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A preparation theorem for codimension-one foliations

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Dedicated to César Camacho for his 60^{th} *birthday*

Abstract

After gluing foliated complex manifolds, we derive a preparation-like theorem for singularities of codimension-one foliations and planar vector fields (in the real or complex setting). Without computation, we retrieve and improve results of Levinson-Moser for functions, Dufour-Zhitomirskii for nondegenerate codimension-one foliations (proving in turn the analyticity), Stróżyna-Żoładek for non degenerate planar vector fields and Bruno-Écalle for saddle-node foliations in the plane.

Introduction

We denote by $(\underline{z}, w)$ the variable of $\mathbb{C}^{n+1}$, $\underline{z} = (z_1, \ldots, z_n)$, for $n \geq 1$. Recall that a germ of (non-identically vanishing) holomorphic 1-form

$$\Theta = f_1(\underline{z}, w)dz_1 + \cdots + f_n(\underline{z}, w)dz_n + g(\underline{z}, w)dw$$

$f_1, \ldots, f_n, g \in \mathbb{C}\{\underline{z}, w\}$, defines a codimension-1 singular foliation $\mathcal{F}$ (regular outside the zero-set of Θ) if, and only if, it satisfies the Frobenius integrability condition $\Theta \wedge d\Theta = 0$. Maybe after division of coefficients of Θ by a common factor, the zero-set of Θ has codimension-2 and the foliation $\mathcal{F}$ extends as a regular foliation outside this sharp singular set.

Our main result is

Theorem 1. *Let* Θ *and* $\mathcal{F}$ *be as above and assume that* $g(\underline{0}, w)$ *vanishes at the order* $k \in \mathbb{N}^*$ *at* 0. *Then, up to analytic change of the* w*-coordinate* $w := \phi(\underline{z}, w)$, *the foliation* $\mathcal{F}$ *is also defined by a* 1*-form*

$$\widetilde{\Theta} = P_1(\underline{z}, w)dz_1 + \cdots + P_n(\underline{z}, w)dz_n + Q(\underline{z}, w)dw$$

for w*-polynomials* $P_1, \ldots, P_n, Q \in \mathbb{C}\{\underline{z}\}[w]$ *of degree* $\leq k$, Q *monic.*

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In new coordinates given by Theorem 1, the singular foliation $\mathcal{F}$ extends analytically along some infinite cylinder $\{|\underline{z}| < r\} \times \overline{\mathbb{C}}$ (where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ stands for the Riemann sphere). To prove this theorem, we just do the converse. Given a germ of foliation, we force its endless analytic continuation in one direction by constructing it in the simplest way, gluing foliated manifolds into a foliated $\overline{\mathbb{C}}$-bundle. This is done in Section 1. The huge degree of freedom encountered during our construction can be used to preserve additional structure equipping the foliation. For instance, starting with the complexification of a real analytic foliation, our gluing construction can be carried out preserving the anti-holomorphic involution $(\underline{z}, w) \mapsto (\overline{\underline{z}}, \overline{w})$ so that our statement agrees with the real setting. In the same way, if one starts with a closed meromorphic 1-form Θ, one can arrange so that Θ extends meromorphically as well along the infinite cylinder (see Section 2) and becomes itself rational in w. In particular, in the case $\Theta = df$ is exact, we derive a short proof of the following alternate Preparation Theorem.

THEOREM 2 (Levinson). *Let $f(\underline{z}, w)$ be a germ of holomorphic function at $(\underline{0}, 0)$ in $\mathbb{C}^{n+1}$ and assume that $f(\underline{0}, w)$ vanishes at the order $k \in \mathbb{N}^*$ at $w = 0$. Then, up to an analytic change of coordinates, the function germ f becomes a monic w-polynomial of degree k,*

$$f(\underline{z}, w) = w^k + f_{k-1}(\underline{z})w^{k-1} + \cdots + f_0(\underline{z}),$$

where $f_0, \ldots, f_{k-1} \in \mathbb{C}\{\underline{z}\}$.

The difference from the Weierstrass Preparation Theorem lies in the fact that the usual invertible factor term (in variables $(\underline{z}, w)$) is normalized to 1 here; the counterpart is that a change of coordinates is needed. This result was previously obtained by N. Levinson in [8] after an iterative procedure and proved again by J. Moser in [15] as an example illustrating KAM fast convergence. Similarly, we obtain that any germ of a meromorphic function is conjugated to a quotient of Weierstrass w-polynomials (see Theorem 2.1).

For $k = 1$, Theorem 1 reads as follows.

COROLLARY 3. *Let Θ and $\mathcal{F}$ be as in Theorem 1 and assume that the linear part of Θ is not tangent to the radial vector field $\sum_{i=1}^{n} z_i\partial_{z_i} + w\partial_w$. Then, there exist local analytic coordinates $(\underline{z}, w)$ in which the foliation $\mathcal{F}$ is defined by*

$$\widetilde{\Theta} = df_0 + wdf_1 + wdw$$

where $f_0, f_1 \in \mathbb{C}\{\underline{z}\}$ satisfy $df_0 \wedge df_1 = 0$.

Following [12], the functions f_i factor into a primitive function f and the foliation $\mathcal{F}$ is actually the lifting of a foliation in the plane by the holomorphic map $\Phi : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}^2, 0); (\underline{z}, w) \mapsto (f(\underline{z}), w)$. This normal form was obtained in [3] by J.-P. Dufour and M. Zhitomirskii after a formal change of coordinates but the convergence was not proved.

