

The L-class of non-Witt spaces

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Abstract

Characteristic classes for oriented pseudomanifolds can be defined using appropriate self-dual complexes of sheaves. On non-Witt spaces, self-dual complexes compatible to intersection homology are determined by choices of Lagrangian structures at the strata of odd codimension. We prove that the associated signature and L-classes are independent of the choice of Lagrangian structures, so that singular spaces with odd codimensional strata, such as e.g. certain compactifications of locally symmetric spaces, have well-defined L-classes, provided Lagrangian structures exist. We illustrate the general results with the example of the reductive Borel-Serre compactification of a Hilbert modular surface.

Contents

1. Introduction
 2. The Postnikov system of Lagrangian structures
 3. The bordism group Ω_*^{SD}
 4. The signature of non-Witt spaces
 5. The L-class of non-Witt spaces
 6. An example
- References

1. Introduction

Finding natural settings for defining characteristic classes has been, and continues to be, an important theme in geometry. The notion of multiplicative sequences allowed Hirzebruch [Hir56] the definition of L-classes in rational cohomology as certain polynomials in the Pontrjagin classes, leading to his beautiful formula stating equality of the signature and L-genus of a smooth oriented manifold. Using this result together with the principle of representing

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cohomology classes by transverse maps to spheres, Thom [Tho58] constructed L-classes for triangulated manifolds which are piecewise linear invariants.

To define L-classes for singular spaces, various approaches have been successful in various settings. In [GM80], Goresky and MacPherson introduce intersection homology theory as a method to recover generalized Poincaré duality for stratified pseudomanifolds. Using the middle perversity groups, one obtains a signature for oriented pseudomanifolds with only even codimensional strata and thus, following the Thom-Pontrjagin-Milnor program, homology L-classes for such spaces. Completely independently, Cheeger discovered from an analytic viewpoint that Poincaré duality can be restored in the context of pseudomanifolds by working on spaces with locally conical metrics and considering the L^2 deRham complex on the incomplete manifold obtained by removing the singular set. The action of the $*$ -operator on harmonic forms induces the Poincaré duality. Cheeger [Che83] obtains a version of the Atiyah-Patodi-Singer index theorem and, as a main application, a local formula for the L-class as a sum over all simplices of a given dimension, with coefficients given by the η -invariants of the links. More generally, both Cheeger's and Goresky-MacPherson's approaches yield characteristic classes for Witt spaces; see [Sie83], [Che83], [GM83]. A stratified pseudomanifold is *Witt*, if the lower middle perversity middle-dimensional intersection homology of all links of strata of odd codimension vanishes. In [GM83], an elegant formulation of intersection homology theory is presented employing differential complexes of sheaves in the derived category, and it is shown that for a Witt space X Poincaré duality is induced by the Verdier-self-duality of the sheaf $\mathbf{IC}_m^\bullet(X)$ of middle perversity intersection chains.

Cappell, Shaneson and Weinberger [CSW91] construct a functor from self-dual sheaves to controlled visible algebraic Poincaré complexes. As some remarkable consequences, one can deduce that any self-dual sheaf has a symmetric signature, and indeed defines a characteristic class in homology with coefficients in visible L-theory whose image under assembly is the symmetric signature. Moreover, the Pontrjagin character of the associated K-homology class equals the L-class of the self-dual sheaf. The latter class is discussed in [CS91], where L-class formulae for stratified maps are obtained.

It is the goal of this paper to define an L-class for oriented compact pseudomanifolds that have odd codimensional strata, but do not satisfy the Witt space condition. Certain compactifications of locally symmetric varieties constitute an interesting class of examples of non-Witt spaces. Concretely, the reductive Borel-Serre compactification — see [Zuc82] or [GHM94] — of a Hilbert modular surface is a real four-dimensional space whose one-dimensional strata are circles (one for each Γ -conjugacy class of parabolic \mathbb{Q} -subgroups) with toroidal links and hence not a Witt space (together with R. Kulkarni we provide a detailed treatment of self-dual sheaves on such compactifications in [BK04]).

Our approach to defining characteristic classes is via Verdier-self-dual complexes of sheaves compatible to intersection homology. On a non-Witt space X , $\mathbf{IC}_{\bar{m}}^\bullet(X)$ is not self-dual, since the canonical morphism $\mathbf{IC}_{\bar{m}}^\bullet(X) \rightarrow \mathbf{IC}_{\bar{n}}^\bullet(X)$ from lower middle perversity (\bar{m}) to upper middle perversity (\bar{n}) intersection chains is not an isomorphism (in the derived category). A theory of self-dual sheaves on non-Witt spaces has been developed in [Ban02]; a brief summary is given in Section 2. It is convenient to organize sheaf complexes on a non-Witt space which satisfy intersection homology type stalk conditions and are self-dual into a category $\text{SD}(X)$. This category may be empty (Example: a space having strata with links a complex projective space $\mathbb{C}P^{2k}$). If it is not empty, then an object $\mathbf{S}^\bullet \in \text{SD}(X)$ defines a signature $\sigma(\mathbf{S}^\bullet) \in \mathbb{Z}$ and by work of Cappell, Shaneson and Weinberger [CSW91], as well as [CS91], homology L-classes

$$L_k(\mathbf{S}^\bullet) \in H_k(X; \mathbb{Q}).$$

The main result of [Ban02] is that $\text{SD}(X)$ can be described by a Postnikov system whose fibers are categories of Lagrangian structures along the strata of odd codimension. Thus a choice of an object $\mathbf{S}^\bullet \in \text{SD}(X)$ is equivalent to choices of Lagrangian structures. The idea of employing Lagrangian subspaces in order to obtain self-duality is present in an L^2 -cohomology setting as J. Cheeger’s “*-invariant boundary conditions;” see [Che79], [Che80] and [Che83], and is also invoked in unpublished work of J. Morgan on the characteristic variety theorem. From the point of view of characteristic classes, the question arises: Do different choices yield the same L-classes? In the present paper, we give a positive answer to this question. We show (Theorem 5.2 in Section 5):

THEOREM. *Let X^n be a closed oriented pseudomanifold. If $\text{SD}(X) \neq \emptyset$, then the L-classes*

$$L_k(X) = L_k(\mathbf{IC}_{\mathcal{L}}^\bullet) \in H_k(X; \mathbb{Q}),$$

$\mathbf{IC}_{\mathcal{L}}^\bullet \in \text{SD}(X)$, *are independent of the choice of Lagrangian structure \mathcal{L} .*

Thus a non-Witt space has a well-defined L-class $L(X)$, provided $\text{SD}(X) \neq \emptyset$.

Although we have only considered explicitly the independence of L-classes under change of Lagrangian structures, our methods imply topological invariance as well. Firstly, stratification independence can be seen by controlling all the choices for all stratifications in terms of those available to the homologically intrinsic stratification. Then topological invariance is a direct consequence of the uniqueness of the object of $\text{SD}(X)$ regarded as a cobordism class (although, not as an object of the derived category) and the connection between cobordism classes of self-dual sheaves and characteristic classes [CSW91]. Doing

this actually gives a more refined conclusion: a topologically invariant definition of a characteristic class in $H^*(X; L(\mathbb{Q}))$. (Compare, in addition [Sie83].) Although we have not explicitly dealt with the issue in this paper, it is also possible to show that the existence of a Lagrangian structure is also topologically invariant.

If X^n is stratified as $X^n = X_n \supset X_{n-2} \supset X_{n-3} \supset \dots \supset X_0$ (strata are indexed by their dimension), then it is rather clear that the L-class is well-defined in the relative groups $H_*(X, X_s)$, where s is maximal so that $n - s$ is odd. We can for instance argue as follows: If $\mathbf{S}_0^\bullet, \mathbf{S}_1^\bullet \in \text{SD}(X)$, we wish to see $i_*L_k(\mathbf{S}_0^\bullet) = i_*L_k(\mathbf{S}_1^\bullet)$, where $i_* : H_k(X) \rightarrow H_k(X, X_s)$. Here $k > 0$ since the information on the signatures $\sigma(\mathbf{S}_0^\bullet), \sigma(\mathbf{S}_1^\bullet)$ is *a priori* lost in $H_0(X, X_s) = 0$. Let Y be the quotient space $Y^n = X/X_s$ and f be the collapse map $f : (X, X_s) \rightarrow (Y, c)$. The space Y inherits a pseudomanifold stratification from X with respect to which f is a stratified map. The key point is that Y has only strata of *even* codimension (assuming n is even; if not, cross X with a circle first and adapt the argument accordingly). Since i_* is the composition $H_k(X) \xrightarrow{f_*} H_k(X/X_s) \cong H_k(X, X_s)$, it suffices to verify $f_*L_k(\mathbf{S}_0^\bullet) = f_*L_k(\mathbf{S}_1^\bullet)$. The axioms for $\text{SD}(X)$ (see Definition 2.1) imply that $\mathbf{S}_0^\bullet|_{X-X_s} \cong \mathbf{IC}_{\bar{m}}^\bullet(X - X_s) \cong \mathbf{S}_1^\bullet|_{X-X_s}$. Using the Cappell-Shaneson L-class formula [CS91], we calculate ($i = 0, 1$):

$$f_*L_k(\mathbf{S}_i^\bullet) = L_k(Y) + L_k(\{c\}; \mathcal{S}_f^{\{c\}}(\mathbf{S}_i^\bullet)) + \sum_Z L_k(\bar{Z}; \mathcal{S}_f^Z(\mathbf{S}_i^\bullet)),$$

where the first term on the right-hand side is the Goresky-MacPherson L-class of Y (with constant coefficients), the second term is associated to the point singularity c and vanishes as $k > 0$, the summation ranges over all components Z of strata of Y of dimension $> s$ and $< n$, and $L_k(\bar{Z}; \mathcal{S}_f^Z(\mathbf{S}_i^\bullet))$ denotes the L-class of the closure \bar{Z} of Z with coefficient system $\mathcal{S}_f^Z(\mathbf{S}_i^\bullet)$, which however depends only on $\mathbf{S}_i^\bullet|_{X-X_s}$ so that $\mathcal{S}_f^Z(\mathbf{S}_0^\bullet) = \mathcal{S}_f^Z(\mathbf{S}_1^\bullet)$. Therefore,

$$\begin{aligned} f_*L_k(\mathbf{S}_0^\bullet) &= L_k(Y) + \sum_Z L_k(\bar{Z}; \mathcal{S}_f^Z(\mathbf{S}_0^\bullet)) \\ &= L_k(Y) + \sum_Z L_k(\bar{Z}; \mathcal{S}_f^Z(\mathbf{S}_1^\bullet)) = f_*L_k(\mathbf{S}_1^\bullet). \end{aligned}$$

