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Isoparametric hypersurfaces with four principal curvatures

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Abstract

Let M be an isoparametric hypersurface in the sphere S^n with four distinct principal curvatures. Münzner showed that the four principal curvatures can have at most two distinct multiplicities m_1, m_2, and Stolz showed that the pair (m_1, m_2) must either be $(2, 2)$, $(4, 5)$, or be equal to the multiplicities of an isoparametric hypersurface of FKM-type, constructed by Ferus, Karcher and Münzner from orthogonal representations of Clifford algebras. In this paper, we prove that if the multiplicities satisfy $m_2 \geq 2m_1 - 1$, then the isoparametric hypersurface M must be of FKM-type. Together with known results of Takagi for the case $m_1 = 1$, and Ozeki and Takeuchi for $m_1 = 2$, this handles all possible pairs of multiplicities except for four cases, for which the classification problem remains open.

1. Introduction

A hypersurface M in a real space-form $\tilde{M}^n(c)$ of constant sectional curvature c is said to be *isoparametric* if it has constant principal curvatures. An isoparametric hypersurface M in $\mathbf{R}^n$ can have at most two distinct principal curvatures, and M must be an open subset of a hyperplane, hypersphere or a spherical cylinder $S^k \times \mathbf{R}^{n-k-1}$. This was shown by Levi-Civita [18] for $n = 3$ and by B. Segre [27] for arbitrary n. Similarly, E. Cartan [3] proved that an isoparametric hypersurface M in hyperbolic space H^n can have at most two distinct principal curvatures, and M must be either totally umbilic or else be an open subset of a standard product $S^k \times H^{n-k-1}$ in H^n (see also [8, pp. 237, 238]). However, Cartan [3]–[6] showed in a series of four papers written in the late 1930's that the situation is much more interesting for isoparametric hypersurfaces in S^n. Cartan proved several general results and found examples with three and four distinct principal curvatures, as well as those with one or two.

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Despite the beauty of Cartan's theory, it was relatively unnoticed for thirty years, until it was revived in the 1970's by Nomizu [23], [24] and Münzner [22].

Cartan showed that isoparametric hypersurfaces come as a family of parallel hypersurfaces, i.e., if $\mathbf{x} : M \to S^n$ is an isoparametric hypersurface, then so is any parallel hypersurface $\mathbf{x}_t$ at oriented distance t from the original hypersurface $\mathbf{x}$. However, if $\lambda = \cot t$ is a principal curvature of M, then $\mathbf{x}_t$ is not an immersion, since it is constant on the leaves of the principal foliation T_λ, and $\mathbf{x}_t$ factors through an immersion of the space of leaves M/T_λ into S^n. In that case, $\mathbf{x}_t$ is a *focal submanifold* of codimension $m+1$ in S^n, where m is the multiplicity of λ.

Münzner [22] showed that a parallel family of isoparametric hypersurfaces in S^n always consists of the level sets in S^n of a homogeneous polynomial F defined on $\mathbf{R}^{n+1}$ satisfying certain differential equations which are listed at the beginning of Section 2. He showed that the level sets of F on S^n are connected, and thus any connected isoparametric hypersurface can be extended to a unique compact, connected isoparametric hypersurface.

Münzner also showed that regardless of the number of distinct principal curvatures of M, there are only two distinct focal submanifolds in a parallel family of isoparametric hypersurfaces, and each isoparametric hypersurface in the family separates the sphere into two ball bundles over the two focal submanifolds. From this topological information, Münzner was able to prove his fundamental result that the number g of distinct principal curvatures of an isoparametric hypersurface in S^n must be $1, 2, 3, 4$, or 6. As one would expect, classification results on isoparametric hypersurfaces have been dependent on the number of distinct principal curvatures.

Cartan classified isoparametric hypersurfaces with $g \leq 3$ principal curvatures. If $g = 1$, then M is umbilic and it must be a great or small sphere. If $g = 2$, then M must be a standard product of two spheres

$$S^k(r) \times S^{n-k-1}(s) \subset S^n, \quad r^2 + s^2 = 1.$$

In the case $g = 3$, Cartan [4] showed that all the principal curvatures must have the same multiplicity $m = 1, 2, 4$ or 8, and the isoparametric hypersurface must be a tube of constant radius over a standard Veronese embedding of a projective plane $\mathbf{F}P^2$ into S^{3m+1}, where $\mathbf{F}$ is the division algebra $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ (quaternions), $\mathbf{O}$ (Cayley numbers) for $m = 1, 2, 4, 8$, respectively. Thus, up to congruence, there is only one such family for each value of m.

The classification of isoparametric hypersurfaces with four or six principal curvatures has stood as one of the outstanding problems in submanifold geometry for some time, and it was listed as Problem 34 on Yau's list of important open problems in geometry in 1992 (see [36] or [15]). In this paper, we will provide a partial solution to this classification problem in the case $g = 4$, but first we will describe the known results in the two cases.

In the case $g = 6$, there exists one homogeneous family with six principal curvatures of multiplicity one in S^7, and one homogeneous family with six principal curvatures of multiplicity two in S^{13} (see Miyaoka [20] for a description). These are the only known examples. Münzner showed that for $g = 6$, all of the principal curvatures must have the same multiplicity m, and then Abresch [1] showed that m must be 1 or 2. In the case $m = 1$, Dorfmeister and Neher [10] showed in 1985 that an isoparametric hypersurface must be homogeneous, but it remains an open question whether this is true in the case $m = 2$.

For $g = 4$, there is a much larger and more diverse collection of known examples. Cartan produced examples of isoparametric hypersurfaces with four principal curvatures in S^5 and S^9. These examples are homogeneous, and have the property that all of the principal curvatures have the same multiplicity. Cartan asked if all isoparametric hypersurfaces must be homogeneous, and if there exists an isoparametric hypersurface whose principal curvatures do not all have the same multiplicity.

Nomizu [23] generalized Cartan's example in S^5 to produce a collection of isoparametric hypersurfaces whose principal curvatures have two distinct multiplicities $(1, k)$, for any positive integer k, thereby answering Cartan's second question in the affirmative. At approximately the same time as Nomizu's work, Takagi and Takahashi [31] used the work of Hsiang and Lawson [17] on submanifolds of cohomogeneity two to determine all homogeneous isoparametric hypersurfaces of the sphere. Takagi and Takahashi showed that every homogeneous isoparametric hypersurface is a principal orbit of the isotropy representation of a rank two symmetric space, and they presented a complete list of examples. This list included some examples with 6 principal curvatures, as well as those with $1, 2, 3$ or 4 distinct principal curvatures.

In a separate paper, Takagi [30] proved that in the case $g = 4$, if one of the principal curvatures of M has multiplicity one, then M must be homogeneous.

In a two-part paper, Ozeki and Takeuchi [25] produced two infinite series of inhomogeneous isoparametric hypersurfaces with multiplicities $(3, 4k)$ and $(7, 8k)$, for any positive integer k. They also classified isoparametric hypersurfaces for which one principal curvature has multiplicity two, proving that they must be homogeneous. In the process, Ozeki and Takeuchi developed a formulation of the Cartan-Münzner polynomial F in terms of the second fundamental forms of the focal submanifolds that is very useful in our work.

Next, Ferus, Karcher and Münzner [13] used representations of Clifford algebras to construct for any positive integer m_1 an infinite series of isoparametric hypersurfaces with four principal curvatures having multiplicities (m_1, m_2), where m_2 is nondecreasing and unbounded in each series. In fact, $m_2 = k\delta(m_1) - m_1 - 1$, where $\delta(m_1)$ is the positive integer such that the Clifford algebra C_{m_1-1} has an irreducible representation on $\mathbf{R}^{\delta(m_1)}$ (see [2]), and k is any positive integer for which m_2 is positive. Isoparametric hypersurfaces obtained by this construction of Ferus, Karcher and Münzner are said to be of FKM-*type*. The FKM-series with multiplicities $(3, 4k)$ and $(7, 8k)$ are precisely those constructed by Ozeki and Takeuchi. For isoparametric hypersurfaces of FKM-type, one of the focal submanifolds is always a Clifford-Stiefel manifold (see Pinkall-Thorbergsson [26]).

The set of FKM-type isoparametric hypersurfaces contains all known examples with $g = 4$ with the exception of two homogeneous examples, with multiplicities (m_1, m_2) equal to $(2, 2)$ and $(4, 5)$ (see [25, part II, p.27] for more detail on these two exceptions). Over the years, many restrictions on the multiplicities were found by Münzner [22], Abresch [1], Grove and Halperin [16], Tang [32] and Fang [12]. This series of papers culminated in the recent work of Stolz [29], who showed that the multiplicities of an isoparametric hypersurface with $g = 4$ must be the same as those in the known examples of Ferus, Karcher and Münzner or the two homogeneous exceptions. This certainly adds weight to the conjecture that the known examples are actually the only isoparametric hypersurfaces with $g = 4$. In this paper, we prove that this conjecture is true, if the two multiplicities satisfy $m_2 \geq 2m_1 - 1$. Specifically, we prove (see Theorem 47):

Classification Theorem. *Let M be an isoparametric hypersurface in the sphere S^n with four distinct principal curvatures, whose multiplicities m_1, m_2 satisfy $m_2 \geq 2m_1 - 1$. Then M is of* FKM-*type.*

Taken together with the classifications of Takagi for the case $m_1 = 1$ and Ozeki and Takeuchi for $m_1 = 2$, this handles all possible pairs (m_1, m_2) of multiplicities, with the exception of $(4, 5)$ and 3 pairs of multiplicities, $(3, 4)$, $(6, 9)$, $(7, 8)$ corresponding to isoparametric hypersurfaces of FKM-type. For these 4 pairs, the classification problem for isoparametric hypersurfaces remains open.

The first part of this work (through §9) gives necessary and sufficient conditions in terms of a natural second order moving frame for an isoparametric hypersurface to be of FKM-type. The second part shows that these conditions are satisfied if $m_2 \geq 2m_1 - 1$.

Next we will provide a detailed outline of the paper. For more information on isoparametric hypersurfaces and the extensive theory of isoparametric submanifolds of codimension greater than one in the sphere, which was introduced by Carter and West [7] and Terng [33], the reader is referred to the excellent survey article by Thorbergsson [35], who proved that all isoparametric submanifolds of codimension greater than one in the sphere are homogeneous [34].

We think of an isoparametric hypersurface as an immersion $\tilde{\mathbf{x}} : M^{n-1} \to S^n$. About any point of M there is a neighborhood U on which there is defined an orthonormal frame field $\tilde{\mathbf{x}}, \tilde{e}_0, e_a, e_p, e_\alpha, e_\mu$ for which $\tilde{e}_0$ is normal to the hypersurface and the other sets of vectors are principal directions for the four respective principal curvatures of $\tilde{\mathbf{x}}$. The index range of a, p has length m, and that of α, μ has length N, where $m = m_1$ and $N = m_2$ are the multiplicities for our isoparametric hypersurface. The dual coframe on U is the set of 1-forms $\theta^a, \theta^p, \theta^\alpha, \theta^\mu$ defined on U by the equation (sum on repeated indices)

$$d\tilde{\mathbf{x}} = \theta^a e_a + \theta^p e_p + \theta^\alpha e_\alpha + \theta^\mu e_\mu \,.$$

The curvature surfaces are the integral submanifolds of the distribution obtained by setting any three sets of these forms equal to zero. The Levi-Civita connection forms of a curvature surface are given, essentially, by the forms $\theta^a_b = de_a \cdot e_b$, $\theta^p_q = de_q \cdot e_p$, etc. The second fundamental tensors of the focal submanifolds are given in terms of our frame field by the four sets of tensors $F^\mu_{\alpha a}$, $F^\mu_{\alpha p}$, F^μ_{pa}, and F^α_{pa} defined in (4.18) in which the coframe field $\omega^a, \omega^p, \omega^\alpha, \omega^\mu$ is defined in (4.13) as constant multiples of $\theta^a, \theta^p, \theta^\alpha, \theta^\mu$, respectively. We derive the identities imposed on these tensors and their derivatives by the Maurer-Cartan structure equations of the orthogonal group $O(n+1)$, the isometry group of S^n.

If our isoparametric hypersurface is of FKM-type, then a simple calculation shows that the following equations hold for an appropriate choice of the Darboux frame field.

$$F^\mu_{\alpha\, a+m} = F^\mu_{\alpha a}, \tag{1.1}$$

$$F^\alpha_{b+m\, a} + F^\alpha_{a+m\, b} = 0, \tag{1.2}$$

$$F^\mu_{b+m\, a} + F^\mu_{a+m\, b} = 0, \tag{1.3}$$

$$\theta^a_b - \theta^{a+m}_{b+m} = L^a_{bc}(\omega^c + \omega^{c+m}), \quad L^a_{bc} = -L^b_{ac} = -L^a_{cb}, \tag{1.4}$$

where $a, b, c = 1, \ldots, m$ and $a+m, b+m$ run through the range of the indices p, q. The matrices of the operators of the Clifford system in terms of our frame field have as entries certain constants and the functions $F^\mu_{\alpha a}$, $F^\mu_{\alpha p}$, F^μ_{pa}, F^α_{pa}, and L^a_{bc}. Thus, using these matrices, we can define these operators for an arbitrary isoparametric hypersurface. If equations (1.1)–(1.4) hold for the isoparametric hypersurface, then by an elementary, but extremely long, calculation we show that these operators form a Clifford system whose FKM construction produces the given isoparametric hypersurface. This calculation is contained in the proof of Theorem 24.

In Proposition 19 we prove that (1.1) implies (1.2)–(1.4) on U provided that $\tilde{\mathbf{x}}$ satisfies the *spanning property* (Definition 8), which is:

(a) There exists a vector $\sum_\alpha x_\alpha e_\alpha$ such that

$$\{\sum_{a,\alpha,\mu} F^\mu_{\alpha a} x_\alpha y_\mu e_a : (y_\mu) \in \mathbf{R}^N\} = \text{span}\,\{e_1, \ldots, e_m\}.$$

(b) There exists a vector $\sum_\mu y_\mu e_\mu$ such that

$$\{\sum_{a,\alpha,\mu} F^\mu_{\alpha a} x_\alpha y_\mu e_a : (x_\alpha) \in \mathbf{R}^N\} = \text{span}\,\{e_1, \ldots, e_m\}.$$

Combining these results, we see that if an isoparametric hypersurface satisfies the spanning property and (1.1) on U, then it is of FKM-type. The next step is to see when (1.1) will be true.

The parallel hypersurface at an oriented distance t from $\tilde{\mathbf{x}}$ is given by $\mathbf{x} = \cos t\, \tilde{\mathbf{x}} + \sin t\, \tilde{e}_0$. Its unit normal vector is $e_0 = -\sin t\, \tilde{\mathbf{x}} + \cos t\, \tilde{e}_0$ and its principal directions are still given by the remaining vectors in the frame field. At some value of t the rank of $\mathbf{x}$ is less than $n-1$, in which case the image of $\mathbf{x}$ is a focal submanifold of the isoparametric family. Any multiple of $\pi/4$ added to this value of t again gives a focal submanifold.

From Münzner's result that there are only two focal submanifolds, it follows that as t changes by a multiple of $\pi/2$, we return to the same focal submanifold. If $\mathbf{x}$ is a focal submanifold, then we may assume that e_0, e_a is a normal frame field along $\mathbf{x}$ and the vectors e_p, e_α, e_μ are the principal vectors for the second fundamental form II_{e_0}, of principal curvatures 0, 1 and -1, respectively. Moving a distance $t = \pi/2$ from $\mathbf{x}$ along the geodesic in the direction of e_0, we arrive at e_0, which must then be a position vector on the same focal submanifold. At e_0, the normal frame field is $\mathbf{x}, e_p$, and the principal vectors, of principal curvatures 0, 1 and -1 are e_a, e_α and e_μ, respectively.

There is a simple relationship between the four sets of tensors at e_0, denoted with the same letters barred, and these tensors at $\mathbf{x}$. For our purposes, the most important is

$$\bar{F}^\mu_{\alpha a} = F^\mu_{\alpha\, a+m}.$$

Use these tensors to define real bihomogeneous polynomials

$$p_a(x,y) = \sum_{\alpha,\mu} F^\mu_{\alpha a} x_\alpha y_\mu, \quad \bar{p}_a(x,y) = \sum_{\alpha,\mu} \bar{F}^\mu_{\alpha a} x_\alpha y_\mu.$$

In Proposition 11 we prove that if $\mathbf{x}$ satisfies the spanning property on U and if at each point of U the $\bar{p}_a$ are contained in the ideal I generated by $p_1, \dots, p_m$ in the polynomial ring $\mathbf{R}[x_\alpha, y_\mu]$, then the frame field can be chosen so that (1.1) holds on U.

The key to linking the set of polynomials $\bar{p}_a$ with the set of polynomials p_a comes from a formula for the isoparametric function derived by Ozeki and Takeuchi [25] (recorded in (10.1) below). In Proposition 27 (see also Proposition 28) we use this formula to prove that the zero locus of $p_1, \dots, p_m$ in $\mathbf{R}P^{N-1} \times \mathbf{R}P^{N-1}$ is identical to that of $\bar{p}_1, \dots, \bar{p}_m$.

Algebraic geometers have developed a substantial body of information about the relationship between two polynomial ideals whose zero varieties coincide. Let I be the ideal generated by $p_1, \dots, p_m$ in the polynomial ring $\mathbf{R}[x_\alpha, y_\mu]$ and let $I^{\mathbf{C}}$ be the ideal they generate in the polynomial ring $\mathbf{C}[x_\alpha, y_\mu]$. For $1 \le s \le m$, define the affine bi-cones

$$V_s = \{(x,y) \in \mathbf{R}^N \times \mathbf{R}^N : p_a(x,y) = 0,\ a = 1, \dots, s\},$$

$$V_s^{\mathbf{C}} = \{(x,y) \in \mathbf{C}^N \times \mathbf{C}^N : p_a(x,y) = 0,\ a = 1,\ldots,s\}.$$

We denote V_m and $V_m^{\mathbf{C}}$, which are in fact what we are after, by V_I and $V_I^{\mathbf{C}}$, respectively. Let J_s be the complex subvariety of $V_s^{\mathbf{C}}$ where the Jacobian matrix of $p_1,\ldots,p_s$ is of rank less than s. In our Classification Theorem 47 we prove the following. Fix a point in U. Assume $N \geq m+2$. If the codimension of J_s is greater than 1 in $V_s^{\mathbf{C}}$ for all $s \leq m$, then, at the point, through an inductive procedure, we establish

(I) $p_1,\ldots,p_m$ form a regular sequence in $\mathbf{C}[x_\alpha, y_\mu]$,

(II) $\dim_{\mathbf{R}} V_I = \dim_{\mathbf{C}} V_I^{\mathbf{C}}$,

(III) $I^{\mathbf{C}}$ is a prime ideal of codimension m,

(IV) The spanning property holds for $\mathbf{x}$.

The primeness (more generally, reducedness) of $I^{\mathbf{C}}$ is precisely the condition which allows us to conclude that the $\bar{p}_a \in I$.

The final step in our argument is then provided by Proposition 46 which states that for $N \geq m+2$, if $N \geq 2m$, then indeed $\operatorname{codim}(J_s) \geq 2$ for all $s \leq m$ at every point of U, so that $I^{\mathbf{C}}$ is prime; as a result, if $N = 2m-1$, then $I^{\mathbf{C}}$ is a reduced ideal. The proof of this estimate requires a detailed analysis of the second fundamental forms II_{e_a} of $\mathbf{x}$. In the case $m = 1$, we give a simpler proof that M is of FKM-type, thereby providing another proof of Takagi's result. Our approach also recovers Ozeki-Takeuchi's result when $m = 2$ and $N \geq 3$.

The paper is very much self-contained, and we have made an effort to make the exposition as clear as possible. We would like to thank N. Mohan Kumar for substantial help with the algebraic geometry and John Little for his comments on previous versions of this paper. We are grateful to the referee, whose many helpful comments have improved the exposition and quality of the paper.

2. Second order frames

An immersed connected oriented hypersurface $\tilde{\mathbf{x}} : M^{n-1} \to S^n$ is called *isoparametric* if $\tilde{\mathbf{x}}$ has constant principal curvatures. Such a hypersurface always occurs as part of a family, the level surfaces of an *isoparametric function* f, which is a smooth function on S^n such that $|\nabla f|^2 = a(f)$ and $\Delta f = b(f)$, for some smooth functions $a, b : \mathbf{R} \to \mathbf{R}$.

Denote the principal curvatures of $\tilde{\mathbf{x}}$ by k_i, with multiplicity m_i, for $i = 1,\ldots,g$, and assume that $k_1 > \cdots > k_g$. Münzner [22, part I] showed that the multiplicities satisfy $m_i = m_{i+2}$ (subscripts mod g). He then showed that the isoparametric function f must be the restriction to S^n of a homogeneous polynomial $F : \mathbf{R}^{n+1} \to \mathbf{R}$ of degree g satisfying the differential equations

$$|\text{grad}\, F|^2 = g^2 r^{2g-2}, \quad r = |\mathbf{x}|,$$
$$\Delta F = \frac{m_2 - m_1}{2} g^2 r^{g-2},$$

where m_1 and m_2 are the two (possibly equal) multiplicities. The polynomial F is called the *Cartan-Münzner polynomial* of the family of isoparametric hypersurfaces, and F takes values between -1 and 1 on the sphere S^n. For $-1 < t < 1$, the level set $F^{-1}(t)$ in S^n is one of the isoparametric hypersurfaces in the family. The level sets $M_+ = F^{-1}(1)$ and $M_- = F^{-1}(-1)$ are the two focal submanifolds of the family, having codimensions $m_1 + 1$ and $m_2 + 1$ in S^n, respectively.

We now develop the local geometry of isoparametric hypersurfaces using the method of moving frames in the sphere. In the process, we will reprove some of the results obtained by Münzner, although this is not our primary goal.

We assume now that $g = 4$, even though many of the results in Sections 2–4 have analogues for arbitrary values of g. Let $\tilde{e}_0$ be the unit normal vector field along $\tilde{\mathbf{x}}$ defining the orientation of M. Any point of M has an open neighborhood U on which there exists a Darboux frame field $\tilde{\mathbf{x}}, e_i, \tilde{e}_0 : U \to \mathrm{SO}(n+1)$, $1 \leq i \leq n-1$, for which each vector e_i is a principal direction. We adopt the index ranges

$$\begin{aligned} &i, j, k \in \{1, \ldots, n-1\}, \\ &a, b, c \in \{1, \ldots, m_1\}, \quad p, q, r \in \{m_1 + 1, \ldots, m_1 + m_3\}, \\ &\alpha, \beta, \gamma \in \{m_1 + m_3 + 1, \ldots, m_1 + m_2 + m_3\}, \\ &\mu, \nu, \sigma \in \{m_1 + m_2 + m_3 + 1, \ldots, n-1\}. \end{aligned} \tag{2.1}$$

Arrange the frame so that the e_a span the principal space for k_1, the e_α span the principal space for k_2, the e_p span the principal space for k_3, and the e_μ span the principal space for k_4. We shall call such a Darboux frame field

$$\tilde{\mathbf{x}}, e_a, e_p, e_\alpha, e_\mu, \tilde{e}_0 \tag{2.2}$$

on U a *second order frame field* along $\tilde{\mathbf{x}}$, (a first order Darboux frame field is one for which $\tilde{e}_0$ is normal and the remaining vectors are tangent, but not necessarily principal directions). For such a frame field

$$d\tilde{\mathbf{x}} = \theta^i e_i \text{ and } de_i = \theta^j_i e_j - \theta^i \tilde{\mathbf{x}} + \theta^0_i \tilde{e}_0 \tag{2.3}$$

where θ^i, $\theta^0_i = -\theta^i_0$, $\theta^i_j = -\theta^j_i$ are 1-forms on U and $\theta^1, \ldots, \theta^{n-1}$ is an orthonormal coframe field on U with respect to the metric induced by $\tilde{\mathbf{x}}$ on M. Notice that $\theta^0 = d\tilde{\mathbf{x}} \cdot \tilde{e}_0 = 0$. We use the Einstein summation convention unless the contrary is stated explicitly. This means that repeated indices in a product are to be summed over the range defined in (2.1). In some instances the repeated indices are both up, or both down, but still they are to be summed as in the standard case of one up and one down. The 1-forms in (2.3) satisfy the Maurer-Cartan structure equations of SO$(n+1)$:

$$\begin{aligned} d\theta^i &= -\theta^i_j \wedge \theta^j, \\ d\theta^0_i &= -\theta^0_j \wedge \theta^j_i, \\ d\theta^i_j &= \theta^i \wedge \theta^j - \theta^i_0 \wedge \theta^0_j - \theta^i_k \wedge \theta^k_j. \end{aligned} \tag{2.4}$$

We also have

$$d\tilde{e}_0 = \theta^i_0 e_i \tag{2.5}$$

where the 1-forms $\theta^i_0 = -\theta^0_i$ are linear combinations of the coframe forms, namely

$$\theta^0_i = h_{ij}\theta^j \tag{2.6}$$

where these coefficient functions on U satisfy $h_{ij} = h_{ji}$ as a consequence of taking the exterior derivative of the equation $\theta^0 = 0$. The second fundamental form of $\tilde{\mathbf{x}}$ is

$$\widetilde{II} = -d\tilde{\mathbf{x}} \cdot d\tilde{e}_0 = h_{ij}\theta^i\theta^j. \tag{2.7}$$

Having chosen the e_i to be principal vectors, we know that the symmetric matrix h_{ij} is a diagonal matrix. In fact, we have

$$\theta^0_a = k_1\theta^a, \quad \theta^0_p = k_3\theta^p, \quad \theta^0_\alpha = k_2\theta^\alpha, \quad \theta^0_\mu = k_4\theta^\mu. \tag{2.8}$$

Set $\theta^i_j = \sum h^i_{jk}\theta^k$, where the smooth function coefficients satisfy $h^i_{jk} = -h^j_{ik}$, for all $i, j, k = 1, \dots, n-1$. Take the exterior differential of equations (2.8), using the structure equations of SO$(n+1)$, to find

$$\begin{aligned} \theta^p_a &= h^p_{a\alpha}\theta^\alpha + h^p_{a\mu}\theta^\mu, \text{ since } h^p_{ab} = 0 = h^p_{aq}, \\ \theta^\alpha_a &= h^\alpha_{ap}\theta^p + h^\alpha_{a\mu}\theta^\mu, \text{ since } h^\alpha_{ab} = -h^a_{\alpha b} = 0 = h^\alpha_{a\beta}, \\ \theta^\mu_a &= h^\mu_{ap}\theta^p + h^\mu_{a\alpha}\theta^\alpha, \text{ since } h^\mu_{ab} = -h^a_{\mu b} = 0 = h^\mu_{a\nu}, \\ \theta^\alpha_p &= h^\alpha_{pa}\theta^a + h^\alpha_{p\mu}\theta^\mu, \text{ since } h^\alpha_{pq} = -h^p_{\alpha q} = 0 = h^\alpha_{p\beta}, \\ \theta^\mu_p &= h^\mu_{pa}\theta^a + h^\mu_{p\alpha}\theta^\alpha, \text{ since } h^\mu_{pq} = -h^p_{\mu q} = 0 = h^\mu_{p\nu}, \\ \theta^\mu_\alpha &= h^\mu_{\alpha a}\theta^a + h^\mu_{\alpha p}\theta^p, \text{ since } h^\mu_{\alpha\beta} = -h^\alpha_{\mu\beta} = 0 = h^\mu_{\alpha\nu}. \end{aligned} \tag{2.9}$$

The coefficient functions further satisfy

$$\begin{aligned} (k_3 - k_1)h^p_{a\alpha} &= (k_2 - k_1)h^\alpha_{ap} = (k_2 - k_3)h^\alpha_{pa}, \\ (k_3 - k_1)h^p_{a\mu} &= (k_4 - k_1)h^\mu_{ap} = (k_4 - k_3)h^\mu_{pa}, \\ (k_2 - k_1)h^\alpha_{a\mu} &= (k_4 - k_1)h^\mu_{a\alpha} = (k_4 - k_2)h^\mu_{\alpha a}, \\ (k_2 - k_3)h^\alpha_{p\mu} &= (k_4 - k_3)h^\mu_{p\alpha} = (k_4 - k_2)h^\mu_{\alpha p}. \end{aligned} \tag{2.10}$$

At a point of M the set of principal vectors for a principal curvature k_i is a subspace of dimension m_i, defined by the equations $\theta^j = 0$, for all j not in the range of the given principal curvature. This m_i-plane distribution on M is called a *curvature distribution* on M.

LEMMA 1. *The curvature distributions are completely integrable. Their integral submanifolds are called curvature surfaces. A curvature surface corresponding to k_j is totally geodesic in M and its induced metric has constant sectional curvature $1+k_j^2$.*

Proof. This is a simple application of the structure equations and the first three equations in (2.9). □

Remark 2. One can show that each curvature surface corresponding to k_j is also totally geodesic in the curvature sphere of M corresponding to k_j (see Theorems 4.11–4.13 of [8, pp. 149, 150]).

Additional conditions are imposed by the structure equations on the coefficients upon the exterior differentiation of equations (2.9).

3. Parallel hypersurfaces

Let $\tilde{\mathbf{x}}, e_a, e_p, e_\alpha, e_\mu, \tilde{e}_0$ be a second order frame field (2.2) along $\tilde{\mathbf{x}}$ on U. We may arrange to have $k_1 > k_2 > k_3 > k_4$. It will be convenient to set $k_i = \cot s_i$, for $i = 1, \ldots, 4$, where $0 < s_1 < s_2 < s_3 < s_4 < \pi$. For any fixed real number t, let

$$\mathbf{x} = \cos t\, \tilde{\mathbf{x}} + \sin t\, \tilde{e}_0. \tag{3.1}$$

From (2.3), (2.5) and (2.8) we have

$$\begin{aligned} d\mathbf{x} =&(\cos t - \sin t \cot s_1)\theta^a e_a + (\cos t - \sin t \cot s_3)\theta^p e_p \\ &+ (\cos t - \sin t \cot s_2)\theta^\alpha e_\alpha + (\cos t - \sin t \cot s_4)\theta^\mu e_\mu. \end{aligned} \tag{3.2}$$

We conclude that $\mathbf{x}$ is an immersion of M except when $t \equiv s_i \mod \pi$, for some $i = 1, 2, 3, 4$. Suppose t is not one of these exceptional values. Then the unit normal vector field along $\mathbf{x}$ preserving the orientation of M is

$$e_0 = -\sin t\, \tilde{\mathbf{x}} + \cos t\, \tilde{e}_0 \tag{3.3}$$

and again from (2.3), (2.5) and (2.8) we have

$$\begin{aligned} de_0 =& -(\sin t + \cos t \cot s_1)\theta^a e_a - (\sin t + \cos t \cot s_3)\theta^p e_p \\ &- (\sin t + \cos t \cot s_2)\theta^\alpha e_\alpha - (\sin t + \cos t \cot s_4)\theta^\mu e_\mu. \end{aligned} \tag{3.4}$$

Since $(\sin t + \cos t \cot s)/(\cos t - \sin t \cot s) = \cot(s-t)$, for any s and t, we find that the second fundamental form of $\mathbf{x}$ is

$$\begin{aligned} II =& -d\mathbf{x} \cdot de_0 \\ =& \cot(s_1 - t)\, \omega^a \omega^a + \cot(s_3 - t)\, \omega^p \omega^p \\ &+ \cot(s_2 - t)\, \omega^\alpha \omega^\alpha + \cot(s_4 - t)\, \omega^\mu \omega^\mu. \end{aligned} \tag{3.5}$$

We conclude that the principal curvatures of $\mathbf{x}$ are constant, equal to $\cot(s_i - t)$ with multiplicity m_i, for $i = 1, 2, 3, 4$, and that

$$\mathbf{x}, e_a, e_p, e_\alpha, e_\mu, e_0 \tag{3.6}$$

is a second order frame field along $\mathbf{x}$ on U.

4. Focal submanifolds

We consider now what happens when t is one of the exceptional values. To be specific, suppose that $t = s_1$. Then $\mathbf{x}$ is as defined in (3.1) and e_0 is as defined in (3.3) with $t = s_1$. For the frame field (3.6) along $\mathbf{x}$ on U, equation (3.2) becomes

$$d\mathbf{x} = \omega^p e_p + \omega^\alpha e_\alpha + \omega^\mu e_\mu \tag{4.1}$$

whose rank is $n - 1 - m_1$ at every point of M and where

$$\omega^p = \frac{\sin(s_3 - s_1)}{\sin s_3}\theta^p, \quad \omega^\alpha = \frac{\sin(s_2 - s_1)}{\sin s_2}\theta^\alpha, \quad \omega^\mu = \frac{\sin(s_4 - s_1)}{\sin s_4}\theta^\mu. \tag{4.2}$$

Therefore, the image $\mathbf{x}(M)$ is a submanifold of codimension $m_1 + 1$ in S^n. It is called the *focal submanifold* for the principal curvature $\cot s_1$. In the same way, there are focal submanifolds for each of the principal curvatures. For a point $\mathbf{v} \in \mathbf{x}(M)$, the set $L = \mathbf{x}^{-1}\{\mathbf{v}\}$ is a curvature surface of $\mathbf{x}$ for the principal curvature $\cot s_1$. Restricted to this curvature surface, the forms θ^a give a coframe field on it.

If e_0 is defined by (3.3), then (4.1) shows that $\mathbf{x}, e_p, e_\alpha, e_\mu, e_a, e_0$ is a Darboux frame field along $\mathbf{x}$, with e_p, e_α, e_μ tangent and e_0, e_a normal vectors. Take a point p in the curvature surface L and let N denote the normal space to $\mathbf{x}$ at p. Let S^{m_1} denote the unit sphere in N. The next lemma shows that $e_0(L)$ covers an open neighborhood of $e_0(p)$ in this sphere.

LEMMA 3. *The rank of* $e_0 : L \to S^{m_1}$ *is* m_1 *at every point of the curvature surface* L*. Therefore,* $e_0(L)$ *covers an open neighborhood of* $e_0(p)$ *in* S^{m_1}*.*

Proof. Consider the frame field $e_0, e_a, \mathbf{x}, e_p, e_\alpha, e_\mu$ along e_0 on L. Since θ^p, θ^α and θ^μ are all zero pulled back to L, it follows from (2.9) that θ^p_0, θ^α_0 and θ^μ_0 are also zero pulled back to L. Therefore, restricted to L, and using (2.8), in which now $k_1 = \cot s_1$, we have

$$de_0 = -\sin s_1\, \theta^a e_a + \cos s_1\, \theta^a_0 e_a = -\csc s_1\, \theta^a e_a \tag{4.3}$$

which has rank equal to m_1 at every point of L. □

We can now calculate the second fundamental form of the submanifold $\mathbf{x}$ at the point $\mathbf{x}(p) = \mathbf{v}$ with respect to any unit normal vector there.

LEMMA 4. *At any point of M and with respect to any unit normal vector at the point, the principal curvatures of the focal submanifold $\mathbf{x}$ are*

$$\cot(s_2 - s_1), \quad \cot(s_3 - s_1), \quad \cot(s_4 - s_1) \tag{4.4}$$

with multiplicities m_2, m_3, m_4, respectively.