In Theorem 1, the $\overline{\mathbb{C}}$-fibration is constructed simultaneously with the extension of the foliation $\mathcal{F}$ by gluing bifoliated manifolds. In dimension 2, when $\mathcal{F}$ is defined by a vector field X, it is still possible to extend X on a 2-dimensional tubular neighborhood M of an embedded sphere $\overline{\mathbb{C}}$ but it is not possible to construct the $\overline{\mathbb{C}}$-fibration at the same time. Here, we need the Rigidity Theorem of V. I. Savelev [17] (see also [21]): *the germ of a 2-dimensional neighborhood of an embedded sphere having zero self-intersection is a trivial $\overline{\mathbb{C}}$-bundle over the disc.* In Section 3, we derive, for nondegenerate singularities of vector fields

THEOREM 4. *Let X be a germ of an analytic vector field vanishing at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$). Assume that its linear part is not radial, i.e. not of the form $\lambda(x\partial_x + y\partial_y)$, $\lambda \in \mathbb{C}$. Then, there exist local analytic coordinates (x, y) in which*

$$X = (y + f(x))\partial_x + g(x)\partial_y$$

where $f, g \in \mathbb{R}\{x\}$ (resp. $f, g \in \mathbb{C}\{x\}$) vanish at 0.

Denote by $\lambda_1, \lambda_2 \in \mathbb{C}$ the eigenvalues of the vector field X: we have $\lambda_1 + \lambda_2 = f'(0)$ and $\lambda_1 \cdot \lambda_2 = -g'(0)$. In the case $\lambda_2 = -\lambda_1$ (including the nilpotent case $\lambda_i = 0$), Theorem 4 was obtained by E. Stróżyna and H. Żoładek [19]. They proved the convergence of an explicit iterative reduction process after long and technical estimates. In the case $\lambda_2/\lambda_1 \notin \mathbb{R}^-$, Theorem 4 becomes just useless since H. Poincaré and H. Dulac gave a unique and very simple polynomial normal form. In the remaining case, taking into account the invariant curve of the vector field X, we can specify our normal form as follows (see Section 3 for a statement including nilpotent singularities).

COROLLARY 5. *Let X be a germ of an analytic vector field in the real or complex plane with eigenratio $\lambda_2/\lambda_1 \in \mathbb{R}^-$. Then, there exist local analytic coordinates in which the vector field X takes the forms:*

(1) *In the saddle case $\lambda_2/\lambda_1 \in \mathbb{R}^-_*$ (with $\lambda_1, \lambda_2 \in \mathbb{R}$ in the real case),*

$$X = f(x+y)\left\{(\lambda_1 x\partial_x + \lambda_2 y\partial_y) + g(x+y)(x\partial_x + y\partial_y)\right\}.$$

(2) *In the saddle-node case, say $\lambda_2 = 0$, $\lambda_1 \neq 0$,*

$$X = f(x)\left\{(\lambda_1 x + y)\partial_x + g(x)y\partial_y\right\}.$$

(3) *In the real center case $\lambda_2 = -\lambda_1 = i\lambda$, $\lambda \in \mathbb{R}$,*

$$X = f(x)\left\{(-\lambda y\partial_x + \lambda x\partial_y) + g(x)(x\partial_x + y\partial_y)\right\}.$$

In each case, $f(0) = 1$ and $g(0) = 0$.

The orbital normal form (i.e. the normal form for the induced foliation) can be immediately derived just by setting $f \equiv 1$: coefficient g stands for the moduli of the foliation. The normal form (3) was also derived in [19].

In case (1), A. D. Bruno proved in [1] that the vector field X is actually analytically linearisable for generic eigenratio $\lambda_2/\lambda_1 \in \mathbb{R}^-$ (in the sense of the Lebesgue measure). In this case, normal form (1) of Corollary 5 becomes just useless. For the remaining exceptional values, the respective works of J.-C. Yoccoz in the diophantine case (see [22] and [16]) and J. Martinet with J.-P. Ramis in the resonant case $\lambda_2/\lambda_1 \in \mathbb{Q}^-$ (see [11]) derive a huge moduli space for the analytic classification of the induced foliations. This suggests that most of the vector fields having such eigenvalues are not polynomial in any analytic coordinates. Moreover, at least in the resonant case, the analytic classification of all vector fields inducing a given foliation gives rise to functional moduli as well (see [7], [13] and [20]). Thus, the functional parameters f and g appearing in our normal form seem necessary in many cases.

Finally, one can shortly derive from (2) a *versal deformation*

$$X_f = x\partial_x + y^2\partial_y + yf(x)\partial_x, \quad f \in \mathbb{C}\{x\},$$

of the saddle-node foliation $\mathcal{F}_0$ defined by $X_0 = x\partial_x + y^2\partial_y$ (see [10]). In other words, any germ of analytic deformation of X_0 without bifurcation of the saddle-node point factor into the family above after analytic change of coordinates and renormalization. Moreover, the derivative of Martinet-Ramis' moduli map at X_0 (see [5]) is bijective. When $f(0) = 0$, one can even show that the form above is unique. This result was announced by A. D. Bruno in [2] and proved by J. Écalle at the end of [4] using *mould theory* in the particular case $f'(0) = 0$. We will detail it in a forthcoming paper [10].

