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On local connectivity for the Julia set of rational maps: Newton's famous example

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Abstract

We show that Newton's cubic methods (famous rational maps) have a locally connected Julia set except in some very specific cases. In particular, when those maps are infinitely renormalizable their Julia set is locally connected and contains small copies of nonlocally connected quadratic Julia sets. This also holds when Newton's method is renormalizable and has Cremer points, unlike the polynomial case. After a dynamical description we show the necessity of the Brjuno condition within this family.

1. Introduction

In this article we are interested in the local connectedness of the Julia set of rational maps acting on the Riemann sphere $\widehat{\mathbf{C}}$. This problem is central in holomorphic dynamics, notably in order to obtain topological models and to approach the famous MLC conjecture; it has been studied a lot in the case of polynomials (see [D-H1], [F], [G-Sm], [G-Sw], [K], [L-vS], [Ly], [McM1], [Pe1], [Ra], [Ri], [So1], [So2], [T-Y],...), and mostly in degree two — but hard questions still remain. Important progress was made by Yoccoz who proved that if a quadratic polynomial has only repelling periodic points (in $\mathbf{C}$) and is not infinitely renormalizable, then its Julia set is locally connected (see [Hu], [M2]). Douady exhibited then striking examples of infinitely renormalizable polynomials having a nonlocally connected Julia set (see [M2] and also [So2]). Several years before, Sullivan had given the first examples of such pathological Julia sets by showing that every polynomial with a Cremer point has a nonlocally connected Julia set (see [M1]). A *Cremer* point is a periodic point in the Julia set whose first return map is tangent to an irrational rotation. As we will see here, rational maps may behave in a completely different way:

Theorem 1. *There exist rational maps which have a locally connected Julia set and Cremer points*; *cubic Newton maps provide such examples.*

The question is whether all cubic Newton maps with a Cremer point have a locally connected Julia set is still open (see Question 8.5 and Conjecture 8.6). On the other hand, THEOREM 2. *Every infinitely renormalizable*[1] *genuine cubic Newton map has a locally connected Julia set.*

A *genuine Newton map* will be a Newton map that is not quasi-conformally conjugated to a polynomial in a neighborhood of its Julia set.

Cubic Newton maps arise as natural examples to look at: besides quadratic, they are the only rational maps with simple critical points which are all fixed except one which is "free" (see Lemma 2.1); moreover they form a family of dynamical systems in which this critical point displays all possible behaviors.

COROLLARY 3. *There exist rational maps with locally connected Julia sets containing a wandering nontrivial continuum* (*and having only repelling periodic points*).

This corollary contrasts with the following result of Levin: for polynomials of the form $z^l + c$, whose periodic points are all repelling and which have a connected Julia set, the local connectedness of the Julia set is equivalent to the nonexistence of wandering continua (see [Le]).

The following theorem strengthens the dictionary between rational maps and Kleinian groups established by D. Sullivan (see [McM2]). Indeed, every known example of finitely generated Kleinian group possesses a locally connected limit set (if it is connected). So, one conjectures that the Julia set of a genuine cubic Newton map is always locally connected. The most complete (but also more technical) result we obtain on this question is given in Proposition 8.3.

THEOREM 4. *A genuine cubic Newton map*, *without Siegel points*, *has a locally connected Julia set provided the orbit of the nonfixed critical point does not accumulate on the boundary of any invariant immediate basin of attraction.*

A *Siegel* point of a rational map R is a periodic point in the neighborhood of which R is conjugated to an irrational rotation (linearizable); the maximal domain of linearization is called a *Siegel disc.*

A. Douady has conjectured that for any rational map, whenever it is linearizable near a fixed point of multiplier $\lambda = e^{2i\pi\alpha}$ then α has to be a Brjuno number (see [D]). It is also conjectured that the nonlinearizability is related to the presence of small cycles (see [PM2]). In our setting we have:

THEOREM 5. *If N possesses a periodic point x of multiplier $\lambda = e^{2i\pi\alpha}$ with $\alpha \in \mathbf{R}$ then*

[1] See Definition 6.1.

(1) *N is renormalizable at x_0;*

(2) *$\alpha \in \mathcal{B}$ if and only if N is linearizable near x;*

(3) *if $\alpha \notin (\mathcal{B} \cup \mathbf{Q})$ there exist periodic cycles ($\neq x$) in any neighborhood of x.*

An irrational α of convergents p_n/q_n (rational approximations obtained by the continued fraction development) is a *Brjuno* number, i.e. $\alpha \in \mathcal{B}$, if $\sum_{n=1}^{\infty} (\log q_{n+1})/q_n$ is finite.

All the results above are consequences of the following fundamental brick:

Theorem 6. *For every genuine cubic Newton map, without Siegel points, the connected components of the Fatou set are Jordan domains.*

Theorem 6 implies Theorem 4 and, subsequently, Theorem 1, Theorem 2 and their corollaries. The proof of Theorem 1 and Theorem 2 is based on the fact that N is renormalizable at x_0 (from Theorem 5) and that the “small” Julia set touches the boundary of the basin of attraction at exactly one point which is repelling. Theorem 5 follows from the existence of puzzles with nondegenerate annuli around the critical point.

The proofs use intensively the technique of puzzles introduced by Branner and Hubbard [B-H] and developed by Yoccoz. The basic idea is to construct invariant connected graphs that divide the Julia set into connected subsets, and then to show that the iterated inverse images of these subsets shrink to points. In the case of quadratic polynomials, the graphs constructed by Yoccoz are closures of a finite number of arcs in the unbounded component of the Fatou set. In our case, instead, some edges of the graphs will have to visit infinitely many components of the basins of attraction.

The paper is organized as follows:

Section 2 gives a rough dynamical description of cubic Newton maps and then reviews the results of Yoccoz on puzzles in the context of rational-like maps.

Section 3 is devoted to finding “cut rays” which will serve as basic bricks in the construction of the puzzles.

Section 4 introduces the “articulated rays” (the main new tool in this paper) which provide nice access to points of the Julia set; periodic articulated rays are also constructed.

Section 5 yields candidates for puzzles.

Section 6 studies the renormalizations of N via puzzle pieces and gives the proof of Theorem 5.

Section 7 gives the proof of Theorem 6 with a new strategy for the renormalizable case (case 2 of Theorem 2.15).

Section 8 describes all the known cases where the Julia set is locally connected.

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2. Newton maps and Yoccoz puzzles

2.1. *Newton maps.* When one considers the dynamical system induced by a rational map f on the Riemann sphere $\widehat{\mathbf{C}}$, the interesting object is the *Julia set* $J(f)$, which is the minimal compact subset, invariant by f and f^{-1}, and containing at least three points. The topological properties of $J(f)$ give information on the dynamics of f (see [M1], [D-H1]). In particular, because of Caratheodory's Theorem on extension of conformal representations, local connectivity deserves special attention (see Remark 2.4). The difficulty of the problem depends on the behavior of critical points, and subsequently on their number.

A "simple case", usually called the *geometrically finite case*, occurs when the post-critical set $P(f)$, i.e. the closure of the critical orbits of f, meets $J(f)$ in a finite set. In this case, the Julia set is locally connected provided it is connected (see [D-H1, M2, T-Y]). To catch more interesting critical behaviors with still a tame complexity, we consider rational maps with exactly one "free" critical point. More precisely, we suppose that all critical points are simple (i.e. of local degree two) and that all are fixed (by f) except one.

LEMMA 2.1. *Let $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a rational map of degree d having all its critical points simple, and fixed except one. Then $d \leq 3$ and f is analytically conjugate to either a quadratic polynomial or to a cubic Newton map, i.e., a map of the form*

$$N(z) = z - \frac{P(z)}{P'(z)} \quad \text{where } P \text{ is a polynomial of degree } 3 \text{ with distinct roots.}$$

Proof. A rational map of degree d possesses $2d-2$ critical points and $d+1$ fixed points. Since the critical points are simple and only one is not fixed, there are $2d-3$ distinct fixed points so that $d \leq 4$. If $d = 4$, every fixed point is critical which contradicts the holomorphic fixed point formula (see [M1]).

We now determine f up to conjugation by a Möbius transformation. If $d = 2$, we can assume that the fixed critical point is at infinity, so that f is a quadratic polynomial. Now, if $d = 3$ and if there are three distinct fixed critical points b_1, b_2 and b_3 (the labelling will be fixed in Notation 2.7), we can assume that the fourth fixed point of f is at infinity. We write f as follows

$$f(z) = z - \frac{P(z)}{Q(z)} = \frac{zQ(z) - P(z)}{Q(z)},$$

where P and Q are relatively prime polynomials. Since f fixes infinity, $\deg(Q) < \deg(zQ - P) \leq 3$ so Q has degree at most 2 and P at most 3. Moreover, $f(b_i) = b_i$, so $P(b_i) = 0$ for $i \in \{1, 2, 3\}$. Finally the condition $f'(b_i) = 0$ implies that $P'(b_i)/Q(b_i) = 1$ so $Q = P'$ for degree reasons. Thus, f is a cubic Newton map. □

The dynamics of a cubic Newton map N can be described as follows.
One can always assume (up to affine conjugacy) that N is associated to a polynomial of the form $P(z) = z^3 + pz + 1$ with $p \in \mathbf{C}$; i.e.,

$$N(z) = N_P(z) = z - \frac{P(z)}{P'(z)} = \frac{2z^3 - 1}{3z^2 + p}.$$

It is a degree 3 rational mapping, fixing ∞ which is repelling.[2] It has four critical points, one at each of the roots b_i of the polynomial P, which is a super-attracting fixed point of N, and one called x_0 at 0 (which is the root of P''). Each of the roots b_i has a *basin of attraction* $\widetilde{B}_i = \left\{x \in \widehat{\mathbf{C}} \mid N^n(x) \xrightarrow[n\to\infty]{} b_i\right\}$. We denote by B_i the *immediate basin* of b_i, i.e., the connected component of $\widetilde{B}_i$ containing b_i.

The Julia set $J(N)$ is always connected, according to results of Shishikura [Sh] (see also [T1, He]). Hence, the connected components of the *Fatou* set $F(N) = \widehat{\mathbf{C}} \setminus J(N)$ are simply connected. They are of two types, the components of the basins $\widetilde{B}_i$ and additional (possibly empty) components due to the presence of the fourth (free) critical point x_0. Very special situations appear when the different kinds of Fatou components mix, i.e., when x_0 belongs to some component of $\widetilde{B}_i$. In this case N is geometrically finite and therefore $J(N)$ is locally connected.

The following two remarks describe in detail two particular cases where N is geometrically finite. We get rid of them through Assumption 1 for the discussion afterward.

Remark 2.2. If the critical point x_0 belongs to B_1, B_2 or B_3, N is quasi-conformally conjugated in a neighborhood of the Julia set to the cubic polynomial $Q(z) = z^3 + \frac{3}{2}z$ (which is geometrically finite).

Proof. Assume that x_0 belongs to B_j for some $j \in \{1, 2, 3\}$. By a classical surgery process (see [CG, Ch. 6]) one conjugates N, in a neighborhood of the Julia set, to a rational map f of degree 3 possessing three fixed critical points, two of multiplicity 1 and one of multiplicity 2 (corresponding to b_j collapsed with x_0). Since the critical point of multiplicity 2, it is also backward invariant, f is conjugated (by a Möbius transformation) to a polynomial with two fixed

[2] A point x of period p is *repelling*, *attracting* or *parabolic*, respectively, if $|(f^p)'(x)| > 1$, $|(f^p)'(x)| < 1$ or $(f^p)'(x) = e^{2i\pi\theta}$, $\theta \in \mathbf{Q}/\mathbf{Z}$.

critical points in $\mathbf{C}$ of multiplicity 1. Now, up to affine conjugacy, one can assume that the polynomial is monic and centered. Hence it can be written $P(z) = z^3 + az$. Since the two finite critical points are fixed $a = \frac{3}{2}$. □

Remark 2.3. If the critical point x_0 is mapped to ∞ (i.e. $N(x_0) = \infty$), N is conformally conjugated to the Newton map N_S of the polynomial $S(z) = z^3 - 1$. It is called the "symmetric case" because N_S is invariant by $z \to e^{2i\pi/3}z$.

Proof. Let P denote the polynomial associated to N (i.e. $N = N_p$). Up to affine conjugacy one can assume that $P(z) = az^3 - a$. Indeed, using a translation one can assume that $x_0 = 0$, so that $N(0) = \infty$ and $P(z) = az^3 + b$; now, using a dilatation one can assume that $P(z) = az^3 - a$. In particular, $N_P = N_S$ where $S(z) = z^3 - 1$. □

The cubic polynomial $Q(z) = z^3 + \frac{3}{2}z$ is studied in [F, Ro] and so throughout the paper we will only work with genuine Newton maps. We take away the symmetric case just for technical reasons but we will dare some comments about it.

Assumption 1. From now on, we assume that the critical point x_0 does not belong to the immediate basins B_i, $i \in \{1, 2, 3\}$ and that $N(x_0) \neq \infty$.

Then the classical Böttcher Theorem [B] provides a unique conformal representation $\phi_j \colon \mathbf{D} \to B_j$ — where $\mathbf{D}$ is the open unit disc — which conjugates N to the map $z \mapsto z^2$. Each ϕ_j induces polar coordinates on B_j. The *ray* of angle $t \in \mathbf{R}/\mathbf{Z}$, denoted by $R_j(t)$, is the arc $\phi_j([0, 1[e^{2i\pi t})$ and the *equipotential* of level $v > 0$ is defined as $E_j(v) = \phi_j(e^{-v}\mathbf{S}^1)$. These objects can be lifted to the other components of $\widetilde{B}_j$ as long as they don't contain the critical point x_0 (see Notation 2.9 below).

Remark 2.4. Once Theorem 6 has been proved, the local connectedness of ∂B_j will give that the map ϕ_j extends continuously to $\overline{\mathbf{D}}$ so that ∂B_j is a curve, by Carathéodory's Theorem. Then Lemma 3.8 will give the injectivity of this extension. Hence ∂B_j is a Jordan curve on which $N\big|_{\partial B_j}$ will be conjugated to the map $\theta \mapsto 2\theta$ on $\mathbf{S}^1$ (by the extension ϕ_j^{-1}).

Recall that a ray $R_j(t)$ is said to *converge* (or to *land*) if the quantity $\phi_j(e^{-v+2i\pi t})$ has a limit when v tends to 0. Douady, Hubbard and Sullivan proved that for every rational angle t, the ray $R_j(t)$ converges to an eventually periodic point in ∂B_j, which is repelling or parabolic, with period dividing p where t is written in the form $t = r/(2^m(2^p - 1))$ (see [D-H1, M1]).

The following lemma shows a dis-symmetry of the Julia set of N:

Lemma 2.5. *The three rays* $R_1(0)$, $R_2(0)$ *and* $R_3(0)$ *converge to* ∞. *On the contrary, only two of the rays of angle* $1/2$ *do converge to the same preimage*

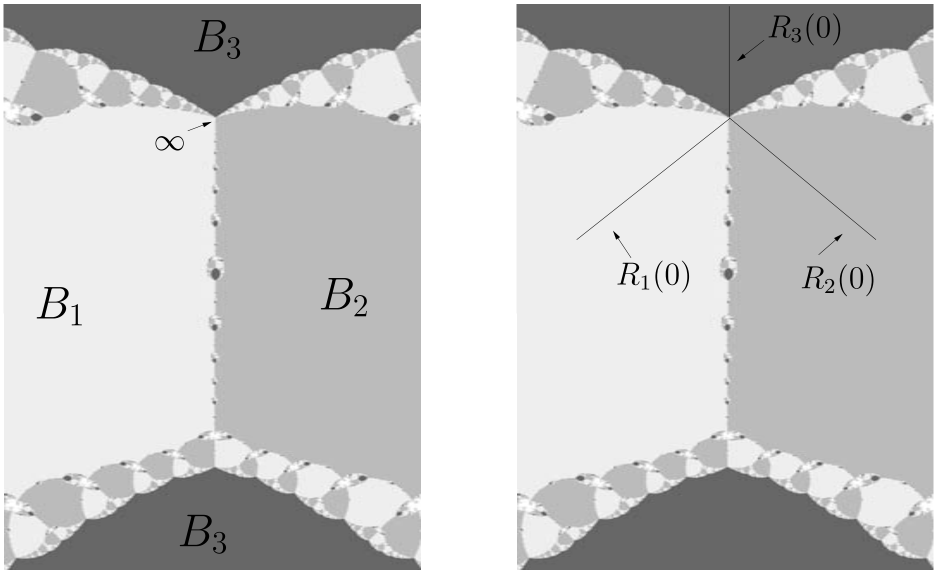

Figure 1: On the left B_1, B_2, B_3 and on the right the rays $R_i(0)$.

of ∞ *(when* $N(x_0) \neq \infty$*, i.e. in the general case). We denote this preimage by* ξ*. The other ray of angle* $1/2$ *converges to* $-\xi$*.*

Note that in the particular case where $N(x_0) = \infty$ *(avoided by assumption* 1*), all the rays of angle* $1/2$ *converge to the preimage of* ∞ *which is* x_0*.*

The following fact will be useful for the proof of Lemma 2.5 and several times later also:

Trivial fact 2.6. *If two distinct rays converge to the same point p and have the same image under N then p is a critical point of N.*

Indeed, N is not injective near p.

Proof of Lemma 2.5. Each ray $R_j(0)$ is fixed by N, and so it converges to a fixed point on ∂B_j that can only be ∞. Each ray of angle $1/2$ lands at a preimage of ∞, distinct from ∞ by the above remark.

If $N(x_0) = \infty$, the map near x_0 is a double cover over a neighborhood of ∞, so that there are two preimages of each ray $R_i(0)$ near x_0. Hence each ray of angle $1/2$ lands at x_0.

If $N(x_0) \neq \infty$, we assume by contradiction that the three rays $R_1(1/2)$, $R_2(1/2)$, $R_3(1/2)$ land at the same inverse image of ∞, denoted by x. Their cyclic order at x is different from the cyclic order of their images $R_1(0)$, $R_2(0)$, $R_3(0)$ at ∞. This contradicts the conformality of N at x. □

We will use the following convention for labelling basins (see Figure 1):

Notation 2.7. In the general case, i.e. if $N(x_0) \neq \infty$, the basins are labelled B_1, B_2, B_3 in such a way that:

- the two rays of angle $1/2$ landing at the same preimage ξ of $N^{-1}(\infty)$ are in B_1 and B_2;
- the rays of angle 0 of B_1, B_2, B_3 land at ∞ in positive cyclic order.

In the symmetric case, i.e. if $N(x_0) = \infty$, the labelling B_1, B_2, B_3 of the basins is without importance because of the symmetry.

Remark 2.8. Under assumption 1, the restriction $N\colon B_j \to B_j$ is a degree two ramified covering. So $N^{-1}(B_j)$ consists of B_j and another connected component which is a topological disc since N can only induce a nonramified covering of degree one on it.

Notation 2.9. We denote by B'_j the connected component of $N^{-1}(B_j)$ disjoint from B_j.

If U is a disc such that $N^p\colon U \to B_j$ is a homeomorphism, we denote by $R_U(t)$ (resp. $E_U(v)$) the ray of angle t in U, i.e. , $(N^p_{|U})^{-1}(R_j(t))$, (resp. the equipotential of level v in U, i.e. , $(N^p_{|U})^{-1}(E_j(v))$). More quickly for B'_j we adopt the notation $R'_j(t)$ and $E'_j(v)$ respectively.

