

On deformations of associative algebras

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Abstract

In a classic paper, Gerstenhaber showed that first order deformations of an associative \mathbb{k} -algebra \mathfrak{a} are controlled by the second Hochschild cohomology group of \mathfrak{a} . More generally, any n -parameter first order deformation of \mathfrak{a} gives, due to commutativity of the cup-product on Hochschild cohomology, a graded algebra morphism $\mathrm{Sym}^\bullet(\mathbb{k}^n) \rightarrow \mathrm{Ext}_{\mathfrak{a}\text{-bimod}}^{2\bullet}(\mathfrak{a}, \mathfrak{a})$. We prove that any extension of the n -parameter first order deformation of \mathfrak{a} to an *infinite order* formal deformation provides a canonical ‘lift’ of the graded algebra morphism above to a dg-algebra morphism $\mathrm{Sym}^\bullet(\mathbb{k}^n) \rightarrow \mathrm{RHom}^\bullet(\mathfrak{a}, \mathfrak{a})$, where the symmetric algebra $\mathrm{Sym}^\bullet(\mathbb{k}^n)$ is viewed as a dg-algebra (generated by the vector space \mathbb{k}^n placed in degree 2) equipped with zero differential.

1. Main result

1.1. Let \mathbb{k} be a field of characteristic zero and write $\otimes = \otimes_{\mathbb{k}}$, $\mathrm{Hom} = \mathrm{Hom}_{\mathbb{k}}$, etc. Given a \mathbb{k} -vector space V , let $V^* = \mathrm{Hom}(V, \mathbb{k})$ denote the dual space.

We will work with unital associative \mathbb{k} -algebras, to be referred to as ‘algebras’. Given such an algebra B , we write $\mathbf{m}_B : B \otimes B \rightarrow B$ for the corresponding multiplication map, and put $\Omega_B := \mathrm{Ker}(\mathbf{m}_B) \subset B \otimes B$. This is a B -bimodule which is free as a *right* B -module; in effect, $\Omega_B \simeq (B/\mathbb{k}) \otimes B$ is a free right B -module generated by the subspace $B/\mathbb{k} \subset \Omega_B$ formed by the elements $b \otimes 1 - 1 \otimes b$, $b \in B$.

Fix a finite dimensional vector space T , and let $\mathcal{O} = \mathbb{k} \oplus T^*$ be the commutative local \mathbb{k} -algebra with unit $1 \in \mathbb{k}$ and with maximal ideal $T^* \subset \mathcal{O}$ such that $(T^*)^2 = 0$. Thus, $\mathcal{O}/T^* = \mathbb{k}$. The algebra \mathcal{O} is Koszul and one has a canonical isomorphism $\mathrm{Tor}_1^{\mathcal{O}}(\mathbb{k}, \mathbb{k}) \cong T^*$.

We are interested in multi-parameter (first order) deformations of a given algebra \mathfrak{a} . Specifically, by an \mathcal{O} -deformation of \mathfrak{a} we mean a free \mathcal{O} -algebra A

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(that is, \mathcal{O} is a *central* subalgebra in A , and A is a free \mathcal{O} -module) equipped with a \mathbb{k} -algebra isomorphism $\psi : A/T^* \cdot A \cong \mathfrak{a}$. Two \mathcal{O} -deformations, (A, ψ) and (A', ψ') , are said to be *equivalent* if there is an \mathcal{O} -algebra isomorphism $\varphi : A \xrightarrow{\sim} A'$ such that its reduction modulo the maximal ideal induces the identity map $\text{Id}_{\mathfrak{a}} : \mathfrak{a} \xrightarrow{\psi} A/T^* \cdot A \xrightarrow{\varphi} A'/T^* \cdot A' \xrightarrow{\psi'} \mathfrak{a}$.

Let (A, ψ) be an \mathcal{O} -deformation of \mathfrak{a} . Reducing each term of the short exact sequence $0 \rightarrow \Omega_A \rightarrow A \otimes A \rightarrow A \rightarrow 0$ (of free right A -modules) modulo T^* on the right, one obtains the following short exact sequence of left A -modules

$$(1.1.1) \quad 0 \rightarrow \Omega_A \otimes_{\mathcal{O}} \mathbb{k} \rightarrow A \otimes \mathfrak{a} \rightarrow \mathfrak{a} \rightarrow 0.$$

Next, we reduce modulo T^* on the left, that is, apply the functor $\text{Tor}_{\bullet}^{\mathcal{O}}(\mathbb{k}, -)$ with respect to the left \mathcal{O} -action. We have $\text{Tor}_1^{\mathcal{O}}(\mathbb{k}, A \otimes \mathfrak{a}) = 0$. Further, since multiplication by T^* annihilates \mathfrak{a} , we get $\text{Tor}_1^{\mathcal{O}}(\mathbb{k}, \mathfrak{a}) = \mathfrak{a} \otimes \text{Tor}_1^{\mathcal{O}}(\mathbb{k}, \mathbb{k}) = \mathfrak{a} \otimes T^*$. Thus, the end of the long exact sequence of Tor-groups corresponding to the short exact sequence (1.1.1) reads

$$(1.1.2) \quad 0 \rightarrow \mathfrak{a} \otimes T^* \xrightarrow{\nu} \mathbb{k} \otimes_{\mathcal{O}} \Omega_A \otimes_{\mathcal{O}} \mathbb{k} \xrightarrow{u} \mathfrak{a} \otimes \mathfrak{a} \xrightarrow{\mathfrak{m}_{\mathfrak{a}}} \mathfrak{a} \rightarrow 0.$$

This is an exact sequence of \mathfrak{a} -bimodules; the map $\nu : \mathfrak{a} \otimes T^* = \text{Tor}_1^{\mathcal{O}}(\mathbb{k}, \mathfrak{a}) \rightarrow \mathbb{k} \otimes_{\mathcal{O}} (\Omega_A \otimes_{\mathcal{O}} \mathbb{k})$ in (1.1.2) is the boundary map which is easily seen to be induced by the assignment $a \otimes t \mapsto ta \otimes 1 - 1 \otimes at \in \Omega_A$, for any $a \in A$ and $t \in T^*$. The map u is induced by the imbedding $\Omega_A \hookrightarrow A \otimes A$.

Interpretation via noncommutative geometry. For any associative algebra A , the bimodule Ω_A is called the bimodule of *noncommutative 1-forms* for A , and there is a geometric interpretation of (1.1.2) as follows.

Let $J \subset A$ be any two-sided ideal, and put $\mathfrak{a} := A/J$. There is a canonical short exact sequence of \mathfrak{a} -bimodules (cf. [CQ, Cor. 2.11]),

$$(1.1.3) \quad 0 \rightarrow J/J^2 \xrightarrow{d} \mathfrak{a} \otimes_A \Omega_A \otimes_A \mathfrak{a} \rightarrow \Omega_{\mathfrak{a}} \rightarrow 0.$$

Here, the map $J/J^2 \rightarrow \mathfrak{a} \otimes_A \Omega_A \otimes_A \mathfrak{a}$ is induced by restriction to J of the de Rham differential $d : A \rightarrow \Omega_A$; cf. [CQ]. The above exact sequence may be thought of as a noncommutative analogue of the *conormal exact sequence* of a subvariety.

We may splice (1.1.3) with the tautological extension (1.1.1), the latter tensored by \mathfrak{a} on both sides. Thus, we obtain the following exact sequence of \mathfrak{a} -bimodules:

$$(1.1.4) \quad 0 \rightarrow J/J^2 \xrightarrow{d} \mathfrak{a} \otimes_A \Omega_A \otimes_A \mathfrak{a} \rightarrow \mathfrak{a} \otimes \mathfrak{a} \xrightarrow{\mathfrak{m}_{\mathfrak{a}}} \mathfrak{a} \rightarrow 0.$$

Let $\text{Ext}_{\mathfrak{a}\text{-bimod}}^i(-, -)$ denote the i -th Ext-group in \mathfrak{a} -bimod, the abelian category of \mathfrak{a} -bimodules. The group $\text{Ext}_{\mathfrak{a}\text{-bimod}}^2(\mathfrak{a}, J/J^2)$ classifies \mathfrak{a} -bimodule extensions of \mathfrak{a} by J/J^2 . The class of the extension in (1.1.4) may be viewed as a noncommutative version of Kodaira-Spencer class.

We return now to the special case where A is an \mathcal{O} -deformation of an algebra \mathfrak{a} . In this case, we have $\mathfrak{a} = A/J$ where $J = \mathfrak{a} \otimes T^*$ and, moreover, $J^2 = 0$. Thus, $J/J^2 = \mathfrak{a} \otimes T^*$, and the long exact sequence in (1.1.4) reduces to (1.1.2). Let

$$\text{deform}(A, \psi) \in \text{Ext}_{\mathfrak{a}\text{-bimod}}^2(\mathfrak{a}, \mathfrak{a} \otimes T^*) = \text{Hom}(T, \text{Ext}_{\mathfrak{a}\text{-bimod}}^2(\mathfrak{a}, \mathfrak{a}))$$

be the class of the corresponding extension.