1. Preparation theorem for codimension-1 foliations

We first prove Theorem 1. Let $\mathcal{F}_0$ denote the germ of singular foliation defined by an integrable holomorphic 1-form at $(\underline{0}, 0) \in \mathbb{C}^{n+1}$:

$$\Theta_0 = f_1(\underline{z}, w)dz_1 + \cdots + f_n(\underline{z}, w)dz_n + g(\underline{z}, w)dw, \quad \Theta_0 \wedge d\Theta_0 = 0,$$

$f_1, \ldots, f_n, g \in \mathbb{C}\{\underline{z}, w\}$ and assume $g(\underline{0}, w) \not\equiv 0$. In particular, for $r > 0$ small enough, the foliation $\mathcal{F}_0$ is well-defined on the vertical disc $\Delta_0 = \{\underline{0}\} \times \{|w| < r\}$, regular and transversal to Δ_0 outside $w = 0$.

Consider in $\mathbb{C}^n \times \overline{\mathbb{C}}$ the vertical line $L = \{\underline{0}\} \times \overline{\mathbb{C}}$ together with the covering given by Δ_0 and another disc, say $\Delta_\infty = \{\underline{0}\} \times \{|w| > r/2\}$. Denote by $C = \Delta_0 \cap \Delta_\infty$ the intersection corona. By the flow-box theorem, there exists a unique germ of a diffeomorphism of the form

$$\Phi : (\mathbb{C}^{n+1}, C) \to (\mathbb{C}^{n+1}, C) \ ; \ (\underline{z}, w) \mapsto (\underline{z}, \phi(\underline{z}, w)), \quad \phi(\underline{0}, w) = w$$

conjugating $\mathcal{F}_0$ to the horizontal foliation $\mathcal{F}_\infty$ (defined by $\Theta_\infty = dw$) at the neighborhood of the corona C. Therefore, after gluing the germs of complex manifolds $(\mathbb{C}^n \times \overline{\mathbb{C}}, \Delta_0)$ and $(\mathbb{C}^n \times \overline{\mathbb{C}}, \Delta_\infty)$ along the corona by means of Φ, we obtain a germ of a smooth complex manifold M, $\dim(M) = n+1$, along a rational curve L equipped with a singular holomorphic foliation $\mathcal{F}$. Moreover, the coordinate $\underline{z}$, which is invariant under the gluing map Φ, defines a germ of a rational fibration $\underline{z} : (M, L) \to (\mathbb{C}^n, \underline{0})$. By [6], there exists a germ of submersion $w : (M, L) \to L \simeq \overline{\mathbb{C}}$ completing $\underline{z}$ into a system of trivializing coordinates $(\underline{z}, w) : (M, L) \to (\mathbb{C}^n, \underline{0}) \times \overline{\mathbb{C}}$. This system is unique up to permissible change

$$(\tilde{\underline{z}}, \tilde{w}) = \left(\phi(\underline{z}), \frac{a(\underline{z})w + b(\underline{z})}{c(\underline{z})w + d(\underline{z})}\right)$$

where $a, b, c, d \in \mathbb{C}\{\underline{z}\}$, $ad - bc \not\equiv 0$, and $\phi \in \mathrm{Diff}(\mathbb{C}^n, \underline{0})$.

In the neighborhood of any point $p \in L$, the foliation $\mathcal{F}$ is defined by a (nonunique) germ of a holomorphic 1-form (respectively Θ_0 or Θ_∞). After division by the coefficient of dw, $\mathcal{F}$ is equivalently defined by a germ of a meromorphic 1-form

$$\Theta = R_1(\underline{z}, w)dz_1 + \cdots + R_n(\underline{z}, w)dz_n + dw,$$

where R_i are meromorphic at p. This normal form is unique and Θ is therefore globally defined on the neighborhood of L. In restriction to each rational fiber $\{\underline{z} = \text{constant}\}$, R_i is a global meromorphic function, thus a rational function by Chow's theorem. In other words, the functions R_i are actually rational in the variable w; i.e. all coefficients R_i are quotients of Weierstrass polynomials.

Choose trivializing coordinates $(\underline{z}, w)$ so that the singular point of $\mathcal{F}$ is still located at $w = 0$. The poles of Θ correspond to tangencies between the foliation $\mathcal{F}$ and the rational fibration (counted with multiplicity). Denote by Σ this divisor. Since $\mathcal{F}_\infty$ is transversal to the rational fibration, those poles come from the first chart, namely from the corresponding tangency divisor

$$\Sigma_0 = \{g(\underline{z}, w) = 0\}.$$

By assumption, the total number of tangencies between $\mathcal{F}$ (or $\mathcal{F}_0$) and a fibre (close to L) is k. It follows that the w-rational coefficients R_i have exactly k poles (counted with multiplicity) in restriction to each fiber. Therefore, if Q denotes the monic w-polynomial of degree k defining Σ and if one lets $R_i = \frac{P_i}{Q}$ for w-polynomials P_i, the transversality of $\mathcal{F}$ with the fibration at $\{w = \infty\}$ implies that the P_i's have at most degree $k+2$ in the variable w. Equivalently, $\mathcal{F}$ is defined by

$$\widetilde{\Theta} = \theta_0 + \theta_1 w + \cdots + \theta_{k+2} w^{k+2} + Q(\underline{z}, w)dw$$

for evident 1-forms $\theta_0, \theta_1, \ldots, \theta_k$ on $(\mathbb{C}^n, \underline{0})$ (depending only on $\underline{z}$).

