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The topological classification of minimal surfaces in $\mathbb{R}^3$

By Charles Frohman and William H. Meeks III*

Abstract

We give a complete topological classification of properly embedded minimal surfaces in Euclidian three-space.

1. Introduction

In 1980, Meeks and Yau [15] proved that properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$ are unknotted in the sense that any two such homeomorphic surfaces are properly ambiently isotopic. Later Frohman [6] proved that any two triply periodic minimal surfaces in $\mathbb{R}^3$ are properly ambiently isotopic. More recently, Frohman and Meeks [9] proved that a properly embedded minimal surface in $\mathbb{R}^3$ with one end is a Heegaard surface in $\mathbb{R}^3$ and that Heegaard surfaces of $\mathbb{R}^3$ with the same genus are topologically equivalent. Hence, properly embedded minimal surfaces in $\mathbb{R}^3$ with one end are unknotted even when the genus is infinite. These topological uniqueness theorems of Meeks, Yau, and Frohman are special cases of the following general classification theorem which was conjectured in [9] and which represents the final result for the topological classification problem of properly embedded minimal surfaces in $\mathbb{R}^3$.

The space of ends of a properly embedded minimal surface in $\mathbb{R}^3$ has a natural linear ordering up to reversal, and the middle ends in this ordering have a parity (even or odd) (see Section 2).

Theorem 1.1 (Topological Classification Theorem for Minimal Surfaces). *Two properly embedded minimal surfaces in $\mathbb{R}^3$ are properly ambiently isotopic if and only if there exists a homeomorphism between the surfaces that preserves the ordering of their ends and preserves the parity of their middle ends.*

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The constructive nature of our proof of the Topological Classification Theorem provides an explicit description of any properly embedded minimal surface in terms of the ordering of the ends, the parity of the middle ends, the genus of each end - zero or infinite - and the genus of the surface. This topological description depends on several major advances in the classical theory of minimal surfaces. First, associated to any properly embedded minimal surface M with more than one end is a unique plane passing through the origin called the limit tangent plane at infinity of M (see Section 2). Furthermore, the ends of M are geometrically ordered over its limit tangent plane at infinity and this ordering is a topological property of the ambient isotopy class of M [8]. We call this result the “Ordering Theorem”. Second, our proof of the classification theorem depends on the nonexistence of middle limit ends for properly embedded minimal surfaces. This result follows immediately from the theorem of Collin, Kusner, Meeks and Rosenberg [2] that every middle end of a properly embedded minimal surface in $\mathbb{R}^3$ has quadratic area growth. Third, our proof relies heavily on a topological description of the complements of M in $\mathbb{R}^3$; this topological description of the complements was carried out by the authors [9] when M has one end and by Freedman [4] in the general case.

Here is an outline of our proof of the classification theorem. The first step is to construct a proper family $\mathcal{P}$ of topologically parallel, standardly embedded planes in $\mathbb{R}^3$ such that the closed slabs and half spaces determined by $\mathcal{P}$ each contains exactly one end of M and each plane in $\mathcal{P}$ intersects M transversely in a simple closed curve. The next step is to reduce the global classification problem to a tractable topological-combinatorial classification problem for Heegaard splittings of closed slabs or half spaces in $\mathbb{R}^3$.

2. Preliminaries

Throughout this paper, all surfaces are embedded and proper. We now recall the definition of the limit tangent plane at infinity for a properly embedded minimal surface $F \subset \mathbb{R}^3$. From the Weierstrass representation for minimal surfaces one knows that the finite collection of ends of a complete embedded noncompact minimal surface Σ of finite total curvature with compact boundary are asymptotic to a finite collection of pairwise disjoint ends of planes and catenoids, each of which has a well-defined unit normal at infinity. It follows that the limiting normals to the ends of Σ are parallel and one defines the limit tangent plane of Σ to be the plane passing through the origin and orthogonal to the normals of Σ at infinity. Suppose that such a Σ is contained in a complement of F. One defines a limit tangent plane for F to be the limit tangent plane of Σ. In [1] it is shown that if F has at least two ends, then F has a unique limit tangent plane which we call *the limit tangent plane at infinity* for F. We say that the limit tangent plane at infinity for F is horizontal if it is the xy-plane.

The main result in [8] is:

THEOREM 2.1 (The Ordering Theorem). *Suppose F is a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end and with horizontal limit tangent plane at infinity. Then the ends of F have a natural linear ordering by their "relative heights" over the xy-plane. Furthermore, this ordering is topological in the sense that if f is a diffeomorphism of $\mathbb{R}^3$ such that $f(F)$ is a minimal surface with horizontal limit tangent plane at infinity, then the induced map on the spaces of ends preserves or reverses the orderings.*

Unless otherwise stated, we will assume that the limit tangent plane at infinity of F is horizontal, so that F is equipped with a particular ordering on its set of ends $\mathcal{E}(F)$. $\mathcal{E}(F)$ has a natural topology which makes it into a compact Hausdorff space. This topology coincides with the order topology coming from the Ordering Theorem. The limit points of $\mathcal{E}(F)$ are called *limit ends* of F. Since $\mathcal{E}(F)$ is compact and the ordering on $\mathcal{E}(F)$ is linear, there exist unique maximal and minimal elements of $\mathcal{E}(F)$ for this ordering. The maximal element is called the *top* end of F. The minimal element is called the *bottom* end of F. Otherwise the end is called a *middle* end of F.

Actually for our purposes we will need to know how the ordering of the ends $\mathcal{E}(F)$ is obtained. This ordering is induced from a proper family $\mathcal{S}$ of pairwise disjoint ends of horizontal planes and catenoids in $\mathbb{R}^3 - F$ that separate the ends of F in the following sense. Given two distinct middle ends e_1, e_2 of F, then for r sufficiently large, e_1 and e_2 have representatives in different components of $\{(x, y, z) \in (\mathbb{R}^3 - \cup\mathcal{S}) \mid x^2 + y^2 \geq r^2\}$. Since the components of $\mathcal{S}$ can be taken to be disjoint graphs over complements of round disks centered at the origin, they are naturally ordered by their relative heights and hence induce an ordering on $\mathcal{E}(F)$ [8].