Remark 2.10. For $j = 1, 2$ the rays $R'_j(0)$ and $R_3(1/2)$ land at the same inverse image ξ' (possibly x_0) of ∞. This follows from Lemma 2.5 and the fact that $R'_k(0)$ and $R_k(1/2)$ (or $R_k(0)$) $k = 1, 2, 3$, cannot land at a common noncritical point because they are all mapped to $R_k(0)$ (trivial fact).

2.2. *Yoccoz Puzzles.* We now describe the technique of puzzles in the convenient framework of rational-like maps.

Definition 2.11. Let X, X' be connected open subsets of $\widehat{\mathbf{C}}$ with finitely many smooth boundary components and such that $\overline{X'} \subset X$. A holomorphic map $f\colon X' \to X$ is called a *rational-like map* (resp. a *simple rational-like map*) if it is proper and has finitely many critical points in X (resp. a single critical point with multiplicity one). We denote by *degree*(f) the topological degree of f and by $K(f) = \{x \in X' \mid \forall n \geq 0, \quad f^n(x) \in X'\} = \bigcap_{n\geq 0} f^{-n}(X)$, the associated *filled Julia set.*

For simply connected domains this is the standard definition of *polynomial-like maps* ([D-H2]).

Example 2.12. Any genuine cubic Newton map N induces a simple rational-like map $N\colon X' \to X$ (see Figure 2) for any potential v, where

$$X = \widehat{\mathbf{C}} \setminus \bigcup_{i=1,2,3} \phi_i(e^{-v}\overline{\mathbf{D}}) \quad \text{and} \quad X' = N^{-1}(X).$$

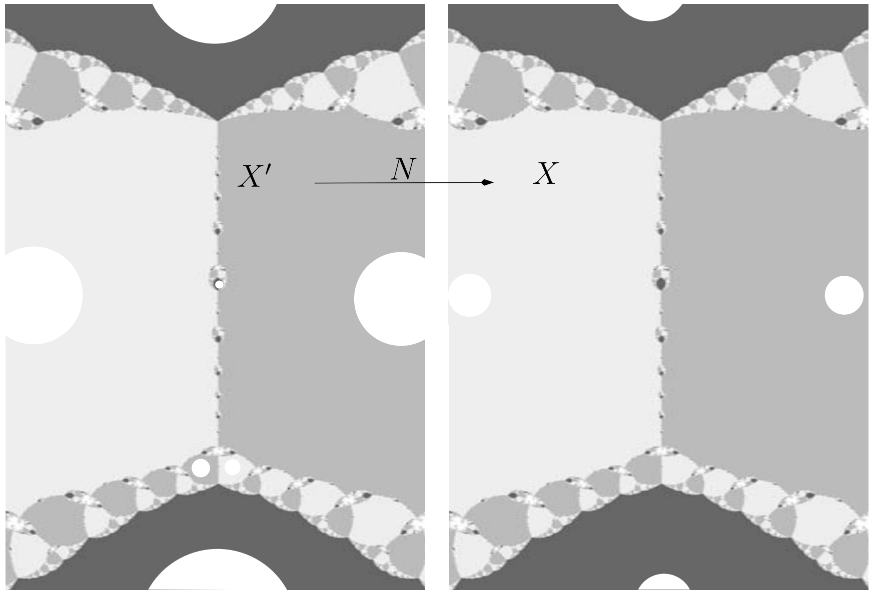

Figure 2: N viewed as a rational like map from X' on the left to X on the right.

Definition 2.13. A finite, connected, graph Γ in $\overline{X}$ is called a *puzzle* if it satisfies the conditions: $\partial X \subset \Gamma$, $f(\Gamma \cap X') \subset \Gamma$, and the orbit of each critical point avoids Γ.

The *puzzle pieces of depth* n are the connected components of $f^{-n}(X \setminus \Gamma)$ and the one containing a point x is denoted by $P_n(x)$. If a point x is contained in a puzzle piece at each depth, i.e. if the orbit of x avoids Γ, the sequence $P_n(x)$ is decreasing and the *impression* of x is defined to be the set $\mathrm{Imp}(x) = \cap \overline{P}_n(x)$. A puzzle is said to be *k-periodic* at x if $f^k\big(P_{n+k}(x)\big) = P_n(x)$ for any sufficiently large n.

Every difference set $A_n = P_n \setminus \overline{P}_{n+1}$ between two nested puzzle pieces P_n, P_{n+1} of consecutive depths is called a *puzzle annulus of depth* n. This "annulus" actually degenerates to a disc if ∂P_n meets ∂P_{n+1}. A point $x \in K(f)$ is said to be *surrounded* at depth n if $f^{n+1}(x) \notin \Gamma$ and $P_n(x) \setminus \overline{P}_{n+1}(x)$ is a nondegenerate (i.e. genuine) annulus. A point is *infinitely surrounded* if it is surrounded at infinitely many depths.

Remark 2.14. One can also consider closed annuli $\overline{P}_n \setminus P_{n+1}$. When such closed annuli degenerate, they still surround the points of P_{n+1} since the closed annuli still have the homotopy type of the circle.

The following theorem is a rewording (see [Ro] for the proof) of Yoccoz' Theorem (see [Hu, M2, Y]). The introduction of different puzzles Γ^i is natural since it is difficult to surround every point of the Julia set with just one puzzle.

THEOREM 2.15. *Let $f\colon X' \to X$ be a simple rational-like map, C a subset of $K(f)$ such that $f(C) \subset C$ and let $\Gamma^0, \dots, \Gamma^r$ be puzzles all surrounding the unique critical point x_0. If every point of C is surrounded by some Γ^j at a uniformly bounded depth, then for every point $x \in C$ there exists $i \in \{0, \dots, r\}$ such that x is infinitely surrounded by Γ^i and the following dichotomy holds:*

1) *If the puzzle Γ^i is not periodic at x_0 then the impression Imp^i for Γ^i satisfies $\mathrm{Imp}^i(x) = \{x\}$;*

2) *If the puzzle Γ^i is k-periodic at x_0 then $f^k\colon P^i_{n+k}(x_0) \to P^i_n(x_0)$ defines a polynomial-like map of degree 2 for some large n and its filled Julia set is $\mathrm{Imp}^i(x_0)$; moreover, depending on whether the orbit of x meets $\mathrm{Imp}^i(x_0)$ or not, $\mathrm{Imp}^i(x)$ is either a conformal copy of $\mathrm{Imp}^i(x_0)$ or $\{x\}$.*

We will apply Theorem 2.15 to $C = \partial B_i$ with the puzzles $\Gamma^0, \dots, \Gamma^r$ constructed in Section 5 and 7, consisting of rays, articulated rays and equipotentials. For any point $x \in C$ such that $\mathrm{Imp}(x) = \{x\}$, the sets $\overline{P}_n(x) \cap C$ form a basis of neighborhoods of x in C so that, if they are all connected, C is locally connected at x.

A useful trick for finding points surrounded by the puzzles is contained in the following remark. It occurs in the proof of our version of Yoccoz' Theorem (Theorem 2.15) and will be crucial in Proposition 5.4 (see [Ro]).

Remark 2.16. If an iterated image $f^k(x)$ of a point x is surrounded by a nondegenerate annulus $P_0(f^k(x)) \setminus \overline{P_1(f^k(x))}$ (for some puzzle Γ) then x itself is surrounded by Γ at depth k and, more precisely, by the nondegenerated annulus $P_k(x) \setminus \overline{P_{k+1}(x)}$. This follows from the fact that f^k induces a proper map $P_k(x) \to P_0(f^k(x))$ taking $P_{k+1}(x)$ to $P_1(f^k(x))$.

3. Cut angles and localization of x_0

Here and below, we identify $\mathbf{S}^1$ with $\mathbf{R}/\mathbf{Z}$, so that $-t$ and $1-t$ coincide. We write $t_1 \le t \le t_2$ to mean that t_1, t, t_2 are in trigonometric order. To compare only two angles, we use their representatives in $(0, 1]$.

A preliminary step for proving that ∂B_i is locally connected is to find small connected subsets in ∂B_i. The following basic examples will be used throughout the paper.

Example 3.1. Let $Q \subset \overline{B}_k$ denote the closure of the sector between two converging rays $R(t_1), R(t_2)$, namely,

$$Q = \overline{\{\phi_k(r\, e^{2i\pi t}),\ r \in [0,1),\ t_1 \le t \le t_2\}}.$$

Then $Q \cap \partial B_k$ is a connected subset of ∂B_k. Indeed it is the decreasing intersection of the compact connected sets $\overline{S}_n$, where

$$S_n = \big\{\phi_k(r\, e^{2i\pi t}),\ r \in [1-1/n, 1),\ t_1 \le t \le t_2\big\} \text{ for } n \ge 1.$$

To construct puzzles, we will use rays of different basins converging to the same point in the Julia set:

Definition 3.2. We say that a ray $R_i(t) \subset B_i$ is a *cut ray* if it converges to the landing point of another ray $R_j(t')$ where $j \neq i$. The angle $t \in S^1$ of such a ray is called a *cut angle* in B_i.

The goal of this section is to determine the set of cut angles. Most properties we establish are proved in [He] and [T1] but under the assumption that $J(N)$ is locally connected. For this reason, detailed arguments are provided here, though most of them are identical. The essential difference resides in Proposition 3.12 and Lemma 3.15.

3.1. *Rays landing at the same point.*

PROPOSITION 3.3. *The only cut angle in B_3 is 0. Furthermore, t is a cut angle in B_1 if and only if $1-t$ is a cut angle in B_2 and, if so, the corresponding cut rays $R_1(t)$ and $R_2(1-t)$ land at the same point.*

Let G denote the set of angles $t \in \mathbf{S}^1$ such that $R_1(t)$ and $R_2(1-t)$ land at the same point. The above proposition asserts that G is the set of cut angles in B_1 and that the set of cut angles in B_2 is $\{1-t,\ t \in G\}$.

Remark 3.4. G contains 0 and 1/2 by Lemma 2.5.

LEMMA 3.5. *For $i \in \{1, 2, 3\}$, the only ray in B_i converging to ∞ is $R_i(0)$.*

Proof. Assume that $R_i(\theta)$ converges to ∞ for some i and some $\theta \in (1/4, 1/2)$. Then the rays $R_i(\theta + 1/2)$, $R_i(1/2)$ and $R_i(2\theta - 1/2)$ all land at points of $N^{-1}(\infty) \setminus \{\infty\}$ (see Trivial Fact 2.6). But $0 < 2\theta - 1/2 < \theta < 1/2$ and $1/2 < 2\theta < \theta + 1/2 < 1$ for the cyclic order, so each of the three connected components of $\widehat{\mathbf{C}} \setminus \delta_i$, where $\delta_i = \overline{R}_i(0) \cup \overline{R}_i(\theta) \cup \overline{R}_i(2\theta)$, contains the landing point of exactly one of those rays. This contradicts the fact that only two preimages of ∞ lie outside of δ_i. Now, if $R_i(\theta)$ lands at ∞ with $\theta \notin (1/4, 1/2)$ and θ nondyadic, some $2^n\theta \bmod 1$ is in $(1/4, 1/2)$ and the above applies. Finally no rays of nonzero dyadic angle converge to ∞ by Trivial Fact 2.6. □

LEMMA 3.6. *Let $\theta \in \mathbf{S}^1$ be such that $2\theta \in G$. If $\theta \notin G$ then the rays $R_1(\theta)$ and $R'_2(1-2\theta)$ converge to the same point, as well as $R_2(1-\theta)$ and $R'_1(2\theta)$. Moreover, the two landing points are distinct. On the other hand, if $\theta \in (0, 1/2)$ belongs to G, the rays $R'_1(2\theta)$ and $R'_2(1-2\theta)$ land at the same point.*

Proof. We assume that $\theta \neq 0$ and 1/2, the special case $\theta = 1/2$ follows from the definition of B_1 and B_2. Since $2\theta \in G$, the rays $R_1(2\theta)$ and $R_2(1-2\theta)$

converge to the same point, denoted x. Hence, each of the six rays $R_1(\theta)$, $R_1(\theta+1/2)$, $R'_1(2\theta)$, $R_2(1-\theta)$, $R_2(1/2-\theta)$, $R'_2(1-2\theta)$ lands at a preimage of x. Opposite rays $R_i(t)$ and $R_i(t+1/2)$ are separated by the Jordan curve $\gamma=\overline{R}_1(0)\cup\overline{R}_1(1/2)\cup\overline{R}_2(0)\cup\overline{R}_2(1/2)$. Since no preimage of x is on this curve (Lemma 3.5), γ separates the six closed rays into two groups. If $x=N(x_0)$, then x has only two preimages and all the rays in each group converge to the same point. If $x\neq N(x_0)$, then x has three preimages and only one preimage of $R_1(2\theta)$ (resp. of $R_2(1-2\theta)$) lands at each of them (Trivial Fact 2.6). Thus if $\theta\notin G$, the only possibility is that $R_1(\theta)$ and $R'_2(1-2\theta)$ land at the same point, as well as $R_2(1-\theta)$ and $R'_1(2\theta)$. In the case where $\theta\in G$ let us assume that $\theta\in(0,1/2)$ (else $\theta+1/2\in(0,1/2)$). Then $R_1(\theta+1/2)$ and $R_2(1/2-\theta)$ are separated from the four other rays by γ (by Remark 2.10 B'_1 and B'_2 are in the other component of $\overline{\mathbf{C}}\setminus\gamma$). Hence they converge to the same point, as well as $R'_1(2\theta)$ and $R'_2(1-2\theta)$. □

Corollary 3.7. *For every $n\in\mathbf{N}$, the angle $1-1/2^n$ belongs to G but there exists a smallest n_0 so that $1/2^{n_0}$ is not in G.*

Proof. An easy induction using Lemma 3.6 shows that $R_1(1-1/2^n)$ and $R_2(1/2^n)$ land at the same point since the curve $\gamma=\overline{R}_1(0)\cup\overline{R}_1(1/2)\cup\overline{R}_2(0)\cup\overline{R}_2(1/2)$ separates $\overline{B}'_2$ and $\overline{R}_1(1-1/2^n)$ for $n\geq 2$.

Assume now that $1/2^n\in G$ for all $n\in\mathbf{N}$. Let D_n be the unbounded connected component of $\widehat{\mathbf{C}}\setminus\big(\overline{R}_1(\pm1/2^n)\cup\overline{R}_2(\pm1/2^n)\cup E_1(2^n)\cup E_2(2^n)\big)$ and $A_n=D_n\setminus\overline{D}_{n+1}$. It is easy to check that N induces a homeomorphism from A_n to A_{n-1}, so that the annuli A_n have equal moduli. Therefore, $\bigcap D_n=\{\infty\}$ (see [A]) which contradicts the fact that $B_3\subset D_n$. □

Lemma 3.8. *Two different rays in the same basin B_i cannot converge to the same point.*

Proof. Given $\theta,\theta'\in[0,1)$ with $\theta<\theta'$, there is some $n\in\mathbf{N}$ such that $2^n\theta$ and $2^n\theta'$ are distinct and belong to $[0,1/2]$ and $[1/2,1]$ respectively. If $R_i(\theta)$ and $R_i(\theta')$ land at the same point, then $R_i(2^n\theta)$ and $R_i(2^n\theta')$ converge to ∞ or $\zeta\in f^{-1}(\infty)$. Indeed, $R_i(2^n\theta)$ and $R_i(2^n\theta')$ lie in different components of $\widehat{\mathbf{C}}\setminus\gamma_i$ where $\gamma_1=\gamma_2=\bigcup_{j=1,2}\overline{R}_j(0)\cup\overline{R}_j(1/2)$. Using the integer n_0 given by Corollary 3.7, we have

$$\gamma_3=\overline{R}_1(0)\cup\overline{R}_1(1/2^{n_0-1})\cup\overline{R}'_2(1-1/2^{n_0})\cup\overline{R}'_2(0)\cup\overline{R}_3(1/2)\cup\overline{R}_3(0)$$

(γ_3 is a curve by Lemma 3.6). By the Trivial Fact 2.6, a nonzero dyadic angle cannot land at ∞, and so, after iterations, the result follows from Lemma 3.5. □

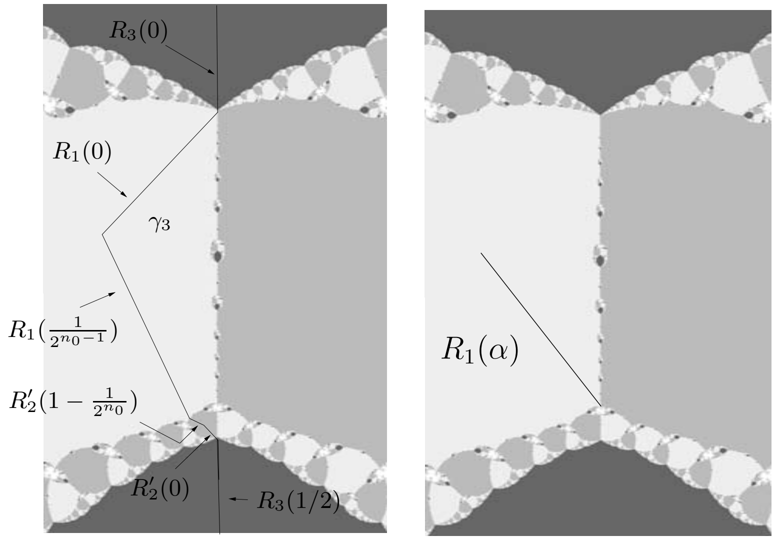

Figure 3: On the left, the curve γ_3 used in Lemma 3.8. On the right Head's angle (Definition 3.11)

COROLLARY 3.9. *Fix some $i \in \{1,2,3\}$ and $m \geq 0$. Let U, V be connected components of $N^{-m}(B_i)$. If two rays $R_U(t)$ and $R_V(t')$ land at the same point then either $U = V$ and $t = t'$, or the landing point is an iterated preimage of the critical point x_0.*

Proof. The rays $N^m(R_U(t))$ and $N^m(R_V(t'))$ both lie in B_i and coincide because they land at the same point (Lemma 3.8). Let $k \leq m$ be the smallest integer such that $N^k(R_U(t))$ is equal to $N^k(R_V(t'))$. If $t \neq t'$ or $U \neq V$ then $k \geq 1$ and the common landing point of $N^{k-1}(R_U(t))$ and $N^{k-1}(R_V(t'))$ is x_0 by the Trivial Fact 2.6. □

Remark 3.10. The proofs of Lemmas 3.5, 3.6, 3.8 and the corollaries 3.7, 3.9 still work in the case where $N(x_0) = \infty$ (with $n_0 = 2$), but not the proof of Proposition 3.3.

Proof of Proposition 3.3. The idea is to study the position of the rays relative to the curve $\gamma = \overline{R}_1(0) \cup \overline{R}_2(0) \cup \overline{R}_1(1/2) \cup \overline{R}_2(1/2)$.

Assume first that $R_1(\theta)$ and $R_2(\theta')$ land at the same point x. As long as $N^n(x)$ avoids γ, the iterated images of the two rays lie in the same connected component of $\widehat{\mathbf{C}} \setminus \gamma$. Hence θ' and $1-\theta$ have the same dyadic expansion. If the point $N^n(x)$ falls in γ for some n then $2^n\theta = 2^n\theta' = 0$ or $1/2$ (Lemma 3.8) and equality $\theta' = 1-\theta$ follows since all previous terms in the dyadic expansion coincide.

