

Finding large Selmer rank via an arithmetic theory of local constants

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Abstract

We obtain lower bounds for Selmer ranks of elliptic curves over dihedral extensions of number fields.

Suppose K/k is a quadratic extension of number fields, E is an elliptic curve defined over k , and p is an odd prime. Let \mathcal{K}^- denote the maximal abelian p -extension of K that is unramified at all primes where E has bad reduction and that is Galois over k with dihedral Galois group (i.e., the generator c of $\text{Gal}(K/k)$ acts on $\text{Gal}(\mathcal{K}^-/K)$ by inversion). We prove (under mild hypotheses on p) that if the \mathbf{Z}_p -rank of the pro- p Selmer group $\mathcal{S}_p(E/K)$ is odd, then $\text{rank}_{\mathbf{Z}_p} \mathcal{S}_p(E/F) \geq [F : K]$ for every finite extension F of K in \mathcal{K}^- .

Introduction

Let K/k be a quadratic extension of number fields, let c be the nontrivial automorphism of K/k , and let E be an elliptic curve defined over k . Let F/K be an abelian extension such that F is Galois over k with dihedral Galois group (i.e., a lift of the involution c operates by conjugation on $\text{Gal}(F/K)$ as inversion $x \mapsto x^{-1}$), and let $\chi : \text{Gal}(F/K) \rightarrow \bar{\mathbf{Q}}^\times$ be a character.

Even in cases where one cannot prove that the L -function $L(E/K, \chi; s)$ has an analytic continuation and functional equation, one still has a conjectural functional equation with a sign $\varepsilon(E/K, \chi) := \prod_v \varepsilon(E/K_v, \chi_v) = \pm 1$ expressed as a product over places v of K of local ε -factors. If $\varepsilon(E/K, \chi) = -1$, then a generalized Parity Conjecture predicts that the rank of the χ -part $E(F)^\chi$ of the $\text{Gal}(F/K)$ -representation space $E(F) \otimes \mathbf{Q}$ is odd, and hence positive. If $[F : K]$ is odd and F/K is unramified at all primes where E has bad reduction, then $\varepsilon(E/K, \chi)$ is independent of χ , and so the Parity Conjecture predicts that if the rank of $E(K)$ is odd then the rank of $E(F)$ is at least $[F : K]$.

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Motivated by the analytic theory of the preceding paragraph, in this paper we prove unconditional parity statements, not for the Mordell-Weil groups $E(F)^\chi$ but instead for the corresponding pro- p Selmer groups $\mathcal{S}_p(E/F)^\chi$. (The Shafarevich-Tate conjecture implies that $E(F)^\chi$ and $\mathcal{S}_p(E/F)^\chi$ have the same rank.) More specifically, given the data $(E, K/k, \chi)$ where the order of χ is a power of an odd prime p , we define (by cohomological methods) local invariants $\delta_v \in \mathbf{Z}/2\mathbf{Z}$ for the finite places v of K , depending only on E/K_v and χ_v . The δ_v should be the (additive) counterparts of the ratios $\varepsilon(E/K_v, \chi_v)/\varepsilon(E/K_v, 1)$ of the local ε -factors. The δ_v vanish for almost all v , and if $\mathbf{Z}_p[\chi]$ is the extension of \mathbf{Z}_p generated by the values of χ , we prove (see Theorem 6.4):

THEOREM A. *If the order of χ is a power of an odd prime p , then*

$$\text{rank}_{\mathbf{Z}_p} \mathcal{S}_p(E/K) - \text{rank}_{\mathbf{Z}_p[\chi]} \mathcal{S}_p(E/F)^\chi \equiv \sum_v \delta_v \pmod{2}.$$

Despite the fact that the analytic theory, which is our guide, predicts the values of the local terms δ_v , Theorem A would be of limited use if we could not actually compute the δ_v 's. We compute the δ_v 's in substantial generality in Section 5 and Section 6. This leads to our main result (Theorem 7.2), which we illustrate here with a weaker version.

THEOREM B. *Suppose that p is an odd prime, $[F : K]$ is a power of p , F/K is unramified at all primes where E has bad reduction, and all primes above p split in K/k . If $\text{rank}_{\mathbf{Z}_p} \mathcal{S}_p(E/K)$ is odd, then $\text{rank}_{\mathbf{Z}_p[\chi]} \mathcal{S}_p(E/F)^\chi$ is odd for every character χ of G , and in particular $\text{rank}_{\mathbf{Z}_p} \mathcal{S}_p(E/F) \geq [F : K]$.*

If K is an imaginary quadratic field and F/K is unramified outside of p , then Theorem B is a consequence of work of Cornut [Co] and Vatsal [V]. In those cases the bulk of the Selmer module comes from Heegner points.

Nekovář [N2, Th. 10.7.17] proved Theorem B in the case where F is contained in a \mathbf{Z}_p -power extension of K , under the assumption that E has ordinary reduction at all primes above p . We gave in [MR3] an exposition of a weaker version of Nekovář's theorem, as a direct application of a functional equation that arose in [MR2] (which also depends heavily on Nekovář's theory in [N2]).

The proofs of Theorems A and B proceed by methods that are very different from those of Cornut, Vatsal, and Nekovář, and are comparatively short. We emphasize that our results apply whether E has ordinary or supersingular reduction at p , and they apply even when F/K is not contained in a \mathbf{Z}_p -power extension of K (but we always assume that F/k is dihedral).

This extra generality is of particular interest in connection with the search for new Euler systems, beyond the known examples of Heegner points. Let $\mathcal{K}^- = \mathcal{K}_{c,p}^-$ be the maximal "generalized dihedral" p -extension of K (i.e.,

the maximal abelian p -extension of K , Galois over k , such that c acts on $\text{Gal}(\mathcal{K}^-/K)$ by inversion). A “dihedral” Euler system \mathbf{c} for $(E, K/k, p)$ would consist of Selmer classes $\mathbf{c}_F \in \mathcal{S}_p(E/F)$ for every finite extension F of K in \mathcal{K}^- , with certain compatibility relations between \mathbf{c}_F and $\mathbf{c}_{F'}$ when $F \subset F'$ (see for example [R] §9.4). A necessary condition for the existence of a nontrivial Euler system is that the Selmer modules $\mathcal{S}_p(E/F)$ are large, as in the conclusion of Theorem B. It is natural to ask whether, in these large Selmer modules $\mathcal{S}_p(E/F)$, one can find elements \mathbf{c}_F that form an Euler system.

Outline of the proofs. Suppose for simplicity that $E(K)$ has no p -torsion. The group ring $\mathbf{Q}[\text{Gal}(F/K)]$ splits into a sum of irreducible rational representations $\mathbf{Q}[\text{Gal}(F/K)] = \bigoplus_L \rho_L$, summing over all cyclic extensions L of K in F , where $\rho_L \otimes \bar{\mathbf{Q}}$ is the sum of all characters χ whose kernel is $\text{Gal}(F/L)$. Corresponding to this decomposition there is a decomposition (up to isogeny) of the restriction of scalars $\text{Res}_K^F E$ into abelian varieties over K

$$\text{Res}_K^F E \sim \bigoplus_L A_L.$$

This gives a decomposition of Selmer modules

$$\mathcal{S}_p(E/F) \cong \mathcal{S}_p((\text{Res}_K^F E)/K) \cong \bigoplus_L \mathcal{S}_p(A_L/K)$$

where for every L , $\mathcal{S}_p(A_L/K) \cong (\rho_L \otimes \mathbf{Q}_p)^{d_L}$ for some $d_L \geq 0$. Theorem B will follow once we show that $d_L \equiv \text{rank}_{\mathbf{Z}_p} \mathcal{S}_p(E/K) \pmod{2}$ for every L . More precisely, we will show (see Section 4 for the ideal \mathfrak{p} of $\text{End}_K(A_L)$, Section 2 for the Selmer groups Sel_p and $\text{Sel}_{\mathfrak{p}}$, and Definition 3.6 for \mathcal{S}_p) that

$$(1) \quad \text{rank}_{\mathbf{Z}_p} \mathcal{S}_p(E/K) \equiv \dim_{\mathbf{F}_p} \text{Sel}_p(E/K) \equiv \dim_{\mathbf{F}_p} \text{Sel}_{\mathfrak{p}}(A_L/K) \equiv d_L \pmod{2}.$$

The key step in our proof is the second congruence of (1). We will see (Proposition 4.1) that $E[p] \cong A_L[\mathfrak{p}]$ as G_K -modules, and therefore the Selmer groups $\text{Sel}_p(E/K)$ and $\text{Sel}_{\mathfrak{p}}(A_L/K)$ are both contained in $H^1(K, E[p])$. By comparing these two subspaces we prove (see Theorem 1.4 and Corollary 4.6) that

$$\dim_{\mathbf{F}_p} \text{Sel}_p(E/K) - \dim_{\mathbf{F}_p} \text{Sel}_{\mathfrak{p}}(A_L/K) \equiv \sum_v \delta_v \pmod{2}$$

summing the local invariants δ_v of Definition 4.5 over primes v of K . We show how to compute the δ_v in terms of norm indices in Section 5 and Section 6, with one important special case postponed to Appendix B.

The first congruence of (1) follows easily from the Cassels pairing for E (see Proposition 2.1). The final congruence of (1) is more subtle, because in general A_L will not have a polarization of degree prime to p , and we deal with this in Appendix A (using the dihedral nature of L/k).

In Section 7 we bring together the results of the previous sections to prove Theorem 7.2, and in Section 8 we discuss some special cases.

Generalizations. All the results and proofs in this paper hold with E replaced by an abelian variety with a polarization of degree prime to p .

If F/K is not a p -extension, then the proof described above breaks down. Namely, if χ is a character whose order is not a prime power, then χ is not congruent to the trivial character modulo any prime of $\bar{\mathbf{Q}}$. However, by writing χ as a product of characters of prime-power order, we can apply the methods of this paper inductively. To do this we must use a different prime p at each step, so it is necessary to assume that if A is an abelian variety over K and R is an integral domain in $\text{End}_K(A)$, then the parity of $\dim_{R \otimes \mathbf{Q}_p} \mathcal{S}_p(A/K)$ is independent of p . (This would follow, for example, from the Shafarevich-Tate conjecture.) To avoid obscuring the main ideas of our arguments, we will include those details in a separate paper.

The results of this paper can also be applied to study the growth of Selmer rank in nonabelian Galois extensions of order $2p^n$ with p an odd prime. This will be the subject of a forthcoming paper.

Notation. Fix once and for all an algebraic closure $\bar{\mathbf{Q}}$ of \mathbf{Q} . A number field will mean a finite extension of \mathbf{Q} in $\bar{\mathbf{Q}}$. If K is a number field then $G_K := \text{Gal}(\bar{\mathbf{Q}}/K)$.

1. Variation of Selmer rank

Let K be a number field and p an odd rational prime. Let W be a finite-dimensional \mathbf{F}_p -vector space with a continuous action of G_K and with a perfect, skew-symmetric, G_K -equivariant self-duality

$$W \times W \longrightarrow \mu_p$$

where μ_p is the G_K -module of p -th roots of unity in $\bar{\mathbf{Q}}$.

THEOREM 1.1. *For every prime v of K , Tate's local duality gives a perfect symmetric pairing*

$$\langle \cdot, \cdot \rangle_v : H^1(K_v, W) \times H^1(K_v, W) \longrightarrow H^2(K_v, \mu_p) = \mathbf{F}_p.$$

Proof. See [T1]. □

Definition 1.2. For every prime v of K let K_v^{ur} denote the maximal unramified extension of K_v . A *Selmer structure* \mathcal{F} on W is a collection of \mathbf{F}_p -subspaces

$$H_{\mathcal{F}}^1(K_v, W) \subset H^1(K_v, W)$$

for every prime v of K , such that $H_{\mathcal{F}}^1(K_v, W) = H^1(K_v^{\text{ur}}/K_v, W^{I_v})$ for all but finitely many v , where $I_v := G_{K_v^{\text{ur}}} \subset G_{K_v}$ is the inertia group. If \mathcal{F} and \mathcal{G} are

Selmer structures on W , we define Selmer structures $\mathcal{F} + \mathcal{G}$ and $\mathcal{F} \cap \mathcal{G}$ by

$$\begin{aligned} H_{\mathcal{F}+\mathcal{G}}^1(K_v, W) &:= H_{\mathcal{F}}^1(K_v, W) + H_{\mathcal{G}}^1(K_v, W), \\ H_{\mathcal{F}\cap\mathcal{G}}^1(K_v, W) &:= H_{\mathcal{F}}^1(K_v, W) \cap H_{\mathcal{G}}^1(K_v, W), \end{aligned}$$

for every v . We say that $\mathcal{F} \leq \mathcal{G}$ if $H_{\mathcal{F}}^1(K_v, W) \subset H_{\mathcal{G}}^1(K_v, W)$ for every v , so in particular $\mathcal{F} \cap \mathcal{G} \leq \mathcal{F} \leq \mathcal{F} + \mathcal{G}$.

We say that a Selmer structure \mathcal{F} is *self-dual* if for every v , $H_{\mathcal{F}}^1(K_v, W)$ is its own orthogonal complement under the Tate pairing of Theorem 1.1.

If \mathcal{F} is a Selmer structure on W , we define the *Selmer group*

$$H_{\mathcal{F}}^1(K, W) := \ker(H^1(K, W) \longrightarrow \prod_v H^1(K_v, W)/H_{\mathcal{F}}^1(K_v, W)).$$

Thus $H_{\mathcal{F}}^1(K, W)$ is the collection of classes whose localizations lie in $H_{\mathcal{F}}^1(K_v, W)$ for every v . If $\mathcal{F} \leq \mathcal{G}$ then $H_{\mathcal{F}}^1(K, W) \subset H_{\mathcal{G}}^1(K, W)$.

For the basic example of the Selmer groups we will be interested in, where W is the Galois module of p -torsion on an elliptic curve, see Section 2.

PROPOSITION 1.3. *Suppose that \mathcal{F}, \mathcal{G} are self-dual Selmer structures on W , and S is a finite set of primes of K such that $H_{\mathcal{F}}^1(K_v, W) = H_{\mathcal{G}}^1(K_v, W)$ if $v \notin S$. Then*

- (i) $\dim_{\mathbf{F}_p} H_{\mathcal{F}+\mathcal{G}}^1(K, W)/H_{\mathcal{F}\cap\mathcal{G}}^1(K, W) = \sum_{v \in S} \dim_{\mathbf{F}_p} H_{\mathcal{F}}^1(K_v, W)/H_{\mathcal{F}\cap\mathcal{G}}^1(K_v, W),$
- (ii) $\dim_{\mathbf{F}_p} H_{\mathcal{F}+\mathcal{G}}^1(K, W) \equiv \dim_{\mathbf{F}_p} (H_{\mathcal{F}}^1(K, W) + H_{\mathcal{G}}^1(K, W)) \pmod{2}.$

Proof. Let

$$B := \bigoplus_{v \in S} (H_{\mathcal{F}+\mathcal{G}}^1(K_v, W)/H_{\mathcal{F}\cap\mathcal{G}}^1(K_v, W))$$

and let C be the image of the localization map $H_{\mathcal{F}+\mathcal{G}}^1(K, W) \rightarrow B$. Since \mathcal{F} and \mathcal{G} are self-dual, Poitou-Tate global duality (see for example [MR1, Th. 2.3.4]) shows that the Tate pairings of Theorem 1.1 induce a nondegenerate, symmetric self-pairing

$$(1.1) \quad \langle \cdot, \cdot \rangle : B \times B \longrightarrow \mathbf{F}_p,$$

and C is its own orthogonal complement under this pairing.