In [2] it is shown that a limit end of F must be a top or a bottom end of the surface. This means that each middle end $m \in \mathcal{E}(F)$ can be represented by a proper subdomain $E_m \subset F$ which has compact boundary and one end. We now show how to assign a parity to m. First choose a vertical cylinder C that contains ∂E_m in its interior. Since m is a middle end, there exist components K_+, K_- in $\mathcal{S}$ which are ends of horizontal planes or catenoids in $\mathbb{R}^3 - F$ with K_+ above E_m and K_- below E_m. By choosing the radius of C large enough, we may assume that $\partial K_+ \cup \partial K_-$ lies in the interior of C. Next consider a vertical line L in $\mathbb{R}^3 - C$ which intersects K_+ and K_-, each in a single point. If L is transverse to E_m, then $L \cap E_m$ is a finite set of fixed parity which we call the parity of E_m. The parity of E_m only depends on m, as it can be understood as the intersection number with $\mathbb{Z}_2$-coefficients of the relative homology class of L, intersected with the region between K_+ and K_- and outside C, with the homology class determined by the locally finite chain which comes from the intersection of E_m with this same region. If we let $A(R)$ denote the area of E_m

in the ball of radius R centered at the origin, then the results in [2] imply that $\lim_{R\to\infty} A(R)/\pi R^2$ is an integer with the same parity as the end m. Thus, the parity of m could also be defined geometrically in terms of its area growth. This discussion proves the next proposition.

PROPOSITION 2.2. *If F is a properly embedded minimal surface in $\mathbb{R}^3$, then each middle end of F has a parity.*

In [9] Frohman and Meeks proved that the closures of the complements of a minimal surface with one end in $\mathbb{R}^3$ are handlebodies; that is, they are homeomorphic to the closed regular neighborhood of a properly embedded connected 1-complex in $\mathbb{R}^3$. Motivated by this result and their ordering theorem, Freedman [4] proved the following decomposition theorem for the closure of a complement of F when F has possibly more than one end.

THEOREM 2.3 (Freedman). *Suppose H is the closure of a complement of a properly embedded minimal surface in $\mathbb{R}^3$. Then there exists a proper collection $\mathcal{D}$ of pairwise disjoint minimal disks $(D_n, \partial D_n) \subset (H, \partial H), n \in \mathbb{N}$, such that the closed complements of $\mathcal{D}$ in H form a proper decomposition of H. Furthermore, each component in this decomposition is a compact ball or is homeomorphic to $A \times [0,1)$, where A is an open annulus.*

3. Construction of the family of planes $\mathcal{P}$

In [9] we proved the Topological Classification Theorem for Minimal Surfaces in the case the minimal surface F has one end. Throughout this section, we assume that F has at least two ends.

LEMMA 3.1. *Let F be a properly embedded minimal surface in $\mathbb{R}^3$ with one or two limit ends and horizontal limit tangent plane. Suppose H_1, H_2 are the two closed complements of F and $\mathcal{D}_1$ and $\mathcal{D}_2$ are the proper families of disks for H_1, H_2, respectively, whose existence is described in Freedman's Theorem. Then there exist a properly embedded family $\mathcal{P}$ of smooth planes transverse to F satisfying*:

1. *Each plane in $\mathcal{P}$ has an end representative which is an end of a horizontal plane or catenoid which is disjoint from F*;
2. *In the slab S between two successive planes in $\mathcal{P}$, F has only a finite number of ends*;
3. *Every middle end of F has a representative in one of the just described slab regions S.*

Proof. Since we are assuming that the surface F has one or two limit ends, the collections $\mathcal{D}_1$ and $\mathcal{D}_2$ of disks are each infinite sets. The disks in $\mathcal{D}_1$ can be chosen to be disks of least area in H_1 relative to their boundaries. In fact the disks used by Freedman in the proof of his theorem have this property. Assume that the disks in $\mathcal{D}_2$ also have this least area property. Suppose W is a closed component of $H_1 - \cup\mathcal{D}_1$ or $H_2 - \cup\mathcal{D}_2$ which is homeomorphic to $A \times [0, 1)$.

Let $\gamma(W)$ be a piecewise smooth simple closed curve in ∂W that generates the fundamental group of W. The curve $\gamma(W)$ bounds two noncompact annuli in ∂W. (Imagine W is the closed outer complement of a catenoid and $\gamma(W)$ is the waist circle of the catenoid.) By choosing $\gamma(W)$ to intersect the interior of one of the disks in $\mathcal{D}_1$ or $\mathcal{D}_2$ on the boundary of W, we can insure that neither annulus in ∂W bounded by $\gamma(W)$ is smooth. Fix one of these annuli and an exhaustion of it by compact annuli $A_1 \subset A_2 \subset \ldots A_n \subset \ldots$ with $\gamma(W) \subset \partial A_1$. By [14] the boundary of W is a good barrier for solving Plateau-type problems in W. Let $\widetilde{A}_n$ denote a least area annulus in W with the same boundary as A_n which is embedded by [14]. The curve $\gamma(W)$ bounds a properly embedded least area annulus $A(W)$ in W, where $A(W)$ is the limit of some subsequence of $\{\widetilde{A}_n\}$; the existence of $A(W)$ depends on local curvature and local area estimates given in a similar construction in [9]. Since the interior of the minimal annulus $A(W)$ is smooth, the maximum principle implies that $A(W)$ intersects ∂W only along $\gamma(W)$. The stable minimal annulus $A(W)$ has finite total curvature [3] and so is asymptotic to the end of a plane or catenoid in $\mathbb{R}^3$. By the maximum principle at infinity [13], the end of $A(W)$ is a positive distance from ∂W. Hence, one can choose the representative end of a plane or catenoid to which $A(W)$ is asymptotic to lie in the interior of W.

Let $\mathcal{S}$ denote the collection of ends of planes and catenoids defined above which arises from the collection of nonsimply connected components W of $H_i - \cup\mathcal{D}_i$. It follows from the proof of the Ordering Theorem in [8] that $\mathcal{S}$ induces the ordering of $\mathcal{E}(F)$. Since the middle ends of F are not limit ends, when F has one limit end, then, after a possible reflection of F across the xy-plane, we may assume that the limit end of F is its top end. Thus, $\mathcal{S}$ will be naturally indexed by the nonnegative integers $\mathbb{N}$ if F has one limit end or by $\mathbb{Z}$ if F has two limit ends with the ordering on the index set $\mathbb{N}$ or $\mathbb{Z}$ coinciding with the natural ordering on $\mathcal{S}$, and the subset of nonlimit ends in $\mathcal{E}(F)$.