Assume now that the rays $R_3(\theta)$ and $R_1(\theta')$ land at the same point, with $(\theta, \theta') \neq (0,0)$. If $2^n\theta' = 1/2$ for some $n \geq 0$, the rays $R_3(2^n\theta)$, $R_2(1/2)$,

$R_1(1/2)$ land at the same preimage ξ of ∞ in a cyclic order different from that of their images $R_3(2^{n+1}\theta)$, $R_2(0)$, $R_1(0)$ — observe that $R_3(2^n\theta)$ and $R_3(2^{n+1}\theta)$ are in the same connected component of $\widehat{\mathbf{C}} \setminus \gamma$. This contradicts assumption 1 so there exists $n \geq 0$ such that $2^n\theta' \bmod 1 \in (1/2, 1)$. The ray $R_1(2^n\theta')$ is separated from B_3 by γ (look at the cyclic order in B_1) and so has to land at ξ or ∞. But θ' would be dyadic (Lemma 3.8). The same argument works for B_2 instead of B_1. □

3.2. *Characterization of cut angles and localization of x_0.*

Definition 3.11. Let $\alpha = \alpha(N)$ be the infimum of G for the order obtained by identifying $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$ with $(0, 1]$. The angle α is called *Head's angle* of N (see [He], [T1]).

If an angle θ belongs to G then $2^n\theta$ is in $[\alpha, 1]$ for every $n \geq 0$ because $2^n\theta \in G$. This property characterizes rational cut angles:

PROPOSITION 3.12. *Let $\theta \in \mathbf{Q}/\mathbf{Z}$. If $2^n\theta$ belongs to $(\alpha, 1]$ for all $n \geq 0$ then θ is a cut angle in B_1, i.e. $\theta \in G$.*

LEMMA 3.13. *If $\theta \in (\alpha, 1]$ and $2\theta \in G$ then $\theta \in G$.*

Proof. Let $t < \theta$ be a point in G. The curve $\overline{R}_1(0) \cup \overline{R}_2(0) \cup \overline{R}_1(t) \cup \overline{R}_2(1-t)$ separates $R_1(\theta)$ and B_2', so θ is in G by Lemma 3.6. □

To prove Proposition 3.12, we will construct open discs $U \subset V$ containing the landing points x_1, x_2 of $R_1(\theta)$, $R_2(1-\theta)$ and such that N^k induces a homeomorphism $U \to V$, where θ is of period k under doubling. We will then conclude that $x_1 = x_2$ by the Schwarz' Lemma. The following localisation Lemma will be used to leave the critical point x_0 out of U:

LEMMA 3.14. *Let $\eta \in G$ and let $\gamma(\eta) = \overline{R}_1(0) \cup \overline{R}_1(\eta) \cup \overline{R}_2(0) \cup \overline{R}_2(1-\eta)$. The critical points x_0 and $\phi_1(]0, \eta[)$ are in the same connected component of $\widehat{\mathbf{C}} \setminus \gamma(\eta)$. More precisely, given $\kappa \in \mathbf{S}^1 \setminus G$ such that $2\kappa \in G$, the set $\gamma(\eta, \kappa) = \overline{R}_1(\eta) \cup \overline{R}_2(-\eta) \cup \overline{R}_1(\kappa) \cup \overline{R}_2(-\kappa) \cup \overline{R}_1'(2\eta) \cup \overline{R}_2'(-2\eta) \cup \overline{R}_1'(2\kappa) \cup \overline{R}_2'(-2\kappa)$ separates x_0 from ∞.*

Proof. The first assertion readily follows from the second one. On the other hand, we may assume $\eta \leq 1/2$ since this just makes the bounded connected component of $\widehat{\mathbf{C}} \setminus \gamma(\eta, \kappa)$ smaller. By Lemma 3.6, the curve $\gamma = \gamma(\eta, \kappa)$ and its image $N(\gamma) = \overline{R}_1(2\eta) \cup \overline{R}_2(1-2\eta) \cup \overline{R}_1(2\kappa) \cup \overline{R}_2(1-2\kappa)$ are Jordan curves. Therefore, the bounded component B of $\widehat{\mathbf{C}} \setminus \gamma$ is a disc and its image $N(B)$ is also a disc (a component of $\widehat{\mathbf{C}} \setminus N(\gamma)$) because $\eta, \kappa \leq 1/2$ and the curve $N^{-1}(N(\gamma)) = \gamma \cup \gamma'$ with $\gamma' = \overline{R}_1(\eta + 1/2) \cup \overline{R}_2(1/2 - \eta) \cup \overline{R}_1(\kappa + 1/2) \cup \overline{R}_2(1/2 - \kappa)$ is disjoint from B. Finally $x_0 \in B$ since $N_{|\partial B}$ has degree two. □

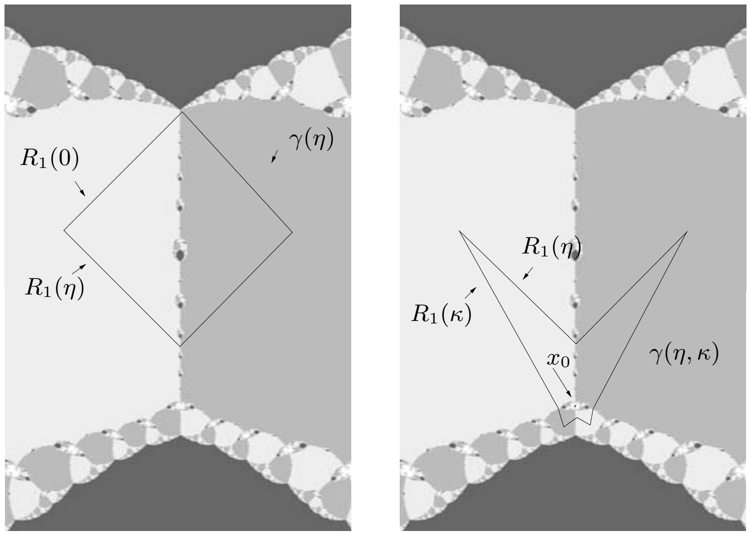

Figure 4: Illustration of Lemma 3.14: on the left, the curve $\gamma(\eta)$, on the right the localization of the critical point x_0 in the bounded component of $\overline{\mathbf{C}}\setminus\gamma(\eta,\kappa)$.

Proof of Proposition 3.12. The angle $\theta \in \mathbf{Q}/\mathbf{Z}$ is (eventually) periodic by $t \to 2t$ since it can be written in the form $r/(2^p(2^k-1))$. Therefore, using Lemma 3.13, we now assume that $2^n\theta$ belongs to $(\alpha, 1)$ for every $n \geq 0$ and that θ is periodic.

1. We first claim that θ is accumulated on both sides by elements of G. Indeed, given $\epsilon > 0$, let n be the first integer such that the interval $2^n[\theta, \theta+\epsilon)$ intersects $(0, \alpha)$. Since $2^n\theta$ lies in $(\alpha, 1)$ there exists $t' \in (\theta, \theta+\epsilon)$ so that $2^n t' = 1$, and hence t' belongs to G (Lemma 3.13). The same argument yields a point $t \in (\theta-\epsilon, \theta) \cap G$.

2. Let k be the period of θ and denote by x_1, x_2 the respective landing points of $R_1(\theta)$, $R_2(1-\theta)$; these points are periodic and their period divides k. Let $t, t' \in G$ be angles surrounding θ (as in point 1.) and let U be the bounded connected component of $\mathbf{C} \setminus \overline{R}_1(t) \cup \overline{R}_1(t') \cup \overline{R}_2(1-t) \cup \overline{R}_2(1-t')$. The disc U contains x_1 and x_2. Moreover for ϵ small enough, $V = N^k(U)$ is also a disc and covers U since multiplication by 2^k is expanding and fixes $\theta = 2^k\theta$. To see that $N^k \colon U \to V$ is a (conformal) homeomorphism, we need to check that x_0 is in no $N^i(U)$ for $0 \leq i < k$. Let $\eta \in G$ be such that $\eta < \inf\{2^i\theta \bmod 1 \mid i \leq k\}$. Then for ϵ small enough $2^i(t,t') \subset (\eta, 1)$ for $i \leq k$, so that $N^i(U)$ is not in the component of $\widehat{\mathbf{C}} \setminus \overline{R}_1(0) \cup \overline{R}_1(\eta) \cup \overline{R}_2(0) \cup \overline{R}_2(1-\eta)$ containing x_0. Injectivity now follows from Lemma 3.14 and $\left(N^k_{|_U}\right)^{-1}$ has only one fixed point so that $x_1 = x_2$. □

Remark. Proposition 3.12 does not show that the Head angle α (assuming it is rational) belongs to G (the above argument fails). This will however follow from the local connectivity of ∂B_i, $i \in \{1,2\}$ (by continuity of the extended Böttcher map).

3.3. *Examples of periodic cut angles.*

Lemma 3.15. *The Head angle α belongs to $(0,1/2)$ (under assumption 1).*

Proof. Assuming $\alpha = 0$, Proposition 3.12 implies that $1/2^n$ is in G for every $n \geq 0$, which contradicts Corollary 3.7. On the other hand, $\alpha \leq 1/2$ since $1/2 \in G$. Suppose now that $\alpha = 1/2$. The critical point x_0 belongs to the bounded connected component U_n of $\widehat{\mathbf{C}} \setminus \gamma_n$, where $\gamma_n = \gamma(1/2, 1/2 - 1/2^n)$ (see Lemma 3.14). Hence $N(x_0)$ belongs to $\cap_{n\in\mathbf{N}} V_n$ where V_n is the unbounded connected component of $N(U_n) \setminus (E_1(2^n) \cup E_2(2^n))$. This is impossible since, as we will now show, $\cap_{n\in\mathbf{N}} V_n = \emptyset$.

Let ψ be a homeomorphism $\overline{V}_1 \to \overline{\mathbf{D}} \cap \{z = x + iy \mid y \geq 0\}$ which is conformal on V_1 and maps $(\overline{R}_1(0) \cup \overline{R}_2(0)) \cap \partial V_1$ to $[-1,1]$. The map N induces a homeomorphism from $\overline{V}_2$ onto $\overline{V}_1$ and we set $g = \psi \circ f \circ \psi^{-1}$ where $f \colon \overline{V}_1 \to \overline{V}_2$ denotes the inverse branch of N. Since g preserves $[-1,1]$, it extends by reflection to a holomorphic map $\tilde{g} \colon \mathbf{D} \to \mathbf{D}$. Now $\tilde{g}(\mathbf{D})$ has compact closure in $\mathbf{D}$, so the intersection of the domains $\tilde{g}^n(\overline{\mathbf{D}})$ is reduced to a point. Similarly, $\cap g^n(\psi(\overline{V}_1))$ is a point and hence $\cap \overline{V}_n = \{\infty\}$. □

Corollary 3.16. *For n large enough (depending on α), the angle $1 - \frac{1}{2^n-1}$ belongs to G.*

Proof. If $\theta_n = 1 - \frac{1}{2^n-1}$, then $2^i\theta_n = 1 - \frac{2^i}{2^n-1}$ is in $(1/2,1)$ for $i < n-1$ and $2^{n-1}\theta_n$ equals $\frac{1}{2} - \frac{1}{2}\frac{1}{2^n-1}$ and so is in $(\alpha,1)$ for n large enough. Hence, by Proposition 3.12, θ_n belongs to G. □

To construct articulated rays in Section 4 (Proposition 4.3), we will need the following lemma:

Lemma 3.17. *For any $\zeta \in G$ with $\zeta < 2\alpha$, there exists $\theta \in G$ dyadic such that $\zeta \leq \theta < 2\alpha$.*

Proof. Let $\beta = \sup\{t \notin G \mid 2t \in G\}$. Note that $\beta \leq \alpha \leq 2\beta$. Moreover, $t > \beta \implies t \in G$ or $2t \notin G$.

1. We prove that if $\beta < \alpha$ then β is dyadic and that $\theta = 2\beta$ works: The interval $2^j(\beta,\alpha)$ does not intersect G provided $2^i(\beta,\alpha) \subset (\beta,1)$ for $1 \leq i \leq j-1$. The first interval $2(\beta,\alpha)$ always lies in $[\alpha,1)$ because $\alpha < 1/2$ and does not intersect G by definition of β. The rest follows by induction. Hence, the first interval $2^n(\beta,\alpha)$ that meets $(0,\beta)$ is disjoint from G, and so is included in $[0,\alpha]$. Assume that $2^n\beta \neq 0 \bmod 1$. By definition of β, there exists a sequence $\beta_k < \beta$ converging to β such that $2\beta_k \in G$ and so $2^n\beta_k \in G$. This contradicts

the fact that for k large enough $0 < 2^n\beta_k < 2^n\beta \leq \alpha$. Finally, $\theta = 2\beta \in G$ by Proposition 3.12 and $2\beta \geq \zeta$ since $\zeta \in G$ and $2(\beta, \alpha)$ does not intersect G.

2. Assume now that $\beta = \alpha$ and let $\beta_k \in (0, \beta)$ be angles converging to β such that $2\beta_k \in G$. Let also $n \geq 1$ be the smallest integer such that $2^n(\beta_k, \beta)$ intersects $(0, \alpha)$. Then either $2^n\beta_k = 0 \bmod 1$ or $1 \in (2^n\beta_k, 2^n\beta)$ (on the circle). Therefore, $G \cap (2\beta_k, 2\beta)$ contains a dyadic angle which, for k large enough, is arbitrarily close to $2\beta = 2\alpha$. □

4. Articulated rays

Cut rays can be used to construct puzzles. Indeed, a typical example of such a puzzle consists of equipotentials $E_i(v)$ (for any $v > 0$), the rays $\overline{R}_i(0)$, with i in $\{1, 2, 3\}$ and some properly chosen periodic cycle of cut rays (see Section 5). However, these graphs all contain the point ∞ and therefore, even taking different cut angles, it seems difficult to surround points close to ∞ in order to apply Theorem 2.15.

The articulated rays considered in this section avoid the point ∞. They are arcs (connecting B_1, B_2 to B_3 away from the rays $R_i(0)$) whose behavior under the dynamics is similar to that of rays. "Periodic" articulated rays will be especially useful to build puzzles. Articulated rays will also reappear in Section 7.

Definition 4.1. An *articulated ray* stemming from B_i with angle θ is a curve L satisfying the following properties:

- $L = \bigcup_{k \geq 0} \bar{l}_k$ where $\bar{l}_k$ is the closure of a converging ray l_k ;
- $l_0 = R_i(\theta)$ and the depth of l_1 is greater than that of l_0 (i.e. is at least 1) ;
- $\bar{l}_k$ and $\bar{l}_{k+1}$ intersect in exactly one point ;
- l_{k+2} is of depth greater than l_k.

A curve $L = \bigcup_{0 \leq k \leq m} \bar{l}_k$ with the same properties will be called a *finite articulated* ray. Articulated rays stemming from any preimage U of B_i are defined similarly, just by replacement of $R_i(\theta)$ by $R_U(\theta)$ in the second item.

Recall that the *depth* of a ray $R_U(\theta)$ is the smallest integer p such that $N^p(U) = B_j$ for some $j \in \{1, 2, 3\}$.

Each articulated ray $L = \bigcup_{k \geq 0} \bar{l}_k$ has a natural parametrization $\rho\colon [0, +\infty) \to \widehat{\mathbf{C}}$: for every $k \geq 0$ and every $t \in [0, 1]$, the points $\rho(2k + t)$ and $\rho(2k + 2 - t)$ belong to l_{2k} and l_{2k+1} respectively, and their Böttcher coordinates have modulus t. We say that L *converges* to a point y if $\rho(t)$ has limit y as t goes to ∞.

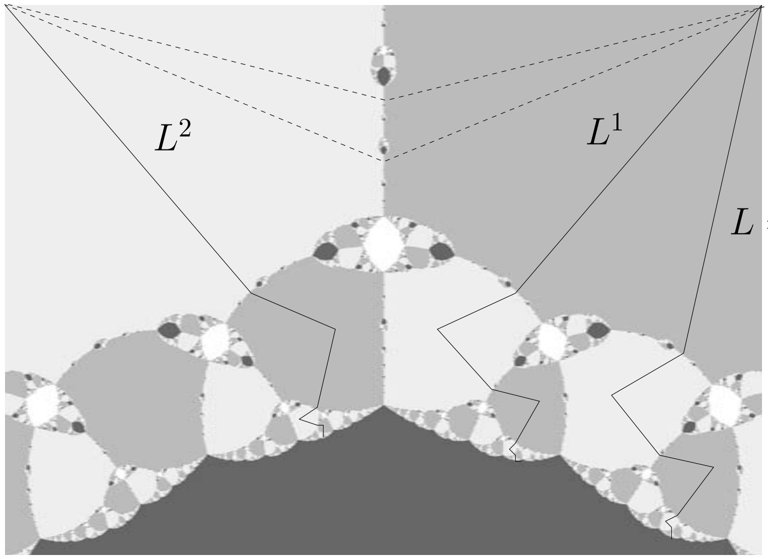

Figure 5: The articulated rays L^0, L^1, L^2 and the image $N^3(L)$ which is L union the dashed rays.

Remark 4.2. Two rays l_i, l_j of an articulated ray L lie in the same Fatou component if and only if $\{i, j\} = \{2k-1, 2k\}$ for some $k \geq 1$. Moreover, L has no self-intersection unless it contains some iterated preimage of x_0 (Lemma 3.8 and Corollary 3.9).

Corollary 3.16 gives a lot of nondyadic angles in $G \cap (\alpha, 2\alpha)$. We use them now to construct a "3-periodic" articulated ray, meaning that $N^3(L)$ differs from L only from a finite number of rays of depth 0.

Proposition 4.3. *Let $\zeta \in G \cap (\alpha, 2\alpha)$ be a nondyadic rational angle and y the landing point of the ray $R_3(1/7)$. There exists a unique articulated ray L stemming from B_2 with angle $-\zeta/4$, converging to y and satisfying the 3-periodicity condition*:

$$N^3(L) = L \cup \left(\overline{R}_1(\zeta) \cup \overline{R}_2(-\zeta) \cup \overline{R}_1(2\zeta) \cup \overline{R}_2(-2\zeta)\right).$$

The method developed below can actually be used to obtain periodic articulated rays converging to any periodic point of ∂B_3, precisely; given a k-periodic point y on ∂B_3, there exists a unique articulated ray L stemming from $B_1 \cup B_2$, landing at y and satisfying $L \subset N^k(L) \subset L \cup \bigcup_{i\geq 0}(R_1(2^i\zeta) \cup R_2(-2^i\zeta))$. The general argument, however, is more tricky and is unnecessary for Theorems 1 and 6, so we will not make it here. The basic idea is to trace the itinerary of the periodic point and to build the articulated ray backward.

Proof of Proposition 4.3. Let $\theta \in G$ be a dyadic angle with $\zeta \leq \theta < 2\alpha$ (Lemma 3.17 provides some) and denote by V the connected component of

$\widehat{\mathbf{C}} \setminus \big(\overline{R}_1(0) \cup \overline{R}_2(0) \cup \overline{R}_1(\theta) \cup \overline{R}_2(-\theta) \cup \overline{R}_3(0) \cup E_3(v)\big)$ $(v > 0)$ which does not contain the critical value $N(x_0)$ (see Lemma 3.14 and Figure 6). Since V is a disc, $N^{-1}(V)$ has three connected components and, for $i \in \{1, 2\}$, only one of them intersects both B_3 and B_i ; we denote it by U_i (see Figure 6) and proceed in four steps.

