_i \in B_{\frac{\eta}{8}}(y_l)} \rho_{l,\eta}(x_i) t_i x_i, \qquad l = 1, \dots, j,$$

and notice that, if η is chosen sufficiently small, the weighted convex combination $\mathcal{X}_l(\sigma)$ is well-defined; see the notation in Section 2. We also set

$$z_i(\sigma) = \frac{8}{\eta}\mathrm{dist}(x_i, y_l) - 1, \qquad \text{for } x_i \in B_{\frac{\eta}{4}}(y_l).$$

Since for all $i \neq l$, $\mathrm{dist}(y_i, y_l) \geq \frac{\varepsilon}{4}$ and since $\eta << \varepsilon$, then for every i there exists at most one point y_l such that $x_i \in B_{\frac{\eta}{4}}(y_l)$. Hence the number $z_i(\sigma)$ is well defined. Now we construct the map $\tilde{T}_j^t(\sigma)$ as follows:

$$\tilde{T}_j^t(\sigma) = \sum_{i=1}^{k} \tilde{t}_i(\sigma, t)\delta_{\tilde{x}_i(\sigma,t)},$$

where the numbers $\tilde{t}_i(\sigma, t)$ and the points $\tilde{x}_i(\sigma, t)$ are given by

$$\begin{array}{ll} \tilde{t}_i(\sigma, t) = (1-t)t_i \text{ and } \tilde{x}_i(\sigma, t) = x_i & \text{if } x_i \in M \setminus \cup_l B_{\frac{\eta}{4}}(y_l); \\ \tilde{t}_i(\sigma, t) = (1-t)t_i \text{ and} & \\ \tilde{x}_i(\sigma, t) = (1-t)x_i + t[z_i(\sigma)x_i + (1-z_i(\sigma))\mathcal{X}_l(\sigma)] & \text{if } x_i \in B_{\frac{\eta}{4}}(y_l) \setminus B_{\frac{\eta}{8}}(y_l); \\ \tilde{t}_i(\sigma, t) = ((1-t) + t\rho_{l,\eta}(x_i))t_i \text{ and} & \\ \tilde{x}_i(\sigma, t) = (1-t)x_i + t\mathcal{X}_l(\sigma) & \text{if } x_i \in B_{\frac{\eta}{8}}(y_l). \end{array}$$

As already mentioned, the numbers $\tilde{t}_i(\sigma, t)$ will in general miss the condition $\sum_i \tilde{t}_i(\sigma, t) = 1$. The next step consists in estimating this sum and correcting the map $\tilde{T}_j^t(\sigma)$ in order to match this condition. For this purpose it is convenient to define

$$\tilde{\mathcal{T}}_l(\sigma, t) = \sum_{x_i \in B_{\frac{\eta}{8}}(y_l)} \tilde{t}_i(\sigma, t); \qquad \tilde{\mathcal{T}}(\sigma, t) = 1 - \sum_{l=1}^{j} \tilde{\mathcal{T}}_l(\sigma, t).$$

Now, finally,

$$T_j^t(\sigma) = \frac{1}{(1-t)\tilde{\mathcal{T}}(\sigma, 0) + \sum_{l=1}^{j} \tilde{\mathcal{T}}_l(\sigma, t)} \sum_{i=1}^{k} \tilde{t}_i(\sigma, t)\delta_{\tilde{x}_i(\sigma,t)}. \tag{62}$$

We notice that the sum of all the coefficients is 1, and that the map is well defined and continuous in both t and σ. We also notice that the properties (i), (iv) and (v) are satisfied, while (ii) follows from (iii). Therefore it only remains to prove (iii). First of all we give an estimate on the quantities $\tilde{\mathcal{T}}_l(\sigma, t)$ and $\tilde{\mathcal{T}}(\sigma, t)$.

We recall that we have taken $\sigma \in \hat{M}_{k,j}^{\hat{\varepsilon},\varepsilon}$, and hence by (59) for any function $f \in C^1(M)$ with $\|f\|_{C^1(M)} \leq 1$ one has $|(\sigma - P_j(\sigma), f)| \leq C_{k,\varepsilon}\hat{\varepsilon}$. We now choose a function f satisfying the following properties

$$f(x) = \begin{cases} \frac{1}{2} & \text{for } x \in \cup_{l=1}^{j} B_{\frac{\eta}{48}}(y_l); \\ \frac{1}{2} + \frac{\eta}{32} & \text{for } x \in M \setminus \cup_{l=1}^{j} B_{\frac{\eta}{16}}(y_l); \\ \|f\|_{C^1(M)} \leq 1. & \end{cases}$$

For this function, $(P_j(\sigma), f) = \sum_{i=1}^{j} s_i f(y_i) = \frac{1}{2}$ and moreover

$$\begin{aligned}(\sigma, f) &= \sum_{x_i \in \cup_{l=1}^{j} B_{\frac{\eta}{16}}(y_l)} t_i f(x_i) + \sum_{x_i \in M \setminus \cup_{l=1}^{j} B_{\frac{\eta}{16}}(y_l)} t_i f(x_i) \\ &\geq \frac{1}{2} \sum_{x_i \in \cup_{l=1}^{j} B_{\frac{\eta}{16}}(y_l)} t_i + \left(\frac{1}{2} + \frac{\eta}{32}\right) \sum_{x_i \in M \setminus \cup_{l=1}^{j} B_{\frac{\eta}{16}}(y_l)} t_i.\end{aligned}$$

Therefore we deduce the following inequality:

$$\frac{\eta}{32} \sum_{x_i \in M \setminus \cup_{l=1}^{j} B_{\frac{\eta}{16}}(y_l)} t_i \leq (\sigma, f) - (P_j(\sigma), f) \leq C_{k,\varepsilon} \hat{\varepsilon}.$$

This estimate implies

$$\tilde{\mathcal{T}}(\sigma, 0) = \sum_{x_i \in M \setminus \cup_{l=1}^{j} B_{\frac{\eta}{16}}(y_l)} t_i \leq 32 \frac{C_{k,\varepsilon} \hat{\varepsilon}}{\eta},$$

and also (since $\rho_{l,\eta} \equiv 1$ in $B_{\frac{\eta}{16}}(y_l)$),

$$\begin{aligned}\tilde{\mathcal{T}}_l(\sigma, t) &= \sum_{x_i \in B_{\frac{\eta}{8}}(y_l) \setminus B_{\frac{\eta}{16}}(y_l)} \left((1-t) + t\rho_{l,\eta}(x_i)\right) t_i \\ &\quad + \sum_{x_i \in B_{\frac{\eta}{16}}(y_l)} \left((1-t) + t\rho_{l,\eta}(x_i)\right) t_i \\ &= \tilde{\mathcal{A}}_l(\sigma, t) + \sum_{x_i \in B_{\frac{\eta}{16}}(y_l)} t_i, \qquad \text{where} \quad \sum_{l=1}^{j} |\tilde{\mathcal{A}}_l(\sigma, t)| \leq 32 \frac{C_{k,\varepsilon} \hat{\varepsilon}}{\eta}.\end{aligned}$$