This and related arguments seem to be insufficient to yield the full statement of Theorem 5.2. To prove the latter, we use the following strategy: Let us illustrate the ideas for the basic case of a two strata space $X^n \supset \Sigma^s$, $X - \Sigma$ is an n -dimensional manifold and Σ^s an s -dimensional manifold, n even, s odd. Given $\mathbf{IC}_{\mathcal{L}_0}^\bullet, \mathbf{IC}_{\mathcal{L}_1}^\bullet \in \text{SD}(X)$, determined by Lagrangian structures $\mathcal{L}_0, \mathcal{L}_1$, respectively, along Σ , the central problem is to prove equality of the signatures $\sigma(\mathbf{IC}_{\mathcal{L}_0}^\bullet) = \sigma(\mathbf{IC}_{\mathcal{L}_1}^\bullet)$, since then the result on L-classes will follow from the fact that they are determined uniquely by the collection of signatures of sub-

varieties with normally nonsingular embedding and trivial normal bundle; see Section 5. To prove equality of the signatures, we use bordism theory: We construct a geometric bordism Y^{n+1} from X to $-X$ and cover its interior with a self-dual sheaf complex \mathbf{S}^\bullet , which, when pushed to the boundary, restricts to $\mathbf{IC}_{\mathcal{L}_0}^\bullet$ on X , and restricts to $\mathbf{IC}_{\mathcal{L}_1}^\bullet$ on $-X$. A topologically trivial h-cobordism $Y^{n+1} = X \times [0, 1]$ already works, but of course not with the natural stratification. Our idea is to “cut” the odd-codimensional stratum at $\frac{1}{2}$, which enables us to “decouple” Lagrangian structures because the stratum of odd codimension then consists of two disjoint connected components. This forces the introduction of a new stratum at $\frac{1}{2}$, but its codimension is even and presents no problem. The stratification of Y with cuts at $\frac{1}{2}$ is thus defined by the filtration $Y^{n+1} \supset Y_{s+1} \supset Y_s$, where $Y_{s+1} - Y_s = \Sigma^s \times [0, \frac{1}{2}) \sqcup \Sigma^s \times (\frac{1}{2}, 1]$ and $Y_s = \Sigma^s \times \{\frac{1}{2}\}$. The sheaf \mathbf{S}^\bullet will be constructible with respect to this stratification. On $Y - Y_{s+1}$, \mathbf{S}^\bullet is $\mathbb{R}_{Y - Y_{s+1}}[n + 1]$, the constant real sheaf in degree $-n - 1$ (indexing conventions after [GM83]). To extend to $Y_{s+1} - Y_s$, we use the Postnikov system 2.1, and the Lagrangian structure whose restriction to $\Sigma^s \times [0, \frac{1}{2})$ is the pull-back of \mathcal{L}_0 under the first factor projection and whose restriction to $\Sigma^s \times (\frac{1}{2}, 1]$ is the pull-back of \mathcal{L}_1 under the first factor projection. Finally, we extend to Y_s by the Deligne-step (pushforward and middle perversity truncation), which produces a self-dual sheaf \mathbf{S}^\bullet , since Y_s is of even codimension.

The paper is organized as follows: Section 2 provides a summary of the definitions and results of [Ban02]. It contains the definition of the category $\text{SD}(X)$ of self-dual sheaves, the definition of the notion of a Lagrangian structure, and some information on the Postnikov system of Lagrangian structures (Theorem 2.1). Section 3 reviews relevant facts about the bordism groups Ω_*^{SD} whose elements are represented by pseudomanifolds carrying a self-dual sheaf. In Section 4, we define the stratification with cuts at $\frac{1}{2}$, and, after some sheaf-theoretic preparation, state and prove our result on the signature of non-Witt spaces (Theorem 4.1). In Section 5, we recall the existence and uniqueness result on L-classes of self-dual sheaves from [CS91] and state and prove the main theorem of this paper on the L-class of non-Witt spaces (Theorem 5.2). We conclude with an illustration of our results for the case of the reductive Borel-Serre compactification of a Hilbert modular surface in Section 6.

2. The Postnikov system of Lagrangian structures

Let X be a stratified oriented topological pseudomanifold without boundary. If X has only strata of even codimension, then $\mathbf{IC}_{\bar{m}}^\bullet(X)$, the intersection chain sheaf with respect to the lower middle perversity \bar{m} , is Verdier self-dual, since $\mathbf{IC}_{\bar{m}}^\bullet(X) = \mathbf{IC}_{\bar{n}}^\bullet(X)$, the intersection chain sheaf with respect to the upper middle perversity \bar{n} . More generally, $\mathbf{IC}_{\bar{m}}^\bullet(X)$ is still self-dual on X if

X is a Witt space. If X is not a Witt space, then the canonical morphism $\mathbf{IC}_{\bar{m}}^\bullet(X) \rightarrow \mathbf{IC}_{\bar{n}}^\bullet(X)$ is not an isomorphism and $\mathbf{IC}_{\bar{m}}^\bullet(X)$ is not self-dual.

The present section reviews results of [Ban02], where a theory of intersection homology type invariants for non-Witt spaces is developed.

Let X^n be an n -dimensional pseudomanifold with a fixed stratification

$$(1) \quad X = X_n \supset X_{n-2} \supset X_{n-3} \supset \dots \supset X_0 \supset X_{-1} = \emptyset$$

such that X_j is closed in X and $X_j - X_{j-1}$ is an open manifold of dimension j . Set $U_k = X - X_{n-k}$ and let $i_k : U_k \hookrightarrow U_{k+1}$, $j_k : U_{k+1} - U_k \hookrightarrow U_{k+1}$ denote the inclusions. Let \bar{m}, \bar{n} be the lower and upper middle perversities, respectively. Throughout this paper we will work with real coefficients.

The intersection chain sheaf $\mathbf{IC}_{\bar{p}}^\bullet(X)$ on X for perversity \bar{p} and constant coefficients is characterized by the following axioms:

(AX0): $\mathbf{IC}_{\bar{p}}^\bullet$ is constructible with respect to stratification (1).

(AX1): Normalization: $\mathbf{IC}_{\bar{p}}^\bullet|_{U_2} \cong \mathbb{R}_{U_2}[n]$.

(AX2): Lower bound: $\mathbf{H}^i(\mathbf{IC}_{\bar{p}}^\bullet) = 0$ for $i < -n$.

(AX3): Stalk vanishing conditions: $\mathbf{H}^i(\mathbf{IC}_{\bar{p}}^\bullet|_{U_{k+1}}) = 0$ for $i > \bar{p}(k) - n, k \geq 2$.

(AX4): Costalk vanishing conditions: $\mathbf{H}^i(j_k^* \mathbf{IC}_{\bar{p}}^\bullet|_{U_{k+1}}) = 0$ for $i \leq \bar{p}(k) - n + 1, k \geq 2$.

We shall denote the derived category of bounded differential complexes of sheaves constructible with respect to (1) by $D^b(X)$. Let us define the category of complexes of sheaves suitable for studying intersection homology type invariants on non-Witt spaces. The objects of this category should satisfy two properties: On the one hand, they should be self-dual, on the other hand, they should be as close to the middle perversity intersection chain sheaves as possible, that is, interpolate between $\mathbf{IC}_{\bar{m}}^\bullet(X)$ and $\mathbf{IC}_{\bar{n}}^\bullet(X)$. Given these specifications, we adopt the following definition:

Definition 2.1. Let $\mathbf{SD}(X)$ be the full subcategory of $D^b(X)$ whose objects \mathbf{S}^\bullet satisfy the following axioms:

(SD1): Normalization: \mathbf{S}^\bullet has an associated isomorphism $\nu : \mathbb{R}_{U_2}[n] \xrightarrow{\cong} \mathbf{S}^\bullet|_{U_2}$.

(SD2): Lower bound: $\mathbf{H}^i(\mathbf{S}^\bullet) = 0$, for $i < -n$.

(SD3): Stalk condition for the upper middle perversity \bar{n} : $\mathbf{H}^i(\mathbf{S}^\bullet|_{U_{k+1}}) = 0$, for $i > \bar{n}(k) - n, k \geq 2$.

(SD4): Self-Duality: \mathbf{S}^\bullet has an associated isomorphism $d : \mathcal{D}\mathbf{S}^\bullet[n] \xrightarrow{\cong} \mathbf{S}^\bullet$ (where \mathcal{D} denotes the Verdier dualizing functor) such that $\mathcal{D}d[n] = d$ and $d|_{U_2}$ is compatible with the orientation under normalization so that

$$\begin{array}{ccc} \mathbb{R}_{U_2}[n] & \xrightarrow[\simeq]{\nu} & \mathbf{S}^\bullet|_{U_2} \\ \text{orient} \uparrow \simeq & & \simeq \uparrow d|_{U_2} \\ \mathbb{D}_{U_2}^\bullet & \xrightarrow[\simeq]{\mathcal{D}\nu^{-1}[n]} & \mathcal{D}\mathbf{S}^\bullet|_{U_2}[n] \end{array}$$

commutes.

