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Lagrangian intersections and the Serre spectral sequence

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Abstract

For a transversal pair of closed Lagrangian submanifolds L, L' of a symplectic manifold M such that $\pi_1(L) = \pi_1(L') = 0 = c_1|_{\pi_2(M)} = \omega|_{\pi_2(M)}$ and for a generic almost complex structure J, we construct an invariant with a high homotopical content which consists in the pages of order ≥ 2 of a spectral sequence whose differentials provide an algebraic measure of the high-dimensional moduli spaces of pseudo-holomorpic strips of finite energy that join L and L'. When L and L' are Hamiltonian isotopic, we show that the pages of the spectral sequence coincide (up to a horizontal translation) with the terms of the Serre spectral sequence of the path-loop fibration $\Omega L \to PL \to L$ and we deduce some applications.

Contents

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1. Introduction

Consider a symplectic manifold (M, ω) which is convex at infinity together with two closed (compact, connected, without boundary) Lagrangian submanifolds L, L' in general position. We fix from now on the dimension of M to be $2n$. Unless otherwise stated we assume in this introduction that

$$\pi_1(L) = \pi_1(L') = 0 = c_1|_{\pi_2(M)} = \omega|_{\pi_2(M)} \tag{1}$$

and we shall keep this assumption in most of the paper.

One of the main tools in symplectic topology is Floer's machinery (see [29] for a recent exposition) which, once a generic almost complex structure compatible with ω is fixed on M, gives rise to a Morse-type chain complex $(CF_*(L, L'), d_F)$ such that $CF_*(L, L')$ is the free $\mathbb{Z}/2$-vector space generated by (certain) intersection points in $L \cap L'$ and d_F counts the number of connecting orbits (also called "Floer trajectories" - in this case they are pseudo-holomorphic strips) joining intersection points of relative (Maslov) index equal to 1 (elements of Floer's construction are recalled in §2). In this construction are only involved 1 and 2-dimensional moduli spaces of connecting trajectories,

The present paper is motivated by the following problem: *extract out of the structure of higher dimensional moduli spaces of Floer trajectories useful homotopical-type data which are not limited to Floer homology* (*or cohomology*).

This question is natural because the properties of Floer trajectories parallel those of negative gradient flow lines of a Morse function (defined with respect to a generic riemannian metric) and the information encoded in the Morse-Smale negative-gradient flow of such a function is much richer than only the homology of the ambient manifold. Indeed, in a series of papers on "Homotopical Dynamics" [2], [3], [4], [5] the second author has described a number of techniques which provide ways to "quantify" algebraically the homotopical information carried by a flow. In particular, in [3] and [5] it is shown how to estimate the moduli spaces arising in the Morse-Smale context when the critical points involved are consecutive in the sense that they are not joined by any "broken" flow line. However, the natural problem of finding a computable algebraic method to "measure" *general*, *high dimensional moduli spaces* of connecting orbits has remained open till now even in this simplest Morse-Smale case. Of course, in the Floer case, a significant additional difficulty is that there is no "ambient" space with a meaningful topology.

We provide a solution to this problem in the present paper. The key new idea can be summarized as follows:

> *In ideal conditions, the ring of coefficients used to define a Morse type complex can be enriched so that the resulting chain complex contains information about high dimensional moduli spaces of connecting orbits.*

Roughly, this "enrichment" of the coefficients is achieved by viewing the relevant connecting orbits as loops in an appropriate space $\tilde{L}$ in which the finite number of possible ends of the orbits are naturally identified to a single point. The "enriched" ring is then provided by the (cubical) chains of the pointed (Moore) loop space of $\tilde{L}$. This ring turns out to be sufficiently rich algebraically such as to encode reasonably well the geometrical complexity of the combinatorics of the higher dimensional moduli spaces. Operating with the new chain complex is no more difficult than using the usual Morse complex. In particular, there is a natural filtration of this complex and the pages of order higher than 2 of the associated spectral sequence (together with the respective differentials) provide our invariant. Moreover, these pages are computable purely algebraically in certain important cases.

This technique is quite powerful and is general enough so that each manifestation of a Morse type complex in the literature offers a potential application. From this point of view, our construction is certainly just a first — and, we hope, convincing — step.

1.1. *The main result.* Fix a path-connected component $\mathcal{P}_\eta(L, L')$ of the space $\mathcal{P}(L, L') = \{\gamma \in C^\infty([0,1], M) : \gamma(0) \in L,\ \gamma(1) \in L'\}$. The construction of Floer homology depends on the choice of such a component. We denote the corresponding Floer complex by $CF_*(L, L'; \eta)$ and the resulting homology by $HF_*(L, L'; \eta)$. In case $L' = \phi_1(L)$ with ϕ_1 the time 1-map of a Hamiltonian isotopy $\phi : M \times [0,1] \to M$ (such a ϕ_1 is called a Hamiltonian diffeomorphism) we denote by $\mathcal{P}(L, L'; \eta_0)$ the path-component of $\mathcal{P}(L, L')$ such that $[\phi_t^{-1}(\gamma(t)] = 0 \in \pi_1(M, L)$ for some (and thus all) $\gamma \in \eta_0$. We omit η_0 in the notation for the Floer complex and Floer homology in this case. Given two spectral sequences $(E^r_{p,q}, d^r)$ and $(G^r_{p,q}, d^r)$ we say that they are *isomorphic up to translation* if there exist an integer k and an isomorphism of chain complexes $(E^r_{*+k,\star}, d^r) \approx (G^r_{*,\star}, d^r)$ for all r. Recall that the path-loop fibration $\Omega L \to PL \to L$ of base L has as total space the space of based paths in L and as fibre the space of based loops. Given two points $x, y \in L \cap L'$ we denote by $\mu(x, y)$ their relative Maslov index and by $\mathcal{M}(x, y)$ the nonparametrized moduli space of Floer trajectories connecting x to y (see §2 for the relevant definitions). We denote by $\mathcal{M}$ the disjoint union of all the $\mathcal{M}(x, y)$'s. We denote by $\mathcal{M}'$ the space of all parametrized pseudo-holomorphic strips. All homology groups below have $\mathbb{Z}/2$-coefficients.

THEOREM 1.1. *Under the assumptions above there exists a spectral sequence*

$$EF(L,L';\eta)=(EF^r_{p,q}(L,L';\eta),d^r_F)\ ,r\geq 1$$

with the following properties:

a. *If* $\phi : M\times[0,1]\to M$ *is a Hamiltonian isotopy, then* $(EF^r_{p,q}(L,L';\eta),d^r)$ *and* $(EF^r_{p,q}(L,\phi_1 L';\phi_1\eta),d^r)$ *are isomorphic up to translation for* $r\geq 2$ *(here* $\phi_1\eta$ *is the component represented by* $\phi_t(\gamma(t))$ *for* $\gamma\in\eta$*).*

b. $EF^1_{p,q}(L,L';\eta)\approx CF_p(L,L';\eta)\otimes H_q(\Omega L)$, $EF^2_{p,q}(L,L';\eta)\approx HF_p(L,L';\eta)\otimes H_q(\Omega L)$.

c. *If* $d^r_F\neq 0$, *then there exist points* $x,y\in L\cap L'$ *such that* $\mu(x,y)\leq r$ *and* $\mathcal{M}(x,y)\neq\emptyset$.

d. *If* $L'=\phi' L$ *with* ϕ' *a Hamiltonian diffeomorphism, then for* $r\geq 2$ *the spectral sequence* $(EF^r(L,L'),d^r_F)$ *is isomorphic up to translation to the* $\mathbb{Z}/2$*-Serre spectral sequence of the path loop fibration* $\Omega L\to PL\to L$.

1.2. *Comments on the main result.* We survey here the main features of the theorem.

1.2.1. *Geometric interpretation of the spectral sequence.* The differentials appearing in the spectral sequence $EF(L,L';\eta)$ provide an algebraic measure of the Gromov compactifications $\overline{\mathcal{M}}(x,y)$ of the moduli spaces $\mathcal{M}(x,y)$ in — roughly — the following sense. Let $\tilde{L}$ be the quotient topological space obtained by contracting to a point a path in L which passes through each point in $L\cap L'$ and is homeomorphic to $[0,1]$. Let $\tilde{M}$ be the space obtained from M by contracting to a point the same path. Clearly, L and $\tilde{L}$ (as well as M and $\tilde{M}$) have the same homotopy type. Each point $u\in\mathcal{M}(x,y)$ is represented by a pseudo-holomorphic strip $u:\mathbb{R}\times[0,1]\to M$ with $u(\mathbb{R},0)\subset L$, $u(\mathbb{R},1)\subset L'$ and such that $\lim_{s\to-\infty}u(s,t)=x, \lim_{s\to+\infty}u(s,t)=y,\ \forall t\in[0,1]$. Clearly, to such a u we may associate the path in L given by $s\to u(s,0)$ which joins x to y. Geometrically, by projecting onto $\tilde{L}$, this associates to u an element of $\Omega\tilde{L}\simeq\Omega L$. The action functional can be used to reparametrize uniformly the loops obtained in this way so that the resulting application extends in a continuous manner to the whole of $\overline{\mathcal{M}}(x,y)$ thus producing a continuous map $\Phi_{x,y}:\overline{\mathcal{M}}(x,y)\to\Omega L$. The space $\overline{\mathcal{M}}(x,y)$ has the structure of a manifold with boundary with corners (see §2 and §3.4.6) which is compatible with the maps $\Phi_{x,y}$. If it happened that $\partial\overline{\mathcal{M}}(x,y)=\emptyset$ one could measure $\overline{\mathcal{M}}(x,y)$ by the image in $H_*(\Omega L)$ of its fundamental class via the map $\Phi_{x,y}$. This boundary is almost never empty so this elementary idea fails. However, somewhat miraculously, the differential $d_F^{\mu(x,y)}$ of $EF(L,L';\eta)$ reflects homologically what is left of $\Phi_{x,y}((\overline{\mathcal{M}}(x,y))$ after "killing" the boundary $\partial\overline{\mathcal{M}}(x,y)$.

From this perspective, it is clear that it is not so important where the spectral sequence $EF(L, L'; \eta)$ converges but rather whether it contains many nontrivial differentials.

1.2.2. *Role of the Serre spectral sequence.* Clearly, point a. of the theorem shows that the pages of order higher than 1 of the spectral sequence together with all their differentials are invariant (up to translation) with respect to Hamiltonian isotopy. Moreover, b. implies that Floer homology is isomorphic to $EF^2_{*,0}(L, L'; \eta)$ and so our invariant extends Floer homology. It is therefore natural to expect to be able to estimate the invariant $EF(L, L'; \eta)$ when L' is Hamiltonian-isotopic to L (and $\eta = \eta_0$) in terms of some algebraic-topological invariant of L. The fact that this invariant is precisely the Serre spectral sequence of $\Omega L \to PL \to L$ is remarkable because, due to the fact that PL is contractible, this last spectral sequence always contains nontrivial differentials. As we shall see this trivial algebraic-topological observation together with the geometric interpretation of the differentials discussed in §1.2.1 leads to interesting applications.

1.3. *Some applications.* Here is an overview of some of the consequences discussed in the paper. It should be pointed out that we focus in this paper only on the applications which follow rather rapidly from the main result. We intend to discuss others that are less immediate in later papers.

We shall only mention in this subsection applications that take place in the case when L and L' are Hamiltonian isotopic and so we make here this assumption.

1.3.1. *Algebraic consequences.* Under the assumption at (1), a first consequence of the theorem is that, if $\mathcal{K} = \bigcup_{x,y}\{\Phi_{x,y}(\overline{\mathcal{M}}(x,y))\} \subset \Omega L$ and $\widehat{\mathcal{K}}$ is the closure of $\mathcal{K}$ with respect to concatenation of loops, then the inclusion $\widehat{\mathcal{K}} \overset{k}{\hookrightarrow} \Omega L$ is surjective in homology. An immediate consequence of this is as follows. Notice first that the space $\mathcal{M}'$ maps injectively onto a subspace $\tilde{\mathcal{M}}$ of $\mathcal{P}(L, L')$ via the map that associates to each pseudo-holomorphic strip $u : \mathbb{R} \times [0,1] \to M$ the path $u(0, -)$. Let $e : \tilde{\mathcal{M}} \to L$ be defined by $e(u) = u(0,0)$. We show that

$$H_*(\Omega e) : H_*(\Omega\tilde{\mathcal{M}}; \mathbb{Z}/2) \to H_*(\Omega L; \mathbb{Z}/2) \text{ is surjective .} \tag{2}$$

This complements a result obtained by Hofer [13] and independently by Floer [7] which claims that $H_*(e)$ is also surjective.

Another easy consequence is that for a generic class of choices of L', the image of the group homomorphism $\Pi = \omega| : \pi_2(M, L \cup L') \to \mathbb{R}$ verifies

$$\mathrm{rk}(\mathrm{Im}(\Pi)) \geq \sum_i \dim_{\mathbb{Z}/2} H_i(L; \mathbb{Z}/2) - 1 \ . \tag{3}$$

1.3.2. *Existence of pseudo-holomorphic "strips"*. A rather immediate consequence of the construction of $EF(L, L')$ is that through each point in $L\backslash L'$ passes at least one strip $u \in \mathcal{M}'$ of Maslov index at most n. By appropriately refining this argument we shall see that we may even bound the energy of these strips which "fill" L by the energy of a Hamiltonian diffeomorphism that carries L to L'. More precisely, denote by $||\phi||_H$ the Hofer norm (or energy; see [14] and equation (29)) of a Hamiltonian diffeomorphism ϕ. We put (as in [1] and [25]):

$$\nabla(L, L') = \inf_{\psi \in \mathcal{H}, \psi(L)=L'} ||\psi||_H$$

where $\mathcal{H}$ is the group of compactly supported Hamiltonian diffeomorphisms. We prove that *through each point of $L\backslash L'$ passes a pseudo-holomorphic strip which is of Maslov index at most n and whose symplectic area is at most* $\nabla(L, L')$. This fact has many interesting geometric consequences. We describe a few in the next paragraph.

1.3.3. *Nonsqueezing and Hofer's energy.* Consider on M the riemannian metric induced by some fixed generic almost complex structure which tames ω. The areas below are defined with respect to this metric. For two points $x, y \in L \cap L'$ let

$$\begin{aligned}(4)\quad \mathcal{S}(x, y) = \{u \in C^{\infty}([0,1] \times [0,1], M) \ : u([0,1], 0) \subset L, u([0,1], 1) \subset L', \\ u(0, [0,1]) = x, u(1, [0,1]) = y\} \ .\end{aligned}$$

Fix the notation:

$$a_{L,L'}(x, y) = \inf\{\text{area}(u) : u \in \mathcal{S}(x, y)\} \ .$$

Let $a_k(L, L') = \min\{a_{L,L'}(x, y) : x, y \in L \cap L' \ , \ \mu(x, y) = k\}$ and, similarly, let $A_k(L, L')$ be the maximum of all $a_{L,L'}(x, y)$ where $x, y \in L \cap L'$ verify $\mu(x, y) = k$.

We prove that:

$$a_n(L, L') \leq \nabla(L, L') \quad .$$

For $x \in L\backslash L'$ let $\delta(x) \in [0, \infty)$ be the maximal radius r of a standard symplectic ball $B(r)$ such that there is a symplectic embedding $e_{x,r} : B(r) \to M$ with $e_{x,r}(0) = x$, $e_{x,r}^{-1}(L) = B(r) \cap \mathbb{R}^n$ and $e_{x,r}(B(r)) \cap L' = \emptyset$. We thank François Lalonde who noticed that, as we shall see, δ_x does not depend on x. Therefore, we introduce the *ball separation energy* between L and L' by

$$\delta(L, L') = \delta_x \ .$$

We show a second inequality

$$\frac{\pi}{2}\ \delta(L, L')^2 \leq A_n(L, L') \tag{5}$$

and also:

$$\frac{\pi}{2}\ \delta(L,L')^2 \leq \nabla(L,L')\ . \tag{6}$$

The results summarized in §1.3.2 as well as the inequalitities (5) and (6) are first proved under the assumption at (1). However, we then show that our spectral sequence may also be constructed (with minor modifications) when L and L' are Hamiltonian isotopic under the single additional assumption $\omega|_{\pi_2(M,L)} = 0$ and as a consequence these three results also remain true in this setting.

The inequality (6) is quite powerful: it implies that $\nabla(-,-)$ (which is easily seen to be symmetric and to satisfy the triangle inequality) is also non-degenerate thus reproving - when $\omega|_{\pi_2(M,L)} = 0$ - a result of Chekanov [1]. The same inequality is of course reminiscent the known displacement-energy estimate in [18] and, indeed, this estimate easily follows from (6) (of course, under the assumption $\omega|_{\pi_2(M)} = 0$) by application of this inequality to the diagonal embedding $M \to M \times M$.

1.4. *The structure of the paper.* In Section 2 we start by recalling the basic notation and conventions used in the paper as well as the elements from Floer's theory that we shall need. We then pass to the main task of the section which is to present the construction of $EF(L,L';\eta)$. A key technical ingredient in this construction is the fact that the compactifications of the moduli spaces of Floer trajectories, $\overline{\mathcal{M}}(x,y)$, have a structure of manifolds with corners. This property is closely related to the gluing properties proven by Floer in his classical paper [8] and is quite similar to more recent results proven by Sikorav in [34]. In fact, this same property also appears to be a feature of the Kuranishi structures used by Fukaya and Ono in [12]. For the sake of completeness we include a complete proof of the existence of the manifold-with-corners structure in the appendix. We then verify the points a., b., c. of Theorem 1.1. In Section 2.4 we prove point d. of Theorem 1.1. This proof is based on one hand on the classical method of comparing the Floer complex to a Morse complex of a Morse function on L and, on the other hand, on a new Morse theoretic result which shows that if in the construction of $EF(L,L')$ the moduli spaces of pseudo-holomorphic curves are replaced with moduli spaces of negative gradient flow lines, then the resulting spectral sequence is the Serre spectral sequence of the statement. The whole construction of $EF(L,L')$ has been inspired by precisely this Morse theoretic result which, in its turn, is a natural but nontrivial extension of some ideas described in [3] and [5].

Finally, Section 3 contains the applications mentioned above as well as various other comments.

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2. The spectral sequence

It turns out that it is more natural to construct a richer invariant than the one appearing in Theorem 1.1. The spectral sequence of the theorem will be deduced as a particular case of this construction.

As before let L, L' be closed lagrangian submanifolds of the fixed symplectic manifold (M,ω). In this section we assume that their intersection is transversal and that $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0 = \pi_1(L) = \pi_1(L')$. As $\pi_2(M) \to \pi_2(M,L)$ is surjective (and similarly for L') we deduce $\omega|_{\pi_2(M,L)} = \omega|_{\pi_2(M,L')} = 0$.

2.1. *Recalls and notation.* We start by recalling some elements from Floer's construction. This machinery has now been described in detail in various sources (for example, [8], [26]) so that we shall only give here a very brief presentation.

We fix a path $\eta \in \mathcal{P}(L,L') = \{\gamma \in C^\infty([0,1],M) : \gamma(0) \in L,\, \gamma(1) \in L'\}$ and let $\mathcal{P}_\eta(L,L')$ be the path-component of $\mathcal{P}(L,L')$ containing η. We also fix an almost complex structure J on M that tames ω in the sense that the bilinear form $X, Y \to \omega(X, JY) = \alpha(X,Y)$ is a Riemannian metric. The set of all the almost complex structures on M that tame ω will be denoted by $\mathcal{J}_\omega$. Moreover, we also consider a smooth Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ and its associated family of Hamiltonian vector fields X_H determined by the equation

$$\omega(X_H^t, Y) = -dH_t(Y) \ , \ \forall Y$$

as well as the Hamiltonian isotopy ϕ_t^H given by

$$\frac{d}{dt}\phi_t^H = X_H^t \circ \phi_t^H \ , \phi_0^H = id \ . \tag{7}$$

The gradient of H, ∇H, is computed with respect to α and it verifies $J\nabla H = X_H$.

We shall also assume that $\phi_1^H(L)$ intersects L' transversely. Moreover, H and all the Hamiltonians considered in this paper are assumed to be constant outside of a compact set.

2.1.1. *The action functional and pseudo-holomorphic strips.* The idea behind the whole construction is to consider the action functional

$$\mathcal{A}_{L,L',H} : \mathcal{P}_\eta(L,L') \to \mathbb{R}\ ,\ x \to -\int \overline{x}^*\omega + \int_0^1 H(t,x(t))dt \tag{8}$$

where $\overline{x}(s,t) : [0,1]\times[0,1] \to M$ is such that $\overline{x}(0,t) = \eta(t)$, $\overline{x}(1,t) = x(t)$, $\forall t \in [0,1]$, $x([0,1],0) \subset L$, $x([0,1],1) \subset L'$. The fact that L and L' are simply connected Lagrangians and ω vanishes on $\pi_2(M)$ implies that $\mathcal{A}_{L,L',H}$ is well-defined. To shorten notation we neglect the subscripts L, L', H in case no confusion is possible. We shall also assume $\mathcal{A}(\eta) = 0$ (this is of course not restrictive).

Given a vector field ξ tangent to TM along $x \in \mathcal{P}(L,L')$ we derive $\mathcal{A}$ along ξ thus getting

$$\begin{aligned} d\mathcal{A}(\xi) &= -\int_0^1 \omega(\xi, \frac{dx}{dt})dt + \int_0^1 dH_t(\xi)(x(t))dt \\ &= \int_0^1 \alpha(\xi, J\frac{dx}{dt} + \nabla H(t,x))dt\ . \end{aligned} \tag{9}$$

This means that the critical points of $\mathcal{A}_{L,L'}$ are precisely the orbits of X_H which start on L, end on L' and which belong to $\mathcal{P}_\eta(L,L')$. Obviously, these orbits are in bijection with a subset of $\phi_1^H(L)\cap L'$ so that they are finite in number. A particular important case is when H is constant. Then these orbits coincide with the intersection points of L and L' which are in the class of η. We denote the set of these orbits by $I(L,L';\eta,H)$. In case H is constant we shall also use the more intuitive notation $L\cap_\eta L'$.

The putative associated equation for the negative L^2-gradient flow lines of $\mathcal{A}$ has been at the center of Floer's work and is:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t,u) = 0 \tag{10}$$

with

$$u(s,t) : \mathbb{R}\times[0,1] \to M\ , u(\mathbb{R},0) \subset L\ ,\ u(\mathbb{R},1) \subset L'\ .$$

When H is constant, the solutions of (10) are called *pseudo-holomorphic strips*. They coincide with the zeros of the operator $\overline{\partial}_J = \frac{1}{2}(d + J\circ d\circ \mathbf{i})$. It is well known that (10) does not define a flow in any convenient sense.

Let $\mathcal{S}(L,L') = \{u \in C^\infty(\mathbb{R}\times[0,1],M) : u(\mathbb{R},0)\subset L\ ,\ u(\mathbb{R},1)\subset L'\}$ and for $u \in \mathcal{S}(L,L')$ consider the energy

$$E_{L,L',H}(u) = \frac{1}{2}\int_{\mathbb{R}\times[0,1]} ||\frac{\partial u}{\partial s}||^2 + ||\frac{\partial u}{\partial t} - X_H^t(u)||^2\ ds\ dt\ . \tag{11}$$

The key point of the whole theory is that, for a generic choice of J, the solutions u of (10) which *are of finite energy*, $E_{L,L',H}(u) < \infty$, do behave very much like (negative) flow lines of a Morse-Smale function when viewed as elements in

$C^\infty(\mathbb{R},\mathcal{P}_\eta(L,L'))$ (in particular, $\mathcal{A}$ is decreasing along such solutions) . The type of genericity needed here sometimes requires that J be time-dependent. In other words $J=J_t$, $t\in[0,1]$ is a one-parameter family of almost complex structures each taming ω. In this case the equation (10) is understood as $\partial u/\partial s+J_t(u)\partial u/\partial t+\nabla H(t,u)=0$. We now put

$$\mathcal{M}'=\{u\in\mathcal{S}(L,L') : u \text{ verifies(10) },\ E_{L,L',H}(u)<\infty\}\ . \tag{12}$$

The translation $u(s,t)\to u(s+k,t)$ obviously induces an $\mathbb{R}$ action on $\mathcal{M}'$ and we let $\mathcal{M}$ be the quotient space. An important feature of $\mathcal{M}'$ is that for each $u\in\mathcal{M}'$ there exist $x,y\in I(L,L';\eta,H)$ such that the (uniform) limits verify

$$\lim_{s\to-\infty}u(s,t)=x(t)\ ,\ \lim_{s\to+\infty}u(s,t)=y(t)\ . \tag{13}$$

We let $\mathcal{M}'(x,y)=\{u\in\mathcal{M}' : u \text{ verifies (13)}\}$ and $\mathcal{M}(x,y)=\mathcal{M}'(x,y)/\mathbb{R}$. Therefore, $\mathcal{M}=\bigcup_{x,y}\mathcal{M}(x,y)$. In case we need to indicate explicitly to which pair of Lagrangians, to what Hamiltonian and to what almost complex structure are associated these moduli spaces we shall add L and L', H, J as subscripts (for example, we may write $\mathcal{M}_{L,L',H,J}(x,y)$).

2.1.2. *Dimension of $\mathcal{M}(x,y)$ and the Maslov index.* Let $\mathcal{L}(n)$ be the set of Lagrangian subspaces in $(\mathbb{R}^{2n},\omega_0)$. It is well-known that $H^1(\mathcal{L}(n);\mathbb{Z})\approx\mathbb{Z}$ has a generator given by a morphism called the Maslov index $\mu:\mathcal{L}(n)\to S^1$ (geometrically it is given as the class dual to the Maslov cycle constituted by the Lagrangian subspaces nontransversal to the vertical Lagrangian). For $x,y\in I(L,L';\eta,H)$ we let (as in (4))

$$\begin{aligned}\mathcal{S}(x,y)=\{u\in C^\infty([0,1]\times[0,1],M)\ &: u([0,1],0)\subset L, u([0,1],1)\subset L',\\ &u(0,t)=x(t),u(1,t)=y(t)\}\ .\end{aligned}$$

and suppose that $u\in\mathcal{S}(x,y)$. Following the work of Viterbo [36], the Maslov index of u, $\mu(u)$, is given as the degree of the map $S^1=\partial([0,1]\times[0,1])\xrightarrow{\gamma}\mathcal{L}(n)\xrightarrow{\mu}S^1$ with the loop γ defined as follows. First notice that u^*TM is a trivial symplectic bundle (and all trivializations are homotopic). We fix such a trivialization. This allows the identification of each space $T_xL\subset T_xM$ to an element of $\mathcal{L}(n)$ (and similarly for T_xL'). We then define the loop $\gamma:S^1\to\mathcal{L}(n)$ as follows. We let γ_0 be the path of Lagrangians $(\phi^H_t)^{-1}_*T_{x(1)}L'$ and we let γ_1 be the path $(\phi^H_t)^{-1}_*T_{y(1)}L'$. We then join $(\phi^H_1)^{-1}_*T_{x(1)}L'$ to $(\phi^H_1)^{-1}_*T_{y(1)}L'$ by a path of Lagrangian subspaces $\gamma'(t)\subset T_{u(t,0)}M$ such that for each t, $\gamma'(t)$ is transversal to $T_{u(t,0)}L$ and let $\gamma=\gamma_0*\gamma'*\gamma_1^{-1}*\gamma''$ where $\gamma''(t)$ is the path $t\to T_{u(1-t,1)}L'$. It is easy to see that such a path γ' does exist and that the degree of the composition $\mu\circ\gamma$ is independent of the choice of γ' as well as of that of the trivialization. Given that L and L' are simply connected and $c_1|_{\pi_2(M)}=0$ we see that for any $u,v\in\mathcal{S}(x,y)$ we have $\mu(u)=\mu(v)$. Therefore, for any $x, y \in I(L, L'; \eta, H)$ we may define

$$\mu(x, y) = \mu(u)\ ,\ u \in \mathcal{S}(x, y)\ .$$

This implies that, in this case, for any three points $x, y, z \in I(L, L'; \eta, H)$ we have

$$\mu(x, z) = \mu(x, y) + \mu(y, z)\ . \tag{14}$$

The fundamental role of the Maslov index in relation to the properties of the action functional is provided by the fact that the linearized operator $D_u^{H,J}$ associated to the operator $\overline{\partial}_J + (1/2)\nabla H$ at u is Fredholm of index $\mu(u)$. In case J is such that $D_u^{H,J}$ is surjective for all $u \in \mathcal{M}'(x, y)$ and all $x, y \in I(L, L'; \eta, H)$ (see 3.4.6), it follows that the spaces $\mathcal{M}'(x, y)$ are smooth manifolds (generally noncompact) of dimension $\mu(x, y)$. Under certain circumstances the theory works in the same way even if L and L' are nontransversal (for example if $L = L'$) but in that case the choice of H needs to be generic. In all cases, we shall call a pair (H, J) regular if the surjectivity condition mentioned above is satisfied. In our setting it is easy to see that for any $x \in I(L, L'; \eta, H)$, the space $\mathcal{M}'(x, x)$ is reduced to the constant solution equal to x. Because of that we will always assume here that in writing $\mathcal{M}(x, y)$ we have $x \neq y$. Thus, $\mathcal{M}(x, y)$ is also a smooth manifold whose dimension is $\mu(x, y) - 1$. The set of regular (H, J)'s is generic and we assume below that we are using such a pair.