Hence, since $\sum_{l=1}^{j} \tilde{\mathcal{T}}_l(\sigma, 0) + \tilde{\mathcal{T}}(\sigma, 0) = 1$,

$$1 = \sum_{x_i \in \cup_{l=1}^{j} B_{\frac{\eta}{16}}(y_l)} t_i + \sum_{l=1}^{j} \tilde{\mathcal{A}}_l(\sigma, 0) + \sum_{x_i \in M \setminus \cup_{l=1}^{j} B_{\frac{\eta}{8}}(y_l)} t_i,$$

from which we deduce

$$\begin{aligned}&\left| \sum_{l=1}^{j} \tilde{\mathcal{T}}_l(\sigma, t) + (1-t)\tilde{\mathcal{T}}(0) - 1 \right| \\ &\qquad = \left| \left(\sum_{l=1}^{j} \left(\tilde{\mathcal{A}}_l(\sigma, t) - \tilde{\mathcal{A}}_l(\sigma, 0) \right) \right) + (1-t) \sum_{x_i \in M \setminus \cup_{l=1}^{j} B_{\frac{\eta}{8}}(y_l)} t_i \right| \\ &\qquad \leq 64 \frac{C_{k,\varepsilon} \hat{\varepsilon}}{\eta} + 32 \frac{C_{k,\varepsilon} \hat{\varepsilon}}{\eta} = 96 \frac{C_{k,\varepsilon} \hat{\varepsilon}}{\eta}.\end{aligned}$$

As a consequence, using a Taylor expansion (recall that $\frac{C_{k,\varepsilon}\hat{\varepsilon}}{\eta} \ll 1$), we find that the coefficient added in the definition of T_j^t, see (62), can be estimated by

$$\left| \frac{1}{\sum_{l=1}^{j} \tilde{T}_l(\sigma,t) + (1-t)\tilde{T}(0)} - 1 \right| \leq 100 \frac{C_{k,\varepsilon}\hat{\varepsilon}}{\eta}.$$

To control the metric distance in (iii), we use the last formula to get, for an arbitrary function $f \in C^1(M)$ with $\|f\|_{C^1(M)} \leq 1$,

$$\begin{aligned} (63) \quad \left|(\sigma,f) - (T_j^t(\sigma),f)\right| &\leq \left|(\sigma,f) - (\tilde{T}_j^t(\sigma),f)\right| + \left|(\tilde{T}_j^t(\sigma),f) - (T_j^t(\sigma),f)\right| \\ &\leq \left|(\sigma,f) - (\tilde{T}_j^t(\sigma),f)\right| + 100\frac{C_{k,\varepsilon}\hat{\varepsilon}}{\eta}. \end{aligned}$$

Hence it is sufficient to estimate the distance between σ and $\tilde{T}_j^t(\sigma)$. We can write

$$\begin{aligned} \left|(\sigma,f) - (\tilde{T}_j^t(\sigma),f)\right| \leq & \sum_{x_i \in M \setminus \cup_{l=1}^j B_{\frac{\eta}{4}}(y_l)} t_i \\ & + \sum_{x_i \in \cup_{l=1}^j (B_{\frac{\eta}{4}}(y_l) \setminus B_{\frac{\eta}{16}}(y_l))} \left|t_i f(x_i) - \tilde{t}_i(\sigma,t) f(\tilde{x}_i(\sigma,t))\right| \\ & + \sum_{x_i \in \cup_{l=1}^j B_{\frac{\eta}{16}}(y_l)} t_i \mathrm{dist}(x_i, \tilde{x}_i(\sigma,t)). \end{aligned}$$

Since $\left|t_i f(x_i) - \tilde{t}_i(\sigma,t) f(\tilde{x}_i(\sigma,t))\right| \leq |t_i - \tilde{t}_i(\sigma,t)| + \tilde{t}_i(\sigma,t)\mathrm{dist}(x_i, \tilde{x}_i(\sigma,t)) \leq 2t_i$ (for η small), we obtain

$$\begin{aligned} \left|(\sigma,f) - (\tilde{T}_j^t(\sigma),f)\right| &\leq 2 \sum_{x_i \in M \setminus \cup_{l=1}^j B_{\frac{\eta}{16}}(y_l)} t_i + \sum_{l=1}^{j} \sum_{x_i \in B_{\frac{\eta}{16}}(y_l)} t_i \mathrm{dist}(x_i, \mathcal{X}_l(\sigma)) \\ &\leq 64 \frac{C_{k,\varepsilon}\hat{\varepsilon}}{\eta} + \sum_{l=1}^{j} \sum_{x_i \in B_{\frac{\eta}{16}}(y_l)} t_i \mathrm{dist}(x_i, \mathcal{X}_l(\sigma)). \end{aligned}$$

In order to estimate the last term, we notice that each point x_i in the homotopy is shifted at most by $\frac{\eta}{2}$; see the comments at the beginning of Section 2. Therefore from (63) and the last expression we get

$$\left|(\sigma,f) - (T_j^t(\sigma),f)\right| \leq 170 \frac{C_{k,\varepsilon}\hat{\varepsilon}}{\eta} + \frac{\eta}{2}.$$

Therefore, choosing $\eta = C_{k,\varepsilon}\sqrt{\hat{\varepsilon}}$, we obtain the desired conclusion. □

Proof of Lemma 4.2. For simplicity, see Section 4, we adopt again the notation $d_i = d_i(y) = \mathrm{dist}(y, x_i)$, $i = 1, \dots, k$, and we consider these as functions of y, for $(x_i)_i$ fixed. By (39) with some straightforward computations we find

$$\nabla \varphi_{\lambda,\sigma} = -\lambda^2 (2\lambda)^4 \frac{\sum_{i=1}^k t_i \nabla(\chi_\delta^2(d_i))(1+\lambda^2\chi_\delta^2(d_i))^{-5}}{\sum_{s=1}^k t_s \left(\frac{2\lambda}{1+\lambda^2\chi_\delta^2(d_s)}\right)^4}, \tag{64}$$

and

$$\begin{aligned} \Delta\varphi_{\lambda,\sigma} &= \lambda^2(2\lambda)^4 \\ &\times \frac{\sum_{i=1}^k t_i (1+\lambda^2\chi_\delta^2(d_i))^{-6}\left[5\lambda^2|\nabla(\chi_\delta^2(d_i))|^2 - \Delta(\chi_\delta^2(d_i))(1+\lambda^2\chi_\delta^2(d_i))\right]}{\sum_{s=1}^k t_s \left(\frac{2\lambda}{1+\lambda^2\chi_\delta^2(d_s)}\right)^4} \\ &-4\lambda^4(2\lambda)^8 \frac{\sum_{i,s=1}^k t_i t_s (1+\lambda^2\chi_\delta^2(d_i))^{-5}(1+\lambda^2\chi_\delta^2(d_s))^{-5}\nabla(\chi_\delta^2(d_i))\cdot\nabla(\chi_\delta^2(d_s))}{\left[\sum_{r=1}^k t_r \left(\frac{2\lambda}{1+\lambda^2\chi_\delta^2(d_r)}\right)^4\right]^2}. \end{aligned} \tag{65}$$