After a permissible change of the w-coordinate, one may assume that the line $\{w = \infty\}$ at infinity is a leaf of the foliation (just straighten one

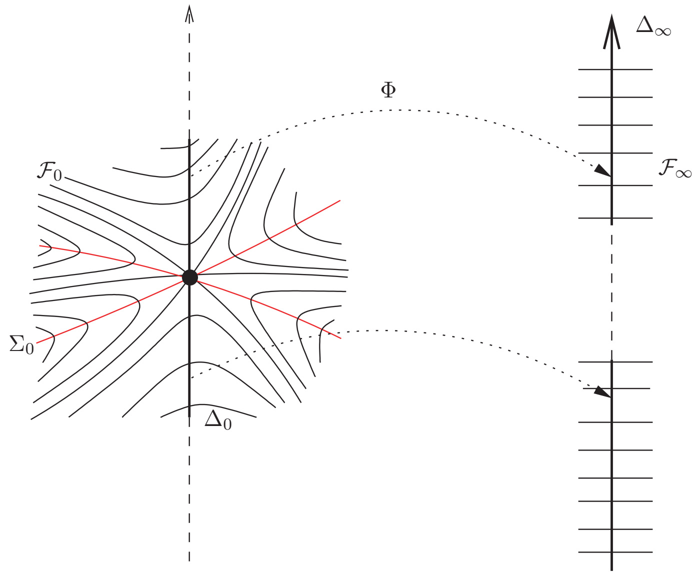

Figure 1: Gluing construction

leaf); i.e., $\theta_{k+2} = 0$. In fact, one may furthermore assume that the contact between $\mathcal{F}$ and the horizontal fibration $\{w = \text{constant}\}$ along the line $\{w = \infty\}$ has multiplicity 2 (there is no linear holonomy along this leaf in the w-coordinate). Indeed, the change of coordinate $\tilde{w} = e^{-\int \theta_{k+1}} w$ (θ_{k+1} is closed by the integrability condition $\widetilde{\Theta} \wedge d\widetilde{\Theta} = 0$). In new coordinates, $\theta_{k+1} = 0$ and Theorem 1 is proved. Notice that we can further simplify the form $\widetilde{\Theta}$ by using the remaining possible changes of coordinates $\underline{\tilde{z}} = \phi(\underline{z})$ and $\tilde{w} = w + b(\underline{z})$.

We now prove Corollary 3. According to the begining of the proof above, if the linear part of Θ_0 is not tangent to the radial vector field, up to a linear change of coordinates, one may assume that the tangency set $\Sigma_0 = \{g(\underline{z}, w) = 0\}$ between the foliation $\mathcal{F}_0$ and the vertical fibration $\{\underline{z} = \text{constant}\}$ is smooth and transverse to the fibration. By the assumption of Theorem 1 with $k = 1$, up to a change of the w-coordinate, one may assume that $\mathcal{F}$ is defined by $\widetilde{\Theta} = \theta_0 + w\theta_1 + (w + f(\underline{z}))dw$ where θ_0 and θ_1 are holomorphic 1-forms depending only on the $\underline{z}$-variable and $f \in \mathbb{C}\{\underline{z}\}$. After translation $w := w + f(\underline{z})$ (notice that $f(\underline{0}) = 0$), one may assume furthermore that $f \equiv 0$ and the integrability condition $\widetilde{\Theta} \wedge d\widetilde{\Theta} = 0$ yields

$$\theta_0 \wedge \theta_1 = 0, \quad d\theta_0 = 0 \quad \text{and} \quad d\theta_1 = 0.$$

After integration, we obtain $\theta_i = df_i$ for functions $f_i \in \mathbb{C}\{\underline{z}\}$ with the tangency condition $df_0 \wedge df_1 = 0$; Corollary 3 is proved. By [12], there exists a primitive function $f \in \mathbb{C}\{\underline{z}\}$ (with connected fibres) through which f_0 and f_1 factor: $f_i = \tilde{f}_i \circ f$ with $\tilde{f}_i \in \mathbb{C}\{z\}$, z a single variable. Notice that we can further

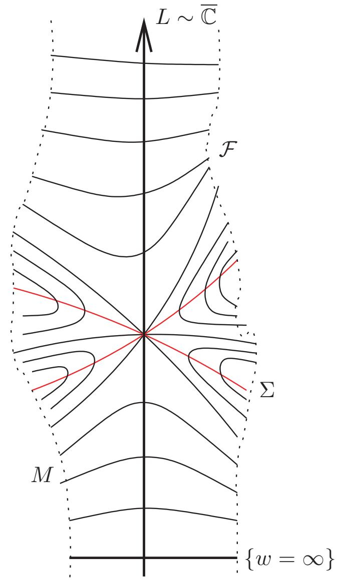

Figure 2: Uniformisation

simplify the form $\widetilde{\Theta}$ by using the remaining possible changes of coordinate $\underline{\tilde{z}} = \phi(\underline{z})$.

