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Cauchy transforms of point masses: The logarithmic derivative of polynomials

By J. M. Anderson and V. Ya. Eiderman*

1. Introduction

For a polynomial

$$Q_N(z) = \prod_{k=1}^{N} (z - z_k)$$

of degree N, possibly with repeated roots, the logarithmic derivative is given by

$$\frac{Q_N'(z)}{Q(z)} = \sum_{k=1}^{N} \frac{1}{z - z_k}.$$

For fixed $P > 0$ we define sets $\mathcal{Z}(Q_N, P)$ and $\mathcal{X}(Q_N, P)$ by

$$\begin{aligned} \mathcal{Z}(Q_N, P) &= \left\{ z : z \in \mathbb{C},\ \left| \sum_{k=1}^{N} \frac{1}{z - z_k} \right| > P \right\}, \\ \mathcal{X}(Q_N, P) &= \left\{ z : z \in \mathbb{C},\ \sum_{k=1}^{N} \frac{1}{|z - z_k|} > P \right\}. \end{aligned} \tag{1.1}$$

Clearly $\mathcal{Z}(Q_N, P) \subset \mathcal{X}(Q_N, P)$. Let $D(z, r)$ denote the disk

$$\{\zeta : \zeta \in \mathbb{C},\ |\zeta - z| < r\}.$$

In [2] it was shown that $\mathcal{X}(Q_N, P)$ is contained in a set of disks $D(w_j, r_j)$ with centres w_j and radii r_j such that

$$\sum_j r_j < \frac{2N}{P}(1 + \log N),$$

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or, as we prefer to state it,

$$M(\mathcal{X}(Q_N, P)) < \frac{2N}{P}(1 + \log N). \tag{1.2}$$

Here M denotes 1-dimensional Hausdorff content defined by

$$M(A) = \inf \sum_j r_j,$$

where the infimum is taken over all coverings of a bounded set A by disks with radii r_j. The question of the sharpness of the bound in (1.2) was left open in [2]. We prove – Theorem 2.3 below – that the estimate (1.2) for $\mathcal{X}$ is essentially best possible.

Obviously, (1.2) implies the same estimate for $M(\mathcal{Z}(Q_N, P))$. It was suggested in [2] that in this case the $(1+\log N)$ term could be omitted at the cost of multiplying by a constant. The above suggestion means that in the passage from the sum of moduli to the modulus of the sum in (1.1) essential cancellation should take place. As a contribution towards this end the authors showed that any straight line L intersects $\mathcal{Z}(Q_N, P)$ in a set F_P of linear measure less than $2eP^{-1}N$. Further information about the complement of F_P under certain conditions on $\{z_k\}$ is obtained in [1]. Clearly we may assume that $N > 1$ and we do so in what follows, for ease of notation.

However, it was shown in [3] that there is an absolute positive constant c such that for all $N \geqslant 3$ one can find a polynomial Q_N of degree N for which the projection Π of $\mathcal{Z}(Q_N, P)$ onto the real axis has measure greater than

$$\frac{c}{P} N (\log N)^{\frac{1}{2}} (\log \log N)^{-\frac{1}{2}}, \quad N \geqslant 3. \tag{1.3}$$

Throughout this paper c will denote an absolute positive constant, not necessarily the same at each occurrence. Marstand suggested in [3] that the best result for $M(\mathcal{Z}(Q_N, P))$ would be obtained by omitting the $\log\log$ -term in (1.3). It is the object of this paper to show that this is indeed the case and that the corresponding result is then, apart from a constant best possible (Theorems 2.1 and 2.2 below). Thus the cancellation mentioned above does indeed occur but in general it is not as “strong” as was suggested in [2].

2. Results

We prove

THEOREM 2.1. *Let z_k, $1 \leqslant k \leqslant N$, $N > 1$, be given points in $\mathbb{C}$. There is an absolute constant c such that for every $P > 0$ there exists a set of disks $D_j = D(w_j, r_j)$ so that*

$$\left| \sum_{k=1}^{N} \frac{1}{z - z_k} \right| < P, \quad z \in \mathbb{C} \setminus \bigcup_j D_j \tag{2.1}$$

and

$$\sum_j r_j < \frac{c}{P} N (\log N)^{\frac{1}{2}}.$$

In other words

$$M(\mathcal{Z}(Q_N, P)) < \frac{c}{P} N (\log N)^{\frac{1}{2}}. \tag{2.2}$$

THEOREM 2.2. *For every* $N > 1$ *and every* $P > 0$ *there are points* $z_1, z_2, \ldots, z_N$ *such that*

$$M(\mathcal{Z}(Q_N, P)) > \frac{c}{P} N (\log N)^{\frac{1}{2}}, \tag{2.3}$$

where

$$Q_N(z) = \prod_{i=1}^{N} (z - z_i),$$

i.e. for every set of disks satisfying (2.1) *we have*

$$\sum_j r_j > \frac{c}{P} N (\log N)^{\frac{1}{2}}.$$

Moreover there is a straight line L *such that* $|\Pi| > \frac{cN}{P} (\log N)^{1/2}$, *where* Π *is the projection of* $\mathcal{Z}(Q_N, P)$ *onto* L *and* $|\cdot|$ *denotes length. Here*, *as always*, c *denotes absolute constants.*

The logarithmic derivative is, of course, an example of a Cauchy transform. For a complex Radon measure ν in $\mathbb{C}$ the Cauchy transform $\mathcal{C}\nu(z)$ is defined by

$$\mathcal{C}\nu(z) = \int_{\mathbb{C}} \frac{d\nu(\zeta)}{\zeta - z}, \qquad z \in \mathbb{C} \backslash \operatorname{supp} \nu.$$

In fact $\mathcal{C}\nu(z)$ is defined almost everywhere in $\mathbb{C}$ with respect to area measure. In analogy with (1.1) we set

$$\mathcal{Z}(\nu, P) = \{z : z \in \mathbb{C}, \quad |\mathcal{C}\nu(z)| > P\}.$$

The proof of Theorem 2.1 is based on results of Melnikov [5] and Tolsa [6], [7]. The important tool is the concept of curvature of a measure introduced in [5].

For the counter example required for the lower estimate in Theorem 2.2 we need a Cantor-type set E_n. We set $E^{(0)} = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and at the ends of $E^{(0)}$ we take subintervals $E_j^{(1)}$ of length $\frac{1}{4}$, $j = 1, 2$. Let $E^{(1)} = \bigcup_{j=1}^{2} E_j^{(1)} = \left[-\frac{1}{2}, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \frac{1}{2}\right]$. We then construct, in a similar manner, two sub-intervals $E_{j,i}^{(2)}$ of length 4^{-2} in each $E_j^{(1)}$ and denote by $E^{(2)}$ the union of the four intervals

$E^{(2)}_{j,i}$. Continuing this process we obtain a sequence of sets $E^{(n)}$ consisting of 2^n intervals of length 4^{-n}. We define

$$E_n = E^{(n)} \times E^{(n)},$$

the Cartesian product, and note that E_n consists of 4^n squares $E_{n,k}$, $k = 1, 2, \dots, 4^n$ with sides parallel to the coordinate axes. The following is the explicit form of Theorem 2.2.

Theorem 2.2′. *Let $P > 0$ be given and set $E = (100P)^{-1} n^{\frac{1}{2}} 4^n E_n$ where E_n is the set defined above. Let ν be the measure formed by 4^{n+1} Dirac masses* (*i.e. unit charges in the language of Potential Theory*) *located at the corners of the squares which form E_n. Then*

$$M(\mathcal{Z}(\nu, P)) > \frac{cN}{P} (\log N)^{\frac{1}{2}} \text{ where } N = 4^{n+1}. \tag{2.4}$$

Moreover, there is a straight line L such that $|\Pi| > \frac{cN}{P}(\log N)^{\frac{1}{2}}$.