2.1.3. *Naturality of Floer's equation.* Let $L'' = (\phi_1^H)^{-1}(L')$. Consider the map $b_H : \mathcal{P}(L, L'') \to \mathcal{P}(L, L')$ defined by $(b_H(x))(t) = \phi_t^H(x(t))$. Let $\eta' \in \mathcal{P}(L, L'')$ be such that $\eta = b_H(\eta')$. Clearly, b_H restricts to a map between $\mathcal{P}_{\eta'}(L, L'')$ and $\mathcal{P}_\eta(L, L')$ and, moreover, by our assumption on ϕ, the intersection of L and L'' is transverse and the same map restricts to a bijection $L \cap_{\eta'} L'' = I(L, L''; \eta', 0) \to I(L, L'; \eta, H)$.

We also have

$$\mathcal{A}_{L,L',H}(b_H(x)) = \mathcal{A}_{L,L'',0}(x)\ .$$

Indeed, let $\overline{x}(s, t) : [0, 1] \times [0, 1] \to M$ be such that $\overline{x}(0, t) = \eta'(t)$, $\overline{x}(1, t) = x(t)$, $\forall t \in [0, 1]$, $x([0, 1], 0) \subset L$, $x([0, 1], 1) \subset L''$ and let $\tilde{x}(s, t) = \phi_t^H(x(s, t))$. We then have (by using (7) and letting $\phi = \phi^H$):

$$\int_{[0,1]\times[0,1]} \tilde{x}^*\omega = \int_{[0,1]\times[0,1]} \overline{x}^*(\phi^*\omega) + \int_0^1 \int_0^1 \omega(\frac{\partial \tilde{x}}{\partial s}, \frac{\partial \phi}{\partial t}) ds dt$$
$$= \int_{[0,1]\times[0,1]} \overline{x}^*\omega + \int_0^1 (\int_0^1 dH(\frac{\partial \tilde{x}}{\partial s}) ds) dt = -\mathcal{A}_{L,L'',0}(x) + \int_0^1 H(b_H(x)(t)) dt\ .$$

Moreover, the map b_H does identify the geometry of the two action functionals. This is due to the fact that for $u : \mathbb{R} \times [0, 1] \to M$ with $u(\mathbb{R}, 0) \subset L$, $u(\mathbb{R}, 1) \subset L''$, $\tilde{u}(s, t) = \phi_t(u(s, t))$, $\tilde{J} = \phi^* J$ we have

$$\phi_*(\frac{\partial u}{\partial s} + \tilde{J}\frac{\partial u}{\partial t}) = \frac{\partial \tilde{u}}{\partial s} + J(\frac{\partial \tilde{u}}{\partial t} - X_H)\ .$$

Therefore, the map b_H induces diffeomorphisms (that we shall denote by the same symbol):

$$b_H : \mathcal{M}_{L,L'',\tilde{J},0}(x,y) \to \mathcal{M}_{L,L',J,H}(x,y)$$

where we have identified $x, y \in L \cap_{\eta'} L''$ with their orbits $\phi_t^H(x)$ and $\phi_t^H(y)$.

2.1.4. *Gromov compactification of $\mathcal{M}(x,y)$.* The noncompactness of $\mathcal{M}(x,y)$ for $x, y \in I(L, L'; \eta, H)$ is only due to the fact that, as in the Morse-Smale case, a sequence of strips $u_n \in \mathcal{M}(x,y)$ might "converge" to a broken strip. For example, it might converge to an element of $\mathcal{M}(x,z) \times \mathcal{M}(z,y)$ for some other $z \in I(L, L'; \eta, H)$. The type of convergence used here has been studied extensively and it is called Gromov convergence. Moreover, there are natural compactifications of the moduli spaces $\mathcal{M}(x,y)$ called Gromov compactifications and denoted by $\overline{\mathcal{M}}(x,y)$ so that each of the spaces $\overline{\mathcal{M}}(x,y)$ is a manifold with boundary and there is a homeomorphism:

$$\partial\overline{\mathcal{M}}(x,y) = \bigcup_{z \in I(L,L';\eta,H)} \overline{\mathcal{M}}(x,z) \times \overline{\mathcal{M}}(z,y) \ . \tag{15}$$

It is shown in the Appendix A, that the manifolds $\overline{\mathcal{M}}(x,y)$ are manifolds with corners. We insist there mainly on the homogenous case, when $H = 0$. However, as the maps b_H constructed in Section 2.1.3 are compatible with equation (15) this result is also true for a general H.

2.2. *Construction of the spectral sequence.*

2.2.1. *Deformed pseudo-holomorphic strips viewed as paths.* To each element $u \in \mathcal{M}'(x,y)$ we associate a continuous path

$$\gamma_u : [0, \mathcal{A}(x) - \mathcal{A}(y)] \to \mathcal{P}_\eta(L, L') \tag{16}$$

in a rather obvious way: $\gamma_u(\mathcal{A}(x) - \mathcal{A}(y)) = y$, $\gamma_u(0) = x$ and for $\tau \in (0, \mathcal{A}(x) - \mathcal{A}(y))$, $\gamma_u(\tau) = u(h_u(-\tau), [0,1])$ where

$$h_u : (\mathcal{A}(y) - \mathcal{A}(x), 0) \to \mathbb{R}$$

is defined by $\mathcal{A}(u(h_u(\tau), [0,1])) = \tau + \mathcal{A}(x)$. In short, γ_u associates to τ the unique element of $\mathcal{P}(L, L')$ which is of the form $u(\xi, -) : [0,1] \to M$ for some $\xi \in \mathbb{R}$ and on which $\mathcal{A}$ has the value $\mathcal{A}(x) - \tau$. The function h_u is well defined because $\mathcal{A}$ is strictly decreasing along u and it is easy to see that γ_u is continuous (we shall use here the compact-open C^0-topology on $\mathcal{P}(L, L')$). Obviously, γ_u only depends on the class of u in $\mathcal{M}(x,y)$ and thus we have a map:

$$\gamma_{x,y} : \mathcal{M}(x,y) \to C^0([0, \mathcal{A}(x) - \mathcal{A}(y)], \mathcal{P}_\eta(L, L')) \ , \ \gamma_{x,y}(u) = \gamma_u \ .$$

To simplify notation let

$$C_{x,y}\mathcal{P} = C^0([0, \mathcal{A}(x) - \mathcal{A}(y)], \mathcal{P}_\eta(L, L'))$$

which is taken to be void in case $\mathcal{A}(x) \leq \mathcal{A}(y)$. The map $\gamma_{x,y}$ is easily seen to be continuous in view of the description of the charts of $\mathcal{M}(x,y)$. Moreover, in view of the definition of Gromov compactness (or by using the description of the small neighbourhoods of broken Floer orbits given in the Appendix A) we see that this map extends to a continuous map

$$\overline{\gamma}_{x,y} : \overline{\mathcal{M}}(x,y) \to C_{x,y}\mathcal{P}.$$

Notice that there exists an obvious continuous composition map given by concatenation of paths

$$\# : C_{x,y}\mathcal{P} \times C_{y,z}\mathcal{P} \to C_{x,z}\mathcal{P} \tag{17}$$

which is associative in the obvious sense. As an immediate consequence of the proof of (15) we also see that for each element $u = (u_1, u_2, \ldots u_k) \in \mathcal{M}(x, z_1) \times \mathcal{M}(z_1, z_2) \times \ldots \times \mathcal{M}(z_{k-1}, y) \subset \partial\overline{\mathcal{M}}(x,y)$ we have:

$$\overline{\gamma}_{x,y}(u) = \gamma_{x,z_1}(u_1)\#\gamma_{z_1,z_2}(u_2)\#\ldots\#\gamma_{z_{k-1},y}(u_k). \tag{18}$$

2.2.2. *Some additional path spaces.* We fix here some more notation. Let w be a path (homeomorphic to [0,1]) embedded in L that joins all points $\{x(0) : x \in I(L, L'; \eta, H)\}$ and let $\tilde{M}, \tilde{L}$ be respectively the quotient topological spaces obtained by contracting w to a point. Obviously, the quotient maps $M \to \tilde{M}$, $L \to \tilde{L}$ are homotopy equivalences. We also have homotopy equivalences $\mathcal{P}(L, L') \to \mathcal{P}(\tilde{L}, L')$, $\mathcal{P}_\eta(L, L') \to \mathcal{P}_\eta(\tilde{L}, L')$. We denote any of these quotient maps by q. We also need the obvious map $l : \mathcal{P}_\eta(\tilde{L}, L') \to \tilde{L}$, $\; l(\gamma) = \gamma(0)$. Notice that the spaces $\tilde{L}, \tilde{M}, \mathcal{P}(\tilde{L}, L')$ have a distinguished base point, $*$, given by the class of the path w and $(l \circ q)(I(L, L'; \eta, H)) = *$.

For any pointed topological space X we recall that ΩX is the space of continuous loops in X that are based at the distinguished point of X and are parametrized by the interval $[0, 1]$. This space is homotopy equivalent to the space of Moore loops on X, $\Omega' X$, which consists of the continuous loops in X that are parametrized by arbitrary intervals $[0, a]$, $a \in [0, \infty)$ (and, again, are based at the distinguished point of X).

The compositions $l \circ q$ induce maps

$$\mathcal{Q}_{x,y} : C_{x,y}\mathcal{P} \to \Omega'\tilde{L}, \; (\mathcal{Q}_{x,y}(a))(\tau) = (l \circ q)(a(\tau)) \; .$$

Concatenation of loops gives Moore loops the structure of a topological monoid. This operation, denoted by $\cdot$, commutes in an obvious way with the maps $\mathcal{Q}_{-,-}$ and the operation $\#$ of (17).

Fix also the notation

$$\Phi_{x,y} = \mathcal{Q}_{x,y} \circ \overline{\gamma}_{x,y} \; . \tag{19}$$

For further use, notice that the space $\mathcal{P}(L, L')$ (and therefore also $\mathcal{P}(\tilde{L}, L')$) is homotopy equivalent to the homotopy pull-back of the two inclusions $L \hookrightarrow M$ and $L' \hookrightarrow M$.

2.2.3. *An algebraic construction.* For a topological space X let $S_*(X)$ be the $\mathbb{Z}/2$-cubical (normalized) chain complex of X . We use cubical chains - that is chains whose domains are unit cubes (see [23] for definitions) - instead of singular chains because in this case, for two spaces X, Y, we have an obvious map $S_k(X) \times S_q(Y) \to S_{k+q}(X \times Y)$ defined by $(\sigma \times \sigma')(x, y) = (\sigma(x), \sigma'(y))$. Moreover, the multiplication $\cdot$ directly induces a natural multiplication denoted again by $\cdot : S_k(\Omega' X) \otimes S_l(\Omega' X) \to S_{k+l}(\Omega' X)$ defined by $(\sigma \cdot \sigma')(x, y) = \sigma(x) \cdot \sigma(y)$ where $x \in [0,1]^k$, $y \in [0,1]^l$.

In particular, this turns $S_*(\Omega' \tilde{L})$ into a differential ring that we shall denote from now on by $\mathcal{R}_*$.

Definition 2.1. A representing chain system for the moduli spaces associated to L, L', J, H, η is a family $\{s_{xy} \in S_{\mu(x,y)-1}(\overline{\mathcal{M}}(x, y)) : x, y \in I(L, L'; \eta, H)\}$ such that:

i. The image of s_{xy} in $S_*(\overline{\mathcal{M}}(x, y), \partial\overline{\mathcal{M}}(x, y))$ is a cycle representing the fundamental class.

ii. With the identifications given by equation (15) we have $\partial s_{xy} = \sum_z s_{xz} \times s_{zy} \in S_*(\overline{\mathcal{M}}(x, y))$.

LEMMA 2.2. *With the assumptions and notation above, there exists a representing chain system for the moduli spaces* $\mathcal{M}_{L,L',J,H,\eta}(-, -)$.

Proof. We construct the s_{xy}'s by induction. Assume the construction accomplished for $\mu(x, y)-1 < k$. Consider now a pair x, y with $\mu(x, y)-1 = k$. We may assume that $\mathcal{M}(x, y)$ is connected (if not we apply the argument below one component at a time). Using the identifications in (15) consider the chain $c_{xy} = \sum_z s_{xz} \times s_{zy} \in S_{k-1}(\partial\overline{\mathcal{M}}(x, y))$. We denote the differential in $S_*(-)$ by ∂ and we compute

$$\begin{aligned}\partial c_{xy} &= \sum_z \partial s_{xz} \times s_{yz} + \sum s_{xz} \times \partial s_{zy} \\ &= \sum_{z,k} (s_{xt} \times s_{tz}) \times s_{yz} + \sum_{z,j} s_{yz} \times (s_{zj} \times s_{jy}) = 2(\sum_{s,r} s_{xs} \times s_{sr} \times s_{rz}) = 0.\end{aligned}$$

The homology class represented by c_{xy} is the fundamental class of $\partial\overline{\mathcal{M}}(x, y)$. This is because the image of this class in any one of

$$H_{k-1}(\overline{\mathcal{M}}(x, z) \times \overline{\mathcal{M}}(z, y), \partial(\overline{\mathcal{M}}(x, z) \times \overline{\mathcal{M}}(z, y)))$$

coincides with the class represented by $s_{xz} \times s_{zy}$ which is the fundamental class. Therefore, $c_{xy} \in \mathrm{Im}(\partial : S_k(\overline{\mathcal{M}}(x, y)) \to S_{k-1}(\overline{\mathcal{M}}(x, y)))$. Let s_{xy} be such that $\partial s_{xy} = c_{xy}$. By construction, property ii. of a representing system is then satisfied. The first property is also satisfied because the image of s_{xy} is a cycle in $S_k(\overline{\mathcal{M}}(x, y), \partial\overline{\mathcal{M}}(x, y))$ and the homology connectant δ of the pair $(\overline{\mathcal{M}}(x,y), \partial\overline{\mathcal{M}}(x,y))$ is an isomorphism in dimension k and it verifies $\delta([s_{xy}]) = [c_{xy}]$. □

Remark 2.3. Representing chain systems appear naturally when the compactified moduli spaces $\overline{\mathcal{M}}(x,y)$ are triangulated (or rather "cubulated") in a way compatible with formula (15): the s_{xy}'s may then be taken to be the sum of the top dimensional cubes. However, the existence of such a triangulation is not obvious. The most direct approach to constructing such a triangulation is to proceed by induction. Assuming that a triangulation of $\partial\overline{\mathcal{M}}(x,y)$ is constructed the induction step is then to extend this triangulation to the whole of $\overline{\mathcal{M}}(x,y)$. For this extension to exist one needs to check that the Kirby-Siebenmann obstruction vanishes - fact which is not *a priori* clear.

We now fix a representing chain system $\zeta = \{s_{xy}\}$ and we define

$$a_{xy} \in \mathcal{R}_{\mu(x,y)-1}\ ,\ a_{xy} = \Phi_{x,y}(s_{xy})\ . \tag{20}$$

Let m be the number of elements of the set $I(L,L';\eta,H)$. Fix one point $z_0 \in I(L,L';\eta,H)$ and for each $x \in I(L,L';\eta,H)$ let $\mu(x) = \mu(x,z_0)$. In view of (14) the function $\mu(-)$ so defined only depends of z_0 up to a translation by a constant. Let a strict ordering $\succ$ of the set $I(L,L';\eta,H)$ be such that we have $\mu(x) > \mu(y) \Rightarrow x \succ y$.

The main algebraic object that we shall be using is the matrix

$$A = (a_{xy})_{\{x,y\in I(L,L';\eta,H)\}} \in M_{m,m}(\mathcal{R}_*)\ . \tag{21}$$

Remark 2.4. Of course, despite our short notation for A, this matrix depends on L, L', H, η, the choice of J and of ζ.

If C is a matrix with coefficients in $\mathcal{R}_*$, then we let ∂C be the matrix whose coefficients are obtained by applying the differential ∂ of $\mathcal{R}_*$ to the coefficients of C.

The key property of A is as follows.

Proposition 2.5. *Under the assumptions above we have*:

$$A^2 = \partial A\ .$$

Proof. This is immediate from the construction of A and from (15) and (18). Indeed, we have the following sequence of equalities

$$\begin{aligned}
\partial a_{xy} &= \partial\Phi_{x,y}(s_{xy}) = \Phi_{x,y}(\partial s_{xy}) = \Phi_{x,y}(\textstyle\sum_z s_{xz}\times s_{zy})\\
&= \mathcal{Q}_{x,y}\circ\overline{\gamma}_{x,y}(\textstyle\sum_z s_{xz}\times s_{zy}) = \mathcal{Q}_{x,y}\circ(\textstyle\sum_z \overline{\gamma}_{x,z}(s_{xz})\#\overline{\gamma}_{z,y}(s_{zy}))\\
&= \textstyle\sum_z(\mathcal{Q}_{x,z}\circ\overline{\gamma}_{x,z})(s_{xz})\cdot(\mathcal{Q}_{z,y}\circ\overline{\gamma}_{z,y})(s_{zy})\\
&= \textstyle\sum_z \Phi_{x,z}(s_{xz})\cdot\Phi_{z,y}(s_{zy})\\
&= \textstyle\sum_z a_{xz}\cdot a_{zy}
\end{aligned}$$

which is valid for any $x, y \in I(L,L';\eta,H)$. □

2.2.4. *The spectral sequence.* We first use the matrix A to define an $\mathcal{R}_*$- chain complex

$$\mathcal{C}^{\eta,J,\zeta}(L,L';H)=(\mathcal{C}_*,d)$$

which should be thought of as an *extended Morse-type chain complex* as discussed in the introduction.

We consider the graded $\mathbb{Z}/2$-vector space $\mathbb{Z}/2\langle I(L,L';\eta,H)\rangle$ where the grading is given by $|x|=\mu(x)$, $\forall x\in I(L,L';\eta,H)$ (recall that the "absolute" Maslov index function $\mu : I(L,L';\eta,H)\to\mathbb{Z}$ from §2.2.3 depends on our choice of a fixed point $z_0\in I(L,L';\eta,H)$ only up to translation by an integral constant).

Now let $\mathcal{C}_*$ be equal to the left $\mathcal{R}_*$-module $\mathcal{R}_*\otimes\mathbb{Z}/2\langle I(L,L';\eta,H)\rangle$. The module operation is so that for $c\in\mathcal{R}_*$ and $a\otimes b\in\mathcal{C}$ we have $c\cdot(a\otimes b)=(c\cdot a)\otimes b$. The differential $d:\mathcal{C}_*\to\mathcal{C}_{*-1}$ is the unique $\mathcal{R}_*$-module derivation (in the sense that $d(a\otimes b)=\partial a\otimes b+a\cdot db$) such that

$$d(x)=\sum_y a_{xy}\otimes y\ ,\ \forall x\in I(L,L';\eta,H)\ .$$

Corollary 2.6. *For $\mathcal{C}^{\eta,J,\zeta}(L,L';H)=(\mathcal{C}_*,d)$ defined as above, $d^2=0$.*

Proof. For any $x\in I(L,L';\eta,H)$ we have:

$$\begin{aligned}d(d(x))=d(\sum_y a_{xy}\otimes y)&=\sum_y\partial a_{xy}\otimes y+\sum_{z,y}a_{xy}\cdot a_{yz}\otimes z\\&=\sum_t(\partial a_{xt}+\sum_s a_{xs}\cdot a_{st})\otimes t\end{aligned}$$

and all these last terms vanish in view of the equality in Proposition 2.5 (and because we work over $\mathbb{Z}/2$). □

Consider the spectral sequence which is associated to the filtration of the complex $\mathcal{C}^{\eta,J,\zeta}(L,L';H)$ defined by:

$$F^k\mathcal{C}=\mathcal{R}_*\otimes\mathbb{Z}/2<x\in I(L,L';\eta,H):\mu(x)\leq k>\ .$$

Clearly, this is a differential filtration and thus it does indeed induce a spectral sequence which we shall denote by

$$EF(L,L';\eta,H,J,\zeta)=(EF^r_{pq}(L,L';\eta,J,H,\zeta),d^r_F).$$

We fix the notation such that an element of bi-degree (p,q) in the spectral sequence is a class coming from an element in $\mathcal{R}_q\otimes\mathbb{Z}/2\langle x:\mu(x)=p\rangle$ (this last vector space being isomorphic to $EF^0_{pq}(L,L';H)$). We shall sometimes omit η, J, ζ in the notation for the spectral sequence.

We denote by $CF_*(L,L';H)$ the Floer chain complex associated to $\mathcal{A}_{L,L',H}$ and by $HF_*(L,L';H)$ the respective Floer homology. The relation of these to our spectral sequence is as follows.

Proposition 2.7. *For the spectral sequence defined above,*

a. $EF^1(L,L';H) \simeq CF_*(L,L';H) \otimes H_*(\Omega L)$.

b. $EF^2(L,L';H) \simeq HF_*(L,L';H) \otimes H_*(\Omega L)$.

c. *If* $d_F^r \neq 0$, *then there exist* $x, y \in I(L,L';\eta,H)$, $\mu(x,y) \leq r$, *such that* $\mathcal{M}(x,y) \neq 0$.

d. *For* $r \geq 1$, $(EF^r_{pq}(L,L';H), d_F^r)$ *is a spectral sequence of* $H_*(\Omega L)$*-modules.*

Proof. The only part of d that counts for the first point is the internal differential in $S_*(\Omega'\tilde{L})$. This expresses the E^1 term as desired. The differential d^1 is horizontal and is generated by the part of d that connects orbits of relative Maslov index equal to 1. This is precisely the Floer (classical) differential and thus implies the second point. The third point is obvious as $d_F^r \neq 0$ implies that there are some $x, y \in I(L,L';\eta,H)$ such that $a_{xy} \neq 0$ and $\mu(x,y) \leq r$. The fourth point is a direct consequence of the fact that the differential d of $\mathcal{C}_*$ verifies $d(a \otimes b) = \partial a \otimes b + a \cdot db$ □

Remark 2.8. Notice that a different choice for z_0 only modifies the resulting spectral sequence by a translation.

The spectral sequence of Theorem 1.1 consists of the terms of order greater than or equal to 1 of $EF(L,L') = EF(L,L';0)$. In particular, Proposition 2.7 implies the points b. and c. of this theorem. We still need to prove the rest of the theorem.

Remark 2.9. It is possible to modify the construction above in such a way as to replace the ring $\mathcal{R}_*$ with the richer ring $S_*(\Omega'\mathcal{P}_\eta(L,L'))$. However, as $\mathcal{R}_*$ is sufficient for the applications discussed in this paper we shall not pursue this extension here.

2.3. *Proof of the main theorem.* I: *Invariance of the spectral sequence.* Our next aim is to prove the point a. of Theorem 1.1. As we shall see this point will follow rapidly from the main result of this subsection which is shown in §2.3.1 below.

2.3.1. *Variation of the Hamiltonian.* Assume that with L, L', η, H, J, ζ as above we additionally have a Hamiltonian $H' : [0,1] \times M \to \mathbb{R}$ which is also constant outside of a compact set. We consider an almost complex structure J' so that the pair (H', J') is regular and so $EF(L,L';\eta,J',H',\zeta')$ is defined with ζ' a representing system of chains for the moduli spaces associated to $L, L', J'H', \eta$. Let

$$\varepsilon(L,L';H,J) = \min\{E_{L,L',H}(u) : u \in \mathcal{M}'_{L,L',J,H}\} \tag{22}$$

(where $E_{L,L',H}$ is the energy as defined in (11)).

Theorem 2.10. *Under the assumptions above*:

a. *There exists a chain morphism*

$$\mathcal{V} : \mathcal{C}^{\eta,J,\zeta}(L,L';H) \to \mathcal{C}^{\eta,J',\zeta'}(L,L';H')$$

of possibly nonzero degree which induces an isomorphism up to translation between $EF^r(L,L';H)$ *and* $EF^r(L,L';H')$ *for* $r \geq 2$.

b. *If* $||H'-H||_0 < \varepsilon(L,L';H,J)/4$, *then there exists a morphism* $\mathcal{V}$ *as before which admits a retract.*

Remark 2.11. A morphism of chain complexes $f : C_* \to D_{*+k}$ is said to admit a retract if there exists another morphism $g : D_* \to C_{*-k}$ such that $g \circ f = id_C$. Clearly, if $\mathcal{V}$ admits a retract, then the same is true for the morphism induced by $\mathcal{V}$ on each page of the spectral sequence. Therefore, point b. of Theorem 2.10 shows, in particular, that $EF^r(L,L';\eta,J,H,\zeta)$ does not depend on J (or ζ) already for $r \geq 1$.

The idea for the proof of Theorem 2.10 is classical in Floer's theory : we adapt the previous construction to the case of the moduli spaces of solutions of an equation similar to (10) but such as to allow for deformations from the Hamiltonian H to the Hamiltonian H'.

Proof. To shorten notation let $I = I(L,L';\eta,H)$, $I' = (L,L';\eta,H')$. We start with some recalls on Floer's comparison method. Take a smooth homotopy $H^{01} : \mathbb{R}\times[0,1]\times M \to \mathbb{R}$ and a homotopy $J^{01} : \mathbb{R}\times M \to \mathrm{End}(TM)$, $J^{01}_s \in \mathcal{J}, \forall s \in \mathbb{R}$ (here $\mathcal{J}$ is the set of almost complex structures on M) such that there exists $R > 0$ with the property that, for $s \geq R$, we have $(H^{01}_s(x), J^{01}_s(x)) = (H(x), J(x))$ and for $s \leq -R$, $(H^{01}_s, J^{01}_s) = (H'(x), J'(x))$, $\forall x \in M$. Moreover, we assume that there exists a compact set such that for all $s \in \mathbb{R}$, H^{01}_s is constant outside this compact set. Consider the equation:

$$\frac{\partial u}{\partial s} + J^{01}(s,u)\frac{\partial u}{\partial t} + \nabla^s_x H^{01}(s,t,u) = 0 \tag{23}$$

where $\nabla^s_x H^{01}(s,t,-)$ is the gradient of the function $H^{01}(s,t,-)$ with respect to the riemannian metric induced by J^{01}_s and $u : \mathbb{R}\times[0,1] \to M$ with $u(\mathbb{R},0) \subset L$ and $u(\mathbb{R},1) \subset L'$. We may define the energy $E_{L,L',H^{01}}$ by replacing H in formula (11) by H^{01}.