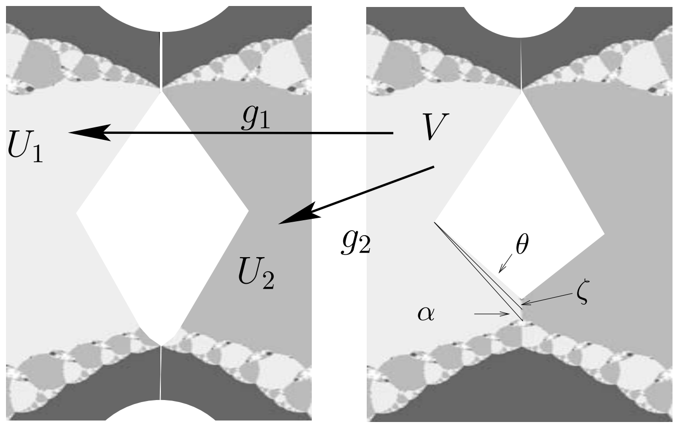

Figure 6: The domains U_1, U_2 and V.

1. *Uniqueness.* Since $\zeta/2 \notin G$, the curve $N(L)$ is an articulated ray (stemming from B_2 with angle $-\zeta/2$) but $N^2(L)$ is not. Indeed, the announced form of $N^3(L)$ shows that the second ray in $N^2(L)$ lies in B_1' or in B_1 but B_1' is ruled out for the following reason: $\zeta > \alpha$, so that there exists $\theta' \in G \cap [\alpha, \zeta)$ and the curve $\overline{R}_1(0) \cup \overline{R}_2(0) \cup \overline{R}_1(\theta') \cup \overline{R}_2(-\theta')$ separates $\overline{R}_2(-\zeta)$ from $\overline{B}_1'$ (Remark 2.10 and the cyclic ordering of angles). Hence, the image $N^2(L)$ equals $\overline{R}_1(\zeta) \cup \overline{R}_2(-\zeta) \cup L^2$ where L^2 is an articulated ray stemming from B_1, and $N(L^2) = \big(\overline{R}_1(\zeta) \cup \overline{R}_2(-\zeta)\big) \cup L$ by the periodicity condition. Set $L^0 = L$, $L^1 = N(L)$ and $L^i = \cup_{k \geq 0} \, \overline{l}_k^i$ for $i \in \{0, 1, 2\}$ (L^2 is as defined before). The relations $N(l_0^1) = N(l_1^2) = R_2(-\zeta)$ and also $N(l_1^1) = N(l_0^2) = R_1(\zeta)$ determine the first rays of L^1 and L^2: $l_0^2 = R_1(\zeta/2)$, $l_0^1 = R_2(-\zeta/2)$, $l_1^2 = R_2'(-\zeta)$, $l_1^1 = R_1'(\zeta)$. The similar identities $N(l_k^0) = l_k^1 = N(l_{k+2}^1)$ and $N(l_{k+2}^2) = l_k^0$ will guarantee uniqueness provided we have inverse branches for N and know which one to choose. This will be shown in the next step by proving that $L^0 \cup L^1$ and L^2 are contained in $U_2 \cup N^{-1}(\{b_1, b_2\})$ and $U_1 \cup N^{-1}(\{b_1, b_2\})$ respectively, and by observing that N is univalent on those sets.

2. *Localization.* First, the articulated rays L^i all lie in $V \cup \{b_1, b_2\}$. Indeed, by step 1, each articulated ray starts with $\overline{l}_0^i$ in $V \cup \{b_1, b_2\}$. Moreover, if L^i crosses ∂V, there exists j so that $2^j\theta = \zeta$ or $2^j\zeta = 0$ (recall that L^i consists only of preimages of $R_1(\zeta)$ and $R_2(-\zeta)$, and that two rays in B_k cannot land at the same point). This is impossible since θ is dyadic but not ζ.

The following relations $N(L^0) = L^1, N(L^1) = L^2 \cup \overline{R}_1(\zeta) \cup \overline{R}_2(-\zeta)$ and $N(L^2) = L \cup \overline{R}_1(\zeta) \cup \overline{R}_2(-\zeta)$ imply that $N(L^i)$ all lie in $V \cup \{b_1, b_2\}$. Therefore L^0 and L^1 lie in $U_2 \cup N^{-1}(\{b_1, b_2\})$ (just because they start there by step 1) while L^2 lies in $U_1 \cup N^{-1}(\{b_1, b_2\})$.

3. *Existence*. Let $g_i \colon V \to U_i$ denote the inverse of the homeomorphism $U_i \to V$ induced by N. The articulated rays L_i are constructed as follow. Their rays of depth 0 and 1 are determined in point 1. The other rays are obtained inductively by applying the relevant inverse branch to each additional relation of step 1: $l^2_{k+2} = g_1(l^1_k)$, $l^0_k = g_2(l^1_k)$, $l^0_{k+2} = g_2(l^2_k)$. The curves L^0, L^1 and L^2 constructed this way are clearly articulated rays.

We will now prove that L^0 converges to y. For this, we rewrite the above relations as $l^i_{k+2} = h_i(l^i_k)$ where $h_0 = g_2 g_1 g_2$, $h_1 = g_2^2 g_1$ and $h_2 = g_1 g_2^2$ are as defined on V. We claim that $\overline{h_i(V)} \subset V$. Indeed, $h_i(V)$ is included in $g_1(U_2)$ or $g_2(U_1)$. Moreover, $\partial g_1(U_2) \cap \partial V \subset \overline{R}_1(0) \cup \overline{R}_3(0)$ because $g_1(U_2) \subset U_1$. But $\overline{R}_1(0)$ is disjoint from ∂U_2 and hence also from $\partial g_1(U_2)$. On the other hand, $\overline{R}_3(0)$ cannot meet $\partial g_1(U_2)$; otherwise the latter would not be connected (the only rays converging to ∞ are the $R_i(0)$). Hence, $\overline{g_1(U_2)} \subset V$ and similarly $\overline{g_2(U_1)} \subset V$.

The map $h_0 \colon V \to V$ contracts the hyperbolic metric by a bounded factor, so that the hyperbolic lengths of the rays l^0_k decrease geometrically and L^0 converges. Moreover, the limit point is a fixed point of h_0 in U_2. Now, h_0 has at most one fixed point and, considering the action of h_0 on the rays of $B_3 \cap V$, we see that $h_0(y) = y$. Therefore, L converges to y. □

Corollary 4.4. *Let L be the articulated ray of Proposition* 4.3. *Then L, $N(L)$ and $N^2(L)$ do not intersect outside* $\{b_2\}$.

Proof. In the previous proof (Proposition 4.3 point 2.) it is shown that L^0 and L^1 lie in $U_2 \cup N^{-1}(\{b_1, b_2\})$ while L^2 lies in $U_1 \cup N^{-1}(\{b_1, b_2\})$. Hence L^1 and L^2, as well as L^0 and L^2, do not intersect. If L^0 and L^1 intersect, their images L^1 and $\overline{R}_1(\zeta) \cup \overline{R}_2(-\zeta) \cup L^2$ also do. However, this is not possible out of b_2. Hence L^0 and L^1 can intersect only at depth 1, i.e. at $N^{-1}(\{b_2\})$. But l^0_1, l^0_2 are not in B_2', so that L^0 and L^1 only intersect at b_2. □

Remark 4.5. The orbit of the articulated ray L avoids the critical point x_0.

This is just by construction: L, L^1, L^2 belong to $U_1 \cup U_2$ and the orbit of $\overline{R}_1(\zeta) \cup \overline{R}_2(-\zeta)$ avoids x_0.

5. Graphs defining puzzles

In this section we define graphs in $\overline{X} = \widehat{\mathbf{C}} \setminus \cup_{i=1,2,3} \phi_i(e^{-v}\overline{\mathbf{D}})$ (for any $v > 0$) that turn out to be useful puzzles for well chosen angles. At the end of the section we establish that the closure of the puzzle pieces intersect ∂B_i under a connected set.

5.1. *Puzzles.*

Definition 5.1. Let $\zeta \in G \cap [\alpha, 2\alpha)$ be a nondyadic rational angle and L the articulated ray constructed in Proposition 4.3 stemming from B_2 with angle $-\zeta/4$. Given θ, η in $G \cap \mathbf{Q}/\mathbf{Z}$, we define two graphs $I(\theta)$ and $II(\zeta, \eta)$ as follows:

$$I(\theta) = \partial X \cup \left(\overline{X} \cap \left(\overline{R}_1(0) \cup \overline{R}_2(0) \cup \overline{R}_3(0) \cup \bigcup_{j \geq 0} \left(\overline{R}_1(2^j\theta) \cup \overline{R}_2(-2^j\theta) \right) \right) \right);$$

$$II(\zeta, \eta) = \partial X \cup \left(\overline{X} \cap \left(\bigcup_{j \geq 0} \left(N^j(L) \cup \overline{R}_1(2^j\eta) \cup \overline{R}_2(-2^j\eta) \cup \overline{R}_3(2^j/7) \right) \right) \right).$$

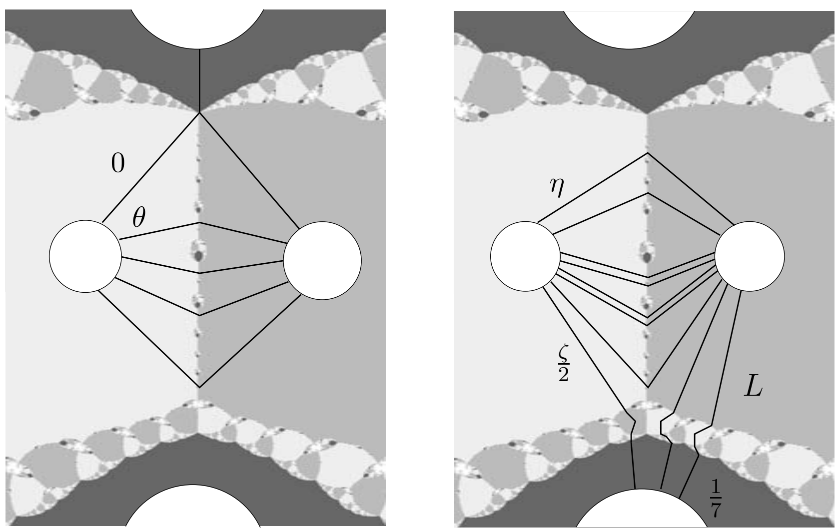

Figure 7: On the left the graph $I(\theta)$, on the right $II(\zeta, \eta)$.

Remark 5.2. The cycle of cut rays generated by $R_1(\eta) \cup R_2(-\eta)$ is added in order to (clearly) surround the critical point for the graph II (see Proposition 5.4).

Remark 5.3. The graphs considered in Definition 5.1 are connected and do not have terminal edges since $\theta, \eta \in G$ (see Figure 7).

For the purpose of the proof of Proposition 5.4 below, we recall some notation:

$$Q_i(t_1,t_2)=\overline{\{\phi_i(r\,e^{2i\pi t}),\ r\in[0,1),t_1\le t\le t_2\}}$$ (see example 3.1);

for $2\kappa\in G$, with $\kappa\notin G$ (resp. $\kappa\in G$), $\gamma(\eta,\kappa)$ is the curve (resp. the two curves)

$$\overline{R}_1(\eta)\cup\overline{R}_2(-\eta)\cup\overline{R}_1(\kappa)\cup\overline{R}_2(-\kappa)\cup\overline{R}'_1(2\eta)\cup\overline{R}'_2(-2\eta)\cup\overline{R}'_1(2\kappa)\cup\overline{R}'_2(-2\kappa)$$

(see Lemma 3.14 and Figure 4); and $\gamma(\theta)=\overline{R}_1(\theta)\cup\overline{R}_2(-\theta)\cup\overline{R}_1(0)\cup\overline{R}_2(0)$ for $\theta\in G$.

PROPOSITION 5.4. *Let $\theta=1-\frac{1}{2^n-1},\eta=1-\frac{1}{2^m-1}$ and $\zeta=\frac{1}{2^i}(1-\frac{1}{2^r-1})$. For n,m,r large enough $\theta,\eta,2^i\zeta$ are in G; and for i properly chosen, the graphs $I(\theta)$ and $II(\zeta,\eta)$ are puzzles surrounding the critical point x_0. Moreover, for $n>m>r$ large enough, every point of $\partial B_1\cup\partial B_2$ is surrounded by $I(\theta)$ or $II(\zeta,\eta)$ at a uniformly bounded depth.*

Proof. We focus here on ∂B_1 (the argument is similar for ∂B_2).

Step 1. *For n,m,r large enough $\theta,\eta,2^i\zeta$ are in G, and for i properly chosen, $I(\theta)$ and $II(\zeta,\eta)$ are puzzles.* Indeed, for n,m,r large enough, θ, η and $1-1/(2^r-1)$ belong to G by Corollary 3.16 and, for some i (depending on r) $\zeta=1/2^i(1-1/(2^r-1))$ lies in $(\alpha,2\alpha)$. By definition the graphs satisfy the condition $\Gamma\cap X'\subset f^{-1}(\Gamma)$. Moreover, if the orbit of the critical point meets one of the graphs, it must necessarily be at the landing point of a ray. The critical point would then follows this periodic ray but this is avoided by changing n,m,r to larger values.

Step 2. *Every point of $X'\setminus N^{-1}(I(\theta))$ sitting in $Q_1(\theta/4,\theta+1/2)$ is surrounded by $I(\theta)$ at depth* 0. Let x be such a point. The piece $P_0(x)$ is the connected component of $X\setminus\gamma(\theta/2)$ intersecting $R_1(1/4)$ and $P_1(x)$ is included in a bounded connected component of $X'\setminus\gamma(\theta+1/2,\theta/4)$ since $x\in Q_1(\theta/4,\theta+1/2)$ (see Figure 8 below).

Hence $\overline{P}_1(x)\subset P_0(x)$ since any point of $\gamma(\theta+1/2,\theta/4)$ which is also on $\partial P_0(x)$ would be critical (Lemma 3.8 and trivial fact 2.6) but there is no periodic critical point on $J(N)$.

Step 3. *For n large enough the critical point x_0 is surrounded by $I(\theta)$.* By Remark 2.16 and Step 2 above, it suffices to show that $N^k(x_0)$ is for some k in D the bounded connected component of $\mathbf{C}\setminus\gamma(\theta+1/2,\theta/4)$. Let k be the smallest integer such that $2^k\alpha\in[1/4,1/2)$. For n large, $2^k\alpha<\theta+1/2<\theta/2<1/2$. Hence, $\alpha<(\theta+1/2)/2^k<\theta/2^{k+1}\le 2\alpha$ and Lemma 3.14 insures that x_0 belongs to the bounded component D_k of $\widehat{\mathbf{C}}\setminus\gamma(\frac{\theta+1/2}{2^k},\frac{\theta}{2^{k+2}})$. Now, looking at the image of the rays in B_1, one sees that N^k maps D_k inside D.

Step 4. *Every point of $X'\setminus N^{-1}(II(\zeta,\eta))$ sitting in $Q_1(\epsilon/2+1/2,\zeta/4)$ is surrounded by $II(\zeta,\eta)$ at depth* 0, *where $\epsilon=\sup_{i\ge 0}\{2^i\zeta \bmod 1,\ 2^i\eta \bmod 1\}$.* Let x be such a point. The piece $P_0(x)$ is the unbounded connected component of

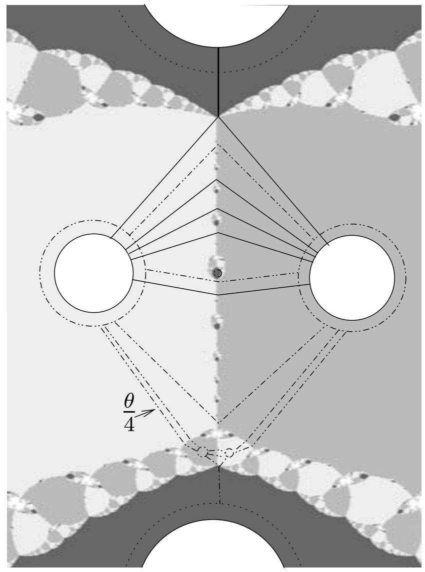

Figure 8: The graph $N^{-1}(I(\theta))$ dashed.

$\widehat{\mathbf{C}} \setminus II(\zeta,\eta)$. It is very easy to see that $\partial P_1(x) \cap \partial P_0(x)$ is empty. Indeed, $P_1(x)$ is the unbounded connected component of $N^{-1}(P_0(x))$. Hence its boundary consists of the rays $R_1(1/2+\epsilon/2)$, $R_2(1/2-\epsilon/2)$, $R_3(1/14)$, $R_3(11/14)$ and the preimage of L stemming from B_2 with angle $-\zeta/8$ together with the preimage of L^2 stemming from B_1 with angle $\zeta/4$ (L^2 is the articulated ray in $N(L)$ and stems from B_1; see Figure 5).

Step 5. *The critical point x_0 is surrounded by $II(\zeta,\eta)$ provided $m > r$.* Let $\zeta_1 = 1 - \frac{1}{2^r-1}$. If $m > r$ then $\zeta_1 < \eta < 1$, so that $\zeta_1/4 < \eta/4 < \zeta+1/2 < \zeta_1/2$. Hence the piece $P_1(x_0)$ is included in D' the bounded connected component of $X' \setminus \gamma(\eta/4, \zeta+1/2)$. It is easy to see that $\overline{D'} \cap II(\zeta,\eta)$ is empty. The end of the argument goes as in Step 3: if k denotes the smallest integer such that $2^k\alpha \in [1/4, 1/2)$, for m, r large enough $N^k(x_0) \in D'$.

Step 6. *For n, m, r, i such that Steps* 1–5 *hold, every point of ∂B_1 is surrounded by $I(\theta)$ or $II(\zeta,\eta)$ at a uniformly bounded depth.* The points of $\partial B_1 \cap I(\theta)$ are mapped to the landing point z of $R_1(\theta)$. For n large, $\epsilon/2+1/2<\theta$, so z belongs to $Q_1(\epsilon/2+1/2, \zeta/4)$ and by Step 4 and Remark 2.16 the points of $\partial B_1 \cap I(\theta)$ are surrounded by $II(\zeta,\eta)$ at uniformly bounded depth . □

To conclude, we now show that ∂B_1 is covered, for some p, by the subsets $N^{-i}(\Delta)$, $0 \le i \le p$, where $\Delta = Q_1(\theta/4, \theta+1/2) \cup Q_1(\epsilon/2+1/2, \zeta/4)$. First, for n large enough, $\epsilon/4 + 1/4 < \theta + 1/2 < 1/2$, so $\Delta \cup N^{-1}(\Delta)$ covers $Q_1(\theta/4, 1/2)$. Let p_0 be such that $0 < \theta/(4.2^{p_0}) < \zeta/4$; then the sets $Q_1(\epsilon/2 + 1/2, \zeta/4)$ and $N^{-i}(Q_1(\theta/4, 1/2))$, $0 \le i \le p_0$, cover $Q_1(\epsilon/2 + 1/2, 1/2)$. Finally for

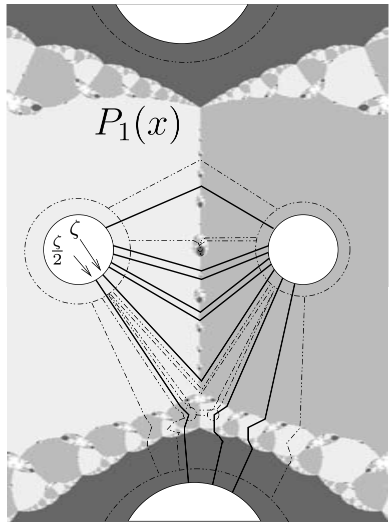

Figure 9: The graph $N^{-1}(II(\zeta,\eta))$ dashed, with the piece $P_1(x)$ of step 4 .

p_1 satisfying $\epsilon/2+1/2 < 1-1/2^{p_1} < 1$, the sets $N^{-i}(Q_1(\epsilon/2+1/2,1/2))$, $0 \le i \le p_1$, cover ∂B_1 entirely. □

5.2. *Connectivity of the neighborhoods.* We establish now the connectivity of $\overline{P}_n(x)\cap\partial B_i$, where P_n is any depth n puzzle piece for $I(\zeta)$ or $II(\zeta,\eta)$.