The constant 100 appearing in Theorem 2.2′ is merely a constant convenient for our proof.

For fixed $N \geqslant 4$ (not necessarily of the form $N = 4^{n+1}$) we can choose n with $4^{n+1} \leqslant N < 4^{n+2}$ to see that (2.4) holds for all $N \in \mathbb{N}$ with a different constant c. To obtain a corresponding measure ν with N Dirac masses we locate the remaining $N - 4^{n+1}$ points sufficiently far from the set E in order to make the influence of these points as small as we want.

A set homothetic to E_n also gives the example which shows the sharpness of the estimate (1.2). We have

Theorem 2.3. *For the set $E = (\sqrt{2}P)^{-1} n 4^n E_n$ and for the measure ν as in Theorem 2.2′ we have*

$$M(\mathcal{X}(Q_N, P)) > \frac{cN}{P} (\log N). \tag{2.5}$$

In Section 5 we give a generalization of Theorem 2.1.

3. Preliminary lemma and notation

Following [5] we define the Menger curvature $c(x, y, z)$ of three pairwise different points $x, y, z \in \mathbb{C}$ by

$$c(x, y, z) = [R(x, y, z)]^{-1},$$

where $R(x, y, z)$ is the radius of the circle passing through x, y, z with $R(x, y, z) = \infty$ if x, y, z lie on some straight line (or if two of these points coincide). For a positive Radon measure μ we set

$$c_\mu^2(x) = \iint c(x,y,z)^2 d\mu(y) d\mu(z)$$

and we define the curvature $c(\mu)$ of μ as

$$c^2(\mu) = \int c_\mu^2(x) d\mu(x) = \iiint c(x,y,z)^2 d\mu(x) d\mu(y) d\mu(z).$$

The analytic capacity $\gamma(E)$ of a compact set $E \subset \mathbb{C}$ is defined by

$$\gamma(E) = \sup \left| f'(\infty) \right|,$$

where the supremum is taken over all holomorphic functions $f(z)$ on $\mathbb{C} \backslash E$ with $|f(z)| \leqslant 1$ on $\mathbb{C} \backslash E$. Here $f'(x) = \lim\limits_{z \to \infty} z(f(z) - f(\infty))$. The capacity γ_+ is defined as follows:

$$\gamma_+(E) = \sup \mu(E),$$

where the supremum runs over all positive Radon measures μ supported in E such that $\mathcal{C}\mu(z) \in L^\infty(\mathbb{C})$ and $\|\mathcal{C}\mu\|_\infty \leqslant 1$. Since $|\mathcal{C}'\mu(\infty)| = \mu(E)$, we have $\gamma_+ \leqslant \gamma$.

Theorem A. *For any compact set $E \subset \mathbb{C}$ we have*

$$\gamma_+(E) \geqslant c \cdot \sup \left\{ [\mu(E)]^{\frac{3}{2}} \left[\mu(E) + c^2(\mu) \right]^{-\frac{1}{2}} \right\}, \tag{3.1}$$

where c is an absolute constant and the supremum is taken over all positive measures μ supported in E such that $\mu(D(z,r)) \leqslant r$ for any disk $D(z,r)$.

The inequality (3.1) with γ instead of γ_+ was obtained by Melnikov [5]. The strengthened form is due to Tolsa [7].

Theorem B ([8, p. 321]). *There is an absolute constant c such that for any positive Radon measure ν and any $\lambda > 0$*

$$\gamma_+ \{ z : z \in \mathbb{C},\ \mathcal{C}_*\nu(z) > \lambda \} \leqslant \frac{c\,\|\nu\|}{\lambda}. \tag{3.2}$$

Here $\mathcal{C}_*\nu(z) = \sup\limits_{\varepsilon > 0} |\mathcal{C}_\varepsilon \nu(z)|$ where $\mathcal{C}_\varepsilon$ denotes the truncated Cauchy transform

$$\mathcal{C}_\varepsilon \nu(z) = \int\limits_{|\zeta - z| > \varepsilon} \frac{d\nu(\zeta)}{\zeta - z}.$$

We apply this result (excepting the proof of Theorem 5.1) only to discrete measures ν with unit charges at the points z_k, $k = 1, 2, \ldots, N$ according to multiplicity. So the support of ν is $\{z_1, z_2, \ldots, z_N\}$ and $\|\nu\| = N$. Also

$$\mathcal{C}_*\nu(z) \geqslant |\mathcal{C}\nu(z)| = \left| \sum_{i=1}^N \frac{1}{z - z_i} \right|,\ z \in \mathbb{C} \backslash \{z_1, z_2, \ldots, z_N\}.$$

For $P > 0$ we set

$$\mathcal{Z}(P) = \mathcal{Z}(\nu, P) = \mathcal{Z}(Q_N, P) = \{z : z \in \mathbb{C},\ |\mathcal{C}\nu(z)| > P\}$$

and put $M(P) = M(\mathcal{Z}(P))$.

LEMMA 3.1. *Suppose that $P > 0$ and z_k, $1 \leqslant k \leqslant N$, are given and that $M(P) > \frac{10N}{P}$. Then there is a family of disks $D_j = D(w_j, r_j)$, $j = 1, 2, \ldots, N_0$ (different from the disks of Theorem 2.1), with the following properties*

1) $N_0 \leqslant N$,
2) $\bar{D}_j \subset \mathcal{Z}\left(\frac{P}{2}\right), \quad j = 1, 2, \ldots, N_0,$
3) $D(w_k, 4r_k) \cap \left(\bigcup_{j \neq k} D_j\right) = \emptyset,\ k = 1, 2, \ldots, N_0,$
4) $\sum_j r_j > cM(P),$
5) *if μ is a positive measure concentrated on $\bigcup_j D_j$ such that $\mu(D_j) = r_j$ and μ is uniformly distributed on each D_j, $j = 1, 2, \ldots, N_0$ (with different densities, of course) then $\mu(D(w, r)) < cr$ for every disk $D \subset \mathbb{C}$.*

Proof. (a) Let $d(z) = \operatorname{dist}(z, S)$ for our set $S = \{z_1, z_2, \ldots, z_N\}$. We apply Lemma 1 in [1] (which is an analogue of Cartan's Lemma) with $H = \frac{N}{P}$, $\alpha = 1$, $n = N$. There is a set of at most N disks $D'_k = D(w'_k, h_k)$ whose radii satisfy the inequality

$$\sum_k h_k \leqslant \frac{2N}{P} \tag{3.3}$$

such that if

$$\mathcal{Z}'(P) = \bigcup_k D'_k,$$

then $\nu(D(z, r)) < Pr$ for all $r > 0$ and all $z \notin \mathcal{Z}'(P)$. One may also obtain this result, with a worse constant, by standard arguments based on the Besicovitch covering lemma. Hence, for $z \notin \mathcal{Z}'(P)$

$$\left|\mathcal{C}'\nu(z)\right| \leqslant \sum_i \frac{1}{|z - z_i|^2} < \sum_{j=1}^{\infty} \left\{ \sum_{(i,j)} \frac{1}{|z - z_i|^2} \right\},$$

where $\sum_{(i,j)}$ denotes summation over the annulus $2^{j-1}d(z) \leqslant |z - z_i| < 2^j d(z)$. This latter sum does not exceed

$$\sum_{j=1}^{\infty} \frac{P 2^j d(z)}{[2^{j-1} d(z)]^2} = \frac{4P}{d(z)} \sum_{j=1}^{\infty} 2^{-j} = \frac{4P}{d(z)}. \tag{3.4}$$

We now set

$$\mathcal{Z}''(P) = \{z : z \in \mathcal{Z}(P),\ \operatorname{dist}(z, \mathcal{Z}'(P)) > (0.1)d(z)\},$$
$$\mathcal{Z}_1(P) = \{z : \operatorname{dist}(z, \mathcal{Z}'(P)) \leqslant (0.1)d(z)\},$$

so that $\mathcal{Z}''(P) = \mathcal{Z}(P)\backslash\mathcal{Z}_1(P)$.