Suppose that F has one limit end and let $\mathcal{S} = \{E_0, E_1, \ldots\}$. Let B_0 be a ball of radius r_0 centered at the origin with $\partial E_0 \subset B_0$ and such that ∂B_0 intersects E_0 transversely in a single simple closed curve γ_0. The curve γ_0 bounds a disk $D_0 \subset \partial B_0$. Attach D_0 to $E_0 - B_0$ to make a plane P_0. Next let B_1 be a ball centered at the origin of radius $r_1, r_1 \geq r_0 + 1$, such that $\partial E_1 \subset B_1$ and ∂B_1 intersects E_1 transversely in a single simple closed curve γ_1. Let D_1 be the disk in ∂B_1 disjoint from P_0. Let P_1 be the plane obtained by attaching D_1 to $E_1 - B_1$. Continuing in this manner we produce planes $P_n, n \in \mathbb{N}$, that satisfy properties 1, 2, 3 in the lemma. These planes can be modified by a small C^0-perturbation so that the resulting planes are smooth.

If F had two limit ends instead of one limit end, then a simple modification of this argument also would give a collection of planes $\mathcal{P}$ satisfying properties 1, 2, 3 in the lemma. □

Remark 3.2. Lemma 3.1 only addresses the case where the surface F has an infinite number of ends. When there are a finite number of ends greater than two, then the proof of the lemma goes through with minor modifications. If F has two ends and is an annulus, then extra care must be taken to find the single plane in $\mathcal{P}$ (see for example, the proof of the Ordering Theorem for this argument).

PROPOSITION 3.3. *There exists a collection of planes $\mathcal{P}$ satisfying the properties described in Lemma* 3.1 *and such that each plane in $\mathcal{P}$ intersects F in a single simple closed curve. Furthermore, in the slab between two successive planes in $\mathcal{P}$, F has exactly one end.*

Proof. Suppose the limit tangent plane to F is horizontal and that $\mathcal{P}$ is finite. Let P_T and P_B be the top and bottom planes in the ordering on $\mathcal{P}$. Since the inclusion of the fundamental group of F into the fundamental group of either complement is surjective [9], the proof of Haken's lemma [10] implies that P_T can be moved by an ambient isotopy supported in a large ball so that the resulting plane P_T' intersects F in a single simple closed curve. Let $\widetilde{P}_B$ be the image of P_B under this ambient isotopy. Consider the part F_B of F that lies in the half space below P_T' and note that the fundamental group of F_B maps onto the fundamental group of each complement of F_B in the half space. The proof of Haken's lemma applied to $\widetilde{P}_B$ in the half space produces a plane P_B' isotopic to $\widetilde{P}_B$ that intersects F_B in a simple closed curve. We may assume that this is an isotopy which is the identity outside of a compact domain in F_B.

Consider the slab bounded by P_T' and P_B'. The following assertion implies that $\{P_T', P_B'\}$ can be expanded to a collection of planes $\mathcal{P}$ satisfying all of the conditions of Proposition 3.3.

ASSERTION 3.4. *Suppose S is a slab bounded by two planes in $\mathcal{P}$ where $\mathcal{P}$ satisfies Lemma* 3.1. *Suppose each of these planes intersects F in a simple closed curve. Then there exists a finite collection of smooth planes in S, each intersecting F in a simple closed curve, which separate S into subslabs each of which contains a single end of F. Furthermore the addition of these planes to $\mathcal{P}$ gives a new collection satisfying Lemma* 3.1.

Proof. Here is the idea of the proof of the assertion. If F has more than one end in S, then there is a plane in S which is topologically parallel to the boundary planes of S and which separates two ends of $F \cap S$. The proof of Haken's lemma then applies to give another such plane with the same end which intersects F in a simple closed curve. This new plane separates S into two slabs each containing fewer ends of F. Since the number of ends of $F \cap S$ is finite, the existence of the required collection of planes follows by induction. $\square$

Assume now that the number of planes in $\mathcal{P}$ satisfying Lemma 3.1 is infinite. We first check that $\mathcal{P}$ can be refined to satisfy the following additional property: If W is a closed complement of either $H_1 - \cup\mathcal{D}_1$ or $H_2 - \cup\mathcal{D}_2$, then W intersects at most one plane in $\mathcal{P}$. We will prove this in the case that F has one limit end. In what follows we will assume our standard conventions: F has a horizontal limit tangent plane at infinity and the limit end is maximal in the ordering of ends. The proof of the case where F has two limit ends is similar.

Let $\mathcal{W}$ be the set of closures of the components of $H_1 - \cup\mathcal{D}_1$ and $H_2 - \cup\mathcal{D}_2$. Given $W \in \mathcal{W}$, let $\mathcal{P}(W)$ be the collection of planes in $\mathcal{P}$ that intersect W. If W is a compact ball, then $\mathcal{P}(W)$ is a finite set of planes since $\mathcal{P}$ is proper. If W is homeomorphic to $A \times [0,1)$, then $\mathcal{P}(W)$ is also finite. To see this choose a plane $P \in \mathcal{P}$ whose end lies above the end of W; the existence of such a plane is clear from the construction of $\mathcal{P}$ in the previous lemma. Note that the closed half space above P intersects W in a compact subset. Hence, only a finite number of the planes above P can intersect W. Since there are an infinite number of planes in $\mathcal{P}$ above P, there exists a plane $\widetilde{P}$ above P so that $\widetilde{P}$ is disjoint from W and any plane in $\mathcal{P}$ above $\widetilde{P}$ is also disjoint from W. Since there are only a finite number of planes below $\widetilde{P}$, only a finite number of planes in $\mathcal{P}$ can intersect W.

We now refine $\mathcal{P}$. First recall that the end of P_0 is contained in a single component of $\mathcal{W}$. Hence, the plane P_0 intersects a finite number of components in $\mathcal{W}$ and each of these components intersects a finite collection of planes in $\mathcal{P}$ different from P_0. Remove this collection from $\mathcal{P}$ and reindex to get a new collection $\mathcal{P} = \{P_0, P_1, \cdots\}$. Note that P_1 does not intersect any component $W \in \mathcal{W}$ that also intersects P_0. Now remove from $\mathcal{P}$ all the planes different from P_1 that intersect some component $W \in \mathcal{W}$ that P_1 intersects. Continuing inductively one eventually arrives at a refinement of $\mathcal{P}$ such that for each $W \in \mathcal{W}$, $\mathcal{P}(W)$ has at most one element. This refinement of $\mathcal{P}$ satisfies the conditions of Lemma 3.1 and so, henceforth, we may assume that $\mathcal{P}(W)$ contains at most one plane for every $P \in \mathcal{P}$.