Proof. From (3.4) we have for $t = s_1$

$$\begin{aligned} de_0 = &- \frac{1}{\sin s_1}\theta^a e_a - \frac{\cos(s_3 - s_1)}{\sin s_3}\theta^p e_p \\ &- \frac{\cos(s_2 - s_1)}{\sin s_2}\theta^\alpha e_\alpha - \frac{\cos(s_4 - s_1)}{\sin s_4}\theta^\mu e_\mu. \end{aligned} \tag{4.5}$$

Combining this with (4.2) we have for the second fundamental form at p with respect to the normal vector e_0

$$\begin{aligned} II_{e_0} &= -d\mathbf{x} \cdot de_0 \\ &= \cot(s_3 - s_1)\omega^p\omega^p + \cot(s_2 - s_1)\omega^\alpha\omega^\alpha + \cot(s_4 - s_1)\omega^\mu\omega^\mu \end{aligned}$$

where $\omega^p, \omega^\alpha, \omega^\mu$, defined in (4.3), form an orthonormal coframe with respect to the metric induced by $\mathbf{x}$ on the focal submanifold for the principal curvature $\cot s_1$. By Lemma 3 we know that $e_0(L)$ covers some open subset of the unit sphere in the normal space to $\mathbf{x}$ at p. Since the characteristic polynomial of $II_{\mathbf{n}}$ is an analytic function of $\mathbf{n}$ in the unit sphere of the normal space, it follows that the eigenvalues of $II_{\mathbf{n}}$ must be given by (4.4) for every unit normal vector at p. (See [8, Proof Cor. 2.2, p. 249]). □

Münzner [22, Part I] proved Lemma 4 and used it to prove the following important consequence (see also [8, p. 249]).

COROLLARY 5. *The angles $s_i = s_1 + (i-1)\pi/4$, for $i = 2, 3, 4$ and the multiplicities satisfy $m_1 = m_3$ and $m_2 = m_4$. To simplify the notation we set $m_1 = m_3 = m$ and $m_2 = m_4 = N$.*

Given these facts, our index conventions (2.1) become

$$\begin{aligned} &i, j, k \in \{1, \dots, n-1\}, \quad a, b, c \in \{1, \dots, m\}, \\ &p, q, r \in \{m+1, \dots, 2m\}, \quad \alpha, \beta, \gamma \in \{2m+1, \dots, 2m+N\}, \\ &\mu, \nu, \sigma \in \{2m+N+1, \dots, n-1\}, \end{aligned} \tag{4.6}$$

so that $2m + 2N = n - 1$, and n must be odd. Combining Lemma 4 and Corollary 5 yields the following.

COROLLARY 6. *At any point of M and with respect to any unit normal vector of $\mathbf{x}$ at the point, the principal curvatures of $\mathbf{x}$ are*

$$1, \quad 0, \quad -1 \tag{4.7}$$

with multiplicities N, m and N, respectively.

In the light of Corollary 5, the principal curvatures $k_i = \cot s_i$ of $\tilde{\mathbf{x}}$ satisfy

$$k_2 = \frac{k_1 - 1}{k_1 + 1}, \quad k_3 = -\frac{1}{k_1}, \quad k_4 = \frac{1 + k_1}{1 - k_1}. \tag{4.8}$$

We will have occasion to use the following differences of these principal curvatures.

$$\begin{aligned} k_2 - k_1 &= -\frac{1 + k_1^2}{1 + k_1}, & k_3 - k_1 &= -\frac{1 + k_1^2}{k_1}, \\ k_4 - k_1 &= \frac{1 + k_1^2}{1 - k_1}, & k_3 - k_2 &= -\frac{1 + k_1^2}{k_1(1 + k_1)}, \\ k_4 - k_2 &= 2\frac{1 + k_1^2}{1 - k_1^2}, & k_4 - k_3 &= \frac{1 + k_1^2}{k_1(1 - k_1)}. \end{aligned} \tag{4.9}$$

We use equations (4.9) to rewrite equations (2.10) as

$$\begin{aligned} h^p_{a\alpha} &= -\frac{1}{1 + k_1} h^\alpha_{pa}, & h^\alpha_{ap} &= -\frac{1}{k_1} h^\alpha_{pa}, \\ h^p_{a\mu} &= \frac{1}{k_1 - 1} h^\mu_{pa}, & h^\mu_{ap} &= \frac{1}{k_1} h^\mu_{pa}, \\ h^\alpha_{a\mu} &= \frac{2}{k_1 - 1} h^\mu_{\alpha a}, & h^\mu_{a\alpha} &= \frac{2}{1 + k_1} h^\mu_{\alpha a}, \\ h^\alpha_{p\mu} &= \frac{2k_1}{1 - k_1} h^\mu_{\alpha p}, & h^\mu_{p\alpha} &= \frac{2k_1}{1 + k_1} h^\mu_{\alpha p}. \end{aligned} \tag{4.10}$$

Now, with $s_i = s_1 + (i-1)\pi/4$, equation (4.1) takes the form

$$d\mathbf{x} = \frac{1}{\sin s_3}\theta^p e_p + \frac{1}{\sqrt{2}\sin s_2}\theta^\alpha e_\alpha + \frac{1}{\sqrt{2}\sin s_4}\theta^\mu e_\mu, \tag{4.11}$$

and with $t = s_1$ equation (3.4) becomes

$$de_0 = -\frac{1}{\sin s_1}\theta^a e_a - \frac{1}{\sqrt{2}\sin s_2}\theta^\alpha e_\alpha + \frac{1}{\sqrt{2}\sin s_4}\theta^\mu e_\mu. \tag{4.12}$$

If we define a new coframe field on $U \subset M$ by

$$\begin{aligned} \omega^a &= -\frac{1}{\sin s_1}\theta^a, & \omega^p &= \frac{1}{k_1 \sin s_1}\theta^p, \\ \omega^\alpha &= \frac{1}{(1 + k_1)\sin s_1}\theta^\alpha, & \omega^\mu &= \frac{1}{(k_1 - 1)\sin s_1}\theta^\mu, \end{aligned} \tag{4.13}$$

then, because

$$\sin s_2 = \frac{1 + k_1}{\sqrt{2}} \sin s_1, \ \sin s_3 = k_1 \sin s_1, \ \sin s_4 = \frac{k_1 - 1}{\sqrt{2}} \sin s_1 \tag{4.14}$$

equations (4.11) and (4.12) become

$$d\mathbf{x} = \omega^p e_p + \omega^\alpha e_\alpha + \omega^\mu e_\mu, \quad de_0 = \omega^a e_a - \omega^\alpha e_\alpha + \omega^\mu e_\mu. \tag{4.15}$$

One conclusion we can draw from (4.15) is that

$$\mathbf{x}, e_0, e_a, e_p, e_\alpha, e_\mu \tag{4.16}$$

is a Darboux frame field along $\mathbf{x}$ on U, with e_0, e_a normal vectors and e_p, e_α, e_μ tangent vectors spanning the principal spaces of curvature 0, 1 and -1, respectively of II_{e_0}. We shall call this a *second order frame field* along the focal submanifold $\mathbf{x}$ on U. For each point of U, define linear subspaces of $\mathbf{R}^{n+1}$ by

$$V_+ = \operatorname{span}\{e_\alpha\}, \quad V_- = \operatorname{span}\{e_\mu\}, \quad V_0 = \operatorname{span}\{e_p\}. \tag{4.17}$$

These are the $+1$, -1 and 0 principal curvature spaces, respectively, for the normal vector e_0 at this point. If we express the Maurer-Cartan forms (2.9) in terms of our coframe field (4.13) as

$$\begin{aligned}
\theta^p_a &= \sum_\alpha F^\alpha_{pa}\omega^\alpha - \sum_\mu F^\mu_{pa}\omega^\mu, & \theta^\alpha_a &= \sum_p F^\alpha_{pa}\omega^p - 2\sum_\mu F^\mu_{\alpha a}\omega^\mu,\\
\theta^\alpha_p &= \sum_a F^\alpha_{pa}\omega^a - 2\sum_\mu F^\mu_{\alpha p}\omega^\mu, & \theta^\mu_a &= -\sum_p F^\mu_{pa}\omega^p - 2\sum_\alpha F^\mu_{\alpha a}\omega^\alpha,\\
\theta^\mu_p &= \sum_a F^\mu_{pa}\omega^a + 2\sum_\alpha F^\mu_{\alpha p}\omega^\alpha, & \theta^\mu_\alpha &= \sum_a F^\mu_{\alpha a}\omega^a + \sum_p F^\mu_{\alpha p}\omega^p,
\end{aligned} \tag{4.18}$$

then comparison with (2.9), using (4.10) and (4.13), gives

$$\begin{aligned}
F^\alpha_{pa} &= -h^\alpha_{pa}\sin s_1, & F^\mu_{pa} &= -h^\mu_{pa}\sin s_1,\\
F^\mu_{\alpha a} &= -h^\mu_{\alpha a}\sin s_1, & F^\mu_{\alpha p} &= h^\mu_{\alpha p}\cos s_1.
\end{aligned} \tag{4.19}$$

Notice that the distribution obtained by setting any three sets of $\{\omega^a\}$, $\{\omega^p\}$, $\{\omega^\alpha\}$ and $\{\omega^\mu\}$ equal to zero is completely integrable and its integral submanifolds are the respective curvature surfaces.

Equations (2.3) become, for the Darboux frame field (4.16),

$$\begin{aligned}
d\mathbf{x} &= \omega^p e_p + \omega^\alpha e_\alpha + \omega^\mu e_\mu,\\
de_0 &= \omega^a e_a - \omega^\alpha e_\alpha + \omega^\mu e_\mu,\\
de_a &= -\omega^a e_0 + \theta^b_a e_b + \theta^q_a e_q + \theta^\alpha_a e_\alpha + \theta^\mu_a e_\mu,\\
de_p &= -\omega^p \mathbf{x} + \theta^b_p e_b + \theta^q_p e_q + \theta^\alpha_p e_\alpha + \theta^\mu_p e_\mu,\\
de_\alpha &= -\omega^\alpha \mathbf{x} + \omega^\alpha e_0 + \theta^a_\alpha e_a + \theta^q_\alpha e_q + \theta^\beta_\alpha e_\beta + \theta^\mu_\alpha e_\mu,\\
de_\mu &= -\omega^\mu \mathbf{x} - \omega^\mu e_0 + \theta^a_\mu e_a + \theta^q_\mu e_q + \theta^\alpha_\mu e_\alpha + \theta^\nu_\mu e_\nu.
\end{aligned} \tag{4.20}$$

The Cartan-Münzner polynomial $F : \mathbf{R}^{n+1} \to \mathbf{R}$ defining the isoparametric function $f = F|_{S^n} : S^n \to [-1, 1]$ has ± 1 as the only two singular values, and focal points at a distance $\pi/2$ along a normal geodesic from each other lie on the same focal submanifold. If our second order Darboux frame field (4.16) is along the focal submanifold

$$\mathbf{x} : U \subset M \to M_+ = f^{-1}\{1\} \subset S^n$$

then the tube (3.1) with $t = \pi/2$ shows that the image of $\bar{x} = e_0 : U \to M_+$ is the same focal submanifold. If we let $\bar{e}_0 = \mathbf{x}$, then by (4.15)

$$\begin{aligned} d\bar{\mathbf{x}} &= de_0 = \omega^a e_a - \omega^\alpha e_\alpha + \omega^\mu e_\mu, \\ d\bar{e}_0 &= d\mathbf{x} = \omega^p e_p + \omega^\alpha e_\alpha + \omega^\mu e_\mu, \end{aligned} \tag{4.21}$$

which shows that e_a, e_α, e_μ are tangent to M_+ at $\bar{\mathbf{x}} = e_0$, while $\bar{e}_0, e_p$ are normal to M_+ at $\bar{\mathbf{x}}$. The second fundamental form at $\bar{\mathbf{x}}$ with respect to $\bar{e}_0$ is

$$\overline{II}_{\bar{e}_0} = -d\bar{\mathbf{x}} \cdot d\bar{e}_0 = -de_0 \cdot d\mathbf{x} = II_{e_0} = \sum \omega^\alpha \omega^\alpha - \sum \omega^\mu \omega^\mu$$

which implies that V_+ is the $+1$ eigenspace and V_- is the -1 eigenspace of $\overline{II}_{\bar{e}_0}$ at $\bar{\mathbf{x}}$. Therefore, the principal curvature spaces of $\bar{e}_0$ at $\bar{\mathbf{x}}$ are

$$\bar{V}_+ = V_+, \quad \bar{V}_- = V_-, \quad \bar{V}_0 = \operatorname{span}\{e_a\}. \tag{4.22}$$

It follows that a second order Darboux frame field along $\bar{\mathbf{x}}$ on U is

$$\bar{\mathbf{x}} = e_0,\ \bar{e}_0 = \mathbf{x},\ \bar{e}_a = e_{a+m},\ \bar{e}_{a+m} = e_a,\ \bar{e}_\alpha = e_\alpha,\ \bar{e}_\mu = e_\mu. \tag{4.23}$$

From (4.21) we see that

$$\bar{\omega}^a = \omega^{a+m}, \quad \bar{\omega}^{a+m} = \omega^a, \quad \bar{\omega}^\alpha = -\omega^\alpha, \quad \bar{\omega}^\mu = \omega^\mu \tag{4.24}$$

is the coframe field dual to (4.23).

Of the forms in (4.18) for the frame field (4.23) and its coframe field (4.24), we consider

$$\begin{aligned} d\bar{e}_\alpha \cdot \bar{e}_\mu = \bar{\theta}^\mu_\alpha &= \bar{F}^\mu_{\alpha a}\bar{\omega}^a + \bar{F}^\mu_{\alpha\, a+m}\bar{\omega}^{a+m} \\ = de_\alpha \cdot e_\mu = \theta^\mu_\alpha &= F^\mu_{\alpha a}\omega^a + F^\mu_{\alpha\, a+m}\omega^{a+m} \end{aligned}$$

to conclude that

$$\bar{F}^\mu_{\alpha a} = F^\mu_{\alpha\, a+m}, \quad \bar{F}^\mu_{\alpha\, a+m} = F^\mu_{\alpha a}. \tag{4.25}$$

Therefore, if $v = \sum(x_\alpha e_\alpha + y_\mu e_\mu) \in V_+ \oplus V_-$, then

$$\bar{p}_a(v) = \sum_{\alpha,\mu} \bar{F}^\mu_{\alpha a} x_\alpha y_\mu = \sum_{\alpha,\mu} F^\mu_{\alpha\, a+m} x_\alpha y_\mu = p_{a+m}(v) \tag{4.26}$$

where the polynomials $\bar{p}_a$ and p_{a+m} are defined by these equations.

5. Consequences of the structure equations

We continue working with a second order frame field (4.16) along the focal submanifold $\mathbf{x}$ defined in (3.1) with $t = s_1$. Equations (4.19) show that differentiating equations (2.9) is equivalent to differentiating equations (4.18), which we now proceed to do. In preparation for this we first take the exterior differential of the coframe field (4.13) to obtain

$$
\begin{aligned}
d\omega^a &= -\theta^a_b \wedge \omega^b - F^\alpha_{pa}\omega^p \wedge \omega^\alpha - F^\mu_{pa}\omega^p \wedge \omega^\mu - 4F^\mu_{\alpha a}\omega^\alpha \wedge \omega^\mu,\\
d\omega^p &= -\theta^p_q \wedge \omega^q + F^\alpha_{pa}\omega^a \wedge \omega^\alpha + F^\mu_{pa}\omega^a \wedge \omega^\mu + 4F^\mu_{\alpha p}\omega^\alpha \wedge \omega^\mu,\\
d\omega^\alpha &= -\theta^\alpha_\beta \wedge \omega^\beta - F^\alpha_{pa}\omega^a \wedge \omega^p + F^\mu_{\alpha a}\omega^a \wedge \omega^\mu - F^\mu_{\alpha p}\omega^p \wedge \omega^\mu,\\
d\omega^\mu &= -\theta^\mu_\nu \wedge \omega^\nu - F^\mu_{pa}\omega^a \wedge \omega^p - F^\mu_{\alpha a}\omega^a \wedge \omega^\alpha + F^\mu_{\alpha p}\omega^p \wedge \omega^\alpha.
\end{aligned}
\tag{5.1}
$$

We define the *covariant derivatives* of the tensors F^α_{pa}, F^μ_{pa}, $F^\mu_{\alpha a}$ and $F^\mu_{\alpha p}$, respectively, to be the 1-forms

$$
\begin{aligned}
F^\alpha_{pai}\omega^i &= dF^\alpha_{pa} - F^\alpha_{qa}\theta^q_p - F^\alpha_{pb}\theta^b_a + F^\beta_{pa}\theta^\alpha_\beta,\\
F^\mu_{pai}\omega^i &= dF^\mu_{pa} - F^\mu_{qa}\theta^q_p - F^\mu_{pb}\theta^b_a + F^\nu_{pa}\theta^\mu_\nu,\\
F^\mu_{\alpha ai}\omega^i &= dF^\mu_{\alpha a} - F^\mu_{\beta a}\theta^\beta_\alpha - F^\mu_{\alpha b}\theta^b_a + F^\nu_{\alpha a}\theta^\mu_\nu,\\
F^\mu_{\alpha pi}\omega^i &= dF^\mu_{\alpha p} - F^\mu_{\beta p}\theta^\beta_\alpha - F^\mu_{\alpha q}\theta^q_p + F^\nu_{\alpha p}\theta^\mu_\nu.
\end{aligned}
\tag{5.2}
$$

Any other second order frame field along $\mathbf{x}$ is given in terms of (4.15) by

$$
\mathbf{x}, e_0, \hat{e}_a, \hat{e}_p, \hat{e}_\alpha, \hat{e}_\mu \tag{5.3}
$$

where

$$
\hat{e}_a = A^b_a e_b,\ \hat{e}_p = A^q_p e_q,\ \hat{e}_\alpha = A^\beta_\alpha e_\beta,\ \hat{e}_\mu = A^\nu_\mu e_\nu \tag{5.4}
$$

with $(A^b_a), (A^q_p) : U \to O(m)$ and $(A^\beta_\alpha), (A^\nu_\mu) : U \to O(N)$ smooth maps. If the coefficients with respect to this new frame field are denoted by the same letters covered by a hat, then the transformation rules are tensorial. For example,

$$
\hat{F}^\alpha_{pa} = A^\alpha_\beta F^\beta_{qb} A^q_p A^b_a,\ \hat{F}^\alpha_{pab} = A^\alpha_\beta F^\beta_{qcd} A^q_p A^c_a A^d_b \tag{5.5}
$$

and so forth. If we take the exterior differential of the equations (4.18) and use (5.1) and (5.2) together with the Maurer-Cartan structure equations (2.4) we obtain the following sets of equations (compare [25, I, p. 536 and II, p. 45]).

$$
\begin{aligned}
F^\alpha_{pa}F^\alpha_{qb} + F^\alpha_{pb}F^\alpha_{qa} - (F^\mu_{pa}F^\mu_{qb} + F^\mu_{pb}F^\mu_{qa}) &= 0,\\
F^\alpha_{pa}F^\beta_{pb} + F^\alpha_{pb}F^\beta_{pa} + 2(F^\mu_{\alpha a}F^\mu_{\beta b} + F^\mu_{\alpha b}F^\mu_{\beta a}) &= \delta_{\alpha\beta}\delta_{ab},\\
F^\alpha_{pa}F^\beta_{qa} + F^\alpha_{qa}F^\beta_{pa} + 2(F^\mu_{\alpha p}F^\mu_{\beta q} + F^\mu_{\alpha q}F^\mu_{\beta p}) &= \delta_{pq}\delta_{\alpha\beta},\\
F^\mu_{pa}F^\nu_{pb} + F^\mu_{pb}F^\nu_{pa} + 2(F^\mu_{\alpha a}F^\nu_{\alpha b} + F^\mu_{\alpha b}F^\nu_{\alpha a}) &= \delta_{ab}\delta_{\mu\nu},\\
F^\mu_{pa}F^\nu_{qa} + F^\mu_{qa}F^\nu_{pa} + 2(F^\mu_{\alpha p}F^\nu_{\alpha q} + F^\mu_{\alpha q}F^\nu_{\alpha p}) &= \delta_{pq}\delta_{\mu\nu},\\
F^\mu_{\alpha a}F^\nu_{\beta a} + F^\mu_{\beta a}F^\nu_{\alpha a} - (F^\mu_{\alpha p}.F^\nu_{\beta p} + F^\mu_{\beta p}F^\nu_{\alpha p}) &= 0.
\end{aligned}
\tag{5.6}
$$

$$
\begin{aligned}
F^\alpha_{pab} &= -F^\mu_{pa}F^\mu_{\alpha b} - 2F^\mu_{pb}F^\mu_{\alpha a},\\
F^\alpha_{paq} &= F^\mu_{pa}F^\mu_{\alpha q} + 2F^\mu_{\alpha p}F^\mu_{qa},\\
F^\alpha_{pa\beta} &= 2F^\mu_{\alpha p}F^\mu_{\beta a} - 2F^\mu_{\beta p}F^\mu_{\alpha a}.
\end{aligned}
\tag{5.7}
$$

$$\begin{aligned} F^\mu_{pab} &= F^\alpha_{pa}F^\mu_{\alpha b} + 2F^\alpha_{pb}F^\mu_{\alpha a},\\ F^\mu_{paq} &= -F^\alpha_{pa}F^\mu_{\alpha q} - 2F^\mu_{\alpha p}F^\alpha_{qa},\\ F^\mu_{pa\nu} &= 2F^\mu_{\alpha p}F^\nu_{\alpha a} - 2F^\mu_{\alpha a}F^\nu_{\alpha p}. \end{aligned} \tag{5.8}$$

$$\begin{aligned} F^\mu_{\alpha ab} &= -\frac{1}{2}F^\mu_{pa}F^\alpha_{pb} + \frac{1}{2}F^\mu_{pb}F^\alpha_{pa},\\ F^\mu_{\alpha a\beta} &= F^\mu_{\alpha p}F^\beta_{pa} + 2F^\mu_{\beta p}F^\alpha_{pa},\\ F^\mu_{\alpha a\nu} &= F^\mu_{\alpha p}F^\nu_{pa} + 2F^\mu_{pa}F^\nu_{\alpha p}. \end{aligned} \tag{5.9}$$

$$\begin{aligned} F^\mu_{\alpha pq} &= \frac{1}{2}F^\mu_{pa}F^\alpha_{qa} - \frac{1}{2}F^\mu_{qa}F^\alpha_{pa},\\ F^\mu_{\alpha p\beta} &= -F^\mu_{\alpha a}F^\beta_{pa} - 2F^\mu_{\beta a}F^\alpha_{pa},\\ F^\mu_{\alpha p\nu} &= -F^\mu_{\alpha a}F^\nu_{pa} - 2F^\mu_{pa}F^\nu_{\alpha a}. \end{aligned} \tag{5.10}$$

$$F^\alpha_{pa\mu} = -F^\mu_{pa\alpha} = -2F^\mu_{\alpha ap} = -2F^\mu_{\alpha pa}. \tag{5.11}$$

6. Second fundamental forms of a focal submanifold

Consider the focal submanifold $\mathbf{x}$ of (3.1) with $t = s_1$ with a second order frame field (4.16) along it on U. For each point of $\mathbf{x}$, Corollary 6 tells us the principal curvatures of the second fundamental forms II_{e_a} of $\mathbf{x}$. In order to derive the consequence of this knowledge, we begin by finding the expression of II_{e_a} of $\mathbf{x}$ in terms of the orthonormal coframe field $\omega^p, \omega^\alpha, \omega^\mu$ and from that obtain the matrices of the corresponding shape operators with respect to the orthonormal tangent frame field e_p, e_α, e_μ. For our frame, equations (2.3) have become, in part,

$$\begin{aligned} d\mathbf{x} &= \omega^p e_p + \omega^\alpha e_\alpha + \omega^\mu e_\mu,\\ de_a &= (k_1 e_0 - \mathbf{x})\theta^a + \theta^b_a e_b + \theta^p_a e_p + \theta^\alpha_a e_\alpha + \theta^\mu_a e_\mu. \end{aligned} \tag{6.1}$$

The shape operator S_a is the symmetric operator on the tangent space at $\mathbf{x}$ given by

$$II_{e_a} = -de_a \cdot d\mathbf{x} = d\mathbf{x} \circ S_a \cdot d\mathbf{x}. \tag{6.2}$$

That is, S_a is the tangential component of $-de_a$. Combining the second equation in (6.1) with (4.18), we find

$$S_a = (2F^\mu_{\alpha a}e_\mu - F^\alpha_{pa}e_p)\omega^\alpha + (2F^\mu_{\alpha a}e_\alpha + F^\mu_{pa}e_p)\omega^\mu + (-F^\alpha_{pa}e_\alpha + F^\mu_{pa}e_\mu)\omega^p.$$

Recall the curvature spaces V_0, V_+, V_- defined in (4.17). Define linear operators

$$\begin{aligned} A_a &= 2F^\mu_{\alpha a}e_\alpha\omega^\mu : V_- \to V_+,\\ B_a &= -F^\alpha_{pa}e_\alpha\omega^p : V_0 \to V_+,\\ C_a &= F^\mu_{pa}e_\mu\omega^p : V_0 \to V_-, \end{aligned} \tag{6.3}$$

and their transposes

$$
\begin{aligned}
{}^tA_a &= 2F^\mu_{\alpha a} e_\mu \omega^\alpha : V_+ \to V_-, \\
{}^tB_a &= -F^\alpha_{pa} e_p \omega^\alpha : V_+ \to V_0, \\
{}^tC_a &= F^\mu_{pa} e_p \omega^\mu : V_- \to V_0.
\end{aligned} \tag{6.4}
$$

With respect to the orthogonal direct sum decomposition $V_+ \oplus V_- \oplus V_0$ of the tangent space to $\mathbf{x}$ at the point, the operator S_a has the block form

$$
S_a = \begin{pmatrix} 0 & A_a & B_a \\ {}^tA_a & 0 & C_a \\ {}^tB_a & {}^tC_a & 0 \end{pmatrix}. \tag{6.5}
$$

Restriction of the second fundamental forms II_{e_0} and II_{e_a} to $V_+ \oplus V_-$ defines quadratic forms

$$
\begin{aligned}
p_0(x,y) &= II_{e_0}((x,y),(x,y)) = \sum_\alpha x_\alpha^2 - \sum_\mu y_\mu^2, \\
p_a(x,y) &= \frac{1}{4} II_{e_a}((x,y),(x,y)) = F^\mu_{\alpha a} x_\alpha y_\mu,
\end{aligned} \tag{6.6}
$$

where $x = x_\alpha e_\alpha \in V_+$ and $y = y_\mu e_\mu \in V_-$.

Note that by Corollary 6, the minimal polynomial of S_a is $x(x^2-1)$, and therefore

$$
S_a = S_a^3 \tag{6.7}
$$

for all a at every point of U.

Proposition 7. *If $m < N$, then the operators A_a in* (6.3) *must be linearly independent at every point of U.*

Proof. Suppose that the operators A_a are linearly dependent at a point $p \in U$. This means that there exists a unit vector $u = (u^a) \in \mathbf{R}^m$ such that

$$
u^a F^\mu_{\alpha a} = 0 \tag{6.8}
$$

for all μ and α, at the point p. Then multiplying the second equation in (5.6) by $u^a u^b$, summing on a and b and using (6.8) gives

$$
2F^\alpha_{pa} u^a F^\beta_{pb} u^b = \delta_{\alpha\beta}.
$$

Therefore,

$$
\{\sqrt{2} \sum_{a,p} F^\alpha_{pa} u^a e_p : \alpha = 2m+1, 2m+2, \ldots, 2m+N\}
$$

is an orthonormal set of N vectors in the m-dimensional subspace V_0 defined in (4.17), which contradicts the assumption that $m < N$. $\square$

We need a condition which is stronger than the linear independence of the A_a.

Definition 8 (Spanning Property). The focal submanifold $\mathbf{x}$ satisfies the *spanning property* at a point of M if

(a) There exists a vector $X = \sum_\alpha x_\alpha e_\alpha \in V_+$ such that the set of vectors $\{\sum_{\alpha,\mu} F^\mu_{\alpha a} x_\alpha e_\mu : a = 1, \dots, m\}$ in V_- are linearly independent; and

(b) There exists a vector $Y = \sum_\mu y_\mu e_\mu \in V_-$ such that the set of vectors $\{\sum_{\alpha,\mu} F^\mu_{\alpha a} y_\mu e_\alpha : a = 1, \dots, m\}$ in V_+ are linearly independent.

Remark 9. Condition (a) is equivalent to

(a′) There exists $X = \sum_\alpha x_\alpha e_\alpha \in V_+$ such that

$$\{\sum_{a,\alpha,\mu} F^\mu_{\alpha a} x_\alpha y_\mu e_a : Y = y_\mu e_\mu \in V_-\} = \operatorname{span}\{e_1, \dots, e_m\}$$

and (b) is equivalent to

(b′) There exists $Y = \sum_\mu y_\mu e_\mu \in V_-$ such that

$$\{\sum_{a,\alpha,\mu} F^\mu_{\alpha a} x_\alpha y_\mu e_a : X = x_\alpha e_\alpha \in V_+\} = \operatorname{span}\{e_1, \dots, e_m\}.$$

Remark 10. If $\mathbf{x}$ satisfies the spanning property at a point of M, then it satisfies it on some open neighborhood of the point by a standard argument on the rank of the $N \times m$ matrix $(F^\mu_{\alpha a} x_\alpha)$.

Let $\mathbf{x}, e_0, e_a, e_p, e_\alpha, e_\mu$ be a second order frame field (4.16) along $\mathbf{x}$ on U, where $\mathbf{x}(U) \subset M_+$ is a focal submanifold. Let the same letters with bars denote the second order frame field (4.23) along $\bar{\mathbf{x}} = e_0$ on U. At each point of U define bihomogeneous polynomials p_a and $\bar p_a$ in $\mathbf{R}[x_\alpha, y_\mu]$ by

$$p_a(x,y) = \sum_{\alpha,\mu} F^\mu_{\alpha a} x_\alpha y_\mu, \quad \bar p_a(x,y) = \sum_{\alpha,\mu} \bar F^\mu_{\alpha a} x_\alpha y_\mu \tag{6.9}$$

where $F^\mu_{\alpha a}$ and $\bar F^\mu_{\alpha a}$ are as defined in (4.18) for the respective frame fields.

Proposition 11. *If at each point of U there exist polynomials f_{ab} in the polynomial ring $\mathbf{R}[x_\alpha, y_\mu]$ such that*

$$\bar p_a = \sum_b f_{ab} p_b \tag{6.10}$$

and if the spanning property holds for $\mathbf{x}$ on U, then there exists a second order frame field $\mathbf{x}, e_0, \hat e_a, \hat e_p, \hat e_\alpha, \hat e_\mu$ along $\mathbf{x}$ on U with respect to which

$$\hat F^\mu_{\alpha\, a+m} = \hat F^\mu_{\alpha a} \tag{6.11}$$

for all a, α, μ, at each point of U.

Proof. If we let $p_{a+m}(x,y) = \sum_{\alpha,\mu} F^\mu_{\alpha\, a+m} x_\alpha y_\mu$, then by (4.26), $p_{a+m} = \bar{p}_a$ and therefore (6.10) implies that at each point of U

$$p_{a+m} = \sum_b f_{ab} p_b. \tag{6.12}$$

If we expand the right side of this equation in terms of the bihomogeneous components of the f_{ab} and collect all terms of the same bi-degrees, then all terms must cancel except those of bi-degree $(1,1)$, since p_{a+m} has bi-degree $(1,1)$. This results in an expression for p_{a+m} as a linear combination of the p_b with constant coefficients, since each p_b has bi-degree $(1,1)$. Hence, we may assume that the f_{ab} in (6.12) are constant polynomials. Now (6.12) implies that

$$F^\mu_{\alpha\, a+m} = \sum_b f_{ab} F^\mu_{\alpha b} \tag{6.13}$$

for all α, μ at each point of U. We claim that the functions $f_{ab} : U \to \mathbf{R}$ are smooth. In fact, if we let $A_{a+m} = 2\sum_{\alpha,\mu} F^\mu_{\alpha\, a+m} e_\alpha \omega^\mu : V_- \to V_+$ and let A_a be the operators defined in (6.3), then (6.13) implies that $A_{a+m} = \sum_b f_{ab} A_b$. The spanning property implies that the operators A_b are linearly independent in $\mathrm{End}(V_-, V_+)$, and therefore at each point of U an inner product can be defined on this space of endomorphisms, depending smoothly on the point of U, such that $\{A_b\}$ is an orthonormal set. Then $f_{ab} = \langle A_{a+m}, A_b\rangle : U \to \mathbf{R}$ is smooth.

Fix $\alpha = \alpha_0$ and for each μ define vectors in $\mathbf{R}^m$

$$W_\mu = {}^t(F^\mu_{\alpha_0\, 1}, \dots, F^\mu_{\alpha_0\, m}), \quad V_\mu = {}^t(F^\mu_{\alpha_0\, m+1}, \dots, F^\mu_{\alpha_0\, m+m}).$$

If we define the $m \times m$ matrix $B = (f_{ab})$, then by (6.13), we have

$$V_\mu = B W_\mu \tag{6.14}$$

for each μ. The sixth equation in (5.6) says that for any μ and ν

$$V_\mu \cdot V_\nu = W_\mu \cdot W_\nu.$$

Combining these equations, we have

$$W_\mu \cdot W_\nu = B W_\mu \cdot B W_\nu \tag{6.15}$$

for all μ, ν. It follows that B is orthogonal, provided that the set $\{W_\mu\}$ spans $\mathbf{R}^m$. By the spanning property, this is true for some choice of α_0, for some choice of frame field. Therefore, assuming we have made that choice, we have a smooth map

$$B = (f_{ab}) : U \to O(m).$$

Alter the second order frame field along $\mathbf{x}$ by

$$\hat{e}_{a+m} = \sum_b e_{b+m} f_{ba}$$

leaving the other vectors in the frame unchanged. If we let $\hat{F}^\mu_{\alpha a}$, etc. be the coefficients with respect to this new frame field, then by (5.5), we have $\hat{F}^\mu_{\alpha a} = F^\mu_{\alpha a}$ and, also using (6.13), we have

$$\hat{F}^\mu_{\alpha\, a+m} = \sum_b F^\mu_{\alpha\, b+m} f_{ba} = \sum_{b,c} f_{bc} F^\mu_{\alpha c} f_{ba} = \sum_c \hat{F}^\mu_{\alpha c}\delta_{ca} = \hat{F}^\mu_{\alpha a}$$

which proves (6.11). □

7. The Ferus-Karcher-Münzner construction

Let $P_0, P_1, \dots, P_m$ be a Clifford system on $\mathbf{R}^{2l}$. Recall that this means that these are symmetric operators on $\mathbf{R}^{2l}$ satisfying

(7.1) $$P_i P_j + P_j P_i = 2\delta_{ij} I, \quad i, j = 0, 1, \dots, m.$$

It follows that each operator P_i is also orthogonal. For this section we modify the index conventions (4.6) by

(7.2) $$i, j, k \in \{0, \dots, m\}$$

and now $N = l - m - 1$ and $n + 1 = 2l$. If $A = (A^j_i) \in \mathrm{SO}(m+1)$, and if we let

(7.3) $$Q_i = A^j_i P_j$$

then $Q_0, Q_1, \dots, Q_m$ is also a Clifford system on $\mathbf{R}^{2l}$. Since $Q_0^2 = I$, the eigenvalues of Q_0 must be ± 1. If $E_\pm$ are the eigenspaces of Q_0, then $\mathbf{R}^{2l} = E_+ \oplus E_-$ is an orthogonal direct sum and $E_\pm$ each has dimension l, because for any a, the operator Q_a interchanges E_+ and E_-.