The finite energy solutions of (23) have properties that are very similar to those of (10). In particular, for each such solution u there exist $x \in I$, $y \in I'$ such that

$$\lim_{s\to-\infty} u(s,-) = x \ , \ \lim_{s\to\infty} u(s,-) = y \ . \tag{24}$$

If the linearized operator asociated to (23), $D^{H^{01},J^{01}}_u$, is surjective for each finite energy solution u we say that the pair (H^{01}, J^{01}) is regular. There is again a generic set of choices of regular such pairs. Again, to insure genericity of regularity one might need to assume that J^{01} is also time dependent. We shall assume from now on that (H^{01}, J^{01}) is regular. We denote by $\mathcal{M}_{H^{01},J^{01}}(x,y)$ the finite energy solutions of (23) that satisfy (24). These spaces are smooth manifolds of dimension $\mu(x,y)$ (the relative Maslov index in this case being defined by a straightforward adaptation of the definition in §2.1.2). Gromov compactifications also exist in this context and we shall denote them by $\overline{\mathcal{M}}_{H^{01},J^{01}}(x,y)$. They are manifolds with boundary and they verify:

(25)
$$\partial\overline{\mathcal{M}}_{H^{01}}(x,y) = \bigcup_{z\in I} \overline{\mathcal{M}}_H(x,z) \times \overline{\mathcal{M}}_{H^{01}}(z,y) \cup \bigcup_{z'\in I'} \overline{\mathcal{M}}_{H^{01}}(x,z') \times \overline{\mathcal{M}}_{H'}(z',y)$$

Moreover, in the same way as the one described in the Appendix A it is possible to show that these manifolds are manifolds with corners.

Another useful remark concerns the functional $\mathcal{A}_{H^{01}}(s,x) : \mathbb{R} \times \mathcal{P}_\eta(L,L') \to \mathbb{R}$ which is defined by the action functional formula (8) but by using H^{01} instead of H. This is clearly a homotopy between $\mathcal{A}_H$ and $\mathcal{A}_{H'}$. Assume now that H^{01} is a monotone homotopy in the sense that $\frac{\partial H^{01}}{\partial s}(s,t,y) \leq 0,\ \forall\, s,t,y \in \mathbb{R}\times[0,1]\times M$. In this case, if we put $a_{H^{01}}(s) = \mathcal{A}_{H^{01}}(s,u(s,-))$ for u a solution of (23), then

$$\frac{da_{H^{01}}}{ds} = d\mathcal{A}^s_{H^{01}}(\frac{\partial u}{\partial s}) + \int_0^1 \frac{\partial H^{01}}{\partial s}(s,t,u(s,t))dt \tag{26}$$

and, by (9), the first term of the sum is negative and the second is negative or null due to monotonicity. In other words, monotone homotopies which have been introduced in the symplectic setting by Floer and Hofer in [11], insure that the relevant action functionals decrease along solutions of (23). Since both H and H' are constant outside of a compact set we see that after possibly adding some positive constant to H we may assume that $H(t,x) > H'(t,x)$ for all $t,x \in [0,1]\times M$. As adding a constant to H does not modify its Hamiltonian flow and only changes $\mathcal{A}_H$ by the addition of the same constant, we may assume that monotone homotopies as above always exist and we fix one such homotopy H^{01} for the rest of this proof. To each element $u \in \mathcal{M}_{H^{01}}(x,y)$ we associate a path $\gamma_u : [0, \mathcal{A}_H(x) - \mathcal{A}_{H'}(y)] \to \mathcal{P}_\eta(L,L')$ defined by the same formula as that used for (16) but with $\mathcal{A}_{H^{01}}$ instead of $\mathcal{A}_H$. We continue the construction in perfect analogy to that described in Section 2.2.1 and we thus get continuous maps

$$\overline{\gamma}_{x,y} : \overline{\mathcal{M}}_{H^{01}}(x,y) \to C_{x,y}\mathcal{P}$$

which are coherent with the maps constructed in §2.2.1 in the sense that an obvious analogue of (18) is verified as implied by (25). To pursue the construction along the lines in Section 2.2.2 we first need to impose an additional restriction on the path w used to construct $\tilde{L}$: we shall assume that

$\{y(0) : y \in I(L, L'; \eta, H')\} \subset w$. With this nonrestrictive assumption and for any $x, y \in I(L, L'; H) \cup I(L, L'; H')$, we define $\Phi_{x,y} = \mathcal{Q}_{x,y} \circ \overline{\gamma}_{x,y}$ as in (19). We pursue the construction with the step described in Section 2.2.3. This construction involves the choice of $z_0 \in I(L, L'; H)$. We shall also need a similar choice: $z'_0 \in I(L, L'; H')$. To insure the compatibility of these choices we take z_0 and z'_0 so that $\mu(z_0, z'_0) = 0$ (it is easy to see that such a couple necessarily exists). With these choices, the construction described in Section 2.2.3 applied to H and to H' produces, respectively, matrices $A = (a_{xy})$ and $A' = (a'_{xy})$ and chain complexes $\mathcal{C}(L, L'; H)$, $\mathcal{C}(L, L'; H')$. There is an obvious analogue $\{\tilde{s}_{xy}\}$ of the representing system of chains for the moduli spaces $\mathcal{M}_{H^{01}}(x, y)$ so that this system is compatible with both $\zeta = \{s_{xy}\}$ and with $\zeta' = \{s'_{x'y'}\}$. The condition ii. in Definition 2.1 is replaced by $\partial \tilde{s}_{xy'} = \sum_z s_{xz} \times \tilde{s}_{zy'} + \sum_{z'} \tilde{s}_{xz'} \times s_{z'y'}$ which reflects equation (25). The existence of such representing chain systems for H^{01} compatible with ζ and ζ' then follows as in Lemma 2.2. Pursuing the construction we obtain a matrix $B = (b_{xy}) \in M_{m,m'}(\mathcal{R}_*)$ where, as in Section 2.2.3, m is the number of elements of $I(L, L'; H)$ and m' is the number of elements in $I(L, L'; H')$ and $b_{xy} = \Phi_{x,y}(\tilde{s}_{xy})$. As in Proposition 2.5 we see that

$$\partial B = A \cdot B + B \cdot A' \ . \tag{27}$$

For $x \in I(L, L'; H)$ we now define

$$\mathcal{V}(x) = \sum_{y \in I(L, L'; H')} b_{xy} \otimes y$$

and extend this to an $\mathcal{R}_*$-morphism. We then have

$$\begin{aligned}
\mathcal{V}(dx) &= \mathcal{V}(\sum_{y'} a_{xy'} \otimes y') \\
&= \sum_{y'} a_{xy'} \otimes \mathcal{V}(y') = \sum_{y', z} a_{xy'} \cdot b_{y'z} \otimes z \\
&= \sum_z (\sum_{y'} a_{xy'} \cdot b_{y'z}) \otimes z = \sum_z (\partial b_{xz} + \sum_v b_{xv} \cdot a'_{vz}) \otimes z \\
&= \sum_z \partial b_{xz} \otimes z + \sum_v (\sum_z b_{xv} \cdot a'_{vz} \otimes z) = \sum_z \partial b_{xz} \otimes z + \sum_v b_{xv} \cdot dv \\
&= d(\sum_v b_{xv} \otimes v) = d\mathcal{V}(x) \ .
\end{aligned}$$

Therefore, the map $\mathcal{V}$ so defined is a morphism of chain complexes which we shall sometimes also denote by $\mathcal{V}_{H^{01}}$ to emphasize the monotone homotopy to which it is associated. If the choices of z_0 and z'_0 are compatible, as above, then this morphism is of degree 0. If z_0 and z'_0 are independent, then this morphism could have a nonzero degree. Assuming for now the compatible choices from above it is obvious that this morphism preserves filtrations and so it induces a morphism of spectral sequences. Moreover, by the definition of the Floer comparison morphism $V_{H^{01}} : CF_*(L, L'; J, H) \to CF_*(L, L'; J', H')$ (induced by the same monotone homotopy) we see that the morphism induced by $\mathcal{V}_{H^{01}}$ at the E^1 term of our spectral sequences is the $H_*(\Omega L)$-module morphism induced by V. But $H_*(V_{H^{01}})$ is an isomorphism so $E^2(\mathcal{V})$ is also an isomorphism and so $E^r(\mathcal{V})$ is an isomorphism for all $r \geq 2$. Obviously, in case the choices for z_0 and z'_0 are not compatible, then this is still an isomorphism up to translation and this proves point a. of the theorem.

For the point b. notice that, for $x, y \in I(L, L'; H)$ and $u \in \mathcal{M}_H(x, y)$ we have

$$\mathcal{A}_H(x) - \mathcal{A}_H(y) = E_{L,L',H}(u) \; . \tag{28}$$

Therefore, $\varepsilon(L, L'; H, J) = \min\{\mathcal{A}_H(x) - \mathcal{A}_H(y) : \mathcal{M}_{H,J}(x, y) \neq \emptyset\}$. It has been proven by the second author together with Andrew Ranicki in §2.1 of [6] that under the assumptions of the theorem and for the case of periodic orbits, the Floer comparison morphism admits a retract. More precisely, there exist monotone homotopies H^{01} and G^{01} so that $V_{G^{01}} \circ V_{H^{01}}$ is an isomorphism whose matrix is upper triangular with 1's on the diagonal. The exact same argument applies also here: the only difference with respect to the proof of Theorem 2.1 in [6] is that we deal with orbits starting in L and ending in L' instead of periodic orbits; everything else remains the same. The fact that the matrix for $V_{G^{01}} \circ V_{H^{01}}$ is as above implies that the matrix for $\mathcal{V}_{G^{01}} \circ \mathcal{V}_{H^{01}}$ is also upper triangular with 1's on the diagonal. Therefore, $\mathcal{V}_{G^{01}} \circ \mathcal{V}_{H^{01}}$ is an isomorphism and this proves the claim. □

2.3.2. *Proof of Theorem* 1.1 a. Point a. of Theorem 1.1 is a simple consequence of Theorem 2.10 and of the naturality property recalled in Section 2.1.3.

In fact, we can as easily prove slightly more. For this we let $\varepsilon(L, L') = \varepsilon(L, L'; 0, J)$ and we recall the setting: L, L' are as before and we have also the Lagrangian L'' which is transversal to L and the almost complex structure J' so that the complexes $\mathcal{C}^J(L, L') = \mathcal{C}^{\eta,J,\zeta}(L, L'; 0)$, $\mathcal{C}^{J'}(L, L'') = \mathcal{C}^{\eta',J',\zeta'}(L, L''; 0)$ are defined as well as the associated spectral sequences $EF(L, L')$ and $EF(L, L'')$. Assume also that we have a Hamiltonian diffeomorphism ϕ such that

$$\phi(L'') = L' \; , \; \eta(t) = \phi(\eta'(t)) \; , \forall t \in [0, 1] \; .$$

We shall assume here that ϕ has a compact support. This is not restrictive for our purposes because L, L' are compact. Denote by $\mathcal{T}$ the set of 1-periodic Hamiltonians on M which are constant outside some compact set and recall the Hofer norm (or energy) [14] of a compactly supported Hamiltonian diffeomorphism:

$$||\phi||_H = \inf_{H \in \mathcal{T} \; , \; \phi_1^H = \phi} (\sup_{x,t} H(t, x) - \inf_{x,t} H(t, x)) \tag{29}$$

Corollary 2.12. *Under the assumptions above*:

a. *There exists a morphism of chain complexes, possibly of nonzero degree*

$$\mathcal{W} : \mathcal{C}^J(L, L') \to \mathcal{C}^{J'}(L, L'')$$

which induces an isomorphism up to translation between the spectral sequences $(EF^r(L, L'), d^r)$ *and* $(EF^r(L, L''), d^r)$ *for* $r \geq 2$.

b. *If* $||\phi||_H < \varepsilon(L, L')/4$, *then* $\mathcal{W}$ *admits a retract.*

Remark 2.13. Point a. of Theorem 1.1 is clearly the same as point a. of Corollary 2.12. In view of the moduli-spaces interpretation of the differentials in $\mathcal{C}^J(L, L')$ we may interpret point b. of the corollary as saying that a small enough Hamiltonian isotopy of L' can only increase the algebraic complexity of the moduli spaces of pseudo-holomorphic strips. A different useful formulation is that, if $\mathcal{C}^J(L, L')$ is not a retract of $\mathcal{C}^{J'}(L, L'')$ (for example if the number of intersection points in $L \cap L''$ is smaller than the number of intersection points in $L \cap L'$), then at least as much energy as $\varepsilon(L, L')/4$ is needed to deform L' into L''.

Proof. Let $H \in \mathcal{T}$ be such that $\phi_1^H = \phi$. Let J_* be the almost complex structure on M which satisfies $\phi^*(J_*) = J'$. Recall from Section 2.1.3 the map $b_H : \mathcal{M}_{L,L'',J',0}(x, y) \to \mathcal{M}_{L,L',J_*,H}(x, y)$ which is defined by $(b_H(u))(s, t) = \phi_t^H(u(s, t))$ and is a homeomorphism respecting the various compactifications. Obviously, this map is also compatible with the maps $\Phi_{x,y}$ and so b_H induces an identification of the two chain complexes (in the sense that it gives a base-preserving isomorphism of chain complexes):

$$\bar{b}_H : \mathcal{C}^{\eta',J',\zeta'}(L, L''; 0) \to \mathcal{C}^{\eta,J_*,\zeta''}(L, L'; H) \tag{30}$$

where ζ'' is the image of ζ' by b_H. Clearly, $\bar{b}_H$ induces an isomorphism up to translation between the respective spectral sequences and as, by Theorem 2.10 a., we also have a morphism

$$\mathcal{V} : \mathcal{C}^{\eta,J,\zeta}(L, L'; 0) \to \mathcal{C}^{\eta,J_*,\zeta''}(L, L'; H)$$

which induces an isomorphism at the level of the spectral sequences, we conclude that the composition $\mathcal{W} = \mathcal{V} \circ (\bar{b}_H)^{-1}$ verifies point a.

Point b. of Theorem 2.10 shows that if $\sup_{x,t} |H(t, x(t))| \leq \varepsilon(L, L')/4$ for all $x \in \mathcal{P}_\eta(L, L')$, $t \in [0, 1]$, then the conclusion at point b. of the corollary holds. We pick a Hamiltonian $H \in \mathcal{T}$ such that $\phi_1^H(L'') = L'$ and $\sup_{x,t} H(t, x) - \inf_{x,t} H(t, x) = ||\phi||_H + \delta$ where δ verifies $||\phi||_H + \delta \leq \varepsilon(L, L')/4$. By adding an appropriate constant to H we may assume $\inf_{x,t} H(t, x) = 0$ and this proves the second point of the corollary. □

2.4. *Proof of the main theorem.* II: *Relation to the Serre spectral sequence.* The purpose of this subsection is to show point d. of Theorem 1.1.

2.4.1. *Elements of classical Morse theory.* We shall fix here a Morse function $f : L \to \mathbb{R}$ and we also fix a Riemannian metric α on L such that the pair (f, α) is Morse-Smale. The Morse-Smale condition means that, if we denote by γ the flow induced by the negative α-gradient of f, $-\nabla f$, then the unstable manifolds

$$W^u(P) = \{x \in L : \lim_{t \to -\infty} \gamma_t(x) = P\}$$

and the stable manifolds

$$W^s(Q) = \{x \in L : \lim_{t \to +\infty} \gamma_t(x) = Q\}$$

intersect transversely for any two critical points $P, Q \in \mathrm{Crit}(f)$. If the index of the critical points P is equal to p, then $W^u(P)$ is diffeomorphic to an open p-disk and $W^s(P)$ is diffeomorphic to an open $(n-p)$-disk. It is easy to see that if $\alpha \in \mathbb{R}$ is a regular value of f such that $f(P) > \alpha > f(Q)$, then the space of γ-flow lines that join P to Q is parametrized by the intersection $W^u(P) \cap f^{-1}(\alpha) \cap W^s(Q)$ which, due to the transversality assumption, is seen to be a manifold of dimension $\mathrm{ind}(P) - \mathrm{ind}(Q) - 1$. This moduli space of negative gradient flow lines will be denoted by $M_{f,\alpha}(P, Q)$ and the space of all the points situated on elements of $M_{f,\alpha}(P, Q)$ will be denoted by $M'_{f,\alpha}(P, Q)$ (to shorten notation we shall sometimes omit the symbol α). These moduli spaces $M_f(-, -)$ have properties that parallel those of the moduli spaces $\mathcal{M}_{L,L',H}(-, -)$ as described in Section 2.1.2 and 2.1.4 but with the set $I(L, L'; H)$ replaced by the set of critical points of f, $\mathrm{Crit}(f)$, and with the difference of Morse indexes $\mathrm{ind}(P) - \mathrm{ind}(Q)$ used instead of the Maslov index $\mu(x, y)$. These properties are much easier to prove for negative-gradient flow lines than for pseudo-holomorphic strips and, in fact, historically the Morse case has preceded and inspired Floer's machinery. From an analytic point of view, the study of the moduli spaces $M_{f,\alpha}(-, -)$ is clearly a simpler version of the study of $\mathcal{M}_{L,L',H}(-, -)$ because negative gradient flow lines are solutions $v : \mathbb{R} \to L$ of the equation

$$\frac{dv}{ds} + \nabla f(v) = 0$$

which may be treated as a simplified version of equation (10). This approach has been developed in detail in [30].

2.4.2. *Morse flow lines and pseudo-holomorphic strips.* There exists another deeper relation between the moduli spaces of Morse trajectories and the moduli spaces of pseudo-holomorphic strips which has been established by Floer [9] and which we now recall. Recall that there exists a neighbourhood of L in M which is symplectically equivalent to the total space of a disk bundle associated to the cotangent bundle T^*L. We shall denote this neighbourhood by DT^*L and consider the Hamiltonian $\overline{f} : DT^*L \to \mathbb{R}$, $\overline{f} = -f \circ \pi$

where $\pi : DT^*L \to L$ is the projection. Notice that if $L_f = \phi_1^{\overline{f}}(L)$, then L_f is precisely the image of $-df$ and $L \cap L_f$ coincides with the set of critical points of f (we assume here that f is small enough so that the image of df is contained in DT^*L). The fact that f is a Morse function is equivalent to the transversality of L_f and L. For any $x, y \in L \cap L_f$, it is natural to define a map $c_f : M'_{f,\alpha}(x,y) \to C^\infty(\mathbb{R} \times [0,1], M)$ by $(c_f(v))(s,t) = \phi_t^{\overline{f}}(v(s))$. Floer's result is that, if f is sufficiently small in C^2-norm, then there exists a (time-dependent) almost complex structure J^f such that the image of this map belongs to $\mathcal{M}'_{L,L_f,J^f,0}(x,y)$ and, moreover, the resulting application $c_f : M'_{f,\alpha}(x,y) \to \mathcal{M}'_{L,L_f,J^f,0}(x,y)$ is a diffeomorphism. The fact that c_f is surjective is in itself highly nontrivial as, *a priori*, $\mathcal{M}'_{L,L_f,J^f,0}(x,y)$ could contain some "long" Floer trajectories which do not belong to DT^*L; however, Gromov compactness together with our assumptions on the lack of bubbling imply that by making f sufficiently small (for example by replacing it with λf with $\lambda > 0$ and small) this does not happen. Obviously, this application induces a diffeomorphism

$$l_f : M_{f,\alpha}(x,y) \to \mathcal{M}_{L,L_f,J^f,0}(x,y)$$

and it is clear that this is compatible with the compactifications and the stratifications on the two sides.

2.4.3. *The Morse spectral sequence.* We now let w be a path in L which is embedded and joins all critical points of f. We then define the quotient map $q : L \to \tilde{L}$ as in §2.2.2. Following the scheme in §2.2 it is easy to see how to build a spectral sequence asociated to the Morse-index filtration of the $\mathcal{R}_*$-chain complex $C^{f,\alpha} = (C_*, d)$ which is defined by $C_k = \bigoplus_{q+p=k} \mathcal{R}_q \otimes \mathbb{Z}/2\langle \mathrm{Crit}_k(f)\rangle$ (where $\mathrm{Crit}_k(f)$ are the critical points of f which are of Morse index equal to k) and

$$dx = \sum_{y \in \mathrm{Crit}(f)} m_{xy} \otimes y \ .$$

As in formula (20), the coefficients $\{m_{xy}\}$ are defined as images of a representing chain system for the moduli spaces $M_{f,\alpha}(x,y)$ by the map $v \in M_f(x,y) \to q \circ s_v \in \Omega' \tilde{L}$ where

$$s_v : [0, f(x) - f(y)] \to L$$

is a reparametrization of v such that $s_v(t) = z \Leftrightarrow f(z) = f(x) - t$. Further, as in §2.2.4, the filtration $F^k C = \mathcal{R}_* \otimes \mathbb{Z}/2\langle \mathrm{Crit}_j(f) : j \leq k\rangle$ induces a spectral sequence which we shall denote by $E(f,\alpha) = (E^r_{pq}(f,\alpha), d^r)$ (again, sometimes we shall omit α to shorten notation). A result similar to Proposition 2.7 is true after we replace the Floer complex with the Morse complex and Floer homology with the usual homology of L.

2.4.4. *Reduction to the Morse case.* We now assume that f is sufficiently C^2-small so that Floer's result mentioned above applies. Clearly, we may extend both $\overline{f}$ and J^f to a Hamiltonian and, respectively, an almost complex structure defined on all of M which shall be denoted by the same respective symbols.

If we let η_0 coincide with z_0 and let both be equal to a minimum of f, then we see that the map l_f of §2.4.2 induces an identification of chain complexes $\overline{l}_f : C^{f,\alpha} \to \mathcal{C}^{\eta_0,J^f}(L, L_f; 0)$. This obviously preserves filtrations and identifies the spectral sequences $E(f,\alpha)$ and $EF(L, L_f; \eta_0, J^f, 0)$.

We now turn to the setting of Theorem 1.1 d. Therefore, L' is Hamiltonian isotopic to L. By Corollary 2.12, we then have that $(EF^r(L, L'; \eta, J, 0), d^r)$ is isomorphic up to translation to $(EF^r(L, L_f; \eta_0, J^f, 0), d^r)$ for $r \geq 2$. At the same time, as discussed above, this last spectral sequence is isomorphic to $E(f,\alpha)$. Thus, to prove Theorem 1.1 d., it suffices to show that $E^r(f,\alpha)$ is isomorphic to the Serre spectral sequence of $\Omega L \to PL \to L$ for $r \geq 2$.

2.4.5. *The Morse and Serre spectral sequences.* The purpose of this subsubsection is to conclude the proof of Theorem 1.1 by showing:

THEOREM 2.14. *Assume that $f : L \to \mathbb{R}$ is a Morse function and α is a riemannian metric on L so that the spectral sequence $E(f,\alpha) = (E^r_{pq}(f,\alpha), d^r)$ is defined as in §2.4.3. For $r \geq 2$ there exist an isomorphism of spectral sequences between $E(f,\alpha)$ and the Serre spectral sequence $E(L) = (E^r_{pq}, d^r)$ of the path loop fibration of base L.*

Proof. We may assume that the function f has just one minimum that we shall denote by B. We also assume that $f(B) = 0$. It is not restrictive to suppose also that f is self-indexed which means that for any critical point x of f we have that $f(x) = \mathrm{ind}_f(x)$. Take ε to be a very small positive constant and let $L_k = f^{-1}(-\infty, k+\varepsilon]$. Of course, by classical Morse theory, L_k is homotopy equivalent to a k-th dimensional skeleton of L. Consider the path-loop fibration $\Omega L \to PL \to L$ and let $\Omega L \to E_k \to L_k$ be the pull-back of this fibration over the inclusion $L_k \hookrightarrow L$. We consider the filtration $\Omega L = E_0 \hookrightarrow \ldots E_k \hookrightarrow E_{k+1} \hookrightarrow \ldots PL$ and the resulting filtration of the cubical chain complex $S_*(PL)$ which is given by the $S_*(E_k)$'s. The spectral sequence associated to this filtration is, by definition, the Serre spectral sequence of the statement [35]. The proof of the theorem consists of the following two steps:

i. There exists a morphism of chain complexes $\xi : C^{f,\alpha} \to S_*(PL)$ so that $\xi(F^k C) \subset S_*(E_k)$. Such a ξ induces a morphism of spectral sequences denoted by $E(\xi) : E(f,\alpha) \to E(L)$.
ii. With ξ as above the morphism $E^2(\xi)$ is an isomorphism.

Before proceeding with the proof we need to make a few adjustments. First, notice that instead of using unit paths in the definition of the path-loop fibration we may as well use Moore paths - these are paths parametrized by arbitrary intervals $[0,a]$. The resulting fibration is denoted by $\Omega' L \to P'L \to L$ and the associated filtration is $\{E'_k\}$. Moreover, as $q : L \to \tilde{L}$ is a homotopy equivalence we may replace the spaces L_k, E'_k by their respective image $\tilde{L}_k$ and $\tilde{E}_k \subset P'\tilde{L}$ in the latter case via the induced map $P'q : P'L \to P'\tilde{L}$ (the two induced spectral sequences being obviously isomorphic). For further use, notice also that there is an obvious action $\cdot : \Omega'\tilde{L} \times P'\tilde{L} \to P'\tilde{L}$ which induces $\mathcal{R}_k \otimes S_q(P'\tilde{L}) \to S_{k+q}(P'\tilde{L})$.

2.4.6. *Blow-up of unstable manifolds.* The first step is based on a geometric construction which, as we shall see, is of independent interest. This construction provides an efficient geometric description for the compactification of the unstable manifolds of f.

We fix $x \in \mathrm{Crit}(f)$. Notice that for each element $v \in \overline{M}_f(x,B)$ there exists some $k \geq 0$ such that $v = (v_1, v_2, \ldots, v_k)$ with $v_1 \in M_f(x,x_1)$, $\ldots$, $v_i \in M_f(x_{i-1},x_i)$, $\ldots$, $v_k \in M_f(x_{k-1},B)$. This writing is of course unique. We recall the parametrizations s_v for the flow lines represented by $v \in M_f(x,B)$ defined as in §2.4.3. Clearly, this parametrization extends in an obvious way to the elements $v = (v_1, v_2, \ldots, v_k) \in \partial\overline{M}_f(x,B)$ and we shall continue to denote the parametrization of these elements by s_v.

We consider the space $\widehat{M}(x)$ which is defined as the topological quotient of the space $\overline{M}_f(x,B) \times [0, f(x)]$ by the equivalence relation induced by:

$$((v_1,\ldots,v_k),t) \sim ((v'_1,\ldots,v'_k),t) \ \text{ if } \ v_i = v'_i \ \ \forall i$$
$$\text{with } \ f(x_{i-1}) > t \ , \ v_i \in M_f(x_{i-1},x_i) \ .$$

In short, two couples $(v,t),(v',t) \in \overline{M}_f(x,B)$ are identified in $\widehat{M}_f(x,B)$ if the (possibly broken) negative gradient trajectories of v and v' coincide above level t. Notice that if $(l_n,t_n) \sim (l'_n,t_n)$, where $l_n, l'_n \in \overline{M}_f(x,B)$ with $l_n \to l \in \overline{M}_f(x,B)$, $l'_n \to l' \in \overline{M}_f(x,B)$, $t_n \to t$, then $(l,t) \sim (l',t)$ and $\widehat{M}(x)$ is Hausdorff.

It is useful to introduce the map $S : \overline{M}_f(x,B) \times [0,f(x)] \to L$ defined by $S(v,\tau) = s_v(f(x)-\tau)$ ($S(v,\tau)$ is thus simply the intersection of the trajectory v with $f^{-1}(\tau)$). This map factors as

$$S : \overline{M}_f(x,B) \times [0,f(x)] \stackrel{k}{\longrightarrow} \widehat{M}(x) \stackrel{o}{\longrightarrow} L$$

where k is the quotient map.