The next step in the proof is to modify each $P \in \mathcal{P}$ so that the resulting plane P' intersects F in a simple closed curve. We will do several modifications of P to obtain P' and the reader will notice that each modification yields a new plane that is a subset of the union of the closed components of $\mathcal{W}$ that intersect the original plane P. This is important to make sure that further modifications can be carried out.

Suppose $P \in \mathcal{P}$ and the end of P is contained in H_1. Let $\mathcal{A}_2$ be the set of components of $\mathcal{W} \cap H_2$ that are homeomorphic to $A \times [0,1)$. For each $W \in \mathcal{A}_2$, let $T(W)$ be a properly embedded half plane in W, disjoint from $\cup\mathcal{D}_2$, such that the geodesic closure of $W - T(W)$ is homeomorphic to a closed half space of $\mathbb{R}^3$. Assume that P intersects transversely the half planes of the form $T(W)$ and the disks in $\mathcal{D}_2$.

We first modify P so that there are no closed curve components in $P \cap (\cup\mathcal{D}_2)$. If $D \in \mathcal{D}_2$ and $P \cap D$ has a closed curve component, then there is an innermost one and it can be removed by a disk replacement. Since the end of P is contained in H_1, there are only a finite number of closed curve components in $\cup\mathcal{D}_2$ and they can be removed by successive innermost disk replacements. In a similar way we can remove the closed curve components in $P \cap (\cup T(W))_{W \in \mathcal{A}_2}$.

We next remove compact arc intersections in $P \cap (\cup\mathcal{D}_2)$ by sliding P over an outermost disk bounded by an outermost arc and into H_1. In a similar way we can remove the finite number of compact arc intersections of P with $\cup T(W)_{W \in \mathcal{A}_2}$. Notice that P already intersects the region that we are pushing it into.

After the disk replacements and slides described above, we may assume that P is disjoint from the disks in $\mathcal{D}_2$ and the half planes in $\mathcal{A}_2$. Let $W \in \mathcal{W}$ be the component which contains the end of P and let $P(*)$ be the component of $P \cap W$ which contains the end of P. Cut H_2 along the disks in $\mathcal{D}_2$ and half planes in $\mathcal{A}_2$. Since every closed component of the result is a compact ball or a closed half space, the boundary curves of $P(*)$, considered as subsets of these components, bound a collection of pairwise disjoint disks in H_2. The union of these disks with $P(*)$ is a plane P'' with $P'' \cap W = P(*)$. If $P(*)$ is an annulus, then we are done. Otherwise, since the fundamental group of W is $\mathbb{Z}$, the loop theorem implies that one can do surgery in W on $P(*) \subset P''$ such that after the surgery, the component with the end of P'' has fewer boundary components. After further surgeries in W we obtain an annulus $P'(*)$ with the same end as $P(*)$ and with boundary curve being one of the boundary curves of $P(*)$. By our previous modifications, $\partial P'(*)$ lies on the boundary of the closure of one of the components of $H_2 - \cup\mathcal{D}_2$ and bounds a disk D in this component. We obtain the required modified plane $P' = P'(*) \cup D$ which intersects F in the curve $\partial P'(*)$.

The above modification of a plane $P \in \mathcal{P}$ can be carried out independently of the other planes since the modified plane is contained in the union of the components of $\mathcal{W}$ that intersect P and when P intersects $W \in \mathcal{W}$, then no other plane in $\mathcal{P}$ intersects W. Now perform these modifications on all of the even indexed planes in $\mathcal{P}$ to form a new collection. Note that the odd indexed planes of $\mathcal{P}$ give rise to a proper collection of slabs with exactly one even indexed plane in each of these slabs. Next remove all of the odd indexed planes from $\mathcal{P}$ and reindex the remaining ones by $\mathbb{N}$ in an order preserving manner.

Finally, applying the Assertion 3.4 allows one to subdivide the slabs between successive planes in $\mathcal{P}$ so that each slab contains at most one end of F. This completes the construction of $\mathcal{P}$ and the proof of Proposition 3.3. $\square$

4. The structure of a minimal surface in a slab or half space

Let F be an orientable surface and let $\mathcal{C}$ be a proper collection of disjoint simple closed curves in $F \times \{0\}$. If H is a three-manifold that is obtained by adding 2-handles to $F \times [0,1]$ along $\mathcal{C}$ and then capping off the sphere components with balls, then H is a *compression body.* Alternatively, if H is an irreducible three-manifold and $\partial_- H$ is a closed proper subsurface of ∂H and Γ is a properly embedded 1-dimensional CW-complex in H so that there is a proper deformation retraction $r\colon H \to \partial_- H \cup \Gamma$, then H is a compression body. The surface $\partial_+ H = \overline{\partial H - \partial_- H}$ is called the *inner boundary component* of H. If $\partial_- H = \emptyset$, then we say H is a *handlebody.* The compression body H is properly embedded in the three-manifold M, if its inclusion map is a proper embedding in the topological sense and $\partial_- H = H \cap \partial M$. A *Heegaard splitting* of a three-manifold M is a pair of compression bodies H_1 and H_2 properly embedded in M so that $M = H_1 \cup H_2$ and the intersection of H_1 and H_2 is exactly their inner boundary components. The surface $\partial_+ H_1 = \partial_+ H_2$ is called a *Heegaard surface.*

The 1-dimensional CW-complex Γ in the definition of compression body is called a *spine* of the compression body. There are many choices of spines for a given compression body. For the sake of combinatorial clarity we will only work with spines whose vertices are all monovalent or trivalent, and the monovalent vertices coincide with $\Gamma \cap \partial_- H$. We can further assume that the restriction of the deformation retraction $r\colon H \to \partial_- H \cup \Gamma$ restricted to $\partial_+ H$ has the property that the inverse image of any point that is in the interior of an edge is a single circle, the inverse image of any monovalent vertex is a circle and the inverse image of any trivalent vertex is a trivalent graph with three edges and two vertices (a theta curve). This leads to a corresponding decomposition of $\partial_+ H$ into pairs of pants, annuli, and a copy of $\partial_- H$ with a disk removed for each monovalent vertex of Γ. There is a pair of pants for each trivalent vertex, and an annulus for each edge that contains no vertex, and the rest of the surface runs parallel to $\partial_- H$. We can reconstruct Γ up to isotopy from this decomposition.