We begin by estimating $\int_M (\Delta\varphi_{\lambda,\sigma})^2 dV_g$. This is the most involved part of the proof, and the result is given in formula (88) below. We notice first that the following pointwise estimate holds true, as one can easily check using (65):

$$|\Delta\varphi_{\lambda,\sigma}| \leq \frac{C}{\lambda^2}.$$

For a large but fixed constant $\Theta > 0$ (and for $\lambda \to +\infty$), the volume of a ball in M of radius $\frac{\Theta}{\lambda}$ is bounded by $C\frac{\Theta^4}{\lambda^4}$. From this we deduce that

$$\int_{\cup_{i=1}^k B_{\frac{\Theta}{\lambda}}(x_i)} (\Delta\varphi_{\lambda,\sigma})^2 dV_g \leq C\Theta^4. \tag{66}$$

Therefore we just need to estimate the integral on the complement of the union of these balls, which we denote by

$$M_{\sigma,\Theta} = M \setminus \cup_{i=1}^k B_{\frac{\Theta}{\lambda}}(x_i). \tag{67}$$

In this set, since we are taking Θ large, the ratio between $1+\lambda^2 d_i^2$ and $\lambda^2 d_i^2$ is very close to 1, and hence we obtain the following estimates

$$(1+\lambda^2\chi_\delta^2(d_i)) = (1+o_{\delta,\Theta}(1))\lambda^2\chi_\delta^2(d_i) \qquad \text{in } M_{\sigma,\Theta}; \tag{68}$$

$$\begin{aligned} 5\lambda^2|\nabla(\chi_\delta^2(d_i))|^2 &- \Delta(\chi_\delta^2(d_i))(1+\lambda^2\chi_\delta^2(d_i)) \\ &= 12(1+o_{\delta,\Theta}(1))\lambda^2\tilde{\chi}_\delta^2(d_i) \qquad \text{in } M_{\sigma,\Theta}, \end{aligned} \tag{69}$$

where $o_{\delta,\Theta}(1)$ tends to zero as δ tends to zero and Θ tends to infinity, and where $\tilde{\chi}_\delta$ is a new cutoff function (which depends on χ_δ) satisfying

$$\begin{cases} \tilde{\chi}_\delta(t) = t, & \text{for } t \in [0,\delta]; \\ \tilde{\chi}_\delta(t) = 0, & \text{for } t \geq 2\delta; \\ \tilde{\chi}_\delta(t) \in [0, 2\delta], & \text{for } t \in [\delta, 2\delta]. \end{cases} \tag{70}$$

Using (65), (68) and (69) one finds that the following estimate holds

$$
(71) \quad \Delta\varphi_{\lambda,\delta} = 12(1+o_{\delta,\Theta}(1)) \frac{\sum_{i=1}^{k} t_i \frac{\tilde{\chi}_\delta^2(d_i)}{\chi_\delta^{12}(d_i)}}{\sum_{s=1}^{k} \frac{t_s}{\chi_\delta^8(d_s)}}
$$

$$
-4(1+o_{\delta,\Theta}(1)) \frac{\sum_{i,s=1}^{k} t_i t_s \frac{\nabla(\chi_\delta^2(d_i))\cdot\nabla(\chi_\delta^2(d_s))}{\chi_\delta^{10}(d_i)\chi_\delta^{10}(d_s)}}{\left[\sum_{r=1}^{k} t_r \frac{1}{\chi_\delta^8(d_r)}\right]^2}
$$

$$
+o_{\delta,\Theta}(1) \frac{\sum_{i,s=1}^{k} t_i t_s \frac{|\nabla(\chi_\delta^2(d_i))|\,|\nabla(\chi_\delta^2(d_s))|}{\chi_\delta^{10}(d_i)\chi_\delta^{10}(d_s)}}{\left[\sum_{r=1}^{k} t_r \frac{1}{\chi_\delta^8(d_r)}\right]^2} \qquad \text{in } M_{\sigma,\Theta}.
$$

To further simplify the last expression, it is convenient to get rid of the cutoff functions χ_δ and $\tilde{\chi}_\delta$. In order to do this, we divide the set of points $\{x_1, \dots, x_k\}$ in a suitable way. Since the number k is fixed, there exists $\hat{\delta}$ and sets $\mathcal{C}_1, \dots, \mathcal{C}_j$, $j \leq k$, with the following properties

$$
(72) \qquad \begin{cases} C_k^{-1}\delta \leq \hat{\delta} \leq \frac{\delta}{16}; & \\ \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_j = \{x_1, \dots, x_k\}; & \\ \operatorname{dist}(x_i, x_s) \leq \hat{\delta} & \text{if } x_i, x_s \in \mathcal{C}_a; \\ \operatorname{dist}(x_i, x_s) > 4\hat{\delta} & \text{if } x_i \in \mathcal{C}_a, x_s \in \mathcal{C}_b, a \neq b, \end{cases}
$$

where C_k is a positive constant depending only on k. Now we define

$$
\hat{\mathcal{C}}_a = \left\{ y \in M \;:\; \operatorname{dist}(y, \mathcal{C}_a) \leq 2\hat{\delta} \right\}; \qquad T_a = \sum_{x_i \in \mathcal{C}_a} t_i, \qquad \text{for } a = 1, \dots, j.
$$

By the definition of $\hat{\delta}$ it follows that

$$
(73) \qquad \chi_\delta(d_i(y)) = \tilde{\chi}_\delta(d_i(y)) = d_i(y), \qquad \text{for } x_i \in \mathcal{C}_a \text{ and } y \in \hat{\mathcal{C}}_a,
$$

and

$$
(74) \qquad \chi_\delta(d_i(y)) \geq 2\hat{\delta}, \qquad \text{for } x_i \in \mathcal{C}_a \text{ and } y \notin \hat{\mathcal{C}}_a.
$$

Furthermore,

$$
(75) \qquad \hat{\mathcal{C}}_a \cap \hat{\mathcal{C}}_b = \emptyset \text{ for } a \neq b.
$$

From (71) and (74) it follows that

$$
(76) \qquad |\Delta\varphi_{\lambda,\sigma}(y)| \leq C_{\hat{\delta}} \qquad \text{for } y \in M \setminus \cup_{a=1}^{j} \hat{\mathcal{C}}_a.
$$

Therefore, by (75), it is sufficient to estimate $\Delta\varphi_{\lambda,\sigma}$ inside each set $\hat{\mathcal{C}}_a$, where (73) holds. We obtain immediately the following two estimates, regarding the first terms in (71),

$$\sum_{i=1}^{k} t_i \frac{\tilde{\chi}_\delta^2(d_i)}{\chi_\delta^{12}(d_i)} = \sum_{x_i \in \hat{\mathcal{C}}_a} \frac{t_i}{d_i^{10}} + O((1-T_a)\hat{\delta}^{-10}); \tag{77}$$

$$\sum_{s=1}^{k} \frac{t_s}{\chi_\delta^{8}(d_s)} = \sum_{x_s \in \hat{\mathcal{C}}_a} \frac{t_s}{d_s^{8}} + \overline{O}((1-T_a)\hat{\delta}^{-8}) \quad \text{in } \hat{\mathcal{C}}_a.$$

Here we have used the symbol $\overline{O}$ to denote a quantity such that

$$\overline{O}(t) \geq C^{-1} t,$$

where C is a large but fixed positive constant (which depends on k, M, but not on $\delta, \hat{\delta}$, λ and $(x_i)_i$). The same dependence on the constants is understood when we use the symbol O when it has as argument $(1-T_a)$, or its powers.