Depending on X , the category $\text{SD}(X)$ may or may not be empty. One can show (cf. Theorem 2.2 in [Ban02]) that if $\mathbf{S}^\bullet \in \text{SD}(X)$, there exist morphisms $\mathbf{IC}_{\bar{m}}^\bullet(X) \xrightarrow{\alpha} \mathbf{S}^\bullet \xrightarrow{\beta} \mathbf{IC}_{\bar{n}}^\bullet(X)$ uniquely determined by $\alpha|_{U_2} = \nu : \mathbb{R}_{U_2}[n] \xrightarrow{\simeq} \mathbf{S}^\bullet|_{U_2}$ and $\beta|_{U_2} = \nu^{-1} : \mathbf{S}^\bullet|_{U_2} \xrightarrow{\simeq} \mathbb{R}_{U_2}[n]$, such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{IC}_{\bar{m}}^\bullet(X) & \xrightarrow{\alpha} & \mathbf{S}^\bullet \\ \simeq \uparrow & & \simeq \uparrow d \\ \mathcal{D}\mathbf{IC}_{\bar{n}}^\bullet(X)[n] & \xrightarrow{\mathcal{D}\beta[n]} & \mathcal{D}\mathbf{S}^\bullet[n] \end{array}$$

(where d is given by (SD4)), which clarifies the relation between intersection chain sheaves and objects of $\text{SD}(X)$.

To understand the structure of $\text{SD}(X)$ (e.g. how can one construct objects in $\text{SD}(X)$?), one introduces the notion of a *Lagrangian structure*. Assume k is odd and $\mathbf{A}^\bullet \in \text{SD}(U_k)$. Note that $\bar{n}(k) = \bar{m}(k) + 1$. We shall use the shorthand notation $\bar{m}\mathbf{A}^\bullet = \tau_{\leq \bar{m}(k)-n} Ri_{k*}\mathbf{A}^\bullet$, $\bar{n}\mathbf{A}^\bullet = \tau_{\leq \bar{n}(k)-n} Ri_{k*}\mathbf{A}^\bullet$, and $s = \bar{n}(k) - n$. The reason why $\bar{m}\mathbf{A}^\bullet$ need not be self-dual is that the ‘‘obstruction-sheaf’’

$$\mathcal{O}(\mathbf{A}^\bullet) = \mathbf{H}^s(Ri_{k*}\mathbf{A}^\bullet)[-s] \in D^b(U_{k+1})$$

need not be trivial. Its support is $U_{k+1} - U_k$, and it is isomorphic to the algebraic mapping cone of the canonical morphism $\bar{m}\mathbf{A}^\bullet \rightarrow \bar{n}\mathbf{A}^\bullet$: We have a distinguished triangle

$$(2) \quad \begin{array}{ccc} \bar{m}\mathbf{A}^\bullet & \longrightarrow & \bar{n}\mathbf{A}^\bullet \\ & \searrow \mathbb{1} & \swarrow \\ & \mathcal{O}(\mathbf{A}^\bullet) & \end{array} .$$

Dualizing (2), one sees that $\mathcal{O}(\mathbf{A}^\bullet)$ is self-dual, $\mathcal{D}\mathcal{O}(\mathbf{A}^\bullet)[n + 1] \cong \mathcal{O}(\mathbf{A}^\bullet)$ (the duality-dimension is one off).

Definition 2.2. A *Lagrangian structure* (along $U_{k+1} - U_k$) is a morphism $\mathcal{L} \rightarrow \mathcal{O}(\mathbf{A}^\bullet)$, $\mathcal{L} \in D^b(U_{k+1})$, which induces injections on stalks and has the property that some distinguished triangle on $\mathcal{L} \rightarrow \mathcal{O}(\mathbf{A}^\bullet)$ is an algebraic nullcobordism (in the sense of [CS91]) for $\mathcal{O}(\mathbf{A}^\bullet)$.

This means the following: Some distinguished triangle on $\phi : \mathcal{L} \rightarrow \mathcal{O}(\mathbf{A}^\bullet)$ has to be of the form

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\phi} & \mathcal{O}(\mathbf{A}^\bullet) \\ & \searrow \gamma & \swarrow \\ & \mathcal{D}\mathcal{L}[n+1] & \end{array}$$

and we require $\mathcal{D}\gamma[n+1] = \gamma[-1]$.

Equivalently, every stalk $\mathcal{L}_x, x \in U_{k+1} - U_k$, is a Lagrangian (i.e. maximally isotropic) subspace of $\mathcal{O}(\mathbf{A}^\bullet)_x$ with respect to the pairing $\mathcal{O}(\mathbf{A}^\bullet)_x \otimes \mathcal{O}(\mathbf{A}^\bullet)_x \rightarrow \mathbb{R}$ induced by the self-duality of $\mathcal{O}(\mathbf{A}^\bullet)$. If $\mathbf{B}^\bullet \in \text{SD}(U_k)$ and $\mathcal{L}_A \rightarrow \mathcal{O}(\mathbf{A}^\bullet)$, $\mathcal{L}_B \rightarrow \mathcal{O}(\mathbf{B}^\bullet)$ are two Lagrangian structures, then a *morphism of Lagrangian structures* is a commutative square in $D^b(U_{k+1})$:

$$\begin{array}{ccc} \mathcal{L}_A & \longrightarrow & \mathcal{O}(\mathbf{A}^\bullet) \\ \downarrow & & \downarrow \mathcal{O}(f) \\ \mathcal{L}_B & \longrightarrow & \mathcal{O}(\mathbf{B}^\bullet) \end{array}$$

where $f \in \text{Hom}_{D^b(U_k)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet)$ and $\mathcal{O}(f) = \mathbf{H}^s(\text{Ri}_{k*}f)[-s]$. Thus Lagrangian structures form a category $\text{Lag}(U_{k+1} - U_k)$. The relevance of $\text{Lag}(U_{k+1} - U_k)$ vis-à-vis $\text{SD}(X)$ is explained as follows:

1. Extracting Lagrangian structures from self-dual sheaves: There exists a covariant functor

$$\Lambda : \text{SD}(U_{k+1}) \longrightarrow \text{Lag}(U_{k+1} - U_k).$$

This means that every self-dual perverse sheaf has naturally associated Lagrangian structures.

2. Lagrangian structures as building blocks for self-dual sheaves: Let

$$\text{SD}(U_k) \rtimes \text{Lag}(U_{k+1} - U_k)$$

denote the twisted product of categories whose objects are pairs $(\mathbf{A}^\bullet, \mathcal{L} \xrightarrow{\phi} \mathcal{O}(\mathbf{A}^\bullet))$, $\mathbf{A}^\bullet \in \text{SD}(U_k)$, $\phi \in \text{Lag}(U_{k+1} - U_k)$, and whose morphisms are pairs with first component a morphism $f \in \text{Hom}_{D^b(U_k)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet)$ and second compo-

nent a commutative square

$$\begin{CD} \mathcal{L}_A @>\phi_A>> \mathcal{O}(\mathbf{A}^\bullet) \\ @VVV @VV\mathcal{O}(f)V \\ \mathcal{L}_B @>\phi_B>> \mathcal{O}(\mathbf{B}^\bullet). \end{CD}$$

There exists a covariant functor

$$\begin{aligned} \boxplus : \text{SD}(U_k) \times \text{Lag}(U_{k+1} - U_k) &\longrightarrow \text{SD}(U_{k+1}), \\ (\mathbf{A}^\bullet, \mathcal{L}) &\longmapsto \mathbf{A}^\bullet \boxplus \mathcal{L}; \end{aligned}$$

that is, a Lagrangian structure along $U_{k+1} - U_k$ naturally gives rise to a self-dual sheaf on U_{k+1} .

It is shown in [Ban02] that

$$\text{SD}(U_k) \times \text{Lag}(U_{k+1} - U_k) \xrightleftharpoons[(i_k^*, \Lambda)]{\boxplus} \text{SD}(U_{k+1})$$

induces an equivalence of categories. Summarizing, one obtains a Postnikov-type decomposition of the category $\text{SD}(X)$:

THEOREM 2.1. *Let $n = \dim X$ be even. There is an equivalence of categories*

$$\begin{aligned} \text{SD}(X) \simeq &\text{Lag}(U_n - U_{n-1}) \times \text{Lag}(U_{n-2} - U_{n-3}) \\ &\times \dots \times \text{Lag}(U_4 - U_3) \times \text{Const}(U_2). \end{aligned}$$

Here, $\text{Const}(U_2)$ denotes the category whose single object is the constant sheaf \mathbb{R}_{U_2} on U_2 and whose morphisms are sheaf maps $\mathbb{R}_{U_2} \rightarrow \mathbb{R}_{U_2}$. The theorem is phrased for even-dimensional spaces owing to the choice of sign in axiom (SD4) of Definition 2.1. The appropriate category $\text{SD}^o(X)$ for odd-dimensional X is obtained by changing (SD4) to $\mathcal{D}d[n] = -d$, and the analog of Theorem 2.1 for $\text{SD}^o(X)$ holds.

3. The bordism group Ω_*^{SD}

We briefly review the construction of the bordism group Ω_*^{SD} ; more details can be found in [Ban02, Ch. 4]. Elements of Ω_*^{SD} are represented by closed pseudomanifolds supporting a self-dual sheaf with stalk conditions. An appropriate notion of cobordism and boundary operator will be defined. A pair (pseudomanifold, self-dual sheaf) has a tautological signature associated to it, namely the signature of the quadratic form on hypercohomology in the middle dimension, induced by the self-duality isomorphism. This signature is a cobordism invariant.