Let $C_{\mathcal{F}}$ (resp. $C_{\mathcal{G}}$) denote the image of $\bigoplus_{v \in S} H_{\mathcal{F}}^1(K_v, W)$ (resp. $\bigoplus_{v \in S} H_{\mathcal{G}}^1(K_v, W)$) in B . Since \mathcal{F} and \mathcal{G} are self-dual, $C_{\mathcal{F}}$ and $C_{\mathcal{G}}$ are each their own orthogonal complements under (1.1). In particular we have

$$\dim_{\mathbf{F}_p} C = \dim_{\mathbf{F}_p} C_{\mathcal{F}} = \dim_{\mathbf{F}_p} C_{\mathcal{G}} = \frac{1}{2} \dim_{\mathbf{F}_p} B.$$

Since

$$C \cong H_{\mathcal{F}+\mathcal{G}}^1(K, W)/H_{\mathcal{F}\cap\mathcal{G}}^1(K, W)$$

and

$$C_{\mathcal{F}} \cong \bigoplus_{v \in S} H_{\mathcal{F}}^1(K_v, W) / H_{\mathcal{F} \cap \mathcal{G}}^1(K_v, W),$$

this proves (i).

The proof of (ii) uses an argument of Howard ([Hb, Lemma 1.5.7]). We have $C_{\mathcal{F}} \cap C_{\mathcal{G}} = 0$ and $C_{\mathcal{F}} \oplus C_{\mathcal{G}} = B$. If $x \in H_{\mathcal{F} + \mathcal{G}}^1(K, W)$, let $x_S \in C \subset B$ be the localization of x , and let $x_{\mathcal{F}}$ and $x_{\mathcal{G}}$ denote the projections of x_S to $C_{\mathcal{F}}$ and $C_{\mathcal{G}}$, respectively.

Following Howard, we define a pairing

$$(1.2) \quad [\ , \] : H_{\mathcal{F} + \mathcal{G}}^1(K, W) \times H_{\mathcal{F} + \mathcal{G}}^1(K, W) \longrightarrow \mathbf{F}_p$$

by $[x, y] := \langle x_{\mathcal{F}}, y_{\mathcal{G}} \rangle$, where $\langle \ , \ \rangle$ is the pairing (1.1). Since the subspaces C , $C_{\mathcal{F}}$, and $C_{\mathcal{G}}$ are all isotropic, for all $x, y \in H_{\mathcal{F} + \mathcal{G}}^1(K, W)$ we have

$$0 = \langle x_S, y_S \rangle = \langle x_{\mathcal{F}} + x_{\mathcal{G}}, y_{\mathcal{F}} + y_{\mathcal{G}} \rangle = \langle x_{\mathcal{F}}, y_{\mathcal{G}} \rangle + \langle x_{\mathcal{G}}, y_{\mathcal{F}} \rangle = [x, y] + [y, x]$$

so the pairing (1.2) is skew-symmetric.

We see easily that $H_{\mathcal{F}}^1(K, W) + H_{\mathcal{G}}^1(K, W)$ is in the kernel of the pairing $[\ , \]$. Conversely, if x is in the kernel of this pairing, then for every $y \in H_{\mathcal{F} + \mathcal{G}}^1(K, W)$

$$0 = [x, y] = \langle x_{\mathcal{F}}, y_{\mathcal{G}} \rangle = \langle x_{\mathcal{F}}, y_S \rangle.$$

Since C is its own orthogonal complement we deduce that $x_{\mathcal{F}} \in C$, i.e., there is a $z \in H_{\mathcal{F} + \mathcal{G}}^1(K, W)$ whose localization is $x_{\mathcal{F}}$. It follows that $z \in H_{\mathcal{F}}^1(K, W)$ and $x - z \in H_{\mathcal{G}}^1(K, W)$, i.e., $x \in H_{\mathcal{F}}^1(K, W) + H_{\mathcal{G}}^1(K, W)$. Therefore (1.2) induces a nondegenerate, skew-symmetric, \mathbf{F}_p -valued pairing on

$$H_{\mathcal{F} + \mathcal{G}}^1(K, W) / (H_{\mathcal{F}}^1(K, W) + H_{\mathcal{G}}^1(K, W)).$$

Since p is odd, a well-known argument from linear algebra shows that the dimension of this \mathbf{F}_p -vector space must be even. This proves (ii). \square

THEOREM 1.4. *Suppose that \mathcal{F} and \mathcal{G} are self-dual Selmer structures on W , and S is a finite set of primes of K such that $H_{\mathcal{F}}^1(K_v, W) = H_{\mathcal{G}}^1(K_v, W)$ if $v \notin S$. Then*

$$\begin{aligned} \dim_{\mathbf{F}_p} H_{\mathcal{F}}^1(K, W) - \dim_{\mathbf{F}_p} H_{\mathcal{G}}^1(K, W) \\ \equiv \sum_{v \in S} \dim_{\mathbf{F}_p} (H_{\mathcal{F}}^1(K_v, W) / H_{\mathcal{F} \cap \mathcal{G}}^1(K_v, W)) \pmod{2}. \end{aligned}$$

Proof. We have (modulo 2)

$$\begin{aligned} \dim_{\mathbf{F}_p} H_{\mathcal{F}}^1(K, W) - \dim_{\mathbf{F}_p} H_{\mathcal{G}}^1(K, W) &\equiv \dim_{\mathbf{F}_p} H_{\mathcal{F}}^1(K, W) + \dim_{\mathbf{F}_p} H_{\mathcal{G}}^1(K, W) \\ &= \dim_{\mathbf{F}_p} (H_{\mathcal{F}}^1(K, W) + H_{\mathcal{G}}^1(K, W)) + \dim_{\mathbf{F}_p} H_{\mathcal{F} \cap \mathcal{G}}^1(K, W) \\ &\equiv \dim_{\mathbf{F}_p} H_{\mathcal{F} + \mathcal{G}}^1(K, W) - \dim_{\mathbf{F}_p} H_{\mathcal{F} \cap \mathcal{G}}^1(K, W) \\ &= \sum_{v \in S} \dim_{\mathbf{F}_p} (H_{\mathcal{F}}^1(K_v, W) / H_{\mathcal{F} \cap \mathcal{G}}^1(K_v, W)), \end{aligned}$$

the last two steps by Proposition 1.3(ii) and (i), respectively. \square

2. Example: elliptic curves

Let K be a number field. If A is an abelian variety over K , and $\alpha \in \text{End}_K(A)$ is an isogeny, we have the usual Selmer group $\text{Sel}_\alpha(A/K) \subset H^1(K, E[\alpha])$, sitting in an exact sequence

$$(2.1) \quad 0 \longrightarrow A(K)/\alpha A(K) \longrightarrow \text{Sel}_\alpha(A/K) \longrightarrow \text{III}(A/K)[\alpha] \longrightarrow 0,$$

where $\text{III}(A/K)$ is the Shafarevich-Tate group of A over K . If p is a prime we let $\text{Sel}_{p^\infty}(A/K)$ be the direct limit of the Selmer groups $\text{Sel}_{p^n}(A/K)$, and then we have

$$(2.2) \quad 0 \longrightarrow A(K) \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \text{Sel}_{p^\infty}(A/K) \longrightarrow \text{III}(A/K)[p^\infty] \longrightarrow 0.$$

Suppose now that E is an elliptic curve defined over K , and p is an odd rational prime. Let $W := E[p]$, the Galois module of p -torsion in $E(\bar{\mathbf{Q}})$. Then W is an \mathbf{F}_p -vector space with a continuous action of G_K , and the Weil pairing induces a perfect G_K -equivariant self-duality $E[p] \times E[p] \rightarrow \mu_p$. Thus we are in the setting of Section 1.

We define a Selmer structure \mathcal{E} on $E[p]$ by taking $H^1_{\mathcal{E}}(K_v, E[p])$ to be the image of $E(K_v)/pE(K_v)$ under the Kummer injection

$$E(K_v)/pE(K_v) \hookrightarrow H^1(K_v, E[p])$$

for every v . By Lemma 19.3 of [Ca2], $H^1_{\mathcal{E}}(K_v, E[p]) = H^1(K_v^{\text{ur}}/K_v, E[p])$ if $v \nmid p$ and E has good reduction at v . With this definition the Selmer group $H^1_{\mathcal{E}}(K, E[p])$ is the usual p -Selmer group $\text{Sel}_p(E/K)$ of E as in (2.1).

If C is an abelian group, we let C_{div} denote its maximal divisible subgroup.

PROPOSITION 2.1. *The Selmer structure \mathcal{E} on $E[p]$ defined above is self-dual, and*

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) \equiv \dim_{\mathbf{F}_p} H^1_{\mathcal{E}}(K, E[p]) - \dim_{\mathbf{F}_p} E(K)[p] \pmod{2}.$$

Proof. Tate’s local duality [T1] shows that \mathcal{E} is self-dual. Let

$$\begin{aligned} d &:= \dim_{\mathbf{F}_p} (\text{Sel}_{p^\infty}(E/K)/(\text{Sel}_{p^\infty}(E/K))_{\text{div}}[p]) \\ &= \dim_{\mathbf{F}_p} (\text{III}(E/K)[p^\infty]/(\text{III}(E/K)[p^\infty])_{\text{div}}[p]). \end{aligned}$$

The Cassels pairing [Ca1] shows that d is even. Further,

$$\begin{aligned} \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) &= \dim_{\mathbf{F}_p} \text{Sel}_{p^\infty}(E/K)_{\text{div}}[p] \\ &= \dim_{\mathbf{F}_p} \text{Sel}_{p^\infty}(E/K)[p] - d \\ &= \text{rank}_{\mathbf{Z}} E(K) + \dim_{\mathbf{F}_p} \text{III}(E/K)[p] - d \end{aligned}$$

by (2.2) with $A = E$. On the other hand, (2.1) shows that

$$\dim_{\mathbf{F}_p} H^1_{\mathcal{E}}(K, E[p]) = \text{rank}_{\mathbf{Z}} E(K) + \dim_{\mathbf{F}_p} E(K)[p] + \dim_{\mathbf{F}_p} \text{III}(E/K)[p]$$

so we conclude

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) = \dim_{\mathbf{F}_p} H_{\mathcal{E}}^1(K, E[p]) - \dim_{\mathbf{F}_p} E(K)[p] - d.$$

This proves the proposition. \square

3. Decomposition of the restriction of scalars

Much of the technical machinery for this section will be drawn from Sections 4 and 5 of [MRS].

Suppose F/K is a finite abelian extension of number fields, $G := \text{Gal}(F/K)$, and E is an elliptic curve defined over K . We let $\text{Res}_K^F E$ denote the Weil restriction of scalars ([W, §1.3]) of E from F to K , an abelian variety over K with the following properties.

PROPOSITION 3.1. (i) *For every commutative K -algebra X there is a canonical isomorphism*

$$(\text{Res}_K^F E)(X) \cong E(X \otimes_K F)$$

functorial in X . In particular, $(\text{Res}_K^F E)(K) \cong E(F)$.

(ii) *The action of G on the right-hand side of (i) induces a canonical inclusion $\mathbf{Z}[G] \hookrightarrow \text{End}_K(\text{Res}_K^F E)$.*

(iii) *For every prime p there is a natural G -equivariant isomorphism, compatible with the isomorphism $(\text{Res}_K^F E)(K) \cong E(F)$ of (i),*

$$\text{Sel}_{p^\infty}((\text{Res}_K^F E)/K) \cong \text{Sel}_{p^\infty}(E/F)$$

where G acts on the left-hand side via the inclusion of (ii).

Proof. Assertion (i) is the universal property satisfied by the restriction of scalars [W], and (ii) is (for example) (4.2) of [MRS]. For (iii), Theorem 2.2(ii) and Proposition 4.1 of [MRS] give an isomorphism

$$(\text{Res}_K^F E)[p^\infty] \cong \mathbf{Z}[G] \otimes E[p^\infty]$$

that is G -equivariant (with G acting on $\text{Res}_K^F E$ via the map of (ii) and by multiplication on $\mathbf{Z}[G]$) and G_K -equivariant (with $\gamma \in G_K$ acting by $\gamma^{-1} \otimes \gamma$ on $\mathbf{Z}[G] \otimes E[p^\infty]$). Then by Shapiro's lemma (see for example Propositions III.6.2, III.5.6(a), and III.5.9 of [Br]), there is a G -equivariant isomorphism

$$(3.1) \quad H^1(K, (\text{Res}_K^F E)[p^\infty]) \xrightarrow{\sim} H^1(F, E[p^\infty]).$$

Using (i) with $X = K_v$, along with the analogue of (3.1) for the local extensions $(F \otimes_K K_v)/K_v$ for every prime v of K , one can show that the isomorphism (3.1) restricts to the isomorphism of (iii). \square

Definition 3.2. Let $\Xi := \{\text{cyclic extensions of } K \text{ in } F\}$, and if $L \in \Xi$ let ρ_L be the unique faithful irreducible rational representation of $\text{Gal}(L/K)$. Then $\rho_L \otimes \bar{\mathbf{Q}}$ is the direct sum of all the injective characters $\text{Gal}(L/K) \hookrightarrow \bar{\mathbf{Q}}^\times$. The correspondence $L \leftrightarrow \rho_L$ is a bijection between Ξ and the set of irreducible rational representations of G . Thus the semisimple group ring $\mathbf{Q}[G]$ decomposes

$$(3.2) \quad \mathbf{Q}[G] \cong \bigoplus_{L \in \Xi} \mathbf{Q}[G]_L$$

where $\mathbf{Q}[G]_L \cong \rho_L$ is the ρ_L -isotypic component of $\mathbf{Q}[G]$. As a field, $\mathbf{Q}[G]_L$ is isomorphic to the cyclotomic field of $[L : K]$ -th roots of unity.