We call the space $\widehat{M}(x)$ the *blow-up* of the unstable manifold $W^u(x)$. As we shall see this is justified by a number of remarkable properties of this space. We start with the most immediate. First, the image of o is included and is onto the closure of $W^u(x)$. Secondly, all the points in $\overline{M}_f(x,B) \times \{f(x)\}$ belong to a unique equivalence class which we shall denote by $*$. Furthermore, define paths $s'_v : [0,f(x)] \to \widehat{M}(x)$ by the formula $s'_v(\tau) = k(v, f(x)-\tau)$. Obviously,

$s'_v(0) = *$ and $s_v = o \circ s'_v$. Moreover, for each $y \in \widehat{M}(x)$ there exists a unique $t \in [0, f(x)]$ and a unique path $\overline{y} : [0,t] \to \widehat{M}(x)$ such that $\overline{y}(t) = y$ and $\overline{y}(\tau) = s'_v(\tau),\ \forall \tau \in [0,t]$ for some $v \in \overline{M}_f(x, B)$. It is easy to see that the map

$$\beta : \widehat{M}(x) \to P'(\widehat{M}(x)),\ \beta(y) = \overline{y}$$

is continuous. As we also have that $\overline{y}(0) = *$ this shows that $\widehat{M}(x)$ is contractible by a contraction that pushes each $y \in \widehat{M}(x)$ along the path $\overline{y}$ till it reaches $*$. We formulate a stronger property next. For this first notice that for all $y \in \mathrm{Crit}(f) \cap \overline{W^u(x)}$ there is a natural inclusion $M_f(x, y) \times \widehat{M}(y) \subset \widehat{M}(x)$ which is induced by the product of inclusions $(M_f(x, y) \times \overline{M}_f(y, B)) \times [0, f(y)] \hookrightarrow \overline{M}_f(x, B) \times [0, f(x)]$.

LEMMA 2.15. *The space $\widehat{M}(x)$ is homeomorphic to a closed disk of dimension equal to $\mathrm{ind}_f(x)$. Moreover,*

$$\partial\widehat{M}(x) = \bigcup_y M_f(x, y) \times \widehat{M}(y)\ .$$

Remark 2.16. a. As we shall see below, the actual proof of Theorem 2.14 only uses that $\widehat{M}(x)$ is a topological manifold with a boundary described as in the statement of the lemma and that $*$ has a neighbourhood homeomorphic to a disk. Of course, the fact that $\widehat{M}(x)$ is a topological manifold is not surprising: this space is obviously homeomorphic to the space of all (appropriately parametrized) possibly broken gradient flow lines that join x to points in L.

b. While the definition of $\widehat{M}(x)$ based on the equivalence relation $\sim$ is new, the space of all geometric, possibly broken, flow lines ending in points of L and originating in $x \in \mathrm{Crit}(f)$ has appeared before in the Morse theoretic literature, for example, in [16] and [17]. The fact that $\widehat{M}(x)$ is homeomorphic to a disk is of independent interest as it immediately implies that the union of the closures of the unstable manifolds of a self-indexed Morse-Smale function has a natural CW-complex structure - the attaching map corresponding to the cell associated to x being simply $o|_{\partial\widehat{M}(x)}$. For completeness we provide here an explicit proof of the existence of a homeomorphism between $\widehat{M}(x)$ and a closed disk. Related arguments appear in the literature in [17], [19] as well as in [20].

Proof of the lemma. We fix $i = \mathrm{ind}_f(x)$ and recall that f is self-indexed. We start by verifying explicitly that $\widehat{M}(x)$ is a topological manifold whose boundary has the description of the statement. We first notice that the restriction of k to $M_f(x, B) \times (0, f(x))$ is a homeomorphism onto its image. Moreover, the definition of the equivalence relation $\sim$ directly implies that the restriction

$$o| : k(\overline{M}_f(x, B) \times [(i-1) + \delta, f(x)]) \to W^u(x) \cap f^{-1}[(i-1) + \delta, +\infty) \tag{31}$$

is a homeomorphism for any small positive δ where o is, as before, the factor of the map S (for further use, notice also that $W^u(x)\cap f^{-1}[(i-1)+\delta,+\infty)$ is homeomorphic to an i-disk).

Consider a point $(v,t)\in \overline{M}_f(x,B)\times[0,f(x)]$ such that $v=(v_1,\dots,v_k)\in M_f(x,x_1)\times\ldots\times M_f(x_{k-1},B)$ and $t>f(x_1)$. We notice that $k(v,t)$ has a neighbourhood homeomorphic to an i-disk. Indeed, for λ sufficiently close to $f(x)$, the point $k(v,\lambda)$ does have such a neighbourhood V because of the homeomorphism at (31). This neighbourhood V verifies $V\subset\bigcup_{y\in C(x_1)}k(M_f(x,y)\times\overline{M}_f(y,B)\times[0,f(x)])$ where $C(x_1)=\{y\in \mathrm{Crit}(f): x_1\in\overline{W^u(x)}\cap\overline{W^s(y)}\}$. But this means that, if V is sufficiently small, we may isotope it by sliding it along the paths $s'_r, r\in V$ till we get a neighbourhood of $k(v,t)$. "Sliding" along the paths s'_v is given by

$$h(k(r,t'),\tau)=s'_r(t'+\tau)$$

and is well defined and an isotopy when restricted to $k(M_f(x,y)\times\overline{M}_f(y,B)\times[s,f(x)])$ as long as $\tau+s>f(y)$. As $y\in C(x_1)$ we have that $f(y)\le f(x_1)$ and thus sliding is indeed possible.

a. *First look at boundary points.* Next, to continue the proof of the lemma, we need to show that each point belonging to some $M_f(x,y)\times\widehat{M}(y)$ has a neighbourhood homeomorphic to a semi-disk. Let $z=k(v,t)$ with $v=(v_1,\dots,v_k)\in M_f(x,x_1)\times M_f(x_1,x_2)\times\ldots M_f(x_{k-1},B)$ so that $f(x_{j-1})>t>f(x_j)$. Because we are only interested in a neighbourhood of z we may assume that the interval $(f(x_j),f(x_{j-1}))$ is regular and, in particular, t is a regular value of f. Recall that the point $o(z)$ is the intersection with $f^{-1}(t)$ of the broken negative gradient flow line of f represented by v.

Let $\tilde{M}_t(x)=\{z\in C^0([0,f(x)-t],M):\exists v\in\overline{M}_f(x,B),\ z=s_v|_{[0,f(x)-t]}\}$. In short, a path in $\tilde{M}_t(x)$ joins the point x to some point in $f^{-1}(t)$ and it coincides geometrically to the part of a negative-gradient (possibly broken) flow line of f which is above (and on) level t. Clearly, the spaces $\tilde{M}_t$ for t such that $f(x_{j-1})>t\ge f(x_j)$ are canonically identified with $\tilde{M}_j=\tilde{M}_{f(x_j)}$. Obviously, for our fixed point $z=k(v,t)$ there exists a unique point $z'\in\tilde{M}_j(x)$ such that $o(z)=z'(f(x)-t)$ (the parametrization used for z' is similar to that used for the paths s_v). In fact, in view of the definition of $\sim$ it is immediate to see that the application $z\to(z',t)$ is a *local* homeomorphism defined on a neighbourhood of $z\in\widehat{M}(x)$ and with values in $\tilde{M}_j(x)\times(f(x_j),f(x_{j-1}))$. Now notice that $\tilde{M}_j(x)$ is a compact topological manifold whose boundary consists as usual of broken trajectories. This means that in case $x_{j-1}\neq x$ the trajectory v is broken at x_{j-1} and thus z is mapped by this local homeomorphism to a point in $\partial\tilde{M}_j(x)\times(f(x_j),f(x_{j-1}))$. Therefore, z has a semi-disk neighbourhood in $\widehat{M}(x)$.

b. *Local study around breaking points.* A slightly different argument is needed for the points $k(v,t)$ with $v=(v_1,\dots,v_k)$ as before but with $t=$

$f(x_{j-1})$. The first such case corresponds to $j = 2$. The key observation is that x_1 has inside $W^u(x_1)$ a neighbourhood U which is homeomorphic to a disk (of dimension $\mathrm{ind}_f(x_1)$). The element v_1 also has a neighbourhood V in $M_f(x, x_1)$ which is homeomorphic to a disk. Together with the definition of $\sim$ this shows that $k(v,t)$ has a neighbourhood in $\widehat{M}(x)$ which is homeomorphic to the product $U \times V \times [0, 1)$.

To see this we study the problem locally in a neighbourhood of x_1 in L. We may assume that f is in normal form around x_1. Let $a = f(x_1)$ and let ε, δ be very small positive constants. Let W be a neighbourhood of x_1 which consists of all the points $x \in f^{-1}[a-\varepsilon, a+\varepsilon]$ that are situated on flow lines of f whose intersection with $f^{-1}(a)$ is at distance less than δ from x_1. We remark that $D = W^s(x_1) \cap W$ is homeomorphic to a disk of dimension $n - \mathrm{ind}_f(x_1)$, similarly $D' = W^u(x_1) \cap W$ is a disk of dimension $\mathrm{ind}_f(x_1)$. We let $S^s = \partial D$ and $S^u = \partial D'$, $A^s = W \cap f^{-1}(a+\varepsilon)$, $A^u = W \cap f^{-1}(a-\varepsilon)$. Notice that A^s, A^u are respectively tubular neighbourhoods of S^s inside $f^{-1}(a+\varepsilon)$ and of S^u inside $f^{-1}(a-\varepsilon)$. Therefore, $A^s = S^s \times D''$, $A^u = D''' \times S^u$ with D'' a disk of dimension $\mathrm{ind}(x_1)$ and D''' a disk of dimension $n - \mathrm{ind}(x_1)$. Moreover, the flow provides a homeomorphism between $A' = A^s \backslash (S^s \times \{0\})$ and $A'' = A^u \backslash (\{0\} \times S^u)$. In view of this we may identify both A' and A'' with $S^s \times S^u \times (0, \delta)$. The set of all paths in W which join A^s to A^u, which are parametrized by the values of f (similarly to the s_v's) and which coincide geometrically to portions of possibly broken flow lines of f is identified with $S^s \times S^u \times [0, \delta)$ (the broken flow lines correspond to $S^s \times S^u \times \{0\}$).

We now consider the space $K(x_1) = (S^s \times S^u \times [0, \delta)) \times [a-\varepsilon, a+\varepsilon]/\sim'$ where $\sim'$ is the analogue of $\sim$ for our paths in W. It is easy to see that the existence of our semi-disk neighbourhood of $k(v,t)$ inside $\widehat{M}(x)$ follows if we show that any point of type $[(x, y, 0), a]$, has a similar semi-disk neighbourhood inside $K(x_1)$. We have $K(x_1) \approx S^s \times (S^u \times [0, \delta) \times [a-\varepsilon, a+\varepsilon]/\sim'')$ where $\sim''$ is the equivalence relation induced by $(x, 0, t) \sim'' (x', 0, t)$ if $t \geq a$. This means that we reduced the problem to studying the space $K'(x_1) = (S^u \times [0, \delta) \times [a-\varepsilon, a+\varepsilon])/\sim''$. Recall that $S^u = \partial D'$. It is easy to check now that $K'(x_1)$ is homeomorphic to the cylinder $D' \times [a-\varepsilon, a+\varepsilon]$ from which has been eliminated the interior and the base of a circular cone of height $[a-\varepsilon, a)$, whose base lies in the interior of $D' \times \{a-\varepsilon\}$ and whose vertex corresponds to $(y, 0, a)$. This shows our claim.

An immediate adaptation of this argument also works when $t = f(x_{j-1})$ even for $j > 2$ and this shows that $\widehat{M}(x)$ is indeed a compact topological manifold with boundary.

c. *Homeomorphism to a disk.* To end the proof of the lemma we still need to show that $\widehat{M}(x)$ is homeomorphic to a disk. The idea is to construct a copy $\partial'\widehat{M}(x)$ of $\partial\widehat{M}(X)$ such that $\partial'\widehat{M}(x)$ is contained in $\widehat{M}(x)$, it is transverse to the paths s'_v and it bounds a topological manifold $\widehat{M}'(x)$ which contains $*$ and is homeomorphic to $\widehat{M}(x)$. Recall that a neighbourhood U of $*$ as in (31) has a boundary that is also transverse to the paths s'_v. By sliding along these paths it follows that $\widehat{M}'(x)$ is homeomorphic to a disk and as this manifold is homeomorphic to $\widehat{M}(x)$ this concludes the proof. Before starting this construction we make explicit the notion of transversality used here: given a separating hypersurface V of a topological manifold N and a path $g : [-a, a] \to N$ such that $g(0) \in V$ we say that g is transversal to V if for some neighbourhood U of V such that $U \backslash V = U_0 \bigsqcup U_1$ there exists $\varepsilon > 0$ and $i \in \{0, 1\}$ such that $\forall t \leq \varepsilon$ we have $g(-t) \in U_i$, $g(t) \in U_{1-i}$.

To construct $\widehat{M}'(x))$ we first fix the notation $\partial(y) = \overline{M}_f(x, y) \times \overline{M}_f(y, B)$, $D(y) = \partial(y) \times [0, f(x)]$ and we let s''_v be the path $v \times [0, f(x)]$ in $\overline{M}_f(x, B) \times [0, f(x)]$. We now intend to construct for each $y \in \mathrm{Crit}(f) \cap \overline{W^u(x)}, y \neq B$ a map

$$f_y : D(y) \to \overline{M}_f(x, B) \times [0, f(x)]$$

which is a homeomorphism onto its image - we shall denote this image by $D'(y)$ — and has the following additional properties: $f_y(v, t) = (v, t)$ if $t \geq f(y)$; $s''_{f_y(v,t)}$ is transverse to $D'(y)$ at the point $f_y(v, t)$ whenever $t < f(y)$; $D'(y)$ together with $\partial(\overline{M}_f(x, B) \times [0, f(x)]) \backslash D_y$ bound a topological manifold with boundary $M'_y \subset \overline{M}_f(x, B) \times [0, f(x)]$ which is homeomorphic to $\overline{M}_f(x, B) \times [0, f(x)]$ and contains $*$; if $(v, t) \sim (v', t)$, then $f_y(v, t) = f_y(v', t)$. The construction of this auxiliary application is as follows. As $\overline{M}_f(x, B)$ is a manifold with corners and $\partial(y)$ is a part of the boundary of $\overline{M}_f(x, B)$ there exists a collar neighbourhood $U(y)$ of $\partial(y)$ inside $\overline{M}_f(x, B)$. In particular, there exists a homeomorphism $f' : \partial(y) \times [0, \varepsilon) \to U(y)$ so that $f'((v, w), \tau) = v\#_\tau w$ where $v\#_\tau w$ is the flow line obtained by gluing v to w at y with gluing parameter τ. Of course, for this we need to choose a particular gluing formula (we may do this as discussed in Appendix A in the obviously harder Floer case) and we choose the gluing parameter in such a way that $v\#_0 w$ coincides with (v, w). More generally, for τ small enough and $v = (v_1, \ldots, v_i) \in M_f(x, x_1) \times \ldots \times M_f(x_{i-1}, y)$, $w = (w_1, \ldots, w_j) \in M_f(y, y_1) \times \ldots \times M_f(y_{j-1}, B)$ we let $v\#_\tau w$ be the element $(v_1, \ldots, v_i\#_\tau w_1, \ldots, w_j) \in M_f(x, x_1) \times \ldots \times M_f(x_i, y_1) \times \ldots \times M_f(y_j, B)$. As a consequence of the parametrization of the corners of $\overline{M}_f(x, B)$ as described in the appendix we obtain that f' so defined is a homeomorphism. We also notice that if $((v, w), t) \sim (v', w'), t)$ with $t < f(y)$, then $(v\#_\tau w, t) \sim (v'\#_\tau w', t)$ and so we also have $(f'((v, w), \tau), t) \sim (f'((v', w'), \tau), t)$. We now let $\varepsilon' < \varepsilon$ and consider a smooth one parameter family of functions $q^s_y : [0, f(x)] \to [0, \varepsilon']$ such that for each $s \in [0, 1]$, q^s_y is decreasing and smooth, $q^s_y|_{[f(y), f(x)]} = 0$, q^s_y is strictly decreasing on $[0, f(y)]$, $q^s_y(0) \leq \varepsilon'$ and, moreover, for any fixed t, $q^-_y(t)$ is a function increasing in s and $q^0 \equiv 0$. We now define $f^s_y(v, t) = (f'(v, q^s_y(t)), t)$ and we let $f_y = f^1_y$.

We pursue with the construction of $\widehat{M}'(x)$. We let $D'_s(y) = \mathrm{Im}(f^s_y)$ (so that $D'_1(y) = D'(y)$, $D'_0(y) = D(y)$), $V^s(y) = \bigcup_{0\leq s'\leq s} D'_{s'}(y)$ (so that the slice of $V^s(y)$ of height t is a tubular neighbourhood of $\partial(y)$ in $\overline{M}_f(x,B) \times \{t\}$), $V^s = \bigcup_{y\neq B} V^s(y)$ and $M'_s = \overline{M}_f(x,B)\times[0,f(x)]\backslash V^s$ (so that $M'_0 = \overline{M}_f(x,B)\times[0,f(x)]$). Notice that, for each $s>0$, and for each $v\in M_f(x,B)$ the path s''_v is transversal to $\partial M'_s$. We now define $\widehat{M}'_s = k(M'_s)$, $E_s(y) = k(D'_s(y)\cap \overline{M}(x,B)\times[0,f(y)])$, $W_s(y) = k(V^s(y)\cap\overline{M}(x,B)\times[0,f(y)])$. By definition recall that $V^s(y)\cap\overline{M}(x,B)\times[f(y),f(x)] = \partial(y)\times[f(y),f(x)] = D'_s(y)\cap\overline{M}(x,B)\times[f(y),f(x)]$. Because of this, as f^s_y respects the relation $\sim$ and as the identifications producing $\overline{M}_f(x,y)\times\widehat{M}(y)\subset\partial\widehat{M}(x)$ occur only on the boundary of $\partial(y)\times[0,f(y)]$, it follows that $E_s(y)$ is a copy of $\overline{M}_f(x,y)\times\widehat{M}(y)$ which verifies $E_s(y)\cap\overline{M}_f(x,y)\times\widehat{M}(y) = \overline{M}_f(x,y)\times *$. Clearly, $W_l(y) = \bigcup_{0\leq s\leq l} E_s(y)$ and $\widehat{M}'_s = \widehat{M}(x)\backslash(\bigcup_{y\neq B} W_s(y))$. The description of $W_s(y)$ shows that, by starting with the y's of lowest index, and proceeding by induction we may isotope $\widehat{M}'_s$ to $\widehat{M}(x)$ and thus these two spaces are homeomorphic.

To conclude the proof we only need to show that the paths s'_v are transverse to the boundary of $\widehat{M}'(s)$ and we may then take $\widehat{M}'(x) = \widehat{M}'(1)$. This transversality is already clear for the points on the "bottom" - the points that belong to $M_f(x,B)\times\widehat{M}(B)$. The transversality of the paths s''_v to $\partial M'_s$ for $v\in M_f(x,B)$ implies that for each such v, the path s'_v is transversal to $\partial\widehat{M}'_s$. As $\bigcup_y \partial(y) = \partial\overline{M}_f(x,B)$, the only case that remains to be discussed is that of transversality at the points $k(v,t)\in\partial\widehat{M}'_s$ with v in some $\partial(y)$. By the description of $E_s(y)$, such a point $k(v,t)$ belongs to $\overline{M}_f(x,y)\times *_y$ (where $*_y$ is the distinguished point in $\widehat{M}(y)$), in particular $t = f(y)$. We notice that, moreover, such a $k(v,t)$ actually belongs to $M_f(x,y)\times *_y$. Indeed, a point $k(v,t)\in\partial\overline{M}_f(x,y)\times *_y$ has the property that there exists x_1, $f(x_1) > f(y)$ so that $k(v,t)\in M_f(x,x_1)\times\widehat{M}(x_1)$. This means that $t < f(x_1)$ which implies $k(v,t)\in W_s(x_1)$ and thus $k(v,t)\notin\widehat{M}'_s$. Therefore, we now consider $k(v,t)\in M_f(x,y)\times *_y$. But from the transversality of s''_v to $M_f(x,y)\times\{w\}\times\{f(y)\}$ for any $v=(v',w)\in M_f(x,y)\times\overline{M}_f(y,B)$ we immediately deduce the transversality of s'_v in this case and this concludes the proof of the lemma.

2.4.7. *Construction of* ξ. We first fix a representing chain system (recall Definition 2.1) $s_{xy}\in S_{\mathrm{ind}(x)-\mathrm{ind}(y)-1}(\overline{M}_f(x,y))$ for the moduli spaces $M_f(x,y)$, $x,y\in\mathrm{Crit}(f)$. By using the description of $\partial\widehat{M}(x)$ and proceeding as in Lemma 2.2, we define, by induction on $\mathrm{ind}_f(x)$, cubical chains $\lambda_x\in$

$S_i(\widehat{M}(x)), x \in \mathrm{Crit}_i(f)$, representing the fundamental class of $(\widehat{M}(x), \partial\widehat{M}(x))$.

$$\partial\lambda_x = \sum_y s_{xy} \times \lambda_y \ . \tag{32}$$

Recall the map $\beta : \widehat{M}(x) \to P'(\widehat{M}(x)),\ y \to \overline{y}$ and let $o' : P'(\widehat{M}(x)) \to P'(L) \to P'(\tilde{L})$ be induced by $\widehat{M}(x) \xrightarrow{o} L \xrightarrow{q} \tilde{L}$. Let also $\beta' = o' \circ \beta$. We now define $\xi : \mathcal{R}_* \otimes \mathbb{Z}/2\langle\mathrm{Crit}(f)\rangle \to S_*(P'\tilde{L})$ by $\xi(x) = \beta'(\lambda_x)$ for each $x \in \mathrm{Crit}(f)$. It is clear that this map respects the relevant filtrations. Due to (32) it is also obvious that ξ so defined is a chain map.

2.4.8. *$E^2(\xi)$ is an isomorphism.* By construction, ξ is a morphism of $\mathcal{R}_*$-modules so it is sufficient to show that $\xi' = E^2_{*,0}(\xi)$ is an isomorphism. For this purpose we notice that there is a natural evaluation map $\Upsilon : P'(\tilde{L}) \to \tilde{L}$. By considering the map $id_L : L \to L$ as a trivial fibration we see that Υ induces an isomorphism $\Upsilon' : E^2_{*,0} \to H_*(\tilde{L})$ and that the composition $\Upsilon \circ \xi$ may be factored as

$$\mathcal{R}_* \otimes \mathbb{Z}/2\langle\mathrm{Crit}(f)\rangle \xrightarrow{r\otimes id} C'(f) \xrightarrow{u} S_*(\tilde{L})$$

where $C'(f)$ is the chain complex defined as $C'(f) = S_*(*) \otimes \mathbb{Z}/2\langle\mathrm{Crit}(f)\rangle$ with differential $\partial' x = \sum_y r(s_{xy})y$ and with $r : S_*(\Omega\tilde{L}) \to S_*(*)$ induced by the projection $\Omega' L \to *$ (as our cubical chains are normalized we have $S_*(*) = \mathbb{Z}/2$). Given that $r \otimes id$ induces an isomorphism $E^2_{*,0}(f) \to H_*(C'(f))$, our proof ends if we show that u induces an isomorphism in homology. Clearly, u is defined by $u(x) = \Upsilon(\beta'(\lambda_x))$. To prove that u induces an isomorphism we proceed by induction. We let C'_k be the subcomplex of $C'(f)$ consisting of elements of degree at most k and we assume that $u_k = u|_{C'_k} : C'_k \to S_*(\tilde{L}_k)$ induces an isomorphism in homology. For each $x \in \mathrm{Crit}(f)$ the chain λ_x represents the fundamental class of $(\widehat{M}(x), \partial\widehat{M}(x))$ and, moreover, we have the homeomorpism indicated in (31). This implies that the couple of maps (u_{k+1}, u_k) induces an isomorphism $H_{k+1}(C'_{k+1}, C'_k) \to H_*(\tilde{L}_{k+1}, \tilde{L}_k)$. By the 5-lemma this shows that u_{k+1} induces an isomorphism and concludes the proof of the theorem. □

3. Applications

As mentioned in the introduction the Serre spectral sequence has many nontrivial differentials. Obviously, in view of Theorem 1.1 d. this shows that there is an abundace of pseudo-holomorphic strips. In this section we make explicit this statement and deduce a number of applications.

We consider here the same setting as before: (M, ω) is fixed as well as the Lagrangian submanifolds L and L' which are in general position and satisfy (1) if not otherwise indicated. This condition is dropped only in Section 3.4

where it will be replaced by requiring that L and L' be Hamiltonian isotopic and $\omega|_{\pi_2(M,L)} = 0$.

We review shortly the other relevant notation to be used in this chapter. In the presence of an almost complex structure which tames ω, $J \in \mathcal{J}_\omega$, we have the moduli spaces $\mathcal{M}_J(x,y)$, and $\mathcal{M}'_J(x,y)$ of, respectively, unparametrized and parametrized pseudo-holomorphic strips joining the intersection points $x, y \in L \cap L'$ as in §2.1.1. Moreover, $\mathcal{M}'_J = \bigcup_{x,y} \mathcal{M}'_J(x,y)$, $\mathcal{M}_J = \bigcup_{x,y} \mathcal{M}_J(x,y)$. In case J is regular (which in the terminology used before in the paper means that the pair $(0, J)$ is regular), then the Gromov compactifications $\overline{\mathcal{M}}_J(x,y)$ satisfy (15) and the spectral sequence $EF(L, L') = EF(L, L'; 0)$ is defined as in §2.2.4. We denote by $\mathcal{J}_{\mathrm{reg}}$ the set of those elements of $\mathcal{J}_\omega$ that are regular. To simplify notation, we drop the index J if no confusion is possible. We recall that $CF(L, L')$ is the usual Floer complex and $\mathcal{C}(L, L')$ is the extended complex constructed in §2.2.4. Recall also from (19) the maps $\Phi_{x,y} : \overline{\mathcal{M}}(x,y) \to \Omega'\tilde{L}$ where $\Omega'\tilde{L}$ is the space of Moore loops and $\tilde{L}$ is obtained from L by contracting to a point an embedded path connecting the points in $L \cap L'$ as in §2.2.2. More explicitly, for $u \in \overline{\mathcal{M}}(x,y)$, $\Phi_{x,y}(u)$ is the curve traced by the strip u on L parametrized by the interval $[0, \mathcal{A}(x) - \mathcal{A}(y)]$ (where $\mathcal{A}$ is the relevant action functional - see 2.2.1) viewed as a loop on $\tilde{L}$. Finally, $\mathcal{R}_* = S_*(\Omega'\tilde{L})$.

3.1. *Global abundance of pseudo-holomorphic strips*: *loop space homology.* In all this subsection we work under the assumption that (1) is satisfied. The point of view here is global. Roughly, we show that much of the algebraic topology of ΩL may be recovered from $\mathcal{M}$. We fix the additional notation

$$\mathcal{K} = \bigcup_{x,y} \mathrm{Im}(\Phi_{x,y}) \subset \Omega'\tilde{L}$$

and we let $\widehat{\mathcal{K}}$ be the submonoid of $\Omega'\tilde{L}$ generated by $\mathcal{K}$. Let $k : \widehat{\mathcal{K}} \to \Omega'\tilde{L}$ be the obvious inclusion.

Corollary 3.1. *If L and L' are Hamiltonian isotopic and $J \in \mathcal{J}_{\mathrm{reg}}$, then the morphism*

$$k_* : H_*(\widehat{\mathcal{K}}; \mathbb{Z}/2) \to H_*(\Omega'\tilde{L} : \mathbb{Z}/2)$$

is surjective.