Let $z \in \mathcal{Z}_1(P)$ and let $D'_k = D(w'_k, h_k)$ be a disk such that $\operatorname{dist}(z, \mathcal{Z}'(P)) = \operatorname{dist}(z, D'_k)$. By the construction in [2], each disk D'_k contains at least one point $z_j \in S$. Hence

$$\operatorname{dist}(z, \mathcal{Z}'(P)) \leqslant (0.1)d(z) \leqslant (0.1)\,|z - z_j| \leqslant (0.1)[\operatorname{dist}(z, \mathcal{Z}'(P)) + 2h_k],$$

so that

$$\operatorname{dist}(z, \mathcal{Z}'(P)) \leqslant \frac{2}{9}h_k,$$

and hence

$$\left|z - w'_k\right| < \operatorname{dist}(z, \mathcal{Z}'(P)) + 2h_k \leqslant \frac{20}{9}h_k.$$

Thus

$$\mathcal{Z}_1(P) \subset \bigcup_k D\left(w'_j, \frac{20}{9}h_k\right).$$

Since $M(P) > \frac{10N}{P}$ we have, using (3.3),

$$\frac{20}{9}\sum_k h_k \leqslant \frac{40}{9}\frac{N}{P} < \frac{4}{9}M(P) < \frac{1}{2}M\left(P\right).$$

Hence

$$\text{(3.5)}\quad M(\mathcal{Z}''(P)) = M[\mathcal{Z}(P)\backslash\mathcal{Z}_1(P)]$$
$$\geqslant M(\mathcal{Z}(P)) - M(\mathcal{Z}_1(P)) \geqslant M(P) - \frac{20}{9}\sum_k h_k > \frac{1}{2}M\left(P\right).$$

For every $j = 1, 2, \ldots, N$ for which the set $\{w : w \in \mathcal{Z}''(P),\ d(w) = |w - z_j|\}$ is not empty we finally choose a point $w_j \in \mathcal{Z}''(P)$ such that $d(w_j) = |w_j - z_j|$ and

$$d(w_j) > \tfrac{3}{4}\sup\left\{d(w) : w \in \mathcal{Z}''(P),\, d(w) = |w - z_j|\right\}.$$

The point is that not only is $|\mathcal{C}\nu(w_j)| > P$ but we can use the estimate (3.4) on the derivative to show that a disk around w_j is contained in $\mathcal{Z}\left(\frac{P}{2}\right)$. So set $r_j = (0.1)d(w_j)$ and consider the disks $D_j = D(w_j, r_j)$. Clearly $D_j \subset \mathbb{C}\backslash\mathcal{Z}'(P)$ and so, for every $z \in D_j$,

$$\text{(3.6)}\quad |\mathcal{C}\nu(z)| = \left|\mathcal{C}\nu(w_j) - \int_z^{w_j} \mathcal{C}'\nu(t)dt\right| > |\mathcal{C}\nu(w_j)| - \int_z^{w_j} \left|\mathcal{C}'\nu(t)\right| |dt|$$
$$> P - \frac{4P}{d(w_j) - |w_j - z|}\cdot|w_j - z| > P - \frac{4P(0.1)d(w_j)}{d(w_j) - (0.1)d(w_j)}$$
$$= \frac{5}{9}P > \frac{P}{2},$$

by (3.4). Hence $\bar{D}_j = \bar{D}(w_j, r_j) \subset \mathcal{Z}\left(\frac{P}{2}\right)$ and conditions 1) and 2) of Lemma 3.1 are satisfied.

We now show that we can extract a subsequence D_{j_i} with the properties 3), 4) and 5). Take any point $z \in \mathcal{Z}''(P)$ and suppose that $d(z) = |z - z_j|$. Then $|z - w_j| \leqslant |z - z_j| + |z_j - w_j| \leqslant \frac{4}{3}d(w_j) + d(w_j) = \frac{70}{3}r_j < 25r_j$, so that

$$\mathcal{Z}''(P) \subset \bigcup_j D(w_j, 25r_j).$$

(b) Denote by D_{j_1} the disk $D(w_j, r_j)$ with maximal r_j. We delete all disks $D_j,\ j \neq j_1$ for which $D_j \cap D(w_{j_1}, 4r_{j_1}) \neq \emptyset$. From the remaining disks $d_j,\ j \neq j_1$ we select the maximal disk $D_{j_2} = D(w_{j_2}, r_{j_2})$ and remove all disks for which $D_j \cap D(w_{j_2}, 4r_{j_2}) \neq \emptyset$, and so on. For all the disks $D(w_j, r_j)$ which we remove on the k'th step, $r_j \leqslant r_{j_k}$ and $|w_j - w_{j_k}| < 5r_{j_k}$. Hence

$$D(w_j, 25r_j) \subset D(w_{j_k}, 30r_{j_k}).$$

For simplicity, henceforth we denote the family of disks $\{D_{j_k}\}$ so obtained also by $\{D_k\}$. Note that $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_{N_1}$, where $N_1 \leqslant N$. We have

$$\mathcal{Z}''(P) \subset \bigcup_k D(w_k, 30r_k), \tag{3.7}$$

and, by (3.5), conditions 3) and 4) are satisfied.

(c) Let μ be a measure satisfying the assumptions of 5). To prove 5) we extract a further subsequence from $\{D_k\}$ with preservation of the property 4). We denote by $\mathcal{Q}(w, \ell)$ the square

$$\mathcal{Q}\,(w, \ell) = \{z = x + iy : |x - a| < \ell,\ |y - b| < \ell\},$$

where $w = a + ib$, and set

$$J(\mathcal{Q}) = \{j : D_j \cap \partial\mathcal{Q} \neq \emptyset\}.$$

We shall show that

$$\mu\left(\mathcal{Q} \cap \{\cup\,(D_j : j \in J(\mathcal{Q}))\}\right) < 4\ell. \tag{3.8}$$

We note that each D_j is contained in a square $\mathcal{Q}(D_j)$ (with sides parallel to the coordinate axes) and with side-length $2r_j$ and all squares $\mathcal{Q}(D_j)$ are disjoint. If $\mathcal{Q}(D_j)$ intersects only one side of $\mathcal{Q}$ then $\mu(\mathcal{Q}(D_j) \cap \mathcal{Q}) \leqslant r_j = \frac{1}{2}\,|\mathcal{Q}(D_j) \cap \partial\mathcal{Q}|$. If, however, $\mathcal{Q}(D_j)$ intersects at least two sides of $\mathcal{Q}$ we suppose that the side-lengths of the rectangle $\mathcal{Q} \cap \mathcal{Q}(D_j)$ are $2\alpha r_j$ and $2\beta r_j$ where $0 \leqslant \alpha, \beta \leqslant 1$. The density of the measure μ in D_j is $(\pi r_j)^{-1}$ and so

$$\mu\left(\mathcal{Q} \cap \mathcal{Q}(D_j)\right) < 4\alpha\beta r_j^2(\pi r_j)^{-1} = 4\alpha\beta r_j(\pi^{-1}).$$

