Annals of Mathematics, **167** (2008), 1109–1117

On the zeros of cosine polynomials: solution to a problem of Littlewood

By P. Borwein, T. Erdélyi, R. Ferguson, and R. Lockhart*

Abstract

Littlewood in his 1968 monograph "Some Problems in Real and Complex Analysis" [12, Problem 22] poses the following research problem, which appears to be still open:

Problem. "*If the n_j are integral and all different, what is the lower bound on the number of real zeros of $\sum_{j=1}^{N} \cos(n_j\theta)$? Possibly $N-1$, or not much less.*"

No progress seems to have been made on this in the last half century. We show that this is false.

Theorem. *There exists a cosine polynomial $\sum_{j=1}^{N} \cos(n_j\theta)$ with the n_j integral and all different so that the number of its real zeros in the period $[-\pi, \pi)$ is $O\left(N^{5/6}\log N\right)$.*

1. Littlewood's 22nd problem

Problem. "*If the n_j are integral and all different, what is the lower bound on the number of real zeros of $\sum_{j=1}^{N} \cos(n_j\theta)$? Possibly $N-1$, or not much less.*"

Here "real zeros" means "zeros in $[-\pi, \pi)$". Note that if T is a real trigonometric cosine polynomial of degree n, then it is of the form $T(t) = \exp(-int)P(\exp(it))$, $t \in \mathbb{R}$, where P is a reciprocal algebraic polynomial of degree $2n$, and if T has only real zeros, then P has all its zeros on the unit circle. So in terms of reciprocal algebraic polynomials one is looking for a reciprocal algebraic polynomial with coefficients in $\{0, 1\}$, with $2N$ terms, and with $N-1$ or fewer zeros on the unit circle. Even achieving $N-1$ is fairly hard. An exhaustive search up to degree $2n_N \le 32$ yields only 10 examples achieving $N-1$ and only one example with fewer. This first example disproving the

*Research of P. Borwein supported in part by NSERC of Canada, IRMACS and MITACS. Research of R. Ferguson supported in part by NSERC of Canada, IRMACS and MITACS. Research of R. Lockhart supported in part by NSERC of Canada.

"possibly $N-1$" part of the conjecture is

$$\sum_{j=0,\, j\notin\{9,10,11,14\}}^{14} (z^j + z^{28-j})$$

which has 8 roots of modulus 1 and corresponds to a cosine sum of 11 terms with 8 roots in $[-\pi,\pi)$. It is hard to see how one might generate infinitely many such examples or indeed why Littlewood made his conjecture. The following is a reciprocal polynomial with 32 terms and exactly 14 zeros of modulus 1:

$$\sum_{j=0,\, j\notin\{10,11,17,19\}}^{19} (z^j + z^{38-j})\,.$$

So it corresponds to a cosine sum of 16 terms with 14 zeros in $[-\pi,\pi)$. In other words the sharp version of Littlewood's conjecture is false again, though barely. The following is a reciprocal polynomial with 280 terms and 52 zeros of modulus 1:

$$\sum_{j=0,\, j\notin\{124,125,126,127,128,134,141,143,145,147,148,151,152\}}^{152} (z^j + z^{304-j})\,.$$

So it corresponds to a cosine sum of 140 terms with 52 zeros in $[-\pi,\pi)$. Once again the sharp version of Littlewood's conjecture is false, though this time by a margin. It was found by a version of the greedy algorithm (and some guessing). There is no reason to believe it is a minimal example.

The interesting feature of this example is how close it is to the Dirichlet kernel $(1+z+z^2+\cdots+z^{304})$. This is not accidental and suggests the approach that leads to our main result.

Littlewood explored many problems concerning polynomials with various restrictions on the coefficients. See [9], [10], and [11], and in particular Littlewood's delightful monograph [12]. Related problems and results may be found in [2] and [4], for example. One of these is Littlewood's well-known conjecture of around 1948 asking for the minimum L_1 norm of polynomials of the form

$$p(z) := \sum_{j=0}^{n} a_j z^{k_j},$$

where the coefficients a_j are complex numbers of modulus at least 1 and the exponents k_j are distinct nonnegative integers. It states that such polynomials have L_1 norms on the unit circle that grow at least like $c\log n$. This was proved by S. Konyagin [7] and independently by McGehee, Pigno, and Smith [13] in 1981. A short proof is available in [5]. It is believed that the minimum, for polynomials of degree n with complex coefficients of modulus at least 1, is attained by $1+z+z^2+\cdots+z^n$, but this is open.