The following theorem is an invariant and multiparameter generalization of the classic result due to Gerstenhaber [G2].

THEOREM 1.1.5. *The map assigning the class*

$$\text{deform}(A, \psi) \in \text{Hom}(T, \text{Ext}_{\mathfrak{a}\text{-bimod}}^2(\mathfrak{a}, \mathfrak{a}))$$

to an \mathcal{O} -deformation (A, ψ) provides a canonical bijection between the set of equivalence classes of \mathcal{O} -deformations of \mathfrak{a} and the vector space

$$\text{Hom}(T, \text{Ext}_{\mathfrak{a}\text{-bimod}}^2(\mathfrak{a}, \mathfrak{a})). \quad \square$$

Gerstenhaber worked in more down-to-earth terms involving explicit cocycles. To make a link with Gerstenhaber’s formulation, observe that, for any deformation (A, ψ) , the composite $A \twoheadrightarrow A/T^* \cdot A \xrightarrow{\psi} \mathfrak{a}$ may be lifted (since A is free over \mathcal{O}) to an \mathcal{O} -module isomorphism $A \cong \mathfrak{a} \otimes \mathcal{O} = \mathfrak{a} \otimes (\mathbb{k} \oplus T^*) = \mathfrak{a} \oplus (\mathfrak{a} \otimes T^*)$ that reduces to ψ modulo T^* . Transporting the multiplication map on A via this isomorphism, we see that giving a deformation amounts to giving an associative truncated ‘star product’:

$$(1.1.6) \quad a \star a' = a \cdot a' + \beta(a, a'), \quad \beta \in \text{Hom}(\mathfrak{a} \otimes \mathfrak{a}, \mathfrak{a} \otimes T^*) = \text{Hom}(T, \text{Hom}(\mathfrak{a} \otimes \mathfrak{a}, \mathfrak{a})).$$

The associativity condition gives a constraint on β saying that β is a Hochschild 2-cocycle (such that $\beta(1, x) = 0$). Changing the isomorphism $A \simeq \mathcal{O} \otimes \mathfrak{a}$ has the effect of replacing β by a cocycle in the same cohomology class.

One can show that $\beta = \text{deform}(A, \psi)$; i.e., the class of the 2-cocycle β in the Hochschild cohomology group $\text{Ext}_{\mathfrak{a}\text{-bimod}}^2(\mathfrak{a}, \mathfrak{a} \otimes T^*)$ represents the class of the extension in (1.1.2). Thus, our cocycle-free construction is equivalent to the one given by Gerstenhaber.

1.2. Next, we consider the total Ext-group

$$\text{Ext}_{\mathfrak{a}\text{-bimod}}^\bullet(\mathfrak{a}, \mathfrak{a}) = \bigoplus_{i \geq 0} \text{Ext}_{\mathfrak{a}\text{-bimod}}^i(\mathfrak{a}, \mathfrak{a}).$$

This is a graded vector space that comes equipped with an associative algebra structure given by Yoneda product. Another fundamental result due to Gerstenhaber [G1] is

THEOREM 1.2.1. *The Yoneda product on $\text{Ext}_{\mathfrak{a}\text{-bimod}}^{\bullet}(\mathfrak{a}, \mathfrak{a})$ is (graded) commutative.* \square

In view of this result, any linear map $T \rightarrow \text{Ext}_{\mathfrak{a}\text{-bimod}}^2(\mathfrak{a}, \mathfrak{a})$, of vector spaces can be uniquely extended, due to commutativity of the algebra $\text{Ext}_{\mathfrak{a}\text{-bimod}}^{2\bullet}(\mathfrak{a}, \mathfrak{a})$, to a graded algebra homomorphism $\text{Sym}(T[-2]) \rightarrow \text{Ext}_{\mathfrak{a}\text{-bimod}}^{2\bullet}(\mathfrak{a}, \mathfrak{a})$, where $\text{Sym}(T[-2])$ denotes the commutative graded algebra freely generated by the vector space T placed in degree 2. We conclude that any \mathcal{O} -deformation of \mathfrak{a} gives rise, by Theorem 1.1.5, to a graded algebra homomorphism

$$(1.2.2) \quad \text{deform} : \text{Sym}(T[-2]) \rightarrow \text{Ext}_{\mathfrak{a}\text{-bimod}}^{2\bullet}(\mathfrak{a}, \mathfrak{a}).$$

The present paper is concerned with the problem of ‘lifting’ this morphism to the level of derived categories. Specifically, we consider the dg-algebra $\text{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$, see Sect. 2.1 below, and also view the graded algebra $\text{Sym}(T[-2])$ as a dg-algebra with trivial differential. We are interested in lifting the graded algebra map (1.2.2) to a dg-algebra map $\text{Sym}(T[-2]) \rightarrow \text{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$.

To this end, one has to consider *infinite order* formal deformations of \mathfrak{a} . Thus, we now let \mathcal{O} be a *formally smooth* local \mathbb{k} -algebra with maximal ideal \mathfrak{m} such that $\mathcal{O}/\mathfrak{m} = \mathbb{k}$. We assume \mathcal{O} to be complete in the \mathfrak{m} -adic topology; that is, $\mathcal{O} \cong \varprojlim_n \mathcal{O}/\mathfrak{m}^n$. The (finite dimensional) \mathbb{k} -vector space $T := (\mathfrak{m}/\mathfrak{m}^2)^*$ may be viewed as the tangent space to $\text{Spec } \mathcal{O}$ at the base point and one has a *canonical* isomorphism $\mathcal{O}/\mathfrak{m}^2 = \mathbb{k} \oplus T^*$. The algebra \mathcal{O} is *noncanonically* isomorphic to $\mathbb{k}[[T]]$, the algebra of formal power series on the vector space T .

Let A be a complete topological \mathcal{O} -algebra, $A \cong \varprojlim_n A/\mathfrak{m}^n A$, such that, for any $n = 1, 2, \dots$, the quotient $A/\mathfrak{m}^n A$ is a free $\mathcal{O}/\mathfrak{m}^n$ -module. Given an algebra \mathfrak{a} and an algebra isomorphism $\psi : \mathfrak{a} \xrightarrow{\sim} A/\mathfrak{m} A$, we say that the pair (A, ψ) is an infinite order formal \mathcal{O} -deformation of \mathfrak{a} .

Clearly, reducing an infinite order deformation modulo \mathfrak{m}^2 , one obtains a first order $\mathcal{O}/\mathfrak{m}^2$ -deformation of \mathfrak{a} . The main result of this paper reads

THEOREM 1.2.3 (Deformation formality). *Any infinite order formal \mathcal{O} -deformation (A, ψ) of an associative algebra \mathfrak{a} provides a canonical lift of the graded algebra morphism (1.2.2), associated with the corresponding first order $\mathcal{O}/\mathfrak{m}^2$ -deformation, to a dg-algebra morphism $\mathfrak{D}\text{eform} : \text{Sym}(T[-2]) \rightarrow \text{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$; see Section 2.1 for explanation.*

Observe that Theorem 1.2.3 says, in particular, that one can map a basis of the vector space $\text{deform}(T[-2]) \subset \text{Ext}_{\mathfrak{a}\text{-bimod}}^2(\mathfrak{a}, \mathfrak{a})$ to a set of *pairwise commuting* elements in $\text{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$. Thus, the above theorem may be seen as a (partial) refinement of Gerstenhaber’s Theorem 1.2.1. Yet, our approach to Theorem 1.2.3 is *totally different* from Gerstenhaber’s proof of his theorem;

indeed, we are unaware of any connection between the commutativity resulting from Theorem 1.2.3 and the Gerstenhaber *brace operation* on Hochschild cochains that plays a crucial role in the proof of Theorem 1.2.1. This ‘paradox’ may be resolved, perhaps, by observing that the notation $\mathrm{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$ stands for a *quasi-isomorphism class* of DG algebras; see Section 2.1 below. Yet, the very notion of commutativity of elements of $\mathrm{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$ only makes sense after one picks a concrete DG algebra in that quasi-isomorphism class. Thus, the commutativity statement resulting from Theorem 1.2.3 implicitly involves a particular DG algebra model for $\mathrm{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$. Now, the point is that the model that we are using as well as our construction of the morphism $\mathfrak{D}\mathfrak{e}\mathfrak{f}\mathfrak{o}\mathfrak{r}\mathfrak{m}$ will both involve the full *infinite order* deformation (A, ψ) , i.e., the full \mathcal{O} -algebra structure on A , and not only the ‘first order’ deformation A/\mathfrak{m}^2A . On the contrary, the statement of Gerstenhaber’s Theorem 1.2.1 is independent of the choice of a DG algebra model; also, the construction of the map $\mathfrak{d}\mathfrak{e}\mathfrak{f}\mathfrak{o}\mathfrak{r}\mathfrak{m}$ in (1.2.2) involves the first order deformation A/\mathfrak{m}^2A only.