If we start with a real analytic foliation $\mathcal{F}_0$, then its complexification is invariant under the anti-holomorphic involution $(\underline{z}, w) \mapsto (\overline{\underline{z}}, \overline{w})$. This involution obviously commutes with $\mathcal{F}_\infty$ and with the gluing map Φ, defining, this way, a germ of anti-holomorphic involution $\Psi : (M, L) \to (M, L)$ on the resulting manifold preserving $\mathcal{F}$. By restriction to the coordinate $\overline{z}$, which is invariant under Φ and well defined on M, Ψ induces the standard involution $\underline{z} \mapsto \overline{\underline{z}}$. Therefore, $\Psi(\underline{z}, w) = (\overline{\underline{z}}, \psi(\underline{z}, w))$ where $\psi(\underline{z}, w)$ is, for fixed $\underline{z}$, a reflection with respect to a real circle. After a holomorphic change of w-coordinate, $\psi(\underline{z}, w) = \overline{w}$ and the constructed foliation $\mathcal{F}$ is actually invariant by the standard involution. The unique meromorphic 1-form defining $\mathcal{F}$,

$$\Theta = R_1(\underline{z}, w)dz_1 + \cdots + R_n(\underline{z}, w)dz_n + dw,$$

satisfies $\Psi^*\Theta = \overline{\Theta}$ and its coefficients are actually real: $R_i \in \mathbb{R}\{\underline{z}\}(w)$. This real form is obtained up to a global change of coordinates commuting with the standard involution; that is,

$$(\underline{\tilde{z}}, \tilde{w}) = \left(\phi(\underline{z}), \frac{a(\underline{z})w + b(\underline{z})}{c(\underline{z})w + d(\underline{z})}\right)$$

where $a, b, c, d \in \mathbb{R}\{\underline{z}\}$, $ad - bc \not\equiv 0$, and $\phi \in \mathrm{Diff}(\mathbb{R}^n, \underline{0})$.

2. Preparation theorem for closed meromorphic 1-forms

For simplicity, we start with the case of (meromorphic) functions:

THEOREM 2.1. *Let f be a germ of a meromorphic function at $(\underline{0}, 0)$ in $\mathbb{C}^{n+1}$ and assume that $f(\underline{0}, w)$ is a well-defined and non constant germ of a meromorphic function. Then, up to analytic change of the w-coordinate $w := \phi(\underline{z}, w)$, the function f becomes a w-rational function*

$$f(\underline{z}, w) = \frac{f_0(\underline{z}) + f_1(\underline{z})w + \cdots + f_{k_0-1}(\underline{z})w^{k_0-1} + w^{k_0}}{g_0(\underline{z}) + g_1(\underline{z})w + \cdots + g_{k_\infty-1}(\underline{z})w^{k_\infty-1} + w^{k_\infty}}$$

where $k_0, k_\infty \in \mathbb{N}$ and $f_i, g_j \in \mathbb{C}\{\underline{z}\}$.

Proof. Denote by $f_0(\underline{z}, w)$ the germ of a meromorphic function above and make a preliminary change of coordinate $\tilde{w} := \varphi(w)$ such that $f_0(\underline{0}, w) = w^l$, $l \in \mathbb{Z}^*$, or $1 + w^l$, $l \in \mathbb{N}^*$. Then, proceed with the underlying foliation $\mathcal{F}_0$ (defined by $f_0 = $ constant) as in the proof of Theorem 1 in Section 1. By construction, the function f_0 will glue automatically with the respective function $f_\infty(\underline{z}, w) = w^l$ or $1 + w^l$ defining $\mathcal{F}_\infty$. Therefore, the global foliation $\mathcal{F}$ is actually defined by a global meromorphic function f on M. Again, f is a quotient of Weierstrass polynomials. In the case $f_0(\underline{0}, w) = w^l$, choose the w-coordinate such that the zero or pole of $f_\infty(\underline{z}, w) = w^l$ still coincides with $\{w = \infty\}$. Therefore, k_0 and k_∞ respectively coincide with the number of zeroes and poles of f_0 restricted to a generic vertical line (close to L). In the other case $f_0(\underline{0}, w) = 1 + w^l$, we add l simple zeroes in the finite part and a pole of order l that can be straightened to $\{w = \infty\}$ as before. In this latter case, $l = k_0 - k_\infty > 0$ and k_∞ is the number of (zeroes or) poles of $f_0(\underline{z}, w)$ restricted to a generic vertical line. In any case, the leading terms f_{k_0} and g_{k_∞} are nonvanishing at $\underline{z} = \underline{0}$ and can be normalized to 1 by division and a further change of coordinate $\tilde{w} = a(\underline{z})w$. □

The proof of Theorem 2 immediately follows when we set $k = k_0 > 0$ and $k_\infty = 0$ in the proof above.

PROPOSITION 2.2. *Let Θ be a germ of a closed meromorphic 1-form at $(\underline{0}, 0) \in \mathbb{C}^{n+1}$ and assume that the vertical line $\{\underline{z} = \underline{0}\}$ is not invariant by the induced foliation. Then, up to analytic change of the w-coordinate $w := \phi(\underline{z}, w)$, the closed form Θ takes the form*

$$\Theta = \frac{P_1(\underline{z}, w)dz_1 + \cdots + P_n(\underline{z}, w)dz_n + P(\underline{z}, w)dw}{Q(\underline{z}, w)}$$

for w-polynomials $P, Q, P_1, \ldots, P_n \in \mathbb{C}\{\underline{z}\}[w]$.