But

$$4\alpha\beta(\pi^{-1})r_j < 2\alpha\beta r_j \leqslant (\alpha + \beta)r_j,$$

and so, again

$$\mu\left(\mathcal{Q}\cap\mathcal{Q}(D_j)\right)\leqslant\tfrac{1}{2}\left|\mathcal{Q}(D_j)\cap\partial\mathcal{Q}\right|.$$

Thus

$$\mu\left(\mathcal{Q}\cap\{\cup\left(D_j : j\in J(\mathcal{Q})\right)\}\right)\leqslant\tfrac{1}{2}\left|\partial\mathcal{Q}\right|=\tfrac{1}{2}\cdot 8\ell=4\ell.$$

We set $\ell_0=10r_{N_1}$ and

$$\mathcal{Q}^{(0)}(k,m)=\mathcal{Q}\left((1+2k)\ell_0+i(1+2m)\ell_0,\,\ell_0\right),\ k,m=0,\pm1,\pm2,\dots\ .$$

Suppose that there are squares

$$\mathcal{Q}_n^{(0)}=\mathcal{Q}^{(0)}(k_n,m_n)$$

and that

$$\mu(\mathcal{Q}_n^{(0)})=\mu\Bigg(\mathcal{Q}_n^{(0)}\cap\Big(\bigcup_j D_j\Big)\Bigg)>6\ell_0.$$

From (3.8) there is at least one disk D_j contained in $\mathcal{Q}_n^{(0)}$. For such disks we have $r_j\leqslant\ell_0$ and $\mu(D_j)=r_j$.

We may, therefore, remove a number of disks D_j contained in $\mathcal{Q}_n^{(0)}$ in such a way that, for the remaining disks D_j,

$$5\ell_0<\mu(\mathcal{Q}_n^{(0)})<6\ell_0.$$

The left inequality, together with (3.8), implies that

$$\sum_j{}^{*}r_j>\ell_0,$$

where the sum extends over those j for which $D_j\subset\mathcal{Q}_n^{(0)}$.

We now set $\ell_1=2\ell_0$ and

$$\mathcal{Q}^{(1)}(k,m)=\mathcal{Q}(\,(1+2k)\ell_1+i(1+2m)\ell_1,\,\ell_1)\,.$$

In a similar manner we remove disks from the corresponding squares

$$\mathcal{Q}_n^{(1)}=\mathcal{Q}^{(1)}(k_n,m_n)$$

for which $\mu(\mathcal{Q}_n^{(1)})>6\ell_1$. Again we obtain

$$5\ell_1<\mu(\mathcal{Q}_n^{(1)})<6\ell_1.$$

Repeating this procedure with $\ell_p=2^p\ell_0$ sufficiently many times we obtain a set of disks $\{D_j\}$ satisfying conditions 1), 2) and 3). Since for every square $\mathcal{Q}^{(p)}(k,m)$ we have

$$\mu(\mathcal{Q}^{(p)}(k,m))<6\ell_p,$$

condition 5) is also satisfied.

To verify 4) we denote by $\tilde{\mathcal{Q}}_n^{(p)}$ those squares $\mathcal{Q}_n^{(p)}$ such that there are no squares $\mathcal{Q}_m^{(q)}$ with $q > p$ containing $\tilde{\mathcal{Q}}_n^{(p)}$. Hence all the squares $\tilde{\mathcal{Q}}_n^{(p)}$ are disjoint and

$$\sum r_j > \ell_p, \tag{3.9}$$

where the sum extends over those j for which $D_j \subset \tilde{\mathcal{Q}}_n^{(p)}$. If $w_n^{(p)}$ denotes the centre of $\tilde{\mathcal{Q}}_n^{(p)}$, so that $\tilde{\mathcal{Q}}_n^{(p)} = \mathcal{Q}(w_n^{(p)}, \ell_p)$, then all disks deleted at stage (c) are contained in $\bigcup_{n,p} \tilde{\mathcal{Q}}_n^{(p)}$. By (3.7),

$$\mathcal{Z}''(P) \subset \left[\bigcup \mathcal{Q}\left(w_n^{(p)}, 30\ell_p\right)\right] \bigcup \left[\bigcup_k D(w_k, 30r_k)\right],$$

where the first union is taken over all squares $\tilde{\mathcal{Q}}_n^{(p)}$. Hence

$$M(P) \leqslant 30\left(\sum \sqrt{2}\ell_p + \sum r_k\right),$$

where, again, the first sum is taken over all squares $\tilde{\mathcal{Q}}_n^{(p)}$. By (3.9)

$$M(P) \leqslant 30\left(\sqrt{2}\sum_k r_k + \sum_k r_k\right) < 75 \sum_k r_k,$$

and the proof of Lemma 3.1 is complete.

4. Another lemma

LEMMA 4.1. *Suppose that a family of disks D_j, $j = 1, 2, \ldots, N_0$, $N_0 > 1$, and a measure μ satisfy the conditions* 3) *and* 5) *in Lemma* 3.1. *Then there exists an absolute constant c so that*

$$c^2(\mu) \leqslant cH \log N_0, \tag{4.1}$$

where

$$H = \sum_{j=1}^{N_0} r_j = \mu(\mathbb{C}).$$

Proof. Suppose that among the N_0 disks $D(w_j, r_j)$ there are N_k disks with $2^{-k}H \leqslant r_j < 2^{-k+1}H$, $k = 2, 3, \ldots, s$ and N_1 disks with $2^{-1}H \leqslant r_j$. Here s is such that $2^{-s}H \leqslant r_j$ for all $j = 1, 2, \ldots, N_0$. Obviously

$$N_1 + N_2 + \cdots + N_s = N_0.$$

Let

$$B_1 = \bigcup_j \left\{D_j : 2^{-1}H \leqslant r_j\right\},$$

$$B_k = \bigcup_j \left\{D_j : 2^{-k}H \leqslant r_j < 2^{-k+1}H\right\}$$

for $k = 2, 3, \ldots, s$. Possibly $N_k = 0$ and $B_k = \emptyset$ for some k.

Now take any $x \in \bigcup_j D_j$ and evaluate $c_\mu^2(x)$. Suppose that $x \in D_j \subset B_k$ and set $\mathcal{F}(x) = \{(y,z) \in \mathbb{C}^2 : |z-x| \leqslant |y-x|\}$. For $(y,z) \in \mathcal{F}(x)$,

$$2R(x,y,z) \geqslant |y-x|\,.$$

Hence

$$c_\mu^2(x) \leqslant 2 \iint\limits_{\mathcal{F}(x)} \frac{1}{R^2(x,y,z)}\, d\mu(y) d\mu(z) \leqslant 8 \iint\limits_{\mathcal{F}(x)} \frac{1}{|y-x|^2}\, d\mu(y) d\mu(z).$$

If we set $\mu_x(r) = \mu(D(x,r))$ then this latter term equals

$$8 \int\limits_{\mathbb{C}} \frac{\mu(D(x,|y-x|))}{|y-x|^2} d\mu(y) = 8 \int_0^\infty \frac{\mu_x(r)}{r^2}\, d\mu_x(r).$$

A related estimate is due to Mattila [4].