Remark. Theorem 1.2.3 was applied in [ABG] to certain natural deformations of quantum groups at a root of unity.

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2. Generalities

2.1. *Reminder on dg-algebras and dg-modules.* Given an integer n and a graded vector space $V = \bigoplus_{i \in \mathbb{Z}} V^i$, we write $V_{<n} := \bigoplus_{i < n} V^i$. Let $[n]$ denote the shift functor in the derived category, and also the grading shift by n , i.e., $(V[n])^i := V^{i+n}$.

Let $B = \bigoplus_{i \in \mathbb{Z}} B^i$ be a dg-algebra. We write $\mathrm{DGM}(B)$ for the homotopy category of all left dg-modules $M = \bigoplus_{i \in \mathbb{Z}} M^i$ over B (with differential $d : M^\bullet \rightarrow M^{\bullet+1}$), and $D(B) := D(\mathrm{DGM}(B))$ for the corresponding derived category obtained by localizing at quasi-isomorphisms. A B -bimodule is the same thing as a left module over $B \otimes B^{\mathrm{op}}$, where B^{op} stands for the opposite algebra. Thus, we write $D(B \otimes B^{\mathrm{op}})$ for the derived category of *dg-bimodules* over B .

Given two objects $M, N \in D(B)$, for any integer i we put $\mathrm{Ext}_B^i(M, N) := \mathrm{Hom}_{D(B)}(M, N[i])$. The graded space $\mathrm{Ext}_B^\bullet(M, M) = \bigoplus_{j \geq 0} \mathrm{Ext}_B^j(M, M)$ has a natural algebra structure, via composition.

Given an exact triangle $\Delta : K \rightarrow M \rightarrow N$, in $D(B)$, we write $\partial_\Delta : N \rightarrow K[1]$ for the corresponding boundary morphism. Thus, $\partial_\Delta \in \mathrm{Hom}_{D(B)}(N, K[1]) = \mathrm{Ext}_B^1(N, K)$.

For a dg-algebra $B = \bigoplus_{i \leq 0} B^i$ concentrated in *nonpositive* degrees, the triangulated category $D(B)$ has a standard *t-structure* $(D^{\tau < 0}(B), D^{\tau \geq 0}(B))$ where $D^{\tau < 0}(B)$, resp. $D^{\tau \geq 0}(B)$, is a full subcategory of $D(B)$ formed by the objects with vanishing cohomology in degrees ≥ 0 , resp., in degrees < 0 ; cf. [BBD]. Write $D(B) \rightarrow D^{\tau < 0}(B)$, $M \mapsto M^{\tau < 0}$, resp., $D(B) \rightarrow D^{\tau \geq 0}(B)$, $M \mapsto M^{\tau \geq 0}$, for the corresponding truncation functors. Thus, for any object $M \in D(B)$, there is a canonical exact triangle $M^{\tau < 0} \rightarrow M \rightarrow M^{\tau \geq 0}$. A triangulated functor $F : D(B_1) \rightarrow D(B_2)$ between two such categories is called *t-exact* if it takes $D^{\tau < 0}(B_1)$ to $D^{\tau < 0}(B_2)$, and $D^{\tau \geq 0}(B_1)$ to $D^{\tau \geq 0}(B_2)$.

An object $M \in \text{DGM}(B)$ is said to be *projective* if it belongs to the smallest full subcategory of $\text{DGM}(B)$ that contains the rank one dg-module B , and which is closed under taking mapping-cones and infinite direct sums. Any object of $\text{DGM}(B)$ is quasi-isomorphic to a projective object, see [Ke] for a proof. (Instead of projective objects, one can use *semi-free objects* considered e.g. in [Dr, Appendices A,B].)

Given $M \in \text{DGM}(B)$, choose a quasi-isomorphic projective object $P \in \text{DGM}(B)$ and write $\text{Hom}'(P, P[n])$ for the space of B -module maps $P \rightarrow P$ which shift the grading by n (but do not necessarily commute with the differential d). The graded vector space $\bigoplus_{n \in \mathbb{Z}} \text{Hom}'(P, P[n])$ has a natural algebra structure given by composition. Super-commutator with the differential $d \in \text{Hom}_{\mathbb{k}}(P, P[-1])$ makes this algebra into a dg-algebra, to be denoted $\text{REnd}_{\bullet}^B(M) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}'(P, P[n])$.

Let DGAlg be the category obtained from the category of dg-algebras and dg-algebra morphisms by localizing at quasi-isomorphisms. The dg-algebra $\text{REnd}_{\bullet}^B(M)$ viewed as an object of DGAlg does not depend on the choice of projective representative P . More precisely, let $\text{QIso}(B)$ denote the groupoid that has the same objects as the category $D(B)$ and whose morphisms are the isomorphisms in $D(B)$. Then, one can show, cf. [Hi] for a similar result, that associating to $M \in D(B)$ the dg-algebra $\text{REnd}_{\bullet}^B(M)$ gives a well-defined functor $\text{QIso}(B) \rightarrow \text{DGAlg}$.

The lift $\mathfrak{Deform} : \text{Sym}(T[-2]) \rightarrow \text{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a}) := \text{REnd}_{\mathfrak{a} \otimes \mathfrak{a}^{\text{op}}}(\mathfrak{a})$, whose existence is stated in Theorem 1.2.3, should be understood as a morphism in DGAlg .

For any dg-algebra morphism $f : B_1 \rightarrow B_2$, we let $f_* : D(B_1) \rightarrow D(B_2)$ be the push-forward functor $M \mapsto B_2 \overset{L}{\otimes}_{B_1} M$, and $f^* : D(B_2) \rightarrow D(B_1)$ the pull-back functor, given by the change of scalars. The functor f^* is clearly *t-exact*; it is the right adjoint of f_* . These functors are triangulated equivalences quasi-inverse to each other, provided the map f is a dg-algebra quasi-isomorphism.

2.2. *Homological algebra associated with a deformation.* Let \mathcal{O} be a formally smooth complete local algebra with maximal ideal \mathfrak{m} . We fix a \mathbb{k} -algebra \mathfrak{a} and let A be an infinite order formal \mathcal{O} -deformation of \mathfrak{a} , as in

Section 1.2. Note that A is a flat \mathcal{O} -algebra. Associated with A and \mathfrak{a} , we have the corresponding ideals $\Omega_A \subset A \otimes A$ and $\Omega_{\mathfrak{a}} \subset \mathfrak{a} \otimes \mathfrak{a}$, respectively.

Set $T := (\mathfrak{m}/\mathfrak{m}^2)^*$. The projection $\mathcal{O} \rightarrow \mathcal{O}/\mathfrak{m}^2$ induces an isomorphism $\mathrm{Tor}_1^{\mathcal{O}}(\mathbb{k}, \mathbb{k}) \xrightarrow{\sim} \mathrm{Tor}_1^{\mathcal{O}/\mathfrak{m}^2}(\mathbb{k}, \mathbb{k}) = T$. It follows, since A is flat over \mathcal{O} , that the exact sequence in (1.1.2) as well as all other constructions of Section 1.1 are still valid in the present setting of formally smooth complete local algebras \mathcal{O} . In particular, we have the canonical morphism $u : \mathbb{k} \otimes_{\mathcal{O}} \Omega_A \otimes_{\mathcal{O}} \mathbb{k} \rightarrow \mathfrak{a} \otimes \mathfrak{a}$, cf. (1.1.2), and the object $\mathrm{Cone}(u) \in D(\mathfrak{a} \otimes \mathfrak{a}^{\mathrm{op}})$. From (1.1.2) we deduce $H^0(\mathrm{Cone}(u)) = \mathfrak{a}$ and $H^{-1}(\mathrm{Cone}(u)) = \mathfrak{a} \otimes T^*$. So, one may view (1.1.2) as an exact triangle

$$(2.2.1) \quad \Delta_u : \mathfrak{a} \otimes T^*[1] = H^{-1}(\mathrm{Cone}(u))[1] \longrightarrow \mathrm{Cone}(u) \longrightarrow H^0(\mathrm{Cone}(u)) = \mathfrak{a},$$

with boundary map $\partial_u : \mathfrak{a} \rightarrow \mathfrak{a} \otimes T^*[2]$. In this language, the bijection of Theorem 1.1.5 assigns to a deformation (A, ψ) the class

$$(2.2.2) \quad \partial_u \in \mathrm{Hom}_{D(\mathfrak{a} \otimes \mathfrak{a}^{\mathrm{op}})}(\mathfrak{a}, \mathfrak{a} \otimes T^*[2]) = \mathrm{Ext}_{\mathfrak{a} \otimes \mathfrak{a}^{\mathrm{op}}}^2(\mathfrak{a}, \mathfrak{a}) \otimes T^*.$$

There is also a different interpretation of the triangle Δ_u . Specifically, apply derived tensor product functor $D(\mathfrak{a} \otimes A^{\mathrm{op}}) \times D(A \otimes \mathfrak{a}^{\mathrm{op}}) \rightarrow D(\mathfrak{a} \otimes \mathfrak{a}^{\mathrm{op}})$ to \mathfrak{a} , viewed as an object of either $D(\mathfrak{a} \otimes A^{\mathrm{op}})$ or $D(A \otimes \mathfrak{a}^{\mathrm{op}})$. This way, we get an object $\mathfrak{a} \overset{L}{\otimes}_A \mathfrak{a} \in D(\mathfrak{a} \otimes \mathfrak{a}^{\mathrm{op}})$.