We prove inductively that for x in ∂B_i, any puzzle piece $\overline{P}_n(x)$ cuts ∂B_i along some $\partial B_i \cap Q_i(t_1,t_2)$. This intersection is connected, as seen in Example 3.1 (where $Q_i(t_1,t_2)$ is defined).

Lemma 5.5. *Let P be an open disc, R any connected component of $N^{-1}(P)$ intersecting B_i, $i\in\{1,2,3\}$. Under condition $(*)$ or $(**)$, $\overline{P}\cap\overline{B}_i = Q_i(\theta_1,\theta_2)$ implies that also $\overline{R}\cap\overline{B}_i = Q_i(\eta_1,\eta_2)$ for $\{\eta_1,\eta_2\}=\{\theta_1/2,\theta_2/2\}$ or $\{\theta_1/2+1/2,\theta_2/2+1/2\}$.*

Condition $(*)$ respectively $(**)$ is that no critical point belongs to R, respectively that $R\cap B_i\subset Q_i(t,t')$ with $|t-t'|\le 1/2$.

Proof. By assumption $\overline{P}\cap\overline{B}_i = Q_i(\theta_1,\theta_2)$, and so the set $R\cap B_i \subset N^{-1}(P)\cap B_i$ lies in $N^{-1}(Q_i(\theta_1,\theta_2))\cap\overline{B}_i$. But $N^{-1}(Q_i(\theta_1,\theta_2))$ intersects $\overline{B}_i$ under the set $\Delta_1 = Q_i(\theta_1/2,\theta_2/2)$ and $\Delta_2 = Q_i(\theta_1/2+1/2,\theta_2/2+1/2)$ since N is conjugated on B_i to $z\mapsto z^2$. So $R\cap B_i\subset\Delta_1\cup\Delta_2$ and R intersects Δ_1 or Δ_2. If R intersects Δ_j it contains $\mathrm{Int}\Delta_j$ since N induces a covering $R\to P$ and $\mathrm{Int}\Delta_j\subset N^{-1}(P)$. But R cannot contain Δ_1 and Δ_2 because of condition $(*)$ or $(**)$. Indeed, under condition $(**)$ Δ_1 and Δ_2 are opposite and under condition $(*)$ $N\colon R\to P$ would be (at least) a double cover over a disc and so ramified in R. Hence $R\cap B_i = \mathrm{Int}\Delta_1$ or $\mathrm{Int}\Delta_2$ so that $\overline{R}\cap\partial B_i = \Delta_1\cap\partial B_i$ or $\Delta_2\cap\partial B_i$. □

COROLLARY 5.6. *For any depth n piece $P_n(x)$ (of the puzzles $I(\theta)$ or $II(\eta,\zeta)$ and $x \in \partial B_i$), $\overline{P}_n \cap \partial B_i$ is connected; more precisely $\overline{P}_n \cap \overline{B}_i = Q_i(\theta_n, \theta'_n) \cap \overline{X}_n$ for $i = 1, 2$ and some θ_n, θ'_n.*

Proof. Every piece $P_n(x)$ satisfies the assumption of Lemma 5.5 under condition $(**)$. Indeed, the pieces $P_n(x)$ are discs and intersect B_i; they are all included in $P_0(x)$ which satisfies condition $(**)$ by choice of θ, η and ζ. □

6. Renormalizations

Definition 6.1. Let $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a rational map and x_0 a critical point of f. Let $k > 1$. The map f is *k-renormalizable at x_0*, if f^k induces, between some discs Y' and Y containing x_0, a quadratic-like map (i.e. a polynomial-like map of degree two) whose filled Julia set $K(f^k)$ is connected (see Definition 2.11). The map $f^k\colon Y' \to Y$ is called a *k-renormalization* of f in x_0. The smallest integer k for which f is k-renormalizable at x_0 is called *the minimal renormalization level.*

Notation 6.2. If $f^k\colon Y' \to Y$ is a k-renormalization of f in x_0, the straightening Theorem (see [D-H2]) shows that f^k is conjugated, by a quasi-conformal map σ, to a unique quadratic polynomial $f_c(z) = z^2 + c$ ($c \in \mathbf{C}$) on a neighborhood of $K = K(f^k)$. The filled Julia set of f_c will be denoted by $K(f_c)$, and $K = \sigma^{-1}\big(K(f_c)\big)$($= K_c$ if there is some ambiguity). If $c \neq 1/4$, the polynomial f_c has exactly one repelling fixed point $\beta(f_c)$ that does not disconnect $K(f_c)$ (see [McM1]), in the terminology of [D-H1] it is the point of external argument 0. Let $\alpha(f_c)$ be the other fixed point ($\beta(f_{1/4}) = \alpha(f_{1/4})$) and $\beta'(f_c) \neq \beta(f_c)$ the other preimage of $\beta(f_c)$. Similarly we note $\beta = \beta_c = \sigma^{-1}(\beta(f_c))$.

6.1. *Linearizability question.*

THEOREM 5. *If N possesses a periodic point x of multiplier $\lambda = e^{2i\pi\alpha}$ with $\alpha \in \mathbf{R}$ then*

(1) *N is renormalizable;*

(2) *$\alpha \in \mathcal{B}$ if and only if N is linearizable near x;*

(3) *if $\alpha \notin \mathcal{B} \cup \mathbf{Q}$ there exist periodic cycles in any neighborhood of x.*

Proof. Let k be the period of x. We first prove that N is k-renormalizable and then that the renormalized polynomial has the desired properties.

For any puzzle Γ of Definition 5.1, the puzzle pieces $P_n(x)$ are well defined since x is not repelling (and so $x \notin \Gamma_n = N^{-n}(\Gamma)$ for all $n \geq 0$). There is some point x' in the orbit of x such that the puzzle pieces $P_n(x')$ contain a critical point x_0, i.e. $P_n(x') = P_n(x_0)$ for $n \geq 0$. Indeed, if for all x' in the orbit of x, the piece $P_n(x')$ is not critical, the restriction $N^k\colon P_{n+k}(x) \to P_n(x)$

would be invertible, and by Schwarz' lemma, x would be an attracting fixed point of the inverse (since for some n, $P_{n+1}(x) \neq P_n(x)$). Let p be such that $\overline{P}_{p+1}(x_0) \subset P_p(x_0)$ (Proposition 5.4 provides graphs satisfying this); the restriction $N^k\colon P_{p+k}(x) \to P_p(x)$ is then a quadratic like map.

Let σ be the straigthening map and $f_c(z) = z^2+c$ the quadratic polynomial conjugated to N^k by σ (see Notation 6.2). The point $\sigma(x)$ is fixed by f_c of multiplier λ since the multiplier of an indifferent fixed point is a quasi-conformal invariant (see [N]). Finally by Yoccoz' result (see [Y]) we obtain that $\alpha \in \mathcal{B}$ if and only if N is linearizable near x and, if $\alpha \notin \mathcal{B}$ there exist small cycles near x. □

6.2. *Renormalization via puzzle pieces.* More generally we prove here (Lemma 6.5) that if N is renormalizable we can take puzzles pieces as sets of renormalization.

LEMMA 6.3. *Let N be k-renormalizable at x_0 with filled Julia set $K = K(N^k)$. If K intersects $\overline{B}_1$ (resp. $\overline{B}_2$) then there exists exactly one ray in B_1 (resp. B_2) accumulating on K. It has angle $p/(2^k-1)$ with $p = p(N) < (2^k-1)/2$ (resp. angle $1 - q(N)/(2^k-1) > 1/2$) and converges to the fixed point β_c of $K = K_c$.*

Proof. We consider the situation where $K \cap \overline{B}_1 \neq \emptyset$. Let V_0 be the connected component of $\widehat{\mathbf{C}} \setminus (\overline{R}_1(0) \cup \overline{R}_1(1/2) \cup \overline{R}_2(0) \cup \overline{R}_2(1/2))$ containing x_0. Then K is included in V_0 since it contains x_0 and cannot cross the eventually fixed ∂V_0. Since $N^k(K) = K$, one can consider for every $n \geq 0$ the connected component V_n of $N^{-k}(V_{n-1})$ containing K. They form a decreasing sequence of open sets: $N^k(\partial V_0) \cap V_0 = \emptyset$; hence $\partial V_0 \cap V_1 = \emptyset$ so that V_1 is included in V_0 and then $V_{n+1} \subset V_n$. After Lemma 5.5 under the condition $(**)$, $\overline{V}_n \cap \overline{B}_1 = Q_1(\theta_n, \theta'_n)$ for some $\theta_n < \theta'_n$ such that $|\theta'_{n+1} - \theta_{n+1}| = 1/2^k |\theta'_n - \theta_n|$. The common limit $\kappa = \lim \theta_n = \lim \theta'_n$ is of the type $p/(2^k-1)$. Indeed, since $N^k(V_{n+1}) = V_n$ for every n, necessarily $\theta_n \leq \{2^k\kappa\} \leq \theta'_n$ (where $\{t\}$ is the fractional part of t), so that $\{2^k\kappa\} = \kappa$. Hence, the ray of angle κ lands at a point which is fixed by N^k. Necessarily this point is β_c for $K = K_c$ since $\beta(f_c)$ is the unique fixed point of f_c where there is an external fixed access (Theorem A of [Pe2]).

The unicity of κ follows from the remark that any ray $R_1(\theta)$ accumulating K_c should belong to V_n so that θ is between θ_n and θ'_n for every n. Hence $\theta = \kappa$. □

COROLLARY 6.4. *If N is k-renormalizable at x_0 and if $K = K(N^k)$ intersects $\overline{B}_1$ and also $\overline{B}_2$ then the unique rays accumulating on K are $R_1(\alpha)$ and $R_2(-\alpha)$ where α is the Head angle.*

Proof. If $R_1(\theta_1)$ and $R_2(\theta_2)$ accumulate on $K = K_c$, they both converge to β_c (Lemma 6.3) so that $\theta_1 = 1-\theta_2 \in G$. If there exists $\theta < \theta_1$ in G, K lies in the bounded component of $\widehat{\mathbf{C}} \setminus (\overline{R}_1(1/2) \cup \overline{R}_2(1/2) \cup \overline{R}_1(\theta) \cup \overline{R}_2(-\theta))$. Indeed, it contains $R_1(\theta_1)$ and the curve cannot accumulate on K after Lemma 6.3. This contradicts Lemma 3.14 which asserts that the critical point x_0 is in the unbounded component (see Figure 4). □

LEMMA 6.5. *If N is k-renormalizable at x_0 then for any puzzle Γ of the form $I(\theta)$ or $II(\eta, \zeta)$ satisfying Proposition* 5.4 *the following holds* (*where $P_n(x_0)$ denote the critical puzzle pieces*):

(1) *For l sufficiently large $N^k(P_{l+k}(x_0)) = P_l(x_0)$ and* $\mathrm{Imp}(x_0) \supset K(N^k)$;

(2) *There exists j dividing k such that, for l large, $N^j\colon P_{l+j}(x_0) \to P_l(x_0)$ is a renormalization of N (in x_0) and* $\mathrm{Imp}(x_0) = K(N^j)$. *The level j is the minimal renormalization level.*

Proof. 1. The filled Julia set $K(N^k)$ is included in $\overline{P}_n(x_0)$. Else the boundary $\partial P_n(x_0)$ would disconnect $K(N^k)$. Then after iterations, two rays of Γ in $B_1 \cup B_2$ would land at the same point of $K(N^k)$ ($\partial B_3 \cap K(N^k) = \emptyset$ after Lemma 3.14) and their closure would also disconnect $K(N^k)$ (after Corollary 3.9 and since the critical orbit is disjoint from Γ). But Lemma 6.3 insures that those rays are fixed by N^k and have to land at β_c. Taking the image by the local homeomorphism σ would imply that $\beta(f_c)$ disconnects the quadratic Julia set $K(f_c)$ which is impossible (see [McM1, Thm 6.10]). Hence $K(N^k) \subset \overline{P}_n(x_0)$ and therefore $K(N^k) \subset \mathrm{Imp}(x_0)$. Moreover $N^k(P_{l+k}(x_0))$ is a piece containing $N^k(K) = K$, of depth l; so it is $P_l(x_0)$.

2. Proposition 5.4 gives an l_0 such that $\overline{P}_{l_0+1}(x_0) \subset P_{l_0}(x_0)$. Let j be the first integer in $[0, k]$ such that $N^j(P_{l_0+j}(x_0)) = P_{l_0}(x_0)$, then the map $N^j\colon P_{l_0+j}(x_0) \to P_{l_0}(x_0)$ is a renormalization of N (in x_0). Taking the inverse images by N, $\overline{P}_{l+j}(x_0) \subset P_l(x_0)$ for all $l \geq l_0$(see Remark 2.16) so that $N^j\colon P_{l+j}(x_0) \to P_l(x_0)$ is also a renormalization of N for l large enough. Finally $\mathrm{Imp}(x_0) = K(N^j)$ by definition of $K(N^j)$.

Assume that j is not the minimal renormalization level. Then, there exists $r < j$ such that $N^r(K(N^r)) = K(N^r)$. From 1, $K(N^r)$ is included in $\mathrm{Imp}(x_0) \subset \overline{P}_l(x_0)$. But this contradicts the minimality in the definition of j since $N^r(P_{l+r}(x_0))$ contains x_0.

If j does not divide k, there exists m such that $k = ij+m$ with $0 < m < j$. Then $N^m\colon P_{l+m}(x_0) \to P_l(x_0)$ would be a renormalization of N in x_0. This contradicts the minimality of j. □

COROLLARY 6.6. *If N is k-renormalizable at x_0 and if k is the minimal level of renormalization, then the images $N^i(K(N^k))$ are disjoint for $0 \leq i \leq k-1$.*

Proof. Let Γ be a puzzle of the form $I(\theta), II(\eta,\zeta)$ satisfying Proposition 5.4. For some large l, $\overline{P}_l(x_0) \subset P_{l-1}(x_0)$. Let $i \leq k-1$ be the first integer such that $N^i(K(N^k))$ intersects $K(N^k)$; then $N^i(P_l(x_0)) = P_{l-i}(x_0)$ and the restriction $N^i\colon P_l(x_0) \to P_{l-i}(x_0)$ is an i-renormalization of N. This contradicts the minimality of k. □

7. The boundaries ∂B_i are Jordan curves

In this section we prove that ∂B_i is locally connected. We will use the previous sections. The case of B_3 is really simpler and treated separately (Section 7.1). We distinguish when N is renormalizable around x_0 or not.

As pointed out in Remark 2.4 (using Remarks 2.2 and 3.10 and Lemma 3.8), the local connectedness of ∂B_i has for consequence that $\partial B_i = \phi_i(\mathbf{S}^1)$ (where ϕ_i denotes also the extension) is a Jordan curve formed by the landing points of the rays, whenever it is not conjugated to the polynomial $z^3+3i/\sqrt{2}z^2$. Hence in this section we prove that ∂B_i is locally connected for a genuine Newton map.

7.1. *The boundary ∂B_3 is a Jordan curve.* The orbit of the critical point x_0 is "far from" $\overline{B}_3$ (Lemma 3.14), so it is rather easy to prove by hand (without Yoccoz' Theorem) the following:

Proposition 7.1. *The boundary of B_3 is a Jordan curve.*

Proof. There exists j such that $t = \frac{1}{2^j} \in [\alpha, 2\alpha)$. Denote by C the curve $R_3(1/2) \cup \bigcup_{i=1,2} \overline{R}_i(\pm t/2) \cup \overline{R}'_i((-1)^{i+1}t) \cup \overline{R}'_i(0)$ and recall that

$$\overline{X} = \widehat{\mathbf{C}} \setminus \bigcup_{i=1,2,3} \phi_i(e^{-v}\overline{\mathbf{D}}) \quad \text{(for any } v>0).$$

1) a) *Local connectivity of ∂B_3 at ∞*: The unbounded connected component P_0 of $X \setminus C$ is a disc (by choice of t). It is easy to check that P_1, the connected component of $N^{-1}(P_0)$ containing ∞, satisfies $\overline{P}_1 \subset P_0$ and that N induces a homeomorphism $P_1 \to P_0$ (Lemma 3.14); let $g\colon P_0 \to P_1$ be its inverse. Since $P_n = g^n(P_0)$ reduces to ∞ (Schwarz Lemma), so does $\overline{P}_n \cap \partial B_3$. Since $P_n \subset P_1$, no critical point belongs to P_n (by Lemma 3.14) so the sequence $\overline{P}_n \cap \partial B_3$ forms a basis of connected neighborhoods of ∞ and condition $(*)$ of Lemma 5.5 is satisfied (the proof is the same as in corollary 5.6 even if P_n is not a puzzle piece).

b) *Local connectivity of ∂B_3 at the preimages of ∞*: For $x \in N^{-k}(\infty)$, let R_n be the connected component of $N^{-k}(P_n)$ containing x, where P_n is as defined in a). For large enough n, the piece P_n satisfies the condition

(∗∗) of Lemma 5.5; therefore the sequence $\overline{R}_n \cap \partial B_3$ is a basis of connected neighborhoods of x (if $n \geq k+1$).

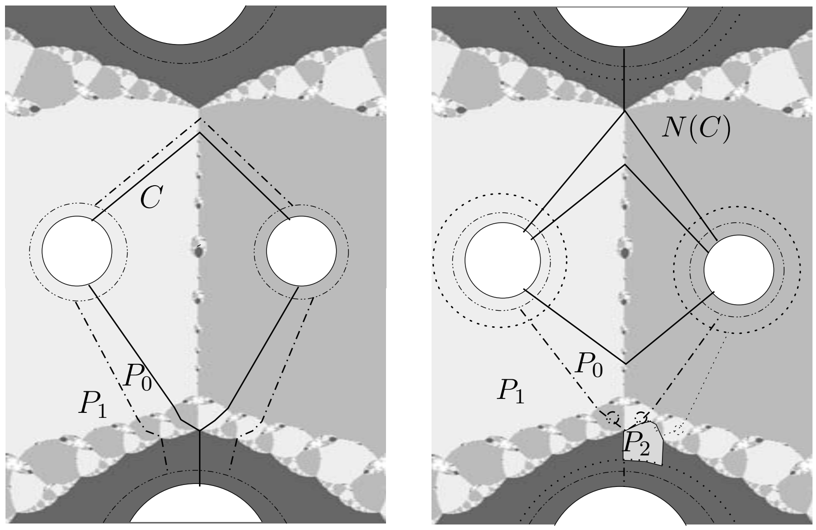

Figure 10: On the left illustration of 1a), on the right of case 2

2) *Local connectivity of ∂B_3 outside the preimages of* ∞: Now we denote by P_0 the connected component of $X \setminus N(C)$ intersecting B_3. It is easy to check that any connected component P_n of $N^{-n}(P_0)$ intersecting B_3 lies in $N^{-1}(P_0)$ by the fact that $\partial N^{-1}(P_0) \cap P_n = \emptyset$ (or equivalently that $N^{n-1}(C) \cap P_0 = \emptyset$) and that $N^{-1}(P_0)$ intersects P_n since $N^{-1}(P_0) \supset B_3 \cap X_1$. Therefore N^n induces a homeomorphism $P_n \to P_0$ since there is no critical point in $N^{-1}(P_0)$ (Lemma 3.14).