To estimate the second and the third term on the right-hand side of (71), we use geodesic coordinates centered at some point $y_a \in \hat{\mathcal{C}}_a$. With an abuse of notation, we identify the points in $\mathcal{C}_a$ with their pre-image under the exponential map. Using these coordinates, we find

$$\nabla(d_i(y))^2 = 2(y-x_i) + o_\delta(1)|y-x_i|, \qquad \text{for } y \in \hat{\mathcal{C}}_a, \text{ and for } x_i \in \mathcal{C}_a,$$

which implies

$$\frac{\nabla(\chi_\delta^2(d_i)) \cdot \nabla(\chi_\delta^2(d_s))}{\chi_\delta^{10}(d_i)\chi_\delta^{10}(d_s)} = 4\frac{(y-x_i)\cdot(y-x_s)}{d_i^{10} d_s^{10}} + o_\delta(1)\frac{1}{d_i^9 d_s^9};$$
$$\text{for } y \in \hat{\mathcal{C}}_a \text{ and for } x_i, x_s \in \mathcal{C}_a.$$

In particular, for $y \in \hat{\mathcal{C}}_a$, we get

$$\begin{aligned} \sum_{i,s=1}^{k} t_i t_s \frac{\nabla(\chi_\delta^2(d_i)) \cdot \nabla(\chi_\delta^2(d_s))}{\chi_\delta^{10}(d_i)\chi_\delta^{10}(d_s)} = {} & 4 \sum_{x_i, x_s \in \mathcal{C}_a} t_i t_s \frac{(y-x_i)\cdot(y-x_s)}{d_i^{10} d_s^{10}} \\ & + o_\delta(1) \sum_{x_i, x_s \in \mathcal{C}_a} \frac{t_i t_s}{d_i^9 d_s^9} + O((1-T_a)\hat{\delta}^{-9}) \sum_{x_i \in \mathcal{C}_a} \frac{t_i}{d_i^9} \\ & + O((1-T_a)^2 \hat{\delta}^{-18}). \end{aligned} \tag{78}$$

We have also (still for $y \in \hat{\mathcal{C}}_a$)

$$\begin{aligned} \sum_{i,s=1}^{k} t_i t_s \frac{|\nabla(\chi_\delta^2(d_i))|\,|\nabla(\chi_\delta^2(d_s))|}{\chi_\delta^{10}(d_i)\chi_\delta^{10}(d_s)} \leq {} & 4 \sum_{x_i, x_s \in \mathcal{C}_a} t_i t_s \frac{(y-x_i)\cdot(y-x_s)}{d_i^{10} d_s^{10}} \\ & + o_\delta(1) \sum_{x_i, x_s \in \mathcal{C}_a} \frac{t_i t_s}{d_i^9 d_s^9}. \end{aligned} \tag{79}$$

Hence from (71), (77), (78) and (79) we deduce (still working in the above coordinates)

$$\begin{aligned}\Delta\varphi_{\lambda,\sigma}(y) = & \, 12(1+o_{\delta,\Theta}(1))\frac{\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{10}}+O((1-T_a)\hat\delta^{-10})}{\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{8}}+\overline{O}((1-T_a)\hat\delta^{-8})}\\ & -16(1+o_{\delta,\Theta}(1))\frac{\left|\sum_{x_i\in\mathcal{C}_a}\frac{t_i(y-x_i)}{d_i^{10}}\right|^2+o_\delta(1)\left|\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{9}}\right|^2}{\left[\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{8}}+\overline{O}((1-T_a)\hat\delta^{-8})\right]^2}\\ & +\frac{O((1-T_a)\hat\delta^{-9})\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{9}}+O((1-T_a)^2\hat\delta^{-18})}{\left[\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{8}}+\overline{O}((1-T_a)\hat\delta^{-8})\right]^2}; \qquad y\in\hat{\mathcal{C}}_a.\end{aligned}$$

Using the inequality $ab\le\varepsilon a^2+C_\varepsilon b^2$ with $a=(1-T_a)\hat\delta^{-9}$ and $b=\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^9}$ we then find

$$\begin{aligned}\Delta\varphi_{\lambda,\sigma}(y) = & \,(1+o_{\delta,\Theta}(1))\left[12\frac{\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{10}}}{\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{8}}+\overline{O}((1-T_a)\hat\delta^{-8})}\right.\\ & \left.-16\frac{\left|\sum_{x_i\in\mathcal{C}_a}\frac{t_i(y-x_i)}{d_i^{10}}\right|^2}{\left[\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{8}}+\overline{O}((1-T_a)\hat\delta^{-8})\right]^2}\right]\\ & +\frac{(o_\delta(1)+O(\varepsilon))\left|\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{9}}\right|^2}{\left[\sum_{x_i\in\mathcal{C}_a}\frac{t_i}{d_i^{8}}+\overline{O}((1-T_a)\hat\delta^{-8})\right]^2}\\ & +O(C_\varepsilon+1)(1-T_a)^2\hat\delta^{-2}; \quad y\in\hat{\mathcal{C}}_a.\end{aligned}\tag{80}$$

Now, given a large and fixed constant $\overline{C}$, we define the set $\mathcal{B}_a^{\overline{C}}$ by

$$\mathcal{B}_a^{\overline{C}}=\Big\{y\in\hat{\mathcal{C}}_a\cap M_{\sigma,\Theta}\text{ s.t. if } x_i\in\mathcal{C}_a \text{ then } d_i(y) \le\left(1+\frac{1}{\overline{C}}\right)\operatorname{dist}(y,\mathcal{C}_a)\text{ or } d_i(y)\ge\overline{C}\operatorname{dist}(y,\mathcal{C}_a)\Big\}.$$

We start by characterizing the points belonging to the complement of $\mathcal{B}_a^{\overline{C}}$ in $M_{\sigma,\Theta}\cap\hat{\mathcal{C}}_a$. By definition, we have

$$\begin{aligned} & y\in\left(M_{\sigma,\Theta}\cap\hat{\mathcal{C}}_a\right)\setminus\mathcal{B}_a^{\overline{C}}\\ & \Rightarrow\qquad\text{there exists } x_i\in\mathcal{C}_a \text{ such that } d_i(y)\in\left(1+\frac{1}{\overline{C}},\overline{C}\right)\operatorname{dist}(y,\mathcal{C}_a).\end{aligned}\tag{81}$$

Given $y\in\left(M_{\sigma,\Theta}\cap\hat{\mathcal{C}}_a\right)\setminus\mathcal{B}_a^{\overline{C}}$, we let $x_{\bar\imath}$ denote one of its closest points in $\mathcal{C}_a$, and let $x_{\bar\jmath}$ denote one of the closest points in $\mathcal{C}_a$ to y, among those which do not realize the infimum of the distance from y. Then, since $\mathrm{dist}(y, x_{\bar{i}}) < \mathrm{dist}(y, x_{\bar{j}})$ and since $\mathrm{dist}(y, x_{\bar{j}}) < \overline{C}\mathrm{dist}(y, x_{\bar{i}})$ (by (81)), we clearly have

$$\frac{1}{\overline{C}}\mathrm{dist}(y, x_{\bar{j}}) < \mathrm{dist}(y, x_{\bar{i}}) < \mathrm{dist}(y, x_{\bar{j}}),$$

that is, y lies in an annulus centered at $x_{\bar{i}}$ whose radii have a ratio equal to $\overline{C}$.