PROPOSITION 2.2.3. (i) *The object $\mathfrak{a} \overset{L}{\otimes}_A \mathfrak{a} \in D(\mathfrak{a} \otimes \mathfrak{a}^{\mathrm{op}})$ is concentrated in nonpositive degrees, and one has a natural quasi-isomorphism $\phi : (\mathfrak{a} \overset{L}{\otimes}_A \mathfrak{a})^{\tau \geq -1} \xrightarrow{\mathrm{qis}} \mathrm{Cone}(u)$, such that the following diagram commutes*

$$\begin{array}{ccc} (\mathfrak{a} \overset{L}{\otimes}_A \mathfrak{a})^{\tau \geq -1} & \xrightarrow{\mathrm{proj}} & H^0(\mathfrak{a} \overset{L}{\otimes}_A \mathfrak{a}) \xlongequal{\quad} \mathfrak{a} \\ \mathrm{qis} \downarrow \phi & & \parallel \mathrm{Id}_{\mathfrak{a}} \\ \mathrm{Cone}(u) & \xrightarrow{\mathrm{proj}} & H^0(\mathrm{Cone}(u)) \xlongequal{\quad} \mathfrak{a} \end{array}$$

(ii) *Thus, associated with a deformation (A, ψ) there is a canonical exact triangle*

$$\Delta_{A,\psi} : \mathfrak{a} \otimes T^*[1] \longrightarrow (\mathfrak{a} \overset{L}{\otimes}_A \mathfrak{a})^{\tau \geq -1} \longrightarrow \mathfrak{a},$$

cf. (2.2.1), with boundary map $\partial_{A,\psi}$, and the bijection of Theorem 1.1.5 reads

$$(2.2.4) \quad (A, \psi) \longmapsto \mathrm{deform}(A, \psi) = \partial_{A,\psi} \in \mathrm{Ext}_{\mathfrak{a} \otimes \mathfrak{a}^{\mathrm{op}}}^2(\mathfrak{a}, \mathfrak{a} \otimes T^*).$$

Proof. Since A is flat over \mathcal{O} , on the category of left A -modules one has an isomorphism of functors $\mathfrak{a} \overset{L}{\otimes}_A (-) = \mathbb{k} \overset{L}{\otimes}_{\mathcal{O}} (-)$. Now, use (1.1.1) to replace

the second tensor factor \mathfrak{a} in $\mathfrak{a} \otimes_A^L \mathfrak{a}$ by $\text{Cone}[(\Omega_A \otimes_{\circ} \mathbb{k})[1] \rightarrow A \otimes \mathfrak{a}]$, a quasi-isomorphic object. We find

$$\begin{aligned} \mathfrak{a} \otimes_A^L \mathfrak{a} &= \mathbb{k} \otimes_{\circ}^L \mathfrak{a} = \mathbb{k} \otimes_{\circ}^L \text{Cone}[(\Omega_A \otimes_{\circ} \mathbb{k})[1] \rightarrow A \otimes \mathfrak{a}] \\ &= \text{Cone}[\mathbb{k} \otimes_{\circ}^L (\Omega_A \otimes_{\circ} \mathbb{k})[1] \rightarrow \mathfrak{a} \otimes \mathfrak{a}]. \end{aligned}$$

The object $\mathbb{k} \otimes_{\circ}^L (\Omega_A \otimes_{\circ} \mathbb{k})$ is concentrated in nonpositive degrees, and we clearly have

$$(\mathbb{k} \otimes_{\circ}^L (\Omega_A \otimes_{\circ} \mathbb{k}))^{\tau \geq 0} = H^0(\mathbb{k} \otimes_{\circ}^L (\Omega_A \otimes_{\circ} \mathbb{k})) = \mathbb{k} \otimes_{\circ} \Omega_A \otimes_{\circ} \mathbb{k}.$$

Thus, we conclude that the object $(\mathfrak{a} \otimes_A^L \mathfrak{a})^{\tau \geq -1}$ is quasi-isomorphic to

$$\begin{aligned} &\text{Cone}[\mathbb{k} \otimes_{\circ}^L (\Omega_A \otimes_{\circ} \mathbb{k})[1] \rightarrow \mathfrak{a} \otimes \mathfrak{a}]^{\tau \geq -1} \\ &= \text{Cone}[(\mathbb{k} \otimes_{\circ}^L (\Omega_A \otimes_{\circ} \mathbb{k}))^{\tau \geq 0}[1] \rightarrow \mathfrak{a} \otimes \mathfrak{a}] \\ &= \text{Cone}[(\mathbb{k} \otimes_{\circ} \Omega_A \otimes_{\circ} \mathbb{k})[1] \rightarrow \mathfrak{a} \otimes \mathfrak{a}] = \text{Cone}(u). \quad \square \end{aligned}$$

2.3. Koszul duality. Fix a finite dimensional vector space T and let $\mathbf{\Lambda} = \wedge^{\bullet}(T^*[1])$ be the exterior algebra of the dual vector space T^* , placed in degree -1 . For each $n = 0, -1, -2, \dots$, we have a graded ideal $\mathbf{\Lambda}_{<n} \subset \mathbf{\Lambda}$. One has a canonical extension of graded $\mathbf{\Lambda}$ -modules

$$(2.3.1) \quad \Delta_{\wedge} : 0 \longrightarrow T^*[1] \longrightarrow \mathbf{\Lambda}/\mathbf{\Lambda}_{<-1} \xrightarrow{\epsilon_{\wedge}} \mathbb{k}_{\wedge} \longrightarrow 0,$$

where we set $\mathbb{k}_{\wedge} := \mathbf{\Lambda}/\mathbf{\Lambda}_{<0}$. We will often view $\mathbf{\Lambda}$ as a dg-algebra concentrated in nonpositive degrees, with zero differential.

Recall that the standard Koszul resolution of \mathbb{k}_{\wedge} provides an explicit dg-algebra model for $\text{REnd}_{\mathbf{\Lambda}}^{\bullet}(\mathbb{k}_{\wedge})$ together with an imbedding of the graded symmetric algebra $\text{Sym}(T[-2])$ as a subalgebra of cocycles in that dg-algebra model. Furthermore, $\mathbf{\Lambda} = \wedge^{\bullet}(T^*[1])$ is a Koszul algebra, cf. [BGG], [GKM], so this imbedding induces a graded algebra isomorphism on cohomology:

$$(2.3.2) \quad \text{koszul} : \text{Sym}(T[-2]) \xrightarrow{\sim} \text{Ext}_{\mathbf{\Lambda}}^{\bullet}(\mathbb{k}_{\wedge}, \mathbb{k}_{\wedge}).$$

Thus, the imbedding yields a dg-algebra quasi-isomorphism

$$(2.3.3) \quad \mathfrak{K}\mathfrak{o}\mathfrak{s}\mathfrak{z}\mathfrak{u}\mathfrak{l} : \text{Sym}(T[-2]) \rightarrow \text{REnd}_{\mathbf{\Lambda}}^{\bullet}(\mathbb{k}_{\wedge}),$$

provided the graded algebra $\text{Sym}(T[-2])$ is viewed as a dg-algebra with zero differential.

From (2.3.2), we get a canonical vector space isomorphism

$$\text{End}_{\mathbb{k}} T = T \otimes T^* \xrightarrow{\text{koszul} \otimes \text{Id}_{T^*}} \text{Ext}_{\mathbf{\Lambda}}^2(\mathbb{k}_{\wedge}, \mathbb{k}_{\wedge}) \otimes T^* = \text{Ext}_{\mathbf{\Lambda}}^2(\mathbb{k}_{\wedge}, T^*).$$

It is immediate from the definition of the Koszul complex that the above isomorphism sends the element $\text{Id}_T \in \text{End}_{\mathbb{k}} T$ to $\partial_{\wedge} \in \text{Ext}_{\mathbf{\Lambda}}^2(\mathbb{k}_{\wedge}, T^*)$, the class

of the boundary map $\mathbb{k}_\wedge \rightarrow T^*[2]$ in the canonical exact triangle Δ_\wedge given by (2.3.1).