Let P_1 be the connected component of $N^{-1}(P_0)$ intersecting $R_3(3/4)$. The connected component P_2 of $N^{-1}(P_1)$ intersecting $R_3(3/8)$ is then compactly contained in P_0 since $\partial P_2 \cap (B_1 \cup B_2) = \emptyset$. Therefore, if we denote by $P_n(x)$ (resp. $P_{n+2}(x)$) the connected component of $N^{-n}(P_0)$ (resp. $N^{-n}(P_2)$) containing x, $A_n(x) = P_n(x) \setminus \overline{P}_{n+2}(x)$ is conformally equivalent to $P_0 \setminus \overline{P}_2$. Assuming that there exists a sequence n_j diverging to ∞ such that $N^{n_j}(x) \in Q = \overline{B}_3 \cap P_2$, we see that the annuli $A_{n_j}(x)$ form a sequence surrounding x and with the same modulus (as $P_0 \setminus \overline{P}_2$) ; one extracts a subsequence to obtain disjoint annuli and by Grötszch inequality $\mathrm{mod}(P_0 \setminus \overline{P}_{n_{k_j}}) \to \infty$, so that $\cap P_n$ reduces to a point.

Now we find the sequence n_j. For $x \in \partial B_3$ and $n \geq 0$, x belongs to some $Q_3(\theta_n, \theta'_n)$ with θ_n, θ'_n of the form $k_n/2^n, (k_n+1)/2^n$, since x is not a preimage of ∞. Hence the common limit $\theta = \lim \theta_n = \lim \theta'_n$ satisfies $\theta_n < \theta < \theta'_n$ for every $n \geq 0$, and so is not dyadic. Moreover $N^n(x)$ and $N^n(R_3(\theta))$ belongs, or avoid, simultaneously, P_2. Since a nondyadic angle θ has infinitely many occurrences of $0, 1$ in its dyadic expansion there exists n_j such that $N^{n_j}(x) \in Q_3(1/4, 1/2)$ and so n_j is in P_2. □

7.2. *The boundary of B_1 and B_2 in the nonrenormalizable case.*

Proposition 7.2. *If the cubic Newton map N is not renormalizable at x_0, the boundaries ∂B_1 and ∂B_2 are Jordan curves.*

Proof. Proposition 5.4 provides graphs $I(\theta)$ and $II(\eta, \zeta)$ satisfying the hypothesis of Yoccoz' Theorem for the set $C = \partial B_1 \cup \partial B_2$ (see Theorem 2.15). Since N is not renormalizable at x_0, the second conclusion of the Theorem does not take place so that $\mathrm{Imp}(x_0) = \bigcap_{n\geq 0} \overline{P}_n(x_0) = \{x_0\}$ and $\mathrm{Imp}(y) = \bigcap_{n\geq 0} \overline{P}_n(y) = \{y\}$ for every $y \in \partial B_1 \cup \partial B_2$ (where $P_n(x)$ designs a piece of depth n for one of the previous graphs). Hence $\overline{P}_n(y) \cap \partial B_i$ form a basis of connected neighborhoods of y in ∂B_i by Corollary 5.6. □

7.3. *The boundary of B_1 and B_2 in the renormalizable case.* We assume now that N is renormalizable at x_0 with minimal renormalization level k. We denote by $N^k \colon Y' \to Y$ a k-renormalization of N in x_0, by $K = K(N^k)$ the connected filled Julia set of this renormalization and by σ the straightening map and by β the fixed point of N^k not disconnecting K (see Notation 6.2). By Lemma 6.5 we assume that Y and Y' are puzzle pieces.

Remark 7.3. If the boundary ∂B_1, or ∂B_2, does not intersect K, it is locally connected.

Proof. From Lemma 6.5 $\mathrm{Imp}(x_0) = K(N^k) = K$. Hence, the orbit of any point $x \in \partial B_i$ will never meet $\mathrm{Imp}(x_0)$ since it stays on ∂B_i. So by Theorem 2.15 $\mathrm{Imp}(x)$ reduces to x. □

For this reason we assume in this section that $\overline{B}_1 \cap K \neq \emptyset$, the case of B_2 is analogous.

We define "sides" of K as follows. Let $R_1(\kappa)$ be the unique ray accumulating on K (Lemma 6.3). Its preimage $R_1'(2\kappa)$ converges to $\beta' \in K \cap N^{-1}(\beta)$ since the other preimage $R_1(\kappa + 1/2)$ cannot accumulate K (by unicity in B_1).

Definition 7.4. Let Δ_1 (resp. Δ_2) be the connected component of $\widehat{\mathbf{C}} \setminus \tilde{K}$ containing B_2 (resp. B_2'), where

$$\tilde{K} = K \cup \overline{R}_1(0) \cup \overline{R}_3(0) \cup \overline{R}_3(1/2) \cup \overline{R}_1'(0) \cup \overline{R}_1'(2\kappa) \cup \overline{R}_1(\kappa).$$

Lemma 7.5. *The connected components Δ_1 and Δ_2 are discs.*

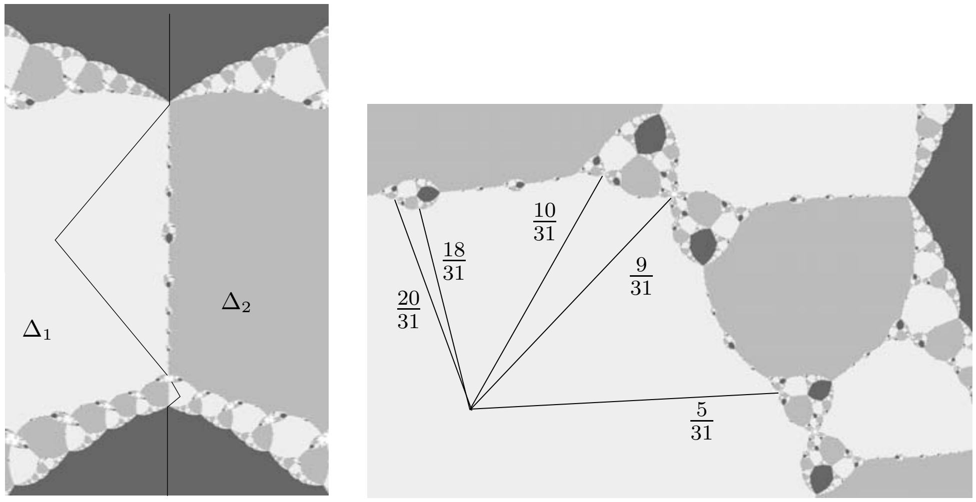

Figure 11: On the left Δ_1, Δ_2. On the right $\bigcup_i N^i(K \cup R_1(9/31))$ for N with Head angle 1/3 for the proof of Proposition 7.6.

Proof. This is a corollary of the following topological Lemma with the open disc $U = \widehat{\mathbf{C}} \setminus (\overline{R}_1(0) \cup \overline{R}_3(0) \cup \overline{R}_3(1/2))$ and the curves $a_1 = \overline{R}'_1(0) \cup \overline{R}'_1(2\kappa)$, $a_2 = \overline{R}_1(\kappa)$. □

Topological Lemma. *Let K be a connected, full compact set in a topological open disc U and $a_1, \dots, a_n$ simple disjoint closed arcs crossing K, as well as ∂U, only at one endpoint. Then the open set $U \setminus (K \cup a_1 \cup \dots \cup a_n)$ is the union of exactly n topological discs.*

Proof. Since $K \subset U$ is a connected, full compact set and U is a disc, there exists a homeomorphism $\varphi\colon U \setminus K \to \mathbf{C} \setminus \{0\}$. The compacts $b_i = \overline{\varphi(a_i \cap (U \setminus K))}$ are arcs of disjoint interior joining 0 to ∞ in $\widehat{\mathbf{C}}$. The open set $\widehat{\mathbf{C}} \setminus \cup_i b_i$ is then the union of n disjoint topological discs (Jordan's theorem) and is homeomorphic to $U \setminus (K \cup a_0 \cup \dots \cup a_n)$. □

The aim of this section is to prove:

Proposition 7.6. *For $i \in \{1, 2\}$ there exists an articulated ray $T_i \subset \Delta_i$ stemming from B'_i and converging to β.*

Corollary 7.7. *For any renormalization $N^j\colon Z' \to Z$ of N near x_0, if $K(N^j)$ intersects ∂B_i, it is in exactly one point.*

Proof. By Lemma 6.5, $K(N^j) \subset K$ so it is enough to prove that K intersects ∂B_i in one point. Recall that there is an angle κ such that the ray $R_1(\kappa)$ converges to the fixed point β of $N^k_{|K}$ (Lemma 6.3). The two articulated

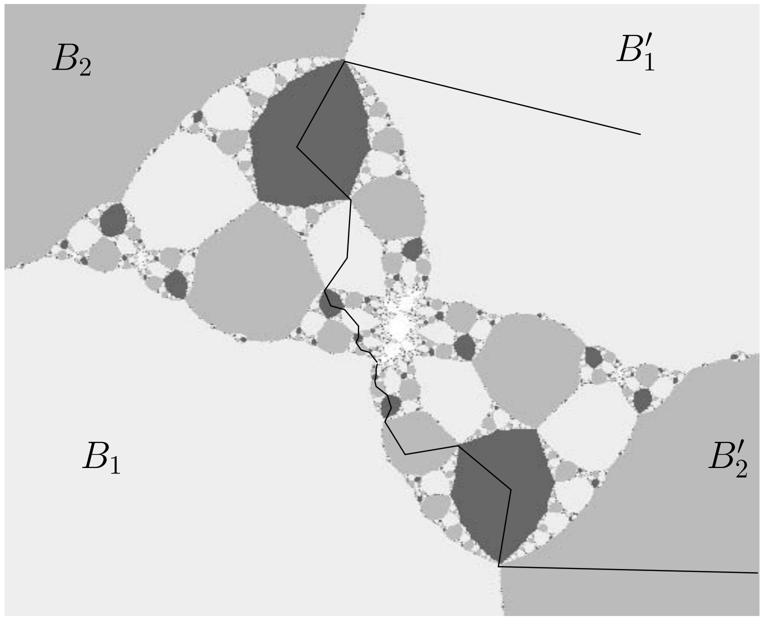

Figure 12: Fixed articulated rays converging to β.

rays T_i, constructed in Proposition 7.6, also land at β. Hence $T = T_1 \cup T_2 \cup \overline{R}'_1(0) \cup \overline{R}'_2(0)$ is a Jordan curve separating $K \setminus \{\beta\}$ from $B_1 \cup B_2$. Indeed, T_i are stemming from B'_i, they are disjoint and stay in Δ_i. Finally, $(\overline{B}_1 \cup \overline{B}_2) \cap K$ is reduced to β. □

Corollary 7.8. *The boundaries ∂B_1 and ∂B_2 are Jordan curves.*

Proof. Proposition 5.4 provides graphs $I(\theta)$ and $II(\eta, \zeta)$ satisfying the hypothesis of Yoccoz' Theorem (Theorem 2.15). Since N is renormalizable at x_0 by Lemma 6.5, we are in the second case of the conclusion of Yoccoz' Theorem for the graphs provided by Proposition 5.4; i.e. $N^k : P_{n+k}(x_0) \to P_n(x_0)$ is a renormalization of N. By Corollary 7.7, $\partial B_i \cap K(N^k)$ is at most one point. Hence for any point $y \in \partial B_i$ whose orbit meets $K(N^k)$, the sequence $\overline{P}_n(y) \cap \partial B_i$ forms in ∂B_i a basis of connected neighborhoods of y. Indeed their intersection reduces to preimages of $K(N^k) \cap \partial B_i$, i.e. to points, and the connectedness follows from Corollary 5.6. On the other hand, if the orbit of $y \in \partial B_i$ does not meet $K(N^k) = \mathrm{Imp}(x_0)$, Yoccoz' Theorem implies that $\mathrm{Imp}(y) = \{y\}$. In both cases ∂B_i is locally connected at y. □

Proof of Proposition 7.6. Let U be the connected component of $\widehat{\mathbf{C}} \setminus (\overline{R}_3(0) \cup \gamma(1/2))$ intersecting B_3. It is a topological disc. Now applying k times the Topological Lemma to $N^j(K)$ and to the arcs $N^j(\overline{R}_1(\kappa))$ (for $0 \le j \le k-1$) one obtains that $V = U \setminus \bigcup_{0 \le j \le k} N^j(K \cup R_1(\kappa))$ is a topological disc.

The proof of the proposition is organized as follows.

(1) Step 1 provides an inverse branch g_i of the renormalization $N^k\colon Y' \to Y$, which is defined on $\Delta_i \cap Y$, but extends to V and satisfies $g_i(V) \subset V \cap \Delta_i$.

(2) In Step 2 we construct a finite articulated ray C_i, joining $b_i' = N^{-1}(b_i) \cap B_i'$ to $g_i(b_i')$ and included in $\Delta_i \cap V$.

(3) In Step 3 we define the articulated ray T_i as the union for $n \geq 0$ of $g_i^n(C_i)$; then $g_i(T_i) \subset T_i$ and T_i has finite length since g_i is a contraction in $Y \cap \Delta_i$.

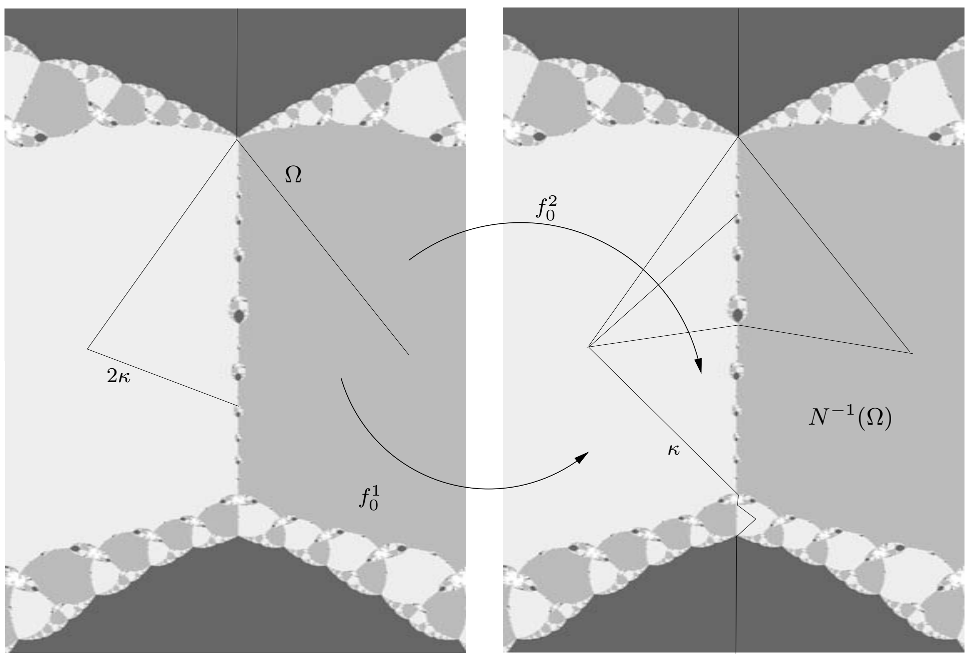

Figure 13: Inverse branches

Step 1. *There exists $g_i\colon V \to V \cap \Delta_i$ an inverse branch of N^k, such that the sequence $(g_i^n)_{n\in\mathbf{N}}$ converges for the open-compact topology of V to a constant $z_i \in K$.*

To define an inverse branch of the renormalization N^k, we first define inverse branches of N on domains containing the orbit of K. There is no critical value of N in the disc $\Omega = \widehat{\mathbf{C}} \setminus \big(N(K \cup \overline{R}_1(\kappa)) \cup_{i=1,2,3} \overline{R}_i(0)\big)$ and since $N(N^j(K)) \subset \Omega$ for $1 \leq j \leq k-1$, let Ω_j be the connected component of $N^{-1}(\Omega)$ containing $N^j(K)$, where $N\colon \Omega_j \to \Omega$ admits an inverse branch called f_j. The case $j = 0$ requires more caution since $N(K)$ is not in Ω. There are exactly three components forming $N^{-1}(\Omega)$, two are stuck to K, we denote by Ω_0^i the one contained in Δ_i ($i = 1, 2$) and by f_0^i the corresponding inverse branch. The maps f_i cannot be composed on Ω (since $f_i(\Omega) \not\subseteq \Omega$) but on $\widehat{\mathbf{C}} \setminus K'$ where $K' = \cup_{j\geq 0} N^j(K) \cup \overline{R}_1(0) \cup \overline{R}_2(0) \cup \overline{R}_3(0)$ (and in particular on $V \subset \widehat{\mathbf{C}} \setminus K'$) since $\widehat{\mathbf{C}} \setminus K' \subset \Omega$ and $f_i(\widehat{\mathbf{C}} \setminus K') \subset \widehat{\mathbf{C}} \setminus K'$, the second inclusion follows from the stability of K': $N(K') \subset K'$ and moreover $f_0^i(\widehat{\mathbf{C}} \setminus K') \subset \widehat{\mathbf{C}} \setminus K'$.

Now we can define g_i as the composition $g_i = f_0^i \circ f_1 \circ \cdots \circ f_{k-1} \colon V \to \Delta_i$; it is an inverse branch of N^k. By construction $g_i(V) \subset \widehat{\mathbf{C}} \setminus C'$ avoids $\gamma(1/2)$ since $f_0^i(\Omega) \subset N^{-1}(\Omega) \subset \widehat{\mathbf{C}} \setminus \gamma(1/2)$, so that $g_i(V) \subset V$ (since it is stuck to $K \subset V$).

Moreover, if $N^k \colon Y' \to Y$ is the k-renormalization in x_0, with Y' small enough, then $g_i(Y \cap V) \subset Y'$. Indeed, $N^j(Y')$ is included in Ω for $2 \leq j \leq k$ and the image $f_j \circ \cdots \circ f_{k-1}(Y)$ is exactly $N^j(Y')$. Finally, since Y' is the only component of $N^{-1}(N(Y'))$ stuck to K, $f_0^i(N(Y') \cap \Omega)$ is included in Y' so that $g_i(V \cap Y') \subset Y' \subset Y$.

Since $g_i \colon V \to V$ is not surjective ($V \cap \Delta_i \neq V$) , the sequence of iterates $(g_i^n)_n$ converges uniformly on every compact set of V to a constant $z_i \in \partial V$ after the Denjoy-Wolff Lemma. The fact that $g_i(Y \cap V) \subset Y'$ forces the limit z_i to be in K. □

Step 2. *There exists a finite articulated ray $C_i \subset V \cap \Delta_i$ joining b_i' and $g_i(b_i')$, for $i = 1, 2$.*

The articulated ray C_1 is $C_1 = f_0^1(R_{k-1})$ where $(R_j)_{j\in\mathbf{N}}$ is a sequence of finite articulated rays stemming from B_1 or B_3 defined as follows (the construction is similar for C_2 with B_2 and B_3). Let $R_0 = R_3(1/2) \cup \overline{R}_1'(0)$ and, for $j \geq 0$, if $f_{k-1-j}(R_j)$ stems from $B_1 \cup B_3$, let R_{j+1} be $f_{k-1-j}(R_j)$. Otherwise take $R_{j+1} = f_{k-1-j}(R_j) \cup \overline{R}_1'(0) \cup R_3(1/2)$ or $R_{j+1} = f_{k-1-j}(R_j) \cup \overline{R}_3'(0) \cup R_1(1/2)$ depending on whether $f_{k-1-j}(R_j)$ stems from B_1' or B_3'. The sequence R_j is well defined since $R_j \subset \Omega$ and consists of finite articulated rays. Moreover, C_1 contains $g_1(b_1')$ by construction and is located in $V \cap \Delta_1$ since it cannot cross the boundary of V nor Δ_1 (after Lemma 3.8 and Lemma 6.3).