Define \mathcal{C}^n (the closed objects) to be the collection of triples $\mathcal{C}^n = \{(X^n, \mathbf{A}^\bullet, d)\}$, where X^n is an n -dimensional closed oriented pseudomanifold, $\mathbf{A}^\bullet \in \text{SD}(X)$ and $d : \mathcal{D}\mathbf{A}^\bullet[n] \cong \mathbf{A}^\bullet$. Disjoint union defines an operation $\mathcal{C}^n \times \mathcal{C}^n \xrightarrow{+} \mathcal{C}^n$. Given $(X^{2k}, \mathbf{A}^\bullet, d)$, d induces a nonsingular pairing on hypercohomology $\mathcal{H}^{-k}(X; \mathbf{A}^\bullet) \otimes \mathcal{H}^{-k}(X; \mathbf{A}^\bullet) \rightarrow \mathbb{R}$. Let $\sigma(X, \mathbf{A}^\bullet, d)$ denote the signature of this pairing and set $\sigma(X^n, \mathbf{A}^\bullet, d) = 0$ for n odd. This defines a map $\sigma : \mathcal{C}^n \rightarrow \mathbb{Z}$. Define Cob^{n+1} (the admissible cobordisms) to be the collection of triples $\text{Cob}^{n+1} = \{(Y^{n+1}, \mathbf{B}^\bullet, \delta)\}$, where Y^{n+1} is an $(n+1)$ -dimensional compact oriented pseudomanifold with boundary, $(\mathbf{B}^\bullet, \delta) \in \text{SD}(\text{int } Y)$, $\delta : \mathcal{D}\mathbf{B}^\bullet[n+1] \xrightarrow{\cong} \mathbf{B}^\bullet$. Again, disjoint sum defines an operation $\text{Cob}^{n+1} \times \text{Cob}^{n+1} \xrightarrow{+} \text{Cob}^{n+1}$. Suppose we are given $(Y^{n+1}, \mathbf{B}^\bullet, \delta) \in \text{Cob}^{n+1}$, $\delta : \mathcal{D}\mathbf{B}^\bullet[n+1] \xrightarrow{\cong} \mathbf{B}^\bullet$. Then δ induces a self-duality isomorphism d for $j^! Ri_! \mathbf{B}^\bullet$ (with $\text{int } Y \xrightarrow{i} Y \xleftarrow{j} \partial Y$ the inclusions):

$$d : \mathcal{D}(j^! Ri_! \mathbf{B}^\bullet)[n] \xrightarrow{\cong} j^! Ri_! \mathbf{B}^\bullet.$$

We call d the *boundary of δ* and write $d = \partial\delta$. In this fashion one defines a boundary map

$$\begin{aligned} \partial : \text{Cob}^{n+1} &\longrightarrow \mathcal{C}^n \\ (Y^{n+1}, \mathbf{B}^\bullet, \delta) &\longmapsto (\partial Y, j^! Ri_! \mathbf{B}^\bullet, \partial\delta). \end{aligned}$$

We have $\partial((Y_1, \mathbf{B}_1^\bullet, \delta_1) + (Y_2, \mathbf{B}_2^\bullet, \delta_2)) = \partial(Y_1, \mathbf{B}_1^\bullet, \delta_1) + \partial(Y_2, \mathbf{B}_2^\bullet, \delta_2)$.

Definition 3.1. Two triples $(X_1, \mathbf{A}_1^\bullet, d_1), (X_2, \mathbf{A}_2^\bullet, d_2) \in \mathcal{C}^n$ are *cobordant* if there exist $(Y_1, \mathbf{B}_1^\bullet, \delta_1), (Y_2, \mathbf{B}_2^\bullet, \delta_2) \in \text{Cob}^{n+1}$ such that

$$(X_1, \mathbf{A}_1^\bullet, d_1) + \partial(Y_1, \mathbf{B}_1^\bullet, \delta_1) \cong (X_2, \mathbf{A}_2^\bullet, d_2) + \partial(Y_2, \mathbf{B}_2^\bullet, \delta_2).$$

Write $[(X, \mathbf{A}^\bullet, d)]$ for the cobordism class of $(X, \mathbf{A}^\bullet, d) \in \mathcal{C}^n$, define

$$\Omega_n^{\text{SD}} = \{[(X, \mathbf{A}^\bullet, d)] : (X, \mathbf{A}^\bullet, d) \in \mathcal{C}^n\}$$

and

$$[(X_1, \mathbf{A}_1^\bullet, d_1)] + [(X_2, \mathbf{A}_2^\bullet, d_2)] = [(X_1, \mathbf{A}_1^\bullet, d_1) + (X_2, \mathbf{A}_2^\bullet, d_2)].$$

This is well defined and $[(X, \mathbf{A}^\bullet, d)] + [(X, \mathbf{A}^\bullet, -d)] = 0$, whence Ω_n^{SD} is an abelian group. In [Ban02, Ch. 4], we prove:

THEOREM 3.1 (Cobordism Invariance of the Signature). *If $(X_i, \mathbf{A}_i^\bullet, d_i) \in \mathcal{C}^n$, $i = 1, 2$, such that $[(X_1, \mathbf{A}_1^\bullet, d_1)] = [(X_2, \mathbf{A}_2^\bullet, d_2)] \in \Omega_n^{\text{SD}}$, then*

$$\sigma(X_1, \mathbf{A}_1^\bullet, d_1) = \sigma(X_2, \mathbf{A}_2^\bullet, d_2).$$

This fact will be used to prove our central Theorem 4.1.

4. The signature of non-Witt spaces

Let X^n be an even-dimensional topological pseudomanifold with stratification

$$X^n = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \dots \supset X_0 \supset X_{-1} = \emptyset,$$

where strata are indexed by their dimension. We denote the pure strata by $V_i = X_i - X_{i-1}$, $i = 0, \dots, n$, and set $V_{-1} = \emptyset$. Consider the open cylinder

$$Y^{n+1} = X \times (0, 1).$$

The natural stratification of Y is obtained by taking $Y_i = X_{i-1} \times (0, 1)$. The crucial idea in the proof of Theorem 4.1 below is to work with the following refinement of the natural stratification: We say that Y is stratified with *cuts* at $\frac{1}{2}$, if it is filtered as

$$Y^{n+1} = Y_{n+1} \supset Y_{n-1} \supset Y_{n-2} \supset \dots \supset Y_0 \supset Y_{-1} = \emptyset,$$

where

$$Y_i = \bigcup_{j=0}^i W_j$$

and

$$\begin{aligned} W_{n+1} &= V_n \times (0, 1), \\ W_n &= \emptyset, \\ W_{n-1} &= V_{n-2} \times (0, 1), \\ W_{n-2} &= V_{n-3} \times (0, \frac{1}{2}) \sqcup V_{n-3} \times (\frac{1}{2}, 1), \\ W_j &= V_j \times \{\frac{1}{2}\} \sqcup V_{j-1} \times (0, \frac{1}{2}) \sqcup V_{j-1} \times (\frac{1}{2}, 1), 0 \leq j \leq n-3. \end{aligned}$$

We continue with a sequence of sheaf-theoretic lemmas (Lemmas 4.1 through 4.6), which prepare the proof of Theorem 4.1.

LEMMA 4.1. *Let Z be a pseudomanifold and $A \subset Z$ a subspace. Given a commutative square*

$$\begin{array}{ccc} A \times (0, 1) & \xrightarrow{i} & Z \times (0, 1) \\ p_0 \downarrow & & p \downarrow \\ A & \xrightarrow{\iota} & Z \end{array}$$

(i, ι inclusions, p, p_0 projections to the first factor), there exist isomorphisms of functors

- (i) $p^! R\iota_* \cong Ri_* p_0^!$,
- (ii) $p^! \tau_{\leq s} R\iota_* \cong \tau_{\leq s-1} Ri_* p_0^!$.

Proof. Note that

$$p^! \cong p^*[1], p_0^! \cong p_0^*[1],$$

since the fiber $(0, 1)$ is nonsingular. Thus (i) follows from the fiber square identity §1.13, (13) of [GM83]. Isomorphism (ii) is obtained from (i) by

$$\begin{aligned} \tau_{\leq s-1}(Ri_*p_0^!) &\cong \tau_{\leq s-1}(p^!R\iota_*) \\ &\cong \tau_{\leq s-1}(p^*R\iota_*[1]) \\ &\cong (\tau_{\leq s}(p^*R\iota_*))[1] \\ &\cong p^*(\tau_{\leq s}R\iota_*[1]) \\ &\cong p^!(\tau_{\leq s}R\iota_*). \end{aligned} \quad \square$$

LEMMA 4.2. *Let X be a pseudomanifold, $i : U \hookrightarrow X$ an open inclusion, $A \subset U$ a subset closed in X . If $\mathbf{A} \in \text{Sh}(U)$ with $\text{supp}(\mathbf{A}) \subset A$, then $i_*\mathbf{A} = i_!\mathbf{A}$.*

Proof. If $j : A \rightarrow U$ denotes the inclusion of A in U , then $\mathbf{A} = j_*j^*\mathbf{A}$, since $A \subset X$ closed implies $A \subset U$ closed. Thus

$$i_*\mathbf{A} = (ij)_*j^*\mathbf{A},$$

and as $ij : A \hookrightarrow X$ is closed, we have $(ij)_* = (ij)_!$. Hence

$$i_*\mathbf{A} = i_!j_!j^*\mathbf{A} = i_!j_*j^*\mathbf{A} = i_!\mathbf{A},$$

since j is closed. □

LEMMA 4.3. *Let X be a topological space and $U_1, U_2 \subset X$ open subsets. Consider the diagram of open inclusions*

$$\begin{array}{ccc} U_1 & \xleftarrow{j|} & U_1 \cap U_2 \\ i \downarrow & & \downarrow i| \\ X & \xleftarrow{j} & U_2. \end{array}$$

If $\mathbf{A} \in \text{Sh}(U_1)$, then

$$j^*i_*\mathbf{A} \cong i|_*j^*\mathbf{A}.$$

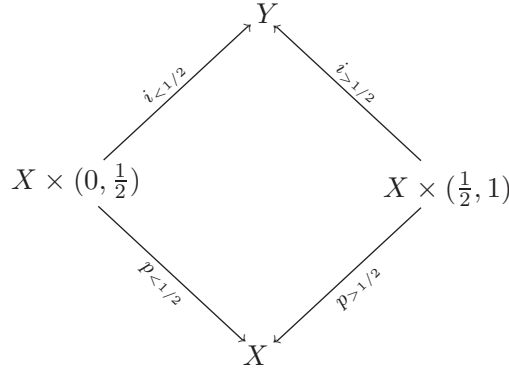
Proof. We show that the two sheaves have isomorphic canonical presheaves. Let $V \subset U_2$ be open in U_2 . As V is then open in X as well, we have

$$\Gamma(V, j^*i_*\mathbf{A}) = \Gamma(V, i_*\mathbf{A}) = \Gamma(V \cap U_1, \mathbf{A}).$$