By conditions 3) and 5) of Lemma 3.1, for $x \in D_j$,

$$\begin{aligned} \mu_x(r) &\leqslant \tfrac{r^2}{r_j}, \quad 0 < r \leqslant 2r_j, \\ &< c\,r, \quad r > 2r_j, \end{aligned}$$

for some absolute constant c. If we define

$$h(r) = \begin{cases} \frac{cr^2}{r_j}, & 0 < r \leqslant 2r_j, \\ 2cr, & 2r_j < r \leqslant \frac{H}{2c}, \\ H, & r > \frac{H}{2c}, \end{cases}$$

then $h(r)$ is a continuous nondecreasing function with $h(r) \geqslant \mu_x(r)$ for $0 < r < \infty$ provided the constant $c \geqslant 1$ is suitably chosen. Now

$$\begin{aligned} \frac{\mu_x(r)}{r} &\leqslant \frac{h(r)}{r} \to 0 \quad \text{as } r \to 0, \\ \frac{\mu_x(r)}{r} &\leqslant \frac{H}{r} \to 0 \quad \text{as } r \to \infty, \end{aligned}$$

and hence, integrating by parts we obtain

$$c_\mu^2(x) \leqslant 8 \int_0^\infty \frac{\mu_x(r)}{r^2}\, d\mu_x(r) = 8 \int_0^\infty \frac{[\mu_x(r)]^2}{r^3}\, dr < 8 \int_0^\infty \frac{h^2(r)}{r^3}\, dr.$$

If $x \in B_k$ this last integral does not exceed

$$c + c \log \frac{H}{r_j} < c + ck,$$

for some c. Thus

$$c^2(\mu) = \sum_{k=1}^{s} \int_{B_k} c_\mu^2(x) d\mu(x) < \sum_{k=1}^{s} (c+ck)\mu(B_k).$$

But $\sum_{k=1}^{s} \mu(B_k) = H$ and

$$\mu(B_k) = \sum\nolimits^* \mu(D_j) = \sum\nolimits^* r_j \leqslant 2HN_k 2^{-k},$$

where the sums extend over those j for which $D_j \subset B_k$. We have

$$c^2(\mu) < cH + cH \sum_{k=1}^{s} kN_k 2^{-k}. \tag{4.2}$$

On the other hand

$$H = \sum_j \mu(B_j) = \sum_{k=1}^{s} \left\{ \sum\nolimits^* \mu(D_j) \right\} \geqslant \sum_{k=1}^{s} 2^{-k} HN_k,$$

so that

$$\sum_{k=1}^{s} 2^{-k} N_k \leqslant 1.$$

Here again, the inner sum $\sum^*$ extends over those j with $D_j \subset B_k$.

We set $K = [\log_2 N_0] + 1$ where $[x]$ denotes the integer part of x. We may suppose that $K < s$; otherwise we set $N_s = N_{s+1} = \cdots = N_K = 0$. Then

$$\begin{aligned} \sum_{k=1}^{\infty} kN_k 2^{-k} &\leqslant \left(\sum_{k=1}^{K} + \sum_{k=K+1}^{\infty} \right) kN_k 2^{-k} \\ &\leqslant K \sum_{k=1}^{K} N_k 2^{-k} + N_0 \sum_{k=K+1}^{\infty} k 2^{-k} < 2K + 2 < c \log N_0, \end{aligned} \tag{4.3}$$

since

$$\sum_{k=K+1}^{\infty} k 2^{-k} = (K+2) 2^{-K} < \frac{K+2}{N_0}.$$

The inequalities (4.2) and (4.3) imply (4.1) and Lemma 4.1 is proved.

5. Proof of Theorem 2.1

If $M(P) \leqslant \frac{10N}{P}$ then (2.2) holds and Theorem 2.1 is proved. So suppose that $M(P) > \frac{10N}{P}$. We set $\lambda = \frac{1}{2}P$. By (3.2)

$$\gamma_+(\mathcal{Z}(\lambda)) \leqslant c \frac{2N}{P}. \tag{5.1}$$

Let $E = \bigcup_j \bar{D}_j$ and put $\mu' = c^{-1}\mu$, where D_j, μ and c are the disks, measure and constant in 5) of Lemma 3.1. Clearly μ' satisfies all the conditions of Theorem A. Moreover, by property 4)

$$\mu'(E) > cM(P)$$

for suitable c. From (3.1), with μ' in place on μ, and (4.1) we have, for suitable constants c,

$$\gamma_+(E) > c(\mu'(E))^{3/2}\left[\mu'(E) + c\mu'(E)\log_2 N\right]^{-\frac{1}{2}} \tag{5.2}$$

$$> c\mu'(E)(\log N)^{-\frac{1}{2}} > cM(P)(\log N)^{-\frac{1}{2}}.$$

The combination of (5.1), (5.2) and 2) in Lemma 3.1 gives

$$c\frac{N}{P} \geqslant \gamma_+(\mathcal{Z}(\lambda)) \geqslant \gamma_+(E) > cM(P)(\log N)^{-\frac{1}{2}},$$

which proves Theorem 2.1.

Remark. Although the same number N appears in the two factors N and $(\log N)^{\frac{1}{2}}$ in (2.2), the meaning in these factors is different. The first factor is the total charge of the measure ν but, in the second factor, N is the number of points and this reflects the complexity of the geometry of $\mathcal{Z}(P)$. More exactly this fact is illustrated by the following generalization of Theorem 2.1.

THEOREM 5.1. *Let points z_k in $\mathbb{C}$ and numbers (generally speaking, complex) ν_k, $1 \leqslant k \leqslant N$, $N > 1$, be given. There is an absolute constant c such that for every $P > 0$*

$$M\left(z : \left|\sum_{k=1}^{N} \frac{\nu_k}{z - z_k}\right| > P\right) < \frac{c}{P}\|\nu\|(\log N)^{\frac{1}{2}},$$

where $\|\nu\| = \sum_{k=1}^{N} |\nu_k|$.

Sketch of the proof. It is claimed in [6, Section 3] that (3.2) holds for any complex Radon measure ν and any $\lambda > 0$. Moreover, one may easily verify that essentially the same arguments as in the proof of Lemma 3.1 work in the more general situation with arbitrary charges ν_k. The required corrections in this case are obvious; for example, we should write $\|\nu\|$ instead of N in the inequality $M(P) > 10N/P$, in (3.3) etc. Thus, the same estimates as above give Theorem 5.1.

6. Proof of Theorem 2.2′

For convenience we consider the set E_n with the normalized measure μ, consisting of 4^{n+1} charges at the corners of $E_{n,k}$ such that each charge is equal to $4^{-(n+1)}$. We denote the centre of $E_{n,k}$ by $z_{n,k}$ and let

$$\mathcal{E} = \left\{E_{n,k} : |\mathrm{Re}\,\mathcal{C}\mu(z_{n,k})| > (0.01)n^{\frac{1}{2}}\right\}.$$

Let $\#F$ denote the number of elements in a set F.

LEMMA 6.1. *There is an absolute positive constant c so that*

$$\#\mathcal{E} > c4^n. \tag{6.1}$$

Assuming this lemma for the moment we show how Theorem 2.2′ follows.