Now, fixing $x_{\bar{i}} \in \mathcal{C}_a$, we consider the following set

$$\mathcal{D}_{\bar{i}} = \left\{ y \in \left(M_{\sigma,\Theta} \cap \hat{\mathcal{C}}_a \right) \setminus \mathcal{B}_a^{\overline{C}} \ : \ d_i(y) = \mathrm{dist}(y, \mathcal{C}_a) \right\},$$

namely the points y in $\left(M_{\sigma,\Theta} \cap \hat{\mathcal{C}}_a \right) \setminus \mathcal{B}_a^{\overline{C}}$ for which $x_{\bar{i}}$ is the closest point to y in $\mathcal{C}_a$. Now, when y varies, there might be different points $x_{\bar{j}}$, chosen as before, which do not realize the distance from y, but anyway their number never exceeds k. This implies that $\mathcal{D}_{\bar{i}}$ is contained in the union of at most k annuli centered at $x_{\bar{i}}$ whose radii $c_{l,\bar{i}}, d_{l,\bar{i}}$ have uniformly bounded ratios, namely

$$\mathcal{D}_{\bar{i}} \subseteq \cup_{l=1}^{k} \left(B_{d_{l,\bar{i}}}(x_{\bar{i}}) \setminus B_{c_{l,\bar{i}}}(x_{\bar{i}}) \right), \qquad \text{with } d_{l,\bar{i}} \leq 2\overline{C} c_{l,\bar{i}}. \tag{82}$$

Clearly we also have

$$\left(M_{\sigma,\Theta} \cap \hat{\mathcal{C}}_a \right) \setminus \mathcal{B}_a^{\overline{C}} = \bigcup_{x_{\bar{i}} \in \mathcal{C}_a} \mathcal{D}_{\bar{i}}. \tag{83}$$

In $\mathcal{D}_{\bar{i}}$,

$$\frac{t_i}{d_i^{10}} \leq \frac{1}{d_{\bar{i}}^2} \frac{t_i}{d_i^8}; \qquad \left| \sum_{x_i \in \mathcal{C}_a} \frac{t_i(y - x_i)}{d_i^{10}} \right| \leq \frac{1}{d_{\bar{i}}} \sum_{x_i \in \mathcal{C}_a} \frac{t_i}{d_i^8}.$$

Then from (80) it follows that

$$|\Delta \varphi_{\lambda,\sigma}| \leq C_{\delta,\Theta,\varepsilon} \left(1 + \frac{1}{d_{\bar{i}}^2} \right), \qquad \text{in } \mathcal{D}_{\bar{i}}. \tag{84}$$

Hence, from (82) and (83), using polar coordinates, we deduce that

$$\begin{aligned} &\int_{\left(M_{\sigma,\Theta} \cap \hat{\mathcal{C}}_a\right) \setminus \mathcal{B}_a^{\overline{C}}} (\Delta \varphi_{\lambda,\sigma})^2 dV_g \\ &\quad \leq \cup_{x_{\bar{i}} \in \mathcal{C}_a} \cup_{l=1}^{k} \int_{\left(B_{d_{l,\bar{i}}}(x_{\bar{i}}) \setminus B_{c_{l,\bar{i}}}(x_{\bar{i}})\right)} C_{\delta,\Theta,\varepsilon} \left(1 + \frac{1}{d_{\bar{i}}^2} \right)^2 dV_g \\ &\quad \leq C_{\delta,\Theta,\varepsilon} \, k \, \mathrm{card}(\mathcal{C}_a) \left(\log \frac{d_{l,\bar{i}}}{c_{l,\bar{i}}} + 1 \right) \leq C_{\delta,\Theta,\varepsilon,\overline{C}}. \end{aligned} \tag{85}$$

At this point, to estimate $\int_M (\Delta \varphi_{\lambda,\sigma})^2 dV_g$, it only remains to consider the contribution inside $\mathcal{B}_a^{\overline{C}}$.

In this set, we call $d_{a,\min}$ the distance of y from $\mathcal{C}_a$, and $d_{a,\text{out}}$ the minimal distance of y from the points x_i in $\mathcal{C}_a$ satisfying $d_i(y) \geq \overline{C}\text{dist}(y, \mathcal{C}_a)$ (see the definition of $\mathcal{B}_a^{\overline{C}}$). Therefore, setting

$$T_{a,in} = \sum_{x_i \in \mathcal{C}_a \,:\, d_i(y) \leq \left(1+\frac{1}{\overline{C}}\right)\text{dist}(y,\mathcal{C}_a)} t_i,$$

from (80) we obtain the estimate

$$\begin{aligned}\Delta\varphi_{\lambda,x}(y) &= 12(1+o_{\delta,\Theta,\varepsilon,\overline{C}}(1))\frac{\frac{T_{a,in}}{d_{a,\min}^{10}} + \frac{O(T_a - T_{a,in})}{d_{a,\text{out}}^{10}}}{\frac{T_{a,in}}{d_{a,\min}^{8}} + \frac{\overline{O}(T_a - T_{a,in})}{d_{a,\text{out}}^{8}} + \overline{O}((1-T_a)\hat{\delta}^{-8})} \\ &\quad - 16(1+o_{\delta,\Theta,\varepsilon,\overline{C}}(1))\frac{\left|\frac{T_{a,in}}{d_{a,\min}^{9}} + \frac{O(T_a - T_{a,in})}{d_{a,\text{out}}^{9}}\right|^2}{\left[\frac{T_{a,in}}{d_{a,\min}^{8}} + \frac{\overline{O}(T_a - T_{a,in})}{d_{a,\text{out}}^{8}} + \overline{O}((1-T_a)\hat{\delta}^{-8})\right]^2} + C_{\delta,\Theta,\varepsilon,\overline{C}}.\end{aligned}$$

Now we notice that for $y \in \mathcal{B}_a^{\overline{C}}$ the following inequalities hold:

$$\begin{aligned}&\frac{T_{a,in}}{d_{a,\min}^{9}} + \frac{O(T_a - T_{a,in})}{d_{a,\text{out}}^{9}} \\ &\qquad \leq \left(C_{\delta,\Theta,\varepsilon,\overline{C}} + \frac{(1+o_{\delta,\Theta,\varepsilon,\overline{C}}(1))}{d_{a,\min}}\right)\left(\frac{T_{a,in}}{d_{a,\min}^{8}} + \frac{\overline{O}(T_a - T_{a,in})}{d_{a,\text{out}}^{8}}\right); \\ &\frac{T_{a,in}}{d_{a,\min}^{10}} + \frac{O(T_a - T_{a,in})}{d_{a,\text{out}}^{10}} \\ &\qquad \leq \left(C_{\delta,\Theta,\varepsilon,\overline{C}} + \frac{(1+o_{\delta,\Theta,\varepsilon,\overline{C}}(1))}{d_{a,\min}^2}\right)\left(\frac{T_{a,in}}{d_{a,\min}^{8}} + \frac{\overline{O}(T_a - T_{a,in})}{d_{a,\text{out}}^{8}}\right); \\ &\frac{T_{a,in}}{d_{a,\min}^{10}} + \frac{O(T_a - T_{a,in})}{d_{a,\text{out}}^{10}} \\ &\qquad \geq \left(-C_{\delta,\Theta,\varepsilon,\overline{C}} + \frac{(1-o_{\delta,\Theta,\varepsilon,\overline{C}}(1))}{d_{a,\min}^2}\right)\left(\frac{T_{a,in}}{d_{a,\min}^{8}} + \frac{\overline{O}(T_a - T_{a,in})}{d_{a,\text{out}}^{8}}\right).\end{aligned}$$