Because $P_0, \dots, P_m$ are linearly independent,

(7.4) $$M_+ = \{\mathbf{x} \in S^{2l-1} \subset \mathbf{R}^{2l} : P_i\mathbf{x} \cdot \mathbf{x} = 0, \quad i = 0, \dots, m\}$$

is a submanifold of S^{2l-1} of codimension $m+1$. If $\mathbf{x} \in M_+$, then $Q_0\mathbf{x}, \dots, Q_m\mathbf{x}$ is an orthonormal set of unit normal vectors to M_+ in S^{2l-1}. Therefore, this is a global frame field for the normal bundle of M_+ and the unit normal bundle of M_+ is isomorphic to the trivial bundle

(7.5) $$M = M_+ \times S^m.$$

Consider the principal bundle

(7.6) $$\begin{aligned} \mathrm{SO}(m+1) &\to S^m \\ A &\mapsto A_0 \end{aligned}$$

where for any $A \in \mathrm{SO}(m+1)$ we let A_i denote the i^{th} column of A. For a section A of (7.6), denote its pull-back to S^m of the Maurer-Cartan form of $\mathrm{SO}(m+1)$ by

(7.7) $$A^{-1} dA = \tau = (\tau^i_j),$$

an $\mathfrak{o}(m+1)$-valued form on S^m. Then $dA_i = A_j\tau_i^j$, and thus, for the Clifford systems

$$Q_i = A_i^j P_j, \tag{7.8}$$

depending on $A \in \mathrm{SO}(m+1)$, we have

$$dQ_i = Q_j\tau_i^j \tag{7.9}$$

for each i. Observe that $\tau_0^1, \dots, \tau_0^m$ is a local coframe field in S^m. For each $(\mathbf{x}, A_0) \in M = M_+ \times S^m$, there is an orthogonal direct sum

$$\mathbf{R}^{2l} = \mathrm{span}\{\mathbf{x}\} \oplus M_+^\perp(\mathbf{x}) \oplus T_0(\mathbf{x}, A_0) \oplus T_+(\mathbf{x}, A_0) \oplus T_-(\mathbf{x}, A_0), \tag{7.10}$$

which is determined by the second fundamental form of M_+ (see Section 4.5 of [13, p. 488]). In Lemmas 12–14 below, we provide the details of the relationship between this decomposition and the second fundamental form of M_+. The subspaces of the decomposition are

$$\begin{aligned}
M_+^\perp(\mathbf{x}) &= \mathrm{span}\{Q_0\mathbf{x}, \dots, Q_m\mathbf{x}\} = \mathrm{span}\{P_0\mathbf{x}, \dots, P_m\mathbf{x}\},\\
T_0(\mathbf{x}, A_0) &= \mathrm{span}\{Q_aQ_0\mathbf{x} : \text{ for all } a\},\\
T_+(\mathbf{x}, A_0) &= E_- \cap T_\mathbf{x}M_+ = \{X \in E_- : X \cdot Q_i\mathbf{x} = 0 \text{ for all } i\}\\
&= \{X \in \mathbf{R}^{2l} : Q_0X = -X \text{ and } X \cdot P_i\mathbf{x} = 0, \text{ for all } i\},\\
T_-(\mathbf{x}, A_0) &= E_+ \cap T_\mathbf{x}M_+ = \{X \in E_+ : X \cdot Q_i\mathbf{x} = 0, \text{ for all } i\}\\
&= \{X \in \mathbf{R}^{2l} : Q_0X = X \text{ and } X \cdot P_i\mathbf{x} = 0, \text{for all } i\}.
\end{aligned} \tag{7.11}$$

Then $\dim M_+^\perp(\mathbf{x}) = m+1$, $\dim T_0(\mathbf{x}, A_0) = m$, $\dim T_+(\mathbf{x}, A_0) = N$ and $\dim T_-(\mathbf{x}, A_0) = N$, where $N = l - (m+1)$. Notice that

$$Q_0 : T_0(\mathbf{x}, A_0) \to M_+^\perp(\mathbf{x}) \tag{7.12}$$

because $Q_0Q_aQ_0\mathbf{x} = -Q_a\mathbf{x} \in M_+^\perp$, for any a.

For any point in $M = M_+ \times S^m$, there is an open neighborhood about it of the form $U \times V$, where $U \subset M_+$ and $V \subset S^m$, such that the section A of (7.6) is defined on V and such that there exist smooth orthonormal bases e_α of $T_+(\mathbf{x}, A_0)$ and e_μ of $T_-(\mathbf{x}, A_0)$ on $U \times V$. This means that at each point of $U \times V$

$$\begin{aligned}
Q_0e_\alpha &= -e_\alpha \text{ and } e_\alpha \cdot Q_i\mathbf{x} = 0,\\
Q_0e_\mu &= e_\mu \quad\ \text{ and } e_\mu \cdot Q_i\mathbf{x} = 0.
\end{aligned} \tag{7.13}$$

Compose $\mathbf{x} : M_+ \to S^{2l-1}$ with the projection $M = M_+ \times S^m \to M_+$ so that we may regard it as a mapping $\mathbf{x} : M \to S^{2l-1}$. Then

$$\mathbf{x},\ e_i = Q_i\mathbf{x},\ e_p = Q_{p-m}Q_0\mathbf{x},\ e_\alpha,\ e_\mu \tag{7.14}$$

is a Darboux frame field along $\mathbf{x}$ on $U \times V$, where the e_i are normal vectors and the rest are tangent to $\mathbf{x}$.

Lemma 12. *For any* $\mathbf{x} \in M_+$

$$Q_i Q_j Q_k \mathbf{x} \cdot \mathbf{x} = 0 \tag{7.15}$$

for all i, j, k *and*

$$L^a_{bc} = Q_a Q_b Q_c e_0 \cdot \mathbf{x} \tag{7.16}$$

is skew-symmetric in a, b, c.

Proof. If i, j, k are distinct, then

$$Q_i Q_j Q_k \mathbf{x} \cdot \mathbf{x} = \mathbf{x} \cdot Q_k Q_j Q_i \mathbf{x} = -\mathbf{x} \cdot Q_i Q_j Q_k \mathbf{x}$$

which implies (7.15). If the indices are not distinct, then the product is a single $\pm Q_i$ and $Q_i \mathbf{x} \cdot \mathbf{x} = 0$ by definition of M_+.

If any two of a, b, c are the same, then the product $Q_a Q_b Q_c$ is a single operator $\pm Q_a$, for some a, and we know that $Q_a e_0 \cdot \mathbf{x} = 0$. If a, b, c are distinct, then $Q_a Q_b Q_c$ changes sign if any two indices are switched. Therefore, L^a_{bc} is skew-symmetric in a, b, c. □

Lemma 13. *For the Darboux frame field* (7.14) *along* $\mathbf{x}$,

$$Q_i \mathbf{x} \cdot \mathbf{x} = 0, \text{ for all } i, \tag{7.17}$$

$$Q_i e_j \cdot e_k = 0, \text{ for all } i, j, k, \tag{7.18}$$

$$Q_i e_p \cdot e_q = 0, \text{ for all } i, p, q, \tag{7.19}$$

$$Q_a e_\alpha \cdot e_\beta = 0, \text{ for all } a, \alpha, \beta, \tag{7.20}$$

$$Q_a e_\mu \cdot e_\nu = 0, \text{ for all } a, \mu, \nu, \tag{7.21}$$

at each point of $U \times V$.

Proof. The first equation follows from the definition of M_+. For the second equation

$$Q_i e_j \cdot e_k = Q_i Q_j \mathbf{x} \cdot Q_k \mathbf{x} = Q_k Q_i Q_j \mathbf{x} \cdot \mathbf{x} = 0$$

by Lemma 12. For the third equation

$$Q_a e_p \cdot e_q = Q_a Q_{p-m} Q_0 \mathbf{x} \cdot Q_{q-m} Q_0 \mathbf{x} = -Q_{q-m} Q_a Q_{p-m} \mathbf{x} \cdot \mathbf{x} = 0$$

by Lemma 12 and

$$Q_0 e_p \cdot e_q = Q_0 Q_{p-m} Q_0 \mathbf{x} \cdot Q_{q-m} Q_0 \mathbf{x} = -\mathbf{x} \cdot Q_{p-m} Q_{q-m} Q_0 \mathbf{x} = 0$$

by Lemma 12. Equations 4 and 5 follow from the observation made above that Q_a interchanges E_- and E_+. □

Lemma 14. *For the frame field* (7.14),

$$d\mathbf{x} = \omega^p e_p + \omega^\alpha e_\alpha + \omega^\mu e_\mu, \tag{7.22}$$

$$de_0 = \omega^a e_a - \omega^\alpha e_\alpha + \omega^\mu e_\mu, \tag{7.23}$$

where $\omega^p, \omega^\alpha, \omega^\mu$ are linearly independent one-forms on U with coefficients being functions on $U \times V$, and

$$\omega^a = \tau_0^a - \omega^{a+m}. \tag{7.24}$$

A smooth coframe field on $U \times V$ is given by $\omega^a, \omega^p, \omega^\alpha, \omega^\mu$.

Proof. The expression (7.22) for $d\mathbf{x}$ follows from the fact that $\mathbf{x} : U \to \mathbf{R}^{2l}$ is an immersion and then $\omega^A = d\mathbf{x} \cdot e_A$, for $A = m+1, \ldots, 2l-1$. Combining this with (7.9), we have

$$\begin{aligned} de_0 &= dQ_0\,\mathbf{x} + Q_0 d\mathbf{x} \\ &= \tau_0^a Q_a \mathbf{x} + \omega^{a+m} Q_0 Q_a Q_0 \mathbf{x} + \omega^\alpha Q_0 e_\alpha + \omega^\mu Q_0 e_\mu \\ &= (\tau_0^a - \omega^{a+m}) e_a - \omega^\alpha e_\alpha + \omega^\mu e_\mu \end{aligned} \tag{7.25}$$

which proves (7.23). □

For $t \in \mathbf{R}$, the tube of radius t about M_+ is given by the immersion

$$\tilde{\mathbf{x}} : M \to S^{2l-1}, \quad \tilde{\mathbf{x}} = \cos t\,\mathbf{x} + \sin t\, e_0. \tag{7.26}$$

A unit normal vector field along $\tilde{\mathbf{x}}$ is

$$\tilde{e}_0 = -\sin t\,\mathbf{x} + \cos t\, e_0 \tag{7.27}$$

and a Darboux frame field along $\tilde{\mathbf{x}}$ is given by

$$\tilde{\mathbf{x}}, e_a, e_p, e_\alpha, e_\mu, \tilde{e}_0. \tag{7.28}$$

From (7.22) we compute

$$\begin{aligned} d\tilde{\mathbf{x}} &= \sin t\,\omega^a e_a + \cos t\,\omega^p e_p \\ &\qquad + (\cos t - \sin t)\omega^\alpha e_\alpha + (\cos t + \sin t)\omega^\mu e_\mu, \\ d\tilde{e}_0 &= \cos t\,\omega^a e_a - \sin t\,\omega^p e_p \\ &\qquad - (\cos t + \sin t)\omega^\alpha e_\alpha + (\cos t - \sin t)\omega^\mu e_\mu, \end{aligned} \tag{7.29}$$

which shows that

$$\begin{aligned} \theta^a &= \sin t\,\omega^a, & \theta^p &= \cos t\,\omega^p, \\ \theta^\alpha &= (\cos t - \sin t)\omega^\alpha, & \theta^\mu &= (\cos t + \sin t)\omega^\mu \end{aligned} \tag{7.30}$$

is an orthonormal coframe field in M for the metric $d\tilde{\mathbf{x}} \cdot d\tilde{\mathbf{x}}$ induced by $\tilde{\mathbf{x}}$. The second fundamental form of $\tilde{\mathbf{x}}$ is then

$$\begin{aligned} II_{\tilde{e}_0} &= -d\tilde{\mathbf{x}} \cdot d\tilde{e}_0 \\ &= -\cot t\,\theta^a\theta^a + \tan t\,\theta^p\theta^p + \frac{\cot t + 1}{\cot t - 1}\theta^\alpha\theta^\alpha - \frac{\cot t - 1}{\cot t + 1}\theta^\mu\theta^\mu \\ &= \cot(-t)\theta^a\theta^a + \cot(\frac{\pi}{2} - t)\theta^p\theta^p + \cot(\frac{\pi}{4} - t)\theta^\alpha\theta^\alpha + \cot(\frac{3\pi}{4} - t)\theta^\mu\theta^\mu \end{aligned}$$

from which we conclude that the principal curvatures are the constants $\cot(-t)$ and $\cot(\pi/2 - t)$, each with multiplicity m and the constants $\cot(\pi/4 - t)$ and $\cot(3\pi/4 - t)$, each with multiplicity N. In addition, the Darboux frame field (7.28) along $\tilde{\mathbf{x}}$ is of second order. Therefore, the $\tilde{\mathbf{x}}$ for $t \in \mathbf{R}$ is an isoparametric family of hypersurfaces in S^{2l-1} and $\mathbf{x}$ is a focal submanifold. This is the *Ferus-Karcher-Münzner construction*, (FKM construction) [13], of an isoparametric hypersurface from a given Clifford system.

We next calculate equations (4.18) for the FKM construction for a given Clifford system.

LEMMA 15. *For the Darboux frame field* (7.14) *along* $\mathbf{x}$, *the coefficients of the forms* $\theta^A_B = de_A \cdot e_B$ *in* (4.18) *are given by*

$$F^\alpha_{pa} = Q_{p-m}Q_a\mathbf{x}\cdot e_\alpha, \quad F^\mu_{pa} = Q_{p-m}Q_a\mathbf{x}\cdot e_\mu,$$
$$F^\mu_{\alpha a} = -\frac{1}{2}Q_a e_\mu \cdot e_\alpha, \quad F^\mu_{\alpha p} = -\frac{1}{2}Q_{p-m}e_\mu\cdot e_\alpha. \tag{7.31}$$

Proof. These coefficients are determined by θ^p_a, θ^α_a and θ^α_p. From (7.9) and (7.22) we have

$$\begin{aligned} de_a &= dQ_a\,\mathbf{x} + Q_a d\mathbf{x} \\ &= -\tau^a_0 e_0 + \tau^b_a e_b + \omega^{b+m}Q_a e_{b+m} + \omega^\alpha Q_a e_\alpha + \omega^\mu Q_a e_\mu \end{aligned} \tag{7.32}$$

and from (7.9) and (7.23) we have

$$\begin{aligned} de_{a+m} &= dQ_a\,e_0 + Q_a de_0 \\ &= -\tau^a_0\mathbf{x} + \tau^b_a e_{b+m} + \omega^b Q_a e_b - \omega^\alpha Q_a e_\alpha + \omega^\mu Q_a e_\mu. \end{aligned} \tag{7.33}$$

Using Lemma 13 and (4.18) we have

$$\begin{aligned} F^\alpha_{b+m\,a}\omega^\alpha - F^\mu_{b+m\,a}\omega^\mu &= \theta^{b+m}_a = de_a\cdot e_{b+m} \\ &= \omega^\alpha Q_a e_\alpha\cdot e_{b+m} + \omega^\mu Q_a e_\mu \cdot e_{b+m}, \end{aligned} \tag{7.34}$$

which implies that

$$\begin{aligned} F^\alpha_{b+m\,a} &= Q_a e_\alpha \cdot e_{b+m} = Q_a e_\alpha \cdot Q_b Q_0\mathbf{x} \\ &= Q_0 e_\alpha \cdot Q_a Q_b\mathbf{x} = -e_\alpha\cdot Q_aQ_b\mathbf{x} = Q_bQ_a\mathbf{x}\cdot e_\alpha \end{aligned}$$

which is the first formula in (7.31), and similarly,

$$-F^\mu_{b+m\,a} = Q_a e_\mu \cdot Q_bQ_0\mathbf{x} = Q_a e_\mu\cdot Q_b\mathbf{x} = e_\mu\cdot Q_aQ_b\mathbf{x}$$

which gives the second formula in (7.31). In the same way,

$$\begin{aligned} F^\alpha_{b+m\,a}\omega^{b+m} - 2F^\mu_{\alpha a}\omega^\mu &= \theta^\alpha_a = de_a\cdot e_\alpha \\ &= \omega^{b+m}Q_a e_{b+m}\cdot e_\alpha + \omega^\mu Q_a e_\mu \cdot e_\alpha \end{aligned} \tag{7.35}$$

which implies that $-2F^\mu_{\alpha a} = Q_a e_\mu \cdot e_\alpha$, which is the third formula in (7.31). Next,

$$\begin{aligned} F^\alpha_{a+m\,b}\omega^b - 2F^\mu_{\alpha a+m}\omega^\mu &= \theta^\alpha_{a+m} = de_{a+m}\cdot e_\alpha \\ &= \omega^b Q_a e_b \cdot e_\alpha + \omega^\mu Q_a e_\mu \cdot e_\alpha \end{aligned} \tag{7.36}$$

which implies that $-2F^\mu_{\alpha a+m} = Q_a e_\mu \cdot e_\alpha$, which is the fourth formula in (7.31). □

Corollary 16. *With respect to a Darboux frame* (7.14) *along an* FKM *construction* $\mathbf{x} : M \to S^{2l-1}$, *the coefficients* (7.31) *satisfy the equations*

$$F^\mu_{\alpha\,a+m} = F^\mu_{\alpha a}, \quad F^\alpha_{a+m\,b} = -F^\alpha_{b+m\,a}, \quad F^\mu_{a+m\,b} = -F^\mu_{b+m\,a}. \tag{7.37}$$

Proof. From (7.31),

$$\begin{aligned} F^\mu_{\alpha\,a+m} &= -\frac{1}{2}Q_a e_\mu \cdot e_\alpha = F^\mu_{\alpha a}, \\ F^\alpha_{a+m\,b} + F^\alpha_{b+m\,a} &= (Q_aQ_b + Q_bQ_a)\mathbf{x}\cdot e_\alpha = 0, \\ F^\mu_{a+m\,b} + F^\mu_{b+m\,a} &= (Q_aQ_b + Q_bQ_a)\mathbf{x}\cdot e_\mu = 0. \end{aligned} \tag{7.38}$$

□

Proposition 17. *For the Darboux frame field* (7.14), *at any point of* $U \times V \subset M$, *the operators* Q_0, Q_a *are given by*

$$\begin{aligned} Q_0\mathbf{x} = e_0, \qquad Q_0 e_0 = \mathbf{x}, \qquad Q_0 e_a = -e_{a+m}, \\ Q_0 e_{a+m} = -e_a, \quad Q_0 e_\alpha = -e_\alpha, \quad Q_0 e_\mu = e_\mu, \end{aligned} \tag{7.39}$$

and for each a

$$\begin{aligned} Q_a\mathbf{x} &= e_a, \\ Q_a e_0 &= e_{a+m}, \\ Q_a e_b &= \delta_{ab}\mathbf{x} - L^c_{ab}e_{c+m} + F^\alpha_{a+m\,b}e_\alpha + F^\mu_{a+m\,b}e_\mu, \\ Q_a e_{b+m} &= \delta_{ab}e_0 + L^c_{ab}e_c + F^\alpha_{b+m\,a}e_\alpha - F^\mu_{b+m\,a}e_\mu, \\ Q_a e_\alpha &= F^\alpha_{a+m\,b}e_b + F^\alpha_{b+m\,a}e_{b+m} - 2F^\mu_{\alpha\,a}e_\mu, \\ Q_a e_\mu &= F^\mu_{a+m\,b}e_b - F^\mu_{b+m\,a}e_{b+m} - 2F^\mu_{\alpha\,a}e_\alpha, \end{aligned} \tag{7.40}$$

where the coefficients are as defined in (7.16) *and* (7.31).

Proof. The expansion (7.39) of Q_0 can be verified by inspection. Also easy are the calculations $Q_a\mathbf{x} = e_a$ and $Q_a e_0 = Q_aQ_0\mathbf{x} = e_{a+m}$. To calculate Q_a on the remaining basis vectors, we use the fact that the basis is orthonormal. In the following calculations we use (7.1), (7.15), (7.14), (7.16) and (7.31).

$$\begin{aligned} Q_a e_b \cdot \mathbf{x} &= Q_aQ_b\mathbf{x}\cdot\mathbf{x} = \delta_{ab}, \\ Q_a e_b \cdot e_0 &= Q_aQ_b\mathbf{x}\cdot Q_0\mathbf{x} = Q_0Q_aQ_b\mathbf{x}\cdot\mathbf{x} = 0, \\ Q_a e_b \cdot e_c &= Q_aQ_b\mathbf{x}\cdot Q_c\mathbf{x} = Q_cQ_aQ_b\mathbf{x}\cdot\mathbf{x} = 0, \end{aligned}$$

$$
\begin{aligned}
Q_a e_b \cdot e_{c+m} &= Q_a Q_b \mathbf{x} \cdot Q_c Q_0 \mathbf{x} = Q_b Q_a Q_c Q_0 \mathbf{x} \cdot \mathbf{x} = L^b_{ac} = -L^c_{ab},\\
Q_a e_b \cdot e_\alpha &= Q_a Q_b \mathbf{x} \cdot e_\alpha = F^\alpha_{a+m\,b},\\
Q_a e_b \cdot e_\mu &= Q_a Q_b \mathbf{x} \cdot e_\mu = F^\mu_{a+m\,b}
\end{aligned}
$$

give the expansion of $Q_a e_b$.

$$
\begin{aligned}
Q_a e_{b+m} \cdot \mathbf{x} &= Q_a Q_b Q_0 \mathbf{x} \cdot \mathbf{x} = 0,\\
Q_a e_{b+m} \cdot e_0 &= Q_a Q_b Q_0 \mathbf{x} \cdot Q_0 \mathbf{x} = \delta_{ab},\\
Q_a e_{b+m} \cdot e_c &= Q_a Q_b Q_0 \mathbf{x} \cdot Q_c \mathbf{x} = Q_c Q_a Q_b Q_0 \mathbf{x} \cdot \mathbf{x} = L^c_{ab},\\
Q_a e_{b+m} \cdot e_{c+m} &= Q_a Q_b Q_0 \mathbf{x} \cdot Q_c Q_0 \mathbf{x} = -Q_c Q_a Q_b \mathbf{x} \cdot \mathbf{x} = 0,\\
Q_a e_{b+m} \cdot e_\alpha &= Q_a Q_b Q_0 \mathbf{x} \cdot e_\alpha = -Q_a Q_b \mathbf{x} \cdot e_\alpha = F^\alpha_{b+m\,a},\\
Q_a e_{b+m} \cdot e_\mu &= Q_a Q_b Q_0 \mathbf{x} \cdot e_\mu = Q_a Q_b \mathbf{x} \cdot e_\mu = -F^\mu_{b+m\,a}
\end{aligned}
$$

give the expansion of $Q_a e_{b+m}$. Using also (7.13), we find

$$
\begin{aligned}
Q_a e_\alpha \cdot \mathbf{x} &= e_\alpha \cdot Q_a \mathbf{x} = 0,\\
Q_a e_\alpha \cdot e_0 &= Q_a e_\alpha \cdot Q_0 \mathbf{x} = Q_a e_\alpha \cdot \mathbf{x} = 0,\\
Q_a e_\alpha \cdot e_b &= Q_a e_\alpha \cdot Q_b \mathbf{x} = e_\alpha \cdot Q_a Q_b \mathbf{x} = F^\alpha_{a+m\,b},\\
Q_a e_\alpha \cdot e_\beta &= 0,\\
Q_a e_\alpha \cdot e_\mu &= Q_a e_\alpha \cdot e_\mu = -2F^\mu_{\alpha a}
\end{aligned}
$$

give the expansion of $Q_a e_\alpha$.

$$
\begin{aligned}
Q_a e_\mu \cdot \mathbf{x} &= e_\mu \cdot Q_a \mathbf{x} = 0,\\
Q_a e_\mu \cdot e_0 &= Q_a e_\mu \cdot Q_0 \mathbf{x} = -Q_a e_\mu \cdot \mathbf{x} = 0,\\
Q_a e_\mu \cdot e_b &= Q_a e_\mu \cdot Q_b \mathbf{x} = e_\mu \cdot Q_a Q_b \mathbf{x} = F^\mu_{a+m\,b},\\
Q_a e_\mu \cdot e_{b+m} &= Q_a e_\mu \cdot Q_b Q_0 \mathbf{x} = e_\mu \cdot Q_a Q_b \mathbf{x} = -F^\mu_{b+m\,a},\\
Q_a e_\mu \cdot e_\alpha &= -2F^\mu_{\alpha a},\\
Q_a e_\mu \cdot e_\nu &= 0
\end{aligned}
$$

give the expansion of $Q_a e_\mu$. □

LEMMA 18. *For the Darboux frame field* (7.14) *along* $\mathbf{x}$,

$$
\begin{aligned}
\theta^b_a &= \tau^b_a + L^b_{ac}\omega^{c+m} + F^\alpha_{a+m\,b}\omega^\alpha + F^\mu_{a+m\,b}\omega^\mu,\\
\theta^{b+m}_{a+m} &= \tau^b_a + L^c_{ab}\omega^c + F^\alpha_{a+m\,b}\omega^\alpha + F^\mu_{a+m\,b}\omega^\mu
\end{aligned}
\tag{7.41}
$$

and therefore

$$
\theta^b_a - \theta^{b+m}_{a+m} = L^b_{ac}(\omega^c + \omega^{c+m}).
\tag{7.42}
$$

Proof. Using (7.9) and (7.22), we find

$$
\begin{aligned}
\theta^b_a &= de_a \cdot e_b = d(Q_a \mathbf{x}) \cdot e_b\\
&= (\tau^i_a Q_i \mathbf{x} + \omega^p Q_a e_p + \omega^\alpha Q_a e_\alpha + \omega^\mu Q_a e_\mu) \cdot e_b\\
&= \tau^b_a + \omega^{c+m} Q_a e_{c+m} \cdot e_b + \omega^\alpha Q_a e_\alpha \cdot e_b + \omega^\mu Q_a e_\mu \cdot e_b
\end{aligned}
\tag{7.43}
$$

which combined with (7.40) gives the first formula in (7.41). The second formula is derived in the same way. □

8. Necessary conditions to be FKM

Let $\tilde{\mathbf{x}}, e_a, e_p, e_\alpha, e_\mu, \tilde{e}_0$ be a second order frame field (2.2) in $U \subset M$ along an isoparametric hypersurface $\tilde{\mathbf{x}} : M \to S^n$. We continue using the index conventions in (4.6). Let $\mathbf{x} = \cos s_1 \, \tilde{\mathbf{x}} + \sin s_1 \, \tilde{e}_0$ be a focal submanifold and let $e_0 = -\sin s_1 \, \tilde{\mathbf{x}} + \cos s_1 \, e_0$ so that $\mathbf{x}, e_0, e_a, e_p, e_\alpha, e_\mu$ is a Darboux frame field (4.16) along $\mathbf{x}$ on U. Let $\omega^a, \omega^p, \omega^\alpha, \omega^\mu$ be its coframe field (4.13) on U. We look for conditions on this Darboux frame field which imply that $\mathbf{x}$ comes from an FKM construction.

Proposition 19. *Suppose that* $\mathbf{x}$ *satisfies the spanning property* (*Def.* 8) *on* U. *If*

(8.1)
$$F^\mu_{\alpha\, a+m} = F^\mu_{\alpha a}$$

on U, *then*

(8.2) $F^\alpha_{b+m\, a} + F^\alpha_{a+m\, b} = 0$

(8.3) $F^\mu_{b+m\, a} + F^\mu_{a+m\, b} = 0$

(8.4) $\theta^a_b - \theta^{a+m}_{b+m} = L^a_{bc}(\omega^c + \omega^{c+m}),$ *where* $L^a_{bc} = -L^b_{ac} = -L^a_{cb}$

on U.

Remark 20. By Corollary 16 and Lemma 18, equations (8.1)–(8.4) hold for the Darboux frame field (7.14) defined along an FKM $\mathbf{x}$.

Proof. The summation convention is not used in this proof. If we subtract the fourth equation in (5.2), with $p = a + m$, from the third equation in (5.2), we obtain

(8.5)
$$\sum_i (F^\mu_{\alpha a i} - F^\mu_{\alpha\, a+m\, i})\omega^i = \sum_b F^\mu_{\alpha b}(\theta^{b+m}_{a+m} - \theta^b_a).$$

Putting (8.1) into the second equation of (5.9) gives

(8.6)
$$F^\mu_{\alpha a \beta} = \sum_b (F^\mu_{\alpha b} F^\beta_{b+m\, a} + 2F^\mu_{\beta b} F^\alpha_{b+m\, a})$$

and putting (8.1) into the second equation of (5.10) gives

(8.7)
$$F^\mu_{\alpha\, a+m\, \beta} = -\sum_b (F^\mu_{\alpha b} F^\beta_{a+m\, b} + 2F^\mu_{\beta b} F^\alpha_{a+m\, b}).$$

Subtracting (8.7) from (8.6) we get

(8.8)
$$F^\mu_{\alpha a \beta} - F^\mu_{\alpha\, a+m\, \beta} = \sum_b \left(F^\mu_{\alpha b}(F^\beta_{b+m\, a} + F^\beta_{a+m\, b}) + 2F^\mu_{\beta b}(F^\alpha_{b+m\, a} + F^\alpha_{a+m\, b}) \right).$$

Likewise, using the third equation in (5.9) and in (5.10), gives

(8.9)
$$F^\mu_{\alpha a\nu} - F^\mu_{\alpha\, a+m\, \nu} = \sum_b \left(F^\mu_{\alpha b}(F^\nu_{b+m\, a} + F^\nu_{a+m\, b}) + 2F^\nu_{\alpha b}(F^\mu_{b+m\, a} + F^\mu_{a+m\, b})\right).$$

Expressing $\theta^a_b - \theta^{a+m}_{b+m}$ in terms of our coframe field, we have

$$\theta^a_b - \theta^{a+m}_{b+m} = \sum_c (L^a_{bc}\omega^c + L^a_{b\, c+m}\omega^{c+m}) + \sum_\alpha L^a_{b\alpha}\omega^\alpha + \sum_\mu L^a_{b\mu}\omega^\mu \tag{8.10}$$

where the coefficients are smooth functions on U, each skew-symmetric in a, b.

By the spanning property, as expressed in (a′) of Remark 9, we may assume the basis of V_+ chosen so that for some α, the set of vectors

$$\{\sum_a F^\mu_{\alpha a} e_a : \text{ all } \mu\}$$

spans V_0. Fix this choice of α. Substitute (8.10) into (8.5) and compare the coefficients of ω^α on each side to obtain

$$F^\mu_{\alpha a\alpha} - F^\mu_{\alpha\, a+m\, \alpha} = \sum_b F^\mu_{\alpha b} L^a_{b\alpha}. \tag{8.11}$$

Compare this to (8.8), in which we set $\beta = \alpha$, to obtain

$$\sum_b F^\mu_{\alpha b}\left(3(F^\alpha_{b+m\, a} + F^\alpha_{a+m\, b}) - L^a_{b\alpha}\right) = 0 \tag{8.12}$$

for all a and μ. By the spanning property, then, the vectors

$$\sum_b (3(F^\alpha_{b+m\, a} + F^\alpha_{a+m\, b}) - L^a_{b\alpha}) e_b$$

for each a and μ, are orthogonal to every vector in V_0. Therefore,

$$3(F^\alpha_{b+m\, a} + F^\alpha_{a+m\, b}) = L^a_{b\alpha}. \tag{8.13}$$

The left side of this equation is symmetric in a, b, while the right side is skew-symmetric in a, b. Therefore, for our choice of α, (8.2) holds and

$$L^a_{b\alpha} = 0 \tag{8.14}$$

for all a, b. Now, (8.8) becomes, for our choice of α and for any β,

$$F^\mu_{\alpha a\beta} - F^\mu_{\alpha\, a+m\, \beta} = \sum_b F^\mu_{\alpha b}(F^\beta_{b+m\, a} + F^\beta_{a+m\, b}). \tag{8.15}$$

Substitute (8.10) into (8.5) and compare the coefficient of ω^β with (8.15) to obtain

$$\sum_b F^\mu_{\alpha b}(F^\beta_{b+m\, a} + F^\beta_{a+m\, b} - L^a_{b\beta}) = 0$$

for all a, β, and μ. Again, the spanning property then implies that

$$F^{\beta}_{b+m\,a} + F^{\beta}_{a+m\,b} = L^{a}_{b\beta}$$

for all a, b, and β. Hence, as before, each side of this equation must be zero. Therefore, (8.2) and (8.14) hold for all a, b, and α.

We can prove (8.3) and

$$L^{a}_{b\mu} = 0 \tag{8.16}$$

for all a, b and μ in a similar way, by first fixing an appropriate μ and comparing coefficients of ω^{μ} in (8.5) after substitution of (8.10) into it. In this case (b′) of the spanning property is used.