Let R_L be the maximal order of $\mathbf{Q}[G]_L$. If $[L : K]$ is a power of a prime p , then R_L has a unique prime ideal above p , which we denote by \mathfrak{p}_L . Also define

$$\mathcal{I}_L := \mathbf{Q}[G]_L \cap \mathbf{Z}[G],$$

so \mathcal{I}_L is an ideal of R_L as well as a G_K -module (where the action of G_K is induced by multiplication on $\mathbf{Z}[G]$).

Definition 3.3. For every $L \in \Xi$ define

$$A_L := \mathcal{I}_L \otimes E$$

as given by Definition 1.1 of [MRS] (see also [Mi, §2]). The abelian variety A_L is defined over K , and its K -isomorphism class is independent of the choice of abelian extension F containing L (see Remark 4.4 of [MRS]). If $L = K$ then $A_K = E$. By Proposition 4.2(i) of [MRS], the inclusion $\mathcal{I}_L \hookrightarrow \mathbf{Z}[G]$ induces an isomorphism

$$(3.3) \quad A_L \cong \bigcap_{\alpha \in \mathbf{Z}[G] : \alpha \mathcal{I}_L = 0} \ker(\alpha : \text{Res}_K^F E \rightarrow \text{Res}_K^F E) \subset \text{Res}_K^F E.$$

Let $T_p(E)$ denote the Tate module $\varprojlim E[p^n]$, and similarly for $T_p(A_L)$. The following theorem summarizes the properties of the abelian varieties A_L that we will need.

THEOREM 3.4. *Suppose p is a prime, $n \geq 1$, and L/K is a cyclic extension of degree p^n . Then:*

- (i) $\mathcal{I}_L = \mathfrak{p}_L^{p^{n-1}}$ in R_L .
- (ii) *The inclusion $\mathbf{Z}[G] \hookrightarrow \text{End}_K(\text{Res}_K^F E)$ of Proposition 3.1(ii) induces (via (3.3)) a ring homomorphism $\mathbf{Z}[G] \rightarrow \text{End}_K(A_L)$ that factors*

$$\mathbf{Z}[G] \twoheadrightarrow R_L \hookrightarrow \text{End}_K(A_L)$$

where the first map is induced by the projection in (3.2).

- (iii) Let M be the unique extension of K in L with $[L : M] = p$. For every commutative K -algebra X , the isomorphism of Proposition 3.1(i) restricts (using (3.3)) to an isomorphism, functorial in X ,

$$A_L(X) \cong \{x \in E(X \otimes_K L) : \sum_{h \in \text{Gal}(L/M)} (1 \otimes h)(x) = 0\}.$$

- (iv) The isomorphism of (iii) with $X = \bar{\mathbf{Q}}$ induces an isomorphism

$$T_p(A_L) \cong \mathcal{I}_L \otimes T_p(E) = \mathfrak{p}_L^{p^n-1} \otimes T_p(E)$$

that is G_K -equivariant, where $\gamma \in G_K$ acts on the tensor products as $\gamma^{-1} \otimes \gamma$, and R_L -linear, where R_L acts on A_L via the map of (ii).

Proof. Assertions (i), (ii), and (iv) are Lemma 5.4(iv), Theorem 5.5(iv), and Theorem 2.2(iii), respectively, of [MRS] ((iv) is also Proposition 6(b) of [Mi]). Assertion (iii) is Theorem 5.8(ii) of [MRS]. □

THEOREM 3.5. *The inclusions $A_L \subset \text{Res}_K^F E$ of (3.3) induce an isogeny*

$$\bigoplus_{L \in \Xi} A_L \longrightarrow \text{Res}_K^F E.$$

Proof. This is Theorem 5.2 of [MRS]; it follows from the fact that $\bigoplus_{L \in \Xi} \mathcal{I}_L$ injects into $\mathbf{Z}[G]$ with finite cokernel. □

Definition 3.6. Define the Pontrjagin dual Selmer vector spaces

$$\begin{aligned} \mathcal{S}_p(E/K) &:= \text{Hom}(\text{Sel}_{p^\infty}(E/K), \mathbf{Q}_p/\mathbf{Z}_p) \otimes \mathbf{Q}_p, \\ \mathcal{S}_p(A_L/K) &:= \text{Hom}(\text{Sel}_{p^\infty}(A_L/K), \mathbf{Q}_p/\mathbf{Z}_p) \otimes \mathbf{Q}_p. \end{aligned}$$

Define $\mathcal{S}_p(E/F)$ similarly for every finite extension F of K .

COROLLARY 3.7. *There is a G -equivariant isomorphism*

$$\mathcal{S}_p(E/F) \cong \bigoplus_{L \in \Xi} \mathcal{S}_p(A_L/K)$$

where the action of G on the right-hand side is given by Theorem 3.4(ii).

Proof. We have $\mathcal{S}_p(E/F) \cong \mathcal{S}_p((\text{Res}_K^F E)/K)$ by (the Pontrjagin dual of) Proposition 3.1(iii), and $\mathcal{S}_p((\text{Res}_K^F E)/K) \cong \bigoplus_{L \in \Xi} \mathcal{S}_p(A_L/K)$ by Theorem 3.5. □

4. The local invariants

Fix an odd prime p and a cyclic extension L/K of degree p^n . We will write simply A for the abelian variety A_L of Definition 3.3, R for the ring R_L of Definition 3.2, \mathfrak{p} for the unique prime \mathfrak{p}_L of R above p , and $\mathcal{I} \subset R$ for the ideal \mathcal{I}_L of Definition 3.2.

PROPOSITION 4.1. *There is a canonical G_K -isomorphism $A[\mathfrak{p}] \xrightarrow{\sim} E[p]$.*

Proof. The action of G on $\mathfrak{p}^{-1}\mathcal{I}/\mathcal{I}$ is trivial, since for every $g \in G$, $g - 1$ lies in the maximal ideal of $\mathbf{Z}_p[G]$. Also, if π and π' are generators of $\mathfrak{p}/\mathfrak{p}^2$, then $\pi/\pi' \in (R/\mathfrak{p})^\times = \mathbf{F}_p^\times$, so $\pi^{p-1} \equiv (\pi')^{p-1} \pmod{\mathfrak{p}^p}$. It follows that π^{p-1} is a canonical generator of $\mathfrak{p}^{p-1}/\mathfrak{p}^p$, so there is a canonical isomorphism $\mathfrak{p}^{a(p-1)}/\mathfrak{p}^{a(p-1)+1} \cong \mathbf{F}_p$ for every integer a . Now using Theorem 3.4(iv) we have G_K -isomorphisms

$$A[\mathfrak{p}] \cong \mathfrak{p}^{-1}T_p(A)/T_p(A) \cong (\mathfrak{p}^{p^{n-1}-1}/\mathfrak{p}^{p^{n-1}}) \otimes T_p(E) \cong \mathbf{F}_p \otimes T_p(E) \cong E[p].$$

□

Remark 4.2. Identifying E with A_K , one can show using (3.3) that

$$E[p] = E \cap A_L = A_L[\mathfrak{p}]$$

inside $\text{Res}_K^F E$. This gives an alternate proof of Proposition 4.1.

Definition 4.3. Recall that in Section 2 we defined a self-dual Selmer structure \mathcal{E} on $E[p]$. We can use the identification of Proposition 4.1 to define another Selmer structure \mathcal{A} on $E[p]$ as follows. For every v define $H_{\mathcal{A}}^1(K_v, E[p])$ to be the image of $A(K_v)/\mathfrak{p}A(K_v)$ under the composition

$$A(K_v)/\mathfrak{p}A(K_v) \hookrightarrow H^1(K_v, A[\mathfrak{p}]) \cong H^1(K_v, E[p])$$

where the first map is the Kummer injection, and the second map is from Proposition 4.1. The first map depends (only up to multiplication by a unit in \mathbf{F}_p^\times) on a choice of generator of $\mathfrak{p}/\mathfrak{p}^2$, but the image is independent of this choice. With this definition the Selmer group $H_{\mathcal{A}}^1(K, E[p])$ is the usual \mathfrak{p} -Selmer group $\text{Sel}_{\mathfrak{p}}(A/K)$ of A , as in (2.1).

PROPOSITION 4.4. *The Selmer structure \mathcal{A} is self-dual.*

Proof. This is Proposition A.7 of Appendix A. (It does not follow immediately from Tate’s local duality as in Proposition 2.1, because A has no polarization of degree prime to p , and hence no suitable Weil pairing.) □

Definition 4.5. For every prime v of K we define an invariant $\delta_v \in \mathbf{Z}/2\mathbf{Z}$ by

$$\delta_v = \delta(v, E, L/K) := \dim_{\mathbf{F}_p}(H_{\mathcal{E}}^1(K_v, E[p])/H_{\mathcal{E} \cap \mathcal{A}}^1(K_v, E[p])) \pmod{2}.$$

We will see in Corollary 5.3 below that δ_v is a purely local invariant, depending only on $K_v, E/K_v$, and L_w , where w is a prime of L above v .

COROLLARY 4.6. *Suppose that S is a set of primes of K containing all primes above p , all primes ramified in L/K , and all primes where E has bad*

reduction. Then

$$\dim_{\mathbf{F}_p} \text{Sel}_p(E/K) - \dim_{\mathbf{F}_p} \text{Sel}_p(A/K) \equiv \sum_{v \in S} \delta_v \pmod{2}.$$

Proof. If $v \notin S$ then both $T_p(E)$ and $T_p(A)$ are unramified at v , so (see for example [Ca2, Lemma 19.3])

$$H_{\mathcal{E}}^1(K_v, E[p]) = H_{\mathcal{A}}^1(K_v, E[p]) = H^1(K_v^{\text{ur}}/K_v, E[p]).$$

Thus the corollary follows from Propositions 2.1 and 4.4 and Theorem 1.4. \square

5. Computing the local invariants

Let $p, L/K, A := A_L$, and $\mathfrak{p} \subset R$ be as in Section 4. Let M be the unique extension of K in L with $[L : M] = p$, and let $G := \text{Gal}(L/K)$ (recall that L/K is cyclic of degree p^n). In this section we compare the local Selmer conditions $H_{\mathcal{E}}^1(K_v, E[p])$ and $H_{\mathcal{A}}^1(K_v, E[p])$ for primes v of K , in order to compute the invariants δ_v of Definition 4.5.

LEMMA 5.1. *Suppose that c is an automorphism of K , and E is defined over the fixed field of c in K . Then for every prime v of K , we have $\delta_{v^c} = \delta_v$.*

Proof. The automorphism c induces isomorphisms

$$E(K_v) \xrightarrow{\sim} E(K_{v^c}), \quad A(K_v) \xrightarrow{\sim} A(K_{v^c}).$$

Therefore the isomorphism $H^1(K_v, E[p]) \xrightarrow{\sim} H^1(K_{v^c}, E[p])$ induced by c identifies

$$H_{\mathcal{E}}^1(K_v, E[p]) \xrightarrow{\sim} H_{\mathcal{E}}^1(K_{v^c}, E[p]), \quad H_{\mathcal{A}}^1(K_v, E[p]) \xrightarrow{\sim} H_{\mathcal{A}}^1(K_{v^c}, E[p]),$$

and the lemma follows directly from the definition of δ_v . \square

For every prime v of K , let $L_v := K_v \otimes_K L = \bigoplus_{w|v} L_w$, and let $G := \text{Gal}(L/K)$ act on L_v via its action on L . Let $M_v := K_v \otimes M$ and let $N_{L/M} : E(L_v) \rightarrow E(M_v)$ denote the norm (or trace) map. The following is our main tool for computing δ_v .

PROPOSITION 5.2. *For every prime v of K , the isomorphism*

$$H_{\mathcal{E}}^1(K_v, E[p]) \cong E(K_v)/pE(K_v)$$

identifies

$$H_{\mathcal{E} \cap \mathcal{A}}^1(K_v, E[p]) \cong (E(K_v) \cap N_{L/M}E(L_v))/pE(K_v).$$

Proof. Fix a generator σ of G , and let π be the projection of $\sigma - 1$ to R under (3.2). Since σ projects to a p^n -th root of unity in R , we see that π is a generator of \mathfrak{p} .

Note that G and G_{K_v} act on $E(\bar{K}_v \otimes L)$ (as $1 \otimes G$ and $G_{K_v} \otimes 1$, respectively). We identify $E(L_v)$, $E(\bar{K}_v)$, $A(K_v)$, and $A(\bar{K}_v)$ with their images in $E(\bar{K}_v \otimes L)$ under the natural inclusions and Theorem 3.4(iii):

$$A(K_v) \subset E(L_v) = E(K_v \otimes L) = E(\bar{K}_v \otimes L)^{G_{K_v}},$$

$$E(\bar{K}_v) = E(\bar{K}_v \otimes K) = E(\bar{K}_v \otimes L)^G, \quad A(\bar{K}_v) \subset E(\bar{K}_v \otimes L).$$

Let $\hat{\pi} := (1 \otimes \sigma) - 1$ on $E(\bar{K}_v \otimes L)$, so $\hat{\pi}$ restricts to π on $A(\bar{K}_v)$ and to zero on $E(\bar{K}_v)$. By Proposition 3.4(iii), $A(\bar{K}_v)$ is the kernel of $N_{L/M} := \sum_{g \in \text{Gal}(L/M)} 1 \otimes g$ in $E(\bar{K}_v \otimes L)$.

If $x \in E(K_v)$, then the image of x in $H^1(K_v, E[p])$ is represented by the cocycle $\gamma \mapsto y^{\gamma^{\otimes 1}} - y$ where $y \in E(\bar{K}_v)$ and $py = x$. Similarly, using the identifications above, if $\alpha \in A(K_v)$ then the image of α in $H^1(K_v, E[p])$ is represented by the cocycle $\gamma \mapsto \beta^{\gamma^{\otimes 1}} - \beta$ where $\beta \in A(\bar{K}_v)$ and $\pi\beta = \alpha$.

Suppose $x \in E(K_v)$, and choose $y \in E(\bar{K}_v)$ such that $py = x$. Then

$$\begin{aligned} &\text{the image of } x \text{ in } H_{\mathcal{E}}^1(K_v, E[p]) \subset H^1(K_v, E[p]) \text{ belongs to } H_{\mathcal{E} \cap \mathcal{A}}^1(K_v, E[p]) \\ &\iff \exists \beta \in A(\bar{K}_v) : \pi\beta \in A(K_v), \beta^{\gamma^{\otimes 1}} - \beta = y^{\gamma^{\otimes 1}} - y \quad \forall \gamma \in G_{K_v} \\ &\iff \exists \beta \in A(\bar{K}_v) : \beta^{\gamma^{\otimes 1}} - \beta = y^{\gamma^{\otimes 1}} - y \quad \forall \gamma \in G_{K_v} \\ &\iff \exists \beta \in E(\bar{K}_v \otimes L) : N_{L/M}\beta = 0, y - \beta \in E(L_v) \\ &\iff N_{L/M}y \in N_{L/M}E(L_v), \end{aligned}$$

where for the second equivalence we use that if $\gamma \in G_{K_v}$ and $\beta^{\gamma^{\otimes 1}} - \beta = y^{\gamma^{\otimes 1}} - y$, then $\hat{\pi}\beta^{\gamma^{\otimes 1}} - \hat{\pi}\beta = \hat{\pi}(y^{\gamma^{\otimes 1}} - y) = 0$, and if this holds for every γ , then $\pi\beta \in A(K_v)$. Since $y \in E(\bar{K}_v) = E(\bar{K}_v \otimes L)^G$, we have $N_{L/M}y = py = x$ and the proposition follows. \square

The following corollary gives a purely local formula for δ_v , depending only on E and the local extension L_w/K_v (where w is a prime of L above v).