Proof. By a preliminary change of the w-coordinate, one can normalize the restriction of Θ to the vertical line into one of the models

$$\Theta|_L = w^k dw \text{ if } k \geq 0,$$
$$\Theta|_L = \lambda \frac{dw}{w} \text{ if } k = -1,$$
$$\Theta|_L = \lambda \frac{dw}{w^k(1-w)} \text{ if } k < -1,$$

where $k \in \mathbb{Z}$ stands for the order of $\Theta|_L$ at $w = 0$ and $\lambda \in \mathbb{C}$ denotes the residue when $k \leq -1$. Then, defining the horizontal foliation $\mathcal{F}_\infty$ by the corresponding model Θ_∞ above (viewed as a 1-form in variables $(\underline{z}, w)$), we proceed gluing the foliations and the 1-forms as we did with functions in the previous proof. If k_0 and k_∞ denote the respective number of zeroes and poles of Θ_0 in restriction to a generic vertical line, then the numerator and denominator have respective degrees k_0 and k_∞ if $k_0 - k_\infty \geq -1$ and k_0 and $k_\infty + 1$ if $k_0 - k_\infty < -1$. □

3. Nondegenerate vector fields in the plane

We prove Theorem 4 and deduce Corollary 5. Let X_0 be a germ of an analytic vector field at $(0,0) \in \mathbb{C}^2$,

$$X_0 = f(z,w)\partial_z + g(z,w)\partial_w,$$

vanishing at $(0,0)$ with a nonradial linear part:

$$\mathrm{lin}(X_0) = (az + bw)\partial_z + (cz + dw)\partial_w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

(in particular, it is assumed that the linear part is not the zero matrix). One can find linear coordinates in which

$$\mathrm{lin}(X_0) = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} + \cdots$$

where $-\alpha$ and β respectively stand for the product and the sum of the eigenvalues λ_1 and λ_2. The eigenvector corresponding to λ_i is $(1, \lambda_i)$; in the case $\lambda_1 = \lambda_2$, we note that the matrix above is not diagonal. After a change of the w-coordinate of the form $w := \varphi(w)$, we may assume that restriction of $f(z,w)$ to the vertical line $\{z = 0\}$ takes the form $f(0,w) = w$. Similarly, to the proof of Theorem 1 in Section 1, we consider in $\mathbb{C} \times \overline{\mathbb{C}}$ the vertical line $L = \{0\} \times \overline{\mathbb{C}}$ together with the covering given by

$$\Delta_0 = \{0\} \times \{|w| < r\} \quad \text{and} \quad \Delta_\infty = \{0\} \times \{|w| > r/2\}.$$

Also we denote by $C = \Delta_0 \cap \Delta_\infty$ the intersection corona.

If $r > 0$ is small enough, the vector field X_0 is well defined on the neighborhood of the closed disc $\overline{\Delta_0}$ and transverse to it outside $w = 0$. By the rectification theorem, there exists a unique germ of a diffeomorphism of the form

$$\Phi : (\mathbb{C}^2, C) \to (\mathbb{C}^2, C), \quad \Phi(0, w) = (0, w)$$

conjugating X_0 to the horizontal vector field $X_\infty = w\partial_z$. After gluing the germs of complex surfaces $(\mathbb{C} \times \overline{\mathbb{C}}, \Delta_0)$ and $(\mathbb{C} \times \overline{\mathbb{C}}, \Delta_\infty)$ along the corona by means of Φ, we obtain a germ of smooth complex surface M along a rational curve L equipped with a meromorphic vector field X. Since the ∂_z-component of X_0 agrees with $w\partial_z$ along L, it follows that the Jacobian of the gluing map Φ takes the form

$$D_{(0,w)}\Phi = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$$

and the embedded rational curve L has zero self-intersection. Following [17], we see that L is the regular fiber of a germ of a trivial fibration on M, i.e. there exist global coordinates $(z, w) : (M, L) \to (\mathbb{C}, \underline{0}) \times \overline{\mathbb{C}}$ sending L onto $\{z = 0\}$. The vector field X has exactly one isolated zero, say $(z, w) = (0, 0)$, and a simple pole along a trajectory (given by $X_\infty = w\partial_z$ in the second chart) that we may assume still given by $\{w = \infty\}$. The tangency divisor Σ between the induced foliation $\mathcal{F}$ and the fibration $\{z = \text{constant}\}$ still is a smooth curve intersecting the fiber $\{z = 0\}$ at the singular point $(z, w) = (0, 0)$ without multiplicity. Indeed, the Jacobian of the change of coordinates (from the first chart to the global coordinates) at the singular point fixes the w-direction, so that the linear part of the vector field takes the form

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \cdots, \quad b \neq 0.$$

As in the proof of Theorem 1, one may choose the (global) w-coordinate so that the foliation has a contact of order 2 with the horizontal foliation $\{w = \text{constant}\}$ along the polar trajectory $\{w = \infty\}$ and the tangency set $\Sigma = \{w = 0\}$ is horizontal as well. Therefore, the vector field X is written

$$X = f(z)w\partial_z + (g_0(z) + wg_1(z))\partial_w$$

for germs $f, g_0, g_1 \in \mathbb{C}\{z\}$. Indeed, the coefficients of $X = P(z, w)\partial_z + Q(z, w)\partial_w$ become automatically rational in the w-variable. Since the unique pole of X is simple and located at $\{w = \infty\}$, P and Q are in fact polynomials of maximal degree 1 and 3 (notice that ∂_w has a double zero at $\{w = \infty\}$). Finally, conditions on tangency and polar sets imply the special form above.