With (8.2) and (8.3) now true, we see that (8.8) and (8.9) become

$$F^{\mu}_{\alpha a\beta} = F^{\mu}_{\alpha\, a+m\,\beta}, \quad F^{\mu}_{\alpha a\nu} = F^{\mu}_{\alpha\, a+m\,\nu} \tag{8.17}$$

and (8.14) and (8.16) substituted into (8.10) give

$$\theta^{a}_{b} - \theta^{a+m}_{b+m} = \sum_{c}(L^{a}_{bc}\omega^{c} + L^{a}_{b\,c+m}\omega^{c+m}). \tag{8.18}$$

Substitute this into (8.5) and compare coefficients of ω^{c} and ω^{c+m} to get

$$\begin{aligned} \sum_{b} F^{\mu}_{\alpha b}L^{a}_{bc} &= F^{\mu}_{\alpha ac} - F^{\mu}_{\alpha\, a+m\, c}, \\ \sum_{b} F^{\mu}_{\alpha b}L^{a}_{b\,c+m} &= F^{\mu}_{\alpha a\, c+m} - F^{\mu}_{\alpha\, a+m\, c+m}. \end{aligned} \tag{8.19}$$

Subtracting gives

$$\sum_{c} F^{\mu}_{\alpha c}(L^{a}_{cb} - L^{a}_{c\,b+m}) = F^{\mu}_{\alpha ab} - F^{\mu}_{\alpha\, a+m\, b} - F^{\mu}_{\alpha a\, b+m} + F^{\mu}_{\alpha\, a+m\, b+m}. \tag{8.20}$$

We want to show now that the right hand side of this equation is zero on U. To that end, we begin with the first equation in (5.9), which says

$$F^{\mu}_{\alpha ab} = -\frac{1}{2}\sum_{c} F^{\mu}_{c+m\,a}F^{\alpha}_{c+m\,b} + \frac{1}{2}\sum_{c} F^{\mu}_{c+m\,b}F^{\alpha}_{c+m\,a}. \tag{8.21}$$

Also, (5.11) says

$$F^{\mu}_{\alpha\, a+m\, b} = \frac{1}{2}F^{\mu}_{a+m\,b\alpha}, \quad F^{\mu}_{\alpha a\, b+m} = \frac{1}{2}F^{\mu}_{b+m\,a\alpha} \tag{8.22}$$

and the first equation in (5.10) states

$$F^{\mu}_{\alpha\, a+m\, b+m} = \frac{1}{2}\sum_{c} F^{\mu}_{a+m\,c}F^{\alpha}_{b+m\,c} - \frac{1}{2}\sum_{c} F^{\mu}_{b+m\,c}F^{\alpha}_{a+m\,c}. \tag{8.23}$$

Hence, by (8.2) and (8.3), the right hand side of (8.20) is

$$\begin{aligned} &F^{\mu}_{\alpha ab} - F^{\mu}_{\alpha\, a+m\, b} - F^{\mu}_{\alpha a\, b+m} + F^{\mu}_{\alpha\, a+m\, b+m} \\ &\quad = -\frac{1}{2}\sum_{c} F^{\mu}_{c+m\,a}F^{\alpha}_{c+m\,b} + \frac{1}{2}\sum_{c} F^{\mu}_{c+m\,b}F^{\alpha}_{c+m\,a} - \frac{1}{2}F^{\mu}_{a+m\,b\alpha} - \frac{1}{2}F^{\mu}_{b+m\,a\alpha} \end{aligned}$$

$$+\frac{1}{2}\sum_c F^{\mu}_{a+m\,c}F^{\alpha}_{b+m\,c} - \frac{1}{2}\sum_c F^{\mu}_{b+m\,c}F^{\alpha}_{a+m\,c} = -\frac{1}{2}(F^{\mu}_{a+m\,b\alpha} + F^{\mu}_{b+m\,a\alpha})$$

$$+\frac{1}{2}\sum_c F^{\mu}_{a+m\,c}(F^{\alpha}_{c+m\,b} + F^{\alpha}_{b+m\,c}) + \frac{1}{2}\sum_c F^{\mu}_{c+m\,b}(F^{\alpha}_{c+m\,a} + F^{\alpha}_{a+m\,c})$$

$$= -\frac{1}{2}(F^{\mu}_{a+m\,b\alpha} + F^{\mu}_{b+m\,a\alpha})$$

and so we want to show that this last term is zero on U when (8.1), (8.2) and (8.3) hold. By the second equation in (5.2),

$$\sum_i F^{\mu}_{a+m\,bi}\omega^i = dF^{\mu}_{a+m\,b} - \sum_c F^{\mu}_{c+m\,b}\theta^{c+m}_{a+m} - \sum_c F^{\mu}_{a+m\,c}\theta^c_b + \sum_\nu F^{\nu}_{a+m\,b}\theta^{\mu}_{\nu}$$

and

$$\sum_i F^{\mu}_{b+m\,ai}\omega^i = dF^{\mu}_{b+m\,a} - \sum_a F^{\mu}_{c+m\,a}\theta^{c+m}_{b+m} - \sum_c F^{\mu}_{b+m\,c}\theta^c_a + \sum_\nu F^{\nu}_{b+m\,a}\theta^{\mu}_{\nu}.$$

Sum these two equations and use (8.2) and (8.3) to get

$$\sum_i (F^{\mu}_{a+m\,bi} + F^{\mu}_{b+m\,ai})\omega^i = \sum_c \left(F^{\mu}_{b+m\,c}(\theta^{c+m}_{a+m} - \theta^c_a) + F^{\mu}_{a+m\,c}(\theta^{c+m}_{b+m} - \theta^c_b)\right).$$

By (8.18), the right hand side of this equation is in the span of the set of 1-forms $\{\omega^c, \omega^{c+m} : c = 1, \ldots, m\}$, and therefore the coefficients of ω^α and ω^μ on the left hand side must vanish, to give

$$F^{\mu}_{a+m\,b\alpha} + F^{\mu}_{b+m\,a\alpha} = 0, \quad F^{\mu}_{a+m\,b\nu} + F^{\mu}_{b+m\,a\nu} = 0 \tag{8.24}$$

and we have finally proved that the right hand side of (8.20) is zero on U, and therefore

$$\sum_b F^{\mu}_{\alpha b}(L^a_{bc} - L^a_{b\,c+m}) = 0 \tag{8.25}$$

on U, for all a, c, α, and μ. Multiplying this equation by the $X = \sum x_\alpha e_\alpha$ of (a) of the spanning property, we conclude that

$$L^a_{bc} - L^a_{b\,c+m} = 0 \tag{8.26}$$

on U for all a, b, c. Substitution of this into (8.18) gives

$$\theta^a_b - \theta^{a+m}_{b+m} = \sum_c L^a_{bc}(\omega^c + \omega^{c+m}). \tag{8.27}$$

To complete the proof of (8.4), it remains to show that

$$L^a_{bc} + L^a_{cb} = 0 \tag{8.28}$$

on U, for all a, b, c. By (5.2), (8.1) and (8.27), and the known skew-symmetry $L^a_{bc} = -L^b_{ac}$,

$$\sum_i F^\mu_{\alpha\, a+m\, i}\omega^i = \sum_i F^\mu_{\alpha a i}\omega^i + \sum_{b,c} F^\mu_{\alpha b} L^b_{ac}(\omega^c + \omega^{c+m}). \tag{8.29}$$

Comparing the coefficients of ω^c, we have

$$F^\mu_{\alpha\, a+m\, c} = F^\mu_{\alpha ac} + \sum_b F^\mu_{\alpha b} L^b_{ac}. \tag{8.30}$$

Interchanging a and c and then summing, we have

$$F^\mu_{\alpha\, a+m\, c} + F^\mu_{\alpha\, c+m\, a} = F^\mu_{\alpha ac} + F^\mu_{\alpha ca} + \sum_b F^\mu_{\alpha b}(L^b_{ac} + L^b_{ca}). \tag{8.31}$$

By the first equation in (5.9),

$$F^\mu_{\alpha ac} + F^\mu_{\alpha ca} = 0. \tag{8.32}$$

Hence

$$F^\mu_{\alpha\, a+m\, c} + F^\mu_{\alpha\, c+m\, a} = \sum_b F^\mu_{\alpha b}(L^b_{ac} + L^b_{ca}) \tag{8.33}$$

on U for all α and μ. In (8.29) compare the coefficients of ω^{c+m} to get

$$F^\mu_{\alpha\, a+m\, c+m} = F^\mu_{\alpha a\, c+m} + \sum_b F^\mu_{\alpha b} L^b_{ac}.$$

Interchange a and c and sum, to get

$$F^\mu_{\alpha\, a+m\, c+m} + F^\mu_{\alpha\, c+m\, a+m} = F^\mu_{\alpha a\, c+m} + F^\mu_{\alpha c\, a+m} + \sum_b F^\mu_{\alpha b}(L^b_{ac} + L^b_{ca}). \tag{8.34}$$

By the first equation in (5.10),

$$F^\mu_{\alpha\, a+m\, c+m} + F^\mu_{\alpha\, c+m\, a+m} = 0$$

and the last equation in (5.11) says that

$$F^\mu_{\alpha a\, c+m} = F^\mu_{\alpha\, c+m\, a} \text{ and } F^\mu_{\alpha c\, a+m} = F^\mu_{\alpha\, a+m\, c}.$$

Therefore, (8.34) is

$$F^\mu_{\alpha\, c+m\, a} + F^\mu_{\alpha\, a+m\, c} = -\sum_b F^\mu_{\alpha b}(L^b_{ac} + L^b_{ca}). \tag{8.35}$$

Combining this with (8.33), we conclude that

$$\sum_b F^\mu_{\alpha b}(L^b_{ac} + L^b_{ca}) = 0 \tag{8.36}$$

for all a, c, α, μ. The spanning property then implies (8.28). □

Resume use of the summation convention.

PROPOSITION 21. *If equations* (8.1) *through* (8.4) *hold on* U, *then*

$$F^{\alpha}_{c+m\,a}L^{c}_{bd} + F^{\alpha}_{c+m\,b}L^{c}_{ad} = 2(F^{\mu}_{\alpha a}F^{\mu}_{d+m\,b} + F^{\mu}_{\alpha b}F^{\mu}_{d+m\,a}), \tag{8.37}$$

$$F^{\mu}_{c+m\,a}L^{c}_{bd} + F^{\mu}_{c+m\,b}L^{c}_{ad} = 2(F^{\alpha}_{b+m\,d}F^{\mu}_{\alpha a} + F^{\alpha}_{a+m\,d}F^{\mu}_{\alpha b}), \tag{8.38}$$

$$F^{\mu}_{\alpha\,b+m\,a} = L^{c}_{ba}F^{\mu}_{\alpha c} - \frac{1}{2}F^{\mu}_{d+m\,b}F^{\alpha}_{d+m\,a} + \frac{1}{2}F^{\mu}_{d+m\,a}F^{\alpha}_{d+m\,b}. \tag{8.39}$$

Proof. These identities come from differentiating (8.1) through (8.3). Using our definition of covariant derivative in (5.2), we have

$$\begin{aligned} dF^{\alpha}_{b+m\,a} + F^{\beta}_{b+m\,a}\theta^{\alpha}_{\beta} - F^{\alpha}_{c+m\,a}\theta^{c+m}_{b+m} - F^{\alpha}_{b+m\,c}\theta^{c}_{a} &= F^{\alpha}_{b+m\,ai}\omega^{i}, \\ dF^{\alpha}_{a+m\,b} + F^{\beta}_{a+m\,b}\theta^{\alpha}_{\beta} - F^{\alpha}_{c+m\,b}\theta^{c+m}_{a+m} - F^{\alpha}_{a+m\,c}\theta^{c}_{b} &= F^{\alpha}_{a+m\,bi}\omega^{i}. \end{aligned} \tag{8.40}$$

Summing these two equations and using (8.2) and (8.4), we get

$$(F^{\alpha}_{c+m\,a}L^{c}_{bd} + F^{\alpha}_{c+m\,b}L^{c}_{ad})(\omega^{d} + \omega^{d+m}) = (F^{\alpha}_{b+m\,ai} + F^{\alpha}_{a+m\,bi})\omega^{i}. \tag{8.41}$$

Equating the coefficients of ω^{d}, we have

$$F^{\alpha}_{c+m\,a}L^{c}_{bd} + F^{\alpha}_{c+m\,b}L^{c}_{ad} = F^{\alpha}_{b+m\,ad} + F^{\alpha}_{a+m\,bd}. \tag{8.42}$$

From (5.7) we see that the right side of (8.42) is

$$\begin{aligned} -F^{\mu}_{b+m\,a}F^{\mu}_{\alpha d} - 2F^{\mu}_{b+m\,d}F^{\mu}_{\alpha a} - F^{\mu}_{a+m\,b}F^{\mu}_{\alpha d} - 2F^{\mu}_{a+m\,d}F^{\mu}_{\alpha b} \\ = 2F^{\mu}_{d+m\,b}F^{\mu}_{\alpha a} + 2F^{\mu}_{d+m\,a}F^{\mu}_{\alpha b} \end{aligned} \tag{8.43}$$

where the last equality comes from using (8.3). Now (8.37) follows from (8.42) and (8.43). Equating the coefficients of ω^{d+m} in (8.41) leads again to (8.37). Equating the other coefficients leads to the identities

$$F^{\alpha}_{b+m\,a\beta} + F^{\alpha}_{a+m\,b\beta} = 0 \text{ and } F^{\alpha}_{b+m\,a\mu} + F^{\alpha}_{a+m\,b\mu} = 0. \tag{8.44}$$

We next find the consequences of taking the covariant derivative of equation (8.3). Again by (5.2), we have

$$\begin{aligned} dF^{\mu}_{b+m\,a} + F^{\nu}_{b+m\,a}\theta^{\mu}_{\nu} - F^{\mu}_{c+m\,a}\theta^{c+m}_{b+m} - F^{\mu}_{b+m\,c}\theta^{c}_{a} &= F^{\mu}_{b+m\,ai}\omega^{i}, \\ dF^{\mu}_{a+m\,b} + F^{\nu}_{a+m\,b}\theta^{\mu}_{\nu} - F^{\mu}_{c+m\,b}\theta^{c+m}_{a+m} - F^{\mu}_{a+m\,c}\theta^{c}_{b} &= F^{\mu}_{a+m\,bi}\omega^{i}. \end{aligned} \tag{8.45}$$

Summing these equations and using (8.3) and (8.4), we get

$$(F^{\mu}_{c+m\,a}L^{c}_{bd} + F^{\mu}_{c+m\,b}L^{c}_{ad})(\omega^{d} + \omega^{d+m}) = (F^{\mu}_{b+m\,ai} + F^{\mu}_{a+m\,bi})\omega^{i}. \tag{8.46}$$

Equating the coefficients of ω^{d} we have

$$F^{\mu}_{c+m\,a}L^{c}_{bd} + F^{\mu}_{c+m\,b}L^{c}_{ad} = F^{\mu}_{b+m\,ad} + F^{\mu}_{a+m\,bd}. \tag{8.47}$$

By (5.8), the right side of (8.47) is

$$F^{\alpha}_{b+m\,a}F^{\mu}_{\alpha d} + 2F^{\alpha}_{b+m\,d}F^{\mu}_{\alpha a} + F^{\alpha}_{a+m\,b}F^{\mu}_{\alpha d} + 2F^{\alpha}_{a+m\,d}F^{\mu}_{\alpha b}. \tag{8.48}$$

Using (8.2) in (8.48), we then arrive at (8.38). Equating coefficients of ω^{d+m} in (8.46) also leads to (8.38). Equating coefficients of ω^α and of ω^μ gives

$$F^\mu_{b+m\,a\alpha} + F^\mu_{a+m\,b\alpha} = 0 \text{ and } F^\mu_{b+m\,a\nu} + F^\mu_{a+m\,b\nu} = 0. \tag{8.49}$$

Finally, substitute the first equation of (5.9) into (8.30) to arrive at (8.39). □

We define the *covariant derivatives* of the L^a_{bc} to be the coefficients L^a_{bci} of the 1-form

$$dL^a_{bc} + L^d_{bc}\theta^a_d - L^a_{dc}\theta^d_b - L^a_{bd}\theta^d_c = L^a_{bci}\omega^i. \tag{8.50}$$

Remark 22. If the L^a_{bc} are skew-symmetric in all three indices, then the functions L^a_{bci} are skew-symmetric in a, b, c.

PROPOSITION 23. *If equations* (8.1) *through* (8.4) *hold, then the* L^a_{bcd} *are skew-symmetric in all four indices, and*

$$\begin{aligned} L^a_{bcd} = {}& \frac{1}{2}(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}) + \frac{1}{2}(L^a_{ce}L^e_{bd} - L^a_{de}L^e_{bc}) \\ & + F^\alpha_{c+m\,a}F^\alpha_{d+m\,b} - F^\mu_{c+m\,b}F^\mu_{d+m\,a}; \end{aligned} \tag{8.51}$$

$$\begin{aligned} L^a_{bc\,d+m} = {}& \frac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + L^a_{be}L^e_{dc} + \frac{1}{2}(L^a_{ce}L^e_{bd} - L^a_{de}L^e_{bc}) \\ & + F^\alpha_{d+m\,a}F^\alpha_{c+m\,b} - F^\mu_{d+m\,b}F^\mu_{c+m\,a}; \end{aligned} \tag{8.52}$$

$$L^a_{bc\alpha} = L^a_{be}F^\alpha_{e+m\,c} + 2(F^\mu_{\alpha a}F^\mu_{c+m\,b} - F^\mu_{\alpha b}F^\mu_{c+m\,a}); \tag{8.53}$$

$$L^a_{bc\mu} = L^a_{be}F^\mu_{e+m\,c} + 2(F^\mu_{\alpha b}F^\alpha_{c+m\,a} - F^\mu_{\alpha a}F^\alpha_{c+m\,b}); \tag{8.54}$$

$$\begin{aligned} 2\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} - \delta_{ab}\delta_{dc} = {}& L^a_{be}L^c_{de} + L^a_{de}L^c_{be} \\ & + 2(F^\alpha_{b+m\,c}F^\alpha_{d+m\,a} + F^\alpha_{b+m\,a}F^\alpha_{d+m\,c}); \end{aligned} \tag{8.55}$$

$$L^a_{bc\,d+m} + L^a_{bd\,c+m} = 0. \tag{8.56}$$

Proof. This proposition is a consequence of taking the exterior derivative of (8.4). Notice that (8.56) follows directly from (8.52).

Using (4.18) and the structure equations (2.4), we find

$$\begin{aligned} d(\theta^a_b - \theta^{a+m}_{b+m}) = {}& \omega^a\wedge\omega^b - \omega^{a+m}\wedge\omega^{b+m} \\ & + (F^\alpha_{c+m\,a}F^\alpha_{d+m\,b} + F^\mu_{c+m\,a}F^\mu_{d+m\,b})(\omega^{c+m}\wedge\omega^{d+m} - \omega^c\wedge\omega^d) \\ & + [L^a_{dc}\theta^d_b - L^d_{bc}\theta^a_d + L^a_{ed}L^e_{bc}(\omega^d + \omega^{d+m}) + 2(F^\mu_{\alpha a}F^\mu_{c+m\,b} - F^\mu_{\alpha b}F^\mu_{c+m\,a})\omega^\alpha \\ & + 2(F^\mu_{\alpha b}F^\alpha_{c+m\,a} - F^\mu_{\alpha a}F^\alpha_{c+m\,b})\omega^\mu] \wedge (\omega^c + \omega^{c+m}). \end{aligned}$$

By (5.1),

$$d(L^a_{bc}(\omega^c+\omega^{c+m})) = (dL^a_{bc} - L^a_{bd}\theta^d_c - L^a_{be}L^e_{dc}\omega^{d+m} - L^a_{bd}F^\alpha_{d+m\,c}\omega^\alpha - L^a_{bd}F^\mu_{d+m\,c}\omega^\mu)\wedge(\omega^c+\omega^{c+m}).$$

The exterior differential of (8.4) is obtained by equating the preceding two equations and using (8.50), to get

$$\begin{aligned} L^a_{bei}\omega^i\wedge(\omega^e+\omega^{e+m}) &= [-\delta_{ae}\omega^b - \delta_{be}\omega^{a+m} \\ &\quad + (L^a_{dc}L^d_{be} + F^\alpha_{e+m\,a}F^\alpha_{c+m\,b} + F^\mu_{e+m\,a}F^\mu_{c+m\,b})\omega^c \\ &\quad + (L^a_{bd}L^d_{ce} + L^a_{dc}L^d_{be} + F^\alpha_{c+m\,a}F^\alpha_{e+m\,b} + F^\mu_{c+m\,a}F^\mu_{e+m\,b})\omega^{c+m} \\ &\quad + (L^a_{bc}F^\alpha_{c+m\,e} + 2F^\mu_{\alpha a}F^\mu_{e+m\,b} - 2F^\mu_{e+m\,a}F^\mu_{\alpha b})\omega^\alpha \\ &\quad + (L^a_{bc}F^\mu_{c+m\,e} + 2F^\alpha_{e+m\,a}F^\mu_{\alpha b} - 2F^\mu_{\alpha a}F^\mu_{e+m\,b})\omega^\mu]\wedge(\omega^e+\omega^{e+m}). \end{aligned} \tag{8.57}$$

Equating the skew-symmetrized coefficients of $\omega^c\wedge\omega^e$ in this equation, we have

$$\begin{aligned} L^a_{bec} - L^a_{bce} &= L^a_{dc}L^d_{be} - L^a_{de}L^d_{bc} - \delta_{ae}\delta_{bc} + \delta_{ac}\delta_{be} \\ &\quad + F^\alpha_{e+m\,a}F^\alpha_{c+m\,b} - F^\alpha_{c+m\,a}F^\alpha_{e+m\,b} \\ &\quad + F^\mu_{e+m\,a}F^\mu_{c+m\,b} - F^\mu_{c+m\,a}F^\mu_{e+m\,b}. \end{aligned} \tag{8.58}$$

Rewrite (8.58) with b and e interchanged and add the result to (8.58). Using the facts that L^d_{be}, L^a_{bec}, $F^\alpha_{e+m\,b}$ and $F^\mu_{e+m\,b}$ are all skew-symmetric in b and e, we get from this sum

$$\begin{aligned} L^a_{bce} + L^a_{ecb} &= L^a_{bd}L^c_{ed} + L^a_{ed}L^c_{bd} + \delta_{ae}\delta_{bc} + \delta_{ab}\delta_{ec} - 2\delta_{ac}\delta_{be} \\ &\quad + F^\alpha_{b+m\,c}F^\alpha_{e+m\,a} + F^\alpha_{b+m\,a}F^\alpha_{e+m\,c} \\ &\quad + F^\mu_{e+m\,a}F^\mu_{b+m\,c} + F^\mu_{b+m\,a}F^\mu_{e+m\,c}. \end{aligned} \tag{8.59}$$

Equating the coefficients of $\omega^c\wedge\omega^{e+m}$ in (8.57), we find

$$L^a_{bec} - L^a_{bc\,e+m} = L^a_{dc}L^d_{be} - L^a_{bd}L^d_{ec} - L^a_{de}L^d_{bc}. \tag{8.60}$$

Rewrite this equation with b and c interchanged and add the result to (8.60). From the skew-symmetry of L^a_{bc} and L^a_{bcd} in a,b,c, it follows from this sum that

$$L^a_{bec} + L^a_{ceb} = 0 \tag{8.61}$$

from which we conclude that L^a_{bcd} is skew-symmetric in all four indices. Putting (8.61) into (8.59), interchanging d and e and using the first equation in (5.6), we arrive at (8.55). Putting (8.61) into (8.58) and using the first equation of (5.6), we get (8.51). Substitute (8.51) into (8.60) to obtain (8.52). Go back to (8.57) and equate coefficients of $\omega^\alpha\wedge\omega^c$ to obtain (8.53), and equate coefficients of $\omega^\mu\wedge\omega^c$ to obtain (8.54). □

9. A sufficient condition to be FKM

Let $\tilde{\mathbf{x}}, e_a, e_p, e_\alpha, e_\mu, \tilde{e}_0$ be a second order frame field (2.2) in $U \subset M$ along an isoparametric hypersurface $\tilde{\mathbf{x}} : M \to S^n \subset \mathbf{R}^{n+1}$. We continue using the index conventions in (4.6). Let $\mathbf{x} = \cos s_1\, \tilde{\mathbf{x}} + \sin s_1\, \tilde{e}_0$ be a focal submanifold and let $e_0 = -\sin s_1\, \tilde{\mathbf{x}} + \cos s_1\, \tilde{e}_0$ so that

$$\mathbf{x}, e_0, e_a, e_p, e_\alpha, e_\mu \tag{9.1}$$

is a Darboux frame field (4.16) along $\mathbf{x}$ on U. Let

$$\omega^a, \omega^p, \omega^\alpha, \omega^\mu \tag{9.2}$$

be its coframe field (4.13) on U.

THEOREM 24. *If* $\mathbf{x}$ *satisfies the spanning property* (*Definition* 8) *and condition* (8.1), $F^\mu_{\alpha\, a+m} = F^\mu_{\alpha a}$, *on* U, *then it comes from an* FKM *construction.*

Proof. It is sufficient to prove the theorem locally, on some open neighborhood, because isoparametric hypersurfaces are algebraic. For each point in U, the vectors of our Darboux frame field (9.1) form an orthonormal basis of $\mathbf{R}^{n+1}$. Linear operators Q_0, Q_a on $\mathbf{R}^{n+1}$, depending on the point in U, can thus be defined by (7.39) and (7.40), which we recopy here for easier reference

$$\begin{aligned} Q_0\mathbf{x} &= e_0, \qquad Q_0 e_0 = \mathbf{x}, \qquad Q_0 e_a = -e_{a+m}, \\ Q_0 e_{a+m} &= -e_a, \quad Q_0 e_\alpha = -e_\alpha, \quad Q_0 e_\mu = e_\mu, \end{aligned} \tag{9.3}$$

and for each a

$$\begin{aligned} Q_a\mathbf{x} &= e_a, \\ Q_a e_0 &= e_{a+m}, \\ Q_a e_b &= \delta_{ab}\mathbf{x} - L^c_{ab} e_{c+m} + F^\alpha_{a+m\, b} e_\alpha + F^\mu_{a+m\, b} e_\mu, \\ Q_a e_{b+m} &= \delta_{ab} e_0 + L^c_{ab} e_c + F^\alpha_{b+m\, a} e_\alpha - F^\mu_{b+m\, a} e_\mu, \\ Q_a e_\alpha &= F^\alpha_{a+m\, b} e_b + F^\alpha_{b+m\, a} e_{b+m} - 2F^\mu_{\alpha\, a} e_\mu, \\ Q_a e_\mu &= F^\mu_{a+m\, b} e_b - F^\mu_{b+m\, a} e_{b+m} - 2F^\mu_{\alpha\, a} e_\alpha, \end{aligned} \tag{9.4}$$

where the coefficients are defined as in (4.18) and (8.4). We first outline the quite elementary proof of the theorem, and then follow that with a proof of the details. The first detail is:

(I). At each point of U these operators are symmetric, orthogonal and satisfy

$$Q_i Q_j + Q_j Q_i = 2\delta_{ij} I, \qquad \text{for } i, j = 0, 1, \ldots, m. \tag{9.5}$$

Given that, next one proves the second detail:

(II). There exist a (constant) Clifford system $P_0, \dots, P_m$ on $\mathbf{R}^{n+1}$ and a smooth map

$$B : U \to \mathrm{SO}(m+1) \tag{9.6}$$

such that at every point of U,

$$Q_j = \sum_{i=0}^{m} B_j^i P_i, \quad \text{for } j = 0, 1, \dots, m \tag{9.7}$$

It will then follow that $\mathbf{x}$ maps U onto an open subset of the focal submanifold M_+ defined in (7.4) by this Clifford system, and that the Darboux frame field (7.14) coming from the FKM construction applied to $P_0, \dots, P_m$ coincides with our frame field (9.1). Therefore, our $\mathbf{x} : U \to S^n$ coincides with the FKM construction applied to this Clifford system.

We turn now to the proof of detail (I). The verification that each Q_i is symmetric can be done almost by inspection. It is equally clear that Q_0 is orthogonal, since it sends the orthonormal basis (9.1) to an orthonormal basis. The operator Q_a sends the orthonormal basis (9.1) to the set of vectors given on the right hand side of (9.4). Among these vectors, $Q_a\mathbf{x}, Q_a e_0$ is an orthonormal pair orthogonal to the remaining vectors because L^a_{bc} are skew-symmetric in a, b, c and $F^\alpha_{a+m\,b}$ and $F^\mu_{a+m\,b}$ are skew-symmetric in a and b.

In the following verification that

$$\{Q_a e_b, Q_a e_{b+m}, Q_a e_\alpha, Q_a e_\mu : b, a, \mu\}$$

is orthonormal, we do not use the Einstein summation convention as a will always be a repeated index which is not summed. We proceed through all the cases.

$$Q_a e_b \cdot Q_a e_d = \delta_{ab}\delta_{ad} + \sum_c L^b_{ac} L^d_{ac} + \sum_\alpha F^\alpha_{a+m\,b} F^\alpha_{a+m\,d} + \sum_\mu F^\mu_{a+m\,b} F^\mu_{a+m\,d} = \delta_{bd}$$

by (8.55) with c changed to b and b changed to a.

$$Q_a e_b \cdot Q_a e_{d+m} = \sum_\alpha F^\alpha_{a+m\,b} F^\alpha_{d+m\,a} - \sum_\mu F^\mu_{a+m\,b} F^\mu_{d+m\,a} = 0$$

by the first equation in (5.6).

$$Q_a e_b \cdot Q_a e_\alpha = \sum_c L^b_{ac} F^\alpha_{c+m\,a} - 2\sum_\mu F^\mu_{a+m\,b} F^\mu_{\alpha a} = 0$$

by (8.37) with d changed to a.

$$Q_a e_b \cdot Q_a e_\mu = -\sum_c L^b_{ac} F^\mu_{c+m\,a} - 2\sum_\alpha F^\alpha_{a+m\,b} F^\mu_{\alpha a} = 0$$

by (8.38) with d changed to a.

$$Q_a e_{b+m}\cdot Q_a e_{d+m} = \delta_{ab}\delta_{ad} + \sum_c L^c_{ab}L^c_{ad} + \sum_\alpha F^\alpha_{b+m\,a}F^\alpha_{d+m\,a} + \sum_\mu F^\mu_{b+m\,a}F^\mu_{d+m\,a} = \delta_{bd}$$

by (8.55) with c changed to b and b changed to a.

$$Q_a e_{b+m}\cdot Q_a e_\alpha = \sum_c L^c_{ab}F^\alpha_{a+m\,c} + 2\sum_\mu F^\mu_{b+m\,a}F^\mu_{\alpha a} = 0$$

by (8.37) with d changed to a.

$$Q_a e_{b+m}\cdot Q_a e_\mu = \sum_c L^c_{ab}F^\mu_{a+m\,c} - 2\sum_\alpha F^\alpha_{b+m\,a}F^\mu_{\alpha a} = 0$$

by (8.38) with d changed to a.

$$Q_a e_\alpha\cdot Q_a e_\beta = 2\sum_b F^\alpha_{b+m\,a}F^\beta_{b+m\,a} + 4\sum_\mu F^\mu_{\alpha a}F^\mu_{\beta a} = \delta_{\alpha\beta}$$

by the second equation in (5.6).

$$Q_a e_\alpha\cdot Q_a e_\mu = \sum_b F^\alpha_{a+m\,b}F^\mu_{a+m\,b} - \sum_b F^\alpha_{b+m\,a}F^\mu_{b+m\,a} = 0$$

by (8.2) and (8.3).

$$Q_a e_\mu\cdot Q_a e_\nu = 2\sum_b F^\mu_{b+m\,a}F^\nu_{b+m\,a} + 4\sum_\alpha F^\mu_{\alpha a}F^\nu_{\alpha a} = \delta_{\mu\nu}$$

by the fourth equation in (5.6). This completes the verification that each Q_i is an orthogonal transformation.

We proceed now to verify (9.5). For this we return to using the Einstein summation convention. Clearly $Q_0^2 = I$. To verify that $Q_0Q_a + Q_aQ_0 = 0$, for all a, we set $S = Q_0Q_a + Q_aQ_0$ and evaluate it on the basis vectors.

$$S\mathbf{x} = Q_0e_a + Q_ae_0 = -e_{a+m} + e_{a+m} = 0.$$

$$Se_0 = Q_0e_{a+m} + Q_a\mathbf{x} = -e_a + e_a = 0.$$

$$\begin{aligned}
Se_b &= Q_0(\delta_{ab}\mathbf{x} + L^b_{ac}e_{c+m} + F^\alpha_{a+m\,b}e_\alpha + F^\mu_{a+m\,b}e_\mu) + Q_a(-e_{b+m})\\
&= \delta_{ab}e_0 - L^b_{ac}e_c - F^\alpha_{a+m\,b}e_\alpha + F^\mu_{a+m\,b}e_\mu\\
&\quad -\delta_{ab}e_0 - L^c_{ab}e_c - F^\alpha_{b+m\,a}e_\alpha + F^\mu_{b+m\,a}e_\mu = 0.\\
Se_{b+m} &= Q_0(\delta_{ab}e_0 + L^c_{ab}e_c + F^\alpha_{b+m\,a}e_\alpha - F^\mu_{b+m\,a}e_\mu) + Q_a(-e_b)\\
&= \delta_{ab}\mathbf{x} - L^c_{ab}e_{c+m} - F^\alpha_{b+m\,a}e_\alpha - F^\mu_{b+m\,a}e_\mu\\
&\quad -\delta_{ab}\mathbf{x} - L^b_{ac}e_{c+m} - F^\alpha_{a+m\,b}e_\alpha - F^\mu_{a+m\,b}e_\mu = 0.\\
Se_\alpha &= Q_0(F^\alpha_{a+m\,b}e_b + F^\alpha_{b+m\,a}e_{b+m} - 2F^\mu_{\alpha a}e_\mu) + Q_a(-e_\alpha)
\end{aligned}$$

$$\begin{aligned}
&= -F^{\alpha}_{a+m\,b}e_{b+m} - F^{\alpha}_{b+m\,a}e_b - 2F^{\mu}_{\alpha a}e_\mu \\
&\quad -F^{\alpha}_{a+m\,b}e_b - F^{\alpha}_{b+m\,a}e_{b+m} + 2F^{\mu}_{\alpha a}e_\mu = 0. \\
Se_\mu &= Q_0(F^{\mu}_{a+m\,b}e_b - F^{\mu}_{b+m\,a}e_{b+m} - 2F^{\mu}_{\alpha a}e_\alpha) + Q_a e_\mu \\
&= -F^{\mu}_{a+m\,b}e_{b+m} + F^{\mu}_{b+m\,a}e_b + 2F^{\mu}_{\alpha a}e_\alpha \\
&\quad +F^{\mu}_{a+m\,b}e_b - F^{\mu}_{b+m\,a}e_{b+m} - 2F^{\mu}_{\alpha a}e_\alpha = 0.
\end{aligned}$$

Therefore, $S = 0$, which is what we wanted to prove.