Proof. In view of the definition of the coefficients $a_{xy} \in S_*(\Omega'\tilde{L})$ as images through the maps $\Phi_{x,y}$ of a representing chain system for the moduli spaces $\mathcal{M}(x,y)$ (as described in §2.2.3) we see that, in fact, $a_{xy} \in S_*(\widehat{\mathcal{K}})$. We denote this ring by $\mathcal{R}'$. Therefore, the chain complex $\mathcal{C}(L, L')$ has coefficients in $\mathcal{R}'$ and, in general, in the construction of the spectral sequence the ring $\mathcal{R}_*$ may be replaced by the smaller ring $\mathcal{R}'$. This produces a spectral sequence $EF'(L, L')$ whose E^2 term is $HF_*(L, L') \otimes H_*(\widehat{\mathcal{K}}; \mathbb{Z}/2)$. There is an obvious natural map

$E(k)$ from this spectral sequence to the spectral sequence $EF(L, L')$. By Theorem 1.1 this last spectral sequence is isomorphic to the Serre spectral sequence $E^r_{p,q}$ of $\Omega L \to PL \to L$. We shall prove that k_* is surjective by induction. We assume already shown that k_* is surjective for $* < i$. Let $a \in H_i(\Omega'\tilde{L}; \mathbb{Z}/2)$. As the Serre spectral sequence converges to the homology of an acyclic space this element viewed in $E^2_{0,i}$ has to verify $[a]_r = d^r[b]_r \in E^r_{0,i}$ for some $r \geq 2$. We shall use again induction here over r. Therefore, assume that for all the elements $a' \in E^2_{0,i}$ which have the property $[a'] \in \mathrm{Im}(d^s)$ with $s < r$ we already know that $a' \in \mathrm{Im}(k_*)$. Let now $b = \sum_j a_j \otimes x_j$ with $a_j \otimes x_j \in \mathcal{R}' \otimes \mathbb{Z}/2\langle L \cap L' \rangle$ (see §2.2.4). We know that $d^0(b) = 0$. Therefore, the a_j's are cycles. As they are of degree stricly less than i it follows that we may assume that they are in the image of k_*. Together with the description of the differential in $\mathcal{C}(L, L')$ this shows that $[a]_r \in \mathrm{Im}(E^r(k))$. But this shows that there exists $c \in E^2_{0,i}$ so that $[c]_r = 0$ and $a + c \in \mathrm{Im}(k_*)$. However, our induction assumption on r implies that $c \in \mathrm{Im}(k_*)$ and so $a \in \mathrm{Im}(k_*)$ and this concludes the proof. □

To state a closely related result, consider the (injective) map

$$i : \mathcal{M}' \to \mathcal{P}(L, L'),\ i(u)(t) = u(0, t) \tag{33}$$

and denote by $\tilde{\mathcal{M}}$ the (compact) image of i. Consider the map $e : \tilde{\mathcal{M}} \to \tilde{L}$, $e(u) = u(0, 0)$.

COROLLARY 3.2. *If L and L' are Hamiltonian isotopic, $J \in \mathcal{J}_{\mathrm{reg}}$, then the map e induces a surjective morphism*

$$H_*(\Omega e) : H_*(\Omega'\tilde{\mathcal{M}}; \mathbb{Z}/2) \to H_*(\Omega'\tilde{L}; \mathbb{Z}/2)\ .$$

Proof. This is immediate from the previous corollary as $k : \widehat{\mathcal{K}} \to \Omega'\tilde{L}$ factors through $\Omega e : \Omega'\tilde{\mathcal{M}} \to \Omega'\tilde{L}$. □

Remark 3.3. a. There exist many examples of maps $w : X \to Y$ so that one of $H_*(w)$, $H_*(\Omega w)$ is surjective but the other is not. Thus, the result at 3.2 is nontrivial.

b. As mentioned in the introduction it has been proven by Floer [7] and Hofer [13] that $H_*(e)$ is also surjective even in the degenerate case. It is likely that the surjectivity at 3.2 remains true in the degenerate case.

c. Both corollaries may be viewed as stability results for moduli spaces of pseudo-holomorphic strips: they are quite immediate for negative gradient flow lines of Morse-Smale functions and therefore they are unsurprising when the isotopy ϕ is small. However the fact that the same properties are preserved even when making ϕ large is nontrivial.

We end this subsection with a different, simple topological consequence. Its content is that, generically, due to the presence of pseudo-holomorphic strips, the form ω "sees" much of $\pi_1(L \cup L')$.

There exists a generic class $\mathcal{L}$ of lagrangians L' which are not only transversal and Hamiltonian isotopic to L as assumed till now but also have the property that the abelian group generated by the obvious map $\mathcal{A}_{L,L'}| : L \cap L' \to \mathbb{R}$ is of maximal rank ($= \#(L \cap L')$). In other words, the action functional $\mathcal{A}_{L,L'}$ takes different values on each of the intersection points $L \cap L'$ and, moreover, these values are linearly independent.

Let $\Pi_{\omega,L,L'} : \pi_2(M, L \cup L') \to \mathbb{R}$ be defined by $I_\omega(u) = \int_{D^2} u^*\omega$.

Corollary 3.4. *For $L' \in \mathcal{L}$, the image of $\Pi_{\omega,L,L'}$ is an abelian group of rank at least $\dim(\overline{H}_*(L; \mathbb{Z}/2))$ (where $\overline{H}(-)$ denotes reduced homology).*

Proof. We fix $J \in \mathcal{J}_{\text{reg}}$. We will prove that there is a set $\{x_1, x_2, \ldots , x_m\} \subset L \cap_\eta L'$ with $m = \dim(\overline{H}_*(L; \mathbb{Z}/2))$ such that for each x_i there exists some y_i and $u_i \in \mathcal{M}(x_i, y_i) \neq \emptyset$. Given the definition of $\mathcal{L}$ this suffices to show the statement because for $u_i \in \mathcal{M}'(x, y)$ we have $\mathcal{A}_{L,L'}(x_i) - \mathcal{A}_{L,L'}(y_i) = \int_{\mathbb{R}\times[0,1]} u_i^*\omega$ which shows that the values $\Pi_{\omega,L,L'}(u_i)$ are linearly independent (over $\mathbb{Z}$).

To simplify notation we shall say that $x \in L \cap L'$ is a strip origin if there exists y such that $\mathcal{M}(x, y) \neq \emptyset$. We now let $a_1, a_2, \ldots a_m$ be a basis for $\overline{H}_*(L, \mathbb{Z}/2)$. We pick chains $z_i \in CF_*(L, L')$ representing respectively the classes a_i. We write $z_i = \sum x^i_j$ where $x^i_j \in L \cap_\eta L'$. Notice that if a point $x \in L \cap L'$ of positive degree is not a strip origin, then its differential in the Floer complex is null so its homology class $[x] \in HF_*(L, L')$ is well-defined. Moreover, this homology class has to be null because $[x]$ viewed as an element of $EF^2_{p,0}(L, L')$ survives to E^∞. Because of this we may assume that each x^i_j appearing in the expression of the chains z_i is a strip origin. This implies the claim because if there are strictly less than m distinct points among the x^i_j's, then the family $\{z_i\}$ is linearly dependent which contradicts the fact that the family $\{a_i\}$ is linearly independent. □

3.2. *Local pervasiveness of pseudo-holomorphic strips.* With our machinery it is not hard to deduce that through each point of $L\backslash L'$ passes some pesudo-holomorphic strip (see for example Corollary 3.6 below; this also follows from the results of Hofer and Floer mentioned in Remark 3.3 and yet another argument has been mentioned to us by Dietmar Salamon). The point of view here is however slightly different: what most interests us is to restrict the type of these strips that "fill" L. We again work under the assumptions at (1) .

We start with a useful, purely algebraic consequence of the construction of the spectral sequence $EF(L, L')$. Assume that $g : L \to X$ is a continuous map. Let $\Omega X \to E_g \to L$ be the pullback fibration $g^*(\Omega X \to PX \to X)$. There is an obvious map of fibrations which is induced by g:

$$\begin{array}{ccc} \Omega L & \xrightarrow{\Omega g} & \Omega X \\ \downarrow & & \downarrow \\ PL & \xrightarrow{\bar{g}} & E_g \\ \downarrow & & \downarrow \\ L & \xrightarrow[\mathrm{id}]{} & L \end{array}$$

This construction may be performed also by using Moore loops instead of the usual ones and we may as well replace L by $\tilde{L}$. Therefore, if we denote the ring $S_*(\Omega' X)$ by $\mathcal{R}''$ we have a change of coefficients map $g^{\#} : \mathcal{R} \to \mathcal{R}''$. We may obviously use this map to define a complex $\mathcal{C}_X(L, L')$ as in §2.2.4 which is obtained from $\mathcal{C}(L, L')$ by this change of coefficients. There is also an associated spectral sequence $EF_X(L, L')$ into which $EF(L, L')$ maps by the map induced by $g^{\#}$. The properties of $EF_X(L, L')$ parallel those of $EF(L, L')$ and have the same proofs. In particular, property d. becomes:

COROLLARY 3.5. *If L and L' are Hamiltonian isotopic and $J \in \mathcal{J}_{\mathrm{reg}}$, then the spectral sequence $EF_X(L, L')$ is defined and its terms of order greater than or equal to 2 are isomorphic up to translation to the corresponding terms of the Serre spectral sequence of the fibration $\Omega X \to E_g \to L$.*

For the next corollary we shall assume that L and L' are Hamiltonian isotopic and $J \in \mathcal{J}_{\mathrm{reg}}$. Recall that we have an isomorphism up to translation between $HF_*(L, L') \approx H_*(L, \mathbb{Z}/2)$. To simplify notation we shall assume this isomorphism to be degree-preserving. We shall denote by $[1] \in HF_0(L, L')$ and $[L] \in HF_n(L, L')$ the generators of the respective homology groups. There is an obvious evaluation map $\mathcal{E} : \mathcal{M}' \to L$ which is defined by $\mathcal{E}(u) = u(0, 0)$ (it verifies $\mathcal{E} = e \circ i$ with i, e as in (33)). For two elements $x = \sum_i c_i x_i \in CF(L, L')$, $y = \sum_j d_j x_j \in CF(L, L')$ we let

$$\mathcal{R}(x, y) = \bigcup_{c_i \neq 0,\ d_j \neq 0} \mathcal{E}(\mathcal{M}'(x_i, x_j)) \ .$$

Let a be a representative of the fundamental class $[L]$. Consider also an element $b \in CF_0(L, L')$ which is the sum of all the intersection points which appear with nonvanishing coefficients in some representatives of the homology class [1].

COROLLARY 3.6. *In the setting above, the set $\mathcal{R}(a, b)$ is dense in L. In particular, each $x \in L \backslash L'$ belongs to some pseudo-holomorphic strip of Maslov index at most n.*

Proof. Assume that the image of $\mathcal{R}(a, b)$ avoids a small open disk $D \subset L$. This implies that $\overline{\mathcal{R}(a, b)} \subset L \backslash D$. We may assume that the path w used to construct $\tilde{L}$ out of L intersects D in just a point and that for all x with $\mu(a,x) < n$, $D \cap \mathcal{R}(a,x) = \emptyset$ ($\mathcal{R}(a,x)$ is nowhere dense in this case). We intend to apply Corollary 3.5 to the map $g : L \to D/\partial D = S^n$ defined by contracting the closure of the complement of D to a point. The basic algebraic fact that we will be using is that in the Serre spectral sequence of the induced fibration $\Omega S^n \to E_g \to L$ the differential d^n verfies $d^n[L] = [1] \otimes l$ where l is the homology class of the bottom sphere $S^{n-1} \hookrightarrow \Omega S^n$ (this inclusion is the adjoint of the identity). By Corollary 3.5 the same statement is true for the spectral sequence $EF_{S^n}(L,L')$. Let $\mathcal{C}'_{S^n}(L,L')$ be the subcomplex of $\mathcal{C}_{S^n}(L,L')$ which is generated by $\{x \in L \cap L' : \mu(a,x) > 0\} \cup \{a\}$. The index filtration defines a spectral sequence $EF'_{S^n}(L,L')$ which obviously maps into $EF_{S^n}(L,L')$ by the map $E(i)$ induced by the inclusion $i : \mathcal{C}'_{S^n}(L,L') \hookrightarrow \mathcal{C}_{S^n}(L,L')$. As a represents the fundamental class in $CF_n(L,L')$ it follows that we also have $d^n([a]) = [1] \otimes [l]$ in $EF'_{S^n}(L,L')$.

Consider also the subcomplex $\mathcal{C}''_{S^n}(L,L')$ which is generated by $\{x \in L \cap L' : \mu(b,x) > 0\}$. The quotient complex $\overline{\mathcal{C}}_{S^n}(L,L') = \mathcal{C}'_{S^n}(L,L')/\mathcal{C}''_{S^n}(L,L')$ is well-defined and it admits an obvious filtration such that the quotient map $p : \mathcal{C}'_{S^n}(L,L') \to \overline{\mathcal{C}}_{S^n}(L,L')$ preserves filtrations. Therefore, it induces a morphism of spectral sequences $E(p) : EF'_{S^n}(L,L') \to \overline{EF}_{S^n}(L,L')$. We notice that $(E(p))^2_{0,0}$ is injective and so, in $\overline{EF}_{S^n}(L,L')$, we have again $d_{S^n}([a]) = [1] \otimes l$. But, if in $\mathcal{C}(L,L')$ we have $da = \sum_i k_{ay} \otimes y$, then the differential of a in $\mathcal{C}_{S^n}(L,L')$ is given by $d_{S^n}a = \sum_i g^{\#}(k_{ay}) \otimes y$ and as by assumption $\overline{\mathcal{R}(a,b)}$ as well as $\mathcal{R}(a,x)$ for $\mu(a,x) < n$ avoid D it follows that all the critical points y which appear with nonzero coefficients in the expression of some representative of 1 have $g^{\#}(k_{ay}) = 0$ which contradicts $d_{S^n}([a]) = [1] \otimes l \neq 0$. $\square$

For the next result recall from the introduction that, for any two Lagrangians $L, L' \subset M$ (not necessarily transversal), we define as in [1], [25] the isotopy energy of L and L' by

$$\nabla(L,L') = \inf_{\phi \in \mathcal{H}, \phi(L)=L'} ||\phi||_H$$

where $\mathcal{H}$ is the group of Hamiltonian diffeomorphisms with compact support and, as before, $||-||_H$ is Hofer's energy (see (29)). In case L and L' are not isotopic we take $\nabla(L,L') = \infty$. It is easy to see that $\nabla(-,-)$ is symmetric and verifies the triangle inequality. Moreover, it has been shown by Chekanov [1] following earlier work by Oh [25] that $\nabla(-,-)$ is nondegenerate for arbitrary compact lagrangians in tame symplectic manifolds thus providing a metric on any (Hamiltonian) isotopy equivalence class of Lagrangians.

Corollary 3.7. *Suppose that $L, L' \subset M$ are transversal, simply-connected lagrangians embedded in the symplectic manifold (M,ω) with $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$. If L and L' are Hamiltonian isotopic, then for any almost* *complex structure* $J \in \mathcal{J}_\omega$ *and for any point* $x \in L \backslash L'$ *there exists a* J*-pseudo-holomorphic strip* $u : \mathbb{R} \times [0,1] \to M$, $u(\mathbb{R}, 0) \subset L$, $u(\mathbb{R}, 1) \subset L'$ *so that* $x \in \mathrm{Im}(u)$,

$$\int_{\mathbb{R}\times[0,1]} u^*\omega \leq \nabla(L, L') \ .$$

Moreover, when $J \in \mathcal{J}_{\mathrm{reg}}$, *there is a strip* u *as above which also verifies* $\mu(u) \leq n$.

Proof. The Gromov compactness theorem applies to sequences u_n of J_n-holomorphic curves where $J_n \in \mathcal{J}_\omega$ is a sequence of almost complex structures which converges towards another almost complex structure $J \in \mathcal{J}_\omega$ [24]. As any almost complex structure belonging to the set $\mathcal{J}_\omega$ may be viewed as the limit of a sequence of regular almost complex structures, this implies that it is sufficient to prove the statement when $J \in \mathcal{J}_{\mathrm{reg}}$ and so we assume this for the rest of the proof.

Recall the definition of the energy $E_{L,L',H}(u)$ of the elements of $u \in \mathcal{M}'_{L,L',J,H}$ from formula (11). If $u \in \mathcal{M}'_{L,L',J,0}$ we have $E_{L,L'}(u) = \int_{\mathbb{R}\times[0,1]} u^*\omega$. Let

$$\mathcal{M}^a_{L,L',J,H} = \{u \in \mathcal{M}'_{L,L',J,H} : E_{L,L',H}(u) \leq a, \mu(u) \leq n\} \ .$$

Our main interest is in $\mathcal{M}^a = \mathcal{M}^a_{L,L',J,0}$. We now assume that ϕ is a Hamiltonian isotopy such that $L' = \phi(L)$. Given that the set of energies of the elements in $\mathcal{M}'$ is discrete it is sufficient for our statement to show that the set $\mathcal{E}(\mathcal{M}^{||\phi||_H})$ is dense in L and this is what we shall show next.

Let $f : L \to \mathbb{R}$ be a Morse function with a single maximum P and a single minimum Q and let $f(P) = \varepsilon > 0$, $f(Q) = 0$. We pick a riemannian metric α so that the pair (f, α) is Morse-Smale. Let $\overline{f}, L_f, J^f$ be as in §2.4.2. As in §2.4.4 we have homeomorphisms of moduli spaces inducing an identification of chain complexes:

$$\bar{l}_f : C^{f,\alpha} \to \mathcal{C}^{J^f}(L, L_f; 0) \ .$$

Notice also that the naturality results in §2.1.3 show that $\mathcal{A}_{L,L_f}(x) = f(x)$ for all $x \in \mathrm{Crit}(f) = L \cap L_f$. Fix some small $\delta > 0$. By taking ε sufficiently small we see that there exists a Hamiltonian $G : [0,1] \times M \to \mathbb{R}$ which is constant outside a compact set and where $L_f = \phi_1^G(L')$ and

$$\mathrm{Var}(G) = \sup_{x,t} G(t,x) - \inf_{x,t} G(t,x) \leq ||\phi||_H + \delta \ .$$

By adding an appropriate constant to G we may assume that $\inf_{x,t} G(t,x) = \varepsilon$. From (30) we see that we have homeomorphisms of moduli spaces $b_G : \mathcal{M}_{L,L',J,0} \to \mathcal{M}_{L,L_f,J'}$ and an identification of chain complexes (which is action-preserving; see §2.1.3):

$$\overline{b}_G : \mathcal{C}^J(L, L'; 0) \to \mathcal{C}^{J'}(L, L_f; G)$$

where $(\phi_1^G)^*(J') = J$.

Thus, in view of the definition of b_G, the image of $\mathcal{M}^a$ by the evaluation map coincides with the image of $\mathcal{M}^a_{L,L_f,J',G}$ and, therefore, to show the claim it is enough to prove that $\mathcal{E}(\mathcal{M}^a_{L,L_f,J',G})$ is dense (recall also that the energy and the action are related by formula (28)).

Let $\Delta = \mathrm{Var}(G)$. We consider also the Hamiltonian $G' = G - \Delta - 2\varepsilon$. We can then define monotone homotopies from G to 0 and from 0 to G'. Let $CF = CF(L, L_f; G)$ be the Floer complex and, similarly, let $CF' = CF(L, L_f; G')$ and $C = CF(L, L_f; 0) = C^{f,\alpha}$.

It follows from equation (26) and as in the proof of Theorem 2.10 that the monotone homotopies mentioned above induce morphisms of chain complexes:

$$\mathcal{V} : \mathcal{C}^{J'}(L, L_f; G) \to \mathcal{C}^{J^f}(L, L_f; 0)\ ,\ V : CF \to C$$

and

$$\mathcal{W} : \mathcal{C}^{J^f}(L, L_f; 0) \to \mathcal{C}^{J'}(L, L_f; G')\ ,\ W : C \to CF'$$

which are not action-increasing.

For an element $x \in CF$, $x = \sum_i c_i x_i$ with $c_i \in \mathbb{Z}/2$ and $x_i \in I(L, L_f; G)$ we let $\overline{\mathcal{A}}_G(x) = \max_{c_i \neq 0} \mathcal{A}_{L,L_f,G}(x_i)$ and $\underline{\mathcal{A}}_G(x) = \min_{c_i \neq 0} \mathcal{A}_{L,L_f,G}(x_i)$. We define similarly $\overline{\mathcal{A}}_{G'}(x')$ for $x' \in CF'$. Let now $a = \sum_i r_i a_i \in CF$ be a chain representative of the fundamental class $[L]$ such that $\overline{\mathcal{A}}_G(a)$ is minimal among all such representatives. Notice at this point that the complexes $\mathcal{C}(L, L_f; G')$ and $\mathcal{C}(L, L_f; G)$ are canonically identified (and similarly for CF and CF'). We distinguish elements of the two complexes by indicating this identification as $\mathcal{C}(L, L_f; G) \ni x \to x' \in \mathcal{C}(L, L_f; G')$ and we clearly have $\mathcal{A}_G(x) = \mathcal{A}_{G'}(x') + \Delta + 2\varepsilon$. Let $c' \in CF'$ be defined by $c' = W(P)$ (we recall that P is the unique maximum point of f). Then c is also a representative of $[L]$ and, therefore, $\overline{\mathcal{A}}_G(c) \geq \overline{\mathcal{A}}_G(a)$. At the same time, as W is not action-increasing, we have $\overline{\mathcal{A}}_{G'}(c') \leq \mathcal{A}_{L,L^f}(P) = f(P) = \varepsilon$ which means that $\overline{\mathcal{A}}_G(a) \leq \Delta + 3\varepsilon \leq ||\phi||_H + \delta + 3\varepsilon$.

For two elements $x, y \in CF$ with $x = \sum c_i x_i$, $y = \sum d_i y_i$ we put $\mathcal{R}_G(x, y) = \bigcup_{i,j} \mathcal{E}(\mathcal{M}'_{L,L_f,G}(x_i, y_j))$. For the proof of the corollary it suffices to show that the element $b = \sum_i b_i \in CF$ defined as the sum of all the generators b_i of CF with $\mu(b_i) = 0$, $\mathcal{A}_G(b_i) \geq 0$ has the property that the set $\mathcal{R}_G(a, b)$ is dense in L. Indeed, each element in $\mathcal{M}'_{L,L_f,G}(a_i, b_j)$ has energy bounded by $\overline{\mathcal{A}}_G(a) - \underline{\mathcal{A}}_G(b) \leq ||\phi||_H + \delta + 3\varepsilon$ and we may take δ and ε arbitrarily small. Because the possible energy values form a discrete set this implies the claim.

To show the density of $\mathcal{R}_G(a, b)$ we proceed in a way similar to that of Corollary 3.6. Therefore, we assume that this set as well as all $\mathcal{R}_G(a, x)$ for $\mu(a, x) < n$ are disjoint from a disk $D \subset L$ and we consider the associated map $g : L \to D/\partial D = S^n$ and the associated change of coefficients $g^{\#} : \mathcal{R} \to \mathcal{R}''$. We use the same conventions as in Corollary 3.5 and to shorten notation we let $\mathcal{C}_1 = \mathcal{C}^{J'}_{S^n}(L, L_f; G)$, $\mathcal{C}(f) = \mathcal{C}^{J^f}_{S^n}(L, L_f; 0)$. Consider the subcomplex $\mathcal{C}_0 \hookrightarrow \mathcal{C}_1$ which is generated by the elements $x \in I(L, L_f; G)$ such that $\mathcal{A}_G(x) < 0$ and consider also the quotient $\mathcal{C}_2 = \mathcal{C}_1/\mathcal{C}_0$. We notice that due to the monotonicity of the homotopy inducing $\mathcal{V}$ the map $\mathcal{V}_{S^n}$ factors as $\mathcal{C}_1 \xrightarrow{\mathcal{V}'} \mathcal{C}_2 \xrightarrow{\mathcal{V}''} \mathcal{C}(f)$ where the first map is the passage to quotient. Both $\mathcal{V}'$ and $\mathcal{V}''$ respect filtrations and thus they induce spectral sequences morphisms $EF(\mathcal{V}') : EF_{S^n}(L, L_f; G) \to \overline{EF}$ (where $\overline{EF}$ is the spectral sequence induced by the degree filtration on $\mathcal{C}_2$) and $EF(\mathcal{V}'') : \overline{EF} \to E_{S^n}(f)$. The composition of these two morphisms is an isomorphism for $r \geq 2$ and as in $E_{S^n}(f)$ we have $d^n[L] = [1]\otimes[l]$ (with l the class of the bottom sphere in $H_{n-1}(\Omega S^n)$ it follows that $d^n[a] = k \otimes [l]$ with $k \neq 0$ in $\overline{EF}$. But the fact that $\mathcal{E}(\mathcal{R}_G(a,b))$ avoids D implies that all the coefficients $g_i \in \mathcal{R}$ of the b_i's in the expression of the differential of $da \in \mathcal{C}^{J'}(L, L_f; G)$ have the property that $g^{\#}(g_i) = 0$ and thus we arrive at a contradiction. □

Remark 3.8. a. There is a certain overlap between Corollary 3.7 and Corollary 3.6. However, the choice of the element a in this last corollary is less restrictive than in the proof of 3.7.

b. Given a manifold N^n, the degree one map $N \to S^n$, produced by collapsing onto $D^n/\partial D^n$ where D^n is a closed disk in N, is the simplest possible example of a Thom-Pontryaguin map. From this point of view, Corollaries 3.6 and 3.7 are truly immediate consequences of the main theorem (compared to this theorem, the only new idea appears in the proof of 3.7).

c. There exist some other methods to produce Floer orbits joining the "top and bottom classes" in the Floer complex. An interesting such approach is provided by Schwarz [32], [31] and is based on the pair-of-pants product. This suggests a relation between our invariant and this product. Obviously, such a relation is also to be expected for purely topological reasons.

3.3. *Nonsqueezing.* In this subsection we shall prove a number of geometric consequences of the previous results.

We consider two closed Lagrangians $L, L' \subset M$. We assume that L and L' intersect transversely and let $J \in \mathcal{J}_\omega$. We do not assume for now that L and L' are Hamiltonian isotopic. Recall from, §1.3.3 and from equation (22) the following notation in which the areas are computed with respect to the riemannian metric induced by J.

- $\mathcal{S}(x,y)$ is the set of C^∞ strips joining $x, y \in L \cap L'$.
- $a_{L,L'}(x,y)$ is the infimum of the areas of the strips in $\mathcal{S}(x,y)$.
- $a_k(L, L'; J)$ is the minimal area of a pseudo-holomorphic strip of index k.
- $A_k(L, L'; J)$ is the maximal area of such a strip (these numbers are taken to be infinite if no such pseudo-holomorphic strips exist).

- $\varepsilon(L, L'; J) = \varepsilon(L, L'; J, 0)$ is the minimal energy of some element in $\mathcal{M}'_{L,L',J}$ (this number is taken to be infinite if $\mathcal{M}'_{L,L',J}$ is void). We define another pair of associated numbers by $\overline{\varepsilon}(L, L') = \sup_{J\in\mathcal{J}_{\text{reg}}} \varepsilon(L, L'; J)$ and $\underline{\varepsilon}(L, L') = \inf_{J\in\mathcal{J}_{\text{reg}}} \varepsilon(L, L'; J)$.
- $\delta(L, L')$ is the maximal radius of a standard symplectic ball $B(r)$ which is symplectically embedded in M with an image disjoint from L' and so that the image of $\mathbb{R}^n \cap B(r)$ is included in L.

These notions are well-defined independently of the connectivity conditions in (1) but for the remainder of this subsection we assume that these conditions are satisfied.

Remark 3.9. a. The area of an element of $\mathcal{M}'_{L,L',J}$ coincides with its energy and also coincides with its symplectic area and, moreover, if $\mathcal{M}_{L,L',J}(x, y) \neq \emptyset$, then $a_{L,L'}(x, y) = E_{L,L',J}(u),\ \forall u \in \mathcal{M}'_{L,L',J}(x, y)$.

b. As has been observed by François Lalonde, the value of $\delta(L, L')$ is not changed if to its definition we add the condition that the image of the center of $B(r)$ is equal to some fixed point $x \in L$. This is because if $e : B(r) \to M$ is an embedding as required and such that $e(0) = y$, then we may find a Hamiltonian isotopy that carries y to x, which is supported in the neighbourhood of a path joining y to x inside L and which sends L to itself. Indeed, as L is compact, by dividing the path joining y to x into small enough pieces, we may assume that both x and y belong to a standard coordinate chart ressembling $(\mathbb{C}^n, \mathbb{R}^n)$, $x, y \in \mathbb{R}^n$, and in this situation the problem is trivial.

c. We have the obvious inequalities: $\underline{\varepsilon}(L, L') \geq \min\{\mathcal{A}_{L,L'}(x) - \mathcal{A}_{L,L'}(y) : x, y \in L \cap L', \mathcal{A}_{L,L'}(x) > \mathcal{A}_{L,L'}(y)\} > 0$.