From the last four formulas and some elementary computations one can deduce that

$$|\Delta\varphi_{\lambda,\sigma}| \leq C_{\delta,\Theta,\varepsilon,\overline{C}} + 4(1+o_{\delta,\Theta,\varepsilon,\overline{C}}(1))\frac{1}{d_{a,\min}^2} \qquad \text{in } \mathcal{B}_a^{\overline{C}}. \tag{86}$$

We notice that, trivially,

$$\mathcal{B}_a^{\overline{C}} = \cup_{x_i \in \mathcal{C}_a}\left(\mathcal{B}_a^{\overline{C}} \cap \{y \;:\; d_i(y) = d_{a,\min}\}\right).$$

Therefore, recalling that $\frac{\Theta}{\lambda} \leq d_i(y) \leq \hat{\delta}$ for every $y \in \mathcal{B}_a^{\overline{C}}$, from the last two formulas, we have (the volume of the three-sphere being $2\pi^2$)

$$\begin{aligned}(87)\quad &\int_{\mathcal{B}_a^{\overline{C}}} (\Delta\varphi_{\lambda,\sigma})^2 dV_g \\ &\leq \sum_{x_i \in \mathcal{C}_a} \int_{\mathcal{B}_a^{\overline{C}} \cap \{y \,:\, d_i(y) = d_{a,\min}\}} \left(C_{\delta,\Theta,\varepsilon,\overline{C}} + 4(1 + o_{\delta,\Theta,\varepsilon,\overline{C}}(1)) \frac{1}{d_{a,\min}^2} \right)^2 dV_g \\ &\leq \sum_{x_i \in \mathcal{C}_a} \int_{B_{\hat{\delta}}(x_i) \setminus B_{\frac{\Theta}{\lambda}}(x_i)} \left(C_{\delta,\Theta,\varepsilon,\overline{C}} + 4(1 + o_{\delta,\Theta,\varepsilon,\overline{C}}(1)) \frac{1}{d_{a,\min}^2} \right)^2 dV_g \\ &\leq \mathrm{card}(\mathcal{C}_a) \left(32\pi^2 (1 + o_{\delta,\Theta,\varepsilon,\overline{C}}(1)) \log \frac{\hat{\delta}\lambda}{\Theta} + C_{\delta,\Theta,\varepsilon,\overline{C}} \right) \\ &\leq \mathrm{card}(\mathcal{C}_a) 32\pi^2 (1 + o_{\delta,\Theta,\varepsilon,\overline{C}}(1)) \log \lambda + C_{\delta,\Theta,\varepsilon,\overline{C}}. \end{aligned}$$

From (66), (76), (85) and (87), considering all the sets $\hat{\mathcal{C}}_a$ and the complement of their union, we finally deduce

$$(88) \qquad \int_M (\Delta\varphi_{\lambda,\sigma})^2 dV_g \leq 32\pi^2 k (1 + o_{\delta,\Theta,\varepsilon,\overline{C}}(1)) \log \lambda + C_{\hat{\delta},\varepsilon,\overline{C},\Theta}.$$

Fixing the values of $\overline{C}, \Theta$ (large, depending on δ) and of ε (small, depending on δ), we obtain the estimate of the term involving the squared Laplacian.

Next, we estimate the term $\int_M |\nabla\varphi_{\lambda,\sigma}|^2 dV_g$. It could be possible to proceed using L^p estimates on $\varphi_{\lambda,\sigma} - \overline{\varphi_{\lambda,\sigma}}$ and interpolation, but having the computations for the Laplacian at hand, it is convenient to work directly. From (64), one finds first the following pointwise estimate

$$|\nabla\varphi_{\lambda,\sigma}| \leq \frac{C}{\lambda},$$

which implies, similarly as before

$$(89) \qquad \int_{\cup_{i=1}^k B_{\frac{\Theta}{\lambda}}(x_i)} |\nabla\varphi_{\lambda,\sigma}|^2 dV_g \leq C \frac{\Theta^4}{\lambda^2}.$$

On the other hand, in the set $M_{\sigma,\Theta}$, using (68) and reasoning as above we obtain

$$\nabla\varphi_{\lambda,\sigma} = -(1 + o_{\delta,\Theta}(1)) \frac{\sum_i t_i \frac{\nabla(\chi_\delta^2(d_i))}{\chi_\delta^{10}(d_i)}}{\sum_s \frac{t_s}{\chi_\delta^8(d_s)}} + o_{\delta,\Theta}(1) \frac{\sum_i t_i \frac{|\nabla(\chi_\delta^2(d_i))|}{\chi_\delta^{10}(d_i)}}{\sum_s \frac{t_s}{\chi_\delta^8(d_s)}}.$$

Taking the square we get

$$|\nabla \varphi_{\lambda,\sigma}|^2 \leq (1 + o_{\delta,\Theta}(1)) \frac{\sum_{i,s} t_i t_s \frac{\nabla(\chi_\delta^2(d_i)) \cdot \nabla(\chi_\delta^2(d_s))}{\chi_\delta^{10}(d_i)\chi_\delta^{10}(d_s)}}{\left[\sum_s \frac{t_s}{\chi_\delta^8(d_s)}\right]^2} + o_{\delta,\Theta}(1) \left(\frac{\sum_i t_i \frac{|\nabla(\chi_\delta^2(d_i))|}{\chi_\delta^{10}(d_i)}}{\sum_s \frac{t_s}{\chi_\delta^8(d_s)}} \right)^2. \tag{90}$$

Using (78) and (79) we deduce (working as before in geodesic coordinates)

$$|\nabla \varphi_{\lambda,\sigma}|^2(y) = 4(1 + o_{\delta,\Theta,\varepsilon}(1)) \frac{\left|\sum_{x_i \in \mathcal{C}_a} \frac{t_i(y - x_i)}{d_i^{10}}\right|^2 + o_{\delta,\Theta,\varepsilon}(1) \left|\sum_{x_i \in \mathcal{C}_a} \frac{t_i}{d_i^9}\right|^2}{\left[\sum_{x_i \in \mathcal{C}_a} \frac{t_i}{d_i^8} + \overline{O}((1 - T_a)\hat{\delta}^{-8})\right]^2} + \frac{C_{\delta,\Theta,\varepsilon} O((1 - T_a)^2 \hat{\delta}^{-18})}{\left[\sum_{x_i \in \mathcal{C}_a} \frac{t_i}{d_i^8} + \overline{O}((1 - T_a)\hat{\delta}^{-8})\right]^2}, \qquad y \in \hat{\mathcal{C}}_a.$$