Next we verify that $Q_aQ_d + Q_dQ_a = 2\delta_{ad}I$ for all a and d. For this verification we let $T = Q_aQ_d + Q_dQ_a$ and evaluate it on the basis vectors.

$$\begin{aligned}
T\mathbf{x} &= Q_a e_d + Q_d e_a = \delta_{ad}\mathbf{x} + L^{d}_{ac}e_{c+m} + F^{\alpha}_{a+m\,d}e_\alpha + F^{\mu}_{a+m\,d}e_\mu \\
&\quad +\delta_{da}\mathbf{x} + L^{a}_{dc}e_{c+m} + F^{\alpha}_{d+m\,a}e_\alpha + F^{\mu}_{d+m\,a}e_\mu = 2\delta_{ad}\mathbf{x}; \\
Te_0 &= Q_a e_{d+m} + Q_d e_{a+m} = \delta_{ad}e_0 + L^{c}_{ad}e_c + F^{\alpha}_{d+m\,a}e_\alpha - F^{\mu}_{d+m\,a}e_\mu \\
&\quad +\delta_{da}e_0 + L^{c}_{da}e_c + F^{\alpha}_{a+m\,d}e_\alpha - F^{\mu}_{a+m\,d}e_\mu = 2\delta_{ad}e_0; \\
Te_b &= Q_a(\delta_{db}\mathbf{x} + L^{b}_{dc}e_{c+m} + F^{\alpha}_{d+m\,b}e_\alpha + F^{\mu}_{d+m\,b}e_\mu) \\
&\quad +Q_d(\delta_{ab}\mathbf{x} + L^{b}_{ac}e_{c+m} + F^{\alpha}_{a+m\,b}e_\alpha + F^{\mu}_{a+m\,b}e_\mu) \\
&= (L^{b}_{da} + L^{b}_{ad})e_0 + (\delta_{bd}\delta_{ae} + \delta_{ba}\delta_{de} + L^{b}_{dc}L^{e}_{ac} + L^{b}_{ac}L^{e}_{dc} \\
&\quad +F^{\alpha}_{d+m\,b}F^{\alpha}_{a+m\,e} + F^{\alpha}_{a+m\,b}F^{\alpha}_{d+m\,e} + F^{\mu}_{d+m\,b}F^{\mu}_{a+m\,e} + F^{\mu}_{a+m\,b}F^{\mu}_{d+m\,e})e_e \\
&\quad +(F^{\alpha}_{d+m\,b}F^{\alpha}_{e+m\,a} + F^{\alpha}_{a+m\,b}F^{\alpha}_{e+m\,d} - F^{\mu}_{d+m\,b}F^{\mu}_{e+m\,a} - F^{\mu}_{a+m\,b}F^{\mu}_{e+m\,d})e_{e+m} \\
&\quad +(L^{b}_{dc}F^{\alpha}_{c+m\,a} + L^{b}_{ac}F^{\alpha}_{c+m\,d} - 2F^{\mu}_{d+m\,b}F^{\mu}_{\alpha a} - 2F^{\mu}_{a+m\,b}F^{\mu}_{\alpha d})e_\alpha \\
&\quad -(L^{b}_{dc}F^{\mu}_{c+m\,a} + L^{b}_{ac}F^{\mu}_{c+m\,d} + 2F^{\alpha}_{d+m\,b}F^{\mu}_{\alpha a} + 2F^{\alpha}_{a+m\,b}F^{\mu}_{\alpha d})e_\mu \\
&= 2\delta_{ad}\delta_{be}e_e = 2\delta_{ad}e_b;
\end{aligned}$$

where the coefficient of e_e comes from (8.55), the coefficient of e_{e+m} is zero by the first equation of (5.6), the coefficient of e_α is zero by (8.37) (with the roles of b and d reversed) and the coefficient of e_μ is zero by (8.38) (with the roles of b and d reversed).

$$\begin{aligned}
Te_{b+m} &= (L^{a}_{db} + L^{d}_{ab})\mathbf{x} \\
&\quad +(F^{\alpha}_{b+m\,d}F^{\alpha}_{a+m\,c} + F^{\alpha}_{b+m\,a}F^{\alpha}_{d+m\,c} - F^{\mu}_{b+m\,d}F^{\mu}_{a+m\,c} - F^{\mu}_{b+m\,a}F^{\mu}_{d+m\,c})e_c \\
&\quad +(\delta_{db}\delta_{ae} + \delta_{ab}\delta_{de} + L^{c}_{db}L^{c}_{ae} + L^{c}_{ab}L^{c}_{de} \\
&\quad +F^{\alpha}_{b+m\,d}F^{\alpha}_{e+m\,a} + F^{\alpha}_{b+m\,a}F^{\alpha}_{e+m\,d} + F^{\mu}_{b+m\,d}F^{\mu}_{e+m\,a} + F^{\mu}_{b+m\,a}F^{\mu}_{e+m\,d})e_{e+m} \\
&\quad +(L^{c}_{db}F^{\alpha}_{a+m\,c} + L^{c}_{ab}F^{\alpha}_{d+m\,c} + 2F^{\mu}_{b+m\,d}F^{\mu}_{\alpha a} + 2F^{\mu}_{b+m\,a}F^{\mu}_{\alpha d})e_\alpha \\
&\quad +(L^{c}_{db}F^{\mu}_{a+m\,c} + L^{c}_{ab}F^{\mu}_{d+m\,c} - 2F^{\alpha}_{b+m\,d}F^{\mu}_{\alpha a} - 2F^{\alpha}_{b+m\,a}F^{\mu}_{\alpha d})e_\mu \\
&= 2\delta_{ad}\delta_{be}e_{e+m} = 2\delta_{ad}e_{b+m},
\end{aligned}$$

where the coefficient of e_{e+m} comes from (8.55), the coefficient of e_c is zero by the first equation of (5.6), the coefficient of e_α is zero by (8.37) (with the roles of b and d reversed) and the coefficient of e_μ is zero by (8.38) (with the roles of b and d reversed).

$$Te_\alpha = (F^{\alpha}_{d+m\,a} + F^{\alpha}_{a+m\,d})\mathbf{x} + (F^{\alpha}_{a+m\,d} + F^{\alpha}_{d+m\,a})e_0$$

$$
\begin{aligned}
&+(F^{\alpha}_{b+m\,d}L^c_{ab} + F^{\alpha}_{b+m\,a}L^c_{db} - 2F^{\mu}_{\alpha d}F^{\mu}_{a+m\,c} - 2F^{\mu}_{\alpha a}F^{\mu}_{d+m\,c})e_c\\
&+(F^{\alpha}_{d+m\,b}L^b_{ac} + F^{\alpha}_{a+m\,b}L^b_{dc} + 2F^{\mu}_{\alpha d}F^{\mu}_{c+m\,a} + 2F^{\mu}_{\alpha a}F^{\mu}_{c+m\,d})e_{c+m}\\
&+2(F^{\alpha}_{b+m\,d}F^{\beta}_{b+m\,a} + F^{\alpha}_{b+m\,a}F^{\beta}_{b+m\,d} + 2F^{\mu}_{\alpha d}F^{\mu}_{\beta a} + 2F^{\mu}_{\alpha a}F^{\mu}_{\beta d})e_{\beta}\\
&+(F^{\alpha}_{b+m\,d}F^{\mu}_{b+m\,a} + F^{\alpha}_{b+m\,a}F^{\mu}_{b+m\,d} - F^{\alpha}_{b+m\,d}F^{\mu}_{b+m\,a} - F^{\alpha}_{b+m\,a}F^{\mu}_{b+m\,d})e_{\mu}\\
=\,&2\delta_{\alpha\beta}\delta_{ad}e_{\beta} = 2\delta_{ad}e_{\alpha},
\end{aligned}
$$

where the coefficients of $\mathbf{x}$ and e_0 are clearly zero, the coefficients of e_c and of e_{c+m} are zero by (8.37) (in which the roles of a, b, c, d are here played by a, d, b, c), the coefficient of e_β comes from the second equation of (5.6) and the coefficient of e_μ is clearly zero.

$$
\begin{aligned}
Te_{\mu} =\,&(F^{\mu}_{d+m\,a} + F^{\mu}_{a+m\,d})\mathbf{x} - (F^{\mu}_{a+m\,d} + F^{\mu}_{d+m\,a})e_0\\
&-(2(F^{\mu}_{\alpha d}F^{\alpha}_{a+m\,b} + F^{\mu}_{\alpha a}F^{\alpha}_{d+m\,b}) + F^{\mu}_{c+m\,d}L^b_{ac} + F^{\mu}_{c+m\,a}L^b_{dc})e_b\\
&-(F^{\mu}_{d+m\,c}L^b_{ac} + 2F^{\mu}_{\alpha d}F^{\alpha}_{b+m\,a} + F^{\mu}_{a+m\,c}L^b_{dc} + 2F^{\mu}_{\alpha a}F^{\alpha}_{b+m\,d})e_{b+m}\\
&+(F^{\mu}_{d+m\,b}F^{\alpha}_{a+m\,b} - F^{\mu}_{b+m\,d}F^{\alpha}_{b+m\,a} + F^{\mu}_{a+m\,b}F^{\alpha}_{d+m\,b} - F^{\mu}_{b+m\,a}F^{\alpha}_{b+m\,d})e_{\alpha}\\
&+2(F^{\mu}_{d+m\,b}F^{\nu}_{a+m\,b} + 2F^{\mu}_{\alpha d}F^{\nu}_{\alpha a} + F^{\mu}_{a+m\,b}F^{\nu}_{d+m\,b} + 2F^{\mu}_{\alpha a}F^{\nu}_{\alpha d})e_{\nu}\\
=\,&2\delta_{ad}\delta_{\mu\nu}e_{\nu} = 2\delta_{ad}e_{\mu},
\end{aligned}
$$

where the coefficients of $\mathbf{x}$ and e_0 are clearly zero, the coefficients of e_b and e_{b+m} are zero by (8.38) (in which the roles of b and d are reversed), the coefficient of e_α is zero by (8.2) and (8.3), and the coefficient of e_μ comes from the fifth equation of (5.6). This completes the proof of detail (I).

In order to prove (II), we must find a Clifford system $P_0, \dots, P_m$ which is related to $Q_0, \dots, Q_m$ by (9.7). We do this by finding the map $B : U \to \mathrm{SO}(m+1)$ of (9.6). Let

$$\nu^a = \omega^a + \omega^{a+m} \tag{9.8}$$

Use (5.1) together with (8.2)–(8.4) to find

$$d\nu^a = -\nu^a_b \wedge \nu^b \tag{9.9}$$

where

$$\nu^a_b = \theta^a_b + L^a_{cb}\omega^{c+m} + F^{\alpha}_{a+m\,b}\omega^{\alpha} + F^{\mu}_{a+m\,b}\omega^{\mu} = -\nu^b_a. \tag{9.10}$$

Set

$$\nu^0_b = -\nu^b_0 = -\nu^b = -(\omega^b + \omega^{b+m}). \tag{9.11}$$

We shall verify below that

$$dQ_j = \sum_{k=0}^{m} Q_k \nu^k_j, \text{ for } j = 0, \dots, m. \tag{9.12}$$

Differentiating this, we find that

$$d\nu^i_j = -\sum_{k=0}^{m} \nu^i_k \wedge \nu^k_j, \text{ for } i, j = 0, \dots m. \tag{9.13}$$

In fact, (9.9) is the case $i = a$, $j = 0$ and also implies the case $i = 0$, $j = a$. To verify the remaining cases in (9.13), we take the exterior derivative of (9.12) when $j = a$, and then use (9.12) and (9.9) to find

$$\begin{aligned} 0 = ddQ_a &= dQ_0 \wedge \nu^0_a + Q_0 d\nu^0_a + dQ_b \wedge \nu^b_a + Q_b d\nu^b_a \\ &= Q_b \nu^b \wedge \nu^0_a + Q_0 \nu^a_b \wedge \nu^b + (Q_0 \nu^0_b + Q_c \nu^c_b) \wedge \nu^b_a + Q_b d\nu^b_a \\ &= Q_b(d\nu^b_a + \nu^b_c \wedge \nu^c_a + \nu^b \wedge \nu^0_a) + Q_0(\nu^a_b \wedge \nu^b + \nu^0_b \wedge \nu^b_a), \end{aligned} \tag{9.14}$$

which implies (9.13) because the coefficient of Q_0 is zero and the Q_b are linearly independent at each point of U, as can be seen from the fact that $Q_b\mathbf{x} = e_b$ are linearly independent at each point. Define the $o(m+1)$-valued 1-form ν to be

$$\nu = \begin{pmatrix} 0 & \nu^0_b \\ \nu^a_0 & \nu^a_b \end{pmatrix}. \tag{9.15}$$

Then (9.9) and (9.13) imply that $d\nu = -\nu \wedge \nu$. Therefore, on a simply connected subset of U, which we continue to call U, there exists a smooth map

$$A : U \to \mathrm{SO}(m+1) \tag{9.16}$$

such that $A^{-1}dA = \nu$. Denote the entries of A by the functions A^i_j, $i, j = 0, \dots, m$, so that the entries of $dA = A\nu$ are given by

$$dA^i_j = \sum_{k=0}^{m} A^i_k \nu^k_j. \tag{9.17}$$

Let

$$P_i = \sum_{j=0}^{m} A^i_j Q_j, \text{ for } i = 0, \dots, m \tag{9.18}$$

which, at each point of U, is a set of symmetric, orthogonal transformations of $\mathbf{R}^{n+1}$ satisfying the conditions $P_iP_j + P_jP_i = 2\delta_{ij}I$, since $Q_iQ_j + Q_jQ_i = 2\delta_{ij}I$ and $A \in \mathrm{SO}(m+1)$. By (9.12) and (9.17),

$$dP_i = \sum_{j=0}^{m}((dA^i_j)Q_j + A^i_j dQ_j) = \sum_{j,k=0}^{m} (A^i_k Q_j \nu^k_j + A^i_j Q_k \nu^k_j) = 0 \tag{9.19}$$

since $\nu^i_j + \nu^j_i = 0$. Therefore, each P_i is constant on U and $P_0, \dots, P_m$ define a Clifford system on $\mathbf{R}^{n+1}$ and (9.7) holds with $B = A^{-1}$.

All that remains of the proof of detail (II) is to verify (9.12), for which we need the Maurer-Cartan equations (4.20) for our Darboux frame field. We first verify (9.12) for $j = 0$, then for $j = a$, in both cases by evaluating each side on the basis vectors. Differentiating equations in (9.3) and using (4.20), we get

$$\begin{aligned}(dQ_0)\mathbf{x} &= d(Q_0\mathbf{x}) - Q_0\, d\mathbf{x}\\ &= de_0 - Q_0(\omega^{a+m}e_{a+m} + \omega^\alpha e_\alpha + \omega^\mu e_\mu)\\ &= \omega^a e_a - \omega^\alpha e_\alpha + \omega^\mu e_\mu + \omega^{a+m}e_a + \omega^\alpha e_\alpha - \omega^\mu e_\mu\\ &= \nu^a e_a = \nu^a Q_a \mathbf{x}.\end{aligned}$$

$$\begin{aligned}(dQ_0)e_0 &= d(Q_0 e_0) - Q_0\, de_0\\ &= d\mathbf{x} - Q_0(\omega^a e_a - \omega^\alpha e_\alpha + \omega^\mu e_\mu)\\ &= \nu^a e_{a+m} = \nu^a Q_a e_0.\end{aligned}$$

$$\begin{aligned}(dQ_0)e_a &= d(Q_0 e_a) - Q_0 de_a = -de_{a+m} - Q_0\, de_a\\ &= \nu^a \mathbf{x} + (\omega^{b+m}_a - \omega^b_{a+m})e_b + (\omega^b_a - \omega^{b+m}_{a+m})e_{b+m}\\ &\quad +(\omega^\alpha_a - \omega^\alpha_{a+m})e_\alpha - (\omega^\mu_{a+m} + \omega^\mu_a)e_\mu\\ &= \nu^b(\delta_{ab}\mathbf{x} + L^c_{ab}e_{c+m} + F^\alpha_{b+m\,a}e_\alpha + F^\mu_{b+m\,a}e_\mu)\\ &= \nu^b Q_b e_a.\end{aligned}$$

$$\begin{aligned}(dQ_0)e_{a+m} &= d(Q_0 e_{a+m}) - Q_0\, de_{a+m} = -de_a - Q_0\, de_{a+m}\\ &= (\omega^a + \omega^{a+m})e_0 + (\omega^{b+m}_{a+m} - \omega^b_a)e_b + (\omega^b_{a+m} - \omega^{b+m}_a)e_{b+m}\\ &\quad +(\omega^\alpha_{a+m} - \omega^\alpha_a)e_\alpha - (\omega^\mu_{a+m} + \omega^\mu_a)e_\mu\\ &= \nu^b(\delta_{ab}e_0 - L^c_{ab}e_c + F^\alpha_{a+m\,b}e_\alpha - F^\mu_{a+m\,b}e_\mu) = \nu^b Q_b e_{a+m}.\end{aligned}$$

$$\begin{aligned}(dQ_0)e_\alpha &= d(Q_0 e_\alpha) - Q_0\, de_\alpha = -de_\alpha - Q_0\, de_\alpha\\ &= (\omega^{a+m}_\alpha - \omega^a_\alpha)e_a + (\omega^a_\alpha - \omega^{a+m}_\alpha)e_{a+m} - 2\omega^\mu_\alpha e_\mu\\ &= \nu^b(F^\alpha_{b+m\,a}e_a - F^\alpha_{b+m\,a}e_{a+m} - 2F^\mu_{\alpha b}e_\mu) = \nu^b Q_b e_\alpha\\ (dQ_0)e_\mu &= d(Q_0 e_\mu) - Q_0\, de_\mu = de_\mu - Q_0\, de_\mu\\ &= (\theta^a_\mu + \theta^{a+m}_\mu)(e_a + e_{a+m}) + 2\theta^\alpha_\mu e_\alpha\\ &= \nu^b(F^\mu_{b+m\,a}(e_a + e_{a+m}) - 2F^\mu_{\alpha b}e_\alpha) = \nu^b Q_b e_\mu.\end{aligned}$$

This completes the verification of (9.12) for the case $j = 0$.

We now verify the equations in (9.12) for the cases $j = a$ by applying each side to the basis vectors. By (9.4) and (4.20),

$$\begin{aligned}(dQ_a)\mathbf{x} &= d(Q_a\mathbf{x}) - Q_a d\mathbf{x} = de_a - Q_a(\omega^{b+m}e_{b+m} + \omega^\alpha e_\alpha + \omega^\mu e_\mu)\\ &= -\nu^a e_0 + (\theta^b_a - L^b_{ac}\omega^{c+m} - F^\alpha_{a+m\,b}\omega^\alpha - F^\mu_{a+m\,b}\omega^\mu)e_b\\ &\quad +(\theta^{b+m}_a - F^\alpha_{b+m\,a}\omega^\alpha - F^\mu_{b+m\,a}\omega^\mu)e_{b+m}\\ &\quad +(\theta^\alpha_a - F^\alpha_{b+m\,a}\omega^{b+m} + 2F^\mu_{\alpha a}\omega^\mu)e_\alpha + (\theta^\mu_a + F^\mu_{b+m\,a}\omega^{b+m} + 2F^\mu_{\alpha a}\omega^\alpha)e_\mu\\ &= -\nu^a_0 e_0 + \nu^b_a e_b = (\nu^0_a Q_0 + \nu^b_a Q_b)\mathbf{x}.\end{aligned}$$

$$\begin{aligned}(dQ_a)e_0 &= d(Q_a e_0) - Q_a de_0 = de_{a+m} - Q_a(\omega^b e_b - \omega^\alpha e_\alpha + \omega^\mu e_\mu)\\ &= (-\omega^{a+m} - \omega^a)\mathbf{x} + (\theta^b_{a+m} + F^\alpha_{a+m\,b}\omega^\alpha - F^\mu_{a+m\,b}\omega^\mu)e_b\\ &\quad +(\theta^{b+m}_{a+m} - L^c_{ab}\omega^c + F^\alpha_{b+m\,a}\omega^\alpha + F^\mu_{b+m\,a}\omega^\mu)e_{b+m}\\ &\quad +(\theta^\alpha_{a+m} - F^\alpha_{a+m\,b}\omega^b + 2F^\mu_{\alpha a}\omega^\mu)e_\alpha\\ &\quad +(\theta^\mu_{a+m} - F^\mu_{a+m\,b}\omega^b - 2F^\mu_{\alpha a}\omega^\alpha)e_\mu\\ &= -\nu^a\mathbf{x} + (\theta^{b+m}_{a+m} - L^c_{ab}\omega^c + F^\alpha_{b+m\,a}\omega^\alpha + F^\mu_{b+m\,a}\omega^\mu)e_{b+m}\\ &= -\nu^a\mathbf{x} + \nu^b_a e_{b+m} = (\nu^0_a Q_0 + \nu^b_a Q_b)e_0,\end{aligned}$$

where the coefficients of e_b, e_α and e_μ are zero by (4.18), and (9.10) is used in the coefficient of e_{b+m}.

In order to verify (9.12) when both sides are applied to e_b, we must verify that

(9.20)
$$\begin{aligned}d(Q_a e_b) - Q_a de_b &= (dQ_a)e_b = \nu^0_a Q_0 e_b + \nu^c_a Q_c e_b\\ &= \nu^b_a\mathbf{x} + (\delta_{bd}\nu^a + L^d_{bc}\nu^c_a)e_{d+m} + F^\alpha_{c+m\,b}\nu^c_a e_\alpha + F^\mu_{c+m\,b}\nu^c_a e_\mu.\end{aligned}$$

Using (9.4) and (4.20), and gathering together the coefficients of each basis vector, we get

$$\begin{aligned}&d(Q_a e_b) - Q_a de_b\\ &\quad= (L^c_{ab}\omega^{c+m} - F^\alpha_{a+m\,b}\omega^\alpha - F^\mu_{a+m\,b}\omega^\mu - \theta^a_b)\mathbf{x}\\ &\qquad + (F^\alpha_{a+m\,b}\omega^\alpha - F^\mu_{a+m\,b}\omega^\mu - \theta^{a+m}_b)e_0\\ &\qquad + (-L^c_{ab}\theta^d_{c+m} + F^\alpha_{a+m\,b}\theta^d_\alpha + F^\mu_{a+m\,b}\theta^d_\mu\\ &\qquad\quad - L^d_{ac}\theta^{c+m}_b - F^\alpha_{a+m\,d}\theta^\alpha_b - F^\mu_{a+m\,d}\theta^\mu_b)e_d\\ &\qquad + (\delta_{ab}\omega^{c+m} - dL^c_{ab} - L^d_{ab}\theta^{c+m}_{d+m} + F^\alpha_{a+m\,b}\theta^{c+m}_\alpha + F^\mu_{a+m\,b}\theta^{c+m}_\mu\\ &\qquad\quad + \delta_{ac}\omega^b + L^c_{ad}\theta^d_b - F^\alpha_{c+m\,a}\theta^\alpha_b + F^\mu_{c+m\,a}\theta^\mu_b)e_{c+m}\\ &\qquad + (\delta_{ab}\omega^\alpha - L^c_{ab}\theta^\alpha_{c+m} + dF^\alpha_{a+m\,b} + F^\beta_{a+m\,b}\theta^\alpha_\beta\\ &\qquad\quad + F^\mu_{a+m\,b}\theta^\alpha_\mu - F^\alpha_{a+m\,c}\theta^c_b - F^\alpha_{c+m\,a}\theta^{c+m}_b + 2F^\mu_{\alpha a}\theta^\mu_b)e_\alpha\\ &\qquad + (\delta_{ab}\omega^\mu - L^c_{ab}\theta^\mu_{c+m} + F^\alpha_{a+m\,b}\theta^\mu_\alpha + dF^\mu_{a+m\,b}\\ &\qquad\quad + F^\nu_{a+m\,b}\theta^\mu_\nu - F^\mu_{a+m\,c}\theta^c_b + F^\mu_{c+m\,a}\theta^{c+m}_b + 2F^\mu_{\alpha a}\theta^\alpha_b)e_\mu.\end{aligned} \tag{9.21}$$

The coefficient of $\mathbf{x}$ is ν^b_a by (9.10). The coefficient of e_0 is zero. Substituting (4.18) into the coefficient of e_d, we get

$$\begin{aligned}&(-F^\alpha_{a+m\,b}F^\alpha_{c+m\,d} - F^\alpha_{a+m\,d}F^\alpha_{c+m\,b} + F^\mu_{a+m\,b}F^\mu_{c+m\,d} + F^\mu_{a+m\,d}F^\mu_{c+m\,b})\omega^{c+m}\\ &-(L^b_{ac}F^\alpha_{c+m\,d} + L^d_{ac}F^\alpha_{c+m\,b} - 2F^\mu_{a+m\,b}F^\mu_{\alpha d} - 2F^\mu_{a+m\,d}F^\mu_{\alpha b})\omega^\alpha\\ &+(L^b_{ac}F^\mu_{c+m\,d} + L^d_{ac}F^\mu_{c+m\,b} + 2F^\alpha_{a+m\,b}F^\mu_{\alpha d} + 2F^\alpha_{a+m\,d}F^\mu_{\alpha b})\omega^\mu\end{aligned}$$

which is zero since the coefficient of ω^{c+m} is zero by the first equation in (5.6), the coefficient of ω^α is zero by (8.37) and the coefficient of ω^μ is zero by (8.38).

Thus, the coefficient of e_d is zero, in agreement with the right hand side of (9.20).

By (4.18), (8.50) and (9.8) with (9.10), the coefficient of e_{c+m} becomes

$$\begin{aligned}
&-L^c_{db}\nu^d_a + \nu^a\delta^c_b \\
&-(L^c_{abd} - L^e_{ab}L^c_{ed} + F^\alpha_{a+m\,b}F^\alpha_{c+m\,d} + F^\mu_{a+m\,b}F^\mu_{c+m\,d} - \delta_{ac}\delta_{bd} + \delta_{bc}\delta_{ad})\omega^d \\
&+(-L^c_{ab\,d+m} + L^c_{eb}L^e_{da} + L^e_{ab}L^c_{ed} - F^\alpha_{c+m\,a}F^\alpha_{d+m\,b} \\
&\qquad -F^\mu_{c+m\,a}F^\mu_{d+m\,b} + \delta_{ab}\delta_{cd} - \delta_{bc}\delta_{ad})\omega^{d+m} \\
&+(-L^c_{ab\alpha} + L^c_{db}F^\alpha_{d+m\,a} - 2F^\mu_{a+m\,b}F^\mu_{\alpha\,c+m} - 2F^\mu_{c+m\,a}F^\mu_{\alpha b})\omega^\alpha \\
&+(-L^c_{ab\mu} + L^c_{db}F^\mu_{d+m\,a} + 2F^\alpha_{a+m\,b}F^\mu_{\alpha\,c+m} + 2F^\alpha_{c+m\,a}F^\mu_{\alpha b})\omega^\mu.
\end{aligned}$$

We now verify that zero is the coefficient of each of ω^d, ω^{d+m}, ω^α, ω^μ.

The coefficient of ω^d can be seen to be zero by taking (8.51) (with indices in the order c, a, b, d) and subtracting half of (8.55) (with indices as is).

The coefficient of ω^{d+m} can be seen to be zero by using (8.60), then adding (8.51) (with indices in the order c, a, b, d), then adding half of (8.55) (with indices as is), and then using (8.55) again (with the roles of d and c reversed).

The coefficient of ω^α can be seen to be zero from (8.53) and (8.37).

The coefficient of ω^μ is zero by (8.54).

Hence, we have shown that the coefficient of e_{c+m} in $(dQ_a)e_b$ is as given in (9.20).

Using (5.2), (8.4) and (9.10), we can rewrite the coefficient of e_α in $(dQ_a)e_b$ (9.21) as

$$\begin{aligned}
F^\alpha_{c+m\,b}\nu^c_a &+ (F^\alpha_{a+m\,bc} + L^b_{ad}F^\alpha_{d+m\,c} - F^\mu_{a+m\,b}F^\mu_{\alpha c} - F^\alpha_{d+m\,b}L^d_{ac})\omega^c \\
&+ (F^\alpha_{a+m\,b\,c+m} - F^\mu_{a+m\,b}F^\mu_{\alpha\,c+m} - 2F^\mu_{\alpha a}F^\mu_{c+m\,b})\omega^{c+m} \\
&+ (F^\alpha_{a+m\,b\beta} + \delta_{ab}\delta_{\alpha\beta} - F^\alpha_{c+m\,a}F^\beta_{c+m\,b} - 4F^\mu_{\alpha a}F^\mu_{\beta b} - F^\alpha_{c+m\,b}F^\beta_{c+m\,a})\omega^\beta \\
&+ (F^\alpha_{a+m\,b\mu} - 2L^b_{ac}F^\mu_{\alpha c} + F^\alpha_{c+m\,a}F^\mu_{c+m\,b} - F^\alpha_{c+m\,b}F^\mu_{c+m\,a})\omega^\mu.
\end{aligned}$$

We see the coefficient of ω^c as zero by using the first equation in (5.7) and then using (8.37). The coefficient of ω^{c+m} is zero by the second equation in (5.7). The coefficient of ω^β is zero by the third equation in (5.7) and then by the second equation in (5.6). The coefficient of ω^μ is seen to be zero by use of (5.11) and then by (8.39) (with the roles of a and b interchanged).

Using (5.2) and (9.10), we can rewrite the coefficient of e_μ in $(dQ_a)e_b$ (9.21) as

$$\begin{aligned}
F^\mu_{c+m\,b}\nu^c_a &+ (F^\mu_{a+m\,bc} + L^b_{ad}F^\mu_{d+m\,c} + F^\alpha_{a+m\,b}F^\mu_{\alpha c} - F^\mu_{d+m\,b}L^d_{ac})\omega^c \\
&+ (F^\mu_{a+m\,b\,c+m} + F^\alpha_{a+m\,b}F^\mu_{\alpha\,c+m} + 2F^\mu_{\alpha a}F^\alpha_{c+m\,b})\omega^{c+m} \\
&+ (F^\mu_{a+m\,b\alpha} + 2L^b_{ac}F^\mu_{\alpha\,c+m} + F^\mu_{c+m\,a}F^\alpha_{c+m\,b} - F^\mu_{c+m\,b}F^\alpha_{c+m\,a})\omega^\alpha \\
&+ (F^\mu_{a+m\,b\nu} + \delta_{ab}\delta_{\mu\nu} - F^\mu_{c+m\,a}F^\nu_{c+m\,b} - 4F^\mu_{\alpha a}F^\nu_{\alpha b} - F^\mu_{c+m\,b}F^\nu_{c+m\,a})\omega^\nu.
\end{aligned}$$

The coefficient of ω^c is seen to be zero by use of the first equation in (5.8) and then by (8.38). The coefficient of ω^{c+m} is zero by the second equation in (5.8). The coefficient of ω^α is zero by (8.39). The coefficient of ω^μ is seen to be zero by use of the third equation in (5.8) and then by the fourth equation in (5.6). This completes the verification of (9.20).

The next case is to verify (9.12) when both sides are applied to e_{b+m}. We must verify that

$$\begin{aligned} d(Q_a e_{b+m}) - Q_a de_{b+m} = (dQ_a)e_{b+m} &= \nu_a^0 Q_0 e_{b+m} + \nu_a^c Q_c e_{b+m} \\ &= \nu^a e_b + \nu_a^b e_0 + L_{db}^c \nu_a^d e_c + F_{b+m\,c}^\alpha \nu_a^c e_\alpha - F_{b+m\,c}^\mu \nu_a^c e_\mu \end{aligned} \tag{9.22}$$

by (9.4). Using (9.4) to compute $Q_a e_{b+m}$ and (4.20) to compute de_{b+m}, we see that the left hand side becomes

$$\begin{aligned} &(-F_{b+m\,a}^\alpha \omega^\alpha + F_{b+m\,a}^\mu \omega^\mu - \theta_{b+m}^a)\mathbf{x} \\ &\quad + (-L_{ab}^c \omega^c + F_{b+m\,a}^\alpha \omega^\alpha + F_{b+m\,a}^\mu \omega^\mu - \theta_{b+m}^{a+m})e_0 \\ &\quad + (\delta_{ab}\omega^c + dL_{ab}^c + L_{ab}^d \theta_d^c + F_{b+m\,a}^\alpha \theta_\alpha^c - F_{b+m\,a}^\mu \theta_\mu^c \\ &\qquad + \delta_{ac}\omega^{b+m} - L_{ad}^c \theta_{b+m}^{d+m} - F_{a+m\,c}^\alpha \theta_{b+m}^\alpha - F_{a+m\,c}^\mu \theta_{b+m}^\mu)e_c \\ &\quad + (L_{ab}^c \theta_c^{d+m} + F_{b+m\,a}^\alpha \theta_\alpha^{d+m} - F_{b+m\,a}^\mu \theta_\mu^{d+m} - L_{ad}^c \theta_{b+m}^c \\ &\qquad - F_{d+m\,a}^\alpha \theta_{b+m}^\alpha + F_{d+m\,a}^\mu \theta_{b+m}^\mu)e_{d+m} \\ &\quad + (-\delta_{ab}\omega^\alpha + L_{ab}^c \theta_c^\alpha + dF_{b+m\,a}^\alpha + F_{b+m\,a}^\beta \theta_\beta^\alpha - F_{b+m\,a}^\mu \theta_\mu^\alpha \\ &\qquad - F_{a+m\,c}^\alpha \theta_{b+m}^c - F_{c+m\,a}^\alpha \theta_{b+m}^{c+m} + 2F_{\alpha a}^\mu \theta_{b+m}^\mu)e_\alpha \\ &\quad + (\delta_{ab}\omega^\mu + L_{ab}^c \theta_c^\mu + F_{b+m\,a}^\alpha \theta_\alpha^\mu - dF_{b+m\,a}^\mu - F_{b+m\,a}^\nu \theta_\nu^\mu \\ &\qquad - F_{a+m\,c}^\mu \theta_{b+m}^c + F_{c+m\,a}^\mu \theta_{b+m}^{c+m} + 2F_{\alpha a}^\mu \theta_{b+m}^\alpha)e_\mu. \end{aligned} \tag{9.23}$$

We want to verify that this is equal to the right side of (9.22), where ν_a^c is given by (9.10). We do this by comparing the coefficients of the basis vectors $\mathbf{x}, e_0, e_c, e_{c+m}, e_\alpha, e_\mu$.

The coefficient of $\mathbf{x}$ is 0 by (4.18).

The coefficient of e_0 is

$$\theta_a^b + L_{ca}^b \omega^{c+m} + F_{b+m\,a}^\alpha \omega^\alpha + F_{b+m\,a}^\mu \omega^\mu - L_{ca}^b(\omega^c + \omega^{c+m}) + (\theta_{a+m}^{b+m} - \theta_a^b) = \nu_a^b$$

by (8.4), (9.8) and (9.10).

The coefficient of e_c in $(dQ_a)e_{b+m}$ in (9.23) is, by (4.18) and (8.50) and the skew-symmetry of L_{bcd}^a in all four indices,

$$\begin{aligned} &L_{db}^c(\theta_a^d + L_{ea}^d \omega^{e+m} + F_{d+m\,a}^\alpha \omega^\alpha + F_{d+m\,a}^\mu \omega^\mu) + (\omega^a + \omega^{a+m})\delta_{bc} \\ &\quad +(L_{bcd}^a + \delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc} + L_{ae}^c L_{bd}^e - F_{a+m\,c}^\alpha F_{b+m\,d}^\alpha - F_{a+m\,c}^\mu F_{b+m\,d}^\mu)\omega^d \\ &\quad +(L_{ab\,d+m}^c + \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} + L_{eb}^c L_{ad}^e + L_{ae}^c L_{bd}^e \\ &\qquad -F_{b+m\,a}^\alpha F_{d+m\,c}^\alpha - F_{b+m\,a}^\mu F_{d+m\,c}^\mu)\omega^{d+m} \end{aligned}$$

$$+(L^c_{ab\alpha} - L^c_{eb}F^\alpha_{e+m\,a} - 2F^\mu_{b+m\,a}F^\mu_{\alpha c} - 2F^\mu_{a+m\,c}F^\mu_{\alpha b+m})\omega^\alpha$$

$$+(L^c_{ab\mu} - L^c_{eb}F^\mu_{e+m\,a} + 2F^\alpha_{b+m\,a}F^\mu_{\alpha c} + 2F^\alpha_{a+m\,c}F^\mu_{\alpha b+m})\omega^\mu$$

$$= L^c_{db}\nu^d_a + \nu^a\delta_{bc}$$

by (9.8) and (9.10) and the following. By (8.45) for L^a_{bcd}, the coefficient of ω^d becomes

$$\delta_{ab}\delta_{cd} - \frac{1}{2}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \frac{1}{2}(L^c_{ae}L^e_{bd} - L^a_{de}L^e_{bc}) + F^\mu_{c+m\,a}F^\mu_{b+m\,d} - F^\mu_{c+m\,b}F^\mu_{d+m\,a}$$

which is zero by (8.49) combined with the first equation of (5.6). In the coefficient of ω^{d+m}, substitute (8.46) for $L^a_{bc\,d+m} = L^c_{ab\,d+m}$, and gather together terms using skew-symmetries, to get

$$\frac{3}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + L^a_{be}L^d_{ce} + \frac{1}{2}L^a_{de}L^e_{bc} + \frac{1}{2}L^a_{ce}L^d_{be} - F^\alpha_{a+m\,d}F^\alpha_{c+m\,b} - F^\alpha_{a+m\,b}F^\alpha_{c+m\,d} + F^\mu_{d+m\,b}F^\mu_{a+m\,c} + F^\mu_{a+m\,b}F^\mu_{d+m\,c}$$

which is zero by the first equation in (5.6) and then by (8.49). The coefficient of ω^α is zero by (8.47). The coefficient of ω^μ is zero by (8.48).