Proof of Theorem 2.2′. We set

$$\begin{aligned} w(n,P) &= (100P)^{-1} n^{\frac{1}{2}} 4^n, & z'_{n,k} &= w(n,P) z_{n,k}, \\ D_{n,k} &= D(z_{n,k}, (0.05)4^{-n}), & D'_{n,k} &= w(n,P) D_{n,k}, \\ \mathcal{Z} &= \{D_{n,k} : E_{n,k} \in \mathcal{E}\}, & \mathcal{Z}' &= w(n,P)\mathcal{Z} = \left\{D'_{n,k} : E_{n,k} \in \mathcal{E}\right\}. \end{aligned}$$

Then, for $E_{n,k} \in \mathcal{E}$,

$$\left|\mathcal{C}\nu(z'_{n,k})\right| = 4^{n+1} w(n,P)^{-1} \left|\mathcal{C}\mu(z_{n,k})\right| = \frac{4^{n+1}100P}{n^{\frac{1}{2}}4^n} \left|\mathcal{C}\mu(z_{n,k})\right| > 4P.$$

Clearly, $\mu(D(z,r)) < cr$ for $r > 0$ and $z \in \mathcal{Z}$. Continuing to scale by $w(n,P)$ we set

$$z' = w(n,P)z, \quad r' = w(n,P)r.$$

If $z \in \mathcal{Z}$ then

$$\nu\left(D(z',r')\right) = 4^{n+1}\mu(D(z,r)) < c4^{n+1}r = cn^{-\frac{1}{2}}Pr' < Pr',$$

if n is sufficiently large. Moreover, if $z' \in D'_{n,k}$ then

$$\left|z' - z'_{n,k}\right| < (0.05)w(n,P)4^{-n} < (0.1)2^{-\frac{1}{2}} w(n,P)4^{-n} = (0.1)\mathrm{dist}(z'_{n,k}, S).$$

Essentially the same estimates as in (3.4) and (3.6) (with $z'_{n,k}$ and z' in place of w_j and z respectively) yield

$$\mathcal{Z}' \subset \mathcal{Z}(\nu, P). \tag{6.2}$$

Clearly, (2.4) follows from the lower bound of $|\Pi|$. To prove the desired inequality, we project onto the line $y = \frac{x}{2}$. We note that the projection of E_0 onto L is equal to the projection of E_1 onto L. Moreover the projections of all four squares $E_{1,k}$ are disjoint apart from the end points. By self similarity the same is true for the projections of E_n. Since, from (6.2) and (6.1), $\mathcal{Z}' \subset \mathcal{Z}(\nu, P)$ and $\#\mathcal{E} > c4^n$ we have

$$|\Pi| > \left|\mathrm{proj}(\mathcal{Z}')\right| = (\#\mathcal{E})\mathrm{diam}(D'_{n,k}) > c4^n \cdot 2w(n,P) \cdot (0.05)4^{-n},$$

as required. Theorem 2.2′ is proved.

7. Proof of Lemma 6.1

This depends on a further lemma. With each square $E_{n,k}$ we associate a sequence of vectors

$$\bar{e}_1^{(k)},\ \bar{e}_2^{(k)},\dots,\bar{e}_n^{(k)},\quad \bar{e}_l^{(k)}=\left(i_l^{(k)},j_l^{(k)}\right),\quad l=1,2,\dots,n,$$

such that every $\bar{e}_l^{(k)}$ is one of the following vectors: $(-1,-1),\ (-1,1),\ (1,-1),$ $(1,1)$. For example, if $\bar{e}_1^{(k)}=(-1,1)$, then the square $E_{n,k}$ lies in the left hand upper square $\mathcal{Q}$ of E_1; $\bar{e}_2^{(k)}=(1,-1)$ means that the square $E_{n,k}$ is in the right hand lower square of $E_2\cap\mathcal{Q}$ and so on. By this means we have a one-to-one correspondence between squares $E_{n,k}$ and couples $(\bar{\imath}^{(k)},\bar{\jmath}^{(k)})$ of multi-indices $\bar{\imath}^{(k)}=\left(i_1^{(k)},\dots,i_n^{(k)}\right)$ and $\bar{\jmath}^{(k)}=\left(j_1^{(k)},\dots,j_n^{(k)}\right)$.

Lemma 7.1. *Suppose that the squares E_{n,k_1} and E_{n,k_2} are such that $\bar{\jmath}^{(k_1)}=\bar{\jmath}^{(k_2)}$ and*

$$i_p^{(k_1)}=-1,\quad i_p^{(k_2)}=1\quad \textit{for some } p;$$
$$i_r^{(k_1)}=i_r^{(k_2)}\quad \textit{for } r\neq p.$$

Then

$$\operatorname{Re}\mathcal{C}\mu(z_{n,k_1})-\operatorname{Re}\mathcal{C}\mu(z_{n,k_2})>0.02. \tag{7.1}$$

Proof. We split the squares $E_{n,k}$ into the following sets:

$$\mathcal{Q}_1=\{E_{n,k}:\bar{e}_r^{(k)}\neq\bar{e}_r^{(k_1)}=\bar{e}_r^{(k_2)}\text{ for some } r<p\},$$
$$\mathcal{Q}_2=\{E_{n,k}:\bar{e}_r^{(k)}=\bar{e}_r^{(k_1)},\ r=1,2,\dots,p\},$$
$$\mathcal{Q}_3=\{E_{n,k}:\bar{e}_r^{(k)}=\bar{e}_r^{(k_2)},\ r=1,2,\dots,p\},$$
$$\mathcal{Q}_4=\{E_{n,k}:\bar{e}_r^{(k)}=\bar{e}_r^{(k_1)},\ r=1,2,\dots,p-1,\ \bar{e}_p^{(k)}=-\bar{e}_p^{(k_1)}\},$$
$$\mathcal{Q}_5=\{E_{n,k}:\bar{e}_r^{(k)}=\bar{e}_r^{(k_1)},\ r=1,2,\dots,p-1,\ \bar{e}_p^{(k)}=-\bar{e}_p^{(k_2)}\}.$$

For simplicity we write $z_{n,k_1}=a,\ z_{n,k_2}=b$, and for $p=1$ we set $\mathcal{Q}_1=\emptyset$. It is easy to see that

$$\int\limits_{Q_2}\frac{d\mu(z)}{z-a}=\int\limits_{Q_3}\frac{d\mu(z)}{z-b},\quad a-b=-\frac{3}{4}4^{-p+1}=-3\cdot 4^{-p}.$$

Thus

$$\mathcal{C}\mu(a)-\mathcal{C}\mu(b)=\int\limits_{Q_1}\frac{(a-b)d\mu(z)}{(z-a)(z-b)}+\int\limits_{Q_4}\frac{(a-b)d\mu(z)}{(z-a)(z-b)}+\int\limits_{Q_5}\frac{(a-b)d\mu(z)}{(z-a)(z-b)}$$
$$+\int\limits_{Q_3}\frac{d\mu(z)}{z-a}-\int\limits_{Q_2}\frac{d\mu(z)}{z-b}=I_1+I_2+I_3+I_4+I_5,$$

say. We examine each integral separately. Let $G_1,\ G_2,\dots,G_{p-1}$ be the following chain of sets: G_{p-1} is the set consisting of the three squares from E_{p-1} which are situated in the same square of E_{p-2} as a and b and which do not contain a and b; G_{p-2} is the set of those three squares from E_{p-2} which are in the same square of E_{p-3} as G_{p-1} and which do not contain G_{p-1}. Continuing in this way we see that

$$\mathcal{Q}_1 = \left\{ E_{n,k} : E_{n,k} \subset \bigcup_{j=1}^{p-1} G_j \right\},$$

$$\mu(G_j) = 3 \cdot 4^{-j}$$

and

$$|z-a| \geqslant 2 \cdot 4^{-j}, \quad |z-b| \geqslant 2 \cdot 4^{-j} \quad \text{for } z \in G_j.$$