2.4. *A dg-algebra.* Let $\mathcal{O} = \mathbb{k}[[T]]$ be the algebra of formal power series, with maximal ideal $\mathfrak{m} \subset \mathcal{O}$ such that $T = (\mathfrak{m}/\mathfrak{m}^2)^*$. There is a standard super-commutative dg-algebra R over \mathcal{O} concentrated in nonpositive degrees and such that

$$(2.4.1) \quad \begin{aligned} (i) \quad & R = \bigoplus_{i \leq 0} R^i \quad \text{and} \quad R^0 = \mathcal{O}, \\ (ii) \quad & H^0(R) = \mathbb{k} \quad \text{and} \quad H^i(R) = 0, \forall i \leq -1, \\ (iii) \quad & R \text{ is a free graded } \mathcal{O}\text{-module.} \end{aligned}$$

To construct R , for each $i = 0, 1, \dots, n = \dim T$, we let $R^{-i} = \mathbb{k}[[T]] \otimes \wedge^i T^*$ be the \mathcal{O} -module of differential forms on the scheme $\text{Spec } \mathcal{O}$. We put $R := \bigoplus_{-n \leq i \leq 0} R^i$. Further, write ξ for the Euler vector field on T . Contraction with ξ gives a differential $d : R^{-i} \rightarrow R^{-i+1}$, and it is well-known that the resulting dg-algebra is acyclic in negative degrees, i.e., property (2.4.1)(ii) holds true. Properties (2.4.1)(i) and (iii) are clear. \square

Until the end of this section, we will use the convention that each time a copy of the vector space T^* occurs in a formula, this copy has grade degree -1 . We form the dg-algebra $R \otimes_{\mathcal{O}} R \simeq \mathbb{k}[[T]] \otimes \wedge^\bullet(T^* \oplus T^*)$. Let $R_\Delta \subset R \otimes_{\mathcal{O}} R$ be the \mathcal{O} -subalgebra generated by the diagonal copy $T^* \subset T^* \oplus T^* = \wedge^1(T^* \oplus T^*)$.

LEMMA 2.4.2. *There is a dg-algebra imbedding $\iota : \mathbf{\Lambda} \hookrightarrow R \otimes_{\mathcal{O}} R$ such that*

- (i) *Multiplication in $R \otimes_{\mathcal{O}} R$ induces a dg-algebra isomorphism $R_\Delta \otimes \iota(\mathbf{\Lambda}) \xrightarrow{\sim} R \otimes_{\mathcal{O}} R$.*
- (ii) *The kernel of multiplication map $\mathbf{m}_R : R \otimes_{\mathcal{O}} R \rightarrow R$ is the ideal in the algebra $R \otimes_{\mathcal{O}} R$ generated by $\iota(\mathbf{\Lambda}_{<0})$.*

Proof. We have $R \otimes_{\mathcal{O}} R \simeq \mathbb{k}[[T]] \otimes \wedge(T^* \oplus T^*) \simeq R_\Delta \otimes \wedge(T^*)$, where the last factor $\wedge(T^*)$ is generated by the anti-diagonal copy $T^* \subset T^* \oplus T^*$. It is clear that this anti-diagonal copy of T^* is annihilated by the differential in the dg-algebra $R \otimes_{\mathcal{O}} R$. We deduce that the subalgebra generated by the anti-diagonal copy of T^* is isomorphic to $\mathbf{\Lambda}$ as a dg-algebra. This immediately implies properties (i)–(ii). \square

Now, let \mathcal{O} be an arbitrary smooth complete local algebra. A pair (R, η) , where $R = \bigoplus_{i \leq 0} R^i$ is a super-commutative dg-algebra concentrated in non-positive degrees and $\eta : \mathcal{O} \rightarrow R^0$ is an algebra homomorphism, will be referred to as an \mathcal{O} -dg-algebra. A map $h : R \rightarrow R'$, between two \mathcal{O} -dg-algebras (R, η) and (R', η') , is said to be an \mathcal{O} -dg-algebra morphism if it is a dg-algebra map such that $h \circ \eta = \eta'$.

Let D denote the standard ‘de Rham dg-algebra of the line’, that is, a free supercommutative \mathcal{O} -algebra with one even generator t of degree -2 and one odd generator dt of degree -1 , and equipped with the \mathcal{O} -linear differential sending t to dt . For any $z \in \mathbb{k}$, the assignment $t \mapsto z, dt \mapsto 0$ gives an \mathcal{O} -dg-algebra morphism $\text{pr}_z : D \rightarrow \mathcal{O}$, where \mathcal{O} is viewed as a dg-algebra with zero differential. A pair $h, g : R \rightarrow R'$, of \mathcal{O} -dg-algebra morphisms, is said to be *homotopic*¹ provided there exists an \mathcal{O} -dg-algebra morphism $\mathbf{h} : R \rightarrow D \otimes_{\mathcal{O}} R'$ such that the composite $R \xrightarrow{\mathbf{h}} D \otimes_{\mathcal{O}} R' \xrightarrow{\text{pr}_z \otimes \text{Id}} \mathcal{O} \otimes_{\mathcal{O}} R' = R'$ is equal to h for $z = 0$, resp. equal to g for $z = 1$. Let $\text{DGCom}(\mathcal{O})$ be the category whose objects are \mathcal{O} -dg-algebras and whose morphisms are obtained from homotopy classes of \mathcal{O} -dg-algebra morphisms by localizing at quasi-isomorphisms.

Write \mathfrak{m} for the maximal ideal in \mathcal{O} and put $T := (\mathfrak{m}/\mathfrak{m}^2)^*$. So we can use all the previously introduced notation, such as $\mathbf{\Lambda} = \wedge^\bullet(T^*)$ (with T^* placed in degree -1).

LEMMA 2.4.3. (i) *There exists an \mathcal{O} -dg-algebra R satisfying the three conditions in (2.4.1) and such that all the statements of Lemma 2.4.2 hold.*

Furthermore, let $R_s, s = 1, 2$, be two such \mathcal{O} -dg-algebras. Then, for $s = 1, 2$, we have

(ii) *There exist a third \mathcal{O} -dg-algebra R , as in (i), and \mathcal{O} -dg-algebra morphisms $h_s : R \xrightarrow{\text{qis}} R_s$; these morphisms are unique up to homotopy.*

Thus, the object of $\text{DGCom}(\mathcal{O})$ arising from any choice of dg-algebra R , as in (i), is uniquely determined up to a canonical (quasi)-isomorphism.

(iii) *Let $\iota : \mathbf{\Lambda} \hookrightarrow R \otimes_{\mathcal{O}} R$ and $\iota_s : \mathbf{\Lambda} \hookrightarrow R_s \otimes_{\mathcal{O}} R_s$, be the corresponding maps of Lemma 2.4.2(i). Then, the dg-algebra morphism $(h_s \otimes h_s) \circ \iota$ is homotopic to ι_s .*

Proof. Any choice of representatives in \mathfrak{m} of some basis of the vector space $T^* = \mathfrak{m}/\mathfrak{m}^2$ provides a topological algebra isomorphism $\mathcal{O} \cong \mathbb{k}[[T]]$. This proves (i).

To prove (ii), choose an identification $\mathcal{O} \cong \mathbb{k}[[T]]$ and let $R := \mathbb{k}[[T]] \otimes \wedge^\bullet T^*$ be the corresponding standard dg-algebra constructed earlier. Since R^1 is a free \mathcal{O} -module, we may find \mathcal{O} -module maps $h_s^1, s = 1, 2$, which make the following diagram commute

$$\begin{array}{ccccc}
 R^1 & \xrightarrow{d} & R^0 = \mathcal{O} & \xrightarrow{\epsilon_R} & \mathbb{k} \\
 \downarrow h_s^1 & & \parallel \text{Id} & & \parallel \text{Id} \\
 R_s^1 & \xrightarrow{d} & R_s^0 = \mathcal{O} & \xrightarrow{\epsilon_{R_s}} & \mathbb{k}.
 \end{array}$$

¹The reader is referred to [BoGu] for an excellent exposition of the homotopy theory of dg-algebras.

Further, R is free as a super-commutative \mathcal{O} -algebra. Hence, the \mathcal{O} -module map $h_s^1 : R^1 \rightarrow R_s^1$ can be uniquely extended, by multiplicativity, to a graded algebra map $h_s : R \rightarrow R_s$. The latter map automatically commutes with the differentials and, moreover, induces isomorphisms on cohomology, since each algebra has no cohomology in degrees $\neq 0$. This proves the existence of quasi-isomorphisms. The remaining statements involving homotopies are proved similarly. \square

3. Proofs

3.1. Fix an associative algebra \mathfrak{a} , a complete smooth local \mathbb{k} -algebra \mathcal{O} with maximal ideal \mathfrak{m} , and an \mathcal{O} -dg-algebra R , as in Lemma 2.4.3(i). Let (A, ψ) be an infinite order formal \mathcal{O} -deformation of \mathfrak{a} .