2. Auxiliary functions

The key is to construct n term cosine sums that are large most of the time. This is the content of this section.

LEMMA 1. *There is an absolute constant c_1 such that for all n and non-negative Lebesgue measurable functions α on $[-\pi,\pi)$ there are coefficients $a_0, a_1, \dots, a_n$ with each $a_j \in \{0,1\}$ such that*

$$\text{meas}\{t \in [-\pi,\pi) : |P_n(t)| \le \alpha(t)\} \le c_1 n^{-1/2} \int_{-\pi}^{\pi} \alpha(u)du,$$

where

$$P_n(t) = \sum_{j=0}^{n} a_j \cos(jt).$$

Proof. We will prove the stronger result that there is an absolute constant c_1 such that for all non-negative Lebesgue measurable α and all n

$$\begin{aligned}\lambda(\alpha) :=& 2^{-(n+1)} \sum_{a_0,a_1,\dots,a_n \in \{0,1\}} \text{meas}\{t \in [-\pi,\pi) : |P_n(t)| \le \alpha(t)\} \\ \le& c_1 n^{-1/2} \int_{-\pi}^{\pi} \alpha(u)du.\end{aligned}$$

If $X_0, X_1, \dots, X_n$ are independent Bernoulli random variables with

$$P(X_j = 0) = P(X_j = 1) = \frac{1}{2}, \qquad j = 0, 1, \dots, n,$$

then the indicated average is an expected value. Let

$$R_n(t) = \sum_{j=0}^{n} X_j \cos(jt)$$

and note that

$$\lambda(\alpha) = \int_{-\pi}^{\pi} P(|R_n(t)| \le \alpha(t))\, dt.$$

Define

$$D_n(t) := \sum_{j=0}^{n} \cos(jt) = \frac{1}{2} + \frac{\sin((n+\frac{1}{2})t)}{2\sin(t/2)}\,.$$

Note that for $0 < |t| < \pi$, we have

$$|D_n(t)| \le \pi/|t|.$$

The expected value of $R_n(t)$ is $\mu_n(t) := D_n(t)/2$; its variance is

$$\sigma_n^2(t) := \frac{1}{4} \sum_{j=0}^{n} \cos^2(jt) = \frac{1}{8}(n+1+D_n(2t)).$$

We now apply a uniform normal approximation to get the desired result. Define the cumulative normal distribution function by

$$\Phi(x) := \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du.$$

Define

$$\begin{aligned} \varrho_2 &:= \frac{1}{n+1}\sum_{j=0}^{n} \operatorname{Var}(X_j \cos(jt)) \\ &= \frac{1}{4(n+1)}\sum_{j=0}^{n} \cos^2(jt) = \frac{1}{8}\left(1 + \frac{D_n(2t)}{n+1}\right), \\ \varrho_3 &:= \frac{1}{n+1}\sum_{j=0}^{n} \mathrm{E}\left(\left|\left(X_j - \frac{1}{2}\right)\cos(jt)\right|^3\right). \end{aligned}$$

We suppress the dependence of each of these on n and t. The Berry-Esseen bound in Bhattacharya and Ranga Rao [1, Theorem 12.4, page 104] is that

$$\left|P(R_n(t) \le c) - \Phi\left(\frac{c - \mu_n(t)}{\sigma_n(t)}\right)\right| \le \frac{11\varrho_3}{4\sqrt{n}\,\varrho_2^{3/2}}.$$