Aside from isotopy there are two moves that we will be using on Γ. They are both variants of the Whitehead move. We alter the graph according to one of the two local operations shown in Figure 1 and Figure 2.

Dually the Whitehead move involves two pairs of pants meeting along a simple closed curve γ which is the inverse image of a point in the interior of the edge to be replaced. If γ' is any simple closed curve lying on that union

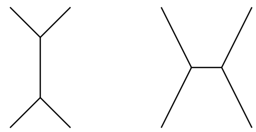

Figure 1: Whitehead move

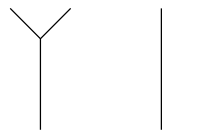

Figure 2: Half Whitehead move with lower vertices on $\partial_- H$

of pants that intersects γ transversely in exactly two points, and separates the boundary components of the two pairs of pants into two sets of two, then we can perform the Whitehead move so that the two new pairs of pants meet along γ'.

The half Whitehead move can occur at a trivalent vertex that is adjacent to a monovalent vertex (lying on $\partial_- H$). You can think of it as collapsing the edge with one endpoint on the boundary and one endpoint at the vertex to the point on $\partial_- H$ and then pulling the ends of the two remaining edges apart.

Suppose that H is a compression body and δ is a simple closed curve on the inner boundary component of H. We can extend δ to a singular surface whose boundary lies in $\Gamma \cup \partial_- H$. First isotope δ so that with respect to the decomposition into annuli, pants and a punctured $\partial_- H$, the part of δ that lies in each component is essential. There is a singular surface with boundary δ obtained by adding "fins" going down to Γ based on the models shown in Figure 3, along with fins in the annuli and near $\partial_- H$.

Figure 3: Extending the disk D to a singular surface.

On a pair of pants there are six isotopy classes of essential proper arcs. For each choice of a pair of boundary components there is an isotopy class of essential arcs joining them, and for each boundary component there is an isotopy class of essential arcs joining that boundary component to itself. We call an essential proper arc *good* if its endpoints lie on distinct boundary components, and *bad* if its endpoints lie in the same boundary component. Two such arcs are *parallel* if they are disjoint and have their endpoints in the same boundary components.

LEMMA 4.1. *Suppose that H is a compression body and δ is a simple closed curve on $\partial_+ H$. Either δ bounds a disk in H or there is a graph Γ so that H is a regular neighborhood of $\Gamma \cup \partial_- H$ such that δ has no bad arcs.*

Proof. The argument will be by induction on a complexity for δ. Let s be the number of bad arcs. Given a bad arc k, the arcs (or arc) of δ adjacent to k lie in the same pair of pants or in the punctured copy of $\partial_- H$. If the two endpoints of the bad arc coincide with the two endpoints of another bad arc, then let $d(k) = 0$. If both arcs lie in the punctured copy of $\partial_- H$, then let $d(k) = 1$. If both arcs lie in the same pair of pants P, then either the two arcs are parallel or not parallel . If they are not parallel, then $d(k) = 1$. If they are parallel, then follow them into the next surface. If the next surface is the punctured copy of $\partial_- H$, then $d(k) = 2$, if the next surface is a pair of pants and the next arcs are not parallel, then $d(k) = 2$, otherwise follow them into the next surface, and keep counting. Let $m = \min_{k \text{ bad}} d(k)$. The complexity of δ is the pair (s, m).

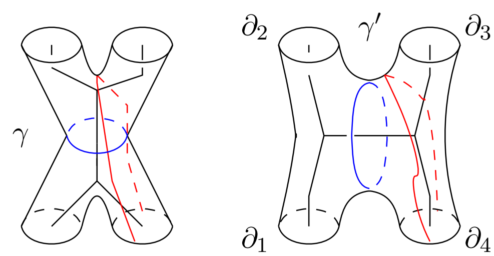

Figure 4: Reducing m when it is greater than 1. On the right-hand side of the figure, P_1 is on the left, P_2 is on the right and $P_1 \cap P_2 = \gamma'$.

If $m > 1$, then we do the Whitehead move to reduce m as follows; see Figure 4. Let k be a bad arc with $d(k) = m$. Let Q be the union of the pair of pants containing k and the pair of pants that contains the adjacent pair of arcs k_1 and k_2. Let γ be the curve that the two pairs of pants meet along. Let $\partial_1, \partial_2, \partial_3, \partial_4$ be the boundary components of Q labeled so that ∂_1 and ∂_4 belong to one pair of pants, ∂_2 and ∂_3 belong to the other pair of pants, and both k_1 and k_2 have an endpoint in ∂_4. Let $a = k \cup k_1 \cup k_2$. There is an arc b of ∂_4 so that a push off γ' of $a \cup b$ lies in Q, misses a and separates the boundary components of Q into two sets of two, say one set is ∂_1 and ∂_2 and the other is ∂_3 and ∂_4. Perform the Whitehead move so that γ' is the intersection of the two new pairs of pants. Denote the new pairs of pants, resulting from the Whitehead move corresponding to γ', by P_1 where $\partial P_1 = \partial_1 \cup \partial_2$ and by P_2 where $\partial P_2 = \partial_3 \cup \partial_4$. Notice that a is a bad arc and $d(a) = m - 1$. To conclude

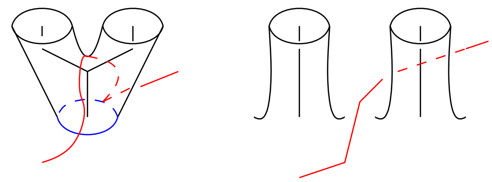

Figure 5: The endpoints of the bad arc are near $\partial_{-}H$.