The differential and the grading on R make the tensor product $R\mathfrak{a} := R \otimes_{\mathcal{O}} A$ a dg-algebra which is concentrated in nonpositive degrees and is such that the subalgebra $A = A \otimes 1 \subset R\mathfrak{a}$ is placed in degree zero. Since A is flat over \mathcal{O} , one has $H^\bullet(R \otimes_{\mathcal{O}} A) = H^\bullet(R) \otimes_{\mathcal{O}} A = \mathbb{k} \otimes_{\mathcal{O}} A$. Thus, we have a natural projection

$$(3.1.1) \quad p : R\mathfrak{a} \rightarrow H^0(R\mathfrak{a}) = \mathbb{k} \otimes_{\mathcal{O}} A = A/\mathfrak{m}A \xrightarrow{\psi} \mathfrak{a}.$$

The map p is a dg-algebra quasi-isomorphism that makes $R\mathfrak{a}$ a *dg-algebra resolution* of \mathfrak{a} . In particular, we have mutually quasi-inverse equivalences $p^*, p_* : D(R\mathfrak{a} \otimes R\mathfrak{a}^{\text{op}}) \rightleftarrows D(\mathfrak{a} \otimes \mathfrak{a}^{\text{op}})$. The first surjection in (3.1.1) may be described alternatively as the map $\epsilon_R \otimes \text{Id}_A : R \otimes_{\mathcal{O}} A \rightarrow \mathbb{k} \otimes_{\mathcal{O}} A$, where $\epsilon_R : R \rightarrow R/R_{<0} \cong R^0 = \mathcal{O} \rightarrow \mathcal{O}/\mathfrak{m} = \mathbb{k}$ is the natural augmentation that makes R a dg-algebra resolution of \mathbb{k} .

The dg-algebra $R \otimes_{\mathcal{O}} R$ is also concentrated in nonpositive degrees. We have the following canonical isomorphisms

$$(3.1.2) \quad (R \otimes_{\mathcal{O}} R) \otimes_{\Lambda} \mathbb{k}_{\wedge} \xrightarrow{L} (R \otimes_{\mathcal{O}} R) \otimes_{\Lambda} \mathbb{k}_{\wedge} \xrightarrow{m_R} R.$$

Here, the last isomorphism is obtained by applying the functor $(-) \otimes_{\Lambda} \mathbb{k}_{\wedge}$ to the isomorphism $R \otimes_{\mathcal{O}} R \cong R_{\Delta} \otimes \Lambda$, provided by Lemmas 2.4.2(i), 2.4.3(i).

Similarly, applying the functor $(-) \otimes_{\mathcal{O}} A$, we obtain

$$(3.1.3) \quad (R \otimes_{\mathcal{O}} R) \otimes_{\mathcal{O}} A \xrightarrow{\sim} (R_{\Delta} \otimes \Lambda) \otimes_{\mathcal{O}} A \xrightarrow{\sim} (R_{\Delta} \otimes_{\mathcal{O}} A) \otimes \Lambda \xrightarrow{\sim} R\mathfrak{a} \otimes \Lambda.$$

Let ξ denote the composite isomorphism in (3.1.3). The isomorphisms (3.1.3), resp. (3.1.2), are incorporated in the top, resp. bottom, row of the following natural commutative diagram of dg-algebra maps

(3.1.4)

$$\begin{array}{ccccccc}
 \mathrm{Ra} \otimes_A \mathrm{Ra} & \xlongequal{\quad} & (\mathrm{R} \otimes_{\mathcal{O}} A) \otimes_A (\mathrm{R} \otimes_{\mathcal{O}} A) & \xlongequal{A \otimes_A A = A} & (\mathrm{R} \otimes_{\mathcal{O}} \mathrm{R}) \otimes_{\mathcal{O}} A & \xlongequal{\xi} & \mathrm{Ra} \otimes_{\Lambda} \Lambda \\
 \downarrow m_{\mathrm{Ra}} & & \downarrow m_{\mathrm{R}} \otimes m_A & & \downarrow \mathrm{Id}_{\mathrm{R} \otimes_{\mathcal{O}} \mathrm{R}} \otimes \epsilon_{\Lambda} & & \downarrow \mathrm{Id}_{\mathrm{Ra}} \otimes \epsilon_{\Lambda} \\
 \mathrm{Ra} & \xlongequal{\quad} & \mathrm{R} \otimes_{\mathcal{O}} A & \xlongequal{(3.1.2) \otimes \mathrm{Id}_A} & ((\mathrm{R} \otimes_{\mathcal{O}} \mathrm{R}) \otimes_{\Lambda} \mathbb{k}_{\Lambda}) \otimes_{\mathcal{O}} A & \xlongequal{\xi \otimes_{\Lambda} \mathbb{k}_{\Lambda}} & \mathrm{Ra} \otimes_{\Lambda} \mathbb{k}_{\Lambda}.
 \end{array}$$

Using the isomorphisms in the top row, we may (and will) further identify $(\mathrm{R} \otimes_{\mathcal{O}} \mathrm{R}) \otimes_{\mathcal{O}} A$ with $\mathrm{Ra} \otimes_A \mathrm{Ra}$. In particular, any $(\mathrm{R} \otimes_{\mathcal{O}} \mathrm{R}) \otimes_{\mathcal{O}} A$ -module may be viewed as an Ra -bimodule, and we may also view Ra and Λ at the upper-right corner as dg-subalgebras in the dg-algebra $\mathrm{Ra} \otimes_A \mathrm{Ra}$.

The algebra R being graded-commutative, any graded left R -module may also be viewed as a right R -module. Thus, any graded left $\mathrm{R} \otimes_{\mathcal{O}} \mathrm{R}$ -module may be viewed as a graded R -bimodule. A key role in our proof of Theorem 1.2.3 will be played by the following push-forward functor

$$\begin{aligned}
 (3.1.5) \quad \Theta : D(\Lambda) &\rightarrow D(\mathrm{Ra} \otimes \mathrm{Ra}^{\mathrm{op}}), \quad M \mapsto ((\mathrm{R} \otimes_{\mathcal{O}} \mathrm{R}) \otimes_{\mathcal{O}} A) \otimes_{\Lambda} M \\
 &= (\mathrm{Ra} \otimes_A \mathrm{Ra}) \otimes_{\Lambda} M.
 \end{aligned}$$

By (3.1.3), for any dg-module M over Λ , we get

$$\Theta(M) = ((\mathrm{R} \otimes_{\mathcal{O}} \mathrm{R}) \otimes_{\mathcal{O}} A) \otimes_{\Lambda} M \simeq (\mathrm{Ra} \otimes \Lambda) \otimes_{\Lambda} M = \mathrm{Ra} \otimes M.$$

Although this isomorphism does *not* exhibit the Ra -bimodule structure on the object on the right, it does imply that formula (3.1.5) gives a well-defined triangulated functor. Moreover, this functor is *t-exact*; indeed, since Ra is quasi-isomorphic to \mathfrak{a} , we find

$$(3.1.6) \quad H^{\bullet}(\Theta(M)) \cong H^{\bullet}(\mathrm{Ra}) \otimes H^{\bullet}(M) \cong \mathfrak{a} \otimes H^{\bullet}(M).$$

PROPOSITION 3.1.7. *For any infinite order deformation (A, ψ) , in $D(\mathfrak{a} \otimes \mathfrak{a}^{\mathrm{op}})$, there is a natural isomorphism $f_{A, \psi} : p_* \circ \Theta(\mathbb{k}_{\Lambda}) \xrightarrow{\sim} \mathfrak{a}$ that makes the following diagram commute:*

(3.1.8)

$$\begin{array}{ccccc}
 \mathrm{Sym}(T[-2]) & \xrightarrow[\sim]{\text{koszul}} & \mathrm{Ext}_{\Lambda}^{\bullet}(\mathbb{k}_{\Lambda}, \mathbb{k}_{\Lambda}) & \xrightarrow{\Theta} & \mathrm{Ext}_{\mathrm{Ra} \otimes \mathrm{Ra}^{\mathrm{op}}}^{\bullet}(\Theta(\mathbb{k}_{\Lambda}), \Theta(\mathbb{k}_{\Lambda})) \\
 \downarrow \text{deform} & & & & \downarrow p_* \text{ qis} \\
 \mathrm{Ext}_{\mathfrak{a} \otimes \mathfrak{a}^{\mathrm{op}}}^{\bullet}(\mathfrak{a}, \mathfrak{a}) & \xleftarrow[\sim]{f_{A, \psi}} & \mathrm{Ext}_{\mathfrak{a} \otimes \mathfrak{a}^{\mathrm{op}}}^{\bullet}(p_* \circ \Theta(\mathbb{k}_{\Lambda}), p_* \circ \Theta(\mathbb{k}_{\Lambda})) & &
 \end{array}$$

Proof. First of all, using the definition of Θ and the isomorphisms in the bottom row of diagram (3.1.4), we find

$$(3.1.9) \quad \Theta(\mathbb{k}_{\Lambda}) = ((\mathrm{R} \otimes_{\mathcal{O}} \mathrm{R}) \otimes_{\mathcal{O}} A) \otimes_{\Lambda} \mathbb{k}_{\Lambda} = \mathrm{Ra}.$$

Write $g : \Theta(\mathbb{k}_\wedge) \xrightarrow{\sim} \mathbf{Ra}$ for the composite, and $\text{can}_p : \mathbf{Ra} \xrightarrow{\text{qis}} p^*\mathbf{a}$ for the map p viewed as a morphism in $D(\mathbf{Ra} \otimes \mathbf{Ra}^{\text{op}})$. We define a morphism $f_{A,\psi}$ to be the following composite:

$$f_{A,\psi} : p_* \circ \Theta(\mathbb{k}_\wedge) \xrightarrow[\sim]{p_*(g)} p_*(\mathbf{Ra}) \xrightarrow[\sim]{p_*(\text{can}_p)} p_*(p^*\mathbf{a}) \xrightarrow[\sim]{\text{adjunction}} \mathbf{a}.$$

We claim that, with this definition of $f_{A,\psi}$, diagram (3.1.8) commutes. To see this, observe first that all the maps in the diagram are clearly algebra homomorphisms. Hence, it suffices to verify commutativity of (3.1.8) on the generators of the algebra $\text{Sym}(T[-2])$. That is, we must prove that for all $t \in T[-2]$ one has $\text{deform}(t) = f_{A,\psi} \circ p_* \circ \Theta \circ \text{koszul}(t)$.