It is elementary that $\varrho_3 \le 1/8$. Moreover there is an absolute constant $c_2 > 0$ such that $\varrho_2 > c_2$ for all $t \in \mathbb{R}$ and all $n = 1, 2, \dots$. Finally the function Φ has derivative bounded by $(2\pi)^{-1/2}$ so that

$$|\Phi(x) - \Phi(y)| \le (2\pi)^{-1/2}|x - y|, \qquad x, y \in \mathbb{R}.$$

It follows that there is an absolute constant c_1 such that

$$P(-\alpha(u) \le R_n(u) \le \alpha(u)) \le c_1 n^{-1/2}\alpha(u). \qquad \square$$

3. The main theorem

THEOREM 1. *There exist a sequence of integers $N_m, m = 1, 2, \cdots$ with N_m/m converging to* 1 *and cosine polynomials $\sum_{j=1}^{N_m}\cos(n_j\theta)$ with the n_j integral and all different so that the number of its real zeros in $[-\pi, \pi)$ is*

$$O\left(N_m^{5/6}\log N_m\right) = O\left(m^{5/6}\log m\right).$$

To prove the theorem we need the following consequence of the Erdős-Turán Theorem [15, p. 278]; see also [6].

LEMMA 2. *Let*

$$S_m(t) = \sum_{j=0}^{m} a_j \cos(jt), \qquad a_j \in \{0, 1\},$$

be not identically zero. Denote the number of zeros of S_m in an interval $I \subset [-\pi, \pi)$ by $\mathcal{N}(I)$. Then

$$\mathcal{N}(I) \le c_3 m|I| + c_3\sqrt{m}\log m\,,$$

where c_3 is an absolute constant and $|I|$ denotes the length of I.

We now prove the theorem.

Proof. Fix any positive integers n and κ. Let χ_ν denote the characteristic function of the interval $J_\nu = [\pi 2^{-\nu}, 2\pi 2^{-\nu})$. Define the function α_κ on $[-\pi, \pi)$ by

$$\alpha_\kappa(t) = \pi \sum_{\nu=1}^{\kappa} 2^\nu \chi_\nu(t).$$

By Lemma 1 there is a trigonometric polynomial $P_{n,\kappa}$ of the form

$$P_{n,\kappa}(t) = \sum_{j=0}^{n} a_j \cos(jt), \qquad a_j \in \{0, 1\},$$

with

$$\begin{aligned}\operatorname{meas}\{t \in [-\pi, \pi) : |P_{n,\kappa}(t)| \le \alpha_\kappa(t)\} &\le c_1 n^{-1/2} \int_{-\pi}^{\pi} \alpha_\kappa(u)du \\ &= c_1 \pi\kappa n^{-1/2}\,.\end{aligned}$$

We construct our desired cosine polynomials in the form

$$S_m(t) := D_m(t) - P_{n,\kappa}(t),$$

where

$$D_m(t) := \sum_{j=0}^{m} \cos(jt) = \frac{1}{2} + \frac{\sin((m+\frac{1}{2})t)}{2\sin(t/2)}\,,$$

and n and κ are chosen depending on m by taking n to be the integer part of $m^{1/3}$ and $2^{\kappa-1} \le m^{1/6} < 2^\kappa$. The resulting polynomial S_m has N_m non-zero coefficients, where

$$m - n \le N_m \le m + 1.$$

The number of zeros of S_m in $(-\pi, \pi)$ is twice the number in $(0, \pi)$. Write

$$\{t \in (0, \pi) : |P_{n,\kappa}(t)| \le \alpha_\kappa(t), 2^\kappa t \ge \pi\} = \bigcup_{\nu=1}^{\kappa} \bigcup_{j=1}^{k_\nu} I_{j,\nu}\,,$$

where the intervals $I_{j,\nu}$ are disjoint and $I_{j,\nu} \subset J_\nu$. The number k_ν is at most 1 plus the number of zeros in J_ν of the trigonometric polynomial $P'_{n,\kappa}$. This polynomial has degree no more than n so that $\sum_{\nu=1}^{\kappa} k_\nu \le 2n + \kappa$. Let

$$I_0 := \{t \in (0, \pi) : |D_m(t)| \ge \pi 2^\kappa\}\,.$$

Note that $I_0 \subset (0, 2^{-\kappa}\pi]$. Since $|D_m(t)| \leq \pi/|t|$ for $0 < t < \pi$, Lemma 1 implies that all zeros of S_m in the interval $(0, \pi)$ actually lie in

$$I_0 \bigcup \left\{ \bigcup_{\nu=1}^{\kappa} \bigcup_{j=1}^{k_\nu} I_{j,\nu} \right\}.$$