From Corollary 3.6 and Remark 3.9 c. we deduce that, if, in addition, L and L' are Hamiltonian isotopic and $J \in \mathcal{J}_{\text{reg}}$ then:

$$\infty > A_n(L, L'; J) \geq a_n(L, L'; J) \geq \varepsilon(L, L'; J) \geq \underline{\varepsilon}(L, L') > 0 \ . \tag{34}$$

Under the same assumptions, we obtain from the proof of Corollary 3.7 that

$$\nabla(L, L') \geq a_n(L, L'; J) \ . \tag{35}$$

More interesting inequalities follow.

COROLLARY 3.10. *Assume L, L' are two simply-connected Lagrangian submanifolds of (M, ω) and suppose $\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}$.*

i. *If L and L' intersect transversely and $J \in \mathcal{J}_{\text{reg}}$, then*

$$\nabla(L, L') \geq \frac{\pi}{2}\ \delta(L, L')^2 \ ,$$

$$A_n(L, L'; J) \geq \frac{\pi}{2}\ \delta(L, L')^2 \ .$$

ii. *Additionally, suppose that* L'' *is another Lagrangian transversal to* L *and assume* J' *is another almost complex structure such that* $\mathcal{C}^{J'}(L, L'')$ *is defined. If* $\mathcal{C}^{J'}(L, L'')$ *does not admit* $\mathcal{C}^{J,\eta}(L, L')$ *as a retract (for some choice of* η*), then*

$$\nabla(L', L'') \geq \varepsilon(L, L'; J)/4 \ .$$

In particular, the energy needed to diminish the number of intersection points between L *and* L' *by a Hamiltonian isotopy is at least* $\underline{\varepsilon}(L, L')/4$.

Proof. The proof of i. is a rapid consequence of Corollary 3.7 combined with an argument classical in symplectic topology since the work of Gromov. We assume L and L' Hamiltonian isotopic by an isotopy ϕ. Fix a (standard) ball $B(r)$ and an embedding $e : B(r) \to M$ as in the definition of $\delta(L, L')$. Thus $e(B(r)) \cap L' = \emptyset$, $e^{-1}(L) = \mathbb{R}^n \cap B(r)$, $e(0) = x \in L$. Fix a small $\delta > 0$. There exists an almost complex structure $J \in \mathcal{J}_\omega$ on M such that $e^*J|_{B(r-\delta)} = J_0$ where J_0 is the canonical almost complex structure on $B(r)$ (in fact, J is constructed by extending the push forward of J_0). Corollary 3.7 shows that there exists a J-pseudo-holomorphic strip $u \in \mathcal{M}'_{L,L',J}$ that passes through x and verifies $\int_{\mathbb{R}\times[0,1]} u^*\omega \leq ||\phi||_H$, $\mu(u) \leq n$ (even with $\mu(u) = n$ after possibly perturbing x in L by an arbitrarily small amount). We now consider $v = e^{-1}(u \cap B(r-\delta))$. This is a J_0-pseudo-holomorphic curve in $B(r-\delta)$ whose boundary lies on $\partial B(r-\delta) \cup \mathbb{R}^n$ and whose area is bounded from above by $||\phi||_H$. By analytic continuation, this curve extends to a J_0-pseudo-holomorphic curve $\overline{v}$ whose boundary is contained in $\partial B(r - \delta)$, which contains 0 and whose area is the double of that of v. By the classical isoperimetric inequality we get that the area of $\overline{v}$ is at least $\pi(r - \delta)^2$. Thus the area of u is at least $\pi(r - \delta)^2/2$ and this shows $||\phi||_H \geq \pi r^2/2$ and implies the inequalities at the first point.

Point ii. is a reformulation of Corollary 2.12. □

Remark 3.11. Point i. implies that $\nabla(-, -)$ is nondegenerate (in our setting): indeed, if $L \neq L''$ then we may find a small symplectic ball $B(r)$ embedded in M as in the definition of $\delta(-, -)$. If L and L'' are not transversal we may perturb L'' to a Lagrangian L''' which is transversal to L without touching $B(r)$. Therefore, using the triangle inequality, we have $\nabla(L, L'') \geq \pi r^2/2 - \delta(L'', L''')$. As the perturbation of L'' can be made as small as needed we get that $\nabla(L, L'') > 0$.

We conjecture that the inequality on the left in Corollary 3.10 i. is true for any pair of closed Lagrangian submanifolds of a closed symplectic manifold. We shall prove it in the next section under weaker assumptions than (1).

3.4. *Relaxing the connectivity conditions.* There are some obvious extensions of our construction - for example by using the orientations of the various moduli spaces involved we could use $\mathbb{Z}$ coefficients instead of $\mathbb{Z}/2$ coefficients.

The purpose of this subsection is to discuss a different extension which is quite useful. This concerns replacing the rather stringent connectivity requirements in (1) by the assumption that L and L' are Hamiltonian isotopic and

$$\omega|_{\pi_2(M,L)} = 0 \ . \tag{36}$$

As we shall see, adapting the construction of our invariant to this setting turns out to be reasonably straightforward even if the result of the construction is less elegant than before (a reason that has made us postpone this variant of the construction till this moment). As a consequence, the proofs of Corollaries 3.7 and 3.10 i. remain valid in this case and therefore we obtain the following strengthening of these two results.

COROLLARY 3.12. *Assume that $L^n \subset (M^{2n}, \omega)$ is a closed Lagrangian submanifold such that $\omega|_{\pi_2(M,L)} = 0$. If $L' \subset M$ is a second Lagrangian in the same Hamiltonian isotopy class as L, then for each point $x \in L$ and each almost complex structure structure $J \in \mathcal{J}_\omega$ there exists a pseudo-holomorphic strip $u \in \mathcal{M}'_{L,L',J}$ such that $x \in \mathrm{Im}(u)$, $\int u^*\omega \leq \nabla(L, L')$ and, additionally, when $J \in \mathcal{J}_{\mathrm{reg}}$, $\mu(u) \leq n$. In particular, $\nabla(L, L') \geq \frac{\pi}{2}\delta(L, L')^2$ and, moreover, if $J \in \mathcal{J}_{\mathrm{reg}}$ we also have $\frac{\pi}{2}\delta(L, L')^2 \leq A_n(L, L')$.*

Remark 3.13. As in Remark 3.11, the inequality in the last corollary recovers (under the assumption $\omega|_{\pi_2(M,L)} = 0$) Chekanov's result claiming that $\nabla(L, L')$ is a distance on the Hamiltonian isotopy class of L. Moreover, the same inequality also implies that for any symplectic manifold M with $\omega|_{\pi_2(M)} = 0$, the disjunction energy of a subset $A \subset M$ is greater than half the Gromov radius of A, a result proven for all symplectic manifolds by Lalonde and McDuff [18]. Indeed, for this last result, assume that ϕ is a Hamiltonian isotopy of M that separates A from itself. Suppose also that A contains a standard ball $B(r)$ or radius r. We may assume that $M' = \mathrm{graph}(\phi) \subset (M \times M, \omega \oplus -\omega)$ intersects the diagonal Δ transversely. The fact that ϕ separates A from itself implies that $B(r) \times B(r)$ is disjoint from M'. Given that $B(r) \times B(r)$ contains a standard $4n$-dimensional ball $B'(r)$ of radius r centered on Δ and such that $B'(r) \cap \Delta$ is included in the image of $\mathbb{R}^{2n}$, Corollary 3.12 implies that $||\phi||_H \geq \frac{\pi}{2}r^2$.

Proof. We start by noting that no bubbling is possible under assumption (36). As in the proof of Corollary 3.7, to prove our claim it is enough to show that for any *regular* pair (H, J) where $J \in \mathcal{J}_\omega$, $H : [0, 1]\times M \to \mathbb{R}$, $\phi_1^H(L) = L'$, and for any $x \in L$ there exists $u \in \mathcal{M}'_{L,L',J,0}$ such that $\int u^*\omega \leq \mathrm{Var}(H)$, $\mu(u) \leq n$ and $x \in \mathrm{Im}(u)$ (the relevant moduli spaces as well as the notion of regular pairs are defined in this setting in the same way as in §2.1).

We fix such a regular pair (H, J) and start to adapt the construction of $EF(L, L')$ to our new setting.

3.4.1. *The action functional.* Consider an additional Hamiltonian $G : [0,1] \times M \to \mathbb{R}$. We first verify that the action functional $\mathcal{A}_{L,L',G}$ from (8) continues to be well defined in our new setting. Indeed, let $\eta_0 = z_0 \in L \cap L'$ such that the path $(\phi_t^H)^{-1}(z_0)$ is null in $\pi_1(M, L)$. We need to show that if $\overline{x} : [0,1] \to \mathcal{P}_{z_0}(L, L')$ is a path joining z_0 to $x \in \mathcal{P}_{z_0}(L, L')$, then the expression in (8) only depends on x and not on $\overline{x}$. This obviously comes down to showing that $\int \overline{x}^*\omega$ only depends on x. For this we use a Hamiltonian isotopy $\phi^{H'}$ inverse of ϕ^H such that $\phi_1^{H'}(L') = L$ and the map $b_{H'} : \mathcal{P}_{z_0}(L, L') \to \mathcal{P}_*(L, L)$ defined by $(b_{H'}(x))(t) = \phi_t^{H'}(x(t))$ as in §2.1.3. By applying the action functional computation in §2.1.3 we see that proving that $\int \overline{x}^*\omega$ only depends on x is equivalent to showing that for any $\overline{x}' : [0,1] \to \mathcal{P}_*(L, L)$ with $\overline{x}'(0) = z_0' = b_{H'}(z_0)$ the integral $\int (\overline{x}')^*\omega$ only depends on $x' = \overline{x}'(1)$. But now as $[z_0'] = 0 \in \pi_1(M, L)$ we consider $d : [0,1] \to \mathcal{P}_*(L, L)$ which contracts z_0' to the constant path and if $\overline{x}''$ is a second path with the same ends as $\overline{x}'$, then we may consider the concatenation $d\#\overline{x}'\#(\overline{x}'')^{-1}\#d^{-1}$ and we notice that this represents geometrically a map defined on a disk whose boundary rests on L. Therefore, the integral of ω on this disk vanishes and this implies that $\int (\overline{x}')^*\omega = \int (\overline{x}'')^*\omega$.

3.4.2. *The Maslov index.* The main other difficulty that remains to be solved is that, when (36) replaces (1), the Maslov index of a strip $u \in \mathcal{M}(x, y)$ as defined in §2.1.2 does not only depend on the ends of the strip, x, y. The space $\mathcal{P}_{z_0}(L, L')$ is not in general simply connected. We let $\Pi = \pi_1(\mathcal{P}_{z_0}(L, L'))$. There is a natural morphism $\mu : \Pi \to \mathbb{Z}$ which is defined as follows: we fix $x \in I(L, L'; G)$ and we consider a C^∞ path $\gamma : [0,1] \to \mathcal{P}_{z_0}(L, L')$ such that $\gamma(0) = x = \gamma(1)$ and $[\gamma] = g \in \Pi$. We then put $\mu(g) = \mu(\gamma)$ (where $\mu(\gamma)$ is computed by viewing γ as a "strip" joining x to x and by using the method described in §2.1.2). It is easy to see that this is well-defined and that it defines a homomorphism. Let π be the image of this homomorphism (obviously π is isomorphic to $\mathbb{Z}$ or trivial) and let $\mathrm{Ker}(\mu)$ be its kernel. There exists a regular covering $\tilde{\mathcal{P}}$ of $\mathcal{P}_{z_0}(L, L')$ with covering projection $p : \tilde{\mathcal{P}} \to \mathcal{P}_{z_0}(L, L')$, covering group equal to π and such that $\pi_1(\tilde{\mathcal{P}}) = \mathrm{Ker}(\mu)$.

Denote $\tilde{I}(L, L', G) = p^{-1}(I(L, L'; \eta_0, G))$ and let $x, y \in \tilde{I}(L, L', G)$. We define $\mu(x, y) = \mu(u)$ where $u \in \mathcal{S}(p(x), p(y))$ verifies $u = p(u')$ with $u' : [0,1] \to \tilde{\mathcal{P}}$ a path joining x to y. Notice that with this definition we have $\mu(x, y) = \mu(gx, gy)$ and $\mu(gx, y) = \mu(g) + \mu(x, y)$ for any $g \in \pi$ (we consider that π acts on $\tilde{\mathcal{P}}$ on the left). Fix $x_0 \in \tilde{I}(L, L', G)$. We also define an absolute Maslov index for the points $y \in \tilde{I}(L, L', G)$ by letting $\mu(y) = \mu(y, x_0)$; clearly this depends on the choice of x_0.

We end this sub-subsection with a remark that will be useful later on. There exists a natural map $j_L : L \to \mathcal{P}_{z_0}(L, L')$ which is defined by $j_L(x) = \phi_t^H(x)$. This map has the property that $l \circ j_L = id_L$ where $l(\gamma) = \gamma(0)$. Therefore, we may view $\pi_1(L)$ as a subgroup of Π. The remark in question is that $j_L(\pi_1(L)) \subset \mathrm{Ker}(\mu)$. To verify this, notice that this property is homotopy invariant and so it is sufficient to check it in case L' is close to L and is the image of some df where $f : L \to R$ is some Morse function which is C^2-small. In this case the relative Maslov index agrees with the relative Morse index and so only depends on the ends of strips and not on the strips themselves.

3.4.3. *The moduli spaces.* We pursue the construction by defining for $x, y \in \tilde{I}(L, L', G)$ the moduli spaces

$$\mathcal{N}_{L,L',J,H}(x, y) = \{u \in C^\infty(\mathbb{R}, \widetilde{\mathcal{P}}) : (p \circ u) \in \mathcal{M}_{L,L',J,G}(p(x), p(y)),\\ u(-\infty) = x,\ u(+\infty) = y\}.$$

These moduli spaces behave in a way perfectly similar to the behaviour of $\mathcal{M}(x, y)$ (when condition (1) is satisfied) as described in §2.1.2, §2.1.3 and §2.1.4. In particular, the manifolds with corner structures of their compactifications remain true (the proof discussed in Appendix A applies in this case without modification). We also have, $0 < E_{L,L',G}(p \circ u) = \mathcal{A}(p(x)) - \mathcal{A}(p(y))$ when $u \in \mathcal{N}(x, y)$, $\mu(x, y) - 1 = \dim(\mathcal{N}(x, y))$, and the formula

$$\partial\overline{\mathcal{N}}(x, y) = \bigcup_z \overline{\mathcal{N}}(x, z) \times \overline{\mathcal{N}}(z, y)$$

remains valid. In particular, we claim that only a finite number of non-trivial terms appear in this union. This is because for any $B > 0$ and any $x, y \in I(L, L'; z_0, G)$ there is at most a finite number of homotopy classes $[u]$ of paths in $\mathcal{P}_{z_0}(L, L')$ that join x to y and are represented by strips $u \in \mathcal{M}(x, y)$ with $E_{L,L',G}(u) \leq B$ (otherwise, by passing to a convergent subsequence of strips, each of a different homotopy class, Gromov compactness would be contradicted). This means that, for any $x \in \tilde{I}(L, L', G)$, $y \in I(L, L'; \eta_0, G)$ there are at most a finite number of points $z \in p^{-1}(y)$ such that $\mathcal{N}(x, z) \neq \emptyset$. As the number of points in $I(L, L'; \eta_0, G)$ is finite this means that, for fixed $x \in \tilde{I}(L, L', G)$, there are only finitely many nonvanishing spaces $\mathcal{N}(x, z)$ which implies the claim.

The spaces $\mathcal{N}(x, y)$ have the additional property that they are equivariant in the sense that the left action of π induces a homeomorphism $\mathcal{N}(x, y) \xrightarrow{g} \mathcal{N}(gx, gy)$ for any $g \in \pi$.

3.4.4. *The extended Floer complex and the spectral sequence.* The next step is to construct an extended Morse complex by following the method in §2.2. We choose elements $\tilde{x} \in p^{-1}(x)$ for each $x \in I(L, L'; \eta_0, G)$ so that all the other elements in $p^{-1}(x)$ can then be uniquely written as $g\tilde{x}$, $g \in \pi$. We want to construct a representing chain system $s_{xy} \in S_{\mu(x,y)-1}\overline{\mathcal{N}}(x,y)$, $x, y \in \tilde{I}(L, L', G)$ for the moduli spaces $\mathcal{N}(x,y)$. We consider a finite, increasing sequence of strictly positive numbers $\Delta_1, \Delta_2, \ldots, \Delta_q$ such that for any $a, b \in I(L, L; G)$ there exists an i verifying $|\mathcal{A}(a) - \mathcal{A}(b)| = \Delta_i$. Assume by induction that the s_{xy} have been constructed for all x, y such that $\mathcal{A}(p(x)) - \mathcal{A}(p(y)) \leq \Delta_k$. We then consider a couple $\tilde{x}, y$ such that $\mathcal{A}(p(\tilde{x})) - \mathcal{A}(p(y)) = \Delta_{k+1}$ and $\mathcal{N}(\tilde{x}, y) \neq \emptyset$ (there are finitely many such couples as mentioned before). We proceed as in Lemma 2.2 to construct $s_{\tilde{x}y}$ and we then define $s_{(g\tilde{x})(gy)}$ to be $g(s_{xy})$.

To continue with the construction we fix a map $s : L \to X$ such that X is simply connected and s carries the 0-ends of the paths in $I(L, L'; G)$ to a distinguished base-point $*$ in X (in the original construction the role of X was played by $\tilde{L}$). In our applications s will be a degree-one map (and so $X = S^n$). We have an obvious map $\tilde{s} : \tilde{\mathcal{P}} \xrightarrow{p} \mathcal{P}_{z_0}(L, L') \xrightarrow{l} L \to X$ (where $l(\gamma) = \gamma(0)$). This map takes the representing chain system s_{xy} to cubical chains $u_{xy} \in S_*(\Omega' X)$. The advantage of using $S_*(\Omega' X)$ with X simply connected is that $\mathbb{Z}[\pi]$ acts trivially on $\Omega' X$. Moreover, this loop space is connected in contrast to ΩL if $\pi_1(L) \neq 0$. We also notice $u_{xy} = u_{(gx)(gy)}$ for all $g \in \pi$.

Denote by $\mathcal{R}^X_*$ the ring $S_*(\Omega' X)$ and let $\tilde{\mathcal{R}} = \mathcal{R}^X_* \otimes \mathbb{Z}[\pi]$ be graded as a tensor product and endowed with the differential coming from the first factor. For $x, y \in I(L, L', G)$ consider $v_{xy} \in \tilde{\mathcal{R}}$ defined by $v_{xy} = \sum u_{\tilde{x}g(\tilde{y})} \otimes g$. The extended Floer complex in this situation will be denoted by $\tilde{\mathcal{C}}^{J,\zeta}(L, L'; G)$. It is a free $\tilde{\mathcal{R}}$- chain complex with generators the elements of $I(L, L', G)$. Its grading is defined as follows. Recall that we have already fixed an absolute index for the points in $\tilde{I}(L, L', G)$. As the generators of $\mathbb{Z}[\pi] \otimes \mathbb{Z}/2\langle I(L, L'; G)\rangle$ are in bijection with the elements in $\tilde{I}(L, L', G)$ this absolute index gives a grading to $\mathbb{Z}[\pi] \otimes \mathbb{Z}/2\langle I(L, L', G)\rangle$ and the grading on $\tilde{\mathcal{C}}$ is the canonical tensor product grading (this is compatible with the action of $\tilde{\mathcal{R}}$ because $\mu(gx, y) = \mu(g) + \mu(x, y)$). Its differential is defined by $dx = \sum_y v_{xy} \otimes y$ (since $u_{xy} = u_{(gx)(gy)}$ it is easily seen that $d^2 = 0$). We denote by $|-|$ the grading defined before and we let $F^k = \{a \in \mathbb{Z}[\pi] \otimes \mathbb{Z}/2\langle I(L, L', G)\rangle : |a| \leq k\}$.

The spectral sequence in this more general setting, $\tilde{E}F(L, L'; G)$, is induced by the filtration $\tilde{F}^k\tilde{\mathcal{C}} = \mathcal{R}^X_* \otimes F^k$. This spectral sequence is not anymore a first quadrant spectral sequence in general but rather an upper semi-plane sequence. It is however a rather well-behaved spectral sequence because π acts by translation parallel to the x-axis on this sequence and, as consequence of the fact that $\tilde{C}$ is a free, finitely generated $\mathcal{R}^X_* \otimes \mathbb{Z}[\pi]$-module, the sequence is equivariant with respect to this action (in the sense that $d^r(ga) = gd^r a$).

3.4.5. *The Morse case.* It is easy to see that all the other properties of $EF(L, L'; G)$ — with the proofs provided in §2.3 — extend to the case of

$\tilde{E}F(L,L';G)$ without difficulty (the key point of course being that the action functional continues to be well-defined and it is equivariant with respect to the action of π). In particular, the pages of order greater than or equal to 2 of the spectral sequence are invariant up to translation and as the isomorphisms in question are also naturally π-equivariant we obtain that $\tilde{E}F^r(L,L'), r \geq 2$ is an invariant up to translation which is π-equivariant. In particular $\tilde{E}F^r(L,L')$ is isomorphic up to translation for $r \geq 2$ to the analogue spectral sequence arising from a Morse function on L. Therefore, the last stage consists in detecting what the output of this construction in the Morse function context (similar to §2.4.5) is. Consider the pull-back covering of L which is induced from $\tilde{\mathcal{P}} \to \mathcal{P}_{z_0}(L,L')$ by the map $j_L : L \to \mathcal{P}_{z_0}(L,L')$. We denote this covering by $p : \hat{L} \to L$. As $j_L(\pi_1(L)) \subset \mathrm{Ker}(\mu)$, this covering is trivial. Therefore, $\hat{L}$ is homeomorphic to $\pi \times L$.

Fix a Morse function $f : L \to \mathbb{R}$ and let $\hat{f} = f \circ p$. Moreover, consider the fibration $\Omega' X \to \hat{E} \to \hat{L}$ which is the pullback of the fibration $\zeta : \Omega' X \to P' X \to X$ over the map $s \circ p$ (recall that we have fixed $s : L \to X$ in Section 3.4.4). This fibration consists simply of π-copies of the fibration $\Omega' X \to E \to L$ which is the pull-back of ζ over s. Let EF_X be the Serre spectral sequence of this last fibration. The argument in §2.4.5 immediately implies that the spectral sequence constructed by following the method above in this Morse case - as described in Section 2.4.3 when (1) is satisfied - is isomorphic up to translation to the spectral sequence $\mathbb{Z}[\pi] \otimes EF_X$.

Therefore, when $r \geq 2$, we have a π-equivariant isomorphism up to translation between $\tilde{E}F^r(L,L')$ and $\mathbb{Z}[\pi] \otimes EF^r_X$.

3.4.6. *End of the proof.* Once the machinery above is constructed we take $X = S^n$ and $s : L \to S^n$ to be a degree-one map and the proof in Corollaries 3.7 and 3.10 i. proceed without change. □

Appendix A.
Structure of manifolds with corners on Floer moduli spaces

A.1. *Introduction.* Let (M, ω) be a symplectic manifold which is convex at infinity and let L and L' be two simply connected compact Lagrangian submanifolds in M. The symplectic form ω will be supposed to vanish on $\pi_2(M)$, so that there are no symplectic spheres in M, nor symplectic disks attached to L or to L'. Similarly, we assume that the first Chern class $c_1(M)$ vanishes on $\pi_2(M)$ so that the Maslov index of Floer trajectories only depends on the ends of the trajectories. Suppose moreover that L and L' intersect transversally.

The purpose of this section is to endow the moduli spaces, $\overline{\mathcal{M}}(x,y)$, of Floer trajectories - pseudo-holomorphic strips, in our case - with the topological structure of “manifolds with corners” (see Definition A.3). In [8], A. Floer introduced the gluing construction to treat the case of relative index 1. His work extends almost verbatim to the case of higher relative indexes, but some particular care is needed when the number of breaking points is bigger than one. In this case, Floer's argument — as he described it in [10] (Proposition 2d.1.) — only provides stratum by stratum homeomorphisms, i.e. local maps of the form

$$\mathcal{M}(x_0, x_{i_1}) \times \cdots \times \mathcal{M}(x_{i_r}, x_k) \times (0,1)^r \xrightarrow{\varphi_{i_1,\ldots,i_r}} \mathcal{M}(x_0, x_k),$$

instead of a map defined up to the boundary, i.e. a local map of the form

$$\mathcal{M}(x_0, x_1) \times \cdots \times \mathcal{M}(x_{k-1}, x_k) \times [0,1)^{k-1} \xrightarrow{\varphi} \overline{\mathcal{M}}(x_0, x_k)$$

where the "k-fold broken" trajectories are identified with elements of the form $(u_1, \ldots, u_k) \times \{0\}$ and the map φ provides a "cornered neighborhood" of these trajectories in the sense that φ preserves the natural stratifications on the two sides. To build this last map out of the former ones, some gluing compatibility conditions have to be fulfilled. Verifying these conditions is not obvious, in essence, because the gluing construction relies on an application of an implicit function theorem. The question of defining some structure on moduli spaces of pseudo-holomorphic curves, at least such as to produce a (virtual) fundamental class, appears as a key point in most applications of pseudo-holomorphic curves to symplectic geometry, and, in particular, in the definition of Gromov-Witten invariants.

This point has been treated (both in the context of Floer homology and Gromov-Witten invariants) by different authors ([22], [28], [12], [21], [33]) in a very general setting (allowing bubbles). However, this goes far beyond what is required for the present paper, and we have not been able to find in the literature a simple and explicit proof of the "manifold with corners" structure for the moduli spaces of pseudo-holomorphic strips. For this reason as well as to make the paper self-contained, we include one here.

We make use of now classical ideas and techniques introduced by different authors ([8], [24], [30], [34], [22]) and we shall follow rather closely the work of J.-C. Sikorav [34] about the gluing construction for compact Riemann surfaces, and adapt it to our strips. The more recent and much more general technique introduced in [15] offers a more conceptual approach to gluing problems of this type.

Recall that the action functional $\mathcal{A}$ is defined as follows : choose a path γ_0 from L to L', let $\mathcal{P}_{\gamma_0}(L, L')$ be the component of γ_0 in $\mathcal{P}(L, L')$ and for $\gamma \in \mathcal{P}_{\gamma_0}(L, L')$, set

$$\mathcal{A}(\gamma) = -\int_{[0,1]^2} \bar{\gamma}^* \omega$$

where $\bar{\gamma}$ is a path from γ_0 to γ in $\mathcal{P}(L, L')$.

Let J be an almost complex structure that tames ω. For $x_+, x_- \in L \cap L'$, a parametrized Floer trajectory from x_- to x_+ is a map $u : \Sigma \to M$, where

$$\Sigma = \{z = s + it \in \mathbb{C}, 0 \leq \mathrm{im}(z) \leq 1\}$$

such that

$$\bar{\partial}_J u := du + J(u) du\, i\ = 0, \tag{37}$$

$$u(\mathbb{R} \times \{0\}) \subset L \quad \text{and} \quad u(\mathbb{R} \times \{1\}) \subset L', \tag{38}$$

$$\lim_{s \to \pm\infty} u(s,t) = x_\pm(t). \tag{39}$$

To shorten notation, the product of a real interval with $[0,1]$ will be denoted by double brackets : $\Sigma =]\!] -\infty, +\infty [\![= (-\infty, +\infty) \times [0,1] \subset \mathbb{C}$.

A nonparametrized Floer trajectory is the orbit of a parametrized one under the action of $\mathbb{R}$ by translation in the s direction.

Let $\mathcal{M}'_J(x_-, x_+)$ be the space of all parametrized Floer trajectories from x_- to x_+:

$$\mathcal{M}'(x_-, x_+) = \{u : \Sigma \to M, (37)(38) \text{ and } (39)\}\,, \tag{40}$$

$$\mathcal{M}(x_-, x_+) = \mathcal{M}'(x_-, x_+)/\mathbb{R}, \tag{41}$$

$$\overline{\mathcal{M}}_J(x_-, x_+) = \text{ Gromov compactification of } \mathcal{M}(x_-, x_+). \tag{42}$$

Remark A.1. For technical reasons, it is sometimes useful to work with "time dependent" almost complex structures $(J_s)_{s\in\mathbb{R}}$. We will use families that are constant near $\pm\infty$. The equation (37) is then naturally replaced by

$$du(s,t) + J(s, u(s,t))\ du(s,t)\ \ i\ = 0. \tag{43}$$

Finally, Hamiltonian perturbations of L and L' are also needed, and to take advantage of particular Hamiltonians, we need to keep track of them. The action functional is then replaced by (8), and the J-holomorphy equation is replaced by the nonhomogeneous one :

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = -\nabla H(t, u(s,t)).$$

We will first deal with the homogeneous case, and add comments about the nonhomogeneous one in the last section.