The coefficient of e_{d+m} in $(dQ_a)e_{b+m}$ in (9.23) is, by (4.18),

$$(-F^\alpha_{b+m\,a}F^\alpha_{d+m\,c} + F^\mu_{b+m\,a}F^\mu_{d+m\,c} - F^\alpha_{d+m\,a}F^\alpha_{b+m\,c} + F^\mu_{d+m\,a}F^\mu_{b+m\,c})\omega^c$$

$$+(L^c_{ba}F^\alpha_{c+m\,d} - 2F^\mu_{a+m\,b}F^\mu_{\alpha d} + L^c_{da}F^\alpha_{c+m\,b} - 2F^\mu_{a+m\,d}F^\mu_{\alpha\,b+m})\omega^\alpha$$

$$+(-L^c_{ba}F^\mu_{c+m\,d} + 2F^\alpha_{b+m\,a}F^\mu_{\alpha\,d+m} - L^c_{da}F^\mu_{c+m\,b} + 2F^\alpha_{d+m\,a}F^\mu_{\alpha\,b+m})\omega^\mu = 0$$

because the coefficient of ω^c is 0 by the first equation in (5.6), the coefficient of ω^α is 0 by (8.37) and (8.1), and the coefficient of ω^μ is 0 by (8.38) and (8.1).

The coefficient of e_α in $(dQ_a)e_{b+m}$ in (9.23) is, by (4.18) and (5.2),

$$F^\alpha_{b+m\,c}(\theta^c_a + L^c_{da}\omega^{d+m} + F^\beta_{c+m\,a}\omega^\beta + F^\mu_{c+m\,a}\omega^\mu)$$

$$+(F^\alpha_{b+m\,ac} + F^\mu_{b+m\,a}F^\mu_{\alpha c} + 2F^\mu_{\alpha a}F^\mu_{b+m\,c})\omega^c$$

$$+(F^\alpha_{b+m\,a\,d+m} + L^c_{ba}F^\alpha_{c+m\,d} + L^c_{da}F^\alpha_{c+m\,b} + F^\mu_{b+m\,a}F^\mu_{\alpha\,d+m})\omega^{d+m}$$

$$+(-\delta_{ab}\delta_{\alpha\beta} + F^\alpha_{b+m\,a\beta} + F^\alpha_{a+m\,c}F^\beta_{b+m\,c} - F^\alpha_{b+m\,c}F^\beta_{c+m\,a} + 4F^\mu_{\alpha a}F^\mu_{\beta\,b+m})\omega^\beta$$

$$+(-2L^c_{ab}F^\mu_{\alpha c} + F^\alpha_{b+m\,a\mu} - F^\alpha_{a+m\,c}F^\mu_{b+m\,c} - F^\alpha_{b+m\,c}F^\mu_{c+m\,a})\omega^\mu$$

$$= F^\alpha_{b+m\,c}\nu^c_a$$

by (9.10), because the other terms are zero as follows. The coefficient of ω^c is zero by the first equation in (5.7). The coefficient of ω^{d+m} is zero by the second equation in (5.7) and (8.37). The coefficient of ω^β is zero by the third equation of (5.7) and the third equation of (5.6). The coefficient of ω^μ is zero by the third equation in (5.11) and (8.39).

Finally, the coefficient of e_μ in $(dQ_a)e_{b+m}$ from (9.23) is, by (4.18) and (5.2)

$$\begin{aligned}
&- F^\mu_{b+m\,c}(\theta^c_a - L^c_{ad}\omega^{d+m} + F^\alpha_{c+m\,a}\omega^\alpha + F^\nu_{c+m\,a}\omega^\nu)\\
&+ (F^\alpha_{b+m\,a}F^\mu_{\alpha c} - F^\mu_{b+m\,ac} + 2F^\mu_{\alpha a}F^\alpha_{b+m\,c})\omega^c\\
&+ (-L^c_{ab}F^\mu_{d+m\,c} - L^c_{ad}F^\mu_{b+m\,c} + F^\alpha_{b+m\,a}F^\mu_{\alpha\,d+m} - F^\mu_{b+m\,a\,d+m})\omega^{d+m}\\
&+ (-2L^c_{ab}F^\mu_{\alpha c} - F^\mu_{b+m\,a\alpha} + F^\mu_{a+m\,c}F^\alpha_{b+m\,c} + F^\mu_{b+m\,c}F^\alpha_{c+m\,a})\omega^\alpha\\
&+ (\delta_{ab}\delta_{\mu\nu} - F^\mu_{b+m\,a\nu} - F^\mu_{a+m\,c}F^\nu_{b+m\,c} + F^\mu_{b+m\,c}F^\nu_{c+m\,a} - 4F^\mu_{\alpha a}F^\nu_{\alpha\,b+m})\omega^\mu\\
=&- F^\mu_{b+m\,c}\nu^c_a
\end{aligned}$$

and by (9.10), because the other terms are zero as follows. The coefficient of ω^c is zero by the first equation in (5.8). The coefficient of ω^{d+m} is zero by the second equation in (5.8) and by (8.38). The coefficient of ω^α is zero by (5.11) and (8.39). The coefficient of ω^ν is zero by the fourth equation in (5.6).

That concludes the verification of (9.22).

The next case is to verify (9.12) when both sides are applied to e_α. We must verify that

$$\begin{aligned}
d(Q_a e_\alpha) - Q_a de_\alpha = (dQ_a)e_\alpha &= \nu^0_a Q_0 e_\alpha + \nu^c_a Q_c e_\alpha\\
&= \nu^a e_\alpha + F^\alpha_{b+m\,c}\nu^b_a e_c + F^\alpha_{c+m\,b}\nu^b_a e_{c+m} - 2F^\mu_{\alpha b}\nu^b_a e_\mu.
\end{aligned} \tag{9.24}$$

Using (9.4) and (4.20), and gathering together the coefficients of each basis vector, we get

$$\begin{aligned}
(dQ_a)e_\alpha = &(-F^\alpha_{b+m\,a}\omega^{b+m} + 2F^\mu_{\alpha a}\omega^\mu - \theta^a_\alpha)\mathbf{x}\\
&+ (-F^\alpha_{a+m\,b}\omega^b + 2F^\mu_{\alpha a}\omega^\mu - \theta^{a+m}_\alpha)e_0\\
&+ (dF^\alpha_{a+m\,c} + F^\alpha_{a+m\,b}\theta^c_b + F^\alpha_{b+m\,a}\theta^c_{b+m} - 2F^\mu_{\alpha a}\theta^c_\mu\\
&\qquad + \delta_{ac}\omega^\alpha - L^c_{ab}\theta^{b+m}_\alpha - F^\beta_{a+m\,c}\theta^\beta_\alpha - F^\mu_{a+m\,c}\theta^\mu_\alpha)e_c\\
&+ (F^\alpha_{a+m\,b}\theta^{c+m}_b + dF^\alpha_{c+m\,a} + F^\alpha_{b+m\,a}\theta^{c+m}_{b+m} - 2F^\mu_{\alpha a}\theta^{c+m}_\mu\\
&\qquad - \delta_{ac}\omega^\alpha + L^c_{ab}\theta^b_\alpha - F^\beta_{c+m\,a}\theta^\beta_\alpha + F^\mu_{c+m\,a}\theta^\mu_\alpha)e_{c+m}\\
&+ (F^\alpha_{a+m\,b}\theta^\beta_b + F^\alpha_{b+m\,a}\theta^\beta_{b+m} - 2F^\mu_{\alpha a}\theta^\beta_\mu - F^\beta_{a+m\,b}\theta^b_\alpha\\
&\qquad - F^\beta_{b+m\,a}\theta^{b+m}_\alpha + 2F^\mu_{\beta a}\theta^\mu_\alpha)e_\beta\\
&+ (F^\alpha_{a+m\,b}\theta^\mu_b + F^\alpha_{b+m\,a}\theta^\mu_{b+m} - 2F^\nu_{\alpha a}\theta^\mu_\nu - 2dF^\mu_{\alpha a}\\
&\qquad - F^\mu_{a+m\,b}\theta^b_\alpha + F^\mu_{b+m\,a}\theta^{b+m}_\alpha + 2F^\mu_{\beta a}\theta^\beta_\alpha)e_\mu.
\end{aligned} \tag{9.25}$$

The coefficient of $\mathbf{x}$ is zero and the coefficient of e_0 is zero, both by (4.18). For the coefficient of e_c, use (4.18), (5.2) and (8.4), and add and subtract appropriate terms, to rewrite it as

$$\begin{aligned} & F^\alpha_{b+m\,c}(\theta^b_a - L^b_{ad}\omega^{d+m} + F^\beta_{b+m\,a}\omega^\beta + F^\mu_{b+m\,a}\omega^\mu) \\ &\quad +(F^\alpha_{a+m\,cd} - F^\alpha_{b+m\,c}L^b_{ad} + F^\alpha_{b+m\,d}L^c_{ab} - F^\mu_{a+m\,c}F^\mu_{\alpha d})\omega^d \\ &\quad +(F^\alpha_{a+m\,c\,d+m} - 2F^\mu_{\alpha\,a+m}F^\mu_{d+m\,c} - F^\mu_{a+m\,c}F^\mu_{\alpha\,d+m})\omega^{d+m} \\ &\quad +(-F^\alpha_{b+m\,c}F^\beta_{b+m\,a} - F^\alpha_{b+m\,a}F^\beta_{b+m\,c} + F^\alpha_{a+m\,c\beta} - 4F^\mu_{\alpha a}F^\mu_{\beta c} + \delta_{ac}\delta_{\alpha\beta})\omega^\beta \\ &\quad +(-F^\alpha_{b+m\,c}F^\mu_{b+m\,a} + F^\alpha_{b+m\,a}F^\mu_{b+m\,c} + F^\alpha_{a+m\,c\mu} - 2L^c_{ab}F^\mu_{\alpha\,b+m})\omega^\mu \\ &= F^\alpha_{b+m\,c}\nu^b_a \end{aligned}$$

by (9.10), because the other terms are zero as follows. The coefficient of ω^d is zero by the first equation of (5.7) and (8.37). The coefficient of ω^{d+m} is zero by the second equation of (5.7). The coefficient of ω^β is zero by the third equation of (5.7) and then the second equation of (5.6). The coefficient of ω^μ is zero by (5.11) and then (8.39).

The coefficient of e_{c+m} in $(dQ_a)e_\alpha$ from (9.25) is, by (4.18),

$$\begin{aligned} & F^\alpha_{c+m\,b}(\theta^b_a + L^b_{da}\omega^{d+m} + F^\beta_{b+m\,a}\omega^\beta + F^\mu_{b+m\,a}\omega^\mu) \\ &\quad +(F^\alpha_{c+m\,ad} + 2F^\mu_{\alpha a}F^\mu_{c+m\,d} + F^\mu_{c+m\,a}F^\mu_{\alpha d})\omega^d \\ &\quad +(F^\alpha_{c+m\,a\,d+m} - L^c_{ab}F^\alpha_{d+m\,b} + F^\mu_{c+m\,a}F^\mu_{\alpha\,d+m} - L^b_{da}F^\alpha_{c+m\,b})\omega^{d+m} \\ &\quad +(F^\alpha_{c+m\,a\beta} + F^\alpha_{a+m\,b}F^\beta_{c+m\,b} + 4F^\mu_{\alpha a}F^\mu_{\beta\,c+m} - \delta_{ac}\delta_{\alpha\beta} - F^\alpha_{c+m\,b}F^\beta_{b+m\,a})\omega^\beta \\ &\quad +(F^\alpha_{c+m\,a\mu} - F^\alpha_{a+m\,b}F^\mu_{c+m\,b} + 2L^c_{ab}F^\mu_{\alpha b} - F^\alpha_{c+m\,b}F^\mu_{b+m\,a})\omega^\mu \\ &= F^\alpha_{c+m\,b}\nu^b_a, \end{aligned}$$

by (9.10), because the other terms are zero as follows. The coefficient of ω^d is zero by the first equation in (5.7). The coefficient of ω^{d+m} is zero by the second equation in (5.7) and then (8.37). The coefficient of ω^β is zero by the third equation in (5.7) and then the second equation in (5.6). The coefficient of ω^μ is zero by (5.11) and then (8.39).

The coefficient of e_β in $(dQ_a)e_\alpha$ in (9.25) is, by (4.18),

$$\begin{aligned} & (F^\alpha_{b+m\,a}F^\beta_{b+m\,c} + 2F^\mu_{\alpha a}F^\mu_{\beta c} + 2F^\mu_{\beta a}F^\mu_{\alpha c} + F^\beta_{b+m\,a}F^\alpha_{b+m\,c})\omega^c \\ &\quad +(F^\alpha_{a+m\,b}F^\beta_{c+m\,b} + 2F^\mu_{\alpha a}F^\mu_{\beta\,c+m} + 2F^\mu_{\beta a}F^\mu_{\alpha\,c+m} + F^\beta_{a+m\,b}F^\alpha_{c+m\,b})\omega^{c+m} \\ &\quad -2(F^\alpha_{a+m\,b}F^\mu_{\beta b} + F^\alpha_{b+m\,a}F^\mu_{\beta\,b+m} + F^\beta_{a+m\,b}F^\mu_{\alpha b} + F^\beta_{b+m\,a}F^\mu_{\alpha\,b+m})\omega^\mu \\ &= \delta_{\alpha\beta}(\omega^a + \omega^{a+m}) = \delta_{\alpha\beta}\nu^a \end{aligned}$$

by (9.8), because the coefficient of ω^c is $\delta_{\alpha\beta}\delta_{ac}$ by the second equation of (5.6), and the coefficient of ω^{c+m} is also $\delta_{\alpha\beta}\delta_{ac}$ by (8.1), (8.2) and the second equation of (5.6); and the coefficient of ω^μ is zero by (8.1) and (8.2).

The coefficient of e_μ from $(dQ_a)e_\alpha$ in (9.25) is, by (4.18) and (5.2),

$$\begin{aligned}
&-2F^\mu_{\alpha b}(\theta^b_a + L^b_{ca}\omega^{c+m} + F^\beta_{b+m\,a}\omega^\beta + F^\nu_{b+m\,a}\omega^\nu)\\
&+(-2F^\mu_{\alpha ac} + F^\alpha_{b+m\,a}F^\mu_{b+m\,c} - F^\mu_{b+m\,a}F^\alpha_{b+m\,c})\omega^c\\
&+(-2F^\mu_{\alpha a\,c+m} + 2F^\mu_{\alpha b}L^b_{ca} - F^\alpha_{a+m\,b}F^\mu_{c+m\,b} + F^\mu_{a+m\,b}F^\alpha_{c+m\,b})\omega^{c+m}\\
&+2(-F^\mu_{\alpha a\beta} + F^\beta_{b+m\,a}F^\mu_{\alpha b} - F^\alpha_{a+m\,b}F^\mu_{\beta b} + F^\alpha_{b+m\,a}F^\mu_{\beta\,b+m})\omega^\beta\\
&+2(-F^\mu_{\alpha a\nu} + F^\mu_{\alpha b}F^\nu_{b+m\,a} - F^\mu_{a+m\,b}F^\nu_{\alpha b} + F^\mu_{b+m\,a}F^\nu_{\alpha\,b+m})\omega^\nu\\
=&-2F^\mu_{\alpha b}\nu^b_a,
\end{aligned}$$

by (9.8), because the coefficient of ω^c is zero by the first equation in (5.9), the coefficient of ω^{c+m} is zero by (5.11) and then (8.39), the coefficient of ω^β is zero by (8.1), (8.2) and the second equation in (5.9); and the coefficient of ω^ν is zero by (8.1), (8.3) and the third equation in (5.9).

This completes the verification of (9.24), which verifies that (9.12) holds when both sides are applied to e_α.

The final case is to verify (9.12) when both sides are applied to e_μ. We must verify that

$$\begin{aligned}
d(Q_a e_\mu) - Q_a(de_\mu) &= (dQ_a)e_\mu = \nu^0_a Q_0 e_\mu + \nu^b_a Q_b e_\mu\\
&= -\nu^a e_\mu + F^\mu_{b+m\,c}\nu^b_a e_c - F^\mu_{c+m\,b}\nu^b_a e_{c+m} - 2F^\mu_{\alpha b}\nu^b_a e_\alpha.
\end{aligned} \tag{9.26}$$

Using (9.4) and (4.20), and gathering together the coefficients of each basis vector, we get for the left hand side

$$\begin{aligned}
(dQ_a)e_\mu = \;&(F^\mu_{b+m\,a}\omega^{b+m} + 2F^\mu_{\alpha a}\omega^\alpha - \theta^a_\mu)\mathbf{x}\\
&-(F^\mu_{a+m\,b}\omega^b + 2F^\mu_{\alpha a}\omega^\alpha + \theta^{a+m}_\mu)e_0\\
&+(dF^\mu_{a+m\,c} + F^\mu_{a+m\,b}\theta^c_b - F^\mu_{b+m\,a}\theta^c_{b+m} - 2F^\mu_{\alpha a}\theta^c_\alpha + \delta_{ac}\omega^\mu\\
&\quad -L^c_{ab}\theta^{b+m}_\mu - F^\alpha_{a+m\,c}\theta^\alpha_\mu - F^\nu_{a+m\,c}\theta^\nu_\mu)e_c\\
&+(F^\mu_{a+m\,b}\theta^{c+m}_b - dF^\mu_{c+m\,a} - F^\mu_{b+m\,a}\theta^{c+m}_{b+m} - 2F^\mu_{\alpha a}\theta^{c+m}_\alpha + \delta_{ac}\omega^\mu\\
&\quad +L^c_{ab}\theta^b_\mu - F^\alpha_{c+m\,a}\theta^\alpha_\mu + F^\nu_{c+m\,a}\theta^\nu_\mu)e_{c+m}\\
&+(F^\mu_{a+m\,b}\theta^\alpha_b - F^\mu_{b+m\,a}\theta^\alpha_{b+m} - 2dF^\mu_{\alpha a} - 2F^\mu_{\beta a}\theta^\alpha_\beta - F^\alpha_{a+m\,b}\theta^b_\mu\\
&\quad -F^\alpha_{b+m\,a}\theta^{b+m}_\mu + 2F^\nu_{\alpha a}\theta^\nu_\mu)e_\alpha\\
&+(F^\mu_{a+m\,b}\theta^\nu_b - F^\mu_{b+m\,a}\theta^\nu_{b+m} - 2F^\mu_{\alpha a}\theta^\nu_\alpha\\
&\quad -F^\nu_{a+m\,b}\theta^b_\mu + F^\nu_{b+m\,a}\theta^{b+m}_\mu + 2F^\nu_{\alpha a}\theta^\alpha_\mu)e_\nu.
\end{aligned} \tag{9.27}$$

The coefficient of $\mathbf{x}$ is zero by (4.18). The coefficient of e_0 is zero by (4.18) and (8.1).

After applying (5.2) and (4.18) and adding and then subtracting some terms in the definition of ν^b_a in (9.10), we can rewrite the coefficient of e_c as

$$\begin{aligned}
&F^\mu_{b+m\,c}(\theta^b_a + L^b_{da}\omega^{d+m} + F^\beta_{b+m\,a}\omega^\beta + F^\nu_{b+m\,a}\omega^\nu)\\
&\quad +(F^\mu_{a+m\,cd} - F^\mu_{b+m\,c}L^b_{ad} + L^c_{ab}F^\mu_{b+m\,d} + F^\alpha_{a+m\,c}F^\mu_{\alpha d})\omega^d\\
&\quad +(F^\mu_{a+m\,c\,d+m} + 2F^\mu_{\alpha a}F^\alpha_{d+m\,c} + F^\alpha_{a+m\,c}F^\mu_{\alpha\,d+m})\omega^{d+m}\\
&\quad +(F^\mu_{a+m\,c\alpha} + F^\mu_{b+m\,a}F^\alpha_{b+m\,c} + 2L^c_{ab}F^\mu_{\alpha\,b+m} - F^\mu_{b+m\,c}F^\alpha_{b+m\,a})\omega^\alpha\\
&\quad +(F^\mu_{a+m\,c\nu} - F^\mu_{b+m\,a}F^\nu_{b+m\,c} - 4F^\mu_{\alpha a}F^\nu_{\alpha c} + \delta_{ac}\delta_{\mu\nu} - F^\mu_{b+m\,c}F^\nu_{b+m\,a})\omega^\nu\\
&= F^\mu_{b+m\,c}\nu^b_a
\end{aligned}$$

by (9.10) and the following. The coefficient of ω^d is zero by the first equation in (5.8) and then (8.38). The coefficient of ω^{d+m} is zero by (8.1) and (5.8). The coefficient of ω^α is zero by (5.11) and then (8.39). The coefficient of ω^μ is zero by the third equation in (5.8) and then (8.1) and the fourth equation in (5.6).

Using (5.2) and (4.18), we can rewrite the coefficient of e_{c+m} in $(dQ_a)e_\mu$ from (9.27) as

$$\begin{aligned}
&-F^\mu_{c+m\,b}(\theta^b_a + L^b_{da}\omega^{d+m} + F^\alpha_{b+m\,a}\omega^\alpha + F^\nu_{b+m\,a}\omega^\nu)\\
&+(-F^\mu_{c+m\,ab} + 2F^\mu_{\alpha a}F^\alpha_{c+m\,b} + F^\alpha_{c+m\,a}F^\mu_{\alpha b})\omega^b\\
&+(L^b_{da}F^\mu_{c+m\,b} - F^\mu_{c+m\,a\,d+m} + L^c_{ab}F^\mu_{d+m\,b} + F^\alpha_{c+m\,a}F^\mu_{\alpha\,d+m})\omega^{d+m}\\
&+(F^\mu_{c+m\,b}F^\alpha_{b+m\,a} - F^\mu_{c+m\,a\alpha} + F^\mu_{a+m\,b}F^\alpha_{c+m\,b} + 2F^\mu_{\alpha b}L^c_{ab})\omega^\alpha\\
&+(F^\mu_{c+m\,b}F^\nu_{b+m\,a} - F^\mu_{c+m\,a\nu} - F^\mu_{a+m\,b}F^\nu_{c+m\,b} - 4F^\mu_{\alpha a}F^\nu_{\alpha\,c+m} + \delta_{ac}\delta_{\mu\nu})\omega^\nu\\
&= -F^\mu_{c+m\,b}\nu^b_a,
\end{aligned}$$

by (9.10) and the following. The coefficient of ω^b is zero by the first equation in (5.8). The coefficient of ω^{d+m} is zero by the second equation in (5.8) and then (8.1) and (8.38). The coefficient of ω^α is zero by (5.11), then (8.2) and (8.3) and (8.39). The coefficient of ω^ν is zero by the third equation in (5.8), then (8.1) and (8.3) and then the fourth equation in (5.6).

Using (5.2) and (4.18), we can rewrite the coefficient of e_α in $(dQ_a)e_\mu$ in (9.27) as

$$\begin{aligned}
&-2F^\mu_{\alpha b}(\theta^b_a + L^b_{ca}\omega^{c+m} + F^\beta_{b+m\,a}\omega^\beta + F^\nu_{b+m\,a}\omega^\nu)\\
&\quad +(-2F^\mu_{\alpha ac} - F^\mu_{b+m\,a}F^\alpha_{b+m\,c} + F^\alpha_{b+m\,a}F^\mu_{b+m\,c})\omega^c\\
&\quad +(-2F^\mu_{\alpha a\,c+m} + 2F^\mu_{\alpha b}L^b_{ca} + F^\mu_{a+m\,b}F^\alpha_{c+m\,b} - F^\mu_{c+m\,b}F^\alpha_{a+m\,b})\omega^{c+m}\\
&\quad +2(-F^\mu_{\alpha a\beta} + F^\mu_{\alpha b}F^\beta_{b+m\,a} - F^\mu_{\beta b}F^\alpha_{a+m\,b} + F^\mu_{\beta\,b+m}F^\alpha_{b+m\,a})\omega^\beta\\
&\quad +2(-F^\mu_{\alpha a\nu} + F^\mu_{\alpha b}F^\nu_{b+m\,a} - F^\mu_{a+m\,b}F^\nu_{\alpha b} + F^\mu_{b+m\,a}F^\nu_{\alpha\,b+m})\omega^\nu\\
&= -2F^\mu_{\alpha b}\nu^b_a,
\end{aligned}$$

by (9.10) and the following. The coefficient of ω^c is zero by the first equation in (5.9). The coefficient of ω^{c+m} is zero by (5.11), then (8.2) and (8.3) and then (8.39). The coefficient of ω^β is zero by the second equation in (5.9) and then (8.1)-(8.3). The coefficient of ω^ν is zero by the third equation in (5.9), then (8.1) and (8.3).

Using (4.18), we can rewrite the coefficient of e_μ in $(dQ_a)e_\mu$ in (9.27) as

$$\begin{aligned}
&(-F^\mu_{b+m\,a}F^\nu_{b+m\,c} - 2F^\mu_{\alpha a}F^\nu_{\alpha c} - F^\nu_{b+m\,a}F^\mu_{b+m\,c} - 2F^\nu_{\alpha a}F^\mu_{\alpha c})\omega^c \\
&\quad +(-F^\mu_{a+m\,b}F^\nu_{c+m\,b} - 2F^\mu_{\alpha a}F^\nu_{\alpha\,c+m} - F^\nu_{a+m\,b}F^\mu_{c+m\,b} - 2F^\nu_{\alpha a}F^\mu_{\alpha\,c+m})\omega^{c+m} \\
&\quad +2(-F^\mu_{a+m\,b}F^\nu_{\alpha b} - F^\mu_{b+m\,a}F^\nu_{\alpha\,b+m} - F^\nu_{a+m\,b}F^\mu_{\alpha b} - F^\nu_{b+m\,a}F^\mu_{\alpha\,b+m})\omega^\alpha \\
&= -\delta_{\mu\nu}(\omega^a + \omega^{a+m}) = -\delta_{\mu\nu}\nu^a
\end{aligned}$$

by (8.1), (8.3) and the fourth equation in (5.6). This completes the verification of (9.12) when both sides are applied to e_μ, and therefore also completes the verification of (9.12). □

10. The quadratic forms

For the remainder of the paper, we will again refer to the two multiplicities as m_1 and m_2, rather than m and N, respectively, and we will no longer use the Einstein summation convention. Our task now is to solve (8.1) through (8.4). It is known that $m_1 = m_2$ only when $m_1 = m_2 = 1$, which is of FKM-type, or $m_1 = m_2 = 2$, which is not of FKM-type [1]. Therefore we assume $m_1 \neq m_2$ henceforth. Our convention is that $m_1 < m_2$ and we denote by M_+ (respectively, M_-) the focal submanifold whose co-dimension is $m_1 + 1$ (respectively, $m_2 + 1$) in the ambient sphere. We change the Cartan-Münzner polynomial F to $-F$ if necessary so that always $M_+ = f^{-1}(1)$ with respect to the isoparametric function f. In view of Theorem 24, we look for conditions on m_1 and m_2 that imply the validity of (8.1) and the spanning property.

As in Section 4 let $\mathbf{x} \in M_+$ and let e_0 be a unit normal vector to M_+ at $\mathbf{x}$ for which the shape operator S_{e_0} assumes the eigenspaces V_0, V_+ and V_- with eigenvalues $0, 1$, and -1, respectively. For an orthonormal basis $e_0, \ldots, e_{m_1}$ of the normal space to M_+ at $\mathbf{x}$ we introduce the quadratic homogeneous polynomials

$$\check{p}_i(\mathbf{y}) := S_{e_i}\mathbf{y} \cdot \mathbf{y}$$

for $0 \le i \le m_1$, where $\mathbf{y}$ is tangent to M_+ at $\mathbf{x}$. When such $\mathbf{y}$ has no V_0 component, we shall write $\mathbf{z}$ instead of $\mathbf{y}$. Regarding $V_+ \oplus V_-$ as a subspace of $\mathbf{R}^{2l}$ by parallel translation, consider the set

$$\check{\mathcal{D}} := \{\mathbf{z} \in V_+ \oplus V_- : |\mathbf{z}| = 1, \check{p}_i(\mathbf{z}) = 0, 0 \le i \le m_1\}.$$

Proposition 25. $\check{\mathcal{D}} = (V_+ \oplus V_-) \cap M_+$.

Proof. This follows from the formula of [25, I, pp.524-526], that reads

$$\begin{aligned} F(t\mathbf{x}+\mathbf{y}+\mathbf{w}) = {} & t^4 + (2|\mathbf{y}|^2 - 6|\mathbf{w}|^2)t^2 + 8(\sum_{i=0}^{m_1} \check{p}_i(\mathbf{y})w_i)t \\ & + |\mathbf{y}|^4 - 2\sum_{i=0}^{m_1}(\check{p}_i(\mathbf{y}))^2 + 8\sum_{i=0}^{m_1} q_i(\mathbf{y})w_i \\ & + 2\sum_{i,j=0}^{m_1}(\nabla \check{p}_i \cdot \nabla \check{p}_j)w_i w_j - 6|\mathbf{y}|^2|\mathbf{w}|^2 + |\mathbf{w}|^4. \end{aligned} \tag{10.1}$$

Here, the homogeneous polynomials of degree three, $q_i(\mathbf{y})$, are the components of the third fundamental form of M_+, $\mathbf{w} = \sum_{i=0}^{m_1} w_i e_i$, and $\mathbf{y}$ is tangent to M_+. For the convenience of the reader, let us briefly recall that Ozeki and Takeuchi expanded $F(t\mathbf{x}+\mathbf{y}+\mathbf{w})$ in terms of t and substituted it into its governing partial differential equations mentioned in Section 2 to get

$$F(t\mathbf{x}+\mathbf{y}+\mathbf{w}) = t^4 + At^2 + Bt + C,$$

where A is derived on p. 525, B is on p. 526, and $C = C_0 + \cdots + C_4$, in which C_s (given on p. 526) is the homogeneous part of C of degree s in the normal coordinates $w_0, \ldots, w_{m_1}$. When one sets $t = 0$, $\mathbf{w} = 0$ and $\mathbf{y} = \mathbf{z} \in V_+ \oplus V_-$ in the formula (10.1) one gets

$$F(\mathbf{z}) - |\mathbf{z}|^4 = -2\sum_{i=0}^{m_1}(\check{p}_i(\mathbf{z}))^2.$$

Hence when $|\mathbf{z}| = 1$, we have $F(\mathbf{z}) = 1$ if and only if $\check{p}_i(\mathbf{z}) = 0$ for $0 \le i \le m_1$. $\square$

Remark 26. It is not obvious that the set $\check{\mathcal{D}}$ is non-empty. In Theorem 47 of Section 12, we will prove that $\check{\mathcal{D}}$ is non-empty when $m_2 \ge 2m_1 - 1$. Proposition 28 below still holds in the case where $\check{\mathcal{D}}$ is empty. In that case, the zero locus of each set of polynomials in Proposition 28 is empty.

In view of Proposition 25 we set p_i to be the restriction of $\frac{1}{4}\check{p}_i$ to the space $V_+ \oplus V_-$ for $1 \le i \le m_1$, and set p_0 to be the restriction of $\check{p}_0$ to this space. These are the quadratic polynomials p_0, p_a defined in (6.6). Recall from (4.26) and (6.6), that relative to a second order Darboux frame we have variables $x = (x_\alpha)$ and $y = (y_\mu)$ in terms of which these polynomials are

$$p_0(x,y) = \sum_{\alpha=1}^{m_2}(x_\alpha)^2 - \sum_{\mu=1}^{m_2}(y_\mu)^2, \quad p_a(x,y) = \sum_{\alpha,\mu=1}^{m_2} F^\mu_{\alpha a} x_\alpha y_\mu. \tag{10.2}$$

For notational ease, as the context should remove any possibility of confusion, we will stick to the range $1 \le \alpha, \mu \le m_2$ for x_α and y_μ from now on even though α and μ live in the designated ranges as given in (4.6).

As mentioned in Section 4, we know e_0 also lies in M_+ with the normal space $\mathrm{span}(\mathbf{x}, e_{m_1+1}, \ldots, e_{2m_1})$. The $0, +1, -1$ eigenspaces of the shape operator $S_{\mathbf{x}}$ at e_0 are, respectively, $\mathrm{span}(e_1, \ldots, e_{m_1})$, V_+ and V_-. With respect to the normal basis $\mathbf{x}, e_p, m_1+1 \leq p \leq 2m_1$, at e_0, we let $\overline{p}_0, \ldots, \overline{p}_{m_1}$ be the counterparts of $p_0, \ldots, p_{m_1}$, respectively, as in (6.9). Then Proposition 25 immediately gives the following simple but crucial observation.

Proposition 27. $\check{\mathcal{D}} = \{\mathbf{z} \in V_+ \oplus V_- : |\mathbf{z}| = 1, \overline{p}_i(\mathbf{z}) = 0, 0 \leq i \leq m_1\}$.

Now $\check{\mathcal{D}}$ can be viewed from a different angle. Observe that all

$$\mathbf{z} = (x_1, \ldots, x_{m_2}, y_1, \ldots, y_{m_2}) \in \check{D}$$

must satisfy $\sum_{\alpha=1}^{m_2}(x_\alpha)^2 + \sum_{\mu=1}^{m_2}(y_\mu)^2 = 1$. It follows that $\mathbf{z} \in S^{m_2-1} \times S^{m_2-1}$ due to the fact that $p_0(\mathbf{z}) = 0$, where S^{m_2-1} is the standard sphere of radius $1/\sqrt{2}$. The real projective variety out of $S^{m_2-1} \times S^{m_2-1}$ is $\mathbf{R}P^{m_2-1} \times \mathbf{R}P^{m_2-1}$. Note that the solution to $p_a = 0$, $1 \leq a \leq m_1$, lives naturally in $\mathbf{R}P^{m_2-1} \times \mathbf{R}P^{m_2-1}$, which is parametrized by $[x_1 : \cdots : x_{m_2}] \times [y_1 : \cdots : y_{m_2}]$. As a consequence the projectivized $\check{D}$ in $\mathbf{R}P^{m_2-1} \times \mathbf{R}P^{m_2-1}$ via the map $S^{m_2-1} \times S^{m_2-1} \longrightarrow \mathbf{R}P^{m_2-1} \times \mathbf{R}P^{m_2-1}$ is exactly

$$\mathcal{D} := \{[\mathbf{z}] \in \mathbf{R}P^{m_2-1} \times \mathbf{R}P^{m_2-1} : p_a(\mathbf{z}) = 0, 1 \leq a \leq m_1\}. \tag{10.3}$$

Note that $\check{\mathcal{D}} \neq \emptyset$ if and only if $\mathcal{D} \neq \emptyset$. Since the $+1$ and -1 eigenspaces of the shape operator $S_{\mathbf{x}}$ at e_0 are V_+ and V_-, respectively, it follows from (10.2) that $\overline{p}_0 = p_0$. Hence, Proposition 27 can be rephrased as follows.

Proposition 28. *The zero locus of* $p_1, \ldots, p_{m_1}$ *in* $\mathbf{R}P^{m_2-1} \times \mathbf{R}P^{m_2-1}$ *is identical with that of* $\overline{p}_1, \ldots, \overline{p}_{m_1}$.

Lemma 29. *If* $m_2 \geq m_1 + 2$, *then the quadratic forms* $p_1, \ldots, p_{m_1}$ *are linearly independent and irreducible, both over the real numbers* $\mathbf{R}$ *and over the complex numbers* $\mathbf{C}$.

Proof. The quadratic form $p_a(x, y)$ is given by

$$4p_a(x, y) = \begin{pmatrix} 0 & A_a \\ {}^tA_a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2A_a y \cdot x$$

where A_a is the matrix with respect to e_α, e_μ of the operator defined in the first equation of (6.3). Recall that the rank of p_a is defined to be the rank of the matrix of the associated bilinear form. Hence $\mathrm{rank}\,(p_a) = 2\,\mathrm{rank}(A_a)$.