To prove that C_1 stems from B_1', it is enough to show that R_{k-1} stems from B_1 with an angle $\theta < 2\kappa$ since C_1 is in Δ_1. Let $r < k$ be the largest integer such that $f_{k-1-(r-1)}(R_{r-1}) = R_r$ stems from B_3. Then for $r < j < k-1$, $f_{k-1-j}(R_j)$ stems from B_1, and precisely R_{r+1} stems from B_1 with $R_1(1/2)$ so that $2^{k-r-2}\theta = 1/2$ since $N^{k-r-2}(R_{k-1}) = R_{r+1}$. By definition of f_{k-1-r}, the curve $\gamma(1/2)$ does not separate $N^{k-1-r}(K)$ from $f_{k-1-r}(R_r)$. Therefore $1/2 < 2^{k-r-1}\kappa < 1$ and $\theta < 2\kappa$. □

Step 3. *The union $T_i = \bigcup_{n\geq 0} g_i^n(C_i)$ is an articulated ray located in $V \cap \Delta_i$. Moreover, T_i stems from b_i', $N^k(T_i) \supset T_i$ and T_i converges to β.*

By construction $N^k(T_i) = T_i \cup N^k(C_i) \supset T_i$. After Step 1 and Step 2, $T_i \subset V \cap \Delta_i$ and T_i is an articulated ray stemming from b_i'; it remains to show that T_i converges to β.

By Step 2, the sequence $(g_i^n)_n$ converges uniformly on any compact of V to a point $z_i \in K$. Since $\overline{C}_i \subset V$, the articulated ray T_i converges to z_i (since any accumulation point of T_i is z_i), and $N^k(z_i) = z_i$.

Let $y_i = \sigma(z_i)$ be the corresponding point in the model f_c, where σ (defined on Y) is the conjugacy between N^k and f_c. The arc $\sigma(T_i)$ (or at least a neighborhood of y_i) is a "fixed" access landing at $y_i \in K(f_c)$ (since $f_c(\sigma(T_i)) \supset \sigma(T_i)$). But only $\beta(f_c)$ can be the landing point of a fixed external access to a periodic point of K, so that T_i lands at β. □

7.4. *Corollaries of the local connectivity of ∂B_i.* Recall that we consider only genuine Newton maps.

LEMMA 7.9. *If N has no Siegel point, its Fatou set is $\tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3$ unless it is geometrically finite.*

Proof. Sullivan's classification theorem gives us that, besides $\tilde{B}_i$, there can only be attracting or parabolic components in the Fatou set. Indeed, there is no Herman ring for $J(N)$ is connected (see [Sh]). Moreover, the cycle defined by a component contains a critical point which is either x_0 or one of the roots b_1, b_2, b_3. In the first case N is geometrically finite, in the second the components are in $\tilde{B}_i$. □

THEOREM 6. *For every cubic Newton map without Siegel disc, the boundary of the connected components of the Fatou set are Jordan curves.*

Proof. If N is geometrically finite, the boundary is locally connected (see [D-H1, M2, T-Y]). Otherwise, the Fatou set is exactly the $\tilde{B}_i$ (Lemma 7.9). Let U be any connected component of $\tilde{B}_i$. Its boundary is locally connected since ∂B_i is locally connected and N is a ramified covering.

We prove now that the boundary of a component U of one $\tilde{B}_i$ is a Jordan curve. The boundary is a curve since the Böttcher coordinate extends (by Carathéodory's Theorem). This curve would not be Jordan if two rays land at the same point. For $U = B_j$ this is not possible after Remark 3.10 and Lemma 3.8. For any inverse image of B_j, iterating until B_j, one obtains that two rays in B_j land at the same point which is not possible.

If N is geometrically finite and U is not a component of $\tilde{B}_i$ then N is renormalizable for any of the puzzles given by Proposition 5.4 and U is included in the sequence of critical puzzle pieces (see the proof of Theorem 5). Then $\sigma(U)$ is a Fatou component of a quadratic Julia set, where σ denotes the straightening map. So two rays in U cannot land at the same point (polynomial filled Julia sets are full). □

Corollary 7.10. *There cannot be a Cremer point on the boundary of a component of the Fatou set* $F(N)$.

Proof. Assume that there is a Cremer point on the boundary of a component of $F(N)$. Then N is not geometrically finite so $F(N)$ is exactly the $\tilde{B}_i$ (Lemma 7.9) or there is a Siegel point. Assume first that there is no Siegel point. Since the Cremer point x is a periodic point, it is on ∂B_i and it is accessible by a ray $R_i(\theta)$ (since ∂B_i is locally connected, Theorem 6 and Remark 2.4). Since two rays in B_i cannot land at the same point, $R_i(\theta)$ is periodic. But the landing point of a periodic ray is necessarily repelling or parabolic (Snail Lemma [M1]). If there is a Siegel point, then as in the proof of Theorem 5, N is renormalizable around x_0 and quasi-conformally conjugated to a quadratic polynomial having a Siegel disc. If there is a Cremer point on the boundary of the Siegel disc then by Naïhul's result there is a Cremer periodic point on the Siegel disc of the quadratic polynomial which is impossible (see [G-M]). □

Corollary 7.11. *N is topological conjugated on $\partial B_1 \cap \partial B_2$ to the multiplication by* 2 *on*:

$$G = \{ \ \theta \in \mathbf{R} \quad | \quad \forall n \geq 0, \quad 2^n\theta \mod 1 \in [\alpha, 1] \ \}.$$

Moreover, the Head angle α belongs to G.

Proof. This is an immediate consequence from the fact that the Böttcher map ϕ_i extends to a homeomorphism between $\overline{D}$ and $\overline{B}_i$. □

8. Local connectivity of the whole Julia set

The aim of this section is to prove Theorem 1, Theorem 2 and Corollary 3, which provide surprising differences between rational maps and polynomials. First of all we prove Theorem 4. Then Proposition 8.3 gives a complete description of the situations where $J(N)$ is locally connected. Finally, Sections 8.2 and 8.3 are devoted to the proof of the technical cases (1 and 2) of Proposition 8.3 that are not used for the previously mentioned results.

8.1. *How rational maps differ from polynomials.*

Theorem 4. *A cubic Newton map, without Siegel point, has a locally connected Julia set provided the orbit of the nonfixed critical point does not accumulate on the boundary of any fixed immediate basin of attraction.*

To prove this Theorem we use the following characterization of locally connected sets, see [W, Th. 4.4, p. 113].

PROPOSITION 8.1. *A connected compact set $J \subset \widehat{\mathbf{C}}$ is locally connected if and only if it satisfies the following properties*:

- *For every $\epsilon > 0$, only a finite number of connected components of $\widehat{\mathbf{C}} \setminus J$ has a spherical diameter greater than ϵ*;
- *The boundary of every connected component of $\widehat{\mathbf{C}} \setminus J$ is locally connected.*

We obtain the first condition by applying the so-called “shrinking Lemma”. See [T-Y, Prop. A.3] or [L-M, Shrinking Lemma, p. 86]. This lemma is also an easy consequence of Mañe’s theorem and Koebe’s distortion theorem. Here the *post-critical set* $P(f)$ is the closure of the forward orbit of the critical points of f.

PROPOSITION 8.2 (Shrinking Lemma). *Let $f : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a rational map and D a topological disc whose closure $\overline{D}$ does not cross the post-critical set of f. Then the following holds*:

- *Either $\overline{D}$ is contained in a Siegel disc* (*i.e. Fatou component containing a Siegel point*) *or a Herman ring* (*i.e. Fatou component which is topologically a ring*);
- *Or*, *for every $\varepsilon > 0$, only a finite number of iterated preimages of D have a spherical diameter greater than ε.*

Proof of Theorem 4. Unless N is geometrically finite, in which case $J(N)$ is locally connected (see [D-H1], [M2], [T-Y]), the Fatou set is only the union of the $\tilde{B}_i$ (Lemma 7.9). Hence the second condition of Proposition 8.1 is satisfied (Theorem 6). Since $J(N)$ is connected (see [Sh]), we can apply Proposition 8.2 to $D = B'_i$ for $i \in \{1, 2, 3\}$ (since the orbit of the x_0 does not accumulate on the boundary of B_1, B_2, B_3 and there is no Siegel point). Hence $J(N)$ fulfills the two conditions for being locally connected (Proposition 8.1). □

PROPOSITION 8.3. *Let N be a cubic Newton method without Siegel periodic points. The Julia set of N is locally connected in any of the following cases*:

1) *N is not renormalizable*;
2) *N is renormalizable* (*in x_0*) *exactly once and has no Cremer point*;
3) *N is renormalizable once and the filled Julia set of its renormalization does not encounter the closure of the immediate basins B_i*;
4) *N is renormalizable at least twice.*

Remark 8.4. The first case does not follow from Theorem 4 and will be proved later, as will the second case which is rather technical.

The dichotomy of this proposition is due to the following question 8.5.

Question 8.5. If N is exactly once renormalizable with a Cremer point, can the positive orbit of x_0 accumulate one of the $\overline{B}_i$?

This is in fact equivalent to the following conjecture concerning quadratic polynomials:

Conjecture 8.6. *If $f_c(z) = z^2 + c$ has a Cremer point, can the positive orbit of* 0 *accumulate the β fixed point* (*i.e.* $\beta(f_c)$)?

Regarding the Newton method one can hope that the following is true:

Conjecture 8.7. *Is the Julia set of a genuine cubic Newton method always locally connected*?

Now we prove points 3 and 4 of Proposition 8.3.

Lemma 8.8. *Assume that N is renormalizable with minimal renormalization level k. Let f_c denotes the quadratic polynomial conjugated to N^k. The following equivalence holds*: *N is renormalizable exactly once if and only if f_c is not renormalizable.*

Proof. Let $N^k\colon Y' \to Y$ be the renormalization of the minimal level of N and let σ denote the conjugacy between N^k and f_c.

Assume that f_c is renormalizable. Then there exists $i \in \mathbf{N}$ and open sets $U' \subset U \subset \sigma(Y')$ such that $f_c^i\colon U' \to U$ is quadratic-like. Since $N^{ik}_{|_{Y'}} = \sigma^{-1} \circ f_c^i \circ \sigma$, the restriction of $(N^k)^i$ on $\sigma^{-1}(U')$ defines a renormalization of N^k.

Conversely if N is j-renormalizable at x_0, with $j \neq k$ then $j = ik$ by Lemma 6.5. Hence N^k is i-renormalizable at x_0 with filled Julia set $K(N^j)$and $K(N^j) \subset K(N^k)$ (Lemma 6.5). So $K(N^j) \subset Y'$ and we can chose an open set U' in Y' for the i-renormalization of N^k. Since the conjugacy σ is defined on Y', $f_c^i\colon \sigma(U') \to \sigma(N^j(U'))$ is a renormalization of f_c.

Proof of Point 3 *of Proposition* 8.3. Case 3) is exactly Theorem 4. Indeed, the orbit of the free critical point x_0 is in the union of $N^r(K_c)$ for $r \geq 0$. Since K_c is periodic, if the orbit of x_0 accumulates $\overline{B}_i$ then $N^r(K_c) \cap \overline{B}_i \neq \emptyset$ for some $r \geq 0$ and so K_c crosses $\overline{B}_i$ (by periodicity).

Proof of Point 4 *of Proposition* 8.3. Case 4) follows from Proposition 8.2 and Proposition 8.1 (as well as Theorem 4). The reasons are that ∂B_i crosses K_c at most on β_c (Corollary 6.4) and since f_c is renormalizable (Lemma 8.8) β_c is not accumulated by the orbit of x_0. More precisely, if K' corresponds to the filled Julia set of a renormalization of f_c, Theorem 7.10 of [McM1] insures that K' does not contain $\beta(f_c)$. Hence the orbit $f_c^n(K')$ (for $n \geq 0$) also avoid $\beta(f_c)$ (since $\beta(f_c)$ is fixed and K' periodic). Thus using the conjugacy σ, we see that the point β_c is not in the closure of the orbit of $N^{kn}(x_0)$, nor is it in the closure of the orbit of x_0 after Corollary 6.6. □

THEOREM 1. *There exist cubic Newton methods possessing a Cremer point and having a locally connected Julia set.*

Proof. Note that these maps are genuine Newton methods.

The idea is to find a cubic Newton map, at least twice renormalizable, possessing a Cremer point. By Proposition 8.3 (4) the Julia set will be locally connected. For this, we use the fact that in the parameter plane of cubic Newton method, there are copies of the Mandelbrot set $M = \{c \mid J(f_c) \text{ is connected}\}$ (here $f_c(z) = z^2 + c$). Indeed, considering the family of Newton maps N_λ associated to $P_\lambda(z) = (z-1)(z-\frac{1}{2}+\lambda)(z-\frac{1}{2}-\lambda)$ for $\lambda \in \mathbf{C}$, Douady and Hubbard ([D-H2, Chap. VI]) proved that there exists a subset M_0 of $\mathbf{C}$ and a surjective map $\chi\colon M_0 \to M$ such that for all $\lambda \in M_0$, N_λ is renormalizable at $x_0 = 0$ and the renormalization is quasi-conformally conjugated in a neighborhood of the Julia set to the quadratic polynomial $z^2 + \chi(\lambda)$. Now since Naïshul's result (see [N]) asserts that a Cremer point for $z^2 + \chi(\lambda)$ will give a Cremer point for N_λ it is enough to find a quadratic polynomial $z^2 + c$ which is renormalizable and possesses a Cremer point. For this we use the existence of a copy M' of M strictly contained in M (see for example Theorem 5 of [D-H2]), i.e. a surjective map $\chi'\colon M' \to M$ which gives the straightening parameter. Then if c is a parameter for which $z^2 + c$ has a Cremer point, $(\chi')^{-1}(c)$ does possess a Cremer point and is renormalizable. □

THEOREM 2. *Every infinitely renormalizable cubic Newton method has a locally connected Julia set.*

Proof. This is a direct consequence of Proposition 8.3(4). Indeed, the Julia set of an infinitely renormalizable quadratic polynomial has empty interior (see [McM1, Th. 8.1]). Hence the Newton method N cannot have a Siegel point. □

COROLLARY 3. *There exist rational maps with locally connected Julia set and wandering continuum within.*

From [Le], a continuum $K \subset J$ is called *wandering* for f if for any $n \neq m$ nonnegative, $f^n(K) \cap f^m(K) = \emptyset$. Recall that for polynomials $g_c = z^l + c$ with connected Julia set G. Levin proved that J is locally connected if and only if no continuum $K \subset J$ is wandering.

Proof. There exist quadratic polynomials $f_c(z) = z^2 + c$ such that f_c is infinitely renormalizable and $J(f_c)$ is not locally connected (see [M2]). For this polynomial, nested Julia sets J_i (containing 0 and obtained by renormalizations of f_c) satisfy $J_\infty = \cap_{i=1}^\infty J_i \supsetneq \{0\}$ (see [Le, McM1]). Since J_i has "periods" increasing with i, J_∞ is wandering. As in the proof (above) of Theorem 1, a Newton method N such that $\chi(N) = c$ provides an example.

8.2. *Proof of Case* 2) *of Proposition* 8.3. We recall the notation 6.2: $N^k \colon Y' \to Y$ denotes the (only) renormalization of N, K_c the filled Julia set, σ the conjugacy to f_c and finally the fixed point $\beta(f_c)$ with preimage $\beta'(f_c)$.

After Lemma 7.9 the Fatou components are the iterated preimages of the B_i (excepted in the geometrically finite case treated by [D-H1, M2, T-Y]). So using Proposition 8.1 and Theorem 6 it remains to prove that only finitely many of these components have diameter greater than ε. If $\overline{B}_i \cap K_c = \emptyset$ this is direct from Proposition 8.2 since the orbit of x_0 is in the orbit of K_c (see the proof of Proposition 8.2.3). This is always the situation of $\overline{B}_3$.

Hence we assume that $\overline{B}_1 \cap K_c \neq \emptyset$ (the proof is similar for B_2) and we show that the set of preimages of $\overline{B}_1'$ has *diameters tending to* 0. It will then also be the case with $\overline{B}_1$ since $N^{-1}(\overline{B}_1) = \overline{B}_1 \cup \overline{B}_1'$. Recall that in this case $\overline{B}_1 \cap K_c = \beta_c$ and $\overline{B}_1' \cap K_c = \beta_c'$ (see Lemma 6.3).

Definition 8.9. A set $\mathcal{E}$ of connected parts of $\widehat{\mathbf{C}}$ is said to *have diameters tending to* 0 if for every $\varepsilon > 0$ only a finite number of elements of $\mathcal{E}$ have diameter greater than ε.

Lemma 8.10. *There exists an arbitrarily small neighborhood Q of $\beta'(f_c)$ such that the set of iterated preimages of Q by f_c has diameters tending to* 0.

Proof. Since this neighborhood is constructed as a puzzle piece for f_c, so we have to recall some facts. On $\mathbf{C} \setminus K(f_c)$ the polynomial f_c is conjugated to the map $z \mapsto z^2$ in $\mathbf{C} \setminus \overline{\mathbf{D}}$. This conjugacy defines external rays and equipotentials (respectively) as the images of rays and circles of $\mathbf{C} \setminus \overline{\mathbf{D}}$. The construction of Yoccoz' puzzle (see also [M2] and [Hu]) requires the second fixed point $\alpha(f_c)$ also to be repelling (which is the case here by assumption). Then there exists q external rays, of angles $2^i p/(2^q - 1)$, $1 \le i \le q$, converging to it and forming, with an external equipotential, a graph for f_c. Assuming f_c is not geometrically finite, this graph will be admissible (see Definition 2.13 where X is the disc bounded by the equipotential).

Then to every point $z \in J(f_c)$ we associate $P_r^*(z)$ the union of the puzzle pieces of depth r whose closure contains z. These closed neighborhoods of z satisfy, after Theorem 2 of [M2, p. 14] (or Theorem 5.7.a of [Hu, p. 483]), that

$$(*) \qquad \bigcap_{r \ge 0} \overline{P}_r^*(z) = \{z\}.$$

Hence, since $\beta(f_c)$ is not on the graph, it belongs to a piece, say P, of arbitrarily large depth and by $(*)$ of arbitrarily small size.

Assume (by contradiction) that there exists $\varepsilon > 0$ and a sequence P_i of the iterated preimages of P by f_c whose depth increases and diameter stands greater than ε. Up to extraction, we can find a sequence $x_i \in P_i$ of preimages of $\beta(f_c)$ converging to a point $x \in K(f_c)$. Hence for each depth r, there exists i large such that the points x_i belong to $P_r^*(x)$. Thus, $P_i \subset P_r^*(x)$ (since the pieces are disjoint or nested). Finally the preimage of P containing $\beta'(f_c)$ gives the announced neighborhood which contradicts $(*)$. □

We choose a neighborhood Q sufficiently small so that $Q \subset \sigma(Y')$ and decompose B_1' in disjoint parts: $B_1' = Q_1' \amalg R_1'$, where Q_1' is the connected component of $B_1' \cap \sigma^{-1}(Y')$ whose closure contains β_c' (it is unique since $\partial B_1'$ is a Jordan curve) and $R_1' = B_1' \setminus Q_1'$.