As $V \cap U_1$ is open in U_1 , we obtain on the other hand

$$\Gamma(V, i|_*j^*\mathbf{A}) = \Gamma(V \cap U_1, j^*\mathbf{A}) = \Gamma(V \cap U_1, \mathbf{A}). \quad \square$$

LEMMA 4.4. *Let X^n be a pseudomanifold with bottom stratum Σ , assumed to be of odd codimension k . Consider $X \times (0, 1)$, its open subset $Y = (X \times (0, 1)) - (\Sigma \times \{\frac{1}{2}\})$ and the following diagram of inclusions and projections:*



Y is a pseudomanifold whose bottom stratum is the disjoint union $\Sigma \times (0, \frac{1}{2}) \sqcup \Sigma \times (\frac{1}{2}, 1)$. Suppose $\mathbf{S}_0^\bullet, \mathbf{S}_1^\bullet \in \text{SD}(X - \Sigma)$ and

$$\mathcal{L}_0 \longrightarrow \mathcal{O}(\mathbf{S}_0^\bullet),$$

$$\mathcal{L}_1 \longrightarrow \mathcal{O}(\mathbf{S}_1^\bullet)$$

are Lagrangian structures for \mathbf{S}_0^\bullet and \mathbf{S}_1^\bullet , respectively, at Σ . Let

$$\mathbf{S}^\bullet \in \text{SD}((X - \Sigma) \times (0, 1))$$

be such that

$$\mathbf{S}^\bullet|_{(X-\Sigma) \times (0, \frac{1}{2})} \cong \hat{p}_{<1/2}^! \mathbf{S}_0^\bullet,$$

$$\mathbf{S}^\bullet|_{(X-\Sigma) \times (\frac{1}{2}, 1)} \cong \hat{p}_{>1/2}^! \mathbf{S}_1^\bullet,$$

where $\hat{p}_{<1/2} : (X - \Sigma) \times (0, \frac{1}{2}) \rightarrow X - \Sigma$, $\hat{p}_{>1/2} : (X - \Sigma) \times (\frac{1}{2}, 1) \rightarrow X - \Sigma$ are projections to the first factor.

In this situation:

(i) $\mathcal{O}(\mathbf{S}^\bullet) \cong i_{<1/2!} p_{<1/2}^! \mathcal{O}(\mathbf{S}_0^\bullet) \oplus i_{>1/2!} p_{>1/2}^! \mathcal{O}(\mathbf{S}_1^\bullet)$, and

(ii) $i_{<1/2!} p_{<1/2}^! \mathcal{L}_0 \oplus i_{>1/2!} p_{>1/2}^! \mathcal{L}_1 \rightarrow i_{<1/2!} p_{<1/2}^! \mathcal{O}(\mathbf{S}_0^\bullet) \oplus i_{>1/2!} p_{>1/2}^! \mathcal{O}(\mathbf{S}_1^\bullet)$ is a Lagrangian structure for \mathbf{S}^\bullet at $\Sigma \times (0, \frac{1}{2}) \sqcup \Sigma \times (\frac{1}{2}, 1)$.

Proof. We shall work with the following diagram of inclusion- and projection-maps:

$$\begin{array}{ccc}
 (X - \Sigma) \times (0, 1) & \xhookrightarrow{i} & Y \\
 \uparrow \hat{i}_{<1/2} & & \uparrow i_{<1/2} \\
 (X - \Sigma) \times (0, \frac{1}{2}) & \xhookrightarrow{\bar{i}_{<1/2}} & X \times (0, \frac{1}{2}) \\
 \downarrow \hat{p}_{<1/2} & & \downarrow p_{<1/2} \\
 X - \Sigma & \xhookrightarrow{\iota} & X
 \end{array}$$

and its counterpart for $(\frac{1}{2}, 1)$:

$$\begin{array}{ccc}
 (X - \Sigma) \times (0, 1) & \xhookrightarrow{i} & Y \\
 \uparrow \hat{i}_{>1/2} & & \uparrow i_{>1/2} \\
 (X - \Sigma) \times (\frac{1}{2}, 1) & \xhookrightarrow{\bar{i}_{>1/2}} & X \times (\frac{1}{2}, 1) \\
 \downarrow \hat{p}_{>1/2} & & \downarrow p_{>1/2} \\
 X - \Sigma & \xhookrightarrow{\iota} & X.
 \end{array}$$

Let us discuss statement (i). Set $s = \bar{n}(k) - n$ so that $\mathcal{O}(\mathbf{S}_0^\bullet) = \mathbf{H}^s(R\iota_*\mathbf{S}_0^\bullet)[-s] \in D^b(X)$. Its pull-back with compact supports to $X \times (0, \frac{1}{2})$ is

$$\begin{aligned}
 p_{<1/2}^! \mathcal{O}(\mathbf{S}_0^\bullet) &\cong p_{<1/2}^* \mathbf{H}^s(R\iota_*\mathbf{S}_0^\bullet)[-s][1] \\
 &\cong \mathbf{H}^{s-1}(p_{<1/2}^* R\iota_*\mathbf{S}_0^\bullet[1])[1-s] \\
 &\cong \mathbf{H}^{s-1}(p_{<1/2}^! R\iota_*\mathbf{S}_0^\bullet)[1-s] \\
 &\cong \mathbf{H}^{s-1}(R\bar{i}_{<1/2*} \hat{p}_{<1/2}^! \mathbf{S}_0^\bullet)[1-s] \text{ (by Lemma 4.1, (i))} \\
 &= \mathcal{O}(\hat{p}_{<1/2}^! \mathbf{S}_0^\bullet)
 \end{aligned}$$

and

$$(3) \quad i_{<1/2} p_{<1/2}^! \mathcal{O}(\mathbf{S}_0^\bullet) \cong i_{<1/2} \mathcal{O}(\hat{p}_{<1/2}^! \mathbf{S}_0^\bullet).$$

Further,

$$\begin{aligned} \mathcal{O}(\hat{p}_{<1/2}^! \mathbf{S}_0^\bullet) &= \mathbf{H}^{s-1}(R\bar{i}_{<1/2*} \hat{i}_{<1/2}^* \mathbf{S}^\bullet)[1-s] \\ &\cong \mathbf{H}^{s-1}(i_{<1/2}^* Ri_* \mathbf{S}^\bullet)[1-s] \text{ (by Lemma 4.3)} \\ &\cong i_{<1/2}^* \mathbf{H}^{s-1}(Ri_* \mathbf{S}^\bullet)[1-s] \\ &= i_{<1/2}^* \mathcal{O}(\mathbf{S}^\bullet), \end{aligned}$$

so that by (3),

$$(4) \quad i_{<1/2!} p_{<1/2}^! \mathcal{O}(\mathbf{S}_0^\bullet) \cong i_{<1/2!} i_{<1/2}^* \mathcal{O}(\mathbf{S}^\bullet)$$

and analogously (by the diagram for $(\frac{1}{2}, 1)$)

$$(5) \quad i_{>1/2!} p_{>1/2}^! \mathcal{O}(\mathbf{S}_1^\bullet) \cong i_{>1/2!} i_{>1/2}^* \mathcal{O}(\mathbf{S}^\bullet).$$

Consider the adjunction morphisms

$$\begin{aligned} i_{<1/2!} i_{<1/2}^* \mathcal{O}(\mathbf{S}^\bullet) &\longrightarrow \mathcal{O}(\mathbf{S}^\bullet), \\ i_{>1/2!} i_{>1/2}^* \mathcal{O}(\mathbf{S}^\bullet) &\longrightarrow \mathcal{O}(\mathbf{S}^\bullet) \end{aligned}$$

and their sum

$$(6) \quad i_{<1/2!} i_{<1/2}^* \mathcal{O}(\mathbf{S}^\bullet) \oplus i_{>1/2!} i_{>1/2}^* \mathcal{O}(\mathbf{S}^\bullet) \longrightarrow \mathcal{O}(\mathbf{S}^\bullet).$$

Now using $\text{supp } \mathcal{O}(\mathbf{S}^\bullet) = \Sigma \times (0, \frac{1}{2}) \sqcup \Sigma \times (\frac{1}{2}, 1)$, $\text{supp}(i_{<1/2}^* \mathcal{O}(\mathbf{S}^\bullet)) = \Sigma \times (0, \frac{1}{2})$, $\text{supp}(i_{>1/2}^* \mathcal{O}(\mathbf{S}^\bullet)) = \Sigma \times (\frac{1}{2}, 1)$, and that $i_{<1/2!}$ and $i_{>1/2!}$ are extension by zero, we see by looking at stalks that (6) is an isomorphism. Thus statement (i) follows in view of (4) and (5).