It will be convenient to work ‘universally’ over T ; that is, to treat the map

$$\text{deform} : T[-2] \rightarrow \text{Ext}_{\mathbf{a} \otimes \mathbf{a}^{\text{op}}}^2(\mathbf{a}, \mathbf{a})$$

as an element $\text{deform}(A, \psi) \in \text{Ext}_{\mathbf{a} \otimes \mathbf{a}^{\text{op}}}^2(\mathbf{a}, \mathbf{a} \otimes T^*)$; see (2.2.4). We also have the element $\text{Id}_T \in \text{End}_{\mathbb{k}} T = \text{Sym}^1(T[-2]) \otimes T^*$. Now, tensoring with T^* , we rewrite the equation that we must prove as

$$\text{deform}(A, \psi) = f_{A,\psi} \circ p_* \circ \Theta \circ \text{koszul}(\text{Id}_T).$$

Both sides here belong to $\text{Ext}_{\mathbf{a} \otimes \mathbf{a}^{\text{op}}}^2(\mathbf{a}, \mathbf{a} \otimes T^*)$. Thus, applying further $p^*(-)$ to each side and using adjunctions, we see that proving Proposition 3.17 amounts to showing that

(3.1.10)

$$p^*(\text{deform}(A, \psi)) = \Theta \circ \text{koszul}(\text{Id}_T) \quad \text{holds in} \quad \text{Ext}_{\mathbf{Ra} \otimes \mathbf{Ra}^{\text{op}}}^2(\mathbf{Ra}, \mathbf{Ra} \otimes T^*).$$

We compute the LHS of this equation using Proposition 2.2.3(ii), which says $\text{deform}(A, \psi) = \partial_{A,\psi}$. Therefore, $p^*(\text{deform}(A, \psi)) = p^*(\partial_{A,\psi})$, is the boundary map for $p^*(\Delta_{A,\psi})$, the pull-back via the equivalence $p^* : D(\mathbf{a} \otimes \mathbf{a}^{\text{op}}) \xrightarrow{\sim} D(\mathbf{Ra} \otimes \mathbf{Ra}^{\text{op}})$ of the canonical triangle $\Delta_{A,\psi}$ that appears in part (ii) of Proposition 2.2.3. Now, using the quasi-isomorphism $p^*\mathbf{a} \cong \mathbf{Ra}$, we can write the triangle $p^*(\Delta_{A,\psi})$ as follows:

$$p^*(\Delta_{A,\psi}) : \mathbf{Ra} \otimes T^*[1] \longrightarrow p^*((\mathbf{a} \overset{L}{\otimes}_A \mathbf{a})^{\tau \geq -1}) \xrightarrow{p^*(\mathbf{m}_\mathbf{a})} \mathbf{Ra}.$$

To describe the middle term in the last triangle we recall that \mathbf{Ra} is free over the subalgebra $A \subset \mathbf{Ra}$. It follows that in $D(\mathbf{Ra} \otimes \mathbf{Ra}^{\text{op}})$ one has $p^*(\mathbf{a} \overset{L}{\otimes}_A \mathbf{a}) \cong \mathbf{Ra} \overset{L}{\otimes}_A \mathbf{Ra} \cong \mathbf{Ra} \otimes_A \mathbf{Ra}$. Hence, we deduce $p^*((\mathbf{a} \overset{L}{\otimes}_A \mathbf{a})^{\tau \geq -1}) = (\mathbf{Ra} \otimes_A \mathbf{Ra})^{\tau \geq -1}$, since the pull-back functor p^* is always t -exact. Thus, we see that our exact triangle takes the following final form

$$(3.1.11) \quad p^*(\Delta_{A,\psi}) : \mathbf{Ra} \otimes T^*[1] \longrightarrow (\mathbf{Ra} \otimes_A \mathbf{Ra})^{\tau \geq -1} \xrightarrow{\mathbf{m}_{\mathbf{Ra}}} \mathbf{Ra}.$$

Next, we analyze the RHS of equation (3.1.10). By §2.3, we have $\text{koszul}(\text{Id}_T) = \partial_\wedge$. Hence, the class $\Theta \circ \text{koszul}(\text{Id}_T) = \Theta(\partial_\wedge)$, in

$$\text{Hom}_{D(\mathbf{Ra} \otimes \mathbf{Ra}^{\text{op}})}(\mathbf{Ra}, \mathbf{Ra} \otimes T^*[2]),$$

is represented by the boundary map for the exact triangle $\Theta(\Delta_\wedge)$, see (2.3.1). The latter triangle reads

$$(3.1.12) \quad \Theta(\Delta_\wedge) : \Theta(T^*[1]) \longrightarrow \Theta(\mathbf{\Lambda}/\mathbf{\Lambda}_{<-1}) \xrightarrow{\Theta(\epsilon_\wedge)} \Theta(\mathbb{k}_\wedge).$$

Here, $\Theta(\mathbb{k}_\wedge) = \text{Ra}$, by (3.1.9); hence $\Theta(T^*) = \text{Ra} \otimes T^*$. Further, by definition we have

$$\Theta(\mathbf{\Lambda}/\mathbf{\Lambda}_{<-1}) = (\text{Ra} \otimes_A \text{Ra}) \otimes_{\mathbf{\Lambda}} (\mathbf{\Lambda}/\mathbf{\Lambda}_{<-1}) = (\text{Ra} \otimes_A \text{Ra})/(\text{Ra} \otimes_A \text{Ra}) \cdot \mathbf{\Lambda}_{<-1}.$$

Now, $\text{Ra} \otimes_A \text{Ra} \cong \text{Ra} \otimes \mathbf{\Lambda}$, by (3.1.3), and we deduce

$$(3.1.13) \quad (\text{Ra} \otimes_A \text{Ra}) \cdot \mathbf{\Lambda}_{<-1} \cong \text{Ra} \otimes \mathbf{\Lambda}_{<-1}.$$

We see that the morphism $\Theta(\epsilon_\wedge)$ in (3.1.12) may be viewed as a map induced by the leftmost vertical arrow in diagram (3.1.4). Thus, the triangle in (3.1.12) takes the following form (note that Lemma 2.4.2(iii) insures that \mathbf{m}_{Ra} maps $(\text{Ra} \otimes_A \text{Ra}) \cdot \mathbf{\Lambda}_{<-1}$ to zero):

$$(3.1.14) \quad \Theta(\Delta_\wedge) : \text{Ra} \otimes T^*[1] \longrightarrow (\text{Ra} \otimes_A \text{Ra})/(\text{Ra} \otimes_A \text{Ra}) \cdot \mathbf{\Lambda}_{<-1} \xrightarrow{\mathbf{m}_{\text{Ra}}} \text{Ra}.$$

To compare the LHS with the RHS of (3.1.10), one has to compare (3.1.11) with (3.1.14). We see that in order to prove (3.1.10) it suffices to show

$$(3.1.15) \quad (\text{Ra} \otimes_A \text{Ra})^{\tau \geq -1} = (\text{Ra} \otimes_A \text{Ra})/(\text{Ra} \otimes_A \text{Ra}) \cdot \mathbf{\Lambda}_{<-1} \quad \text{in } D(\text{Ra} \otimes \text{Ra}^{\text{op}}).$$

To this end, we write an exact triangle

$$(3.1.16) \quad (\text{Ra} \otimes_A \text{Ra}) \cdot \mathbf{\Lambda}_{<-1} \rightarrow \text{Ra} \otimes_A \text{Ra} \rightarrow (\text{Ra} \otimes_A \text{Ra})/(\text{Ra} \otimes_A \text{Ra}) \cdot \mathbf{\Lambda}_{<-1}.$$

Here, the dg-vector space on the right is isomorphic to $\Theta(\mathbf{\Lambda}/\mathbf{\Lambda}_{<-1})$, hence, has no cohomology in degrees < -1 , by (3.1.6). Similarly, the dg-vector space on the left is isomorphic to $\text{Ra} \otimes \mathbf{\Lambda}_{<-1}$, see (3.1.13), hence, has no cohomology in degrees ≥ -1 . Therefore, the triangle in (3.1.16) must be isomorphic to the canonical exact triangle $(\text{Ra} \otimes_A \text{Ra})^{\tau < -1} \rightarrow \text{Ra} \otimes_A \text{Ra} \rightarrow (\text{Ra} \otimes_A \text{Ra})^{\tau \geq -1}$. This proves (3.1.15). \square