By Lemma 2 we have

$$\mathcal{N}(I_{j,\nu}) \leq c_3 m |I_{j,\nu}| + c_3 \sqrt{m} \log m\,, \qquad j = 1, 2, \dots, k_\nu\,, \nu = 0, 1, \dots, \kappa\,,$$

and

$$\mathcal{N}(I_0) \leq c_3 m |I_0| + c_3 \sqrt{m} \log m \leq c_4 m 2^{-\kappa} + c_4 \sqrt{m} \log m$$

with an absolute constant c_4. So

$$\begin{aligned}\mathcal{N}([-\pi, \pi)) &\leq 1 + 2\mathcal{N}(I_0) + 2 \sum_{\nu=1}^{\kappa} \sum_{j=0}^{k_\nu} \mathcal{N}(I_{j,\nu}) \\ &\leq c_5 \left(m\kappa n^{-1/2} + \sqrt{m} \log m\,(n + \kappa) + m 2^{-\kappa} \right).\end{aligned}$$

The choices of n and κ given above complete the proof. □

4. Average number of real zeros

Why did Littlewood make this conjecture? He might have observed that the average number of zeros a trigonometric polynomial of the form

$$0 \neq T(t) = \sum_{j=1}^{n} a_j \cos(jt)\,, \qquad a_j \in \{0, 1\}\,,$$

has in $[-\pi, \pi)$ is at least cn. This is what we elaborate in this section. Associated with a polynomial P of degree exactly n with real coefficients we introduce $P^*(z) := z^n P(1/z)$.

THEOREM 2. *Let*

$$S(t) := \sum_{j=1}^{n} a_j \cos(jt) \qquad \textit{and} \qquad \widetilde{S}(t) := \sum_{j=1}^{n} a_{n+1-j} \cos(jt)\,,$$

where each of the coefficients a_j *is real and* $a_1 a_n \neq 0$*. Let* w_1 *be the number of zeros of* S *in* $[-\pi, \pi)$*, and let* w_2 *be the number of zeros of* $\widetilde{S}$ *in* $[-\pi, \pi)$*. Then* $w_1 + w_2 \geq 2n$*.*

Proof. Let $P(z) = \sum_{j=1}^{n} a_j z^j$. Without loss of generality we may assume that P does not have zeros on the unit circle; the general case follows by a simple limiting argument with the help of Rouché's Theorem. Note that if P

has exactly k zeros in the open unit disk then $zP^*(z)$ has exactly $n-k$ zeros in the open unit disk. Also,

$$2S(t) = \mathrm{Re}(P(e^{it})) \qquad \text{and} \qquad 2\widetilde{S}(t) = \mathrm{Re}(e^{it}P^*(e^{it}))\,.$$

Hence the theorem follows from the Argument Principle. Note that if a continuous curve goes around the origin k times then it crosses the imaginary axis at least $2k$ times. □

Theorem 2 has some interesting consequences. As an example we can state and easily see the following.

THEOREM 3. *The average number of zeros of trigonometric polynomials in the class*

$$\left\{ \sum_{j=1}^{n} a_j \cos(jt), \;\; a_j \in \{-1,1\} \right\}$$

in $[-\pi,\pi)$ *is at least* n. *The average number of zeros of trigonometric polynomials in the class*

$$\left\{ 0 \neq \sum_{j=1}^{n} a_j \cos(jt), \;\; a_j \in \{0,1\} \right\}$$

in $[-\pi,\pi)$ *is at least* $n/4$.