Let $S_a = U + V$ be the shape operator given in (6.5), where U is the matrix that retains the A_a and ${}^tA_a$ blocks and is zero elsewhere. Then $\mathrm{rank}\, S_a \leq \mathrm{rank}\, U + \mathrm{rank}\, V$. Since $\mathrm{rank}\, S_a = 2m_2$, $\mathrm{rank}\, U = 2\,\mathrm{rank}\, A_a$ and $\mathrm{rank}\, V \leq 2m_1$, we get

$$2(m_2 - m_1) \leq 2\,\mathrm{rank}\, A_a = \mathrm{rank}\,(p_a)$$

for all a, as proved by Ozeki and Takeuchi [25, II, p45].

If p_a is reducible, then $p_a = fg$ is a product of linear forms $f = a_\alpha x_\alpha + a_\mu y_\mu$ and $g = b_\alpha x_\alpha + b_\mu y_\mu$. If we let $\mathbf{a} = {}^t(a_\alpha\, a_\mu)$ and $\mathbf{b} = {}^t(b_\alpha\, b_\mu) \in \mathbf{R}^{2m_2}$ then the symmetric matrix of the quadratic form $4p_a(x,y)$ must be $(\mathbf{a}^t\mathbf{b} + \mathbf{b}^t\mathbf{a})/2$, which has rank ≤ 2, as each column is a linear combination of $\mathbf{a}$ and $\mathbf{b}$. In particular, if $m_2 - m_1 \geq 2$, then $\mathrm{rank}\,(p_a) \geq 4 > 2$ and hence, p_a is irreducible over $\mathbf{R}$. Notice that this discussion is unchanged if we work over the complex numbers, which shows that they are irreducible over $\mathbf{C}$ as well. Linear independence of $p_1, \ldots, p_{m_1}$ over $\mathbf{R}$ is equivalent to linear independence of $A_1, \ldots, A_{m_1}$, which follows under our hypotheses from Proposition 7. Being real polynomials, they are also linearly independent over $\mathbf{C}$. □

11. Commutative algebra and algebraic geometry

We will explore in more depth the fact that $p_a, 1 \leq a \leq m_1$, are irreducible when $m_2 \geq m_1 + 2$ and are bihomogeneous, i.e., are homogeneous in $x_1, \ldots, x_{m_2}$ and in $y_1, \ldots, y_{m_2}$, of bi-degree $(1,1)$ in this section. We shall pursue commutative algebra only to the extent that serves our need, and shall stress the geometry behind the algebra. A few ad hoc proofs and examples will be given to convey to the reader, who might be unfamiliar with the subject, some intuition about the concepts encountered. Henceforth, n is just an index that has nothing to do with the dimension of the ambient sphere in which the isoparametric hypersurface sits.

Definition 30. Let $\mathbf{F}$ be either $\mathbf{R}$ or $\mathbf{C}$ and let $\mathbf{F}[x_1, \ldots, x_s, y_1, \ldots, y_s]$ be the polynomial ring in variables $x_1, \ldots, x_s, y_1, \ldots, y_s$ over $\mathbf{F}$. Given bi-homogeneous polynomials $p_1, \ldots, p_n$, we say that the ideal $I := (p_1, \ldots, p_n)$ in $\mathbf{F}[x_1, \ldots, x_s, y_1, \ldots, y_s]$ is *reduced* if

(i) The bi-projective variety

$$\mathbf{P}_b V_I := \{([x],[y]) \in \mathbf{F}P^{s-1} \times \mathbf{F}P^{s-1} : p_a(x,y) = 0, 1 \leq a \leq n\}$$

is not empty, and

(ii) Whenever $f \in \mathbf{F}[x_1, \ldots, x_s, y_1, \ldots, y_s]$ satisfies $f|_{\mathbf{P}_b V_I} \equiv 0$ then

$$f = p_1 f_1 + \cdots + p_n f_n$$

for some $f_1, \ldots, f_n \in \mathbf{F}[x_1, \ldots, x_s, y_1, \ldots, y_s]$.

We call the affine variety $V_I := \{(x,y) \in \mathbf{C}^s \times \mathbf{C}^s : p_a(x,y) = 0, 1 \leq a \leq n\}$ a *bi-affine cone*.

For instance, when $\mathbf{F} = \mathbf{C}$, the radical of I, denoted by $\mathrm{rad}(I)$, is always reduced, if $\mathbf{P}_b V_I \neq \emptyset$. This is Hilbert's Nullstellensatz indeed [11]. In particular, since a prime ideal equals its radical, the ideal I will be reduced if I is a prime ideal. $\mathbf{P}_b V_I$ is not empty automatically in this case, because otherwise $V_I = (\mathbf{C}^s \times \{0\}) \cup (\{0\} \times \mathbf{C}^s)$ would not be irreducible. We will extensively probe the primeness of I subsequently. (See [14] and [21] for bi-projective geometry.) Before we proceed, let us introduce some notation. When p is a real polynomial, we denote by $p^{\mathbf{C}}$ the same polynomial whose variables are over the complex numbers. We call $p^{\mathbf{C}}$ the *complexification* of p. Likewise, when $p_1, \cdots, p_n$ are bi-homogeneous in $\mathbf{R}[x_1, \ldots, x_s, y_1, \ldots, y_s]$, we denote by V the resulting real bi-affine cone and by $V^{\mathbf{C}}$ the complex bi-affine cone defined by the complexifications of $p_1, \cdots, p_n$.

LEMMA 31. *Suppose V is a bi-affine cone in $\mathbf{R}^s \times \mathbf{R}^s$ defined by the real polynomials $p_1, \cdots, p_n$, such that its complex counterpart $V^{\mathbf{C}}$ is irreducible and such that $\dim_{\mathbf{R}}(V) = \dim_{\mathbf{C}}(V^{\mathbf{C}})$. If a real polynomial $p(x_1, \ldots, x_s, y_1, \ldots, y_s)$ satisfies $p|_V \equiv 0$, then $p^{\mathbf{C}}|_{V^{\mathbf{C}}} \equiv 0$.*

Here, by the dimension of V we mean the maximal dimension of all the irreducible components of V.

Proof. Suppose $p^{\mathbf{C}}|_{V^{\mathbf{C}}}$ is not identically zero on $V^{\mathbf{C}}$. Then $p^{\mathbf{C}}$ cuts out a subvariety X, all of whose irreducible components are of co-dimension 1 in $V^{\mathbf{C}}$ [28, p. 59]. Clearly, $V \subset X$. Then we have

$$\dim_{\mathbf{R}}(V) \leq \dim_{\mathbf{C}}(X) = \dim_{\mathbf{C}}(V^{\mathbf{C}}) - 1,$$

in contradiction to the assumption that $\dim_{\mathbf{R}}(V) = \dim_{\mathbf{C}}(V^{\mathbf{C}})$. The inequality holds true because any real analytic parametrization $\sigma : t = (t_1, \cdots, t_k) \longmapsto (x_1, \cdots, x_s, y_1, \cdots, y_s) \in V$ around a smooth point, at $t = 0$, of V, satisfies $p_1(\sigma(t)) = \cdots = p_n(\sigma(t)) = p(\sigma(t)) = 0$. The convergent power series defining σ remain so when $t_1, \cdots, t_k$ are allowed to be complex variables, and then $\sigma(t)$ is a holomorphic map, nonsingular at $t = 0$, such that $p_1^{\mathbf{C}}(\sigma(t)) = \cdots = p_n^{\mathbf{C}}(\sigma(t)) = p^{\mathbf{C}}(\sigma(t)) = 0$ because a holomorphic function vanishing on the real part is identically zero. That is, $\sigma(t)$, with t complex, is a holomorphic map, nonsingular at $t = 0$, into X. Therefore, we conclude that $\dim_{\mathbf{C}}(X) \geq \dim_{\mathbf{R}}(V)$. $\square$

PROPOSITION 32. *If $p_1, \ldots, p_n \in \mathbf{R}[x_1, \ldots, x_s, y_1, \ldots, y_s]$ are bihomogeneous polynomials of positive degree in each set of variables, and if $p_1^{\mathbf{C}}, \ldots, p_n^{\mathbf{C}}$, their complexifications, are such that*

(1) $V^{\mathbf{C}} := \{z \in \mathbf{C}^s \times \mathbf{C}^s : p_a^{\mathbf{C}}(z) = 0, 1 \leq a \leq n\}$ *is irreducible,*

(2) $\mathrm{rad}(I) = I$, *where* $I := (p_1^{\mathbf{C}}, \ldots, p_n^{\mathbf{C}})$, *and*

(3) $\dim_{\mathbf{R}}(V)=\dim_{\mathbf{C}}(V^{\mathbf{C}})$, *where* $V:=\{z\in\mathbf{R}^s\times\mathbf{R}^s : p_a(z)=0, 1\leq a\leq n\}$,

then the real ideal $(p_1,\dots,p_n)$ *is reduced.*

Proof. I is a prime ideal by the first two assumptions. Therefore, the remark immediately after Definition 30 ensures that $\mathbf{P}_bV^{\mathbf{C}}$ is not empty. Moreover, $\dim_{\mathbf{C}}(V^{\mathbf{C}})>s$ by the first assumption and the fact that the reducible $(\mathbf{C}^s\times\{0\})\cup(\{0\}\times\mathbf{C}^s)$ is contained in $V^{\mathbf{C}}$. Hence $\mathbf{P}_bV$ is not empty either by the third assumption. So the first condition in Definition 30 holds. Let f be a real polynomial vanishing on $\mathbf{P}_bV$ so that f vanishes on V as well; by Lemma 31 its complexification $f^{\mathbf{C}}$ vanishes on $V^{\mathbf{C}}$. It follows from the reducedness of I that there are complex bi-homogeneous polynomials $h_1,\dots,h_n$ such that

$$f^{\mathbf{C}}=p_1^{\mathbf{C}}h_1+\cdots+p_n^{\mathbf{C}}h_n.$$

Let $f_1,\dots,f_n$ be, respectively, the real parts of $h_1,\dots,h_n$ when they are restricted to the real variables. Since f and $p_1,\dots,p_n$ are real, $f=p_1f_1+\cdots+p_nf_n$. □

We now review some important notions and properties from commutative algebra, leaving detailed expositions to [11] and [19].

Definition 33. Let R be a commutative ring with identity. We say that n elements $x_1,\dots,x_n\in R$ form a *regular sequence* if $(x_1,\dots,x_n)\neq R$, x_1 is not a zero divisor in R and x_{i+1} is not a zero divisor in the quotient ring R/I_i, where I_i is the ideal $(x_1,\dots,x_i)$, for $1\leq i\leq n-1$.

Example 34. A single nonconstant $p\in\mathbf{C}[z_1,\dots,z_L]$ clearly forms a regular sequence.

Example 35. Let p_1 and p_2 in $\mathbf{C}[z_1,\dots,z_L]$ be relatively prime homogeneous polynomials of degree ≥ 1. Then p_1 and p_2 form a regular sequence. This follows simply from the fact that $p_2f=p_1g$ implies $f=p_1h$ for some h. Moreover, (p_1,p_2) is not the entire polynomial ring due to the homogeneity of p_1 and p_2.

Definition 36. Let $\mathcal{P}$ be a prime ideal in a commutative ring R with identity. We define the *codimension* of $\mathcal{P}$ to be

$$\operatorname{codim}(\mathcal{P})=\sup\{s: \text{there is a prime chain } \mathcal{P}_s\subset\cdots\subset\mathcal{P}_1\subset\mathcal{P}_0=\mathcal{P}\},$$

where the set inclusions are all proper. For an arbitrary ideal I we define

$$\operatorname{codim}(I)=\inf_{I\subset\mathcal{P}}\{\operatorname{codim}(\mathcal{P})\},$$

and define the *depth* of I to be

$$\operatorname{depth}(I)=\sup\{n: \text{there is a regular sequence } x_1,\dots,x_n\in I\}.$$

We define the *dimension* of R to be

$$\dim(R) = \sup\{s : \text{there is a prime chain } \mathcal{P}_s \subset \cdots \subset \mathcal{P}_1 \subset \mathcal{P}_0 \subset R\}.$$

Lastly, R is *Cohen-Macaulay* if, for every maximal ideal $\mathcal{M}$ of R (and such ideals are necessarily prime), we have

$$\text{depth}(\mathcal{M}) = \text{codim}(\mathcal{M}).$$

Example 37. Consider $R := \mathbf{C}[x, y, z]$ with $p_1 = xz$ and $p_2 = yz$. The ideal $I := (p_1, p_2)$ has the property $\text{rad}(I) = I$ so that R/I is the coordinate ring of the zero locus of p_1 and p_2, which is made up of the (x, y)-plane and the z-axis. It is not hard to see that $\dim(R/I) = 2 \neq 1$, the ambient dimension minus the number of equations. So the ring R/I is not Cohen-Macaulay. In fact, at the origin the maximal ideal $\mathcal{M} = (x, y, z)/I$ is the first term in a maximal descending prime chain $(x, y, z)/I$, $(y, z)/I$ and $(z)/I$ so that $\text{codim}(\mathcal{M}) = 2$. However, $\text{depth}(\mathcal{M}) = 1$, since $x + z \mod(I)$, for instance, forms a maximal regular sequence in $\mathcal{M}$.

The following ingredient, on the other hand, generates many Cohen-Macaulay rings.

FACT([11, p. 455]). If $p_1, \ldots, p_n$ form a regular sequence in the ring $R := \mathbf{C}[z_1, \ldots, z_L]$ with ideal $I = (p_1, \ldots, p_n)$, then $\text{codim}(I) = n$, the ring R/I is Cohen-Macaulay, and $\dim(R/I) = L - n$.

Remark 38. The **FACT** can be interpreted geometrically. In the case when $\text{rad}(I) = I$, for instance, the quotient ring R/I is the coordinate ring of an affine variety. This quotient ring being Cohen-Macaulay says that each point of the affine variety is the zero locus of $L - n$ coordinate functions from R/I (technically, in a maximal regular sequence vanishing at the point), and thus the codimension in the variety of each point is the expected value $L - n$. The affine variety is then called a *complete intersection*. It is of dimension $L - n$ on all of its irreducible components.

We now come to the major recipe for inductively constructing Cohen-Macaulay rings in this paper.

PROPOSITION 39. *If $p_1, \ldots, p_n$ are linearly independent homogeneous polynomials of equal degree ≥ 1 in the ring $\mathbf{C}[z_1, \ldots, z_L]$ such that the ideal $(p_1, \ldots, p_{n-1})$ is prime and such that $p_1, \ldots, p_{n-1}$ form a regular sequence, then $p_1, \cdots, p_n$ form a regular sequence. In particular, the* **FACT** *above implies that the quotient ring $\mathbf{C}[z_1, \ldots, z_L]/(p_1, \ldots, p_n)$ is Cohen-Macaulay.*

Proof. We know V_{n-1} is irreducible since $I_{n-1} := (p_1, \cdots, p_{n-1})$ is prime. Thus p_n cannot vanish identically on V_{n-1}. Otherwise the Nullstellensatz applied to p_n on the prime I_{n-1} would imply

$$p_n = p_1 f_1 + \cdots + p_{n-1} f_{n-1}$$

for some $f_1, \ldots, f_{n-1} \in \mathbf{C}[z_1, \ldots, z_L]$. As shown in Proposition 11, we may assume that $f_1, \ldots, f_{n-1}$ are constant polynomials, because all of $p_1, \ldots, p_n$ are homogeneous of the same degree ≥ 1. But this would imply that $p_1, \ldots, p_n$ are linearly dependent, which is not the case by assumption.

Suppose there are $f, f_1, \ldots, f_{n-1} \in \mathbf{C}[z_1, \ldots, z_L]$ such that

$$p_n f = p_1 f_1 + \cdots + p_{n-1} f_{n-1}.$$

Then $f|_{V_{n-1}} \equiv 0$ since p_n does not vanish identically on the irreducible V_{n-1}. So once more the Nullstellensatz applied to f on I_{n-1} implies that

$$f = p_1 g_1 + \cdots + p_{n-1} g_{n-1}$$

for some $g_1, \ldots, g_{n-1} \in \mathbf{C}[z_1, \ldots, z_L]$.

Lastly, $(p_1, \ldots, p_n) \neq \mathbf{C}[z_1, \ldots, z_L]$ since $p_1, \ldots, p_n$ are all homogeneous of the same degree ≥ 1. This confirms that $p_1, \ldots, p_n$ form a regular sequence. □

For our later applications on the *variety* level, Proposition 39 is not quite sufficient, because the ring $\mathbf{C}[z_1, \ldots, z_L]/(p_1, \ldots, p_n)$ in the proposition, though being Cohen-Macaulay, may have nilpotent elements, in which case the ring is not the coordinate ring of an affine variety. If the ring contains no nilpotent elements, then it is called *reduced*.

Example 40. Let $p_1 = y - x^2$ and $p_2 = y$ in $\mathbf{C}[x, y]$. The zero locus of p_1 and p_2 is $\{(0, 0)\}$. However, the Cohen-Macaulay ring $\mathbf{C}[x, y]/(p_1, p_2)$ has a nilpotent element, namely, $x \bmod((p_1, p_2))$. Geometrically, the parabola $y = x^2$ intersects $y = 0$ with multiplicity 2.

What we must do now is to find conditions under which the quotient ring in Proposition 39 is reduced, in which case the variety associated with the ring is called a *Cohen-Macaulay variety.*

PROPOSITION 41. *Let J_n be the subvariety of the variety*

$$V_n := \{z \in \mathbf{C}^L : p_1(z) = 0, \ldots, p_n(z) = 0\}$$

where the Jacobian matrix of $p_1, \ldots, p_n$ is not of rank n. If $\operatorname{codim}(J_n) \geq 1$ *in V_n, then the affine coordinate ring $\mathbf{C}[z_1, \ldots, z_L]/(p_1, \ldots, p_n)$ is reduced.*

Proof. This is just Serre's criterion of reducedness [11, p. 457]. □

Remark 42. If we assume in Proposition 39 that J_{n-1}, the subvariety of $V_{n-1} = \{z : p_1(z) = \cdots = p_{n-1}(z) = 0\}$ where the Jacobian of $p_1, \ldots, p_{n-1}$

is not of rank $n-1$, is of codimension ≥ 2 in V_{n-1}, then we can give a somewhat more geometric account of Proposition 41 as follows. (In fact, in our applications to follow, $\operatorname{codim}(J_{n-1}) \geq 2$ always holds true.) Let $R = \mathbf{C}[z_1, \ldots, z_L]$, let $I = (p_1, \ldots, p_{n-1})$ and let $J = (p_n)$. We must show $R/(I+J)$ has no nilpotents. That is, whenever $f \in R$ satisfies

$$f^k = p_1 f_1 + \cdots + p_n f_n \in I + J$$

for some k and $f_1, \ldots, f_n$, we must have $f \in I + J$. We may assume f^k is not in I, or else we are done since then $f \in I$ by the primeness of I. It follows that f is nonzero on V_{n-1} and is zero on V_n.

Let $V_n = W_1 \cup \cdots \cup W_s$ be the irreducible decomposition of V_n in V_{n-1}. We know $\operatorname{codim}(W_i) = 1$ in V_{n-1} for all i. Then by $\operatorname{codim}(J_n) \geq 1$ in V_n the polynomial p_n cuts out W_i with multiplicity 1 for each i (it comes down to the implicit function theorem in calculus). That is, $p_n = 0$ defines the *divisor* $W_1 + \cdots + W_s$ in V_{n-1}.

Now since f vanishes on V_n, the divisor defined by $f = 0$ assumes multiplicity ≥ 1 on each W_i. At this point the principle that says that the poles get cancelled by the zeros seems to suggest that the rational function f/p_n is regular everywhere on V_{n-1}. This is certainly true if V_{n-1} is smooth [28, p. 129], because the germs of local regular functions on V_{n-1} then form a unique factorization domain; more generally, the *normality* of the variety suffices for the conclusion [28, p. 111]. From this it follows that $(f/p_n)|_{V_{n-1}} = g$ for some regular g on V_{n-1}. In other words, $(f - p_n g)|_{V_{n-1}} \equiv 0$. Therefore,

$$f - p_n g = p_1 g_1 + \cdots + p_{n-1} g_{n-1} \in I$$

by the primeness of I. We conclude that $f \in I + J$, proving the reducedness of $R/(I+J)$.

It remains to ensure the normality of V_{n-1}, which is true if the codimension of J_{n-1} is at least 2. This is a consequence of Serre's criterion of primeness [11, p. 457], because V_{n-1} is a Cohen-Macaulay variety since $\operatorname{codim}(I) = n-1$. In any event we resort to Serre's criterion one way or another.

The next proposition plays a vital role in the applications to follow.

PROPOSITION 43. *Assume the notation in Proposition* 41. *If furthermore* $\operatorname{codim}(J_n) \geq 2$ *in* V_n *and* V_n *is connected, then* $(p_1, \ldots, p_n)$ *is a prime ideal.*

Proof. Proposition 41 asserts that V_n is a connected Cohen-Macaulay variety. Now X_n, the complement of J_n in V_n, is smooth on the one hand. On the other hand, X_n is also connected on account of Hartshorne's connectedness theorem [11, p. 454], that says that a connected Cohen-Macaulay variety remains connected when a subvariety of codimension ≥ 2 is removed. Being both smooth and connected, X_n must be irreducible. However, since $\mathrm{codim}(J_n) \geq 2$, J_n cannot be an irreducible component of V_n due to the fact that a Cohen-Macaulay variety is of equal dimension on all of its irreducible components. V_n is then irreducible. As a consequence $(p_1, \ldots, p_n)$ is a prime ideal because Proposition 41 establishes the reducedness of $(p_1, \ldots, p_n)$. □

Example 44. This example shows that $\mathrm{codim}(J_n) \geq 2$ in V_n is a must in Proposition 43. Let $p_1 = z$ and $p_2 = x^2 - y^2 + z^2$ in $\mathbf{C}[x, y, z]$. Then $V_2 = \{(x, \pm x, 0)\}$ and $J_2 = \{(0, 0, 0)\}$, which is of codimension 1 in V_2. But V_2 is reducible albeit connected. It also illustrates that the codimension 2 condition in Hartshorne's connectedness theorem cannot be improved to codimension 1.

12. The classification theorem

We now return to the isoparametric case. For a given second order Darboux frame field (4.16) along $\mathbf{x}$ on $U \subset M$, recall that we have, for $1 \leq a \leq m_1$, bihomogeneous polynomials

$$p_a = \sum_{\alpha,\mu=1}^{m_2} F^\mu_{\alpha a} x_\alpha y_\mu$$

of bi-degree $(1, 1)$ in the polynomial ring $\mathbf{R}[x_1, \ldots, x_{m_2}, y_1, \ldots, y_{m_2}]$, irreducible and linearly independent if $m_2 \geq m_1 + 2$ by Lemma 29. Before proving the theorem, we first introduce a generalized spanning property. For $n = 1, \ldots, m_1$, we define the linear map $S^x_n : \mathbf{R}^{m_2} \to \mathbf{R}^n$

$$S^x_n(y) = (p_1(x, y), \ldots, p_n(x, y)) \tag{12.1}$$

for a fixed x, and the linear map $S^y_n : \mathbf{R}^{m_2} \to \mathbf{R}^n$

$$S^y_n(x) = (p_1(x, y), \ldots, p_n(x, y)) \tag{12.2}$$

for a fixed y.

Definition 45. We say that the *n-spanning property* holds if there is an $x \in \mathbf{R}^{m_2}$ such that S^x_n is surjective and there is a $y \in \mathbf{R}^{m_2}$ such that S^y_n is surjective.

Note that when $n = m_1$, this definition agrees with that of the spanning property in Definition 8 for the second fundamental form (see Remark 9). As for the spanning property, the n-spanning property is an open condition.

We now set up an induction procedure toward our solution to (8.1) and the spanning property.

Induction hypothesis $\mathcal{S}(n)$.

(I) $p_1, \ldots, p_n, n \leq m_1$, being irreducible and linearly independent imply that $p^{\mathbf{C}}_1, \ldots, p^{\mathbf{C}}_n$ form a regular sequence.

(II) $V_n := \{z = (x, y) \in \mathbf{R}^{m_2} \times \mathbf{R}^{m_2} : p_a(z) = 0, a = 1, \dots, n\}$ and $V_n^{\mathbf{C}} := \{z = (x, y) \in \mathbf{C}^{m_2} \times \mathbf{C}^{m_2} : p_a^{\mathbf{C}}(z) = 0, a = 1, \dots, n\}$ satisfy $\dim_{\mathbf{R}}(V_n) = \dim_{\mathbf{C}}(V_n^{\mathbf{C}}) = 2m_2 - n$, where $\dim_{\mathbf{R}} V_n$ is the maximal dimension of all the irreducible components of V_n.

(III) $I_n := (p_1^{\mathbf{C}}, \dots, p_n^{\mathbf{C}})$ is a prime ideal.

(IV) The n-spanning property is true.

Let J_n be the subvariety of $V_n^{\mathbf{C}}$ where the Jacobian matrix of $p_1^{\mathbf{C}}, \dots, p_n^{\mathbf{C}}$ is of rank $< n$. Proposition 43 points out that $\operatorname{codim}(J_n) \geq 2$ plays a decisive role in determining the primeness of I_n. We will establish in the next section the following estimate.

Proposition 46. *Assume $m_2 \geq m_1 + 2$. If $m_2 \geq 2m_1$, then* $\operatorname{codim}(J_n) \geq 2$ *for all $n \leq m_1$. If $m_2 = 2m_1 - 1$, then* $\operatorname{codim}(J_n) \geq 2$ *for all $n \leq m_1 - 1$ whereas* $\operatorname{codim}(J_{m_1}) \geq 1$.

Assuming this proposition for the time being, let us prove the classification theorem of this paper.

Theorem 47 (Classification). *If $m_2 \geq 2m_1 - 1$, then the isoparametric hypersurface is of* FKM-*type.*

Proof. When $m_1 = 1$, then $a = 1$, $p = 2$ and equations (5.6) through (5.10) simplify sufficiently that one easily shows that there exists a second order frame field for which

$$F^{\mu}_{\alpha\, a+m_1} = \delta_{\alpha+m_2\, \mu} = F^{\mu}_{\alpha a}, \quad F^{\alpha}_{pa} = 0 = F^{\mu}_{pa}$$

for all α, μ. The first line of these equations implies (8.1) and the spanning property. Hence, Theorem 24 implies Takagi's result [30] that all such isoparametric hypersurfaces are of FKM-type.

Suppose $m_2 \geq \max(m_1 + 2, 2m_1)$. Our strategy is to show that the induction procedure can be completed for $n \leq m_1$. When $n = m_1$ what we achieve out of the induction is that (8.1) and the spanning property hold true. It follows from Theorem 24 that the isoparametric hypersurface is of FKM-type.

$\mathcal{S}(1)$ is true. (I) holds because $p_1^{\mathbf{C}}$ is irreducible by Lemma 29, and $p_1^{\mathbf{C}}$ cannot generate the polynomial ring since it is of degree 2. (II) is valid because p_1 is bihomogeneous of bi-degree (1,1), and so one can easily solve for one variable in terms of the remaining ones regardless of whether the variables are real or complex. (III) is verified because $(p_1^{\mathbf{C}})$ is a prime ideal due to the irreducibility of $p_1^{\mathbf{C}}$. (IV) is also clear since $p_1 \neq 0$.

Suppose $\mathcal{S}(n-1)$ is true for $n - 1 \leq m_1$. We show $\mathcal{S}(n)$ is true if $n \leq m_1$. Now, (I) comes from Proposition 39, so that the same proposition allows us to conclude that $\mathbf{C}[x_1, \dots, x_{m_2}, y_1, \dots, y_{m_2}]/(p_1^{\mathbf{C}}, \dots, p_n^{\mathbf{C}})$ is Cohen-Macaulay.

We wish to establish (II) next. To this end, note first that $V_n^{\mathbf{C}}$ is of equal dimension $2m_2 - n$ on all irreducible components, because $V_n^{\mathbf{C}}$ is the intersection of the irreducible $V_{n-1}^{\mathbf{C}}$ and the irreducible hypersurface defined by $p_n^{\mathbf{C}} = 0$. It follows that the real variety V_n has the property

$$\dim_{\mathbf{R}}(V_n) \leq \dim_{\mathbf{C}}(V_n^{\mathbf{C}}) = 2m_2 - n,$$

because as established in Lemma 31, V_n is a real subvariety of $V_n^{\mathbf{C}}$ and any real subvariety is of dimension at most half the (real) dimension of $V_n^{\mathbf{C}}$. We claim that there is a component of V_n having dimension $2m_2 - n$ so that

$$\dim_{\mathbf{R}}(V_n) = \dim_{\mathbf{C}}(V_n^{\mathbf{C}}),$$

which will establish (II). To prove the claim, consider

$$V_n \xrightarrow{\iota} \mathbf{R}^{m_2} \times \mathbf{R}^{m_2} \xrightarrow{\pi_1} \mathbf{R}^{m_2},$$

where ι is the natural embedding and π_1 is the projection onto the first summand. Note that $(x, y) \in (\pi_1 \circ \iota)^{-1}(x)$ precisely when y belongs to the kernel of the linear map S_n^x, which has dimension $\geq m_2 - n > 0$; in particular, $\pi_1 \circ \iota$ is surjective. The set $\mathcal{L}$ of x where the dimension of the kernel of S_n^x achieves the minimum value t is Zariski open. Since $\pi_1 \circ \iota$ is surjective, one of the irreducible components W of V_n must be mapped onto an open subset of $\mathcal{L}$ by Sard's theorem. Around a regular value x of $\pi_1 \circ \iota$ in $\mathcal{L}$ we know V_n is a product with fiber $\mathbf{R}^t$, which is therefore contained in the irreducible W. Then since $t \geq m_2 - n$,

$$\dim(W) = m_2 + t \geq m_2 + m_2 - n = 2m_2 - n.$$

Therefore

$$\dim_{\mathbf{R}}(V_n) = 2m_2 - n = \dim_{\mathbf{C}}(V_n^{\mathbf{C}}),$$

which proves (II).

Now that $\dim W = 2m_2 - n$, the fact that V_n is a product with fiber $\mathbf{R}^t$ around the regular value x gives

$$\dim((\pi_1 \circ \iota)^{-1}(x)) = m_2 - n.$$

That is, S_n^x spans $\mathbf{R}^n$. Likewise, there is some $y \neq 0$ in $\mathbf{R}^{m_2}$ such that S_n^y spans $\mathbf{R}^n$ if we consider the projection $\pi_2 : \mathbf{R}^{m_2} \times \mathbf{R}^{m_2} \longrightarrow \mathbf{R}^{m_2}$ onto the second summand. In conclusion, we have shown that (IV) is true.

To finish the induction, we must show that I_n is a prime ideal so that (III) holds. Proposition 43 and Proposition 46 tell us that this is true if $V_n^{\mathbf{C}}$ is connected, which is the case because $V_n^{\mathbf{C}}$ is a cone. In fact, if z and w are any two points in $V_n^{\mathbf{C}}$, then the real lines from z to the origin and from the origin to w are in $V_n^{\mathbf{C}}$, thus showing that $V_n^{\mathbf{C}}$ is path connected.

Thus, by Propositions 43 and 46, the induction procedure is completed.

Setting $n = m_1$ in the induction, we obtain the spanning property in Definition 8 by induction item (IV). Note also that $P_bV_{m_1}$ is $\mathcal{D}$ defined in (10.3), and $P_bV_{m_1}$ is not empty by induction item (II), which says that $\dim_{\mathbf{R}}(V_{m_1}) = 2m_2 - m_1 > m_2$, and thus V_{m_1} contains more than $\{0\} \times \mathbf{R}^{m_2} \cup \mathbf{R}^{m_2} \times \{0\}$.

We are only left with handling (8.1). By Proposition 28 we know $\overline{p}_a, 1 \leq a \leq m_1$, vanish on $\mathbf{P}_bV_{m_1}$ so that $\overline{p}_a|_{V_{m_1}} \equiv 0$, which warrants that $\overline{p}_a^{\mathbf{C}}|_{V_{m_1}^{\mathbf{C}}} \equiv 0$ in view of the induction item (II) and Lemma 31, so that $\overline{p}_a^{\mathbf{C}} \in I_{m_1}$ by the induction item (III). Hence there are complex polynomials $\tau_{ab}, 1 \leq a, b \leq m_1$, such that

$$\overline{p}_a^{\mathbf{C}} = \sum_{b=1}^{m_1} \tau_{ab} p_b^{\mathbf{C}}.$$

As shown in the proof of Proposition 11, we may assume that the τ_{ab} are constant polynomials, since each of the polynomials $\overline{p}_a^{\mathbf{C}}$ and $p_b^{\mathbf{C}}$ is of bi-degree $(1,1)$. Restricting to the real variables we obtain

$$\overline{p}_a = \sum_{b=1}^{m_1} f_{ab} p_b$$

for some real constants f_{ab}. The above argument establishes this at every point of the open set U on which the frame is defined. By Proposition 11, after a possible change of second order frame field along $\mathbf{x}$ on U, equation (8.1) holds on U. Theorem 24 then finishes the proof in the case $m_2 \geq \max(m_1+2, 2m_1)$.

When $m_2 \geq m_1 + 2$ and $m_2 = 2m_1 - 1$, we can only conclude that $V_{m_1}^{\mathbf{C}}$ is a reduced variety since $\operatorname{codim}(J_{m_1}) \geq 1$. Now, $p_1^{\mathbf{C}}, \ldots, p_{m_1}^{\mathbf{C}}$ is still a regular sequence. From the proof of (II) above, the (real) V_{m_1} is of dimension $2m_2 - m_1$. Let W be an irreducible component of $V_{m_1}^{\mathbf{C}}$ that contains an irreducible component V of V_{m_1} of dimension $2m_2 - m_1$. By Proposition 28 all $\overline{p}_i$ vanish on V. Then all $\overline{p}_i^{\mathbf{C}}$ vanish on W by Lemma 31. Hence, we may pick a generic smooth point z of W for the Nullstellensatz to be true at z. That is, W is (transversally) cut out by the ideal $(p_1^{\mathbf{C}}, \ldots, p_{m_1}^{\mathbf{C}})$ localized and still reduced at z, because, in algebraic terms, localization at the maximal ideal corresponding to p in the polynomial ring preserves Cohen-Macaulayness [11, p. 456]. In other words, we obtain $\overline{p}_i^{\mathbf{C}} = \sum_{j=1}^{m_1} s_{ij} p_j^{\mathbf{C}}$ for some local functions s_{ij} at z, i.e., $s_{ij} = r_{ij}/q_i$ with r_{ij} and q_i polynomials and $q_i(z) \neq 0$. Equivalently,

$$\overline{p}_i^{\mathbf{C}} q_i = \sum_{j=1}^{m_1} r_{ij} p_j^{\mathbf{C}}.$$

Let the (x,y)-coordinates of z be $(h_1, \ldots, h_{m_2}, k_1, \ldots, k_{m_2})$ and set $X_\alpha = x_\alpha - h_\alpha$ and $Y_\mu = y_\mu - k_\mu$. Now, since $p_a = \sum_{\alpha,\mu} F^\mu_{\alpha a} x_\alpha y_\mu$, we substitute (X_α, Y_μ) into the above Nullstellensatz equation to compare the 1st-order terms of (X_α, Y_μ) to conclude $k_\mu \overline{F}^\mu_{\alpha a} = \sum_b r_{\mu\alpha b} F^\mu_{\alpha b}$ and $h_\alpha \overline{F}^\mu_{\alpha a} = \sum_b s_{\mu\alpha b} F^\mu_{\alpha b}$ for some constants $r_{\mu\alpha b}$ and $s_{\mu\alpha b}$. We may assume that none of the h_α or k_μ are zero by performing a generic linear transformation. Then, one more time

$$\overline{F}^\mu_{\alpha a} = \sum_{b=1}^{m_1} f_{ab} F^\mu_{\alpha b}$$

for some constants f_{ab}.