COROLLARY 5.3. *Suppose v is a prime of K and w is a prime of L above v . If $L_w \neq K_v$ then let L'_w be the unique subfield of L_w containing K_v with $[L_w : L'_w] = p$, and otherwise let $L'_w := L_w = K_v$. Let N_{L_w/L'_w} denote the norm map $E(L_w) \rightarrow E(L'_w)$. Then*

$$\delta_v \equiv \dim_{\mathbf{F}_p} E(K_v)/(E(K_v) \cap N_{L_w/L'_w}E(L_w)) \pmod{2}.$$

In particular if $N_{L_w/L'_w} : E(L_w) \rightarrow E(L'_w)$ is surjective (for example, if v splits completely in L/K) then $\delta_v = 0$.

Proof. By Proposition 5.2

$$H_{\mathcal{E}}^1(K_v, E[p])/H_{\mathcal{E} \cap \mathcal{A}}^1(K_v, E[p]) \cong E(K_v)/(E(K_v) \cap N_{L/M}E(L_v)),$$

and δ_v is the \mathbf{F}_p -dimension (modulo 2) of the left-hand side. Since L/K is

cyclic, L'_w is the completion of M at the prime below w , so we have

$$E(K_v) \cap N_{L/M}E(L_v) = E(K_v) \cap N_{L_w/L'_w}E(L_w).$$

This proves the corollary. □

By *local field* we mean a finite extension of \mathbf{Q}_ℓ for some rational prime ℓ .

LEMMA 5.4. *If \mathcal{K} is a local field with residue characteristic different from p , and E is defined over \mathcal{K} , then $E(\mathcal{K})/pE(\mathcal{K}) = E(\mathcal{K})[p^\infty]/pE(\mathcal{K})[p^\infty]$ and in particular*

$$\dim_{\mathbf{F}_p} E(\mathcal{K})/pE(\mathcal{K}) = \dim_{\mathbf{F}_p} E(\mathcal{K})[p].$$

Proof. There is an isomorphism of topological groups

$$E(\mathcal{K}) \cong E(\mathcal{K})[p^\infty] \oplus C \oplus D$$

with a finite group C of order prime to p and a free \mathbf{Z}_ℓ -module D of finite rank, where ℓ is the residue characteristic of v . Since $E(\mathcal{K})[p^\infty]$ is finite, the lemma follows easily. □

LEMMA 5.5. *Suppose \mathcal{L}/\mathcal{K} is a cyclic extension of degree p of local fields and E is defined over \mathcal{K} . Let ℓ denote the residue characteristic of \mathcal{K} .*

- (i) *If \mathcal{L}/\mathcal{K} is unramified and E has good reduction, then $N_{\mathcal{L}/\mathcal{K}}E(\mathcal{L}) = E(\mathcal{K})$.*
- (ii) *If \mathcal{L}/\mathcal{K} is ramified, $\ell \neq p$, and E has good reduction, then*

$$E(\mathcal{K})/pE(\mathcal{K}) \rightarrow E(\mathcal{L})/pE(\mathcal{L})$$

is an isomorphism and $N_{\mathcal{L}/\mathcal{K}}E(\mathcal{L}) = pE(\mathcal{K})$.

Proof. The first assertion is Corollary 4.4 of [M].

Suppose now that $\ell \neq p$, \mathcal{L}/\mathcal{K} is ramified, and E has good reduction. Then $\mathcal{K}(E[p^\infty])/\mathcal{K}$ is unramified, so $\mathcal{K}(E(\mathcal{L})[p^\infty]) = \mathcal{K}$, i.e., $E(\mathcal{K})[p^\infty] = E(\mathcal{L})[p^\infty]$. Now (ii) follows from Lemma 5.4. □

THEOREM 5.6. *Suppose that $v \nmid p$ and E has good reduction at v . Let w be a prime of L above v . If L_w/K_v is nontrivial and totally ramified, then*

$$\delta_v \equiv \dim_{\mathbf{F}_p} E(K_v)[p] \pmod{2}.$$

Proof. Let L'_w be the intermediate field $K_v \subset L'_w \subset L_w$ with $[L_w : L'_w] = p$, as in Corollary 5.3. Applying Lemma 5.5(ii) to L_w/L'_w and to L'_w/K_v shows that

$$N_{L_w/L'_w}E(L_w) = pE(L'_w) \quad \text{and} \quad E(K_v) \cap pE(L'_w) = pE(K_v),$$

so by Corollary 5.3 and Lemma 5.4 we have

$$\delta_v \equiv \dim_{\mathbf{F}_p} E(K_v)/pE(K_v) \equiv \dim_{\mathbf{F}_p} E(K_v)[p] \pmod{2}. \quad \square$$

THEOREM 5.7. *Suppose that E is defined over $\mathbf{Q}_p \subset K_v$ with good supersingular reduction at p . If $p = 3$ assume further that $|E(\mathbf{F}_3)| = 4$.*

If K_v contains the unramified quadratic extension of \mathbf{Q}_p , then $\delta_v = 0$.

Proof. Under these hypotheses $|E(\mathbf{F}_p)| = p + 1$, so the characteristic polynomial of Frobenius on E/\mathbf{F}_p is $X^2 + p$. It follows that the characteristic polynomial of Frobenius on E/\mathbf{F}_{p^2} is $(X + p)^2$. In other words, multiplication by $-p$ reduces to the Frobenius endomorphism of E/\mathbf{F}_{p^2}

Let $\mathbf{Q}_{p^2} \subset K_v$ denote the unramified quadratic extension of \mathbf{Q}_p , and \mathbf{Z}_{p^2} its ring of integers. Let \hat{E} denote the formal group over \mathbf{Z}_{p^2} giving the kernel of reduction on E , and $[-p](X) \in \mathbf{Z}_p[[X]]$ the power series giving multiplication by $-p$ on \hat{E} . Then $[-p](X) \equiv -pX \pmod{X^2}$, and since $-p$ reduces to Frobenius, we have $[-p](X) \equiv X^{p^2} \pmod{p}$. In other words, \hat{E} is a Lubin-Tate formal group of height 2 over \mathbf{Z}_{p^2} , for the uniformizing parameter $-p$.

It follows that $\mathbf{Z}_{p^2} \subset \text{End}(\hat{E})$. Therefore $\hat{E}(K_v)$ is a \mathbf{Z}_{p^2} -module, and since E has supersingular reduction, $E(K_v)/pE(K_v) \cong \hat{E}(K_v)/p\hat{E}(K_v)$ is a vector space over $\mathbf{Z}_{p^2}/p\mathbf{Z}_{p^2} = \mathbf{F}_{p^2}$. Similarly, if w is a prime of L above v then $\hat{E}(L_w)$ is a $\mathbf{Z}_{p^2}[\text{Gal}(L_w/K_v)]$ -module and $E(L_w)/pE(L_w)$ is an \mathbf{F}_{p^2} -vector space. Hence $E(K_v)/(E(K_v) \cap N_{L_w/L'_w} E(L_w))$ is an \mathbf{F}_{p^2} -vector space, so its \mathbf{F}_p -dimension δ_v is even. \square

6. Dihedral extensions

Keep the notation of the previous sections. For cyclic extensions L of K in F , Proposition 2.1 relates $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K)$ to $\dim_{\mathbf{F}_p} \text{Sel}_p(E/K)$, and Corollary 4.6 relates $\dim_{\mathbf{F}_p} \text{Sel}_p(E/K)$ to $\dim_{\mathbf{F}_p} \text{Sel}_{\mathfrak{p}}(A_L/K)$. Next we need to relate $\dim_{\mathbf{F}_p} \text{Sel}_{\mathfrak{p}}(A_L/K)$ to $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(A_L/K)$. For this we need an additional hypothesis.

Suppose now that c is an automorphism of order 2 of K , let $k \subset K$ be the fixed field of c , and suppose that E is defined over k . Fix a cyclic extension L/K of degree p^n , and let $A := A_L$, $R := R_L$, $\mathfrak{p} \subset R$ the maximal ideal, etc., as in Section 5. We assume further that L is Galois over k with dihedral Galois group, i.e., c acts by inversion on $G := \text{Gal}(L/K)$.

THEOREM 6.1. $\dim_{\mathbf{F}_p} (\text{III}(A/K)/\text{III}(A/K)_{\text{div}})[\mathfrak{p}]$ is even.

Theorem 6.1 will be proved in Appendix A.

Remark 6.2. Theorem 6.1 is essential for our applications. Without it, the formula in Proposition 6.3 below would not hold, and our approach would fail. The proof of Theorem 6.1 depends heavily on the fact that L/k is a dihedral extension. Stein [S] has given examples with $K = \mathbf{Q}$ where L/\mathbf{Q} is abelian, $\text{III}(A/\mathbf{Q})$ is finite and $\dim_{\mathbf{F}_p} \text{III}(A/\mathbf{Q})[p]$ is odd.

If A had a polarization of degree prime to p , then Theorem 6.1 would follow directly from Tate's generalization of the Cassels pairing [T2]. However, Howe [He] showed that (under mild hypotheses) every polarization of A has degree divisible by p^2 .

Let $R_p := R \otimes \mathbf{Z}_p$.

PROPOSITION 6.3.

$$\text{corank}_{R_p} \text{Sel}_{p^\infty}(A/K) \equiv \dim_{\mathbf{F}_p} H_{\mathcal{A}}^1(K, E[p]) - \dim_{\mathbf{F}_p} E(K)[p] \pmod{2}.$$

Proof. The proof is identical to that of the formula for

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K)$$

in Proposition 2.1, using Theorem 3.4(ii) to view $R \subset \text{End}_K(A)$, using Theorem 6.1 in place of the Cassels pairing, and using Proposition 4.1 to identify $A(K)[p]$ with $E(K)[p]$. \square

THEOREM 6.4. *Suppose that S is a set of primes of K containing all primes above p , all primes ramified in L/K , and all primes where E has bad reduction. Then*

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) - \text{corank}_{R_p} \text{Sel}_{p^\infty}(A/K) \equiv \sum_{v \in S} \delta_v \pmod{2}.$$

Proof. This follows directly from Corollary 4.6 and Propositions 2.1 and 6.3. \square

LEMMA 6.5. *Suppose v is a prime of K and $v = v^c$. Let w be a prime of L above v . Then*

- (i) L_w/K_v is totally ramified (we allow $L_w = K_v$),
- (ii) if $v \nmid p$ and $L_w \neq K_v$ then v is unramified in K/k .

Proof. Let w be a prime of L above v , and u the prime of k below v . Since $v = v^c$, the group $\text{Gal}(L_w/k_u)$ is dihedral. The inertia subgroup $I \subset \text{Gal}(L_w/k_u)$ is normal with cyclic quotient, and the only subgroups with this property are $\text{Gal}(L_w/k_u)$ and $\text{Gal}(L_w/K_v)$. This proves (i).

Suppose now that v is ramified in K/k , and let ℓ be the residue characteristic of K_v . By (i), the inertia group I is a dihedral group of order $2[L_w : K_v]$. On the other hand, the Sylow ℓ -subgroup of I is normal with cyclic quotient (the tame inertia group). The maximal abelian quotient of I has order 2, so $[L_w : K_v]$ must be a power of ℓ , so $\ell = p$. \square

LEMMA 6.6. *If v is a prime of K where E has good reduction, $v \nmid p$, $v = v^c$, and v is ramified in L/K , then $\dim_{\mathbf{F}_p} E(K_v)[p]$ is even.*

Proof. Suppose $v \nmid p$, $v = v^c$, and v ramifies in L/K . Fix a prime w of L above v , and let u be the prime of k below v . Let κ_+ and κ denote the residue fields of k_u and K_v , respectively. Note that K_v/k_u is quadratic since $v = v^c$, and unramified by Lemma 6.5(ii). Let ϕ be the Frobenius generator of $\text{Gal}(K_v^{\text{ur}}/k_u)$, so ϕ^2 is the Frobenius of $\text{Gal}(K_v^{\text{ur}}/K_v)$.

By Lemma 6.5(i), L_w/K_v is totally, tamely ramified. A standard result from algebraic number theory gives a $\text{Gal}(\kappa/\kappa_+)$ -equivariant injective homomorphism $\text{Gal}(L_w/K_v) \hookrightarrow \kappa^\times$. Since c acts by inversion on $\text{Gal}(L_w/K_v)$, which is a nontrivial p -group by assumption, it follows that ϕ acts as inversion on $\mu_p \subset \kappa^\times$.

Let $\alpha, \beta \in \bar{\mathbf{F}}_p^\times$ be the eigenvalues of ϕ acting on $E[p]$. The Weil pairing and the action of ϕ on μ_p show that $\alpha\beta = -1$. If $\alpha \neq \pm 1$, then 1 is not an eigenvalue of ϕ^2 acting on $E[p]$, so $E(K_v)[p] = E[p]^{\phi^2=1} = 0$. If $\alpha = \pm 1$, then $\{\alpha, \beta\} = \{1, -1\}$, the action of ϕ on $E[p]$ is diagonalizable, ϕ^2 is the identity on $E[p]$, and so $E(K_v)[p] = E[p]^{\phi^2=1} = E[p]$. In either case, $\dim_{\mathbf{F}_p} E(K_v)[p]$ is even. □

THEOREM 6.7. *If $v \mid p$ and E has good ordinary reduction at v , then $\delta_v = 0$.*

Proof. Let w be a prime of L above v . The theorem follows directly from Corollary 5.3 and either Proposition B.3 of Appendix B (if L_w/K_v is totally ramified) or Lemma 5.5(i) (if not). □

7. The main theorems

Fix a quadratic extension K/k with nontrivial automorphism c , an elliptic curve E defined over k , and an odd rational prime p . Recall that if F is an extension of K then $\mathcal{S}_p(E/F) := \text{Hom}(\text{Sel}_{p^\infty}(E/F), \mathbf{Q}_p/\mathbf{Z}_p) \otimes \mathbf{Q}_p$. If L is a cyclic extension of K in F , let R_L and A_L be as defined in Definitions 3.2 and 3.3.