Once more we fix notation: the linearization of the operator $\bar{\partial}_J : u \mapsto du + J du\, i$ at a map u such that $\bar{\partial}_J u = 0$ does not depend on the connection used to compute it: this defines an operator $D_u : \mathrm{L}^{1,p}(u^*TM) \to \mathrm{L}^p(\Lambda^{01} u^*TV)$. The J-holomorphic map u is said to be *regular*, if D_u is onto.

Remark A.2. The topological assumptions $\omega(\pi_2(M)) = 0$ and $c_1(\pi_2(M))$ are only used to obtain a global structure on the moduli spaces, but the underlying "gluing" construction is purely local and only relies on the regularity of the trajectories under consideration.

We start with a simple definition:

Definition A.3. A topological space X is said to have a structure of a manifold with corners if there is a partition $X = \sqcup_{i\in I} X_i$ of X into manifolds X_i of dimension i such that for each point $x \in X_i$ there is a neighbourhood U_x of x in X, and a local homeomorphism $\phi : U_x \to V_0 \subset \mathbb{R}^i \times [0,1)^{n-i}$ whose restriction to each $U_x \cap X_j$ is a local diffeomorphism to the j-dimensional stratum of $\mathbb{R}^i \times [0,1)^{n-i}$ (which is defined to be the disjoint union of the products of the form $\mathbb{R}^i \times I_1 \times I_2 \times \ldots \times I_{n-i}$ where I_k is either $\{0\}$ or $(0,1)$ and there are precisely $j-i$ nonzero terms).

We are interested in such a structure to obtain a homological representation of the stratification of $\overline{\mathcal{M}}(x,y)$, as described in Lemma 2.2.

THEOREM A.4. *For a generic choice of J, and two intersection points x and y with $x \neq y$, the space $\overline{\mathcal{M}}_J(x,y)$ admits a structure of a manifold with corners of dimension $\mu(x,y)-1$, whose k co-dimensional stratum is the space of trajectories broken at k intermediate points.*

This statement requires a topology on $\overline{\mathcal{M}}(x,y)$ which we recall now.

For convenience, if $(x_0,\ldots,x_k)$ is a sequence of intersection points of decreasing indexes, let

$$\mathcal{M}(x_0,\ldots,x_k) = \mathcal{M}(x_0,x_1) \times \cdots \times \mathcal{M}(x_{k-1},x_k)\ . \tag{44}$$

Consider a curve $C_\infty \in \mathcal{M}(x_0,\ldots,x_k)$, and a sequence C_n of curves in $\mathcal{M}(x_0,x_k)$.

To express the convergence of C_n to C_∞ in the sense of the Gromov topology, pick some parametrizations $u_\infty : \Sigma_\infty = \sqcup_{i=1}^k \Sigma_{\infty,i} \to M$ of C_∞ and $u_n : \Sigma \to M$ of C_n. Then, for each $i \in \{1,\ldots,k\}$, there is a unique $s_{n,i} \in \Sigma$ such that

$$\mathcal{A}(u_n(s_{n,i},\cdot)) = \mathcal{A}(u_{\infty,i}(0,\cdot)).$$

The sequence C_n is said to converge to C_∞ in the Gromov topology if, for all $i \in \{1,\ldots,k\}$, $u_n(\cdot - s_{n,i},\cdot)$ C^0 converges to $u_{\infty,i}$ on all compact subsets of Σ.

This definition naturally extends to broken trajectories: a sequence of broken trajectories converges, if the topology of the domain stabilizes and each smooth component converges in the previous sense.

A.2. *Sketch of the construction.* Let

$$C_\infty \in \mathcal{M}(x_0 \ldots x_k), \quad C_\infty = (C_{\infty,1},\ldots,C_{\infty,k})$$

be a regular curve, by which we mean that the linearization of the Cauchy Riemann equation (37) on each component is onto. Theorem A.4, will be proved by constructing a local chart centered at C_∞:

$$\phi : \begin{array}{ccc} \overline{\mathcal{M}}(x_0, x_k) & \to & (1, +\infty]^{k-1} \times T_{C_\infty}\mathcal{M}(x_0 \dots x_k) \\ C & \mapsto & (\rho(C), \pi_\rho(C)) \end{array} \tag{45}$$

satisfying the conditions of Definition A.3.

The first component of this map will be called the gluing parameter and is a small perturbation of the parameter ρ defined as follows: we choose one regular level a_i ($a_i = \frac{\mathcal{A}(x_{i-1})+\mathcal{A}(x_i)}{2}$ for instance) of the action functional between each pair of critical values $\mathcal{A}(x_i)$. Then we measure the time a trajectory needs to run from one level to the next one by setting

$$\rho : \begin{array}{ccc} \overline{\mathcal{M}}(x_0, x_k) & \to & (1, +\infty]^{k-1} \\ C & \mapsto & (2(s_2 - s_1), \dots, 2(s_k - s_{k-1})) \end{array} \tag{46}$$

(the only purpose of this 2 is to simplify future notation) where s_i is defined by $\mathcal{A}(u(s_i, \cdot)) = a_i$ for some parametrization $u : \bigsqcup_\alpha \Sigma_\alpha \to M$ of C, with the convention that $s_i - s_{i-1} = +\infty$ if s_i and s_{i-1} do not belong to the same component. Of course, each s_i depends on the choice of u, but the differences $s_i - s_{i-1}$ do not.

Remark A.5. Because of their geometric meaning, our gluing parameters will tend to $+\infty$ when the curve splits, and hence 0 is replaced by $+\infty$ and $[0, 1)$ by $(1, +\infty]$ in Definition A.3. As a consequence, a gluing parameter $\rho = (\rho_1, \dots, \rho_{k-1})$ will be considered as "large" if *all* its components are large, or equivalently, if $e^{-\rho}$ is small. In view of this, we define $|\rho| = \inf \rho_i$.

The second component of the local chart is less explicit. Let $\mathcal{M}_\rho$ denote the fiber of ρ. Then, it is easy to see that, for each ρ large enough, $\mathcal{M}_\rho$ is locally diffeomorphic to $T_{C_\infty}\mathcal{M}(x_0, \dots, x_k)$. The main difficulty is to control the dependence of these diffeomorphisms with respect to ρ. In particular, they need to be constructed in such a way that the map $\phi(C) = (\rho(C), \pi_\rho(C))$ is a homeomorphism on its image.

To achieve this we proceed as follows:

(1) We intend to define an inverse to ϕ. For this we start by defining a pre-gluing map $\rho \mapsto w_\rho$: use a gluing parameter ρ and cutoff functions to glue the different components of C_∞ into a map w_ρ, that is "approximately" a Floer trajectory matching the transit times ρ. Check that any "exact" Floer trajectory C, close enough to C_∞ and matching the transit times ρ, is in fact an $\mathrm{L}^{1,p}$-small perturbation of w_ρ.

(2) We then set up the J-holomorphy equation for perturbations $\xi \in \Gamma^{1,p}(w_\rho^* TM)$ of w_ρ as a nonlinear PDE, $\tilde{\partial}_{w_\rho}\xi = 0$. We check that for $\xi = 0$, $\tilde{\partial}_{w_\rho}(0)$ vanishes as $|\rho|$ tends to $+\infty$, and that the linearization D_ρ of this operator at $\xi = 0$ has a uniformly bounded right inverse. An

implicit function theorem "near infinity" ensures then that, for ρ large enough, the equation $\tilde{\partial}_{w_\rho}\xi = 0$ is regular and provides local diffeomorphisms ζ_ρ between $\mathcal{M}_\rho$ and $\ker D_\rho$.

(3) One difficulty at this point is that the topology of the base of $w_\rho^* TM$ strongly depends on ρ, making the spaces $\ker D_\rho$ difficult to compare. However, these bundles are almost the same "away from the nodes": restricted to compact sets that avoid the nodes, and pushed by parallel transport, their sections can be considered sections of one constant bundle. An L^2 projection on $\ker D_\infty = T_{C_\infty}\mathcal{M}(x_0,\dots,x_k)$ induces then an isomorphism $\pi : \ker D_\rho \to \ker D_\infty = T_{C_\infty}\mathcal{M}(x_0,\dots,x_k)$.

(4) Define $\pi_\rho = \pi \circ \zeta_\rho$. Check that the map (ρ, π_ρ) and its inverse, defined by gluing verify the conditions in Definition A.3.

Remark A.6. We identify below perturbations of w_ρ to sections of $w_\rho^*(TM)$ via the exponential diffeomorphism associated to a fixed metric $g_{L,L'}$ for which L and L' are totally geodesic.

A.3. *Pre-gluing.* We first introduce some notation for the gluing operation at the source level, which is summarized in Figure 1. Then the "pre"-gluing operation will be described.

A.3.1. *Gluing strips.* Let $\Sigma_\infty = \Sigma_{\infty,1} \sqcup \cdots \sqcup \Sigma_{\infty,k}$ denote the disjoint union of k copies of the standard strip, $\Sigma =]\!] -\infty, +\infty[\![$.

Consider now a gluing parameter $\rho = (\rho_1, \dots, \rho_{k-1}) \in (1, +\infty]^{k-1}$. We define $\Sigma_\rho = \Sigma_{\rho,1} \sqcup \cdots \sqcup \Sigma_{\rho,\alpha}$ as the disjoint union of $\alpha = 1 + \#\{i, 1 \le i \le k-1, \rho_i = +\infty\}$ copies of the standard strip which are obtained by gluing together pieces of Σ_∞ as described in 1. Explicitly, if $\rho_i \neq \infty$ we paste $\Sigma_{\infty,i} \backslash (]\!]\rho_i, +\infty]\!])$ and $\Sigma_{\infty,i+1} \backslash ([\![-\infty, -\rho_i]\!])$ by identifying $\{\rho_i\} \times [0,1] \subset \Sigma_{\infty,i}$ with $\{-\rho_i\} \times [0,1] \subset \Sigma_{\infty,i+1}$. The pasting function is the obvious translation on the first coordinate in $]\!] -\infty, +\infty[\![$ and the identity on the second. In case $\rho_i = \infty$ no gluing occurs between $\Sigma_{\infty,i}$ and $\Sigma_{\infty,i+1}$. In particular, if both ρ_i and ρ_{i+1} are infinite, then $\Sigma_{\infty,i+1}$ itself represents a component of Σ_ρ. We let $\theta_{\rho,i} :]\!] -\rho_{i-1}, \rho_i [\![\subset \Sigma_{\infty,i} \to \Sigma_\rho$ be the obvious inclusion and we put $s_i = \theta_{\rho,i}(0)$. Moreover, for $\delta > 0$, the neighbourhood of size δ of the "gluing" region will be denoted by $A_i^{(\delta)}$:

$$A_i^{(\delta)} =]\!] s_i + \rho_i - \delta, s_{i+1} - \rho_i + \delta [\![\tag{47}$$

(in other words, if $\rho_i = +\infty$, then $A_i^{(\delta)} = \emptyset$; otherwise $A_i^{\delta} =]\!] s_i + \rho_i - \delta, s_i + \rho_i + \delta [\![$).

A.3.2. *Pre-gluing maps.* Let $u_\infty : \Sigma_\infty \to M$ be the parametrization of our regular curve C_∞ which reaches level a_i at time 0 on each component. We now use the usual technique of cut-off functions to define an "almost" trajectory

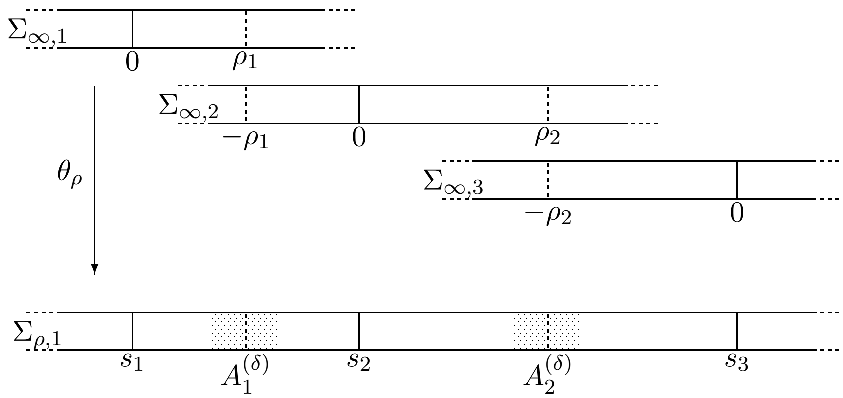

Figure 1: The strip Σ_ρ constructed from pieces of $\Sigma_{\infty,i}$.

$w_\rho : \Sigma_\rho \to M$ which is "close" to C_∞, and agrees with the transit times ρ. The construction is summarized in Figure 2.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\chi(t) = 1$ if $t \leq 0$ and $\chi(t) = 0$ if $t \geq 1$; given two distinct real numbers a and b, we define the function $\chi_b^a(t) = \chi(\frac{t-a}{b-a})$.

For $|s|$ large enough, $u_{\infty,i}(s,t)$ can be written in the form

$$\begin{cases} u_{\infty,i}(s,t) = \exp_{x_{i-1}}(\xi^-_{\infty,i}(s,t)) & (s << 0) \\ u_{\infty,i}(s,t) = \exp_{x_i}(\xi^+_{\infty,i}(s,t)) & (s >> 0) \end{cases}$$

where $\xi^+_{\infty,i} \in T_{x_i}M$, and $\xi^-_{\infty,i} \in T_{x_{i-1}}M$.

Then, for ρ large enough, we define $w_\rho : \Sigma_\rho \to M$ by setting, if $z = \theta_i(s,t)$:

$$w_\rho(z) = \begin{cases} \exp_{x_{i-1}}(\chi^{-(\rho_{i-1})/2+1}_{-(\rho_{i-1})/2}\ \xi^-_{\infty,i}(s,t)) & \text{if } s \leq -\frac{\rho_{i-1}}{2}+1 \\ u_{\infty,i}(s,t) & \text{if } -\frac{\rho_{i-1}}{2}+1 \leq s \leq \frac{\rho_i}{2}-1 \\ \exp_{x_i}(\chi^{\rho_i/2-1}_{\rho_i/2}\ \xi^+_{\infty,i}(s,t)) & \text{if } s \geq \frac{\rho_i}{2}. \end{cases} \tag{48}$$

Notice that, with this definition, w_ρ is constant around the gluing region, namely for $s_i + \rho_i/2 \leq s \leq s_{i+1} - \rho_i/2$.

This map w_ρ is as smooth as u_∞. It is J-holomorphic in the exterior of $\cup A_i^{(\rho_i/2+1)}$ and, because of the exponential decay [27] of u_∞ near each x_i, there are nonnegative constants A and λ such that

$$\|\bar\partial_J w_\rho(z)\|_{1,p} \leq A e^{-\lambda|\rho|} \tag{49}$$

A.4. *Holomorphic perturbations of w_ρ.*

A.4.1. *Some analytic properties of the standard Cauchy Riemann equation.* All the analytic results needed in the sequel concern the standard Cauchy

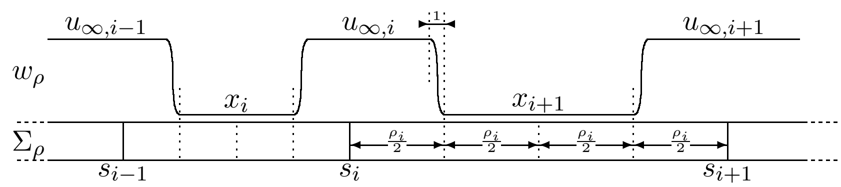

Figure 2: The pre-gluing construction.

Riemann equation applied to Σ in $\mathbb{C}^n$ with boundary conditions imposed by two transversal Lagrangian linear subspaces $\Lambda, \Lambda' \subset \mathbb{C}^n$. We gather these results here.

We first recall the following lemma from [32] (Theorem 3.1.13). It is relatively easy for $p = 2$ but much more delicate for $p > 2$:

Lemma A.7. *For all $p \geq 2$, there is a bounded operator P:*

$$\mathrm{L}^p(\Sigma, \mathbb{C}^n) \xrightarrow{P} \mathrm{L}^{1,p}((\Sigma, \partial\Sigma), (\mathbb{C}^n, \Lambda, \Lambda')) \tag{50}$$

such that $\bar{\partial} \circ P = \mathrm{Id}$. In particular, there exists a constant $c(p)$ such that for all $f \in \mathrm{L}^{1,p}((\Sigma, \partial\Sigma), (\mathbb{C}^n, \Lambda, \Lambda'))$:

$$\|f\|_{1,p} \leq c(p)\|\bar{\partial} f\|_p. \tag{51}$$

The proof of this lemma in [32] is given for tubes $\mathbb{R} \times S^1$ instead of strips, with appropriate deformation of the Cauchy-Riemann equation on the ends, but the boundary case is a strict analogue, where the invertibility assumption for the asymptotic operators on the ends is replaced by the transversality of Λ and Λ'.

We will also have to estimate the "growth" of holomorphic strips. The following lemma is a corollary of [27] in the particular case of an integrable almost complex structure. However we recall the proof to fix any ambiguity about constant dependency.

Lemma A.8. *Let Λ and Λ' be two transversal Lagrangian linear subspaces in $\mathbb{C}^n$, and $f : [0, +\infty) \times [0, 1] \to \mathbb{C}^n$ an $\mathrm{L}^{1,p}$ holomorphic strip such that $f([0, +\infty) \subset \Lambda$, $f([0, +\infty) + i) \subset \Lambda'$. Then there are constants C and $\delta > 0$, depending only on the relative positions of Λ and Λ' such that:*

$$\forall (s, t) \in [0, +\infty) \times [0, 1],\ \|f(s + it)\| \leq C\|f\|_{1,p} e^{-\delta s}.$$

Proof. First, note that $A = i\,\frac{\partial}{\partial t} : \mathrm{L}^{1,2}([0, 1], \mathbb{C}^n, \Lambda, \Lambda') \to \mathrm{L}^2([0, 1], \mathbb{C}^n)$ is a self-adjoint operator. Let $\alpha(s) = \int \|f(s, t)\|^2 dt$. Since both f and $\frac{\partial f}{\partial s}$ belong to $\mathrm{L}^{1,2}([0,1],\mathbb{C}^n,\Lambda,\Lambda')$, we have $\dot\alpha(s) = -2\langle f, Af\rangle$ and $\ddot\alpha(s) = 4\int_0^1 \|\frac{\partial f}{\partial t}\|^2 dy$. Moreover, since Λ and Λ' intersect transversally, A is bijective with bounded inverse (namely $A^{-1}(g) = \int_0^t -i\,g(x)dx - \pi_\Lambda(\int_0^1 -i\,g(x)dx)$ where π_Λ is the projection on Λ in the direction of Λ'). Hence there is a constant δ such that $\int_0^1 \|\frac{\partial g}{\partial y}\|^2 dy \geq \delta^2 \int_0^1 \|g\|^2 dy$ for all functions $g \in \mathrm{L}^{1,2}([0,1],\mathbb{C}^n,\Lambda,\Lambda')$.

From $\ddot\alpha \geq 4\delta^2\alpha$, we derive $\dot\alpha + 2\delta\alpha \leq 0$: otherwise, if $\beta(s_0) = \dot\alpha(s_0) + 2\delta\alpha(s_0) > 0$, then, as $\dot\beta \geq 2\delta\beta$, we have $\beta(s) > 0\ \forall s \geq s_0$, and $\beta(s) \geq \beta(s_0)e^{2\delta(s-s_0)}$. Then $\alpha(s) \geq Ke^{2\delta s} + B$ for some $K > 0$ which is impossible.

Therefore, $e^{2\delta s}\alpha$ is decreasing, and for $s \geq 1$, we have $\alpha(s) \leq \alpha(1)e^{-2\delta(s-1)}$, i.e. $\|f(s,\cdot)\|_2 \leq \|f(1,\cdot)\|_2 e^{-\delta(s-1)}$. The same argument applied to $\frac{\partial f}{\partial s}$ (which is also holomorphic and verifies the needed boundary conditions) leads to the estimate

$$\|f(s,\cdot)\|_{1,2} \leq \|f(1,\cdot)\|_{1,2}\ e^{-\delta(s-1)}.$$

Now, using Sobolev embedding, we have $\|f(s,t)\| \leq K_1\|f(s,\cdot)\|_{1,2}$, and on the other hand, since f is holomorphic in $[\![0,2]\!]$, Schwarz' lemma implies $\|f(1,\cdot)\|_{1,2} \leq K_2\|f\|_{\infty,[\![0,2]\!]} \leq K_2\|f\|_{1,p}$ with a uniform constant K_2. Finally, there is a uniform constant C such that, for $s \geq 1$:

$$\|f(s,t)\| \leq C\|f\|_{1,p}\ e^{-\delta s}.$$

The existence of such constants for $0 \leq s \leq 1$ is obvious and so this ends the proof of the lemma. □

We will also need a bounded version of this lemma, which is a sort of maximum principle:

LEMMA A.9. *Let Λ and Λ' be two transversal Lagrangian linear subspaces in $\mathbb{C}^n$, and $f : [\![a,b]\!] \to \mathbb{C}^n$ a holomorphic strip such that $f([a,b]) \subset \Lambda$, $f([a,b]+i) \subset \Lambda'$, $b-a > 2$. Then there are constants C and $\delta > 0$, depending only on the relative positions of Λ and Λ' such that*:

$$\forall (s,t) \in [a,b]\times[0,1],\ \|f(s+it)\| \leq C\|f\|_{\infty([\![a,a+2]\!]\cup[\![b-2,b]\!])}e^{-\delta\min(s-a,b-s)}.$$

Proof. As before, let $\alpha(s) = \int_0^1 \|f(s,t)\|^2 dt$, and $\beta_+ = \dot\alpha + 2\delta\alpha$. Let $a' = \inf\{s \in [a,b], \beta_+(s) > 0\} \cup \{b\}$. Then $\beta_+ \leq 0$ on $[a,a']$ and α has an exponential decay: $\forall s \leq a', \alpha(s) \leq \alpha(a)e^{-2\delta(s-a)}$. In the same way, let $\beta_- = \dot\alpha - 2\delta\alpha$. Then $\dot\beta_- \geq -2\delta\dot\beta_-$, so that if $\beta_-(s_0) < 0$ then $\beta_-(s) < 0$ for all $s < s_0$, and when $b' = \sup\{s, \beta_-(s) < 0\} \cup \{a\}$, α has an exponential growth on $[b',b]$: $\forall s \in [b',b], \alpha(s) \leq e^{-2\delta(b-s)}\alpha(b)$. On $[a,a'] \cup [b',b]$, we have

$$\alpha(s) \leq e^{-2\delta(s-a)}\alpha(a) + e^{-2\delta(b-s)}\alpha(b). \tag{52}$$

If $a' < b'$, we still have to deal with $[a',b']$: recall that $e^{-2\delta s}\beta_+(s)$ is increasing, so that $\beta_+(s) \leq e^{-2\delta(b'-s)}\beta_+(b')$, and integrating once more, $e^{2\delta s}\alpha(s) \leq e^{2\delta a'}\alpha(a') + \frac{e^{-2\delta(b'-2s)}}{4\delta}\beta_+(s)$. Finally, $\alpha(s) \leq e^{-2\delta(s-a')}\alpha(a') + \frac{e^{-2\delta(b'-s)}}{4\delta}\beta_+(b')$.

Moreover, on $[a', b']$ we have $-2\delta\alpha \leq \dot{\alpha} \leq 2\delta\alpha$, and hence $\beta_+(b') \leq 4\delta\alpha(b')$, and (52) still holds for $s \in [a', b']$.

Finally, the same argument applied to $\frac{\partial f}{\partial s}$ gives the estimate

$$\|f(s,\cdot)\|_{1,2} \leq \|f(a,\cdot)\|_{1,2}\, e^{-\delta(s-a)} + \|f(b,\cdot)\|_{1,2}\, e^{-\delta(b-s)} \ . \tag{53}$$

The proof then ends as in the previous lemma, when we replace a and b by $a+1$ and $b-1$, and using Schwarz lemma. □

A.4.2. $\mathrm{L}^{1,p}$-*smallness.* Let C be a Floer trajectory close enough to C_∞ and let $\rho = \rho(C)$. Let $u : \Sigma_\rho \to M$ be the parametrization of C such that, on each component, the first level a_i which is encoutered is reached at time $s = 0$. Then, u can be written in the form:

$$u(s,t) = \exp_{w_\rho(s,t)}(\xi(s,t))$$

where ξ is a section of $w_\rho^* TM$, satisfying appropriate boundary conditions, $\mathcal{A}\big((\exp_{w_\rho}\xi)(s_i,\cdot)\big) = a_i$, and is small in L^∞ norm. We want to prove that ξ is also small in $\mathrm{L}^{1,p}$-norm.

LEMMA A.10. *For all $\varepsilon > 0$, there exist constants $R, \eta, \eta' > 0$ such that if $C \in V_{\eta,\eta',R}(C_\infty)$ (i.e. $|\rho| > R$, $\|\xi\|_\infty < \eta$ and $\|\xi\|_{\mathrm{C}^1(\theta_{\rho,i}([\![-R,R]\!]))} < \eta'$), then $\|\xi\|_{1,p} < \varepsilon$.*

Proof. Consider a small neighbourhood $U_\eta = \cup U_i$ of the points x_i, and a large $R > 0$ such that $u_{\infty,i}(]\!]-\infty,-R[\![) \subset U_{i-1}$ and $u_{\infty,i}(]\!]R,+\infty[\![) \subset U_i$. For η' small enough, we have

$$\|\xi_{|\theta_{\rho,i}([\![-R,R]\!])}\|_{1,p} < \varepsilon.$$

So we restrict attention now to the neighbourhood U_i of x_i: $\|\xi\|_{1,p}$ has to be estimated on $[\![s_i + R, s_{i+1} - R]\!]$, or after a translation by $-s_i - \rho_i$, on $A_i^{(\rho_i - R)} = [\![-\rho_i + R, \rho_i - R]\!]$ (we suppose $\rho_i < +\infty$; the other case is very similar).

Using a local chart, U_i can be identified with a ball B of $\mathbb{C}^n$ so that L and L' are identified with two Lagrangian linear subspaces intersected with B and, moreover, the corresponding induced almost complex structure J coincides with the standard almost complex structure at the origin. Indeed, it is standard that there is a symplectic chart for U_i which identifies $L \cap U_i$ and $L' \cap U_i$ with $B \cap \mathbb{R}^n$ and $B \cap i\mathbb{R}^n$, respectively. By composing with a linear symplectic map we insure that the condition on J is satisfied and $L \cap U_i$, $L' \cap U_i$ are still identified with linear Lagrangians (obviously, not orthogonal in general). In such a chart, the (almost complex) Cauchy-Riemann equation becomes

$$\bar{\partial} u + q(u)\partial u = 0, \tag{54}$$

where $q = (J + i\,)^{-1}(J - i\,)$ satisfies $q(0) = 0$ and $\bar{\partial}$ is, of course, associated to the standard complex structure in $\mathbb{C}^n$, i.

After possibly rescaling a smaller neighbourhood to the unit ball in $\mathbb{C}^n$, we may assume that $\|q\|_{\mathrm{C}^1}$ is as small as needed.