When $(m_1, m_2) = (2,3)$, Ozeki-Takeuchi [25, II] proved that p_1, p_2 are still irreducible and relatively prime, so that they form a regular sequence. Moreover, we will show in Remark 53 that $\mathrm{codim}(J_2) = 1$. We are done by the preceding arguments. □

Remark 48. In contrast, for $m_1 = m_2 = 2$ of non-FKM-type, we have two pairs of (p_1, p_2) depending on which one of the two focal submanifolds is referred to as M_+. One pair of $(p_1, p_2) = (0,0)$. The other pair is $(2x_2y_1 - 2x_1y_2, -2x_1y_1 - 2x_2y_2)$, out of which the real bi-projective variety $\mathbf{P}_b V_2$ is empty whereas the complex bi-projective variety $\mathbf{P}_b V_2^{\mathbf{C}}$ consists of four points $[1 : \pm\sqrt{-1}] \times [1 : \pm\sqrt{-1}]$. This case fails miserably to satisfy Proposition 32 .

13. The estimate

We now prove Proposition 46 to complete the classification theorem in the preceding section. Recall for $V_n^{\mathbf{C}}$, its subvariety J_n is where the Jacobian matrix of $p_1^{\mathbf{C}}, \ldots, p_n^{\mathbf{C}}$ fails to be of rank n. From now on S_n^x and S_n^y in (12.1) and (12.2) will be set in the complex category.

LEMMA 49. *Notation is as in* (6.5). *For any choice of* $a \in \{1, \ldots, m_1\}$, *there is an orthonormal basis in* V_+ *and an orthonormal basis in* V_- *such that relative to these bases,*

(1) $B_a = C_a$ *with rank* $= r \le m_1$, *and*

(2) $A_a = \begin{pmatrix} I & 0 \\ 0 & \Delta \end{pmatrix}$, *where* Δ *is an* $r \times r$ *matrix in the block form* $\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3, \cdots)$, *in which* $\Delta_1 = 0$ *and* $\Delta_i, i \ge 2$, *are nonzero skew-symmetric matrices in the block form* $\Delta_i = \mathrm{diag}(\Theta_i, \Theta_i, \cdots)$ *with* Θ_i *a 2-by-2 matrix of the form* $\begin{pmatrix} 0 & f_i \\ -f_i & 0 \end{pmatrix}$.

Proof. We know $B_a : V_0 \longrightarrow V_+$, so that $B_a{}^t B_a : V_+ \longrightarrow V_+$. Pick an orthonormal basis $X_1, \ldots, X_{m_2 - r}, Y_1, \ldots, Y_r$ of V_+ for some r such that

$$B_a{}^t B_a : X_t \longmapsto 0, \qquad : Y_s \longmapsto (\sigma_s)^2 Y_s, \tag{13.1}$$

where $1 \leq t \leq m_2 - r$, $1 \leq s \leq r$, and $\sigma_s > 0$. Now ${}^tB_a(X_t) = 0$ because $\mathrm{Ker}(B_a) \cap \mathrm{Im}({}^tB_a) = 0$; hence $X_t \in \mathrm{Ker}({}^tB_a)$. That is, $\mathrm{Ker}({}^tB_a)$ is the eigenspace of $B_a{}^tB_a$ with eigenvalue zero. On the other hand, we know $(\mathrm{Ker}({}^tB_a))^\perp = \mathrm{Im}(B_a)$. So the eigenspace decomposition of $B_a{}^tB_a$ is

$$V_+ = \mathrm{Ker}({}^tB_a) \oplus \mathrm{Im}(B_a)$$

with $X_1, \ldots, X_{m_2-r}$ spanning the first summand and $Y_1, \ldots, Y_r$ spanning the second. As a result, it follows that $r = \mathrm{rank}(B_a)$. Likewise,

$$V_0 = \mathrm{Ker}(B_a) \oplus \mathrm{Im}({}^tB_a).$$

Knowing from above that ${}^tB_a(X_t) = 0$ we set

$$ {}^tB_a : Y_s \longmapsto \sigma_s W_s \tag{13.2}$$

for some W_s. An easy calculation shows $W_i \cdot W_j = \delta_{ij}$ so that $W_1, \ldots, W_r$ form an orthonormal basis of $\mathrm{Im}({}^tB_a)$. In conclusion,

$$V_0 = \mathrm{Ker}(B_a) \oplus \mathrm{Im}({}^tB_a),$$

where $W_1, \ldots, W_r$ span the second summand and we let $Z_1, \ldots, Z_{m_1-r}$ be an orthonormal basis generating the first. We find by (13.1) that

$$\begin{aligned} B_a &: Z_t \longmapsto 0, \\ &: W_s \longmapsto \sigma_s Y_s. \end{aligned} \tag{13.3}$$

We calculate to see that ${}^tB_aB_a : V_0 \longrightarrow V_0$ satisfies

$$\begin{aligned} {}^tB_aB_a &: Z_t \longmapsto 0, \\ &: W_s \longmapsto (\sigma_s)^2 W_s. \end{aligned} \tag{13.4}$$

Now consider

$$C_a : V_0 \longrightarrow V_-.$$

In the same manner as above for B_a, we get $V_0 = \mathrm{Ker}(C_a) \oplus \mathrm{Im}({}^tC_a)$ with

$$\begin{aligned} C_a &: Z_t^* \longmapsto 0, \\ &: W_s^* \longmapsto \sigma_s^* Y_s^*, \end{aligned} \tag{13.5}$$

where $Z_1^*, \ldots, Z_{m_1-p}^*$ span $\mathrm{Ker}(C_a)$ and $W_1^*, \ldots, W_p^*$ span $\mathrm{Im}({}^tC_a)$ for some p. However,

$${}^tC_aC_a = {}^tB_aB_a$$

by the first equation of (5.6); we thus obtain $\mathrm{Ker}(B_a) = \mathrm{Ker}(C_a)$ and $\mathrm{Im}({}^tB_a) = \mathrm{Im}({}^tC_a)$. In particular, $p = r$ and we may take $Z_1, \ldots, Z_{m_1-r}$ to be identical to $Z_1^*, \ldots, Z_{m_1-r}^*$, and $W_1, \ldots, W_r$ to be identical to $W_1^*, \ldots, W_r^*$. Therefore (13.3) and (13.5) imply that we can pick a basis of V_+ and a basis of V_-

relative to which the matrices of these operators, denoted by the same letters as the operators, satisfy

$$B_a = C_a, \tag{13.6}$$

because from

$$\begin{aligned} {}^tC_aC_a : Z_t^* &\longmapsto 0, \\ : W_s^* &\longmapsto (\sigma_s^*)^2 W_s^* \end{aligned}$$

and $W_s = W_s^*$, we know $(\sigma_s)^2 = (\sigma_s^*)^2$, and hence we may assume $\sigma_s = \sigma_s^*$ by adjusting the basis in V_-.

The second and the fourth equations of (5.6) together with (13.6) yield

$$A_a{}^tA_a = {}^tA_aA_a = I - 2B_a{}^tB_a. \tag{13.7}$$

We have three more equations

$$B_a{}^tB_a{}^tA_a + A_aB_a{}^tB_a = 0, \tag{13.8}$$

$$B_a{}^tB_aA_a + {}^tA_aB_a{}^tB_a = 0, \tag{13.9}$$

$${}^tB_a{}^tA_aB_a + {}^tB_aA_aB_a = 0, \tag{13.10}$$

which can be derived from (13.6) and the three diagonal blocks of (6.7). Let

$$A_a = \begin{pmatrix} \alpha & \beta \\ \gamma & \mu \end{pmatrix}$$

where α is of size $(m_2 - r) \times (m_2 - r)$ and μ is of size $r \times r$. Let $\sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_r)$ be the diagonal matrix with the indicated diagonal entries so that by (13.2) and (13.3), B_a and ${}^tB_a$ are of the same form

$$\begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}, \tag{13.11}$$

with $B_a{}^tB_a = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix}$ of the same block sizes as A_a. From (13.8) we obtain

$$\beta = \gamma = 0, \tag{13.12}$$

$$\sigma^2({}^t\mu) = -\mu\sigma^2. \tag{13.13}$$

Moreover from (13.7) we see

$$\alpha{}^t\alpha = I, \tag{13.14}$$

$$\mu{}^t\mu = {}^t\mu\mu = I - 2\sigma^2. \tag{13.15}$$

Similarly, (13.9) yields

$$\sigma^2\mu = -{}^t\mu\sigma^2, \tag{13.16}$$

and (13.10) gives

$$\sigma{}^t\mu\sigma = -\sigma\mu\sigma. \tag{13.17}$$

With (13.13) and (13.16) we deduce

$$\mu_{ij} = -(\sigma_i/\sigma_j)^2 \mu_{ji},$$

and

$$\mu_{ji} = -(\sigma_i/\sigma_j)^2 \mu_{ij}.$$

We therefore conclude

$$\mu_{ij} = 0 \qquad \text{if } \sigma_i \neq \sigma_j,$$

and

$$\mu_{ij} = -\mu_{ji} \quad \text{if } \sigma_i = \sigma_j.$$

In other words,

$$A_a = \begin{pmatrix} \alpha & 0 \\ 0 & \mu \end{pmatrix}$$

with $\alpha^t \alpha = I$ and μ is in blocked form

$$\mu = \operatorname{diag}(\Delta_1, \Delta_2, \Delta_3, \cdots),$$

where all the Δ_i are skew-symmetric such that the number of Δ_i is the number of different non-zero eigenvalues of $B_a{}^t B_a$. Then (13.17) is automatically satisfied. Now by the skew-symmetry of μ and (13.15) we derive

$$\Delta_i^2 = -(1 - 2\sigma_i^2) I. \tag{13.18}$$

In view of (13.14) and the skew-symmetry of μ we can perform an orthonormal basis change so that $\alpha = I$ and

$$\Delta_i = \operatorname{diag}(\begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_2 \\ -r_2 & 0 \end{pmatrix}, \cdots).$$

Thus (13.18) implies $r_1^2 = r_2^2 = \cdots = 1 - 2\sigma_i^2$, and so

$$\Delta_i = \sqrt{1 - 2\sigma_i^2} \ \operatorname{diag}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots)$$

if $1 - 2\sigma_i^2 > 0$. We set $\Delta_1 \equiv 0$ so that $\sigma_1 = 1/\sqrt{2}$. We are done. □

Corollary 50. $\dim(\operatorname{Ker}(A_a)) = \dim(\Delta_1) \leq r = \operatorname{rank}(B_a) \leq m_1$.

Remark 51. When $(m_1, m_2) = (2, m_2), m_2 \geq 3$, Ozeki and Takeuchi showed [25, II, p. 49], that r as given in Lemma 49 is 1, essentially by exploring the fact that p_1 and p_2 form a regular sequence in the spirit of Example 35 above. It follows immediately from Lemma 49 that we have $\Delta = 0$ and so $A_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ as given in [25, II, p. 51]. With this it is not hard to see that $A_2 = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ of the same block sizes as A_1 with $B = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, where I in B is of size $l \times l$ and $m_2 = 2l + 1$.

Proof of Proposition 46. We must estimate the codimension in $V_n^{\mathbf{C}}$ of

$$J_n = \{(x,y) \in V_n^{\mathbf{C}} : dp_1^{\mathbf{C}} \wedge \cdots \wedge dp_n^{\mathbf{C}} = 0\}.$$

We first estimate the dimension of the subvariety Z_n of $\mathbf{C}^{m_2} \times \mathbf{C}^{m_2}$, defined to be the locus of points where the Jacobian matrix of $p_1^{\mathbf{C}}, \ldots, p_n^{\mathbf{C}}$ is of rank $< n$. At $(x,y) \in Z_n$, the differentials $dp_1^{\mathbf{C}}, \ldots, dp_n^{\mathbf{C}}$ are linearly dependent, i.e., there exists $[c_1 : \cdots : c_n] \in \mathbf{C}P^{n-1}$, depending on (x,y), such that

$$0 = \sum_{a=1}^{n} c_a dp_a^{\mathbf{C}} = \sum_{\alpha} (\sum_{a,\mu} c_a F_{\alpha a}^{\mu} y_\mu) dx_\alpha + \sum_{\mu} (\sum_{a,\alpha} c_a F_{\alpha a}^{\mu} x_\alpha) dy_\mu,$$

which requires that the coefficients of dx_α be zero and that the coefficients of dy_μ also be zero. Thus

$$Z_n = \{(x,y) \in \mathbf{C}^{m_2} \times \mathbf{C}^{m_2} : \exists [c_1 : \cdots : c_n], \sum_a c_a\, {}^t\!A_a x = \sum_a c_a A_a y = 0\}.$$

In order to estimate $\dim Z_n$, let us define, for a fixed $[c_1 : \cdots : c_n] \in \mathbf{C}P^{n-1}$,

$$Z_{(c_1,\ldots,c_n)} := \{(x,y) \in \mathbf{C}^{m_2} \times \mathbf{C}^{m_2} : \sum_a c_a\, {}^t\!A_a x = \sum_a c_a A_a y = 0\}.$$

Consider the incidence space Y_n in $\mathbf{C}P^{n-1} \times \mathbf{C}^{m_2} \times \mathbf{C}^{m_2}$ given by

$$Y_n = \{([c_1 : \cdots : c_n], x, y) : (x,y) \in Z_{(c_1,\ldots,c_n)}\}. \tag{13.19}$$

The standard projection of Y_n to $\mathbf{C}^{m_2} \times \mathbf{C}^{m_2}$ maps Y_n onto Z_n. Let π be the standard projection of Y_n to $\mathbf{C}P^{n-1}$. Then with respect to π we have

$$\dim(Z_n) \le \dim(Y_n) \le \dim(\text{base}) + \dim(\text{fiber}), \tag{13.20}$$

where dim(fiber) is the maximal dimension of all fibers. We first estimate the dimension of the fibers $\pi^{-1}\{[c_1 : \cdots : c_n]\} = Z_{(c_1,\ldots,c_n)}$. In fact, it comes down to estimating the dimension of

$$T_{(c_1,\ldots,c_n)} := \{y \in \mathbf{C}^{m_2} : \sum_a c_a A_a y = 0\}$$

for a fixed $[c_1 : \cdots : c_n]$, because

$$\dim(\ker(\sum_a c_a\, {}^t\!A_a)) = \dim(\ker(\sum_a c_a A_a)), \tag{13.21}$$

thus giving us the estimate

$$\dim(Z_{(c_1,\ldots,c_n)}) \le 2\dim(T_{(c_1,\ldots,c_n)}).$$

Remark 52. Let us examine the case $(m_1, m_2) = (2, m_2), m_2 \ge 3$, before we proceed. By the above standard matrix form of A_1 and of A_2 in Remark 51 we see that for ${}^t y = ({}^t z, s) \in \mathbf{C}^{m_2}$, where $s \in \mathbf{C}$,

$$A_1\, {}^t(z,s) = {}^t(z,0) \quad A_2\, {}^t(z,s) = {}^t(Bz,0).$$

Hence $\sum_{a=1}^{n=2} c_a A_a y = 0$ precisely when $z = 0$ or z is an eigenvector of B, with eigenvalue $\varepsilon\sqrt{-1}$, where ε is $\pm$. In other words, when $[c_1 : c_2] = [\varepsilon\sqrt{-1} : 1]$ in $\mathbf{C}P^1$, then

$$Z_{(c_1,c_2)} = \{((u, -\varepsilon\sqrt{-1}u, t), (v, \varepsilon\sqrt{-1}v, s)) : u, v \in \mathbf{C}^l, s, t \in \mathbf{C}\}, \tag{13.22}$$

and $Z_{(c_1,c_2)} = \{((0,0,t),(0,0,s)) : s, t \in \mathbf{C}\}$ for other values of $[c_1 : c_2]$. Thus

$$Z_2 = Z_{(\sqrt{-1},1)} \cup Z_{(-\sqrt{-1},1)}, \tag{13.23}$$

and so that

$$\dim(Z_2) = 2l + 2 = m_2 + 1. \tag{13.24}$$

We continue on now to estimate the dimension of $Z_{(c_1,\ldots,c_n)}$.

Case (1). $c_1, \ldots, c_n$ are either all real or all purely imaginary. Say it is the latter, so that $c_k = \sqrt{-1}d_k$ with d_k real. Then for $y \in T_{(c_1,\ldots,c_n)}$,

$$\sum_{k=1}^{n} d_k A_k y = 0.$$

However, the second fundamental form S has the property

$$d_1 S_{e_1} + \cdots + d_n S_{e_n} = \sqrt{d_1^2 + \cdots + d_n^2}\, S_e,$$

where

$$e = (d_1 e_1 + \cdots + d_n e_n)/\sqrt{d_1^2 + \cdots + d_n^2}.$$

We may therefore rename e to be e_1 in the normal basis, and so by restricting to the A-block in the matrix of S we see that $S_e y = 0$ comes down to, after the renaming, $A_1 y = 0$. Corollary 50 then establishes that

$$\dim(T_{(c_1,\ldots,c_n)}) \le r \le m_1$$

and

$$\dim(Z_{(c_1,\ldots,c_n)}) \le 2\dim T_{(c_1,\ldots,c_n)} \le 2m_1.$$

Case (2). $c_1, \ldots, c_n$ are not all real and not all purely imaginary. Write

$$c_k = \alpha_k + \sqrt{-1}\beta_k,$$

where not all α_k and not all β_k are zero. Then

$$c_1 S_{e_1} + \cdots + c_n S_{e_n} = (\alpha_1 S_{e_1} + \cdots + \alpha_n S_{e_n}) + \sqrt{-1}(\beta_1 S_{e_1} + \cdots + \beta_n S_{e_n}).$$

As in Case (1), we know $\alpha_1 S_{e_1} + \cdots + \alpha_n S_{e_n}$ is a multiple of S_e for some unit vector e. Hence without loss of generality we may assume, after renaming e to be e_1, that

$$c_1 S_{e_1} + \cdots + c_n S_{e_n} = \alpha_1 S_{e_1} + \sqrt{-1}(\beta_1 S_{e_1} + \cdots + \beta_n S_{e_n}).$$

On the other hand $\beta_2 S_{e_2} + \cdots + \beta_n S_{e_n}$ is a multiple of S_f for some unit vector f perpendicular to e_1. We rename f to be e_2 so that we may assume without loss of generality that

$$c_1 S_{e_1} + \cdots + c_n S_{e_n} = (\alpha_1 + \sqrt{-1}\beta_1) S_{e_1} + \sqrt{-1}\beta_2 S_{e_2}.$$

By restricting to the A-block in S again we see that $(\sum_a c_a A_a) y = 0$ is reduced to

$$\beta_2 A_2 y = \sqrt{-1}(\alpha_1 + \sqrt{-1}\beta_1) A_1 y.$$

We may assume both coefficients are nonzero, or else we would be back to Case (1). Hence we are now handling

$$(A_2 - z A_1) y = 0 \tag{13.25}$$

for some nonzero $z \in \mathbf{C}$. By Lemma 49, we may assume $A_1 = \begin{pmatrix} I & 0 \\ 0 & \Delta \end{pmatrix}$. Write

$$A_2 = \begin{pmatrix} \Theta & \Lambda \\ \Omega & \Gamma \end{pmatrix}$$

of the same block sizes as A_1. By the second equation of (5.6), which is

$$A_2 {}^t\!A_1 + A_1 {}^t\!A_2 + 2(B_2 {}^t\!B_1 + B_1 {}^t\!B_2) = 0,$$

we obtain

$$\Theta + {}^t\Theta = 0 \tag{13.26}$$

when we invoke (13.11). If we write

$$y = {}^t(u, v), \quad u \in \mathbf{C}^{m_2 - r}, \quad v \in \mathbf{C}^r$$

then part of (13.25) reads,

$$(zI - \Theta) u = \Lambda v. \tag{13.27}$$

Consider the map $G : \mathbf{C}^{m_2} \longrightarrow \mathbf{C}^{m_2 - r}$ given by

$$G : (u, v) \longmapsto (zI - \Theta) u - \Lambda v.$$

The kernel of G consists of all $y = {}^t(u, v)$ satisfying (13.27). If z is not an eigenvalue of Θ, then the rank of G is at least the rank of $zI - \Theta$, which is $m_2 - r$. Thus, the rank of G is $m_2 - r$, so that the kernel of G has dimension r. On the other hand if z is an eigenvalue of Θ, then because Θ is skew-symmetric by (13.26), the rank of $zI - \Theta$ is at least $(m_2 - r)/2$ due to the fact that a nonzero eigenvalue of Θ is purely imaginary, and its conjugate is also an eigenvalue of Θ. It follows that the rank of G is no less than $(m_2 - r)/2$, so that its kernel is of dimension $\leq (m_2 + r)/2$. The upshot is that, since $r \leq m_1$ and since $\dim(T_{c_1,\ldots,c_n})$ is an integer, we have arrived at the estimate

$$\dim(T_{(c_1,\ldots,c_n)}) \leq [(m_2 + r)/2] \leq [(m_2 + m_1)/2] = (m_2 + m_1 - 1)/2,$$

where $[p]$ is the greatest integer in the number p; the last equality is true because m_2+m_1 is an odd number when $2 \leq m_1 < m_2$ by a result of Münzner [22, II]. Therefore,

$$\dim(\text{fiber}) = \dim(Z_{(c_1,\ldots,c_n)}) \leq 2\dim(T_{(c_1,\ldots,c_n)}) \leq m_2 + m_1 - 1. \tag{13.28}$$

This estimate is sharp in light of (13.24). Note that m_2+m_1-1 is greater than the upper bound $2m_1$ for $\dim(Z_{(c_1,\ldots,c_n)})$ in Case (1), since $m_2 \geq m_1+2$, by assumption.

We next stratify the incidence space Y_n of (13.19) in another way as follows. We let $s \leq m_2$ be the largest integer for which $\sum_{i=1}^n c_i A_i$ is of rank s for some, and hence for generic, $[c_1 : \cdots : c_n]$, the set of which constitutes a Zariski open set U of $\mathbf{C}P^{n-1}$. A look at Corollary 50 shows that $s \geq m_2 - m_1$, so that for $[c_1 : \cdots : c_n]$ in U,

$$\operatorname{rank}(\sum_{i=1}^n c_i A_i) = s \geq m_2 - m_1,$$

and thus, by (13.21),

$$\begin{aligned}\dim(\text{fiber}) = \dim(Z_{(c_1,\ldots,c_n)}) &= \dim(\ker(\sum_1^n c_i A_i)) + \dim(\ker(\sum_1^n c_i {}^t A_i))\\ &= 2(m_2 - \operatorname{rank}(\sum_1^n c_i A_i)) = 2(m_2 - s) \leq 2m_1.\end{aligned}$$

It follows that over U, (13.20) extends to

$$\dim(\text{fiber}) + \dim(\text{base}) \leq 2m_1 + (n-1). \tag{13.29}$$

On the other hand, over a subvariety W, contained in $\mathbf{C}P^{n-1}$, of dimension $\leq n-2$, the rank of $\sum_{i=1}^n c_i A_i$ is less than s. In view of (13.28), we have that over W

$$\begin{aligned}&\dim(\text{fiber}) + \dim(\text{base}) \leq \dim(\text{fiber}) + n - 2\\ &\leq m_1 + m_2 - 1 + n - 2 = m_1 + m_2 + n - 3.\end{aligned} \tag{13.30}$$

The part of Y_n over U, call it A, is irreducible because each fiber over U is a Euclidean space of a fixed dimension, whereas the part over W, call it B, is Zariski closed in Y_n. It follows that the closure of A, call it $\overline{A}$, in Y_n is an irreducible component of Y_n, and the closure of B not in $\overline{A}$ constitutes the remaining irreducible components in Y_n. Therefore, the larger of the two upper bounds in (13.29) and (13.30) will be an upper bound for the dimension of Y_n. However, $2m_1 + n - 1 \leq m_1 + m_2 + n - 3$, because $m_2 \geq m_1 + 2$. We conclude that over $\mathbf{C}P^{n-1}$

$$\dim(Y_n) \leq m_1 + m_2 + n - 3 \tag{13.31}$$

if $m_2 \geq m_1 + 2$.

Now, Lemma 29 says that $p_1^{\mathbf{C}}, \cdots, p_n^{\mathbf{C}}$ are linearly independent. Consider the map

$$f : (x, y) \in \mathbf{C}^{m_2} \times \mathbf{C}^{m_2} \mapsto (p_1^{\mathbf{C}}(x, y), \ldots, p_n^{\mathbf{C}}(x, y)) \in \mathbf{C}^n.$$

Note that Z_n is the singular point set of f and $J_n = f^{-1}(0) \cap Z_n$.

Let us make a general remark about a refined version of Sard's theorem before proceeding. In the following, the irreducible objects X in the projectivized domain of f, which is $\mathbf{C}P^{m_2-1} \times \mathbf{C}P^{m_2-1}$ with $\mathbf{P}V_n^{\mathbf{C}}$ removed, are all quasi-projective, i.e., are all Zariski open subsets of projective varieties. $f|_X$ can be considered as a rational map into $\mathbf{C}P^{n-1}$. So, by [21, p. 50], there is a Zariski open set $\mathcal{O}$ of $f(X)$ in $\mathbf{C}^n$ (with the origin excluded) such that $\dim(f|_X^{-1}(y)) = \dim(X) - \dim(f(X))$ is a constant for all $y \in \mathcal{O}$; we call it the *generic fiber dimension* of $f|_X$. Furthermore, $\operatorname{codim}(f(X) \setminus \mathcal{O}) \geq 2$ in $f(X)$.

Recall from (13.19) that the projection $\Pi : Y_n \to \mathbf{C}^{m_2} \times \mathbf{C}^{m_2}$ is Z_n. Observe that at (x, y), the dimension of the kernel of the Jacobian matrix of $p_1^{\mathbf{C}}, \ldots, p_n^{\mathbf{C}}$ at (x, y) is 1 more than the dimension of the projective space $\Pi^{-1}((x, y))$. $\mathbf{C}^{m_2} \times \mathbf{C}^{m_2}$ is stratified into locally closed sets (i.e., Zariski open sets in their respective closures) $X_{-1}, X_0, X_1, \cdots, X_{n-1}$ such that df has rank $n - j - 1$ on X_j (X_j may be empty). Note that Π has fiber dimension j over X_j. Let k be the first $j \geq 0$ for X_j to be nonempty. Then $Z_n = \cup_{j=k}^{n-1} X_j$ and $\cup_{j=k+1}^{n-1} X_j$ is a Zariski closed set of Z_n. Let U_0 be an irreducible component of Z_n. The smooth part of U_0 consists of irreducible components of X_k. Then the generic rank of $f|_{U_0}$ is $n - k - 1$, so that $\dim(f(U_0)) = n - k - 1$. Set $\mathcal{S} := U_0 \cap (\cup_{j=k+1}^{n-1} X_j)$. $\mathcal{S}$ is Zariski closed of codimension at least 1 in U_0. The generic fiber $\mathcal{F}_0$ of Π over U_0 has dimension k.

We now use an inductive procedure. Suppose U_i of codimension i in U_0 has been defined and the generic hyperplanes L_i chosen (L_0 is the empty set), in such a way that $\mathcal{S}_i := U_i \cap \mathcal{S}$ is of codimension at least 1 in U_i, so that the generic fiber dimension of Π over U_i is k. Let W be an irreducible component of U_i. Observe that since $\dim(W) = \dim(U_i)$ so that $W \cap \mathcal{S}_i$ is of codimension at least 1 in W, we have that $f(W \cap \mathcal{S}_i)$ is of codimension at least 1 in $f(W)$; or else the generic fiber dimension of $f|_W$ over $f(W)$ would be reduced to a smaller number.

Now, if $f(W) \neq \{0\}$, we pick a generic hyperplane L_{i+1}, transversal to $L_j, 0 \leq j \leq i$, through the origin and transversal to $f(W)$ and $f(W \cap \mathcal{S}_i)$. (This is possible. Since f and the hyperplanes $L_1, L_2, \cdots, L_i$ are all homogeneous, we may consider the cuts to be done in the projective setting, and thereby get that L_{i+1} is a hyperplane through the origin.) This warrants that the cone $L_{i+1}(p_1, \cdots, p_n) = 0$ intersects W and $W \cap \mathcal{S}_i$ transversally to cut out Q_W of codimension 1 in W and $Q_W \cap \mathcal{S}_i$ of codimension at least 1 in Q_W, as we go through each of the irreducible components W with $f(W) \neq 0$. Let U_{i+1} be the union of all such Q_W. U_{i+1} is of codimension $i + 1$ in U_0. Furthermore,

$S_{i+1} = U_{i+1} \cap \mathcal{S} = \cup_W (Q_W \cap \mathcal{S}_i)$ is of codimension at least 1 in U_{i+1}. Therefore, the generic fiber $\mathcal{F}_{i+1}$ of Π over U_{i+1} has dimension $k = \dim(\mathcal{F}_0)$.

On the other hand, if an irreducible component L of U_i satisfies $f(L) = 0$, then $\operatorname{codim}(L) = i$ in U_0. We claim the generic fiber of Π over L is of dimension $n - i - 1$. This is because when $i = 0$, $f(U_0) = 0$ implies df has rank 0, so that Π has generic fiber dimension $n - 1$ over U_0; $k = n - 1$ in this case. If $f(U_0) \neq 0$, we go to U_1. If $f(L) = 0$ for some irreducible component of U_1, then since $df = 0$ on L at its generic points, which are also generic in U_0, and since $df \neq 0$ on U_0 generically, we see df has rank 1 at a generic point of L. That is, Π has generic fiber dimension $n - 2$ over L; $k = n - 2$ in this case. Then we move to U_2, etc. Accordingly, we set T_i to be the union of all such L; we have $k = n - i - 1$ for a nonempty T_i. In particular, T_i are all empty for $0 \leq i < n - k - 1$. The first possibly nontrivial one is thus T_{n-k-1}.

Continuing in this fashion, the next-to-last $f(U_{n-k-2})$ consists of finitely many lines through the origin. Then the last cut by the generic hyperplane L_{n-k-1} picks up the origin of $\mathbf{C}^n$. But then $f(U_{n-k-1}) = 0$ means $T_{n-k-1} = U_{n-k-1}$. The cutting procedure ends.

Consequently, with $\dim(\mathcal{F}_{n-k-1}) = k$ and $\operatorname{codim}(U_{n-k-1}) = n - k - 1$ in U_0, we have

$$\begin{aligned}\dim(U_{n-k-1}) &\leq \dim(Z_n) - (n - k - 1)\\ &\leq \dim(Y_n) - \dim(\mathcal{F}_{n-k-1}) - (n - k - 1)\\ &\leq m_1 + m_2 - 2\end{aligned}$$

by (13.31). Now, since the variety J_n is the union of all U_{n-k-1} as U_0 goes through all the irreducible components of Z_n, we deduce $\dim(J_n) \leq m_1 + m_2 - 2$. Hence, if $m_2 \geq m_1 + n$ (respectively, $m_2 \geq m_1 + n - 1$), then

$$\dim(J_n) \leq m_1 + m_2 - 2 \leq 2m_2 - n - 2 \leq \dim(V_n^{\mathbf{C}}) - 2$$

(respectively, $\leq \dim(V_n^{\mathbf{C}}) - 1$). So, if $m_2 \geq 2m_1$, then J_n is of codimension at least 2 for all $n \leq m_1$. Further, if $m_2 = 2m_1 - 1$, then J_n is of codimension at least 2 for all $n \leq m_1 - 1$, and J_{m_1} is of codimension at least 1. This implies the statements of Proposition 46. □

The classification result Theorem 47 is therefore established.

Remark 53. The standard matrix form of A_1 and of A_2 in the case $(m_1, m_2) = (2, m_2), m_2 \geq 3$, given in Remark 51, leads to

$$p_1 = 2\sum_{j=1}^{l}(x_j y_j + x_{l+j} y_{l+j}), \quad p_2 = -2\sum_{j=1}^{l}(x_j y_{l+j} - x_{l+j} y_j),$$

where $m_2 = 2l + 1$. Then $J_2 = V_2^{\mathbf{C}} \cap Z_2$, which by (13.22) and (13.23) is

$$J_2 = \{((u, -\varepsilon\sqrt{-1}u, t), (v, \varepsilon\sqrt{-1}v, s)) \in Z_2 : \sum_{j=1}^{l} u_j v_j = 0\}, \tag{13.32}$$

where $u, v \in \mathbf{C}^l$, $t, s \in \mathbf{C}$, and $\varepsilon = \pm$. It follows that $\dim(J_2) = \dim(Z_2) - 1 = m_2$, by (13.24). Thus, $\operatorname{codim}(J_2) \geq 2$ in $V_2^{\mathbf{C}}$ (which is of dimension $2m_2 - 2$), provided $m_2 \geq 4$.

For $m_2 = 3$, J_2 has codimension 1 in $V_2^{\mathbf{C}}$. Indeed, in this case the bi-projective variety $\mathbf{P}_b V_2^{\mathbf{C}}$ defined by $p_1 = p_2 = 0$ in $\mathbf{C}P^2 \times \mathbf{C}P^2$ is made up of four irreducible components,

$$\begin{aligned} \mathbf{P}_b V_2^{\mathbf{C}} = \mathbf{C}P^1_+ \times \mathbf{C}P^1_+ \cup \mathbf{C}P^1_- \times \mathbf{C}P^1_- \\ \cup \{[0:0:1]\} \times \mathbf{C}P^2 \cup \mathbf{C}P^2 \times \{[0:0:1]\} \end{aligned}$$

where $\mathbf{C}P^1_\varepsilon \hookrightarrow \mathbf{C}P^2$ by $[u:s] \mapsto [u : \varepsilon\sqrt{-1}u : s]$. Hence (p_1, p_2) is not a prime ideal in $\mathbf{C}[x_1, x_2, y_1, y_2]$ and Proposition 43 says then that $\operatorname{codim}(J_2) \leq 1$ in $V_2^{\mathbf{C}}$. In fact, in this case $\operatorname{codim}(J_2) = 1$ in $V_2^{\mathbf{C}}$, and $x_2(y_1^2 + y_2^2) \in (p_1, p_2)$, but neither x_2 nor $y_1^2 + y_2^2$ is in the ideal.

In view of the known classification of Takagi [30] for $m_1 = 1$, Ozeki-Takeuchi [25, II] for $m_1 = 2$, and Stolz's result [29] on the multiplicities $m_1 \leq m_2$ that states that $(m_1, m_2) \neq (2,2)$ or $(4,5)$ must be that of an isoparametric hypersurface of FKM-type, we obtain from Theorem 47 that all isoparametric hypersurfaces with four principal curvatures in spheres, whose multiplicities are not $(2,2)$ or $(4,5)$, are of FKM-type, except possibly for those whose multiplicities are one of the following 3 pairs $(3,4)$, $(6,9)$, $(7,8)$. The $(4,5)$ case also remains unclassified.

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