THEOREM 7.1. *Suppose F is an abelian p -extension of K , dihedral over k (i.e., F is Galois over k and c acts by inversion on $\text{Gal}(F/K)$). Define*

$$\mathfrak{S} := \{\text{primes } v \text{ of } K : v \text{ ramifies in } F/K \text{ and } v = v^c\},$$

and suppose that for every $v \in \mathfrak{S}$, one of the following three conditions holds:

- (a) $v \nmid p$ and E has good reduction at v ,
- (b) $v \mid p$ and E has good ordinary reduction at v ,

- (c) $v \mid p$, E is defined over $\mathbf{Q}_p \subset K_v$ with good supersingular reduction at p (and if $p = 3$, then $|E(\mathbf{F}_3)| = 4$), and K_v contains the unramified quadratic extension of \mathbf{Q}_p .

Then:

- (i) For every cyclic extension L of K in F ,

$$\text{corank}_{R_L \otimes \mathbf{Z}_p} \text{Sel}_{p^\infty}(A_L/K) \equiv \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) \pmod{2}.$$

- (ii) If Ξ is the set of cyclic extensions L of K contained in F , $G = \text{Gal}(F/K)$, and $\mathbf{Q}[G] \cong \bigoplus_{L \in \Xi} \mathbf{Q}[G]_L$ is the decomposition (3.2) of $\mathbf{Q}[G]$ into its isotypic components, then there an isomorphism of $\mathbf{Q}_p[G]$ -modules

$$\mathcal{S}_p(E/F) \cong \bigoplus_{L \in \Xi} (\mathbf{Q}[G]_L \otimes \mathbf{Q}_p)^{d_L}$$

where for every L ,

$$d_L := \text{corank}_{R_L \otimes \mathbf{Z}_p} \text{Sel}_{p^\infty}(A_L/K) \equiv \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) \pmod{2}.$$

Proof. Suppose that L is a cyclic extension of K in F , and let $R_p := R_L \otimes \mathbf{Z}_p$ as in Section 6.

Let v be a prime of K . If $v \neq v^c$ then $\delta_v + \delta_{v^c} \equiv 0 \pmod{2}$ by Lemma 5.1. If $v = v^c$ and v is unramified in L/K , then v splits completely in L/K by Lemma 6.5(i), so $\delta_v = 0$ by Corollary 5.3. Therefore by Theorem 6.4 we have

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) - \text{corank}_{R_p} \text{Sel}_{p^\infty}(A_L/K) \equiv \sum_{v \in \mathfrak{S}} \delta_v \pmod{2}.$$

We will show that if $v \in \mathfrak{S}$ then $\delta_v = 0$, which will prove (i).

Case 1: $v \nmid p$. Then (a) holds, so E has good reduction at v . If w is a prime of L above v , then L_w/K_v is totally ramified by Lemma 6.5(i). Thus if $L_w = K_v$ then $\delta_v = 0$ by Corollary 5.3, and if $L_w \neq K_v$ then Theorem 5.6 and Lemma 6.6 show that $\delta_v \equiv \dim_{\mathbf{F}_p} E(K_v)[p] \equiv 0 \pmod{2}$.

Case 2: $v \mid p$. Then either (b) or (c) must hold. If (b) holds then $\delta_v = 0$ by Theorem 6.7, and if (c) holds then $\delta_v = 0$ by Theorem 5.7. This proves (i).

By Corollary 3.7, $\mathcal{S}_p(E/F) \cong \bigoplus_{L \in \Xi} \mathcal{S}_p(A_L/K)$. By Theorem 3.4(ii), $\mathcal{S}_p(A_L/K)$ is a vector space over the field $\mathbf{Q}[G]_L \otimes \mathbf{Q}_p = R_L \otimes \mathbf{Q}_p$, and by (i) its dimension d_L is congruent to $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K)$ modulo 2. This proves (ii). □

THEOREM 7.2. *Suppose F/k and E satisfy the hypotheses of Theorem 7.1.*

If $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K)$ is odd, then $\mathcal{S}_p(E/F)$ has a submodule isomorphic to $\mathbf{Q}_p[\text{Gal}(F/K)]$, and in particular

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/F) \geq [F : K].$$

Proof. In Theorem 7.1(ii) we have $d_L \geq 1$ for every L , and the theorem follows. \square

THEOREM 7.3. *Suppose F is an abelian p -extension of K , dihedral over k , and all three of the following conditions are satisfied:*

- (a) *every prime $v \nmid p$ of K that ramifies in F/K satisfies $E(K_v)[p] = 0$,*
- (b) *every prime v of K where E has bad reduction splits completely in F/K ,*
- (c) *for every prime v of K dividing p , E has good ordinary reduction at v and if κ is the residue field of K_v , then $E(\kappa)[p] = 0$.*

If $\text{Sel}_{p^\infty}(E/K) \cong \mathbf{Q}_p/\mathbf{Z}_p$ (for example, if $\text{rank}_{\mathbf{Z}} E(K) = 1$ and $\text{III}(E/K)[p] = 0$), then $\mathcal{S}_p(E/F) \cong \mathbf{Q}_p[\text{Gal}(F/K)]$, and in particular $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/F) = [F : K]$.

Proof. Note that the hypotheses of this theorem are stronger than those of Theorem 7.1, so we can apply Theorem 7.1.

Suppose L is a nontrivial cyclic extension of K in F , and $K \subset M \subset L$ with $[L : M] = p$. We will show that for every prime v of K and w of L above v ,

$$(7.1) \quad E(K_v) \subset N_{L_w/M_w} E(L_w).$$

Assume this for the moment. Then $H_{\mathcal{A}}^1(K_v, E[p]) = H_{\mathcal{E}}^1(K_v, E[p])$ for every v by Proposition 5.2, so if \mathfrak{p}_L is the prime above p in $R_L \subset \text{End}(A_L)$, we have

$$\text{Sel}_{\mathfrak{p}_L}(A_L/K) = H_{\mathcal{A}}^1(K, E[p]) = H_{\mathcal{E}}^1(K, E[p]) = \text{Sel}_p(E/K).$$

Let $d_L := \text{corank}_{R_L \otimes \mathbf{Z}_p} \text{Sel}_{p^\infty}(A_L/K)$. Using (2.1) and (2.2) (or the proof of Proposition 2.1) and Proposition 4.1, we have

$$d_L \leq \dim_{\mathbf{F}_p} \text{Sel}_{\mathfrak{p}_L}(A_L/K) - \dim_{\mathbf{F}_p} A_L[\mathfrak{p}_L] = \dim_{\mathbf{F}_p} \text{Sel}_p(E/K) - \dim_{\mathbf{F}_p} E[p] = 1.$$

But by Theorem 7.1(i), d_L is odd, so $d_L = 1$. This holds for every L (including $L = K$), so the theorem follows directly from Theorem 7.1(ii).

It remains to prove (7.1).

Case 1: $v \nmid p$, E has good reduction at v , v is unramified in L/K . In this case (7.1) holds by Lemma 5.5(i).

Case 2: $v \nmid p$, E has good reduction at v , v is ramified in L/K . In this case $E(K_v) = pE(K_v)$ by assumption (a) and Lemma 5.4, so (7.1) holds.

Case 3: $v \mid p$, E has bad reduction at v . In this case $L_w = M_w$ by assumption (b), so (7.1) holds.

Case 4: $v \mid p$. If L_w/K_v is not totally ramified, then L_w/M_w is unramified and (7.1) holds by Lemma 5.5(i). If L_w/K_v is totally ramified, then (7.1) holds by Proposition B.3 of Appendix B and assumption (c). This completes the proof. \square

8. Special cases

8.1. *Odd Selmer corank.* In general it can be very difficult to determine the parity of $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K)$. We now discuss some general situations in which the corank can be forced to be odd.

Fix an elliptic curve E defined over \mathbf{Q} , and let N_E be its conductor. Fix a Galois extension K of \mathbf{Q} such that $\text{Gal}(K/\mathbf{Q})$ is dihedral of order $2m$ with m odd, $m \geq 1$. Let M be the quadratic extension of \mathbf{Q} in K , Δ_M the discriminant of M , and χ_M the quadratic Dirichlet character attached to M . Let c be one of the elements of order 2 in $\text{Gal}(K/\mathbf{Q})$, and let k be the fixed field of c .

LEMMA 8.1. $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) \equiv \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/M) \pmod{2}$.

Proof. The restriction map $\mathcal{S}_p(E/M) \rightarrow \mathcal{S}_p(E/K)^{\text{Gal}(K/M)}$ is an isomorphism, so in the \mathbf{Q}_p -representation $\mathcal{S}_p(E/K)/\mathcal{S}_p(E/M)$ of $\text{Gal}(K/\mathbf{Q})$, neither of the two one-dimensional representations occurs. Since all other representations of $\text{Gal}(K/\mathbf{Q})$ have even dimension, we have that

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) - \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/M) = \dim_{\mathbf{Q}_p}(\mathcal{S}_p(E/K)/\mathcal{S}_p(E/M))$$

is even. \square

The following proposition follows from the ‘‘parity theorem’’ for the p -power Selmer group proved by Nekovář [N1] and Kim [K].

PROPOSITION 8.2. *Suppose that $p > 3$ is a prime, and that p , Δ_M , and N_E are pairwise relatively prime. Then $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K)$ is odd if and only if $\chi_M(-N_E) = -1$.*

Proof. Let E' be the quadratic twist of E by χ_M , and let w, w' be the signs in the functional equation of $L(E/\mathbf{Q}, s)$ and $L(E'/\mathbf{Q}, s)$, respectively. Since Δ_M and N_E are relatively prime, a well-known formula shows that $ww' = \chi_M(-N_E)$.

Using Lemma 8.1 we have

$$\begin{aligned} \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) &\equiv \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/M) \pmod{2} \\ &= \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/\mathbf{Q}) + \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E'/\mathbf{Q}). \end{aligned}$$

By a theorem of Nekovář [N1] (if E has ordinary reduction at p) or Kim [K] (if E has supersingular reduction at p), we have that $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/\mathbf{Q})$ is even

if and only if $w = 1$, and similarly for E' and w' . Thus $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K)$ is odd if and only if $w = -w'$, and the proposition follows. \square

For every prime p , let $\mathcal{K}_{c,p}^-$ be the maximal abelian p -extension of K that is Galois and dihedral over k , and unramified (over K) at all primes dividing N_E that do not split in M/\mathbf{Q} . (Note that if a rational prime ℓ splits in M , then every prime of k above ℓ splits in K/k since $[K : M]$ is odd.)

THEOREM 8.3. *Suppose $p > 3$ is prime, and p, Δ_M , and N_E are pairwise relatively prime. If $\chi_M(-N_E) = -1$, then for every finite extension F of K in $\mathcal{K}_{c,p}^-$,*

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/F) \geq [F : K].$$

Proof. This will follow directly from Theorem 7.2 and Proposition 8.2, once we show that the hypotheses of Theorem 7.1 are satisfied. By definition of $\mathcal{K}_{c,p}^-$, the set \mathfrak{S} of Theorem 7.1 contains only primes above p , and since $p \nmid N_E \Delta_M$ either (b) or (c) holds for every $v \in \mathfrak{S}$. \square

If $m = 1$, so $K = M$, and if M is imaginary, then $\mathcal{K}_{c,p}^-$ contains the anticyclotomic \mathbf{Z}_p -extension of K , and thanks to [Co] and [V] we know that the bulk of the contribution to the Selmer groups in Theorem 8.3 comes from Heegner points.

If $m = 1$ and M is real, then there is no \mathbf{Z}_p -extension of K in $\mathcal{K}_{c,p}^-$. However, $\mathcal{K}_{c,p}^-$ is still an infinite extension of K , and (for example) every finite abelian p -group occurs as a quotient of $\text{Gal}(\mathcal{K}_{c,p}^-/K)$.

More generally, for arbitrary m , if M is imaginary then $\mathcal{K}_{c,p}^-$ contains a \mathbf{Z}_p^d -extension of K with $d = (m+1)/2$, and if M is real then K is totally real so $\mathcal{K}_{c,p}^-$ is infinite but contains no \mathbf{Z}_p -extension of K . Except for Heegner points in special cases (such as when $m = 1$ and M is imaginary), it is not known where the Selmer classes in Theorem 8.3 come from.

8.2. Split multiplicative reduction at p . Suppose now that K/k is a quadratic extension, and F is a finite abelian p -extension of K , dihedral over k . Suppose that E is an elliptic curve over k , and v is a prime of K above p , inert in K/k , where E has split multiplicative reduction. If F/K is ramified at v then Theorems 7.1 and 7.2 do not apply. We now study this case more carefully.

LEMMA 8.4. *Suppose v is a prime of K above p such that $v = v^c$, u is the prime of k below v , and E has split multiplicative reduction at u . If L is a nontrivial cyclic extension of K in F , v is totally ramified in L/K , $K \subset L' \subset L$ with $[L : L'] = p$, and w is a prime of L above v , then*

$$[E(K_v) : E(K_v) \cap N_{L/L'}E(L_w)] = p.$$

Proof. Let \mathfrak{m}_u denote the maximal ideal of k_u . Since E has split multiplicative reduction, there is a nonzero $q \in \mathfrak{m}_u$ such that $E(L_w) \cong L_w^\times/q^{\mathbf{Z}}$ as $\text{Gal}(L_w/k_u)$ -modules.

Since $v = v^c$, L_w/k_v is dihedral so the maximal abelian extension of k_v in L_w is K_v . Thus local class field theory gives an identity of norm groups

$$N_{K_v/k_v} K_v^\times = N_{L_w/k_v} L_w^\times \subset N_{L_w/L'_w} L_w^\times.$$

Since $q^2 \in N_{K_v/k_v} K_v^\times$ and $[(L'_w)^\times : N_{L_w/L'_w} L_w^\times] = [L_w : L'_w] = p$ is odd, we see that $q \in N_{L_w/L'_w} L_w^\times$, and so

$$(8.1) \quad [E(K_v) : E(K_v) \cap N_{L/L'} E(L_w)] = [K_v^\times : K_v^\times \cap N_{L/L'} L_w^\times].$$

Let $[L : K] = p^n$. If $[\cdot, \cdot]$ denotes the Artin map of local class field theory, then $K_v^\times \cap N_{L/L'} L_w^\times$ is the kernel of the map $K_v^\times \rightarrow \text{Gal}(L_w/K_v)$ given by

$$x \mapsto [x, L_w/L'_w] = [N_{L'/K} x, L_w/K_v] = [x^{p^{n-1}}, L_w/K_v] = [x, L_w/K_v]^{p^{n-1}}.$$

Since $x \mapsto [x, L_w/K_v]$ maps K_v^\times onto a cyclic group of order p^n , we conclude that the index (8.1) is p , as desired. \square

Let \mathfrak{S}_p be the set of primes v of K above p such that $v = v^c$ and neither of the hypotheses (b) or (c) of Theorem 7.1 hold for v .