Moreover, $|z-a| \geqslant (3-\frac{1}{4})4^{-j}$ for z lying in the four squares from E_{j+1} situated in G_j. Altogether G_j contains 12 squares from E_{j+1}. Also $|z-a| \geqslant 2\sqrt{2}\cdot 4^{-j}$ for z in three such squares and $|z-a| \geqslant (3-\frac{1}{4})\sqrt{2}\cdot 4^{-j}$ in one such square. The same inequalities hold also for $|z-b|$. Hence

$$\begin{aligned} |I_1| &< 3 \cdot 4^{-p} \sum_{j=1}^{p-1} \int_{G_j} \frac{d\mu(z)}{|z-a|\,|z-b|} \\ &< 3 \cdot 4^{-p} \sum_{j=1}^{p-1} 4^{-j-1} \left\{ 4(2\cdot 4^{-j})^{-2} + 4\left[\left(3-\frac{1}{4}\right)4^{-j}\right]^{-2} \right. \\ &\qquad \left. + 3(2\sqrt{2}\cdot 4^{-j})^{-2} + \left[\left(3-\frac{1}{4}\right)\sqrt{2}\cdot 4^{-j}\right]^{-2} \right\} \\ &= 3 \sum_{j=1}^{p-1} \left\{ \frac{1}{4} + \left(\frac{4}{11}\right)^2 + \frac{3}{32} + \left(\frac{4}{11}\right)^2 \frac{1}{8} \right\} 4^{j-p} \\ &< 3 \cdot 0.4926 \sum_{l=1}^{\infty} 4^{-l} = 0.4926. \end{aligned} \tag{7.2}$$

For $z \in Q_4$ we have

$$\begin{aligned} \arctan\tfrac{1}{2} \leqslant |\arg(z-a)| \leqslant \arctan 2, \\ \arctan 2 \leqslant |\arg(z-b)| \leqslant \pi - \arctan 2. \end{aligned}$$

Moreover, $\arg(z-a)$ and $\arg(z-b)$ have the same sign. Hence $\frac{\pi}{2} \leqslant |\arg(z-a)(z-b)| \leqslant \pi$. Since $a-b<0$ we see that

$$\operatorname{Re} I_2 > 0.$$

Similarly, $\pi \leqslant |\arg(z-a)(z-b)| \leqslant \frac{3\pi}{2}$ for $z \in Q_5$, and $\operatorname{Re} I_3 > 0$.

To estimate $\operatorname{Re} I_4$ we note that, for $z \in Q_3$, $|\operatorname{Im}(z-a)| \leqslant 4^{-p}$. If $t = |z-a|^2$ then

$$\operatorname{Re}\left(\frac{1}{z-a}\right) = \frac{\operatorname{Re}(z-a)}{|z-a|^2} \geqslant \frac{(t-4^{-2p})^{\frac{1}{2}}}{t}$$

and this function decreases for $t \geqslant 2{\cdot}4^{-2p}$. The square $\mathcal{Q}_3$ contains four squares from E_{p+1} where, if $p=n$, we consider, instead, the four vertices. Each of these supports a measure 4^{-p-1}. For two of these squares $t \leqslant \left[\frac{3}{4}4^{-p+1} + \frac{1}{4}4^{-p}\right]^2 + (4^{-p})^2 = 4^{-2p}\left(\left(\frac{13}{4}\right)^2+1\right)$, while for the other two squares $t \leqslant 4^{-2p+2} + (4^{-p})^2 = 17 \cdot 4^{-2p}$. Thus

$$\operatorname{Re} I_4 > 2 \cdot 4^{-p-1}\left[4^{-2p}\left(\frac{13}{4}\right)^2\right]^{\frac{1}{2}} \cdot 4^{2p}\left[\left(\frac{13}{4}\right)^2+1\right]^{-1} + 2 \cdot 4^{-p-1}(16 \cdot 4^{-2p})^{\frac{1}{2}} 4^{2p} \cdot \frac{1}{17} = \frac{26}{185} + \frac{2}{17} > 0.258.$$

Similarly

$$\operatorname{Re} I_5 > 0.258$$

and so from (7.2),

$$\operatorname{Re}\mathcal{C}\mu(a) - \operatorname{Re}\mathcal{C}\mu(b) > 2 \cdot 0.258 - 0.4926 > 0.02$$

and Lemma 7.1 is proved.

We continue the proof of Lemma 6.1. Denote by p_k, q_k the number of positive and negative components of $\bar{\imath}^{(k)}$ respectively, and set $i(n) = [\sqrt{n}+1]$. For $\bar{j}$ fixed we introduce the following sets of squares (or, equivalently, sets of multi-indices $\bar{\imath}^{(k)}$):

$$\begin{aligned}
\mathcal{E}^1(\bar{j}) &= \{E_{n,k} : \bar{j}^{(k)} = \bar{j},\ |\operatorname{Re}\mathcal{C}\mu(z_{n,k})| > (0.01)\sqrt{n}\},\\
\mathcal{F}(\bar{j}) &= \{E_{n,k} : \bar{j}^{(k)} = \bar{j},\ E_{n,k} \notin \mathcal{E}^1(\bar{j})\},\\
\mathcal{E}(\bar{j}, l) &= \{E_{n,k} : \bar{j}^{(k)} = \bar{j},\ p_k = l\}, \quad l = 0, 1, 2, \ldots, n.
\end{aligned}$$

Then all the sets $\mathcal{E}(\bar{j}, l)$ are disjoint and we shall prove that, for $\left[\frac{n}{2}\right] - 2i(n) \leqslant l < \left[\frac{n}{2}\right] - i(n)$ we have

$$\#\left(\mathcal{E}^1(\bar{j}) \cap \mathcal{E}(\bar{j}, l)\right) + \#\left(\mathcal{E}^1(\bar{j}) \cap \mathcal{E}(\bar{j}, l + i(n))\right) \geqslant \#\mathcal{E}(\bar{j}, l). \tag{7.3}$$

If $\mathcal{E}(\bar{j}, l) \subset \mathcal{E}^1(\bar{j})$ then (7.3) is trivial. Suppose that

$$\mathcal{E}(\bar{j}, l) \cap \mathcal{F}(\bar{j}) \neq \emptyset$$

for some $l \in \left[\left[\frac{n}{2}\right] - 2i(n),\ \left[\frac{n}{2}\right] - i(n)\right)$. For simplicity we omit the fixed indices $\bar{j}$, n and set

$$A_l = \mathcal{E}(\bar{j}, l) \cap \mathcal{F}(\bar{j}).$$

For $\bar{\imath} \in A_l$ let $B_l(\bar{\imath})$ be the set of all multi-indices $\bar{\imath}'$ in $\mathcal{E}(\bar{j}, l + i(n))$ such that for all l positive components of $\bar{\imath}$ are also positive components of $\bar{\imath}'$, but $\bar{\imath}'$ has a further $i(n)$ positive components among the $n - l$ negative components of $\bar{\imath}$. Thus

$$\#B_l\,(\bar{\imath}) = \binom{n-l}{i(n)} \quad \text{for each } \bar{\imath} \in A_l.$$

We set $B_l = \cup B_l(\bar{\imath})$ where the union is over all $\bar{\imath} \in A_l$ and consider the following set of couples

$$D_l = \left\{(\bar{\imath}, \bar{\imath}') : \bar{\imath} \in A_l,\ \bar{\imath}' \in B_l(\bar{\imath})\right\}.$$