We prove statement (ii): As $\mathcal{L}_0 \rightarrow \mathcal{O}(\mathbf{S}_0^\bullet)$, $\mathcal{L}_1 \rightarrow \mathcal{O}(\mathbf{S}_1^\bullet)$ are Lagrangian structures, they come with distinguished triangles in $D^b(X)$:

$$\begin{array}{ccc} \mathcal{L}_0 & \longrightarrow & \mathcal{O}(\mathbf{S}_0^\bullet) \\ & \searrow \scriptstyle \cong & \swarrow \\ & \mathcal{DL}_0[n+1] & \end{array}, \quad \begin{array}{ccc} \mathcal{L}_1 & \longrightarrow & \mathcal{O}(\mathbf{S}_1^\bullet) \\ & \searrow \scriptstyle \cong & \swarrow \\ & \mathcal{DL}_1[n+1] & \end{array}.$$

These induce distinguished triangles

$$\begin{array}{ccc} i_{<1/2!} p_{<1/2}^! \mathcal{L}_0 & \longrightarrow & i_{<1/2!} p_{<1/2}^! \mathcal{O}(\mathbf{S}_0^\bullet) \\ & \searrow \scriptstyle \cong & \swarrow \\ & i_{<1/2!} p_{<1/2}^! \mathcal{DL}_0[n+1] & \end{array}$$

and

$$\begin{array}{ccc}
 i_{>1/2!}p_{>1/2}^!\mathcal{L}_1 & \longrightarrow & i_{>1/2!}p_{>1/2}^!\mathcal{O}(\mathbf{S}_1^\bullet) \\
 & \swarrow \scriptstyle (i) & \searrow \\
 & & i_{>1/2!}p_{>1/2}^!\mathcal{D}\mathcal{L}_1[n+1]
 \end{array}$$

We obtain for the duals

$$\begin{aligned}
 i_{<1/2!}p_{<1/2}^!\mathcal{D}\mathcal{L}_0[n+1] &\cong i_{<1/2!}\mathcal{D}(p_{<1/2}^*\mathcal{L}_0)[n+1] \\
 &\cong i_{<1/2!}\mathcal{D}(p_{<1/2}^!\mathcal{L}_0[-1])[n+1] \\
 &\cong i_{<1/2!}\mathcal{D}(p_{<1/2}^!\mathcal{L}_0)[n+2] \\
 &\cong \mathcal{D}(Ri_{<1/2*}p_{<1/2}^!\mathcal{L}_0)[n+2] \\
 &\cong \mathcal{D}(i_{<1/2!}p_{<1/2}^!\mathcal{L}_0)[n+2], && \text{using Lemma 4.2} \\
 &&& \text{(observing that } \Sigma \times (0, \frac{1}{2}) \text{ is} \\
 &&& \text{closed in } Y)
 \end{aligned}$$

and similarly

$$i_{>1/2!}p_{>1/2}^!\mathcal{D}\mathcal{L}_1[n+1] \cong \mathcal{D}(i_{>1/2!}p_{>1/2}^!\mathcal{L}_1)[n+2].$$

Taking direct sums yields the distinguished triangle

$$\begin{array}{ccc}
 i_{<1/2!}p_{<1/2}^!\mathcal{L}_0 \oplus i_{>1/2!}p_{>1/2}^!\mathcal{L}_1 & \longrightarrow & i_{<1/2!}p_{<1/2}^!\mathcal{O}(\mathbf{S}_0^\bullet) \oplus i_{>1/2!}p_{>1/2}^!\mathcal{O}(\mathbf{S}_1^\bullet) \\
 & \swarrow \scriptstyle (i) & \searrow \\
 & & \mathcal{D}(i_{<1/2!}p_{<1/2}^!\mathcal{L}_0)[n+2] \oplus \mathcal{D}(i_{>1/2!}p_{>1/2}^!\mathcal{L}_1)[n+2].
 \end{array}$$

Now, \mathcal{D} is an additive functor and applying statement (i) of the lemma, we get the distinguished triangle

$$\begin{array}{ccc}
 i_{<1/2!}p_{<1/2}^!\mathcal{L}_0 \oplus i_{>1/2!}p_{>1/2}^!\mathcal{L}_1 & \longrightarrow & \mathcal{O}(\mathbf{S}^\bullet) \\
 & \swarrow \scriptstyle (i) & \searrow \\
 & & \mathcal{D}(i_{<1/2!}p_{<1/2}^!\mathcal{L}_0 \oplus i_{>1/2!}p_{>1/2}^!\mathcal{L}_1)[n+2]
 \end{array}$$

which exhibits $i_{<1/2!}p_{<1/2}^!\mathcal{L}_0 \oplus i_{>1/2!}p_{>1/2}^!\mathcal{L}_1$ as a Lagrangian structure for \mathbf{S}^\bullet along $\Sigma \times (0, \frac{1}{2}) \sqcup \Sigma \times (\frac{1}{2}, 1)$. □

LEMMA 4.5. *Let X be a pseudomanifold, $\mathbf{A}^\bullet \in D^b(X)$, $p : X \times (0, 1) \rightarrow X$ the projection to the first factor and i, j the inclusions*

$$X \times (0, 1) \xrightarrow{i} X \times (0, 1] \xleftarrow{j} X \times \{1\}.$$

Then

$$j^! Ri_* p^! \mathbf{A}^\bullet \cong \mathbf{A}^\bullet.$$

Proof. Given a diagram

$$\begin{array}{ccc} X \times (0, 1) & \xrightarrow{i} & X \times (0, 1] \\ & \searrow p & \swarrow \alpha \\ & X & \end{array}$$

we have the identity

$$Ri_* p^* \mathbf{A}^\bullet \cong q^* \mathbf{A}^\bullet;$$

see e.g. [B⁺84, V, 10.22 (4)]. This implies

$$j^* Ri_* p^* \mathbf{A}^\bullet \cong (qj)^* \mathbf{A}^\bullet \cong \mathbf{A}^\bullet,$$

since $qj = 1_X$. Using $p^*[1] \cong p^!$ and $j^* Ri_* \cong j^! Ri_![1]$ we obtain the result. \square

LEMMA 4.6. *Let X be a stratified pseudomanifold, $Y \subset X$ a closed union of components of strata such that Y is an m -dimensional manifold, $i : X - Y \hookrightarrow X$ and p, q integers with $p+q = -m$. If $\mathbf{A}^\bullet, \mathbf{B}^\bullet \in D^b(X - Y)$ and $\mathcal{D}\mathbf{A}^\bullet \cong \mathbf{B}^\bullet$, then there is an induced isomorphism*

$$\mathcal{D}(\tau_{\leq p-1}^Y Ri_* \mathbf{A}^\bullet) \cong \tau_{\leq q-1}^Y Ri_* \mathbf{B}^\bullet.$$

Proof. See [GM83, §9]. \square

THEOREM 4.1. *Let X^n be an even-dimensional closed oriented pseudo-manifold. If $\text{SD}(X) \neq \emptyset$, then the signature*

$$\sigma(X) = \sigma(\mathbf{IC}_{\mathcal{L}}^\bullet),$$

$\mathbf{IC}_{\mathcal{L}}^\bullet \in \text{SD}(X)$, is independent of the choice of Lagrangian structure \mathcal{L} .

Proof. Let $\mathbf{IC}_{\mathcal{L}_0}^\bullet, \mathbf{IC}_{\mathcal{L}_1}^\bullet \in \text{SD}(X)$ be self-dual complexes of sheaves determined by the Lagrangian structures

$$\mathcal{L}_0 = (\mathcal{L}_0^1, \mathcal{L}_0^3, \dots, \mathcal{L}_0^{n-3}) \in \text{Lag}(V_1) \times \text{Lag}(V_3) \times \dots \times \text{Lag}(V_{n-3}) \times \text{Const}(V_n)$$

and

$$\mathcal{L}_1 = (\mathcal{L}_1^1, \mathcal{L}_1^3, \dots, \mathcal{L}_1^{n-3}) \in \text{Lag}(V_1) \times \text{Lag}(V_3) \times \dots \times \text{Lag}(V_{n-3}) \times \text{Const}(V_n),$$

respectively. We have to show that

$$\sigma(\mathbf{IC}_{\mathcal{L}_0}^\bullet) = \sigma(\mathbf{IC}_{\mathcal{L}_1}^\bullet).$$

This will be done by showing that the elements $[(X, \mathbf{IC}_{\mathcal{L}_0}^\bullet)]$ and $[(X, \mathbf{IC}_{\mathcal{L}_1}^\bullet)]$ in the bordism group Ω_n^{SD} are equal, invoking Theorem 3.1. Let Y^{n+1} be the open cylinder

$$Y = X \times (0, 1).$$

Its compactification $\bar{Y} = X \times [0, 1]$ will provide the underlying geometric bordism from X to itself. This bordism is, of course, topologically trivial, but its stratification will *not* be taken to be a product, and it will be covered with a non-trivial sheaf. Thus let Y be stratified with cuts at $\frac{1}{2}$. We shall inductively construct a self-dual sheaf $\mathbf{S}^\bullet \in \text{SD}(Y)$. Set $T_k = X - X_{n-k}$,

$$\begin{aligned} U_2 &= W_{n+1}, \\ U_3 &= U_2 \cup W_{n-1}, \end{aligned}$$

and for $3 \leq k \leq n$,

$$U_{k+1} = \begin{cases} U_k \cup (V_{n-k} \times (0, \frac{1}{2}) \sqcup V_{n-k} \times (\frac{1}{2}, 1)), & k \text{ odd} \\ U_k \cup (V_{n-k+1} \times \{\frac{1}{2}\} \cup V_{n-k} \times (0, 1)), & k \text{ even} \end{cases}$$

so that in closed form ($k \geq 3$)

$$U_k = \begin{cases} T_k \times (0, 1), & k \text{ odd} \\ T_{k-1} \times (0, 1) \cup (V_{n-k+1} \times (0, \frac{1}{2}) \sqcup V_{n-k+1} \times (\frac{1}{2}, 1)), & k \text{ even}, \end{cases}$$

and let $i_k : U_k \hookrightarrow U_{k+1}$ be inclusions. For each k , U_k contains both $T_k \times (0, \frac{1}{2})$ and $T_k \times (\frac{1}{2}, 1)$ as subsets. Define

$$\mathbf{S}_2^\bullet = \mathbb{R}_{U_2}[n+1] \in \text{SD}(U_2).$$

For any subset $A \subset X$, let $p_{<1/2} : A \times (0, \frac{1}{2}) \rightarrow A$ and $p_{>1/2} : A \times (\frac{1}{2}, 1) \rightarrow A$ be generic notation for the first factor projections. We note that

$$\begin{aligned} \mathbf{S}_2^\bullet|_{T_2 \times (0, \frac{1}{2})} &\cong p_{<1/2}^! \mathbf{IC}_{\mathcal{L}_0}^\bullet|_{T_2}, \\ \mathbf{S}_2^\bullet|_{T_2 \times (\frac{1}{2}, 1)} &\cong p_{>1/2}^! \mathbf{IC}_{\mathcal{L}_1}^\bullet|_{T_2} \end{aligned}$$

by the normalization axiom. Assume inductively that

$$\mathbf{S}_k^\bullet \in \text{SD}(U_k)$$

has been constructed such that

$$(7) \quad \mathbf{S}_k^\bullet|_{T_k \times (0, \frac{1}{2})} \cong p_{<1/2}^! \mathbf{IC}_{\mathcal{L}_0}^\bullet|_{T_k},$$

$$(8) \quad \mathbf{S}_k^\bullet|_{T_k \times (\frac{1}{2}, 1)} \cong p_{>1/2}^! \mathbf{IC}_{\mathcal{L}_1}^\bullet|_{T_k}$$

hold.