THEOREM 8.5. *Suppose that F is a finite abelian p -extension of K that is dihedral over k and unramified at all primes $v \nmid p$ of bad reduction that do not split in K/k . Suppose further that for every prime $v \in \mathfrak{S}_p$, E has split multiplicative reduction at v and v is totally ramified in F/K . Then:*

(i) *If $\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) + |\mathfrak{S}_p|$ is odd, then*

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/F) \geq \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) + [F : K] - 1.$$

(ii) *If $\text{Sel}_{p^\infty}(E/K)$ is finite and $|\mathfrak{S}_p|$ is odd, then*

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/F) \geq [F : K] - 1.$$

(iii) *Suppose that $|\mathfrak{S}_p| = 1$, and the hypotheses (a), (b), or (c) of Theorem 7.3 hold for every prime v of K not in \mathfrak{S}_p . If $\text{Sel}_{p^\infty}(E/K) = 0$, then*

$$\text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/F) = [F : K] - 1.$$

Proof. The proof is identical to that of Theorems 7.2 and 7.3, except that we use Lemma 8.4 to compute the δ_v for $v \in \mathfrak{S}_p$.

Suppose L is a nontrivial cyclic extension of K in F . Exactly as in Theorem 7.1, we have $\sum_{v \notin \mathfrak{S}_p} \delta_v \equiv 0 \pmod{2}$. If $v \in \mathfrak{S}_p$, then $\delta_v = 1$ by Lemma

8.4 and Corollary 5.3. Thus we conclude that $\sum_v \delta_v \equiv |\mathfrak{S}_p| \pmod{2}$. Exactly as in Theorem 7.1 we conclude using Theorem 6.4 that

$$(8.2) \quad \mathcal{S}_p(E/F) \cong \bigoplus_{L \in \Xi} (\mathbf{Q}[G]_L \otimes \mathbf{Q}_p)^{d_L}$$

where $d_L \equiv \text{corank}_{\mathbf{Z}_p} \text{Sel}_{p^\infty}(E/K) + |\mathfrak{S}_p| \pmod{2}$ for every $L \neq K$. Assertion (i) now follows exactly as in the proof of Theorem 7.2, and (ii) is a special case of (i).

For (iii), it follows exactly as in the proof of Theorem 7.3 that $H_{\mathcal{A}}^1(K_v, E[p]) = H_{\mathcal{E}}^1(K_v, E[p])$ for every $v \notin \mathfrak{S}_p$. Thus if $\mathfrak{S}_p = \{v_0\}$, there is an exact sequence

$$(8.3) \quad 0 \rightarrow H_{\mathcal{E} \cap \mathcal{A}}^1(K, E[p]) \rightarrow H_{\mathcal{A}}^1(K, E[p]) \rightarrow H_{\mathcal{A}}^1(K_{v_0}, E[p]) / H_{\mathcal{E} \cap \mathcal{A}}^1(K_{v_0}, E[p]).$$

By Lemma 8.4 and Proposition 5.2,

$$\dim_{\mathbf{F}_p} H_{\mathcal{A}}^1(K_{v_0}, E[p]) = \dim_{\mathbf{F}_p} H_{\mathcal{E}}^1(K_{v_0}, E[p]) = \dim_{\mathbf{F}_p} H_{\mathcal{E} \cap \mathcal{A}}^1(K_{v_0}, E[p]) + 1$$

(the first equality holds because \mathcal{A} and \mathcal{E} are self-dual), so it follows from (8.3) that

$$\begin{aligned} \dim_{\mathbf{F}_p} \text{Sel}_{\mathfrak{p}_L}(A_L/K) &= \dim_{\mathbf{F}_p} H_{\mathcal{A}}^1(K, E[p]) \leq \dim_{\mathbf{F}_p} H_{\mathcal{E}}^1(K, E[p]) + 1 \\ &= \dim_{\mathbf{F}_p} E[p] + 1 = \dim_{\mathbf{F}_p} A[\mathfrak{p}] + 1. \end{aligned}$$

Therefore $d_L := \text{corank}_{R_L \otimes \mathbf{Z}_p} \text{Sel}_{p^\infty}(A_L/K) \leq 1$. The proof of (i) showed that d_L is odd, so $d_L = 1$. Hence in (8.2) we have $d_L = 1$ if $L \neq K$, and $d_K = 0$. This proves (iii). \square

Remark 8.6. In the case where $K = M$ is imaginary quadratic and F is a subfield of the anticyclotomic \mathbf{Z}_p -extension, Bertolini and Darmon [BD] give a construction of Heegner-type points that account for most of the Selmer classes in Theorem 8.5.

Appendix A. Skew-Hermitian pairings

In this appendix we prove Proposition 4.4 and Theorem 6.1. Let p be an odd prime, L/K be a cyclic extension of number fields of degree p^n , $G := \text{Gal}(L/K)$, and $\mathcal{R} := R_L \otimes \mathbf{Z}_p$, where R_L is given by Definition 3.2. We view \mathcal{R} as a G_K -module by letting G_K act *trivially* (not the action induced from the action on R_L). Then \mathcal{R} is the cyclotomic ring over \mathbf{Z}_p generated by p^n -th roots of unity (see for example [MRS, Lemma 5.4(ii)]).

Let ι be the involution of R_L (resp., \mathcal{R}) induced by $\zeta \mapsto \zeta^{-1}$ for p^n -th roots of unity $\zeta \in R_L$ (resp., $\zeta \in \mathcal{R}$). If W is an \mathcal{R} -module, we let W^ι be the \mathcal{R} -module whose underlying abelian group is W , but with \mathcal{R} -action twisted by ι .

Definition A.1. Suppose W is an \mathcal{R} -module and B is a \mathbf{Z}_p -module. We say that a \mathbf{Z}_p -bilinear pairing

$$\langle \cdot, \cdot \rangle : W \times W \rightarrow B$$

is ι -adjoint if $\langle rx, y \rangle = \langle x, r^\iota y \rangle$ for every $r \in \mathcal{R}$ and $x, y \in W$. We say that a pairing

$$\langle \cdot, \cdot \rangle : W \times W \rightarrow \mathcal{R} \otimes_{\mathbf{Z}_p} B$$

is \mathcal{R} -semilinear if $\langle rx, y \rangle = r \langle x, y \rangle = \langle x, r^\iota y \rangle$ for every $r \in \mathcal{R}$ and $x, y \in W$, and we say $\langle \cdot, \cdot \rangle$ is skew-Hermitian if it is \mathcal{R} -semilinear and $\langle y, x \rangle = -\langle x, y \rangle^{\iota \otimes 1}$ for every $x, y \in W$.

We say that $\langle \cdot, \cdot \rangle$ is nondegenerate (resp., perfect) if the induced map $W \rightarrow \text{Hom}_{\mathbf{Z}_p}(W^\iota, B)$ (or $\text{Hom}_{\mathcal{R}}(W^\iota, \mathcal{R} \otimes_{\mathbf{Z}_p} B)$, depending on the context) is injective (resp., an isomorphism).

Definition A.2. Let ζ be a primitive p^n -th root of unity in R_L , and let $\pi := \zeta - \zeta^{-1}$. Then π is a generator of the prime \mathfrak{p}_L of R_L above p , and π is also a generator of the maximal ideal \mathfrak{p} of \mathcal{R} , and $\pi^\iota = -\pi$. Let $d := \pi^{p^{n-1}(pn-n-1)}$, so d is a generator of the inverse different of R_L/\mathbf{Z} and of \mathcal{R}/\mathbf{Z}_p , and $d^\iota = -d$. Define a trace pairing

$$t_{\mathcal{R}/\mathbf{Z}_p} : \mathcal{R} \times \mathcal{R} \rightarrow \mathbf{Z}_p, \quad t_{\mathcal{R}/\mathbf{Z}_p}(r, s) := \text{Tr}_{\mathcal{R}/\mathbf{Z}_p}(d^{-1}rs^\iota).$$

This pairing is ι -adjoint, perfect, and (since $d^\iota = -d$) skew-symmetric. Define $\tau : \mathcal{R} \rightarrow \mathbf{Z}_p$ by $\tau(r) := t_{\mathcal{R}/\mathbf{Z}_p}(1, r) = -\text{Tr}_{\mathcal{R}/\mathbf{Z}_p}(d^{-1}r)$.

LEMMA A.3. Suppose that W is an $\mathcal{R}[G_K]$ -module and B is a $\mathbf{Z}_p[G_K]$ -module. Composition with $\tau \otimes 1 : \mathcal{R} \otimes_{\mathbf{Z}_p} B \rightarrow B$ gives an isomorphism of G_K -modules

$$\text{Hom}_{\mathcal{R}}(W, \mathcal{R} \otimes_{\mathbf{Z}_p} B) \xrightarrow{\sim} \text{Hom}_{\mathbf{Z}_p}(W, B).$$

Proof. We will construct an inverse to the map in the statement of the lemma. Suppose $f \in \text{Hom}_{\mathbf{Z}_p}(W, B)$. Fix a \mathbf{Z}_p -basis $\{\nu_1, \dots, \nu_b\}$ of \mathcal{R} , and let $\{\nu_1^*, \dots, \nu_b^*\}$ be the dual basis with respect to $t_{\mathcal{R}/\mathbf{Z}_p}$, i.e., $t_{\mathcal{R}/\mathbf{Z}_p}(\nu_i, \nu_j^*) = \delta_{ij}$. For $x \in W$ define

$$\hat{f}(x) := \sum_{i=1}^b \nu_i^* \otimes f(\nu_i^\iota x) \in \mathcal{R} \otimes_{\mathbf{Z}_p} B.$$

Then for every j and x ,

$$(\tau \otimes 1)(\nu_j^\iota \hat{f}(x)) = \sum_{i=1}^b t_{\mathcal{R}/\mathbf{Z}_p}(1, \nu_j^\iota \nu_i^*) f(\nu_i^\iota x) = \sum_{i=1}^b t_{\mathcal{R}/\mathbf{Z}_p}(\nu_j, \nu_i^*) f(\nu_i^\iota x) = f(\nu_j^\iota x).$$

Since the ν_j are a basis of \mathcal{R} , we conclude that

$$(A.1) \quad (\tau \otimes 1)(r \hat{f}(x)) = f(rx) \text{ for every } r \in \mathcal{R}.$$

Thus if $s \in \mathcal{R}$ then for every r

$$(\tau \otimes 1)(r\hat{f}(sx)) = f(rsx) = (\tau \otimes 1)(rs\hat{f}(x)).$$

Since $t_{\mathcal{R}/\mathbf{Z}_p}$ is perfect and \mathcal{R} is free over \mathbf{Z}_p , it follows that $\hat{f}(sx) = s\hat{f}(x)$, so $\hat{f} \in \text{Hom}_{\mathcal{R}}(W, \mathcal{R} \otimes_{\mathbf{Z}_p} B)$.

By (A.1) with $r = 1$, $(\tau \otimes 1) \circ \hat{f} = f$ so $\text{Hom}_{\mathcal{R}}(W, \mathcal{R} \otimes_{\mathbf{Z}_p} B) \xrightarrow{\circ(\tau \otimes 1)} \text{Hom}_{\mathbf{Z}_p}(W, B)$ is surjective. The injectivity follows from the fact that $t_{\mathcal{R}/\mathbf{Z}_p}$ is perfect and \mathcal{R} is free over \mathbf{Z}_p . The G_K -equivariance is clear (recall that G_K acts trivially on \mathcal{R}). \square

PROPOSITION A.4. *Suppose that W is an \mathcal{R} -module and B is a \mathbf{Z}_p -module. Composition with $\tau \otimes 1 : \mathcal{R} \otimes_{\mathbf{Z}_p} B \rightarrow B$ gives a bijection between the set of \mathcal{R} -semilinear pairings $W \times W \rightarrow \mathcal{R} \otimes_{\mathbf{Z}_p} B$, and the set of ι -adjoint pairings $W \times W \rightarrow B$.*

If $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ maps to $\langle \cdot, \cdot \rangle_{\mathbf{Z}_p}$ under this bijection, then $\langle \cdot, \cdot \rangle_{\mathbf{Z}_p}$ is perfect (resp., G_K -equivariant) if and only if $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ is perfect (resp., G_K -equivariant).

Proof. By Lemma A.3, composition with $\tau \otimes 1$ induces a G_K -isomorphism

$$(A.2) \quad \text{Hom}_{\mathcal{R}}(W, \text{Hom}_{\mathcal{R}}(W^\iota, \mathcal{R} \otimes_{\mathbf{Z}_p} B)) \xrightarrow{\sim} \text{Hom}_{\mathcal{R}}(W, \text{Hom}_{\mathbf{Z}_p}(W^\iota, B)).$$

The left-hand side is the set of \mathcal{R} -semilinear pairings $W \times W \rightarrow \mathcal{R} \otimes_{\mathbf{Z}_p} B$, and the right-hand side is the set of ι -adjoint pairings $W \times W \rightarrow B$.

Since composition with $\tau \otimes 1$ identifies the isomorphisms in (A.2), we see that $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ is perfect if and only if $\langle \cdot, \cdot \rangle_{\mathbf{Z}_p}$ is perfect. Since (A.2) is G_K -equivariant, $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ is G_K -equivariant if and only if $\langle \cdot, \cdot \rangle_{\mathbf{Z}_p}$ is G_K -equivariant. This completes the proof of the proposition. \square

Let A be the abelian variety A_L of Definition 3.3. Recall (Definitions A.2 and 3.2 and Theorem 3.4(i)) that π is a generator of the prime \mathfrak{p}_L of R_L , $\pi\mathcal{R} = \mathfrak{p}$, and $\mathcal{I}_L = \mathfrak{p}_L^{p^n-1}$.