Clearly $\#D_l = (\#A_l)\binom{n-l}{i(n)}$. On the other hand, in order to obtain the corresponding indices $\bar{\imath}$ for given $\bar{\imath}' \in B_l$, we must choose certain $\bar{\imath}(n)$ positive components from among the $l + i(n)$ positive components of $\bar{\imath}'_n$ and replace them by negative ones. Hence, for every $\bar{\imath}' \in B_l$ the number of couples $(\bar{\imath}, \bar{\imath}')$ in D_l does not exceed $\binom{l+i(n)}{i(n)}$. Therefore $\#D_l \leqslant (\#B_l)\binom{l+i(n)}{i(n)}$ and so

$$(\#A_l)\binom{n-l}{i(n)} \leqslant (\#B_l)\binom{l+i(n)}{i(n)}.$$

Since $(n-l) - (l + i(n)) = n - 2l - i(n) > n - (n - 2i(n)) - i(n) \geqslant i(n) > 0$ we see that

$$\#A_l \leqslant \#B_l.$$

Now if $\bar{\imath}' \in B_l$ we let $\bar{\imath} = \bar{\imath}^{(k)}$ be any multi-index in A_l such that $(\bar{\imath}, \bar{\imath}') \in D_l$. Since $\bar{\imath}^{(k)} \in \mathcal{F}(\bar{j})$,

$$|\mathrm{Re}\,\mathcal{C}\mu(z_{n,k})| \leqslant (0.01)\sqrt{n}.$$

In order to obtain $\bar{\imath}'$ from $\bar{\imath}^{(k)}$ we replace a negative component by a positive one $i(n)$ times. We apply (7.1) $i(n)$ times to deduce that, for the point $z_{n,k'}$ which corresponds to $\bar{\imath}'$,

$$\mathrm{Re}\,\mathcal{C}\mu(z_{n,k'}) < (0.01)\sqrt{n} - (0.02)i(n) \leqslant -(0.01)\sqrt{n}.$$

Thus

$$|\mathrm{Re}\,\mathcal{C}\mu(z_{n,k'})| > (0.01)\sqrt{n},$$

and hence $B_l \subset \mathcal{E}^1(\bar{j})$ and so in $(\mathcal{E}^1(\bar{j}) \cap \mathcal{E}(\bar{j}, l + i(n))$.

Moreover, $\#(\mathcal{E}^1(\bar{j}) \cap \mathcal{E}(\bar{j}, l)) = \#\mathcal{E}(\bar{j}, l) - \#A_l$. Since $\#A_l \leqslant \#B_l$ we obtain (7.3). Now $\#\mathcal{E}(\bar{j}, l) = \binom{n}{l}$ and we show that for $\frac{n}{2} - 2i(n) \leqslant l < \frac{n}{2}$,

$$\binom{n}{l} \approx cn^{-\frac{1}{2}}2^n. \tag{7.4}$$

This is an elementary consequence of Stirling's formula. Indeed

$$\binom{n}{l} \approx (2\pi)^{-\frac{1}{2}}2^n\left(\frac{n}{l(n-l)}\right)^{\frac{1}{2}}\left(\frac{n}{2n-2l}\right)^n\left(\frac{n-l}{l}\right)^l,$$

and $l(n-l)$ is maximal when $l = \frac{n}{2}$. Thus

$$\left(\frac{n}{l(n-l)}\right)^{\frac{1}{2}} > \frac{2}{\sqrt{n}}.$$

For the last two factors we set $t = \frac{1}{2} - \frac{l}{n}$, i.e. $l = \frac{n}{2} - nt$. Then $0 < t \leqslant 2i(n) \leqslant 2n^{-\frac{1}{2}} + 2n^{-1}$. Now an easy computation shows that

$$\log\left\{\left(\frac{n}{2n-2l}\right)^n \left(\frac{n-l}{l}\right)^l\right\} = O(nt^2) = O(1) \quad \text{as } n \to \infty$$

and hence (7.4) is established. Inequality (7.4) is obviously related to the Law of Large Numbers.

We note that the sets $(\mathcal{E}^1(\bar{j}) \cap \mathcal{E}(\bar{j}, l))$ and $(\mathcal{E}^1(\bar{j}) \cap \mathcal{E}(\bar{j}, l + i(n)))$ are all disjoint since $\left[\frac{n}{2}\right] - 2i(n) \leqslant l < \left[\frac{n}{2}\right] - i(n)$. Summing the inequalities (7.3) over those l, we have

$$\#\mathcal{E}^1(\bar{j}) \geqslant c2^n.$$

This inequality holds for all multi-indices $\bar{j}$. But there are 2^n different such multi-indices $\bar{j}$ and $\mathcal{E} = \bigcup_{\bar{j}} \mathcal{E}^1(\bar{j})$. We conclude that

$$\#\mathcal{E} \geqslant c4^n.$$

Thus Lemma 6.1 and hence Theorem 2.2′ are proved.

8. Proof of Theorem 2.3

For a fixed point $z \in E_n$ let

$$\mathcal{Q}^{(n)} \subset \mathcal{Q}^{(n-1)} \subset \cdots \subset \mathcal{Q}^{(0)}$$

be the chain of squares such that $z \in \mathcal{Q}^{(n)}$ and

$$\mathcal{Q}^{(j)} \subset E_j, \quad j = 0, 1, 2, \ldots, n.$$

Clearly

$$\begin{aligned} \operatorname{dist}(z, \zeta) \leqslant \sqrt{2} \cdot 4^{-(j-1)} \qquad &\text{for all} \quad \zeta \in \mathcal{Q}^{(j-1)} \setminus \mathcal{Q}^{(j)}, \\ \mu(\mathcal{Q}^{(j-1)} \setminus \mathcal{Q}^{(j)}) = 3 \cdot 4^{-j}, \qquad &j = 1, 2, \ldots, n, \end{aligned}$$

where μ is the normalized measure at the beginning of Section 6. Hence,

$$\int_{E_n} \frac{d\mu(\zeta)}{|\zeta - z|} > \sum_{j=1}^{n} \frac{3 \cdot 4^{-j}}{\sqrt{2} \cdot 4^{-(j-1)}} = \frac{3}{4\sqrt{2}} n.$$

For the set $E = (\sqrt{2}P)^{-1} n 4^n E_n$ and $z' = (\sqrt{2}P)^{-1} n 4^n z$ and for the corresponding measure ν we have

$$\sum_{k=1}^{N} \frac{1}{|z' - z_k|} = \frac{\sqrt{2} P 4^{n+1}}{n 4^4} \int \frac{d\mu(z)}{|\zeta - z|} > 3P.$$

Thus $E \subset \mathcal{X}(Q_N, P)$. Since $\mathcal{Z} \subset E_n$ for $\mathcal{Z}$ defined in Section 6 and $M(\mathcal{Z}) \geqslant c > 0$ (by (6.3)) we have that $M(E_n) \geqslant c > 0$ and hence

$$M(\mathcal{X}(Q_N, P)) \geqslant M(E) = \frac{n4^n}{\sqrt{2}P} M(E_n) > \frac{cn4^n}{P},$$

as required. Theorem 2.3 is proved.

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Note in proof. Extensions of the results of this paper have been obtained by the second author. A summary will appear in *Dokl. Akad. Nauk.* **407** (2006), no. 5 (English translation in *Dokl. Math.*) with complete proofs appearing elsewhere later.

University College, London, United Kingdom
E-mail address: maths@ucl.ac.uk

Moscow State Civil Engineering University, Moscow, Russia
E-mail address: eiderman@orc.ru

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