By a change of z-coordinate, we may furthermore assume $f(z) \equiv 1$ ($f(0) = 1 \neq 0$). Automatically, the linear part of X in the new coordinates is

$$X = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} + \cdots$$

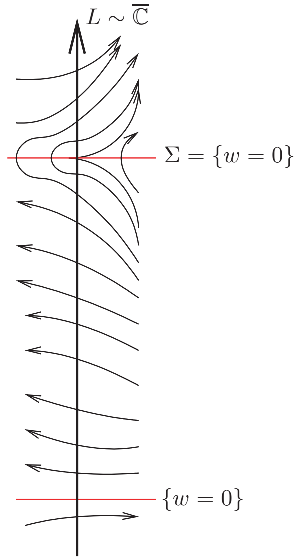

Figure 3: Normalisation

i.e. $g_0(z) = \alpha z + \cdots$ and $g_1(z) = \beta + \cdots$ where dots mean higher order terms. Finally, the form $X = (w + f(z))\partial_z + g(z)\partial_w$ is derived after the last change of coordinate $\tilde{w} := w - f$ where $f'(z) = g_1(z)$, $f(0) = 0$.

If we start with a real vector field X_0, then the anti-holomorphic involution $(z, w) \mapsto (\overline{z}, \overline{w})$ commutes with the gluing map Φ (mind that $X_\infty = w\partial_z$ is also real) and induces a germ of anti-holomorphic involution $\Psi : (M, L) \to (M, L)$ on the resulting surface satisfying $\overline{\Psi^* X} = X$. By Blanchard's argument, Ψ permutes the rational fibration: for any line L' close to L, the restriction of y along the image $\Psi(L')$ is an anti-holomorphic map from a compact manifold into a bounded domain; therefore, $y|_{\Psi(L')}$ is constant and $\Psi(L')$ is actually a fiber of y. In restriction to the coordinate z, Ψ is a regular anti-holomorphic involution and is obviously holomorphically conjugated to the standard one $z \mapsto \overline{z}$. Finally, after holomorphic change of w-coordinate, $\Psi(z, w) = (\overline{z}, \overline{w})$ and X has real coefficients.

Corollary 3.1. *Let X be a germ of an analytic vector field as in Theorem 4. Then, by a further change of (complex or real) analytic coordinates, one of the following cases holds*:

(1) *X has an invariant curve of the form $C : \{w^2 - z^k = 0\}$ and*

$$X = f(z)(2w\partial_z + kz^{k-1}\partial_w) + g(z)z^l(2z\partial_z + kw\partial_w), \quad l + 1 \geq \frac{k}{2} \geq 1,$$

(2) *X has an invariant curve of the form $C : \{w = 0\}$ and*

$$X = f(z)(w + z^k)\partial_z + g(z)z^l w\partial_w, \quad l + 1 \geq k \geq 1,$$

(3) *X is a real center or focus and*

$$X = f(z)(-w\partial_z + kz^{2k-1}\partial_w) + g(z)z^l(z\partial_z + kw\partial_w), \quad l+1 \geq k \geq 1,$$

where, in every case, $f(0) \neq 0$.

Saddles and saddle-nodes respectively correspond to cases 1 and 2. For a complete discussion on the possible invariant curve, we refer to the preliminary version [9, §7] of this paper.

Proof of Corollaries 5 *and* 3.1. We go back to the preliminary form

$$X = w\partial_z + (g_0(z) + g_1(z)w)\partial_w$$

(see proof of Theorem 4). Following [14] (see also [9]), the foliation $\mathcal{F}$ either admit an invariant curve of the form $C : \{w^2 + a(x)w + b(x) = 0\}$, where $a(z)$ and $b(z)$ are (real or complex) analytic functions vanishing at 0, or admit a smooth (real or complex) analytic invariant curve transversal to the fibration $\{w = \text{constant}\}$. We want to simplify this invariant curve by a change of coordinates of the form $(z, w) := (\varphi(z), w+\phi(z))$. The vector field will therefore take the more general form

$$X = (f_0(z) + f_1(z)w)\partial_z + (g_0(z) + g_1(z)w)\partial_w.$$

In the former case, the invariant curve is a 2-fold covering of the z-variable. One can use a vertical translation $w := w + \phi(z)$ so that C becomes invariant by the involution $(z, w) \mapsto (z, -w)$, i.e. $C = \{w^2 = \tilde{b}(z)\}$. Then, by a change of the z-coordinate, one can normalize $\tilde{b}(z) = z^k$ (or $\tilde{b}(z) = -z^k$ when k is even in the real setting). In these new coordinates, letting each of the vector fields $X \pm i_*X$ vanish identically along the curve $t \mapsto (t^k, t^2)$, we deduce that X takes the form (1) (or (3) when k is even in the real setting) of Corollary 3.1.

In the saddle case, we have $k = 2$. We set $f(z) := \frac{g_0(z)}{g_0(0)}$ and $g(z) := g_1(z) - \frac{g_1(0)}{g_0(0)}g_0(z)$ so that $f(0) = 1$, $g(0) = 0$ and the vector field X is written

$$X = f(z)X_1 + g(z)(z\partial_z + w\partial_w) \quad \text{with} \quad X_1 = \begin{pmatrix} g_1(0) & g_0(0) \\ g_0(0) & g_1(0) \end{pmatrix}$$

$(g_0(0) = \pm(\lambda_2 - \lambda_1) \neq 0)$. Finally, after a rotation $(z, w) := (z - w, z + w)$, we obtain normal form (1) of Corollary 5 for saddles.