Reasoning as for (84) and (86), one then finds

$$|\nabla \varphi_{\lambda,\sigma}|^2 \leq C_{\delta,\Theta,\varepsilon,\overline{C}} \left(1 + \frac{1}{d_{a,\min}^2}\right) \qquad \text{in } \hat{\mathcal{C}}_a \cap M_{\sigma,\Theta},$$

which implies

$$\int_{\hat{\mathcal{C}}_a} |\nabla \varphi_{\lambda,\sigma}|^2 dV_g \leq C_{\delta,\Theta,\varepsilon,\overline{C}}.$$

On the other hand,

$$|\nabla \varphi_{\lambda,\sigma}(y)|^2 \leq C_{\hat{\delta}} \qquad \text{for } y \in M \setminus \cup_{a=1}^j \hat{\mathcal{C}}_a.$$

Therefore from the last two formulas we deduce

$$\int_M |\nabla \varphi_{\lambda,\sigma}|^2 dV_g \leq \hat{C}_{\delta,\Theta,\varepsilon,\overline{C}}. \tag{91}$$

From (10) it follows that

$$\langle P_g \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \rangle \leq \int_M (\Delta \varphi_{\lambda,\sigma})^2 dV_g + C \int_M |\nabla \varphi_{\lambda,\sigma}|^2 dV_g.$$

Hence, from (88) and (91) we finally obtain, fixing as before the values of the constants Θ, ε and $\overline{C}$,

$$\langle P_g \varphi_{\lambda,\sigma}, \varphi_{\lambda,\sigma} \rangle \leq 32k\pi^2(1 + o_\delta(1)) \log \lambda + C_\delta.$$

This concludes the proof. □

Proof of Lemma 5.2. If P_g is nonnegative, for $8(k+1)\pi^2 > \rho' \geq \rho > 8k\pi^2$ (resp. for $8\pi^2 > \rho' \geq \rho$) we clearly have

$$\frac{II_\rho(u)}{\rho} - \frac{II_{\rho'}(u)}{\rho'} = \left(\frac{1}{\rho} - \frac{1}{\rho'}\right) \langle P_g u, u \rangle \geq 0,$$

and the conclusion follow immediately for $C = 0$. Therefore from now on we consider the case in which P_g possesses some negative eigenvalues. The last formula in this case yields

$$\frac{II_{\rho'}(u)}{\rho'} \leq \frac{II_{\rho}(u)}{\rho} - \frac{(\rho' - \rho)}{\rho\rho'} \langle P_g \hat{u}, \hat{u} \rangle, \tag{92}$$

where $\hat{u}$ is the V-component of u; see (18).

Fixing $\rho \in [1 - \rho_0, 1 + \rho_0]$ and $\varepsilon > 0$, we consider a map $\pi_{\rho,\varepsilon} \in \overline{\Pi}_{\overline{S},\overline{\lambda}}$ such that

$$\sup_{m \in \widehat{A_{k,\overline{k}}}} II_\rho(\pi_{\rho,\varepsilon}(m)) < \overline{\Pi}_\rho + \varepsilon. \tag{93}$$

We can also assume that each element of the form $u = \pi_{\rho,\varepsilon}(m)$ satisfies the normalization condition $\int_M e^{4u} dV_g = 1$. Now, considering the V-part $\hat{u}$ of all these elements, we fix three numbers $\theta > 0$ (small, depending on $\pi_{\rho,\varepsilon}$), and $C_0, C_1 > 0$ (depending on M and ρ_0, with $C_1 \gg C_0 \gg 1$), and we define a new map $\tilde{\pi}_{\rho,\varepsilon}$ in the following way

$$\tilde{\pi}_{\rho,\varepsilon}(m) = \pi_{\rho,\varepsilon}(m) + \eta_\theta(m) \tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}(m)}) \widehat{\pi_{\rho,\varepsilon}(m)}; \qquad m \in \widehat{A_{k,\overline{k}}}. \tag{94}$$

Here $\widehat{\pi_{\rho,\varepsilon}(m)}$ denotes the V-component of $\pi_{\rho,\varepsilon}(m)$ (see Section 2), the function $\eta_\theta(m)$, $m = (m_1, t) \in A_{k,\overline{k}} \times [0,1]$, is defined as

$$\eta_\theta(m) = \begin{cases} 1, & \text{for } t \in [0, 1-\theta]; \\ \frac{1}{\theta}(1-t), & \text{for } t \in [1-\theta, 1], \end{cases}$$

and $\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}(m)})$ is given by

$$\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}(m)}) = \begin{cases} 0, & \text{for } \|\widehat{\pi_{\rho,\varepsilon}(m)}\| \in [0, C_0]; \\ \frac{1}{C_1 - C_0}(\|\widehat{\pi_{\rho,\varepsilon}(m)}\| - C_0), & \text{for } \|\widehat{\pi_{\rho,\varepsilon}(m)}\| \in [C_0, C_1]; \\ 1, & \text{for } \|\widehat{\pi_{\rho,\varepsilon}(m)}\| \geq C_1. \end{cases}$$

When $\eta_\theta(m) = 1$, by the normalization of $\pi_{\rho,\varepsilon}$ we have the following upper bound on $II_\rho(\tilde{\pi}_{\rho,\varepsilon}(m))$

$$\begin{aligned} II_\rho(\tilde{\pi}_{\rho,\varepsilon}) = {} & \langle P_g \pi_{\rho,\varepsilon}, \pi_{\rho,\varepsilon} \rangle + \left(2\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) + (\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}))^2\right) \langle P_g \widehat{\pi_{\rho,\varepsilon}}, \widehat{\pi_{\rho,\varepsilon}} \rangle \\ & + 4\rho \int_M Q_g(\pi_{\rho,\varepsilon} + \tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) \widehat{\pi_{\rho,\varepsilon}}) dV_g - 4\rho k_P \int_M e^{4\pi_{\rho,\varepsilon} + 4\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) \widehat{\pi_{\rho,\varepsilon}}} dV_g \\ \leq {} & II_\rho(\pi_{\rho,\varepsilon}) + \left(2\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) + (\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}))^2\right) \langle P_g \widehat{\pi_{\rho,\varepsilon}}, \widehat{\pi_{\rho,\varepsilon}} \rangle + \tilde{C}_0 \tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) \|\widehat{\pi_{\rho,\varepsilon}}\|, \end{aligned} \tag{95}$$

where $\tilde{C}_0$ is a fixed constant depending only on M and on ρ_0.