Since $\overline{R}_1'$ is disjoint from $\cup_{0\leq i\leq k} N^i(K_c)$, that contains the post-critical set of N, the set of its iterated preimages have diameters tending to 0 (Proposition 8.2).

We now concentrate on $\mathcal{Q}$ the set of preimages of Q_1':

Let $\mathcal{Q}^i$ $(0 \leq i \leq k-1)$ be the set of iterated preimages of Q_1' by N whose closure touches $N^i(K_c)$. If U is a connected component of $\mathcal{Q}$ not in any $\mathcal{Q}^i$, then for some $j \geq 0$, $N^j(U)$ belongs to a component of $N^{-1}(N^i(Y'))$ disjoint from the post-critical set; that is, U is in

$$Y^{-1} = N^{-1}\left(\bigcup_{0\leq i\leq k-1} N^i(Y')\right) \setminus \bigcup_{0\leq i\leq k-1} N^i(Y').$$

As above $\overline{Y^{-1}}$ is disjoint from $\cup_{0\leq i\leq k} N^i(K_c)$ so its iterated preimages have diameters tending to 0 (Proposition 8.2), and also $\mathcal{Q}\setminus\cup_{0\leq i\leq k-1}\mathcal{Q}^i$ have diameters tending to 0. Hence it remains to show it for $\cup_{0\leq i\leq k-1}\mathcal{Q}^i$.

Remark 8.11. 1) If g denotes the restriction of $N^k\colon Y' \to Y$ then for Y small enough $\mathcal{Q}^0$ coincides with the set of iterated preimages of Q_1' by g. This follows directly from the fact that if Y is small enough, the sets $(N^i(Y))_{0\leq i\leq k-1}$ are disjoint (Corollary 6.6).

2) Every connected component of $\mathcal{Q}^i$ is also in $(N^{k-i})^{-1}(\mathcal{Q}^0)$.

After Remark 8.11.1), every preimage of Q by f_c is the image by σ of a component of $\mathcal{Q}^0$. Hence, Lemma 8.10 and Lemma 8.12 (below) insure that $\mathcal{Q}^0$ has diameters tending to 0. Finally, Remark 8.11.2) and Lemma 8.12 allow us to conclude that the $\mathcal{Q}^i$ also have diameters tending to 0.

Lemma 8.12. *Let Z, Z' be two open sets of $\widehat{\mathbf{C}}$, $\rho\colon Z' \to Z$ a (topological) ramified covering map of degree d and $\mathcal{E}$ (resp. $\mathcal{E}'$) the set of the relatively compact connected open sets of Z (resp. of Z'). If $\mathcal{E}$ has diameters tending to*

0 and if ρ sends every open set of $\mathcal{E}'$ inside an open set of $\mathcal{E}$, then $\mathcal{E}'$ also has diameters tending to 0.

Proof. The proof is by contradiction. We assume that there exists $\varepsilon > 0$ and U_i' of $\mathcal{E}'$ with diameter greater than ε. Hence, in each U_i', one can find $d+1$ points z_i^j, $1 \le j \le d+1$, at (mutual) distance greater than $\varepsilon/(d+1)$. One extracts $d+1$ subsequences converging to points z^j at distances at least $\varepsilon/(d+1)$. We show now that those $d+1$ points have the same image under ρ.

Let U_i be the sequence of images $\rho(U_i')$ and take any $n > 0$. For i large enough, the diameter of U_i is less than $1/n$. Hence, the (mutual) distances between the points $\rho(z_i^j)$ are less than $1/n$. Taking the limit over i, one sees that the mutual distances between the images $\rho(z^j)$ are less than $1/n$. Since this is the case for any $n > 0$, the points $\rho(z^j)$ collapse, which contradicts the fact that ρ is of degree d. □

8.3. *Proof of Case* 1) *of Proposition* 8.3. This part is very technical and deserves only the *a priori* easiest case: the nonrenormalizable case. We use the theory of Yoccoz' puzzle and precisely Theorem 2.15, not only for ∂B_i as in Section 7 but for the whole Julia set. Therefore we need the construction of a new kind of graph. In opposition to Section 7, the difficulty here is to satisfy the hypothesis of Theorem 2.15, since for the conclusion we choose the good one, namely the nonrenormalizable case.

Definition 8.13. Let $\eta, \tau \in G \cap \mathbf{Q}$, $\zeta \in G \cap [\alpha, 2\alpha[$ be a nondyadic rational number and L be the articulated ray constructed in Proposition 4.3 stemming from B_2 with angle $-\zeta/4$. We define a new type of graph:

$$III(\zeta,\eta,\tau) = \partial X \cup \left(X \cap \left(\bigcup_{j\ge 0} (N^j(L) \cup \overline{R}_3(2^j/7) \bigcup_{t=\eta,\tau} (\overline{R}_1(2^j t) \cup \overline{R}_2(-2^j t))) \right) \right).$$

The type III graph corresponds to the type II graph with in addition the orbit of $\overline{R}_1(\tau)$ and $\overline{R}_2(-\tau)$. For the graph of type II, the points surrounded by a nondegenerate annulus of depth 0 were limitated by the preimages of L and of $N^3(L)$. Here while taking the preimages of $R_1(\tau), R_2(-\tau)$ we enlarge this zone almost up to $N^3(L)$.

Since the type III graphs (as type II) are dis-symmetric, we consider the symmetric one: $III^*(\zeta,\eta,\tau)$ constructed as $III(\zeta,\eta,\tau)$ but instead of the images $N^j(L)$ we take the images $N^j(L^*)$, L^* being the articulated ray (symmetric to L) defined as follows:

Proposition 8.14. *Let y be the landing point of $R_3(6/7)$ and $\zeta \in G \cap\,]\alpha, 2\alpha[$ be a nondyadic angle. There exists a unique articulated ray L^* stem-*

ming from B_1 with angle $\zeta/4$ such that:

$$N^3(L^*) = L^* \cup \overline{R}_1(\zeta) \cup \overline{R}_2(-\zeta) \cup \overline{R}_1(2\zeta) \cup \overline{R}_2(-2\zeta)$$

and L^* converges to y.

The proof is the same as that of Proposition 4.3.

As in Section 5 we will only consider graphs of type I, III, III^* with angles of the form ($\diamondsuit$):

$$\begin{cases} \theta = 1 - \dfrac{1}{2^p - 1}, \eta = 1 - \dfrac{1}{2^n - 1}, \zeta = \dfrac{1}{2^l}\left(1 - \dfrac{1}{2^m - 1}\right), \tau = \dfrac{1}{2^l}\left(1 - \dfrac{1}{2^r - 1}\right) \\ \text{such that} : \dfrac{1}{2} > \dfrac{\eta}{2} > 2^{l-1}\zeta > 2^{l-1}\tau > 2^l\tau + \dfrac{1}{2} > 2^{l-1}\alpha > \dfrac{1}{4} \text{ and } \theta - 1/2 > \alpha. \end{cases}$$

For this it is enough to take p large, l such that $1/2 > 2^{l-1}\alpha \geq 1/4$ and $n > m > r$, with r large enough so that $\frac{1}{2} > \frac{1}{2}(1 - \frac{1}{2^r - 1}) > 2^{l-1}\alpha$. In particular we obtain that $\tau, \zeta \in]\alpha, 2\alpha[$.

Proposition 8.15. *There exist an integer δ and a finite number of puzzles of type I, III and III^* such that x_0, resp. every point of the Julia set, is surrounded by each, resp. one, of these puzzles and at depth less than δ.*

Proof. We will refer intensively to the proof of Proposition 5.4, in particular to Remark 5.2 and 5.3 and to the notation namely of the articulated rays $L^0 = L,\ L^1 \subset N(L)$ and $L^2 \subset N^2(L)$.

*Step*1. *For ζ, η, τ satisfying ($\diamondsuit$) and for r sufficiently large, the graphs $III(\zeta, \eta, \tau)$ and $III^*(\zeta, \eta, \tau)$ are puzzles and surround x_0.*

Such graphs are admissible since the critical orbit cannot have arbitrarily large period ($\geq r$). Then the argument goes as in Step 5 of Proposition 5.4. After Lemma 3.14, x_0 belongs to the bounded connected component of $\widehat{\mathbf{C}} \setminus \gamma(\frac{2^l\tau + 1/2}{2^{l-1}}, \frac{\eta}{2^{l+1}})$ since $\alpha < \frac{2^l\tau+1/2}{2^{l-1}} < \eta/2^l < 2\alpha$ by ($\diamondsuit$).

Notation. Let L', resp. L'', L'^* and L''^* be the articulated rays stemming from B_1, resp. B_2', B_2 and B_1' such that $N(L') = L^2$, resp. $N(L'') = L^1$, $N(L'^*) = L^{2^*}$ and $N(L''^*) = L^{1^*}$.

Step 2. *The bounded component of*

$$X' \setminus \left(L' \cup L'' \cup \overline{R}_1\,(\tau/2) \cup \overline{R}_2'\,(-\tau) \cup \overline{R}_3\,(9/14) \cup \overline{R}_3\,(11/14)\right),$$

is a piece of depth 1 *of the puzzle $III(\zeta, \eta, \tau)$, with angles satisfying ($\diamondsuit$) and its boundary is disjoint from $III(\zeta, \eta, \tau)$ (similarly with $III^*(\zeta, \eta, \tau)$, $L'^*, \dots$).*

The image of this connected component is the piece of depth 0 located between $N(L)$, $N^2(L)$, $R_1(\tau), R_2(-\tau)$ etc. Its boundary consists at depth 0

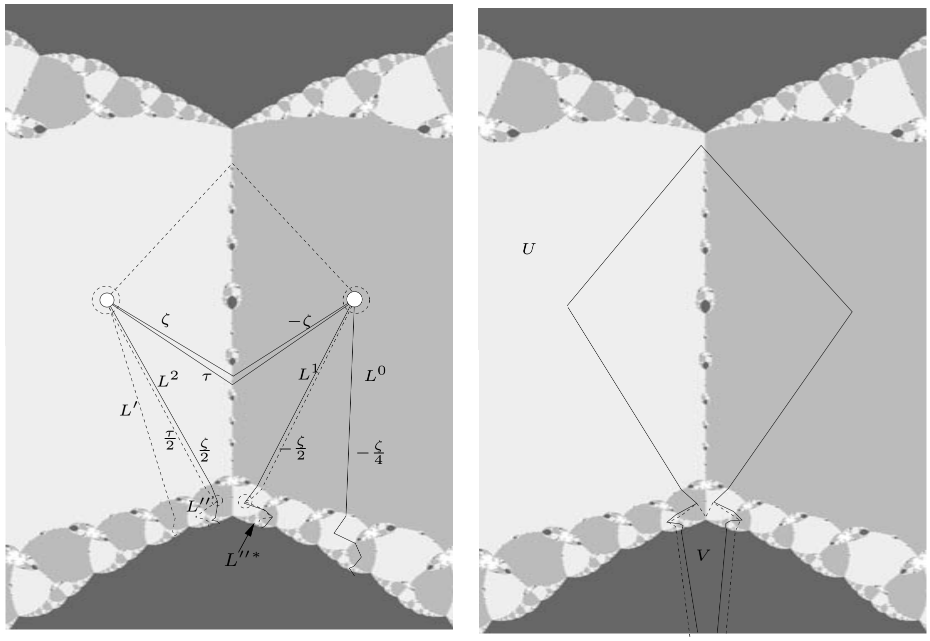

Figure 14: On the left useful parts of the graph $III(\zeta,\eta,\tau)$ and some dashed preimages. On the right U and in dashed V.

and 1 of rays of angles in B_1, resp. B'_2, strictly between 0 and $\zeta/2$, resp. between $-\zeta$ and $-\zeta/4$. Moreover, at depth greater than 1, L'' and L^2 cannot cross, else $N(L)$ and L would cross at depth greater than 0.

Notation. For $(\theta_j,\zeta_j,\eta_j,\tau_j)_{j=1,2}$, let $\Omega(\theta_1)$ be the bounded component of $\widehat{\mathbf{C}}\setminus\bigcup_{i=1,2}\overline{R}_i((-1)^{i+1}\theta_1+1/2)\cup\overline{R}_i((-1)^{i+1}\theta_2/4)\cup\overline{R}'_i((-1)^{i+1}\theta_2/2)\cup\overline{R}'_i(0)$ and for $j=1,2$, let U_j be the unbounded connected component of the complement of $L''_j\cup L^{*\prime\prime}_j\cup\overline{R}_3\left(\frac{\pm 5}{14}\right)\bigcup_{i=1,2}\overline{R}_i\left(\frac{(-1)^{i+1}\eta_j+1}{2}\right)\cup\overline{R}_i\left(\frac{(-1)^{i+1}\tau_j}{2}\right)\cup\overline{R}'_i\left((-1)^{i+1}\tau_j\right)$. ($L_j$ is the articulated ray constructed with ζ_j etc....)

Step 3. *For $\theta_1>\theta_2$ and $\eta>\zeta_1>\tau_1>\zeta_2>\tau_2$ satisfying ($\diamondsuit$) with the same l, the points of $J(N)\cap(U_1\cup U_2\cup\Omega(\theta_1))$ are surrounded by a nondegenerated annulus of depth less than* 2 *for $III(\zeta_j,\eta,\tau_j)$ or $III^*(\zeta_j,\eta,\tau_j)$ or $I(\theta_j)$ with $j\in\{1,2\}$.*

The unbounded pieces of depth 0 and 1 are the same for the graphs of type II or III. By Step 4 of Proposition 5.4 the unbounded piece of depth 1 is the central disc of a nondegenerated annulus of depth 0 (defined by $III(\zeta_j,\eta,\tau_j)$ with $j\in\{1,2\}$). Using Step 2 and the similar result for $III^*(\zeta_j,\eta,\tau_j)$ we obtain the part $U_1\cup U_2$. Indeed, the points which are on $J\cap U_1\cap N^{-1}(III(\zeta_1,\eta,\tau_1))$ are in $U_2\setminus N^{-1}(III(\zeta_2,\eta,\tau_2)\cup III^*(\zeta_2,\eta,\tau_2))$. Moreover the sets $X'\setminus\gamma(\theta_j+$

$1/2, \theta_j/4)$ and $X' \setminus \bigcup_{i=1,2} \overline{R}'_i((-1)^{i+1}\theta_j/2) \cup \overline{R}'_i(0)$ are unions of depth 1 pieces for $I(\theta_j)$, compactly contained in a depth 0 piece (proof of Proposition 5.4, Step 2). Hence by taking two values of the angles $\theta_1 > \theta_2$ we include in the domain the points of the graph and obtain $\Omega(\theta_1)$.

Step 4. *The points of $J(N) \setminus (U_1 \cup U_2 \cup N^{-1}(U_1 \cup U_2))$ are in the bounded connected component of $\widehat{\mathbf{C}} \setminus \widetilde{C}$ where $\widetilde{C} = \gamma(\frac{\tau_1}{2}, \frac{1+\eta}{2})$.*

Let V_j be the bounded connected component of $N^{-1}(U_j)$ intersecting ∂B_3. We will prove the statement with V_j instead of $N^{-1}(U_j)$. For this we study the intersections of the boundary of U_1, V_1, U_2, V_2, concentrating on the rays and articulated rays by depth. As the picture is symmetric, it is enough to study the boundaries on one side for instance near $B_2, B'_1 \dots$. At depth 0, the rays are exactly those involved in $\widetilde{C}$. At depth greater than 3 the boundaries do not cross, else by iteration the articulated rays L^{*0} and L^2 would also cross at depth greater than 0. Hence V_j covers ∂U_j at depth greater than 3 since the articulated ray of the boundary of V_j, resp. U_j, converges to the same points as $R_3(9/28)$, resp. $R_3(5/14)$.

At depth 1, U_2, resp. V_1, contains at least in B'_1 the part between the angles $\zeta_2/2$ and τ_2, resp. 0 and $\tau_1/2$ so that V_1 contains $R'_1(\zeta_2/2)$ since $\tau_1 > \zeta_2$. At depth 2 the situation is similar for the rays touching the previous one. Moreover, U_1, resp. V_2, contains at least the part between the angles $-\zeta_1/2$ and 1, resp. $-\tau$ and $-\zeta_2/2$ so that V_2 covers U_1 at least on the ray of angle $-\zeta_1/2$ since $\zeta_1 > \zeta_2$. At depth 3 the situation is identical for the consecutive rays. Finally, the next rays of depth 3 have angle $\zeta_j/2$ in ∂U_j and ζ_j in ∂V_j so that U_j covers V_j along this ray.

Step 5. *There exist θ_1 and $\delta \in \mathbf{N}$ such that $J(N) \subset N^{-\delta}(\Omega(\theta_1) \cup U_1 \cup U_2)$.*

The inverse image of $U_1 \cup U_2 \cup V_1 \cup V_2$ containing $R_1(1/2)$ covers the bounded complement of $\cup_{i=1,2}\overline{R}_i(1/2) \cup \overline{R}_i((1+(-1)^{i+1}\eta)/4)$. Choosing θ_1 such that $1/2 > \theta_1 - 1/2 > (\eta+1)/4$, we take p large, $N^{-1}(U_1 \cup U_2 \cup V_1 \cup V_2) \cup \Omega(\theta_1)$ covers Q_1 the bounded connected component of $\widehat{\mathbf{C}} \setminus \cup_{i=1,2}\overline{R}_i(1/2) \cup \overline{R}_i(\theta_1/4)$.

If $\widehat{\mathbf{C}} \setminus \gamma(1/2, \alpha/2)$, Q_2 denotes by the bounded connected component, then every point of Q_2 is sent by an iterate of N (less than $l+1$) into Q_1 (to see this, it is enough to cut the component in "sectors" of angles $(2^t\alpha/2, 2^{t+1}\alpha/2)$). Moreover, $\Omega(\theta_1)$ contains the bounded complement of $\cup_{i=1,2}\overline{R}'_i(0) \cup \overline{R}'_i(1/2)$ as well as the rays $\overline{R}'_i(1/2)$ (because $\theta_2/2 < 1/2$). Hence $Q_2 \cup \Omega(\theta_1) \supset Q_3$ the bounded complement of

$$\cup_{i=1,2}\overline{R}_i((-1)^{i+1}\alpha/2) \cup \overline{R}_i(1/2) \cup \overline{R}'_i((-1)^{i+1}\alpha) \cup \overline{R}'_i(0).$$

By Step 4 and since $\tau_1 > \alpha$, the union $Q_4 = \Omega(\theta_1) \cup \cup_{i=1,2}U_i \cup N^{-1}(U_i) \cup Q_i$ contains the unbounded complement of $\cup_{i=1,2}\overline{R}_i((1+\eta)/2) \cup \overline{R}_i(1/2)$ (excepted maybe some points outside J).

Let q be the first integer such that $1-1/2^q > (\eta+1)/2$. Every bounded connected component of $\widehat{\mathbf{C}} \setminus \bigcup_{i=1,2} \overline{R}_i((-1)^i 1/2^t) \cup \overline{R}_i((-1)^i 1/2^{t+1})$ is for $1 \leq t < q$ sent by N^t into Q_4. □

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