Definition A.5. Define a pairing $f : \mathcal{I}_L \times \mathcal{I}_L \rightarrow R_L$ by

$$f(\alpha, \beta) := \pi^{-2p^{n-1}} \alpha\beta^\iota.$$

Theorem 3.4(iv) gives a G_K -isomorphism $T_p(A) \cong \mathcal{I}_L \otimes T_p(E)$, and using this identification we define

$$\langle \cdot, \cdot \rangle_{\mathcal{R}} := f \otimes e : T_p(A) \times T_p(A) \longrightarrow \mathcal{R} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1)$$

where e is the Weil pairing on E . In other words, if $\alpha, \beta \in \mathcal{I}_L$ and $x, y \in T_p(E)$, we set

$$\langle \alpha \otimes x, \beta \otimes y \rangle_{\mathcal{R}} := (\pi^{-2p^{n-1}} \alpha\beta^\iota) \otimes e(x, y).$$

LEMMA A.6. *The pairing $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ of Definition A.5 is perfect, G_K -equivariant, and skew-Hermitian.*

Proof. The Weil pairing is perfect and skew-symmetric, and the pairing f is perfect and Hermitian (since $\pi^t = -\pi$). Thus $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ is perfect and skew-Hermitian. If $\alpha, \beta \in \mathcal{I}_L$, $x, y \in T_p(E)$, and $\gamma \in G_K$ then

$$\begin{aligned} \langle (\alpha \otimes x)^\gamma, (\beta \otimes y)^\gamma \rangle_{\mathcal{R}} &= \langle \alpha\gamma^{-1} \otimes \gamma x, \beta\gamma^{-1} \otimes \gamma y \rangle_{\mathcal{R}} \\ &= \pi^{-2p^{n-1}} (\alpha\gamma^{-1})(\beta\gamma^{-1})^t \otimes e(\gamma x, \gamma y) \\ &= \pi^{-2p^{n-1}} (\alpha\gamma^{-1})(\beta^t\gamma) \otimes e(x, y)^\gamma \\ &= f(\alpha, \beta) \otimes e(x, y)^\gamma = \langle \alpha \otimes x, \beta \otimes y \rangle_{\mathcal{R}}^\gamma \end{aligned}$$

since the Weil pairing is G_K -equivariant and G_K acts trivially on \mathcal{R} . □

The following is Proposition 4.4.

PROPOSITION A.7. *The Selmer structure \mathcal{A} on $E[p]$ of Definition 4.3 is self-dual.*

Proof. Using Proposition A.4, we let

$$\langle \cdot, \cdot \rangle_{\mathbf{Z}_p} : T_p(A) \times T_p(A) \longrightarrow \mathbf{Z}_p(1)$$

be the pairing corresponding under Proposition A.4 to the pairing $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ of Definition A.5, with $B = \mathbf{Z}_p(1)$. It follows from Proposition A.4 and Lemma A.6 that $\langle \cdot, \cdot \rangle_{\mathbf{Z}_p}$ is perfect, G_K -equivariant, and ι -adjoint.

By a generalization of Tate duality due to Bloch and Kato (see Proposition 3.8 and Example 3.11 of [BK]), for every prime v of K , the pairing $\langle \cdot, \cdot \rangle_{\mathbf{Z}_p}$ induces a perfect, ι -adjoint cup-product pairing

$$\lambda : H^1(K_v, T_p(A)) \times H^1(K_v, T_p(A) \otimes \mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p,$$

and under this pairing the image of $A(K_v) \rightarrow H^1(K_v, T_p(A))$ and the image of $A(K_v) \otimes \mathbf{Q}_p/\mathbf{Z}_p \rightarrow H^1(K_v, T_p(A) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$ are orthogonal complements of each other.

The pairing λ induces a pairing

$$\lambda_{\mathfrak{p}_L} : H^1(K_v, T_p(A)/\mathfrak{p}_L T_p(A)) \times H^1(K_v, (T_p(A) \otimes \mathbf{Q}_p/\mathbf{Z}_p)[\mathfrak{p}_L]) \longrightarrow \mathbf{F}_p.$$

We have isomorphisms (the first one uses the chosen generator π of \mathfrak{p}_L)

$$T_p(A)/\mathfrak{p}_L T_p(A) \cong A[\mathfrak{p}_L] \cong (T_p(A) \otimes \mathbf{Q}_p/\mathbf{Z}_p)[\mathfrak{p}_L].$$

Along with the identification $A[\mathfrak{p}_L] \cong E[p]$ of Proposition 4.1, this transforms $\lambda_{\mathfrak{p}_L}$ into a pairing $H^1(K_v, E[p]) \times H^1(K_v, E[p]) \rightarrow \mathbf{F}_p$, and one can check directly from the definition of λ that this pairing is the same as the local cup product pairing on $H^1(K_v, E[p])$ coming from the Weil pairing as in Section 1 and Section 2.

A couple of straightforward diagram chases (see for example Lemma 1.3.8 and Proposition 1.4.3 of [R]) show that the image of

$$(A.3) \quad A(K_v) \rightarrow H^1(K_v, T_p(A)) \rightarrow H^1(K_v, T_p(A)/\mathfrak{p}_L T_p(A)) \xrightarrow{\sim} H^1(K_v, E[p])$$

and the inverse image of $A(K_v) \otimes \mathbf{Q}_p/\mathbf{Z}_p$ under

$$H^1(K_v, E[p]) \xrightarrow{\sim} H^1(K_v, (T_p(A) \otimes \mathbf{Q}_p/\mathbf{Z}_p)[\mathfrak{p}_L]) \rightarrow H^1(K_v, T_p(A) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$$

are equal and are orthogonal complements under $\lambda_{\mathfrak{p}_L}$. By definition the image of (A.3) is $H^1_{\mathcal{A}}(K_v, E[p])$, so this proves that \mathcal{A} is self-dual. \square

It remains to prove Theorem 6.1, and for that we need to be in the dihedral setting of Section 6. We assume now that K has an automorphism c of order 2, that E is defined over the fixed field k of K , that L is Galois over k , and that c acts by inversion on $G := \text{Gal}(L/K)$.

We begin by fixing a model of A defined over k .

Definition A.8. Fix a lift of c to G_k , and denote this lift by \mathbf{c} . Then $\text{Gal}(L/k)$ is the semidirect product $G \rtimes H$, where H is the group of order 2 generated by the restriction of \mathbf{c} . Let $\mathcal{J}_L := (1 + \mathbf{c})\mathcal{I}_L$, where $\mathcal{I}_L \subset \mathbf{Z}[G] \subset \mathbf{Z}[\text{Gal}(L/k)]$ is the ideal of $\mathbf{Z}[G]$ given in Definition 3.2. Then \mathcal{J}_L is a right ideal of $\mathbf{Z}[\text{Gal}(L/k)]$, and we define an abelian variety A' over k by

$$A' := \mathcal{J}_L \otimes E$$

as in Definition 1.1 (and §6) of [MRS].

PROPOSITION A.9. (i) *Left multiplication by $(1 + \mathbf{c})$ is an isomorphism of right G_K -modules from \mathcal{I}_L to \mathcal{J}_L .*

(ii) *The isomorphism of (i) induces an isomorphism $A \cong A'$ defined over K .*

Proof. The first assertion is easily checked, and the second follows by Corollary 1.9 of [MRS]. See also Theorem 6.3 of [MRS]. \square

From now on we view A as defined over k , by using the model A' of A and Proposition A.9(ii). We extend the G_K -action on $T_p(A)$, \mathcal{I}_L , and \mathcal{R} to a G_k -action by identifying $T_p(A)$ with $T_p(A')$, \mathcal{I}_L with \mathcal{J}_L as in Proposition A.9(i), and letting \mathbf{c} act on (the trivial G_K -module) \mathcal{R} by ι . The actions on $T_p(A)$ and \mathcal{I}_L depend on the choice of \mathbf{c} .

PROPOSITION A.10. *With the conventions above, the pairing*

$$\langle \cdot, \cdot \rangle_{\mathcal{R}} : T_p(A) \times T_p(A) \rightarrow \mathcal{R} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1)$$

of Definition A.5 is G_k -equivariant.

Proof. By Theorem 2.2(iii) of [MRS], there is a G_k -isomorphism $T_p(A') \cong \mathcal{I}_L \otimes T_p(E)$. With the conventions above, this says that the isomorphism $T_p(A) \cong \mathcal{I}_L \otimes T_p(E)$, which was used to construct the pairing $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ in Definition A.5, is G_k -equivariant. The proposition follows from this exactly as in Lemma A.6, using the fact that for $\alpha \in \mathcal{I}_L$, $(1 + \mathbf{c})\alpha\mathbf{c} = (1 + \mathbf{c})\mathbf{c}\alpha^t = (1 + \mathbf{c})\alpha^t$. \square

Let $D_p := \mathcal{R} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p$.

PROPOSITION A.11. *Suppose that W is an \mathcal{R} -module of finite cardinality and a $\text{Gal}(K/k)$ -module, and suppose that there is a nondegenerate, skew-Hermitian, $\text{Gal}(K/k)$ -equivariant pairing*

$$[\cdot, \cdot] : W \times W \longrightarrow D_p.$$

Then W has isotropic \mathcal{R} -submodules M, M' such that $M \cong M'$ and $W = M \oplus M'$. In particular $\dim_{\mathbf{F}_p} W[\mathfrak{p}]$ is even.

Proof. Define a pairing $[\cdot, \cdot]' : W \times W \rightarrow D_p$ by $[v, w]' := [v, cw]$. It is straightforward to check that the pairing $[\cdot, \cdot]'$ is nondegenerate, \mathcal{R} -bilinear, and skew-symmetric. The proposition now follows by a well-known argument. \square

We will now deduce Theorem 6.1 from a (slight generalization of a) result of Flach. Let $\text{III}_{/\text{div}} := \text{III}(A/K)[p^\infty]/\text{III}(A/K)[p^\infty]_{\text{div}}$.

THEOREM A.12 (Flach [F]). *Suppose that*

$$\{ \cdot, \cdot \}_{\mathcal{R}} : T_p(A) \times T_p(A) \rightarrow \mathcal{R} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1)$$

is a perfect, G_k -equivariant, skew-Hermitian pairing. Then there is a perfect, $\text{Gal}(K/k)$ -equivariant, skew-Hermitian pairing,

$$[\cdot, \cdot]_{\mathcal{R}} : \text{III}_{/\text{div}} \times \text{III}_{/\text{div}} \rightarrow D_p.$$

Proof. This is essentially Theorems 1 and 2 of [F]. We sketch here the minor modifications to the arguments of [F] needed to prove Theorem A.12.

Given a G_K -equivariant pairing $T_p(A) \times T_p(A) \rightarrow \mathbf{Z}_p(1)$, Flach constructs a pairing $\text{III}_{/\text{div}} \times \text{III}_{/\text{div}} \rightarrow \mathbf{Q}_p/\mathbf{Z}_p$. The definition ([F] p. 116) is given explicitly in terms of cocycles. Since G_K acts trivially on \mathcal{R} , we have canonical isomorphisms

$$(A.4) \quad H^i(K, \mathcal{R} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1)) \cong \mathcal{R} \otimes_{\mathbf{Z}_p} H^i(K, \mathbf{Z}_p(1))$$

for every i , and similarly with K replaced by any of its completions K_v and/or with $\mathbf{Z}_p(1)$ replaced by $\mathbf{Q}_p/\mathbf{Z}_p(1)$. The isomorphisms (A.4) come from analogous isomorphisms on modules of cocycles. Using this, starting with our

pairing $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ and following Flach’s construction verbatim produces a pairing

$$[\cdot, \cdot]_{\mathcal{R}} : \mathbb{III}_{/\text{div}} \times \mathbb{III}_{/\text{div}} \rightarrow D_p.$$

We need to show that $[\cdot, \cdot]_{\mathcal{R}}$ is perfect, $\text{Gal}(K/k)$ -equivariant, and skew-Hermitian.

The fact that $[\cdot, \cdot]_{\mathcal{R}}$ is $\text{Gal}(K/k)$ -equivariant follows directly from the definition in [F], as each step is canonical and Galois-equivariant.

Similarly, following the definition in [F] and using that $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ is skew-Hermitian, one sees directly that $[rx, y]_{\mathcal{R}} = r[x, y]_{\mathcal{R}} = [x, r^{\iota}y]_{\mathcal{R}}$ for every $r \in R$, $x, y \in \mathbb{III}_{/\text{div}}$. The fact that $[y, x]_{\mathcal{R}} = -[x, y]_{\mathcal{R}}^{\iota}$ is proved exactly as Theorem 2 of [F], which proves the skew-symmetry of the pairing in Flach’s setting.

It remains only to show that $[\cdot, \cdot]_{\mathcal{R}}$ is perfect, or equivalently (since $\mathbb{III}_{/\text{div}}$ is finite) $[\cdot, \cdot]_{\mathcal{R}}$ is nondegenerate. Let $\langle \cdot, \cdot \rangle_{\mathbf{Z}_p} : T_p(A) \times T_p(A) \rightarrow \mathbf{Z}_p(1)$ (resp., $[\cdot, \cdot]_{\mathbf{Z}_p} : \mathbb{III}_{/\text{div}} \times \mathbb{III}_{/\text{div}} \rightarrow \mathbf{Q}_p/\mathbf{Z}_p$) be the pairing corresponding to $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ (resp., $[\cdot, \cdot]_{\mathcal{R}}$) under the correspondence of Proposition A.4.

By Proposition A.4, since $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ is perfect, $\langle \cdot, \cdot \rangle_{\mathbf{Z}_p}$ is perfect. One can check from the definition that $[\cdot, \cdot]_{\mathbf{Z}_p}$ is the pairing Flach constructs from $\langle \cdot, \cdot \rangle_{\mathbf{Z}_p}$, and thus Flach’s Theorem 1 shows that $[\cdot, \cdot]_{\mathbf{Z}_p}$ is perfect. Now Proposition A.4 shows that $[\cdot, \cdot]_{\mathcal{R}}$ is perfect. This completes the proof of the theorem. \square

Proof of Theorem 6.1. We apply Theorem A.12, using the pairing $\langle \cdot, \cdot \rangle_{\mathcal{R}}$ of Definition A.5 (along with Lemma A.6 and Proposition A.10) to produce a perfect, $\text{Gal}(K/k)$ -equivariant, skew-Hermitian pairing $[\cdot, \cdot]_{\mathcal{R}} : \mathbb{III}_{/\text{div}} \times \mathbb{III}_{/\text{div}} \rightarrow D_p$. By Proposition A.11 we conclude that $\dim_{\mathbf{F}_p}(\mathbb{III}(A/K)/\mathbb{III}(A/K)_{\text{div}})[\mathfrak{p}_L]$ is even. This is Theorem 6.1. \square

Remark A.13. It is tempting to try to simplify the arguments of this appendix by using the pairing of Definition A.5 along with the construction at the end of the proof of Theorem A.12, to try to produce a perfect, skew-symmetric, G_K -equivariant pairing $T_p(A) \times T_p(A) \rightarrow \mathbf{Z}_p(1)$. If so, Theorems 1 and 2 of [F] would give us directly a skew-symmetric perfect pairing on $\mathbb{III}_{/\text{div}}$. Unfortunately, because $\pi^{\iota} = -\pi$ and the different of \mathcal{R}/\mathbf{Z}_p is an odd power of \mathfrak{p} , one can produce in this way (as in the proof of Proposition A.7) a perfect *symmetric* pairing, but not a skew-symmetric one.

Appendix B. The local norm map in the ordinary case

In this appendix we study the cokernel of the local norm map when E has ordinary reduction, following and expanding on the proof from [LR] of some of the results of [M]. Our main result is Proposition B.3, which is used to prove Theorem 6.7.