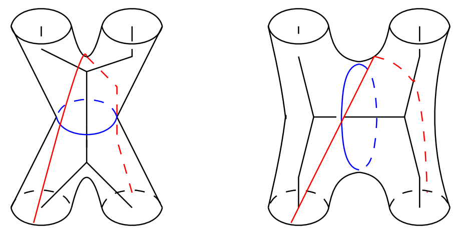

Figure 6: Reducing the number of bad arcs when $m = 1$. In the figure on the left, the waist is the curve γ and the arc b lies in the lower pair of pants.

that we have simplified the picture we need to see that we have not increased the number of bad arcs. If l is a bad arc in $P_1 \cup P_2$ and it has its endpoints in some ∂_i, then it contains some bad arc of the original picture. If l has its endpoints in γ' and lies in P_2, as δ is embedded it is trapped in the annulus between γ' and a and is hence inessential. If l has its endpoints in γ' and is contained in P_1, once again the arc is trapped by a and hence there must be two arcs in P_2 having one endpoint each in common with l and the other in b; but this means l is contained inside a bad arc from the original picture. Hence we did not increase s. On the other hand we have decreased m by 1.

If $m = 1$, then there are two cases. The first is when an adjacent pair of arcs lies in the part of the surface parallel to $\partial_{-}H$. In this case we do a half Whitehead move to reduce the number of bad arcs; see Figure 5.

The other case is when an adjacent pair of arcs is contained in an adjacent pair of pants. Once again, a Whitehead move can be applied to reduce the number of bad arcs; see Figure 6. Let Q be the union of the two pairs of pants that contain k, and let k_1 and k_2 be the arcs of δ adjacent to k and lying in the other pair of pants. Let γ be the circle that the pants intersect along. Let b be an arc in the other pair of pants that contains the endpoints of k, only intersects k_1 and k_2 in the endpoints they share with k, and is transverse to the other components of $\delta \cap Q$. Let γ' be a push off of $b \cup k$ such that it intersects k in a single point, is disjoint from k_1 and k_2, and such that during the push off, the related arcs b_t stay transverse to δ and the related arcs k_t are disjoint from δ for $t \neq 0$. Notice that the arc $k_1 \cup k \cup k_2$ gets separated into two good arcs by γ'. Hence, if we have not created any new bad arcs, then we have reduced the total number of bad arcs. If a bad arc enters and leaves the new picture through a boundary component of Q, then it is either contained in or contains a bad arc of the old picture. Hence, we only need to worry about bad arcs with their endpoints in γ'. Since δ is embedded, such an arc misses $k_1 \cup k \cup k_2$. The result of cutting Q along the arc $k_1 \cup k \cup k_2$ is a pair of pants and γ' gives rise to an arc of this pair of pants that has both its endpoints in the same boundary component of the pair of pants. The only proper arcs that intersect the arc corresponding to γ' in an essential manner in two points must have both their endpoints in the same boundary component of the pair of pants. This implies that such a bad arc is contained inside a bad arc from the original picture.

Finally, when $m = 0$, there are two arcs joined end to end, and the disk inside the regular neighborhood of Γ is readily visible; see Figure 7. $\square$

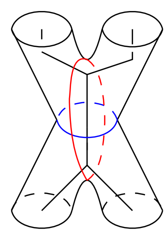

Figure 7: The interior disk.

Suppose S is a flat 3-manifold in $\mathbb{R}^3$ that is homeomorphic to $\mathbb{R}^2 \times [0,1]$. Denote the components of ∂S by $\partial_0 S$ and $\partial_1 S$. Assume further that there are simple closed curves $C_0 \subset \partial_0 S$ and $C_1 \subset \partial_1 S$ so that $\partial_i S$ is a union of two minimal surfaces sharing C_i as their joint boundary. As $\partial_i S$ is a plane, one of these surfaces is a disk D_i and the other is a once-punctured disk A_i. Finally, assume that F is a properly embedded minimal surface in S having one end, boundary $C_0 \cup C_1$ and such that in each closed complement of F in S, the interior angles along ∂F are less than π. Up until the end of this section, we will assume these properties hold for S and F.

Proposition 4.2. *The surface F separates S into two compression bodies H_1 and H_2, having F as their inner boundary components. That is, F is a Heegaard surface.*

Proof. We outline the idea for the sake of completeness. First consider the region H_1 and suppose that ∂H_1 has one end. In this case, by [9], H_1 is a handlebody. Assume now that ∂H_1 has two ends. By Freedman's theorem applied to H_1, there exists a proper family of compressing disks $\mathcal{D}_1$ which can be chosen to have their boundary components disjoint from ∂S. After possibly restricting to a subcollection of $\mathcal{D}_1$, we see that the result of cutting

H_1 along $\mathcal{D}_1$ is connected and homeomorphic to $A \times [0,1)$. But $A \times [0,1)$ is homeomorphic to $\Sigma \times [0,1]$ where Σ is a proper once-punctured disk on one of the boundary planes of S with boundary being one of the two boundary components of F. In this case H_1 is a compression body. Similarly, if ∂H_1 has three ends, then one can choose the collection $\mathcal{D}_1$ so that cutting H_1 along $\mathcal{D}_1$ is homeomorphic to $\Sigma \times [0,1]$ where Σ consists of the two once-punctured disks in ∂S bounded by ∂F. Similarly, H_2 is a compression body, and so F is a Heegaard surface in S. □

The proof of the topological classification theorem will require the examination of three kinds of surfaces with one end.

Type 1. The topology of $F \subset S$ is finite. This means that F is homeomorphic to the result of removing a single point from a compact surface with two boundary components. In this case F separates S into two compression bodies. One of the compression bodies has boundary $D_0 \cup F \cup A_1$ and the other has boundary $D_1 \cup F \cup A_0$. Since A_0 and A_1 lie in different components of the complement of F, any arc joining A_0 to A_1 has $\mathbb{Z}_2$-intersection number 1 with F. Hence the end is odd.

Type 2. F has infinite genus and any arc joining A_0 to A_1 has $\mathbb{Z}_2$-intersection number 1 with F. Once again F separates S into two compression bodies, one with boundary $D_0 \cup F \cup A_1$ and the other with boundary $D_1 \cup F \cup A_0$. This is an odd end.

Type 3. Any arc joining A_0 to A_1 has $\mathbb{Z}_2$-intersection number 0 with F. In this case F separates S into a handlebody with boundary $F \cup D_0 \cup D_1$ and a compression body with boundary $F \cup A_0 \cup A_1$. This end is even and is necessarily of infinite genus.