In the case $\mathcal{F}$ admits a smooth analytic invariant curve transverse to the fibration $\{w = \text{constant}\}$, we first use a vertical translation $w := w + \phi(z)$ to straighten it onto the horizontal axis and then use a change of z-coordinate to send the tangency set Σ between the foliation $\mathcal{F}$ and the vertical fibration onto the line $\{w = z\}$. We immediately obtain normal form (2) of Corollary 3.1 (resp. of Corollary 5 in the saddle-node case $k = 1$). □

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References

[1] A. D. Bruno, The analytical form of differential equations, *Trans. Moscow Math. Soc.* **25** (1971), 131–288; **26** (1972), 199–239.

[2] ———, Analytic invariants of a differential equation, *Soviet Math. Dokl.* **28** (1983), 691–695.

[3] J. P. Dufour and M. Zhitomirskii, Singularities and bifurcations of 3-dimensional Poisson structures, *Israel J. Math.* **121** (2001), 199–220; Nambu structures and integrable 1-forms, *Letters Math. Phys.* **66** (2002), 1–13.

[4] J. Écalle, Les fonctions résurgentes. Tome III. L'équation du pont et la classification analytique des objects locaux, Pub. Math. d'Orsay, 85–5, Université de Paris-Sud, Orsay (1985).

[5] P. M. Elizarov, Tangents to moduli maps, in *Nonlinear Stokes Phenomena*, *Adv. Soviet Math.* **14**, 107–138, A. M. S., Providence, RI (1993).

[6] W. Fischer and H. Grauert, Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten, *Nachr. Akad. Wiss. Göttingen Math.-Phys.* **II** (1965), 89–94.

[7] A. A. Grintchy and S. M. Voronin, An analytic classification of saddle resonant singular points of holomorphic vector fields in the complex plane, *J. Dynam. Control Systems* **2** (1996), 21–53.

[8] N. Levinson, A canonical form for an analytic function of several variables at a critical point, *Bull. Amer. Math. Soc.* **66** (1960), 68–69; Transformation of an analytic function of several variables to a canonical form, *Duke Math. J.* **28**8 (1961), 345–353.

[9] F. Loray, Analytic normal forms for nondegenerate singularities of planar vector fields, Preprint C.R.M. 545, Bellaterra (2003); http://www.crm.es/Publications/preprints2003.html.

[10] ———, Versal deformation of the analytic saddle-node, in *Analyse Complexe, Systèmes Dynamiques, Sommabilité des Séries Divergentes et Théories Galoisiennes* (II), Volume en l'honneur de Jean-Pierre Ramis (Michèle Loday-Richaud, ed.), *Astérisque* **297** (2004), 167–187.

[11] J. Martinet and J.-P. Ramis, Problèmes de modules pour des équations différentielles non linéaires du premier ordre, *Inst. Hautes Études Sci. Publ. Math.* **55** (1982), 63–164; Classification analytique des équations différentielles non linéaires résonnantes du premier ordre, *Ann. Sci. École Norm. Sup.* **16** (1983), 571–621.

[12] J.-F. Mattei and R. Moussu, Holonomie et intégrales premières, *Ann. Sci. École Norm. Sup.* **13** (1980), 469–523.

[13] Yu. I. Meshcheryakova and S. M. Voronin, Analytic classification of generic degenerate elementary singular points of germs of holomorphic vector fields on the complex plane (Russian), *Izv. Vyssh. Uchebn. Zaved. Mat.* **1** (2002), 13–16.

[14] R. Meziani, Classification analytique d'équations différentielles $y\,dy + \cdots = 0$ et espace de modules, *Bol. Soc. Brasil. Mat.* **27** (1996), 23–53.

[15] J. Moser, A rapidly convergent iteration method and nonlinear differential equations. II, *Ann. Scuola Norm. Sup. Pisa* **20** (1966), 499–535.

[16] R. Pérez Marco and J.-C. Yoccoz, Germes de feuilletages holomorphes à holonomie prescrite, in *Complex Analytic Methods in Dynamical Systems* (Rio de Janeiro, 1992), *Astérisque* **222** (1994), 345–371.

[17] V. I. Savelev, Zero-type imbedding of a sphere into complex surfaces, *Moscow Univ. Math. Bull.* **37** (1982), 34–39.

[18] E. Stróżyna, The analytic and formal normal form for the nilpotent singularity, The case of generalized saddle-node, *Bull. Sci. Math.* **126** (2002), 555–579.

[19] E. Stróżyna and H. Żoładek, The analytic and formal normal form for the nilpotent singularity, *J. Differential Equations* **179** (2002), 479–537.

[20] L. Teyssier, Analytical classification of singular saddle-node vector fields, *J. Dynam. Control Systems* **10** (2004), 577–605.

[21] T. Ueda, On the neighborhood of a compact complex curve with topologically trivial normal bundle, *J. Math. Kyoto Univ.* **22** (1982/83), 583–607.

[22] J.-C. Yoccoz, Linéarisation des germes de difféomorphismes holomorphes de $(C, 0)$, *C. R. Acad. Sci. Paris Sér.* I *Math.* **306** (1988), 55–58.

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