We will construct $\mathbf{S}_{k+1}^\bullet \in \text{SD}(U_{k+1})$: There are two cases to consider. In the first case we assume that k is even. Then $U_{k+1} - U_k = V_{n-k+1} \times \{\frac{1}{2}\} \cup V_{n-k} \times (0, 1)$, and we will extend \mathbf{S}_k^\bullet to U_{k+1} in two steps:

$$\mathbf{S}_k^\bullet \in \text{SD}(U_k) \rightsquigarrow \text{SD}(U_k \cup V_{n-k+1} \times \{\frac{1}{2}\}) \rightsquigarrow \text{SD}(U_{k+1}).$$

Since $V_{n-k+1} \times \{\frac{1}{2}\}$ is a closed union of components of strata of $U_k \cup V_{n-k+1} \times \{\frac{1}{2}\}$ and a submanifold, Lemma 4.6 with $\mathbf{A}^\bullet = \mathbf{S}_k^\bullet$, $\mathbf{B}^\bullet = \mathbf{S}_k^\bullet[-n-1]$, $p = \bar{m}(k) - n$, $q = \bar{m}(k) + 1$ (thus $p + q = k - n - 1$) shows that

$$\mathbf{R}^\bullet = \tau_{\leq \bar{m}(k) - n - 1} Ri'_* \mathbf{S}_k^\bullet$$

is an object of $\text{SD}(U_k \cup V_{n-k+1} \times \{\frac{1}{2}\})$, where $i' : U_k \hookrightarrow U_k \cup V_{n-k+1} \times \{\frac{1}{2}\}$ (note $\tau_{\leq \bar{m}(k)}(Ri'_* \mathbf{S}_k^\bullet[-n-1]) = (\tau_{\leq \bar{m}(k) - n - 1} Ri'_* \mathbf{S}_k^\bullet)[-n-1]$).

Now $V_{n-k} \times (0, 1)$ is a closed union of components of strata of U_{k+1} and a submanifold, so that an application of Lemma 4.6 with $\mathbf{A}^\bullet = \mathbf{R}^\bullet$, $\mathbf{B}^\bullet = \mathbf{R}^\bullet[-n-1]$ and p, q as before yields that

$$\mathbf{S}_{k+1}^\bullet = \tau_{\leq \bar{m}(k) - n - 1} Ri''_* \mathbf{R}^\bullet$$

($i'' : U_k \cup V_{n-k+1} \times \{\frac{1}{2}\} \hookrightarrow U_{k+1}$) is an object of $\text{SD}(U_{k+1})$ (in particular, \mathbf{S}_{k+1}^\bullet is constructible on U_{k+1} with respect to the stratification with cuts at $\frac{1}{2}$). Consider the commutative diagram

$$\begin{array}{ccc} U_k \cup V_{n-k+1} \times \{\frac{1}{2}\} & \xrightarrow{i''} & U_{k+1} \\ \uparrow i' j_k & & \uparrow j_{k+1} \\ T_k \times (0, \frac{1}{2}) & \xrightarrow{\hat{i}_k} & T_{k+1} \times (0, \frac{1}{2}) \\ \downarrow \hat{p}_{<1/2} & & \downarrow p_{<1/2} \\ T_k & \xrightarrow{\iota_k} & T_{k+1}. \end{array}$$

We have $T_{k+1} = T_k \cup V_{n-k}$ and

$$(U_k \cup V_{n-k+1} \times \{\frac{1}{2}\}) \cap (T_{k+1} \times (0, \frac{1}{2})) = T_k \times (0, \frac{1}{2}).$$

Thus by Lemma 4.3,

$$j_{k+1}^* Ri''_* \cong R\hat{i}_{k*}(i' j_k)^*$$

and we calculate ($s = \bar{m}(k) - n - 1$)

$$\begin{aligned}
 \mathbf{S}_{k+1}^\bullet|_{T_{k+1} \times (0, \frac{1}{2})} &= j_{k+1}^* \tau_{\leq s} Ri''_* \tau_{\leq s} Ri'_* \mathbf{S}_k^\bullet \\
 &\cong \tau_{\leq s} (j_{k+1}^* Ri''_*) \tau_{\leq s} Ri'_* \mathbf{S}_k^\bullet \\
 &\cong \tau_{\leq s} R\hat{\iota}_{k*} j_k^* i'^* \tau_{\leq s} Ri'_* \mathbf{S}_k^\bullet \\
 &\cong \tau_{\leq s} R\hat{\iota}_{k*} j_k^* (\tau_{\leq s} i'^* Ri'_* \mathbf{S}_k^\bullet) \\
 &\cong \tau_{\leq s} R\hat{\iota}_{k*} j_k^* (\tau_{\leq s} \mathbf{S}_k^\bullet) \\
 &\cong \tau_{\leq s} R\hat{\iota}_{k*} j_k^* \mathbf{S}_k^\bullet \\
 &\cong \tau_{\leq s} R\hat{\iota}_{k*} p_{<1/2}^! (\mathbf{IC}_{\mathcal{L}_0}^\bullet|_{T_k}) && \text{(induction hypothesis)} \\
 &\cong p_{<1/2}^! (\tau_{\leq \bar{m}(k)-n} R\hat{\iota}_{k*} (\mathbf{IC}_{\mathcal{L}_0}^\bullet|_{T_k})) && \text{(Lemma 4.1, (ii))} \\
 &\cong p_{<1/2}^! (\mathbf{IC}_{\mathcal{L}_0}^\bullet|_{T_{k+1}}),
 \end{aligned}$$

and similarly

$$\mathbf{S}_{k+1}^\bullet|_{T_{k+1} \times (\frac{1}{2}, 1)} \cong p_{>1/2}^! \mathbf{IC}_{\mathcal{L}_1}^\bullet|_{T_{k+1}}.$$

In the second case we assume that k is odd, so that

$$U_{k+1} - U_k = V_{n-k} \times (0, \frac{1}{2}) \sqcup V_{n-k} \times (\frac{1}{2}, 1)$$

is of odd codimension in Y . Along the bottom stratum V_{n-k} of T_{k+1} , we have the Lagrangian structure

$$\mathcal{L}_0^{n-k} \longrightarrow \mathcal{O}(\mathbf{IC}_{\mathcal{L}_0}^\bullet|_{T_k})$$

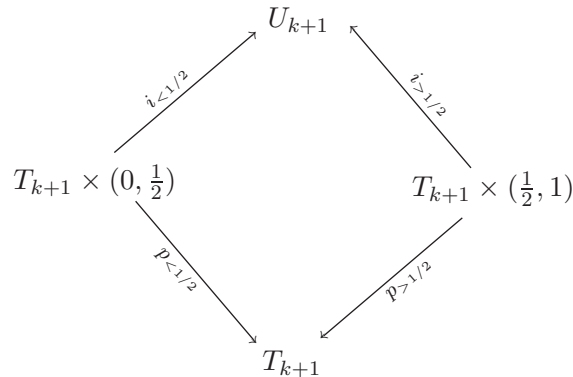
for $\mathbf{IC}_{\mathcal{L}_0}^\bullet|_{T_k}$ and

$$\mathcal{L}_1^{n-k} \longrightarrow \mathcal{O}(\mathbf{IC}_{\mathcal{L}_1}^\bullet|_{T_k})$$

for $\mathbf{IC}_{\mathcal{L}_1}^\bullet|_{T_k}$.

The bottom stratum of U_{k+1} is the disjoint union $V_{n-k} \times (0, \frac{1}{2}) \sqcup V_{n-k} \times (\frac{1}{2}, 1)$.

Thus, in view of (7), (8) and using the notation



Lemma 4.4 tells us that

$$\begin{aligned} \mathcal{L} &= i_{<1/2!}p_{<1/2}^!\mathcal{L}_0^{n-k} \oplus i_{>1/2!}p_{>1/2}^!\mathcal{L}_1^{n-k} \\ &\longrightarrow i_{<1/2!}p_{<1/2}^!\mathcal{O}(\mathbf{IC}_{\mathcal{L}_0}^\bullet|_{T_k}) \oplus i_{>1/2!}p_{>1/2}^!\mathcal{O}(\mathbf{IC}_{\mathcal{L}_1}^\bullet|_{T_k}) \\ &\cong \mathcal{O}(\mathbf{S}_k^\bullet) \end{aligned}$$

is a Lagrangian structure for \mathbf{S}_k^\bullet along $U_{k+1} - U_k$. Set

$$\mathbf{S}_{k+1}^\bullet = \mathbf{S}_k^\bullet \boxplus \mathcal{L} \in \text{SD}(U_{k+1}).$$

The Postnikov equivalence of categories 2.1 supplies us with isomorphisms

$$\begin{aligned} \mathbf{S}_{k+1}^\bullet|_{T_{k+1} \times (0, \frac{1}{2})} &\cong p_{<1/2}^!\mathbf{IC}_{\mathcal{L}_0}^\bullet|_{T_{k+1}}, \\ \mathbf{S}_{k+1}^\bullet|_{T_{k+1} \times (\frac{1}{2}, 1)} &\cong p_{>1/2}^!\mathbf{IC}_{\mathcal{L}_1}^\bullet|_{T_{k+1}}. \end{aligned}$$