3.2. *Proposition 3.1.7 implies Theorem 1.2.3.* To see this, we observe that the functors Θ and p^* provide us not only with maps between the Ext-groups which occur in diagram (3.1.8), but also with lifts of those maps to morphisms in DGAAlg:

$$(3.2.1) \quad \text{REnd}_{D(\mathbf{\Lambda})}^\bullet(\mathbb{k}_\wedge) \xrightarrow{p_* \circ \Theta} \text{REnd}_{D(\mathfrak{a} \otimes \mathfrak{a}^{\text{op}})}^\bullet((p_* \circ \Theta(\mathbb{k}_\wedge))) \xrightarrow[\sim]{f_{A,\psi}} \text{REnd}_{D(\mathfrak{a} \otimes \mathfrak{a}^{\text{op}})}^\bullet(\mathfrak{a}).$$

Let $\mathfrak{D}\text{eform} := p_* \circ \Theta \circ \mathfrak{K}\mathfrak{o}\mathfrak{s}\mathfrak{z}\mathfrak{u}\mathfrak{l}$ be the composite of the dg-algebra morphism $\text{Sym}(T[-2]) \rightarrow \text{REnd}_{\mathbf{\Lambda}}^\bullet(\mathbb{k}_\wedge)$, see (2.3.3), followed by the morphisms in (3.2.1). Thus, in DGAAlg, we get a morphism $\mathfrak{D}\text{eform} : \text{Sym}(T[-2]) \rightarrow \text{REnd}_{D(\mathfrak{a} \otimes \mathfrak{a}^{\text{op}})}^\bullet(\mathfrak{a})$. Further, the induced map of cohomology

$$\begin{aligned} H^\bullet(\mathfrak{D}\mathfrak{e}\mathfrak{f}\mathfrak{o}\mathfrak{r}\mathfrak{m}) &= H^\bullet(p_* \circ \Theta \circ \mathfrak{K}\mathfrak{o}\mathfrak{s}\mathfrak{z}\mathfrak{u}\mathfrak{l}) \\ &= H^\bullet(p_* \circ \Theta) \circ \mathfrak{k}\mathfrak{o}\mathfrak{s}\mathfrak{z}\mathfrak{u}\mathfrak{l} : \text{Sym}(T[-2]) \rightarrow \text{Ext}_{\mathfrak{a} \otimes \mathfrak{a}^{\text{op}}}^\bullet(\mathfrak{a}) \end{aligned}$$

is equal to the map $\mathfrak{d}\mathfrak{e}\mathfrak{f}\mathfrak{o}\mathfrak{r}\mathfrak{m}$, by Proposition 3.1.7. Thus, the morphism $\mathfrak{D}\mathfrak{e}\mathfrak{f}\mathfrak{o}\mathfrak{r}\mathfrak{m}$ yields a morphism in $\text{DGA}\mathfrak{l}\mathfrak{g}$ as required by Theorem 1.2.3.

Our construction of the functor Θ , hence of the morphism $\mathfrak{D}\mathfrak{e}\mathfrak{f}\mathfrak{o}\mathfrak{r}\mathfrak{m}$, was based on the choice of an \mathcal{O} - $\mathfrak{d}\mathfrak{g}$ -algebra \mathfrak{R} . To show independence of such a choice, let \mathfrak{R}_s , $s = 1, 2$, be two \mathcal{O} - $\mathfrak{d}\mathfrak{g}$ -algebras, as in Lemma 2.4.3, and write $\mathfrak{R}\mathfrak{a}_s := \mathfrak{R}_s \otimes_{\mathcal{O}} \mathfrak{a}$. Part (ii) of the lemma implies that there exists a canonical isomorphism $h : \mathfrak{R}\mathfrak{a}_1 \xrightarrow{\text{qis}} \mathfrak{R}\mathfrak{a}_2$, in $\text{DGA}\mathfrak{l}\mathfrak{g}$, which is compatible with the augmentations $p_s : \mathfrak{R}\mathfrak{a}_s \xrightarrow{\text{qis}} \mathfrak{a}$. The isomorphism h induces a triangulated equivalence $h_* : D(\mathfrak{R}\mathfrak{a}_1 \otimes \mathfrak{R}\mathfrak{a}_1^{\text{op}}) \xrightarrow{\sim} D(\mathfrak{R}\mathfrak{a}_2 \otimes \mathfrak{R}\mathfrak{a}_2^{\text{op}})$, and also a canonical isomorphism h_{End} , in $\text{DGA}\mathfrak{l}\mathfrak{g}$, between the two $\mathfrak{d}\mathfrak{g}$ -algebra models for the object $\text{REnd}_{D(\mathfrak{a} \otimes \mathfrak{a}^{\text{op}})}^\bullet(\mathfrak{a}) \in \text{DGA}\mathfrak{l}\mathfrak{g}$, constructed using \mathfrak{R}_1 and \mathfrak{R}_2 , respectively.

Now, let $\Theta_s : D(\Lambda) \rightarrow D(\mathfrak{R}\mathfrak{a}_s \otimes \mathfrak{R}\mathfrak{a}_s^{\text{op}})$, $s = 1, 2$, be the corresponding two functors defined as in (3.1.5). Lemma 2.4.3(iii) yields a canonical isomorphism of functors $\Phi : \Theta_2 \xrightarrow{\sim} h_* \circ \Theta_1$. This way, in $\text{DGA}\mathfrak{l}\mathfrak{g}$, we obtain the following isomorphisms

$$\text{REnd}((p_1)_* \circ \Theta_1(\mathbb{k}_\Lambda)) \xrightarrow{\sim} \text{REnd}(h_* \circ (p_1)_* \circ \Theta_1(\mathbb{k}_\Lambda)) \xrightarrow{\sim} \text{REnd}((p_2)_* \circ \Theta_2(\mathbb{k}_\Lambda))$$

where we have used shorthand notation $\text{REnd} = \text{REnd}_{D(\mathfrak{a} \otimes \mathfrak{a}^{\text{op}})}^\bullet$. Let Υ denote the composite isomorphism, and write $f_{A,\psi,s}$ for the isomorphism of Proposition 3.1.7 corresponding to the $\mathfrak{d}\mathfrak{g}$ -algebra \mathfrak{R}_s , $s = 1, 2$. It is straightforward to check that our construction insures commutativity of the following diagram in $\text{DGA}\mathfrak{l}\mathfrak{g}$

$$\begin{array}{ccccc} \text{REnd}_{D(\Lambda)}^\bullet(\mathbb{k}_\Lambda) & \xrightarrow{(p_1)_* \circ \Theta_1} & \text{REnd}_{D(\mathfrak{a} \otimes \mathfrak{a}^{\text{op}})}^\bullet((p_1)_* \circ \Theta_1(\mathbb{k}_\Lambda)) & \xrightarrow{f_{A,\psi,1}} & \text{REnd}_{D(\mathfrak{a} \otimes \mathfrak{a}^{\text{op}})}^\bullet(\mathfrak{a}) \\ \parallel \text{Id} & & \downarrow \Upsilon & & \downarrow h_{\text{End}} \\ \text{REnd}_{D(\Lambda)}^\bullet(\mathbb{k}_\Lambda) & \xrightarrow{(p_2)_* \circ \Theta_2} & \text{REnd}_{D(\mathfrak{a} \otimes \mathfrak{a}^{\text{op}})}^\bullet((p_2)_* \circ \Theta_2(\mathbb{k}_\Lambda)) & \xrightarrow{f_{A,\psi,2}} & \text{REnd}_{D(\mathfrak{a} \otimes \mathfrak{a}^{\text{op}})}^\bullet(\mathfrak{a}). \end{array}$$

It follows from commutativity of the diagram that, for the resulting two morphisms $\mathfrak{D}\mathfrak{e}\mathfrak{f}\mathfrak{o}\mathfrak{r}\mathfrak{m}_s : \text{Sym}(T[-2]) \rightarrow \text{REnd}_s$, $s = 1, 2$, in $\text{DGA}\mathfrak{l}\mathfrak{g}$, we have $h_{\text{REnd}} \circ \mathfrak{D}\mathfrak{e}\mathfrak{f}\mathfrak{o}\mathfrak{r}\mathfrak{m}_1 = \mathfrak{D}\mathfrak{e}\mathfrak{f}\mathfrak{o}\mathfrak{r}\mathfrak{m}_2$. Thus, the morphisms $\mathfrak{D}\mathfrak{e}\mathfrak{f}\mathfrak{o}\mathfrak{r}\mathfrak{m}$ arising from various choices of \mathfrak{R} all give the same morphism in $\text{DGA}\mathfrak{l}\mathfrak{g}$. Therefore this morphism is *canonical*. \square

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