Notice that the relation

$$\exp_{w_\rho} \xi = w_\rho + \xi'$$

defines a new map $\xi' : [\![-\rho_i + R, \rho_i - R]\!] \to \mathbb{C}^n$ still satisfying appropriate boundary conditions. Since estimating $\|\xi\|_{1,p}$ as a section of $w_\rho^* TM$ is equivalent to estimating $\|\xi'\|_{1,p}$ (as a $\mathbb{C}^n$-valued function), we still denote this ξ' by ξ in the sequel. Thus, ξ is now seen as a map to $\mathbb{C}^n$ instead of a section of $w_\rho^* TM$ and $w_\rho + \xi$ is J-holomorphic. Multiplying ξ by appropriate cut-off functions, we obtain

$$\hat{\xi} = \chi_{-\rho+R}^{-\rho+R+1} \chi_{\rho-R}^{\rho-R-1} \xi$$

which is defined on the whole strip Σ_ρ, satisfies the boundary conditions and belongs to $\mathrm{L}^{1,p}(]\!] -\infty, +\infty[\![)$. Lemma A.7 gives the estimate:

$$\|\hat{\xi}\|_{1,p} \leq c \|\bar{\partial} \hat{\xi}\|_p. \tag{55}$$

Moreover, ξ and $\hat{\xi}$ coincide on $A_i^{(\rho_i - R - 1)}$ away from a neighbourhood of the ends, where $\|\xi\|_{\mathrm{C}^1}$ is controlled by η': therefore $\|\xi - \hat{\xi}\|_{1,p} \leq 2\|\chi\|_{\mathrm{C}^1}\eta' \leq 4\eta'$, so that

$$\|\xi\|_{1,p} \leq c\|\bar{\partial}\xi\|_p + C\eta'. \tag{56}$$

To estimate $\|\bar{\partial}\xi\|_p$, write (54) for $u = w_\rho + \xi$, and compare with $\alpha = \bar{\partial} w_\rho + q(w_\rho)\partial w_\rho$:

$$\bar{\partial}\xi + \big(q(w_\rho + \xi) - q(w_\rho)\big)\partial w_\rho + q(w_\rho + \xi)\partial\xi = -\alpha \tag{57}$$

Recall from (49) that α is small in L^p- norm for ρ large enough. Developing $q(w_\rho + \xi) - q(w_\rho)$ in the form $a(z)\xi$, where $\|a\|_\infty$ is controlled by $\|q\|_{\mathrm{C}^1}$, and observing that $\|\partial w_\rho\|_\infty$ is uniformly bounded, (57) becomes

$$\|\bar{\partial}\xi\|_p \leq \|\alpha\|_p + a\|\xi\|_{1,p} \tag{58}$$

where a is a constant as small as desired. Collecting (58) and (56), we obtain:

$$\big(1 - a\,c\big)\|\xi\|_{1,p} \leq \|\beta\| + C\eta'. \tag{59}$$

Choosing our neighbourhoods U_i small enough so that $a\,c < 1$, we obtain the desired estimate

$$\|\xi\|_{1,p} \leq \varepsilon \tag{60}$$

for η, η', R small/large enough. □

A.4.3. *The Floer equation for perturbations of w_ρ.* Let $\Gamma_\rho^{1,p}(w_\rho^* TM)$ be the linear Banach space of $\mathrm{L}^{1,p}$ sections ξ of $w_\rho^* TM$ satisfying the boundary conditions and so that for all i

$$\int_0^1 \omega(\frac{dw_\rho}{dt}(s_i,t), \xi(s_i,t))dt = 0 \tag{61}$$

(this condition is the linear version of $\mathcal{A}(\exp_{w_\rho} \xi(s_i,\cdot)) = a_i$). Let $\mathcal{H}_i$ be the (local) exponential image of those C^1 paths ξ (depending on t only) which verify (61). This is a smooth hypersurface in $\mathcal{P} = C^1_{L,L'}([0,1],M)$ (independent of ρ) and, by using the implicit function theorem in a C^1 setting and on a bounded portion of the strip, we see that each Floer trajectory C in a sufficiently small, fixed neighbourhood of C_∞ crosses these hypersurfaces, thus defining "linear" transit times $\tilde\rho(C)$ so that $|\tilde\rho(C) - \rho(C)| \le K\|\xi\|_{C^1(\theta_{\rho,i}([\![-R,R]\!]))}$ for constants K and R independent of C. Thus, for each such curve there is one and only one $\tilde\rho(C) > 0$ and $\xi \in \Gamma^{1,p}_{\tilde\rho}(w^*_{\tilde\rho}TM)$ such that $u = \exp_{w_{\tilde\rho}}(\xi)$ is a parametrization of C.

Moreovoer, a section $\xi \in \Gamma^{1,p}_\rho(w^*_\rho TM)$ defines a Floer trajectory if and only if it satisfies a nonlinear PDE, $\tilde\partial\xi = 0$, which is the translation in terms of ξ of the usual Floer equation (37) for $u = \exp_{w_\rho}\xi$:

$$\bar\partial_J[\exp_{w_\rho}\xi] = 0.$$

This expression takes values in $\Gamma(\Omega^{0,1}u^*TM)$. Using a J-hermitian connection, parallel transport along geodesics of $g_{L,L'}$ defines an isomorphism $\Pi_J : \Gamma(\Omega^{0,1}u^*TM) \to \Omega^{0,1}(w^*_\rho TM)$. Finally, $\tilde\partial_{w_\rho}$ is defined as:

$$\tilde\partial_{w_\rho} : \begin{array}{ccc} \Gamma_\rho(w^*_\rho TM) & \to & \Omega^{0,1}(w^*_\rho TM) \\ \xi & \mapsto & \Pi_J\Big(\bar\partial_J\big(\exp_{w_\rho}\xi\big)\Big). \end{array}$$

This map $\tilde\partial_{w_\rho}$ is as smooth as J, and one easily checks that all its derivatives depend only on the derivatives of J and g, and are bounded independently of ρ. In particular, there is a constant A_2 such that, for all ρ large enough:

$$\|\tilde\partial_{w_\rho}\|_{C^2} \le A_2. \tag{62}$$

Moreover, for $\xi = 0$, (49) translates to:

$$\|\tilde\partial_{w_\rho} 0\| \le Ae^{-\lambda|\rho|}. \tag{63}$$

Let D_ρ denote the linearisation of $\tilde\partial_{w_\rho}$ at $\xi = 0$. Then D_ρ is Fredholm, and $\mathrm{ind}D_\rho = \mathrm{ind}(C_\infty)$. Finally, since the initial curve C_∞ is supposed to be regular, for $\rho = (+\infty,\dots,+\infty)$, D_∞ is onto.

A.4.4. *Uniformly bounded right inverse.*

Proposition A.11. *For ρ large enough, the operator D_ρ has a right inverse P_ρ, uniformly bounded with respect to ρ:*

$$\exists C > 0,\ \forall\rho > 0, \forall\alpha \in \Omega^{0,1}(w^*_\rho TM) \quad \|P_\rho\alpha\|_{1,p} \le C\|\alpha\|_p. \tag{64}$$

Proof. To construct a right inverse for D_ρ, we want to look at it as a perturbation of D_∞. Unfortunately, D_ρ acts on $w_\rho^* TM$ and the base of these bundles strongly depends on ρ. To work around this difficulty, D_ρ has first to be brought into $w_\infty^* TM$, where it will be compared to D_∞.

To this end, consider the map:

$$u_\rho : \Sigma_\infty \to M$$

obtained by multiplying u_∞ by appropriate cutoff-functions to make it constant away from $[\![\frac{\rho_{i-1}}{2}, \frac{\rho_i}{2}]\!]$ on each component:

$$u_{\rho,i}(s,t) = \begin{cases} \exp_{x_{i-1}}(\chi_{-(\rho_{i-1})/2}^{-(\rho_{i-1})/2+1} \xi_i^-) & \text{if } s < -\frac{\rho_{i-1}}{2} + 1 \\ u_{\infty,i}(s,t) & \text{if } -\frac{\rho_{i-1}}{2} + 1 \le s \le \frac{\rho_i}{2} - 1 \\ \exp_{x_{i+1}}(\chi_{\rho_i/2}^{\rho_i/2-1} \xi_i^+) & \text{if } s > \frac{\rho_i}{2} - 1. \end{cases}$$

Notice that u_ρ and w_ρ coincide where they are not constant. As before, the Cauchy-Riemann equation for perturbations of u_ρ (satisfying (61) and the boundary conditions) leads to a PDE:

$$\tilde{\partial}_{u_\rho} : \Gamma_\rho(u_\rho^* TM) \to \Omega^{0,1}(u_\rho^* TM).$$

Let $\tilde{D}_\rho$ be its linearization at 0. For $\rho = \infty$ (i.e. $\rho_i = +\infty, \forall i$), $\tilde{D}_\infty$ is just the usual linearisation D_∞ of $\bar{\partial}_J$ at u_∞. Therefore, it is onto (for a generic choice of J). For ρ large enough, parallel transport (using $g_{L,L'}$ on the left to preserve boundary conditions and a J-hermitian connection on the right to preserve $(0,1)$ forms) induces isomorphisms

$$\begin{array}{ccc} \Gamma_\rho(u_\rho^* TM) & \xrightarrow{\tilde{D}_\rho} & \Omega^{0,1}(u_\rho^* TM) \\ \Pi_{L,L'} \big\downarrow & & \big\downarrow \Pi_J \\ \Gamma_\infty(u_\infty^* TM) & \xrightarrow{\tilde{D}_\infty} & \Omega^{0,1}(u_\infty^* TM) \end{array} \tag{65}$$

and $\tilde{D}_\rho$ becomes a continuous family of operators on $\Gamma(u_\infty^* TM)$. Thus, for ρ large enough, $\tilde{D}_\rho$ has a right inverse $\tilde{R}_\rho$, which is uniformly bounded and continuous in ρ.

Now we want to come back to D_ρ. Consider the map R_ρ:

$$\Omega^{0,1}(w_\rho^* TM) \xrightarrow{\text{cut}_\rho} \Omega^{0,1}(u_\rho^* TM) \xrightarrow{\tilde{R}_\rho} \Gamma_\rho(u_\rho^* TM) \xrightarrow{\text{glue}_\rho} \Gamma_\rho(w_\rho^* TM)$$

where cut_ρ is the extension by 0 away from $\rho_{i-1} < s < \rho_i$ on each component:

$$\forall (s,t) \in \Sigma_{\infty,i} : \ \text{cut}_\rho(\alpha)(s,t) = \begin{cases} \alpha(\theta_\rho^{-1}(s,t)) & \text{if } \rho_{i-1} \le s \le \rho_i \\ 0 & \text{otherwise,} \end{cases}$$

and glue_ρ is the following gluing operation, where the different components overlap on some region: for $(s,t) = \theta_\rho(z) \in \Sigma_{\infty,i}$, set

$$\text{glue}_\rho(\xi)(z) = \begin{cases} \xi_i(s,t) + \chi_{-(\rho_{i-1})/2}^{-(\rho_{i-1})/2-1}\ \xi_{i-1}(s',t) & \text{if } -\rho_{i-1} \le s \le -\rho_{i-1}/2 \\ \xi_i(s,t) & \text{if } -(\rho_{i-1})/2 \le s \le \rho_i/2 \\ \xi_i(s,t) + \chi_{\rho_i/2}^{\rho_i/2+1}\xi_{i+1}(s'',t) & \text{if } \rho_i/2 \le s \le \rho_i(z - t_{i+1}) \end{cases}$$

where $s' = s + s_i - 2\rho_{i-1} - s_{i-1}$ and $s'' = s + s_i + 2\rho_i - s_{i+1}$, so that (s',t) and (s'',t) are the value of z seen in $\Sigma_{\infty,i-1}$, and $\Sigma_{\infty,i+1}$ (see Figure 3).

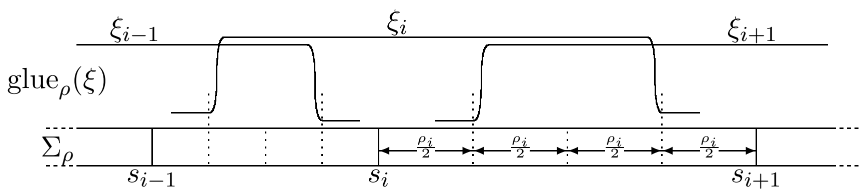

Figure 3: The glue_ρ map.

Since the three operators cut_ρ, $\tilde{R}_\rho$ and glue_ρ are uniformly bounded in ρ, so is R_ρ.

The proof of the proposition is finished by the next lemma.

Lemma A.12. *The operator R_ρ is a quasi inverse for D_ρ, in the sense that*:

$$\lim_{|\rho|\to+\infty} \|D_\rho \circ R_\rho - \text{Id}\| = 0.$$

Proof. Let $\alpha \in \Omega^{01}(w_\rho^* TM)$ and $\beta = D_\rho R_\rho \alpha - \alpha$. We have to estimate $\|\beta\|_p$. Notice that β is supported on the gluing regions $[\![s_i + \rho_i/2, s_{i+1} - \rho_i/2]\!]$. Let us focus on one half of such a region $[\![s_i + \rho_i/2, s_i + \rho_i]\!]$. Let $(\alpha_1, \dots, \alpha_k) = \text{cut}_\rho \alpha$ and $\eta = (\eta_1, \dots, \eta_k) = \tilde{R}_\rho(\text{cut}_\rho(\alpha))$. We have:

$$\begin{aligned} \beta(z) &= D_\rho(\text{glue}_\rho \eta)(z) - \alpha(z) \\ &= D_\rho\big(\eta_i(s,t) + \chi_i \eta_{i+1}(s'',t)\big) - \alpha(z). \end{aligned}$$

Notice that on the domain under consideration (and modulo appropriate translation) w_ρ, $u_{i,\rho}$, and $u_{i+1,\rho}$ are all constant, so that $D_\rho = \tilde{D}_\rho = \bar{\partial}$, where $\bar{\partial}$ is the usual Cauchy-Riemann operator associated to the complex vector space $(T_{x_{i+1}}M, J(x_{i+1}))$. Finally, we obtain

$$\beta(z) = \alpha_i(s,t) + (\bar{\partial}\chi)\eta_{i+1}(s'',t) + \chi\alpha_{i+1}(s'',t) - \alpha(z)$$

But $\alpha_i(s,t) = \alpha(z)$ and $\alpha_{i+1}(s'',t) = 0$. Hence: $\beta(z) = (\bar{\partial}\chi)\eta_{i+1}(s'',t)$ and

$$\|\beta\|_p \le A\|\eta_{i+1}\|_{\infty,[\![-\rho_i/2,-\rho_i/2+1]\!]}. \tag{66}$$

Moreover, we have $\bar\partial\eta_{i+1} = 0$ on $[\![-\infty, 0]\!]$, so that applying Lemma A.8 to η_{i+1}, we obtain that $\|\beta\|_p \leq C\|\eta_{i+1}\|_{1,p}e^{-\delta\rho_i/2} \leq C'\|\alpha\|_p e^{-\delta\rho_i/2}$ with uniform constants C and δ. Gathering all these inequalities, we obtain uniform constants C and δ, such that

$$\|D_\rho R_\rho\alpha - \alpha\|_p \leq Ce^{-\delta|\rho|}\ \|\alpha\|_p. \qquad \square$$

This ends the proof of Proposition A.11, since, for ρ large enough, $D_\rho \circ R_\rho$ is uniformly invertible and we can set $P_\rho = R_\rho \circ (D_\rho \circ R_\rho)^{-1}$. $\square$

A.4.5. *Isomorphism from* $\ker D_\rho$ *to* $\ker D_\infty$. Finally, we need to identify all $\ker D_\rho$ to the constant space $\ker D_\infty$. To this end, consider a small neighbourhood U of the intersection points x_i, and a compact subset $K = \bigsqcup_{i=1}^k [\![-R, R]\!]_i \subset \Sigma_\infty$ with R large enough to have $u_\infty(\Sigma_\infty \setminus K) \subset U$. Then we have ([34]):

Proposition A.13. *Let* $\pi : \Gamma(w_\rho^* TV) \to \Gamma(w_\rho^* TV_{|_K}) \to \ker D_\infty$ *be the* L^2 *orthogonal projection on* $\ker D_\infty$. *Then, for* $|\rho|$ *large enough, the restriction of* π *to* $\ker D_\rho$ *is an isomorphism, and there is a constant* C, *uniform with respect to* ρ, *such that*

$$\|\xi\|_{1,p} \leq C\|\pi(\xi)\|_2.$$

Proof. Suppose there is a sequence (ρ_n, ξ_n) such that $D_{\rho_n}\xi_n = 0$, $\|\xi_n\|_{1,p} = 1$ and $\lim \pi(\xi_n) = 0$. Then a subsequence of ξ_n converges on all compacts to a section $\xi \in \ker D_\infty$. On the other hand, $\pi(\xi_n)$ converges to $\pi(\xi)$, and hence $\xi_{|_K} \in (\ker D_\infty)^\perp$. Hence, $\xi_{|_K} = 0$ and ξ_n converges to 0 uniformly on K in the C^∞ topology.

We will derive a contradiction from Lemma A.9, which implies that $\|\xi_n\|_{1,p}$ on the complement of K is controlled by the behaviour of ξ on the boundary and, therefore, should tend to 0 as n goes to infinity.

To make this explicit, let us focus on one component $[-\rho_{n,i} + R, \rho_{n,i} - R]$ of the complement of K. On this piece of strip, w_ρ takes values in a small neighbourhood of x_i where we choose coordinates in which L and L' are linear spaces and $J(0) = i$ as in the proof of A.10. The Cauchy-Riemann equation takes the form

$$\bar\partial u + q(u)\partial u = 0.$$

As before, we can replace ξ_n by ξ'_n with $\exp_{w_{\rho_n}} \xi_n = w_{\rho_n} + \xi'_n$, keeping control on the $\mathrm{L}^{1,p}$ norm. Then ξ_n satisfies an equation of the form:

$$\bar\partial\xi_n + q_n(z)\partial\xi_n + a_n(z)\xi_n = 0$$

where $q_n(z) = q(w_{\rho_n})$ and $a_n(z)(\cdot) = D_{w_{\rho_n}} q(\cdot)\partial w_{\rho_n}$ are uniformly small.

Using the operator P defined in Lemma A.7, let $\eta_n = \xi_n + P(q_n(z)\partial\xi_n + a_n(z)\xi_n)$ (notice that the operator P of the lemma is defined on the full strip, so to be more precise, we first extend $q_n\partial\xi_n + a_n\xi_n$ by 0 and then consider the restriction of its image under P to our piece of strip). This section η_n is holomorphic, and takes values in L and L' on the boundary. Moreover, we obtain a uniformly small constant κ such that, on the relevant piece of strip:

$$\|\eta_n - \xi_n\|_{1,p} \leq \kappa\|\xi_n\|_{1,p}.l \tag{67}$$

As a consequence, near the ends of our piece of strip $[\![-\rho_{n,i} + R, \rho_{n,i} - R]\!]$, $\|\eta_n\|_\infty$ is arbitrarily small. According to Lemma A.9, we get

$$\|\eta_n\|_\infty \leq C\|\eta_n\|_{\infty,A}e^{-\delta(\rho_n - R - |s|)},$$

where

$$A = [\![-\rho_{n,i} + R, -\rho_{n,i} + R + 2]\!] \cup [\![\rho_{n,i} - R - 2, \rho_{n,i} - R]\!].$$

Integrating this, we get $\|\eta_n\|_p \leq C\|\eta_n\|_{\infty,A}$ for some uniform constant C. In the same way, the Schwartz lemma provides a control of $\partial\eta_n$ near the ends by means of $\|\eta_n\|_\infty$, from where we derive a similar estimate $\|\partial\eta_n\|_p \leq C\|\eta_n\|_{\infty,A}$.

Finally, $\|\eta_n\|_{1,p}$ is arbitrarily small on $[\![-\rho_{n,i}+R, \rho_{n,i}-R]\!]$, which, in view of (67), contradicts $\|\xi_n\|_{1,p} = 1$. □

A.4.6. *End of the proof.* We will use the following version of the implicit function theorem:

PROPOSITION A.14. *Let $(f_\lambda : E_\lambda \to F_\lambda)_{\lambda\in[0,1[^m}$ be a family of maps between Banach spaces such that*

(1) *for all $\lambda > 0$, f_λ is of class C^2, and $\|f_\lambda\|_{\mathrm{C}^2}$ is uniformly bounded,*

(2) $\lim_{\lambda\to 0} f_\lambda(0) = 0$,

(3) *$Df_\lambda(0)$ is uniformly invertible: $\exists R_\lambda \in L(F_\lambda, E_\lambda), Df_\lambda(0) \circ R_\lambda = \mathrm{Id}$ and $\exists C > 0, \forall\lambda\ \|R_\lambda\| \leq C$. Let $H_\lambda = R_\lambda(F_\lambda)$.*

Then there exists $\varepsilon > 0$ such that for all λ with $|\lambda| < \varepsilon$, there are "uniform" open subsets $U_\lambda \subset \ker Df_\lambda(0)$ and $V_\lambda \subset E_\lambda$, and a diffeomorphism $\varphi_\lambda : U_\lambda \to H_\lambda$ such that, in the decomposition $E_\lambda = \ker Df_\lambda \oplus H_\lambda$, one has:

$$f_\lambda(x, y) = 0 \Leftrightarrow y = \psi_\lambda(x).$$

Here, uniform means that the U_λ all contain a ball whose radius is independent of λ.

Proof. Since $Df_\lambda(0)$ has a right inverse, Df_λ remains onto on a neighbourhood V_λ of 0 whose size is controlled by $\|R_\lambda\|$ and $\|f_\lambda\|_{\mathrm{C}^2}$. The conditions 1 and 3 imply that this size is uniform in λ. Finally, the Newton algorithm proves that $f_\lambda^{-1}(0)\cap V_\lambda \neq \emptyset$ for λ small enough, and the usual implicit function theorem gives the result. □

The three conditions in this proposition have been checked for

$$\tilde{\partial}_\rho : \Gamma^{1,p}_\rho(w^*_\rho TM) \to \Gamma^p(\Lambda^{0,1}\Sigma_\rho \otimes w^*_\rho TM)$$

in (62) and (63) and Proposition A.11.

However, to have better control on the spaces H_ρ, we will use a slightly modified right inverse for D_ρ. Recall the compact K used in Proposition A.13, and consider the restriction map $L^{1,p}(\Sigma_\rho) \xrightarrow{r} L^{1,p}(K)$ and the projection $L^{1,p}(K) \xrightarrow{\pi_\rho} \ker D_\rho$ in the (constant) direction of the L^2-orthogonal of $r(\ker D_\infty)$. The operator $P'_\rho = P_\rho - \pi_\rho(r(P_\rho))$ is then still a right inverse of D_ρ, it is still uniformly bounded, but has the additional property that the corresponding space $H_\rho = \mathrm{rk}(P'_\rho)$ is such that $\pi_\rho(r(H_\rho)) = 0$.

We obtain a one-to-one map $\varphi_\rho = \mathrm{id} \oplus \psi_\rho$ from a neighbourhood of 0 in $\ker D_\rho$ to the space of sections in $\Gamma^{1,p}_\rho(w^*_\rho TM)$ associated to Floer trajectories in a neighbourhood of C_∞ in $\overline{\mathcal{M}}(x_k, x_0)$.

Composing φ_ρ^{-1} with the map $\pi : \ker D_\rho \to \ker D_\infty$, constructed in the previous section, we obtain a map $\phi = (\rho, \pi \circ \varphi_\rho^{-1})$ between neighbourhoods of C_∞ in $\overline{\mathcal{M}}(x_k, x_0)$ and $(\infty, \dots, \infty, 0)$ in $(1, +\infty]^{k-1} \times T_{C_\infty}\mathcal{M}(x_k, \dots, x_0)$ with ρ the linear transit times as described at the beginning of §A.4.3:

$$\phi : \overline{\mathcal{M}}(x_k, x_0), C_\infty \xrightarrow{(\rho, \pi\circ\varphi_\rho^{-1})} (1, +\infty]^{k-1} \times T_{C_\infty}\mathcal{M}(x_k, \dots, x_0), ((\infty, \dots, \infty), 0).$$

This map is one-to-one. For the continuity of ϕ^{-1} consider a converging sequence $(\rho_n, \xi_n) \to (\rho, \xi)$. By possibly extracting a subsequence we may assume that $C_n = (\phi^{-1}(\rho_n, \xi_n))$ converges to a curve C' corresponding to $\xi' \in \ker D_\rho$. The curves C_n may be viewed as sections $x_n + y_n \in L^{1,p}(w^*_{\rho_n}(TM)) = \ker D_{\rho_n} \oplus H_{\rho_n}$, and, similarly, the curves $C = \phi^{-1}(\rho, \xi)$ and C' as sections $x + y$ and $x' + y'$.

We now have $\pi(r(y_n)) = \pi(\pi_\rho(r(y_n))) = 0$, so that $\pi(r(x_n + y_n)) = \pi(r(x_n)) = \xi_n$. On the other hand, this has to converge to $\pi(r(x' + y')) = \xi'$, so that $\xi = \xi'$, and finally $C = C'$.

Conversely, if $C_n \to C$ is a converging sequence of trajectories, then $\rho(C_n)$ converges to $\rho(C)$ and ξ_n is bounded in the finite dimensional space $\ker D_\infty$. On the other hand, a converging subsequence of (ρ_n, ξ_n) has to converge to $(\rho, \xi) = \phi(C)$, and hence $\phi(C_n)$ converges to $\phi(C)$.

Let us turn to the behaviour of ϕ with respect to the stratification: ϕ clearly respects the stratifications since a trajectory C is broken at x_i if and only if $\rho_i(C) = +\infty$. Within each stratum now, all the fiber-bundles $w^*_\rho TM$ are topologically equivalent, and we can locally move smoothly from one to another. The whole family $\pi \circ \varphi_\rho^{-1}$ depends then smoothly on ρ and ϕ is a local diffeomorphism.

This ends the proof of Theorem A.4.

A.5. *Hamiltonian perturbations.* We now shortly discuss the nonhomogeneous case. In this case the usual Cauchy-Riemann equation (37) is modified by a Hamiltonian term:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = -\nabla H_t(u) \tag{68}$$

or, equivalently:

$$\bar{\partial}_J u = du + J(u)du\, i\ = b_u, \tag{69}$$

where b_u is the $\mathbb{C}$ anti-linear map defined by $b_u(z)\ \xi = -\bar{\xi}\ \nabla H(u(z))$.

The homogeneous and nonhomogeneous situations differ in many respects. In particular, the intersection points are replaced by orbits of the Hamiltonian flow ϕ^t of H starting on L and reaching L' at time 1 (the transversality assumption being replaced by requiring that $\phi_1^H(L)$ be transverse to L'). Moreover, the "breaks" do not arise in the neighbourhood of a point, but along a curve, making the analysis a bit more technical.

A.5.1. *Reduction to the standard nonhomogeneous equation.* However, using the naturality maps used in §2.1.3, this case reduces to the one in which the basic equation is homogenous but the almost complex structure depends on the variable t. This is the case that we discuss below. Thus, the model equation of Lemmas A.7, A.8 and A.9 is now replaced by

$$\frac{\partial f}{\partial s} + J(t)\frac{\partial f}{\partial t} = 0.$$

Considering a path Φ_t of symplectic matrices such that $J(t) = \Phi_t^{-1}\, i\, \Phi_t$, and letting $g(s,t) = \Phi_t f(s,t)$, we end up with the equation:

$$\bar{\partial} g = b(t) g$$

and the boundary conditions become $g(s,0) \in \Lambda$ and $g(s,1) \in \Lambda'' = \Phi_1 \Lambda'$. Notice that the transversality assumption on Λ and Λ' is now replaced by requiring that the differential equation for $\gamma : [0,1] \to (\mathbb{C}^n, \Lambda, \Lambda'')$:

$$\dot{\gamma} = \ i\, b(t)\gamma$$

has no nontrivial solution.

Finally, Lemma A.7 in this setting is again a boundary version of Theorem 3.1.13 from [32]; as for Lemmas A.8 and A.9, observe that b is self-adjoint, and replacing the operator $A = i\,\frac{\partial}{\partial t}$ in the proof of these lemmas by

$$A = i\,\frac{\partial}{\partial t} + b$$

we get a self-adjoint, injective operator, so that $\alpha(s) = \int_0^1 \|f\|^2 dt$ satisfies the same differential inequality as before.

This discussion also covers the case when dependence of s is required (for example, to study Floer chain maps) because s-dependence has compact support, where the convergence of the curves is well-controlled.

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