Since $\widehat{\pi_{\rho,\varepsilon}}$ belongs to the space V, where P_g is negative-definite, if C_0 is sufficiently large (depending only $\tilde{C}_0$ which, in turn, depends only on M

and ρ_0), then one has

$$(96)\quad \left(2\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) + (\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}))^2\right) \langle P_g \widehat{\pi_{\rho,\varepsilon}}, \widehat{\pi_{\rho,\varepsilon}} \rangle + \tilde{C}_0 \tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) \|\widehat{\pi_{\rho,\varepsilon}}\| \leq 0 \qquad \text{for } \|\widehat{\pi_{\rho,\varepsilon}}(m)\| \geq C_0.$$

Having fixed this value of C_0, from (92) and knowing that $\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}(m)}) = 0$ for $\|\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}(m)})\| \leq C_0$, we see that

$$(97)\quad \frac{II_{\rho'}(\tilde{\pi}_{\rho,\varepsilon})}{\rho'} \leq \frac{II_{\rho}(\pi_{\rho,\varepsilon})}{\rho} - \frac{\rho'-\rho}{\rho\rho'} \langle P_g \widehat{\pi_{\rho,\varepsilon}}, \widehat{\pi_{\rho,\varepsilon}} \rangle \leq \frac{\overline{\Pi}_\rho + \varepsilon}{\rho} + \hat{C}_0(\rho'-\rho); \quad \|\widehat{\pi_{\rho,\varepsilon}}(m)\| \leq C_0, \eta_\theta(m) = 1,$$

where $\hat{C}_0$ depends only on M and ρ_0.

Now we fix also the value of C_1. We choose ρ_0 so small and $C_1 > 0$ (depending only on M and ρ_0) so large that

$$(98)\quad \frac{3}{\rho}\left(1 - \frac{4}{3}\frac{\rho'-\rho}{\rho'}\right) \langle P_g \hat{v}, \hat{v} \rangle + \frac{\tilde{C}_0}{\rho}\|\hat{v}\| \leq \langle P_g \hat{v}, \hat{v} \rangle \leq -2\overline{L} - L \qquad \text{for all } \hat{v} \in V \text{ with } \|\hat{v}\| \geq C_1,$$

where L and $\overline{L}$ are the constants given in (56) and (57). From (92), (95) and (96) we immediately find (still for $\eta_\theta(m) = 1$)

$$(99)\quad \frac{II_{\rho'}(\tilde{\pi}_{\rho,\varepsilon})}{\rho'} \leq \frac{II_{\rho}(\pi_{\rho,\varepsilon})}{\rho} - \frac{\rho'-\rho}{\rho\rho'} \langle P_g \widehat{\pi_{\rho,\varepsilon}}, \widehat{\pi_{\rho,\varepsilon}} \rangle \leq \frac{\overline{\Pi}_\rho + \varepsilon}{\rho} + \hat{C}_1(\rho'-\rho); \quad C_0 \leq \|\widehat{\pi_{\rho,\varepsilon}(m)}\| \leq C_1,$$

where $\hat{C}_1$ depends only on M and ρ_0.

By (92) and (95), since $\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) = 1$ when $\|\widehat{\pi_{\rho,\varepsilon}}\| \geq C_1$ (which implies $\hat{\tilde{\pi}} = 2\hat{\pi}$), we obtain

$$(100)\quad \frac{II_{\rho}(\tilde{\pi}_{\rho,\varepsilon})}{\rho'} \leq \frac{II_{\rho}(\pi_{\rho,\varepsilon})}{\rho} + \frac{3}{\rho}\left(1 - \frac{4}{3}\frac{\rho'-\rho}{\rho'}\right) \langle P_g \widehat{\pi_{\rho,\varepsilon}}, \widehat{\pi_{\rho,\varepsilon}} \rangle + \frac{\tilde{C}_0}{\rho}\|\widehat{\pi_{\rho,\varepsilon}}\|; \quad \|\widehat{\pi_{\rho,\varepsilon}(m)}\| \geq C_1, \eta_\theta(m) = 1.$$

Then (98) implies (see (57) and (93))

$$(101)\quad \frac{II_{\rho'}(\tilde{\pi}_{\rho,\varepsilon})}{\rho'} \leq \frac{\overline{\Pi}_\rho}{\rho}, \qquad \text{for } \|\hat{\pi}\| \geq C_1.$$

From (97), (99) and (101) we deduce

$$(102)\quad \frac{II_{\rho'}(\tilde{\pi}_{\rho,\varepsilon})}{\rho'} \leq \frac{\overline{\Pi}_\rho + \varepsilon}{\rho} + (\hat{C}_0 + \hat{C}_1)(\rho'-\rho) \qquad \text{for } \eta_\theta(m) = 1.$$

Therefore it remains to consider the case in which $\eta_\theta(m) \neq 1$, that is, for $t > 1 - \theta$ (recall that $m = (m_1, t)$ with $m_1 \in A_{k,\overline{k}}$). This is where the choice of θ enters. Reasoning as for (95) we find

$$II_{\rho'}(\tilde{\pi}_{\rho,\varepsilon}) \leq II_{\rho'}(\pi_{\rho,\varepsilon}) + 2\eta_\theta(m)\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}})\langle P_g \widehat{\pi_{\rho,\varepsilon}}, \widehat{\pi_{\rho,\varepsilon}} \rangle + \tilde{C}_0 \eta_\theta(m) \tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) \|\widehat{\pi_{\rho,\varepsilon}}\|.$$

Recall that the map $\pi_{\rho,\varepsilon}$ belongs to $\Pi_{\overline{S},\overline{\lambda}}$, and hence it satisfies $\pi_{\rho,\varepsilon}(t,\cdot) \to \Phi_{\overline{S},\overline{\lambda}}(\cdot)$ in $C^0(A_{k,\overline{k}})$. Since ρ is varying in the small interval $[1-\rho_0, 1+\rho_0]$, we have estimates of the form (51) (with ρk_P replacing k_P) uniformly for ρ in this interval. Thus from the last formula we deduce that, for $\eta_\theta(m) < 1$,

$$\begin{aligned} II_{\rho'}(\tilde{\pi}_{\rho,\varepsilon}(m)) \leq & \, II_{\rho'}(\Phi_{\overline{S},\overline{\lambda}}(m_1)) \\ & + o_\theta(1) + 2\eta_\theta(m)\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}})\langle P_g \widehat{\pi_{\rho,\varepsilon}}, \widehat{\pi_{\rho,\varepsilon}}\rangle + \tilde{C}_0 \eta_\theta(m)\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) \, \|\widehat{\pi_{\rho,\varepsilon}}\| \\ \leq & \left(32k\pi^2 - 4\rho' k_P + o_\delta(1)\right)\log\lambda \\ & - |\lambda_{\overline{k}}||s|^2\overline{S}^2 + O(|s|\overline{S}) + C_\delta + O(1) + o_\theta(1) \\ & + 2\eta_\theta(m)\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}})\langle P_g \widehat{\pi_{\rho,\varepsilon}}, \widehat{\pi_{\rho,\varepsilon}}\rangle + \tilde{C}_0 \eta_\theta(m)\tilde{\eta}(\widehat{\pi_{\rho,\varepsilon}}) \, \|\widehat{\pi_{\rho,\varepsilon}}\| \\ \leq & \left(32k\pi^2 - 4\rho' k_P + o_\delta(1)\right)\log\lambda \\ & - |\lambda_{\overline{k}}||s|^2\overline{S}^2 + O(|s|\overline{S}) + C_\delta + O(1) < -\frac{3}{2}L, \end{aligned}$$

if L is chosen sufficiently large (see (56)) and θ is chosen sufficiently small. Now the conclusion follows from (102) and the last estimate. □

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