Our task is to show that in the first case, the surface is classified up to topological equivalence by its genus, and any two surfaces of the second type (or third type) are topologically equivalent. Let D denote a topological disk, and let A denote $S^1 \times [0,1)$.

THEOREM 4.3. *If $F \subset S$ and $F' \subset S'$ are two minimal surfaces with one end of finite type, the same genus and boundary consisting of circles C_0, C_1 and C_0', C_1' (respectively), then there is a homeomorphism $h\colon S \to S'$ with $h(\partial_i S) = \partial_i S'$ and $h(F) = F'$.*

Proof. We need only check that the embedding of F in S is the standard one. We will assume that we have chosen a homeomorphism between S and $\mathbb{R}^2 \times [0,1]$ and work in those coordinates. It is possible to find a large solid cylinder $D \times [0,1]$ whose boundary cylinder intersects F in a single simple closed curve in $\partial D \times [0,1]$ so that:

1. $S - D \times [0,1]$ is homeomorphic to $A \times [0,1]$;
2. The pair $(S - D \times [0,1], F - D \times [0,1])$ is topologically equivalent to the pair $(A \times [0,1], A \times \{1/2\})$.

This follows quite easily from the fact that F is a Heegaard surface. As F has finite type, there is a *compact* 1-dimensional CW-complex Γ so that F is isotopic to the frontier of a regular neighborhood of $\Gamma \cup \mathbb{R}^2 \times \{0\}$. Since Γ is compact, its projection to $\mathbb{R}^2$ is bounded. Hence there is a large D in $\mathbb{R}^2$ that contains its image. The set $D \times [0,1]$ satisfies the conditions above. (Similarly we could find $D' \times [0,1]$ having the same properties with respect to $F' \subset S'$.)

The existence of the disk D above implies that we can simultaneously compactify S and F by adding a single circle at infinity so that the compactification of S is homeomorphic to the three-ball and the closure of F is a Heegaard surface. The fact that F completes to a surface follows from the second property above. To see that F is a Heegaard surface, note that the natural maps on fundamental groups induced by inclusion of the surface into its complements are surjective. This implies that the compactified surface is a Heegaard splitting of the three-ball. In [7] it was proved that such surfaces are classified up to homeomorphisms of the ball by their boundary and their genus. Hence, if F and F' have the same genus, then we can find a homeomorphism of the compactifications of S and S' taking the compactification of F to the compactification of F'. By restricting the homeomorphism, we get a homeomorphism of S to S' having the desired properties. $\square$

Let M be a manifold and suppose that F is a Heegaard surface of M with compact boundary. We say that F is *infinitely reducible* if there is a properly embedded family of balls that are disjoint from one another, so that each ball intersects F in a surface of genus greater than zero having a single boundary component, and so that every end representative of M has nonempty intersection with the family of balls. It is a good exercise in the application of the Reidemeister–Singer theorem to prove that any two infinitely reducible Heegaard splittings of M which agree on the boundary of M are topologically equivalent via a homeomorphism of M that is the identity on the boundary. This result appears in [5] and it can also be seen to hold from a proof analysis of [9].

Hence, in order to prove that up to topology there is only one surface in types 2 and 3, it suffices to show that a minimal surface in a slab S and with one end of infinite topology with boundary C_0, C_1 is infinitely reducible. For this purpose we use a simple extension of a lemma from [6] to Heegaard surfaces with boundary.

Lemma 4.4. *Suppose that F is the Heegaard surface of the irreducible manifold M, and there are a 1-dimensional* CW*-complex* Γ *in M and a subsurface A of ∂M so that F is properly ambiently isotopic to a regular neighborhood of $\Gamma \cup A$. Suppose further that there is a ball B embedded in M so that there is a nontrivial cycle of Γ contained in the interior of B. Then F is reducible.*

Proof. Let C be the nontrivial cycle of Γ contained in the interior of B. Notice that F is a Heegaard surface for a splitting of the complement of a regular neighborhood of C. Apply Haken's lemma to find a sphere intersecting F in a single circle. The sphere cuts off a subsurface of F having genus greater than zero. Since the sphere bounds a ball in M, F is reducible. $\square$

Theorem 4.5. *If F is a Heegaard surface of S with one end, infinite genus and boundary consisting of two circles $C_i \subset \mathbb{R}^2 \times \{i\}$, then the corresponding Heegaard splitting is infinitely reducible.*

Proof. Recall the coordinatization $S = \mathbb{R}^2 \times [0, 1]$. Let Γ be a 1-dimensional CW-complex so that it is a spine of one of the compression bodies making up the Heegaard splitting. Hence, F is the frontier of a regular neighborhood of Γ and a subsurface Σ of $\partial(\mathbb{R}^2 \times [0, 1])$. Up to proper isotopy we can make this regular neighborhood as thin as we like, so that if we are intersecting F with a proper surface P, we can make Γ transverse to P and assume that the intersection of F with the surface consists of small circles about the intersection of Γ with P, and one manifolds that run parallel to the part of ∂P that lies in Σ.

By Proposition 2.2 of [9], there is an exhaustion of S by compact submanifolds K_i so that the part of F lying outside of each K_i is a Heegaard surface for the complement of K_i. For any K_i there is $D_i \times [0, 1]$ that contains K_i so that its frontier is transverse to Γ. Choose a half plane HP_i whose boundary consists of an arc in $\partial D_i \times [0, 1]$ and two rays, one each in $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$, that cuts the complement of $D_i \times [0, 1]$ into a half space. If there is a cycle of the graph Γ in this half space, then there is a reducing ball outside K_i. We assume that the intersection of F with $\partial D_i \times [0, 1]$ is effected as above so that the part of the compression body containing Γ lying in the complement of $D_i \times [0, 1]$ is a compression body. We further assume that the intersection of F with HP_i is also of this form.

Our goal now is to prove that there is a reducing sphere outside of K_i. Since F has infinite genus, there is a compressing disk E for F in the complement of the compression body and that lies outside of $D_i \times [0, 1]$. By Lemma 4.1, we can choose a decomposition of F into pants, annuli and a surface parallel to Σ minus some disks so that the boundary of E has no bad arcs, or there is a disk inside the compression body with boundary ∂E. In the second case the two disks form a sphere, which bounds a ball in the complement of

$D_i \times [0,1]$ containing a cycle of the graph. Hence there is a reducing sphere outside of K_i.