This finishes case two and concludes the induction step. The sought self-dual sheaf on Y is

$$\mathbf{S}^\bullet = \mathbf{S}_{n+1}^\bullet \in \text{SD}(U_{n+1}) = \text{SD}(Y),$$

which satisfies by construction

$$\begin{aligned} \mathbf{S}^\bullet|_{X \times (0, \frac{1}{2})} &\cong p_{<1/2}^!\mathbf{IC}_{\mathcal{L}_0}^\bullet, \\ \mathbf{S}^\bullet|_{X \times (\frac{1}{2}, 1)} &\cong p_{>1/2}^!\mathbf{IC}_{\mathcal{L}_1}^\bullet. \end{aligned}$$

Consider the following inclusions into the compactification \bar{Y} of Y :

$$\begin{aligned} X \times (0, \frac{1}{2}) &\xrightarrow{i_0} X \times [0, \frac{1}{2}] \xleftarrow{j_0} X \times \{0\}, \\ X \times (\frac{1}{2}, 1) &\xrightarrow{i_1} X \times (\frac{1}{2}, 1] \xleftarrow{j_1} X \times \{1\}. \end{aligned}$$

By Lemma 4.5,

$$\begin{aligned} j_0^! Ri_0^! p_{<1/2}^!\mathbf{IC}_{\mathcal{L}_0}^\bullet &\cong \mathbf{IC}_{\mathcal{L}_0}^\bullet, \\ j_1^! Ri_1^! p_{>1/2}^!\mathbf{IC}_{\mathcal{L}_1}^\bullet &\cong \mathbf{IC}_{\mathcal{L}_1}^\bullet, \end{aligned}$$

and thus

$$\begin{aligned} j_0^! Ri_0^! (\mathbf{S}^\bullet|_{X \times (0, \frac{1}{2})}) &\cong \mathbf{IC}_{\mathcal{L}_0}^\bullet, \\ j_1^! Ri_1^! (\mathbf{S}^\bullet|_{X \times (\frac{1}{2}, 1)}) &\cong \mathbf{IC}_{\mathcal{L}_1}^\bullet. \end{aligned}$$

In terms of Ω_n^{SD} this means that $(\bar{Y}, \mathbf{S}^\bullet)$ is an admissible cobordism such that

$$\partial(\bar{Y}, \mathbf{S}^\bullet) = (X, \mathbf{IC}_{\mathcal{L}_0}^\bullet) + (-X, \mathbf{IC}_{\mathcal{L}_1}^\bullet),$$

whence $[(X, \mathbf{IC}_{\mathcal{L}_0}^\bullet)] = [(X, \mathbf{IC}_{\mathcal{L}_1}^\bullet)] \in \Omega_n^{\text{SD}}$. □

5. The L-class of non-Witt spaces

Let us recall the existence and uniqueness result on L-classes of self-dual sheaves as stated in [CS91]: Let X^n be a compact oriented stratified pseudo-manifold and let

$$j : Y^m \hookrightarrow X^n$$

be a normally nonsingular inclusion of an oriented stratified pseudomanifold Y^m . Consider an open neighborhood $E \subset X$ of Y , the total space of an \mathbb{R}^{n-m} -vector bundle over Y , and put $E_0 = E - Y$, the total space with the zero-section removed. Let $u \in H^{n-m}(E, E_0)$ denote the Thom class. If $\pi : E \rightarrow Y$ denotes the projection, then the composition

$$H_k(X) \xrightarrow{i_*} H_k(X, X - Y) \xrightarrow{e_*} H_k(E, E_0) \xrightarrow{u \cap -} H_{k-n+m}(E) \xrightarrow{\pi_*} H_{k-n+m}(Y)$$

defines a map

$$j^! : H_k(X) \longrightarrow H_{k-n+m}(Y).$$

THEOREM 5.1 ([CS91]). *Let $\mathbf{S}^\bullet \in D^b(X)$ be a self-dual complex of sheaves. There exist unique classes $L_k(\mathbf{S}^\bullet) \in H_k(X; \mathbb{Q})$ such that if $j : Y^m \hookrightarrow X^n$ is a normally nonsingular inclusion with trivial normal bundle, then*

$$j^! L_{n-m}(\mathbf{S}^\bullet) = \sigma(j^! \mathbf{S}^\bullet).$$

In particular $\mathbf{IC}_{\mathcal{L}}^\bullet \in \text{SD}(X)$ has L-classes $L_k(\mathbf{IC}_{\mathcal{L}}^\bullet) \in H_k(X; \mathbb{Q})$. Generalizing Theorem 4.1 on the signature $\sigma(\mathbf{IC}_{\mathcal{L}}^\bullet) = L_0(\mathbf{IC}_{\mathcal{L}}^\bullet)$, we obtain

THEOREM 5.2. *Let X^n be a closed oriented pseudomanifold. If $\text{SD}(X) \neq \emptyset$, then the L-classes*

$$L_k(X) = L_k(\mathbf{IC}_{\mathcal{L}}^\bullet) \in H_k(X; \mathbb{Q}),$$

$\mathbf{IC}_{\mathcal{L}}^\bullet \in \text{SD}(X)$, are independent of the choice of Lagrangian structure \mathcal{L} .

Proof. Let $\mathbf{IC}_{\mathcal{L}_0}^\bullet, \mathbf{IC}_{\mathcal{L}_1}^\bullet \in \text{SD}(X)$ be self-dual sheaves, determined by Lagrangian structures $\mathcal{L}_0, \mathcal{L}_1$, respectively. For a normally nonsingular inclusion $j : Y^m \hookrightarrow X^n$ with trivial normal bundle we have $j^! \mathbf{IC}_{\mathcal{L}_0}^\bullet, j^! \mathbf{IC}_{\mathcal{L}_1}^\bullet \in \text{SD}(Y)$ by a straightforward check of axioms (SD1)–(SD4). Thus by Theorem 4.1,

$$\sigma(j^! \mathbf{IC}_{\mathcal{L}_0}^\bullet) = \sigma(j^! \mathbf{IC}_{\mathcal{L}_1}^\bullet).$$

Therefore, the associated L-classes satisfy

$$j^! L_{n-m}(\mathbf{IC}_{\mathcal{L}_0}^\bullet) = \sigma(j^! \mathbf{IC}_{\mathcal{L}_0}^\bullet) = \sigma(j^! \mathbf{IC}_{\mathcal{L}_1}^\bullet) = j^! L_{n-m}(\mathbf{IC}_{\mathcal{L}_1}^\bullet)$$

and so $L_k(\mathbf{IC}_{\mathcal{L}_0}^\bullet) = L_k(\mathbf{IC}_{\mathcal{L}_1}^\bullet)$ for all k by the uniqueness statement of Theorem 5.1. □

6. An example

We illustrate the general result with the special situation of the reductive Borel-Serre compactification of a Hilbert modular surface, following our joint work [BK04] with Rajesh Kulkarni. Let K be a real quadratic number field and \mathcal{O}_K the ring of algebraic integers in K . Consider the Hilbert modular surface $X = (H \times H)/\Gamma$, where H is the upper half plane and $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K)$ the Hilbert modular group. This complex surface is not compact, and various compactifications have been studied. The reductive Borel-Serre compactification — [Zuc82] or [GHM94] — has the advantage that the Hecke operators extend to this compactification. We represent the Hilbert modular surface as

$$X = \Gamma \backslash G(\mathbb{R})/K,$$

with G the algebraic group $G = \mathrm{SO}^\circ(Q)$ (Q an appropriate quadratic form), $G(\mathbb{R})$ the real points of G , K the maximal compact subgroup. The reductive Borel-Serre compactification \bar{X} adds a stratum X_P for each Γ -conjugacy class of parabolic subgroups P of G . It is known that every X_P is topologically a circle and the link L of X_P is a 2-torus. In particular, \bar{X} is a real 4-dimensional pseudomanifold which is not a Witt space. There exists a Lagrangian subspace in $H^1(L)$. An analysis of the monodromy using the theorems of Kostant and Nomizu-van Est shows that there exists a Lagrangian structure along the circle. Thus (Theorem 2.1), we obtain the result that the category $\mathrm{SD}(\bar{X})$ of self-dual sheaves on \bar{X} compatible with intersection chain sheaves is nonempty. Consequently, Theorem 5.2 shows that the reductive Borel-Serre compactification of any Hilbert modular surface has a well-defined L-class $L_i(\bar{X}) \in H_i(\bar{X}; \mathbb{Q})$. This can be established by an alternative argument which uses the Baily-Borel-Satake compactification, as follows: In [BK04], we moreover investigate the relationship between cohomology theories on \bar{X} and on the Baily-Borel-Satake compactification \hat{X} of X . Let $\pi : \bar{X} \rightarrow \hat{X}$ be the canonical (collapse) map. Our result is:

THEOREM 6.1. *If $\mathbf{IC}_{\mathcal{L}}^\bullet \in \mathrm{SD}(\bar{X})$ and*

$$\mathbf{IC}_{\bar{m}}^\bullet(\bar{X}) \xrightarrow{\alpha} \mathbf{IC}_{\mathcal{L}}^\bullet \xrightarrow{\beta} \mathbf{IC}_{\bar{n}}^\bullet(\bar{X})$$

are the canonical morphisms (cf. Section 2), then

- (i) $R\pi_*\alpha$ and $R\pi_*\beta$ are isomorphisms, and
- (ii) $R\pi_*\mathbf{IC}_{\bar{m}}^\bullet(\bar{X}) \cong \mathbf{IC}_{\bar{m}}^\bullet(\hat{X})$. In particular, also $R\pi_*\mathbf{IC}_{\mathcal{L}}^\bullet \cong \mathbf{IC}_{\bar{m}}^\bullet(\hat{X})$ and $R\pi_*\mathbf{IC}_{\bar{n}}^\bullet(\bar{X}) \cong \mathbf{IC}_{\bar{m}}^\bullet(\hat{X})$.

This implies that

$$\sigma(\mathbf{IC}_{\mathcal{L}}^\bullet) = \sigma(R\pi_*\mathbf{IC}_{\mathcal{L}}^\bullet) = \sigma(\mathbf{IC}_{\bar{m}}^\bullet(\hat{X})),$$

and the latter integer does not depend on \mathcal{L} , giving an independent verification of Theorem 4.1 for the reductive Borel-Serre compactification of a Hilbert modular surface.

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