If \mathcal{K} is an algebraic extension of \mathbf{Q}_p and E is an elliptic curve over \mathcal{K} with good ordinary reduction, let $E_1(\mathcal{K})$ denote the kernel of reduction in $E(\mathcal{K})$, and let $U_1(\mathcal{K})$ denote the units in the ring of integers of \mathcal{K} congruent to 1 modulo the maximal ideal. We can identify $E_1(\mathcal{K})$ (resp., $U_1(\mathcal{K})$) with the maximal ideal of \mathcal{K} under the operation given by the formal group of E (resp., the formal multiplicative group).

Suppose now that \mathcal{K} is a finite extension of \mathbf{Q}_p , with residue field κ . Let $u \in \mathbf{Z}_p^\times$ be the unit eigenvalue of Frobenius acting on the ℓ -adic Tate module of E , for $\ell \neq p$. Following [M], we say that E has anomalous reduction if $E(\kappa)[p] \neq 0$, or equivalently if $u \equiv 1 \pmod{p}$.

Fix a totally ramified cyclic extension \mathcal{L}/\mathcal{K} of degree p^n . Let ϕ denote the Frobenius generator of $\text{Gal}(\mathcal{L}^{\text{ur}}/\mathcal{L})$; the restriction of ϕ is the Frobenius generator of $\text{Gal}(\mathcal{K}^{\text{ur}}/\mathcal{K})$. Let $I_{\mathcal{L}/\mathcal{K}} \subset \mathbf{Z}[\text{Gal}(\mathcal{L}/\mathcal{K})]$ denote the augmentation ideal.

LEMMA B.1. *There is a commutative diagram with exact rows and columns*

$$\begin{array}{ccccccc}
 & & & 0 & & & 0 \\
 & & & \downarrow & & & \downarrow \\
 & & & E_1(\mathcal{L})/(E_1(\mathcal{L}) \cap I_{\mathcal{L}/\mathcal{K}}U_1(\mathcal{L}^{\text{ur}})) & \xrightarrow{N_{\mathcal{L}/\mathcal{K}}} & E_1(\mathcal{K}) & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \text{Gal}(\mathcal{L}/\mathcal{K}) & \longrightarrow & U_1(\mathcal{L}^{\text{ur}})/I_{\mathcal{L}/\mathcal{K}}U_1(\mathcal{L}^{\text{ur}}) & \xrightarrow{N_{\mathcal{L}/\mathcal{K}}} & U_1(\mathcal{K}^{\text{ur}}) \longrightarrow 0 \\
 & & \downarrow 1-u & & \downarrow \phi-u & & \downarrow \phi-u \\
 0 & \longrightarrow & \text{Gal}(\mathcal{L}/\mathcal{K}) & \longrightarrow & U_1(\mathcal{L}^{\text{ur}})/I_{\mathcal{L}/\mathcal{K}}U_1(\mathcal{L}^{\text{ur}}) & \xrightarrow{N_{\mathcal{L}/\mathcal{K}}} & U_1(\mathcal{K}^{\text{ur}}) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Proof. This is proved on page 239 of [LR], using an identification

$$E_1(\mathcal{L}) \cong \{x \in U_1(\mathcal{L}^{\text{ur}}) : x^\phi = x^u\}$$

(see the lemma on page 237 of [LR]). □

PROPOSITION B.2. *Suppose $\mathcal{K} \subset \mathcal{M} \subset \mathcal{L}$ and $[\mathcal{L} : \mathcal{M}] = p$. Then*

$$\dim_{\mathbf{F}_p}(E_1(\mathcal{K})/(E_1(\mathcal{K}) \cap N_{\mathcal{L}/\mathcal{M}}E_1(\mathcal{L}))) = \begin{cases} 1 & \text{if } E \text{ has anomalous reduction,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $G := \text{Gal}(\mathcal{L}/\mathcal{K})$ and $H := \text{Gal}(\mathcal{L}/\mathcal{M})$. There is a commutative diagram

$$(B.1) \quad \begin{array}{ccc} E_1(\mathcal{M})/N_{\mathcal{L}/\mathcal{M}}E_1(\mathcal{L}) & \xrightarrow{\sim} & H/(1-u)H \\ \uparrow & & \uparrow \text{Tr} \\ E_1(\mathcal{K})/N_{\mathcal{L}/\mathcal{K}}E_1(\mathcal{L}) & \xrightarrow{\sim} & G/(1-u)G \end{array}$$

where the horizontal isomorphisms are Corollaries 4.30 and 4.37 of [M], (proved in [LR] by applying the Snake Lemma to the diagram of Lemma B.1 for \mathcal{L}/\mathcal{M} and \mathcal{L}/\mathcal{K}), the left-hand vertical map is induced by the inclusion of \mathcal{K} into \mathcal{M} , and the right-hand vertical map is induced by the transfer map $G \rightarrow H$. The commutativity of the diagram follows from Lemma B.1 and the commutativity of

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \longrightarrow & U_1(\mathcal{L}^{\text{ur}})/I_{\mathcal{L}/\mathcal{M}}U_1(\mathcal{L}^{\text{ur}}) & \xrightarrow{N_{\mathcal{L}/\mathcal{M}}} & U_1(\mathcal{M}^{\text{ur}}) \longrightarrow 0 \\ & & \uparrow \text{Tr} & & \uparrow N_{\mathcal{M}/\mathcal{K}} & & \uparrow \\ 0 & \longrightarrow & G & \longrightarrow & U_1(\mathcal{L}^{\text{ur}})/I_{\mathcal{L}/\mathcal{K}}U_1(\mathcal{L}^{\text{ur}}) & \xrightarrow{N_{\mathcal{L}/\mathcal{K}}} & U_1(\mathcal{K}^{\text{ur}}) \longrightarrow 0 \end{array}$$

(see the proof of Lemma 2 of [LR]).

If E has nonanomalous reduction, then $1-u \in \mathbf{Z}_p^\times$ so the top isomorphism of (B.1) shows that $N_{\mathcal{L}/\mathcal{M}}E_1(\mathcal{L}) = E_1(\mathcal{M}) \supset E_1(\mathcal{K})$.

If E has anomalous reduction, then $(1-u)H \subset pH = 0$. Since G is cyclic, the transfer map is surjective. Therefore (B.1) shows $E_1(\mathcal{M})/N_{\mathcal{L}/\mathcal{M}}E_1(\mathcal{L})$ has order p , and is generated by the image of $E_1(\mathcal{K})$. The proposition follows. \square

PROPOSITION B.3. *Suppose that E is defined and has good reduction over a subfield $\mathcal{K}^+ \subset \mathcal{K}$ such that $[\mathcal{K} : \mathcal{K}^+] = 2$, $\mathcal{L}/\mathcal{K}^+$ is Galois, and $\text{Gal}(\mathcal{L}/\mathcal{K}^+)$ is dihedral. If $\mathcal{K} \subset \mathcal{M} \subset \mathcal{L}$ and $[\mathcal{L} : \mathcal{M}] = p$, then*

$$\dim_{\mathbf{F}_p}(E(\mathcal{K})/(E(\mathcal{K}) \cap N_{\mathcal{L}/\mathcal{M}}E(\mathcal{L}))) = \begin{cases} 2 & \text{if } E \text{ has anomalous reduction,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let κ denote the common residue field of \mathcal{K} , \mathcal{M} , and \mathcal{L} . We have a commutative diagram

$$(B.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & E_1(\mathcal{L}) & \longrightarrow & E(\mathcal{L}) & \longrightarrow & E(\kappa) \longrightarrow 0 \\ & & \downarrow N_{\mathcal{L}/\mathcal{M}} & & \downarrow N_{\mathcal{L}/\mathcal{M}} & & \downarrow p \\ 0 & \longrightarrow & E_1(\mathcal{M}) & \longrightarrow & E(\mathcal{M}) & \longrightarrow & E(\kappa) \longrightarrow 0. \end{array}$$

If E has nonanomalous reduction, then $E(\kappa)$ has order prime to p and the proposition follows from Proposition B.2.

Suppose now that E has anomalous reduction. Let $H := \text{Gal}(\mathcal{L}/\mathcal{M})$, and fix \mathcal{L}^+ with $\mathcal{K}^+ \subset \mathcal{L}^+ \subset \mathcal{L}$, $[\mathcal{L} : \mathcal{L}^+] = 2$. Let $\mathcal{M}^+ = \mathcal{M} \cap \mathcal{L}^+$.

Replacing E/\mathcal{K}^+ by its quadratic twist by $\mathcal{K}/\mathcal{K}^+$ if necessary, we may suppose that E has anomalous reduction over \mathcal{K}^+ . We will show that

$$(B.3) \quad N_{\mathcal{L}/\mathcal{M}} : E_1(\mathcal{L}^+) \rightarrow E_1(\mathcal{M}^+) \text{ is surjective.}$$

Assuming this for the moment, choose $x \in E(\mathcal{K}^+)$ such that the reduction of x has order p in $E(\kappa)$. Then $\mathcal{N}_{\mathcal{L}/\mathcal{M}}(x) = px \in E_1(\mathcal{M}^+)$ so we can find $y \in E_1(\mathcal{L}^+)$ such that $N_{\mathcal{L}/\mathcal{M}}(y) = \mathcal{N}_{\mathcal{L}/\mathcal{M}}(x)$. Then $N_{\mathcal{L}/\mathcal{M}}(x - y) = 0$ and the reduction of $x - y$ is nontrivial. Therefore, since $E(\kappa)[p]$ is cyclic of order p , the Snake Lemma applied to (B.2) gives an exact sequence

$$(B.4) \quad 0 \rightarrow E_1(\mathcal{M})/N_{\mathcal{L}/\mathcal{M}}E_1(\mathcal{L}) \rightarrow E(\mathcal{M})/N_{\mathcal{L}/\mathcal{M}}E(\mathcal{L}) \rightarrow E(\kappa)/pE(\kappa) \rightarrow 0.$$

Using the natural injections

$$E_1(\mathcal{K})/(E_1(\mathcal{K}) \cap N_{\mathcal{L}/\mathcal{M}}E_1(\mathcal{L})) \hookrightarrow E_1(\mathcal{M})/N_{\mathcal{L}/\mathcal{M}}E_1(\mathcal{L})$$

and

$$E(\mathcal{K})/(E(\mathcal{K}) \cap N_{\mathcal{L}/\mathcal{M}}E(\mathcal{L})) \hookrightarrow E(\mathcal{M})/N_{\mathcal{L}/\mathcal{M}}E(\mathcal{L}),$$

(B.4) restricts to an exact sequence

$$\begin{aligned} 0 \rightarrow E_1(\mathcal{K})/(E_1(\mathcal{K}) \cap N_{\mathcal{L}/\mathcal{M}}E_1(\mathcal{L})) \\ \rightarrow E(\mathcal{K})/(E(\mathcal{K}) \cap N_{\mathcal{L}/\mathcal{M}}E(\mathcal{L})) \rightarrow E(\kappa)/pE(\kappa) \rightarrow 0. \end{aligned}$$

Now the proposition follows from Proposition B.2.

It remains to prove (B.3). We consider two cases.

Case 1: $\mathcal{K}/\mathcal{K}^+$ is unramified. Let v be the unit eigenvalue of Frobenius over \mathcal{K}^+ , so $v^2 = u$. Since E has anomalous reduction over \mathcal{K}^+ , $v \equiv 1 \pmod{p}$. Let ψ denote the Frobenius generator of $\text{Gal}(\mathcal{L}^{\text{ur}}/\mathcal{L}^+)$ (note that $(\mathcal{L}^+)^{\text{ur}} = \mathcal{L}^{\text{ur}}$), so $\psi^2 = \phi$. As in Lemma B.1, there is a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & & & 0 & & & 0 \\ & & & \downarrow & & & \downarrow \\ & & & E_1(\mathcal{L}^+)/(E_1(\mathcal{L}^+) \cap I_{\mathcal{L}/\mathcal{M}}U_1(\mathcal{L}^{\text{ur}})) & \xrightarrow{N_{\mathcal{L}/\mathcal{M}}} & E_1(\mathcal{M}^+) & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & H & \longrightarrow & U_1(\mathcal{L}^{\text{ur}})/I_{\mathcal{L}/\mathcal{M}}U_1(\mathcal{L}^{\text{ur}}) & \xrightarrow{N_{\mathcal{L}/\mathcal{M}}} & U_1(\mathcal{M}^{\text{ur}}) \longrightarrow 0 \\ & & \downarrow -1-v & & \downarrow \psi-v & & \downarrow \psi-v \\ 0 & \longrightarrow & H & \longrightarrow & U_1(\mathcal{L}^{\text{ur}})/I_{\mathcal{L}/\mathcal{M}}U_1(\mathcal{L}^{\text{ur}}) & \xrightarrow{N_{\mathcal{L}/\mathcal{M}}} & U_1(\mathcal{M}^{\text{ur}}) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0. \end{array}$$

The proof is the same as the proof in [LR] of Lemma B.1. The only point to notice is the map $-1 - v$ on the left, which arises because if π is a uniformizing parameter of \mathcal{L}^+ and $h \in H$, then $\psi h \psi^{-1} = h^{-1}$ on \mathcal{L} , so

$$(\pi^{h-1})^{1+\psi} = \pi^{h-1+\psi h^{-1}-\psi} = \pi^{h+h^{-1}-2} = (\pi^{h-1})^{1-h^{-1}} \in I_{\mathcal{L}/\mathcal{M}}U^1(\mathcal{L}).$$

Since the left-most horizontal maps send $h \mapsto \pi^{h-1}$, this shows that the left-hand square commutes (see [LR] page 239). Since $p \neq 2$, $-1 - v \in \mathbf{Z}_p^\times$, and (B.3) now follows from the Snake Lemma in this case.

Case 2: $\mathcal{K}/\mathcal{K}^+$ is ramified. In this case $\mathcal{L}^{\text{ur}}/(\mathcal{L}^+)^{\text{ur}}$ is a quadratic extension. Taking $\text{Gal}(\mathcal{L}^{\text{ur}}/(\mathcal{L}^+)^{\text{ur}})$ -invariants in the diagram of Lemma B.1 (applied to \mathcal{L}/\mathcal{M}) gives a new diagram with exact rows and columns. The top row of the new diagram is

$$E_1(\mathcal{L}^+)/ (E_1(\mathcal{L}^+) \cap I_{\mathcal{L}/\mathcal{M}}U_1((\mathcal{L}^+)^{\text{ur}})) \xrightarrow{N_{\mathcal{L}/\mathcal{M}}} E_1(\mathcal{M}^+),$$

and the left-hand column is $0 \rightarrow 0$ since $\text{Gal}(\mathcal{L}^{\text{ur}}/(\mathcal{L}^+)^{\text{ur}})$ acts on H by -1 . Now the Snake Lemma applied to this new diagram proves (B.3) in this case. This completes the proof of the proposition. \square

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