We now consider the case where E has no bad arcs. First make E transverse to HP_i. We can isotope E (and the graph Γ) so that there are no simple closed curves in $E \cap HP_i$. Let k be an arc of $E \cap HP_i$ that is outermost in E. We will show that we can either alter the cycle which is the boundary of E so that it intersects HP_i in fewer points or we can find a nontrivial cycle of Γ contained in the singular disk extending E. In the case that we reduce the number of points, we continue on. Either we find a nontrivial cycle or we pull E completely off of HP_i, in which case there is a nontrivial cycle of Γ disjoint from HP_i in the desired region.

There are two cases.

1. *The two endpoints of k lie in the same boundary component of the same pair of pants.*

The arc of the boundary of the disk extending E lying in Γ defines a cycle. As ∂E does not ever enter and leave a pair of pants through the same boundary component, this cycle is nontrivial. As the disk is outermost, there is a nontrivial cycle of Γ in the result of cutting the complement of $D_i \times [0,1]$ along HP_i. As HP_i is a half space, it is easy to see there is a cycle of Γ contained in a ball. Hence there is a trivial handle of F lying outside K_i.

2. *The two endpoints of k lie in distinct boundary components of pairs of pants.*

The first thing to notice is that the arc of the boundary of the disk extending E in Γ is embedded. If not, then it would contain a cycle, and as the disk is outermost that cycle would live in a ball. Let l be the number of pairs of pants that the arc passes through.

It remains to show that if $l > 1$, we can reduce it. If $l > 1$, we reduce it via a sequence of Whitehead moves on Γ so as not to make any arcs of ∂E bad. Let Q be a union of two pairs of pants so that one of them has a boundary component on HP_i and an arc of k runs across Q from that boundary component to another component that belongs in a separate pair of pants from the first. Let γ' be the frontier of a regular neighborhood in Q of the union of the arc of k with the two boundary components. It is easy to check that γ' can be used to perform a Whitehead move on Q that does not create any new bad arcs, and reduces l by 1.

If $l = 1$, we can then use the outermost disk as a guide to isotope Γ so as to reduce the number of points of intersection of that part of the graph in the boundary of the singular disk which is the extension of E.

After finitely many steps, we have either found a cycle in a ball or pulled the singular disk, which is the boundary of E, off of HP_i. In case 2 above, because E was a compressing disk, there is a cycle of Γ contained in the boundary of the singular disk, and it is disjoint from HP_i meaning that we have a cycle in a ball and this ball lies outside of K_i as desired. This ball is contained in some $K_j, j > i$, and so we can reproduce our arguments to find a cycle contained in a ball outside K_j. It follows that the Heegaard splitting is infinitely reducible. $\square$

Remark 4.6. Results similar to Proposition 4.2, Theorem 4.3 and Theorem 4.5 hold if S is a topological half space of R^3 with one boundary plane consisting of the union of an annulus A and a compact disk D, both of surfaces having least area in the respective closed complements of a properly embedded minimal surface $F \subset \mathbb{R}^3$, and $\partial S \cap F = \partial D = \partial A$, and $F \cap S$ has one end. This situation arises when $F \cap S$ represents a top or bottom end of F. The halfspace case is somewhat easier to analyze than the slab case and we leave the details to the reader to verify.

Proof of Theorem 1.1. Let $\mathcal{P} = \{P_i | i \in I\}$ be the collection of smooth planes given in the statement of Proposition 3.3. We now check that there exists a similar collection of piecewise smooth planes $\widetilde{\mathcal{P}}$ with the same ordered indexing set I and such that each plane in $\widetilde{\mathcal{P}}$ is the union of a proper minimal annulus and a minimal disk, each of which intersects F only along their common boundary. The conclusion of Proposition 3.3 is true even when F has finitely many ends.

Let $\{\gamma_i = P_i \cap F \mid P_i \in \mathcal{P},\ i \in I\}$. Let $\widetilde{D}_i$ be the related least area disk bounded by γ_i in P_i. The arguments in the proof of Lemma 3.1 show that we can replace the annular component A_i of $P_i - \gamma_i$ by a properly embedded annulus $\widetilde{A}_i$ which has least area in a closed complement of F and which is homotopic to A_i in this complement. If $\widetilde{A}_i$ does not intersect F only along its boundary, then it is contained in F by the maximum principle and it must represent a top or bottom annular end of F. Since a top or bottom annular end of F is easily seen to be standardly embedded in its related half space (formed by a small push off of the plane $\widetilde{A}_i \cup \widetilde{D}_i$ which then intersects the end A_i in a simple closed curve), in the next paragraph we assume that $\widetilde{A}_i$ is not contained in F.

Define $\widetilde{\mathcal{P}} = \{P_i = \widetilde{A}_i \cup \widetilde{D}_i \mid i \in I\}$. We claim that this is a proper collection of planes. Otherwise, there is subsequence $\widetilde{P}_{i(n)}$ of these planes, each of which intersects a fixed ball B. Since the curves $\gamma_{i(n)}$ eventually leave any compact ball of $\mathbb{R}^3$, then Schoen's curvature estimates for stable minimal surfaces with boundary [16] imply that there exists a flat plane P passing through B, which lies in the limit set of the sequence of the planes $\widetilde{P}_{i(n)}$. Since P lies in a closed complement of F, we contradict the Halfspace Theorem in [11]. Hence, the collection $\widetilde{\mathcal{P}}$ is proper.

Suppose that F and F' are two properly embedded minimal surfaces and there exists a homeomorphism $h\colon F \to F'$ that preserves the ordering and parity of the ends. By the above discussion, we can find systems of planes that separate space into slabs (and one half space if $I = \mathbb{N}$) and the parts of F and F' lying in the respective slabs (or one or two half spaces if they exist) are Heegaard surfaces. The parity and order-preserving homeomorphism implies that there is a correspondence between the slabs so that the parts of F and F' lying in the corresponding slabs have the same parity. After shifting some handles around so that finite genus surfaces have the same genus, we can then apply the classification theorem for surfaces in a slab to build a homeomorphism of $\mathbb{R}^3$ that takes F to F'. □

University of Iowa, Iowa City, IA
E-mail address: frohman@math.uiowa.edu

University of Massachusetts, Amherst, MA
E-mail address: bill@math.umass.edu

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