Proof. Most of the cosine sums in both classes naturally break into pairs with a large combined total number of real zeros in $[-\pi,\pi)$. □

5. Conclusion

Let $0 \leq n_1 < n_2 < \cdots < n_N$ be integers. A cosine polynomial of the form $T_N(\theta) = \sum_{j=1}^{N} \cos(n_j\theta)$ (other than $T_N \equiv 1$) must have at least one real zero in $[-\pi,\pi)$. This is obvious if $n_1 \neq 0$, since then the integral of the sum on $[-\pi,\pi)$ is 0. The above statement is less obvious if $n_1 = 0$, but for sufficiently large N it follows from Littlewood's Conjecture simply. Here we mean the Littlewood's Conjecture proved by S. Konyagin [7] and independently by McGehee, Pigno, and Smith [13] in 1981. See also [5] for a book proof. It is not difficult to prove the statement in general even in the case $n_1 = 0$. One way is to use the identity, valid if $n_1 = 0$ and $N > 1$,

$$\sum_{j=1}^{n_N} T_N((2j-1)\pi/n_N) = 0\,.$$

See [8], for example. Another way is to use Theorem 2 of [14]. So there is certainly no shortage of possible approaches to prove the starting observation of our conclusion even in the case $n_1 = 0$. It seems likely that the number of zeros of the above sums in $[-\pi, \pi)$ must tend to infinity with N. This does not appear to be easy. The case when the sequence $0 \le n_1 < n_2 < \cdots$ is fixed was handled in [3].

Acknowledgment. We thank the referees for suggestions which led to an improved rate calculation in our main theorem.

Department of Mathematics, Simon Fraser University, Burnaby, B.C., Canada (P.B. and R.F.)
E-mail addresses: pborwein@cecm.sfu.ca
rferguson@pims.math.ca

Department of Mathematics, Texas A&M University College Station, TX (T.E.)
E-mail address: terdelyi@math.tamu.edu

Department of Statistics and Actuarial Science, Simon Fraser University, Burnaby, B.C., Canada (R.L.)
E-mail address: lockhart@stat.sfu.ca

References

[1] R. N. Bhattacharya and R. Ranga Rao, *Normal Approximation and Asymptotic Expansions*, John Wiley & Sons, New York, 1976.

[2] P. Borwein, *Computational Excursions in Analysis and Number Theory*, *CMS Books in Mathematics* **10**, Springer-Verlag, New York, 2002.

[3] P. Borwein and T. Erdélyi, Lower bounds for the number of zeros of cosine polynomials in the period: a problem of Littlewood, *Acta Arith.* **128** (2007), 377–384.

[4] B. Conrey, A. Granville, B. Poonen, and K. Soundararajan, Zeros of Fekete polynomials, *Ann. Inst. Fourier* (*Grenoble*) **50** (2000), 865–889.

[5] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, New York, 1993.

[6] P. Erdős and P. Turán, On the distribution of roots of polynomials, *Ann. of Math.* **51** (1950), 105–119.

[7] S. V. Konjagin, On a problem of Littlewood, *Mathematics of the* USSR, *Izvestia* **18** (1981), 205–225.

[8] S. V. Konyagin and V. F. Lev, Character sums in complex half-planes, *J. Théor. Nombres Bordeaux* **16** (2004), 587–606.

[9] J. E. Littlewood, On the mean values of certain trigonometrical polynomials, *J. London Math. Soc.* **36** (1961), 307–334.

[10] ———, On the real roots of real trigonometrical polynomials (II), *J. London Math. Soc.* **39** (1964), 511–532.

[11] ———, On polynomials $\sum \pm z^m$, $\sum^n e^{\alpha_m i} z^m$, $z = e^{\theta i}$, *J. London Math. Soc.* **41** (1966), 367–376.

[12] ———, *Some Problems in Real and Complex Analysis*, *Heath Mathematical Monographs*, D. C. Heath and Co., Lexington, Mass., 1968.

[13] O. C. McGehee, L. Pigno, and B. Smith, Hardy's inequality and the L_1 norm of exponential sums, *Ann. of Math.* **113** (1981), 613–618.

[14] I. D. Mercer, Unimodular roots of special Littlewood polynomials, *Canad. Math. Bull.* **49** (2006), 438–447.

[15] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, *Topics in Polynomials*: *Extremal Problems*, *Inequalities*, *Zeros*, World Scientific Publ. Co., River Edge, NJ, 1994.

(Received November 26, 2006)